From Poincaré to Whittaker to Ford

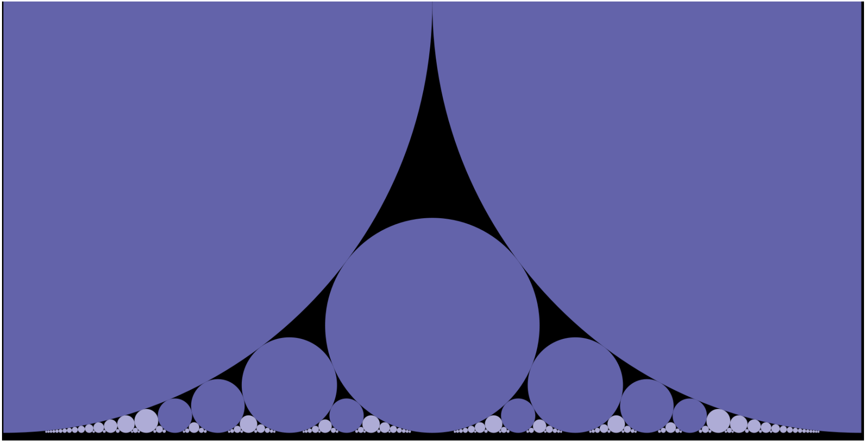
John Stillwell

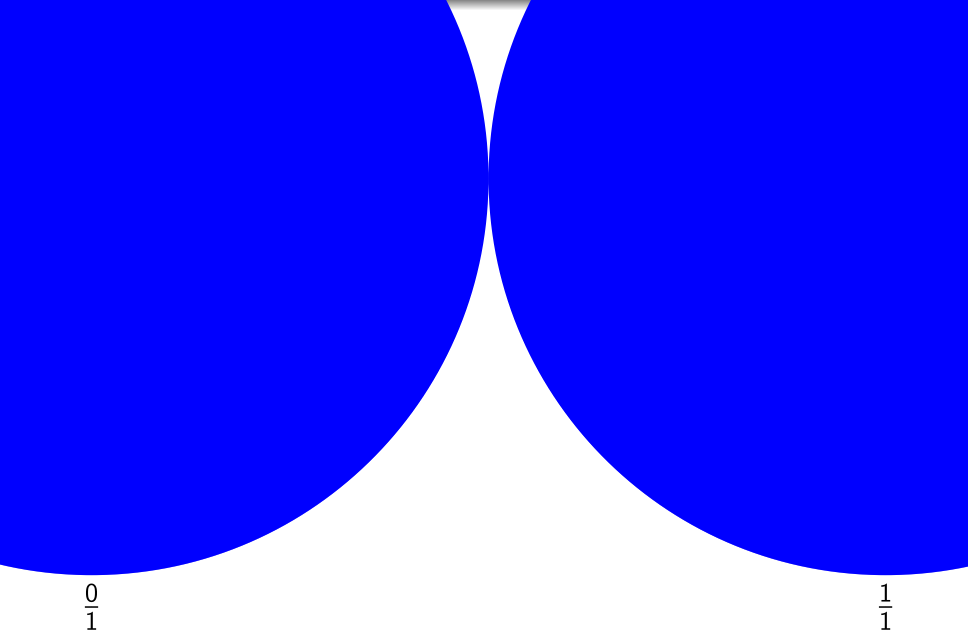
University of San Francisco

May 22, 2012

Ford circles

Here is a picture, generated from two equal tangential circles and a tangent line, by repeatedly inserting a maximal circle in the space between two tangential circles and the line.

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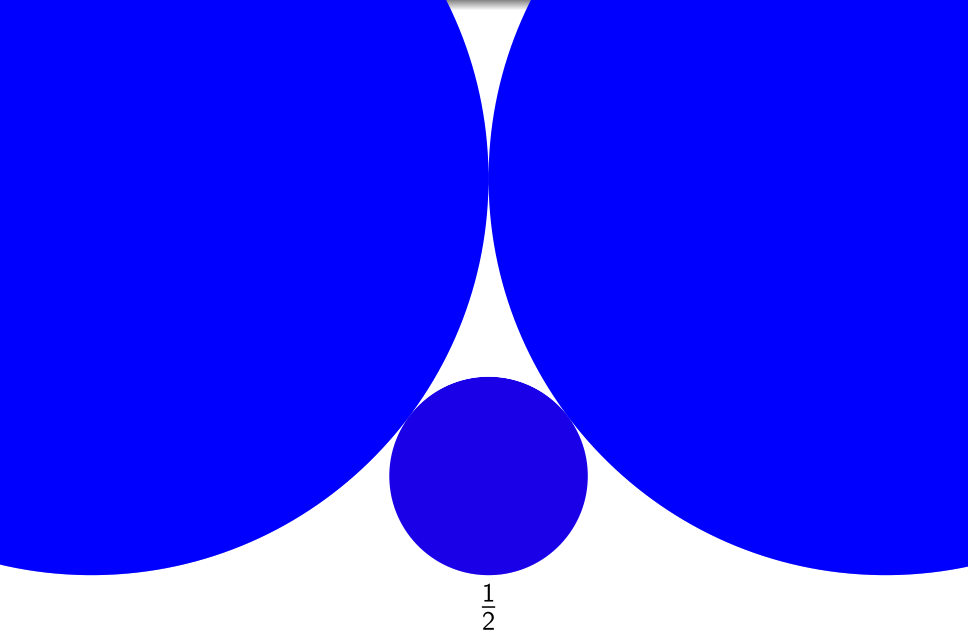
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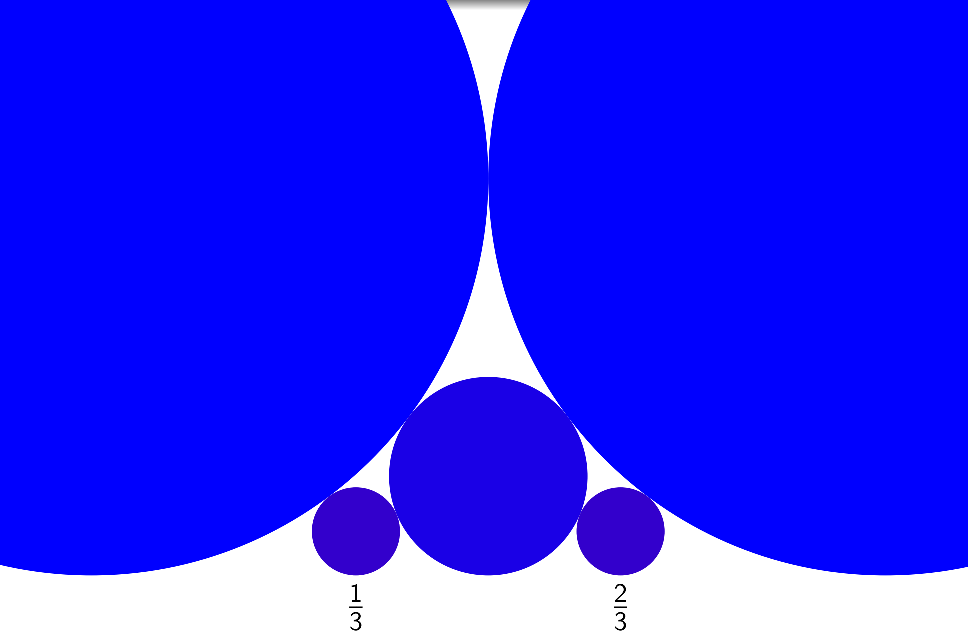
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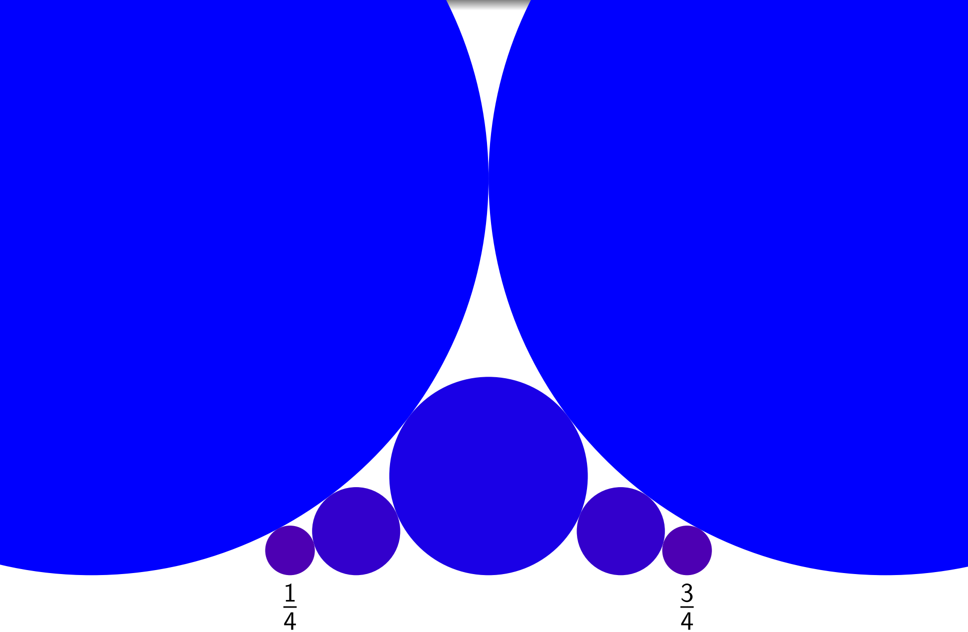
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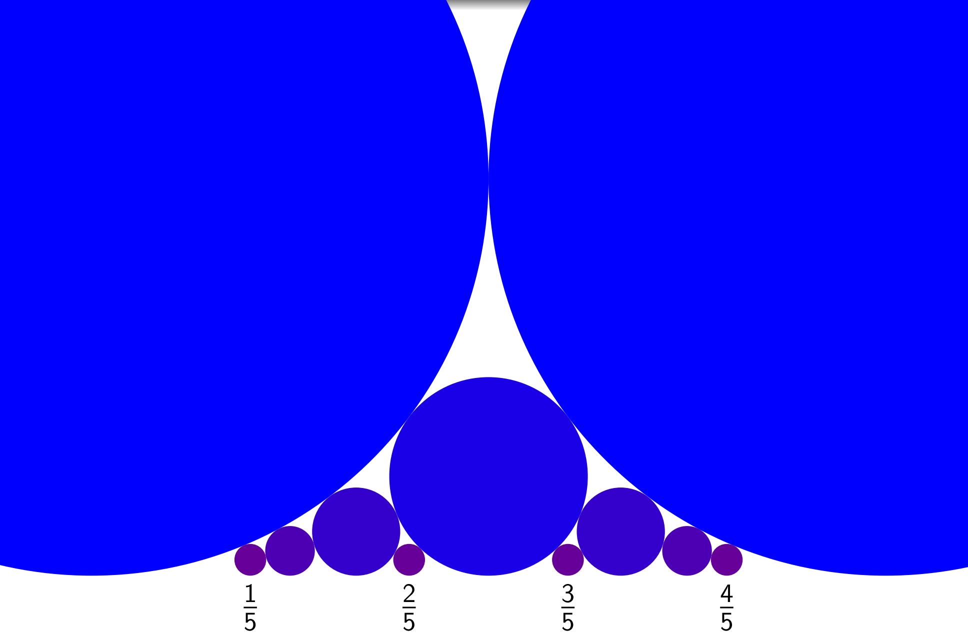
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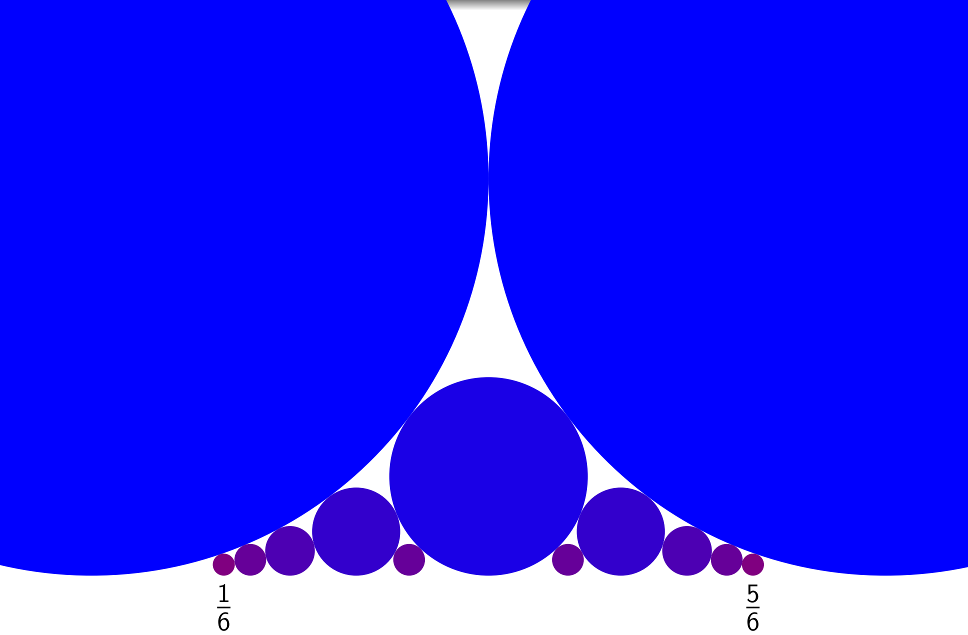
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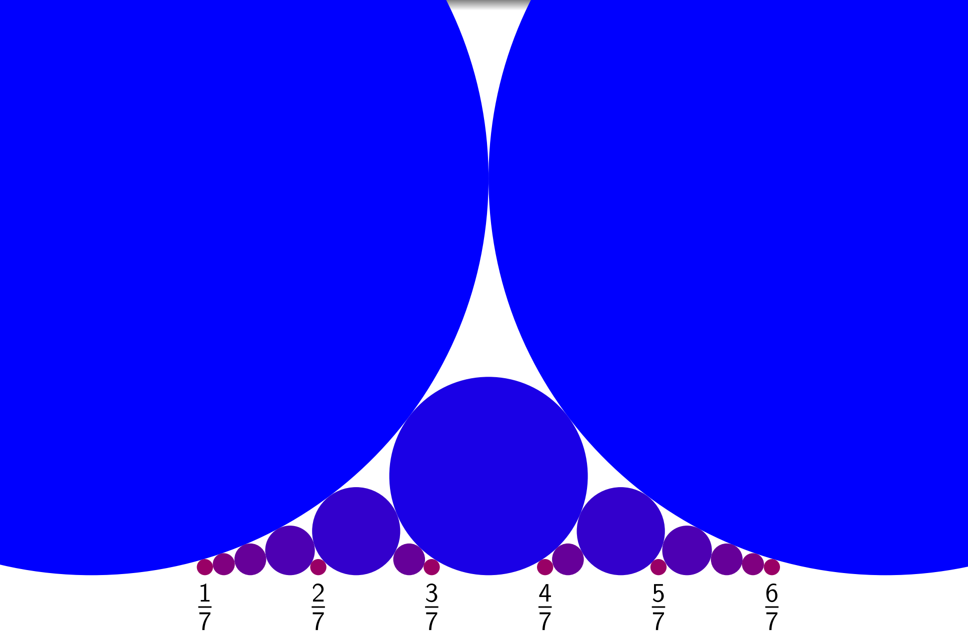
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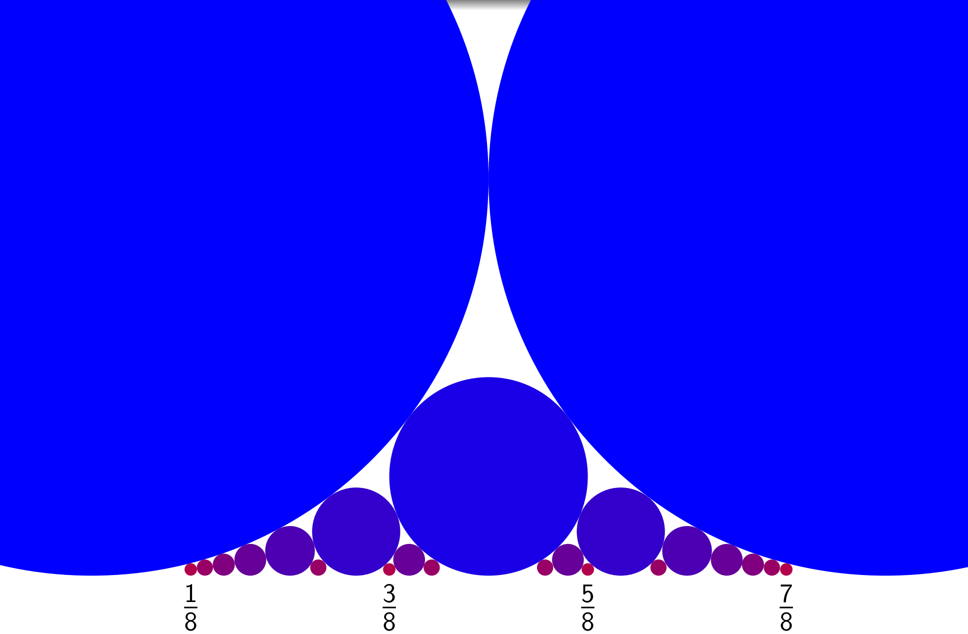
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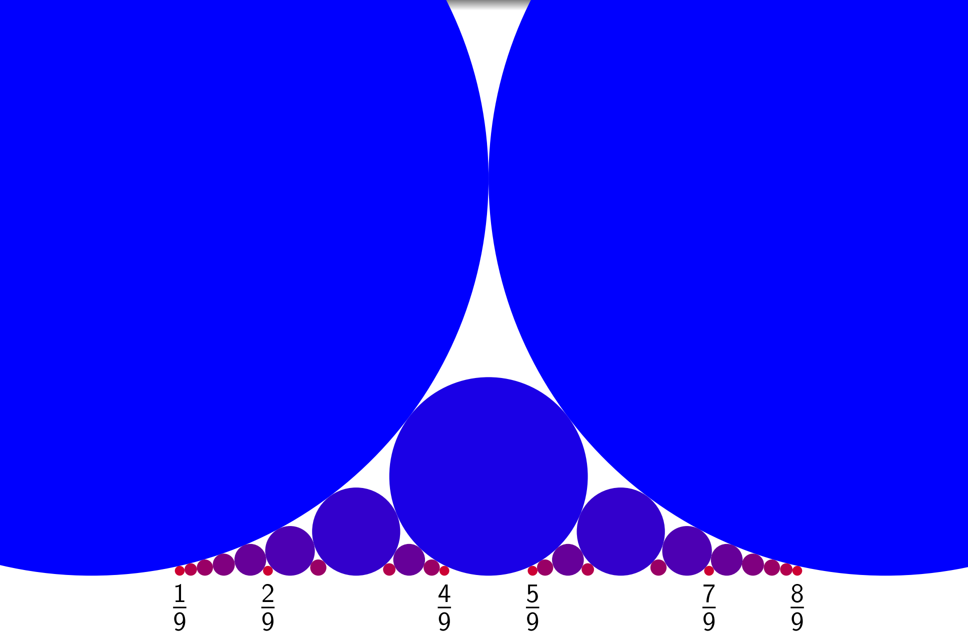

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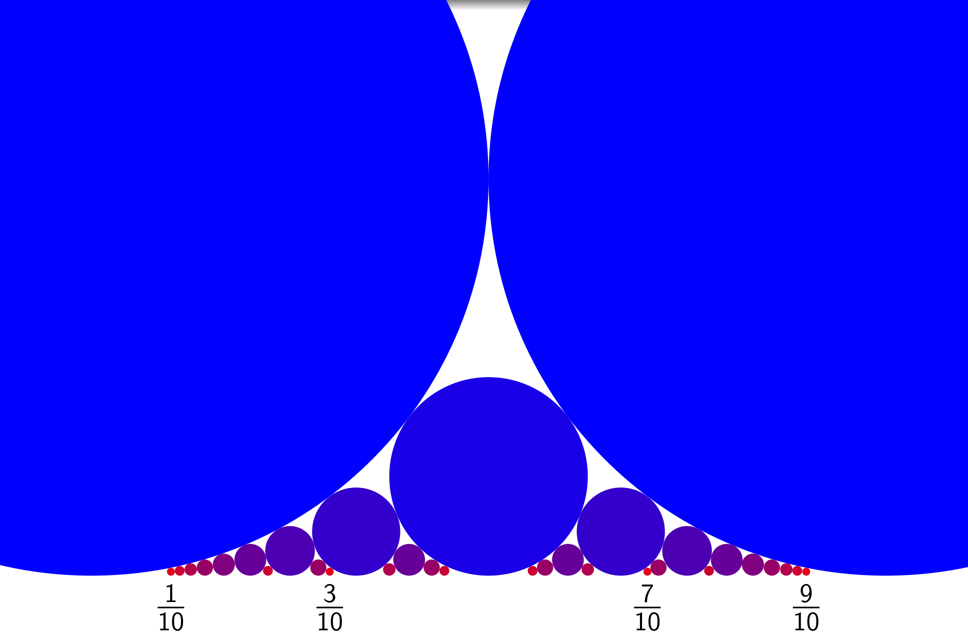
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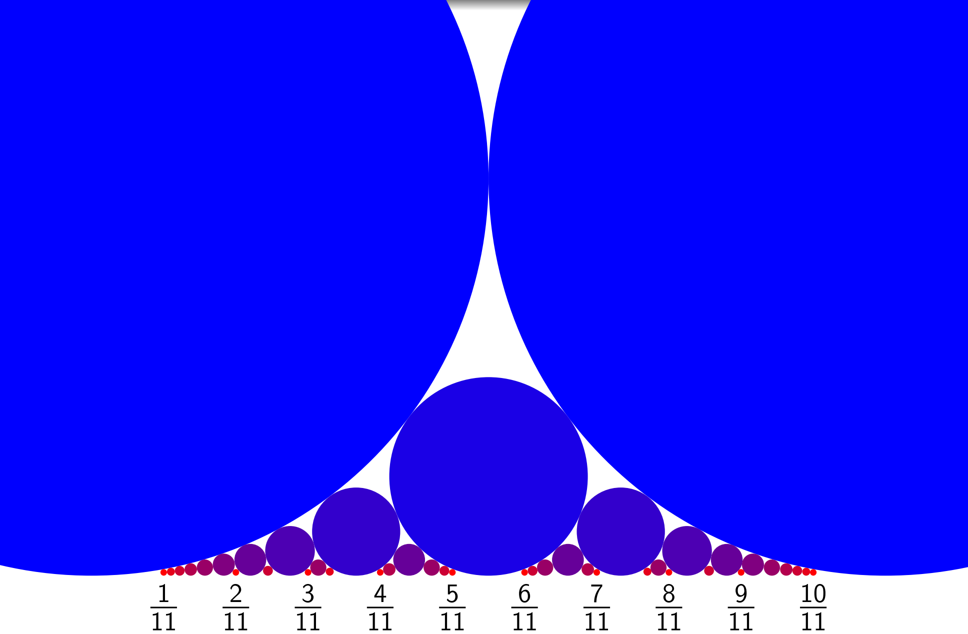
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The Ford circles and fractions

Thus the Ford circles, when generated in order of size, generate all *reduced fractions*, in order of their denominators.

The first n stages of the Ford circle construction give the so-called Farey sequence of order n—all reduced fractions between 0 and 1 with denominator $\leq n$.

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The LADIES *Diary:*
OR, THE
Woman's ALMANACK,
For the YEAR of our LORD, 1747.
Being the *Third* after *Bissextile*, or LEAP-YEAR.
Containing many Delightful and Entertaining PARTICULARS
Peculiarly adapted for the *Use* and *Diversion* of the
FAIR-SEX.

Being the *Forty Fourth* ALMANACK publish'd of this Kind.

YE *British* FAIR, with ev'ry Grace refin'd,
Born to delight, and to subdue Mankind!

2. Vouchsafe in *Arts*, your sprightlier Parts to lend,
And Wit and Sense their Empire shall extend:

3. The florid PEN more than the *Needle* prize,
One pleasures *Hearts!* the Other pleases *Eyes!*

4. Thus o'er the World, LADIES, assert your Sway,
Beauty commands, and *Science* leads the Way:
Proceed and conquer by your matchless Charms,
And lead Whole *Nations* captive to your *Arms!*

Printed by *A. Wilde*, for the Company of STATIONERS, 1747.

The Farey sequence has a long history, going back to a question in the *Ladies Diary* of 1747.

III. QUESTION 281, *by Mr. J. May, jun. of Amsterdam.*

It is required to find (by a general theorem) the number of fractions of different values, each less than unity, so that the greatest denominator be less than 100?

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How did Ford come to discover its geometric interpretation?

Henri Poincaré (1854–1912)

Poincaré in 1889

Poincaré made contributions to many fields of mathematics, from algebraic topology to celestial mechanics.

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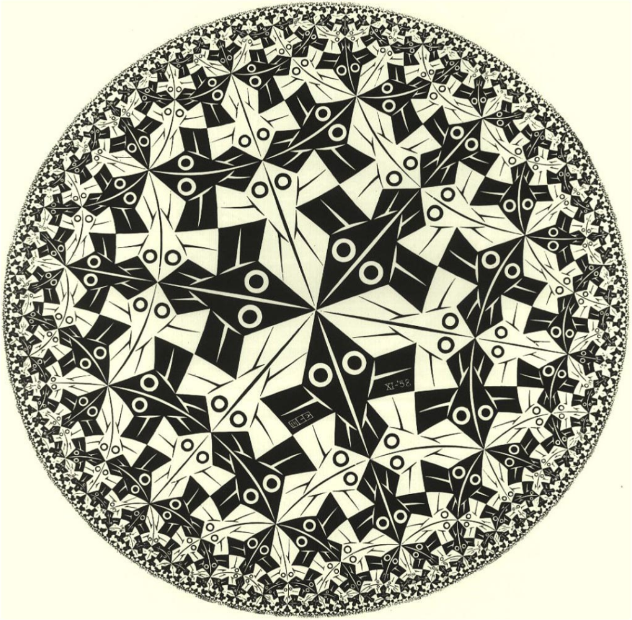
In particular, in the case of the disk (or half-plane) he discovered the role of *non-Euclidean geometry* in the study of periodicity.

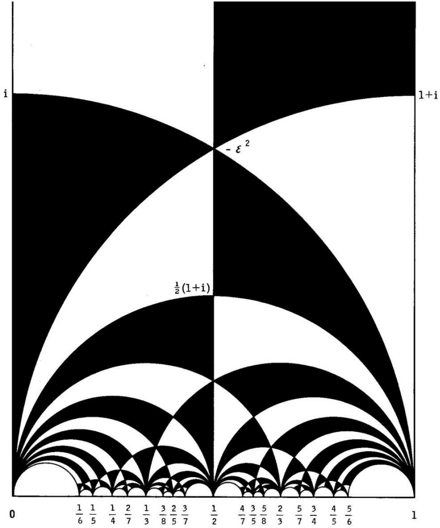


Escher's
Circle limit I

Non-Euclidean View of the Half-Plane

Half-plane version of Escher's *Circle Limit I*, showing fish that are congruent according to the non-Euclidean metric.

Non-Euclidean Periodicity before Poincaré



From Patrick du Val: *Elliptic Functions and Elliptic Curves*

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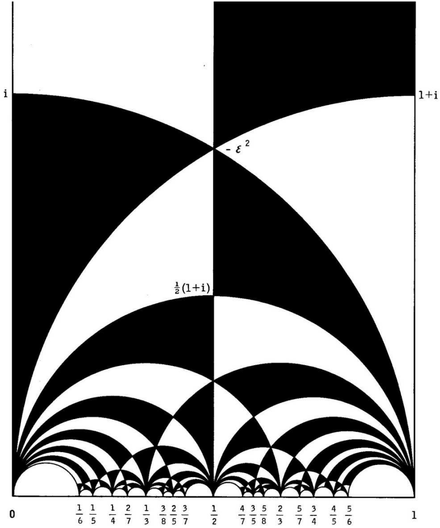

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The values of j repeat in each region consisting of a black and white triangle.

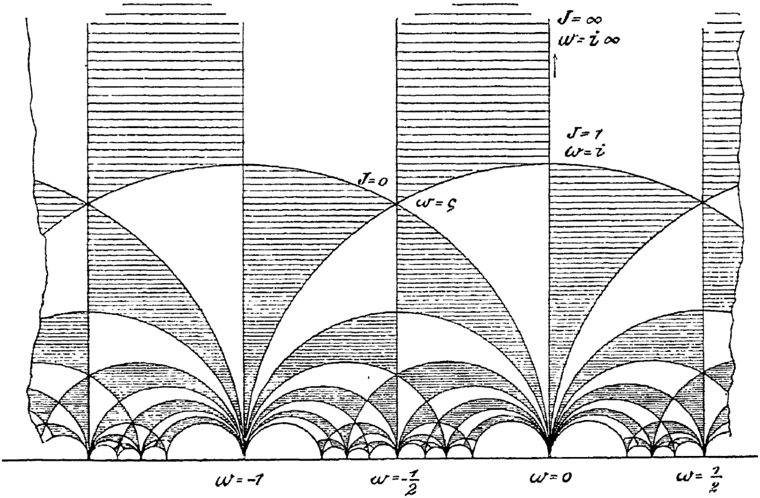
More precisely,

$$j\left(\frac{az+b}{cz+d}\right) = j(z)$$

for $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$.

Classical picture of the modular tessellation

From Klein and Fricke *Theorie der Elliptischen Modulfunction*



The Modular Group

The tessellation above is generated from a single tile, consisting of any adjacent black and white region, by repeatedly applying the transformations

$$z \mapsto z + 1 \quad \text{and} \quad z \mapsto -1/z.$$

These two generate all the transformations

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All members of the modular group are *isometries* of the half-plane under the metric given by

$$ds = \frac{|dz|}{y} \quad \text{where} \quad z = x+iy,$$

which makes the half-plane a model of the *non-Euclidean plane* of Bolyai and Lobachevsky.

Uniformization

A major goal of Poincaré was uniformization (i.e., parametrization) of algebraic curves. The two classical examples of uniformization are those for genus 0 and genus 1:

Genus 0. The circle $x^2 + y^2 = 1$ is parametrized by the rational functions

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}.$$

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Poincaré believed that any algebraic curve of higher genus could be parametrized by automorphic functions, but he was unable to prove this until 1907.

Edmund Whittaker (1873–1956)

Whittaker in 1933

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With G.N. Watson he wrote the classic *Modern Analysis* in 1927 (updating his own *Modern Analysis* of 1902).

Here are the top wranglers of 1895.

MR. J. H. GRACE

Bracketed Second Wrangler

Photo by Scott and Wilkinson, Cambridge

John Hilton Grace, bracketed Second Wrangler, is the son of Mr. William Grace, and was born at Haleswood, Lancashire in 1873. He was educated at the Liverpool Institute, and entered at Peterhouse in 1892, having gained a foundation scholarship

MR. T. J. BROMWICH

Senior Wrangler

Photo by T. Bromwich, Bridgnorth

Mr. Thomas John I'Anson Bromwich, who is Senior Wrangler, is the son of Mr. John I'Anson Bromwich, and was born at Wolverhampton in 1875. He was educated at Wolverhampton Grammar School and the High School, Durban, South Africa. He entered at St. John's College in 1892, where he gained a foundation scholarship and exhibition. While at Durban, he gained the Natal Government scholarship

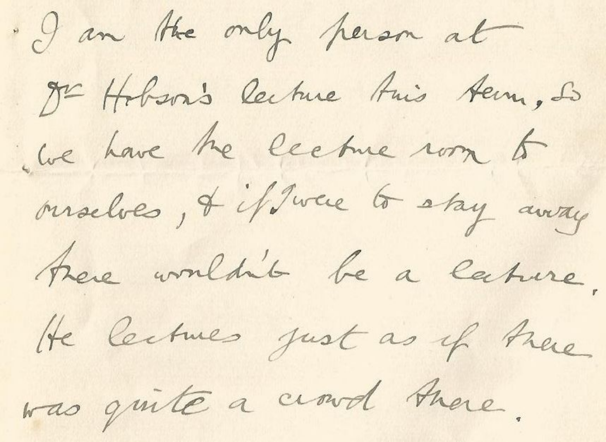
MR. E. T. WHITTAKER

Bracketed Second Wrangler

Photo by Scott and Wilkinson, Cambridge

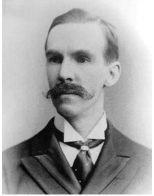
Mr. Edward Taylor Whittaker, bracketed Second Wrangler, is the son of Mr. John Whittaker, of Birkdale, Southport. He was born in 1873. He was educated at Manchester Grammar School. He won a small scholarship at Trinity College in 1892, and at the end of his first term he gained a better one. He also won the Sheepshanks Astronomical Exhibition last December

CAMBRIDGE UNIVERSITY MATHEMATICAL TRIPOS


I am the only person at Dr Hobson's lecture this term, so we have the lecture room to ourselves, & if I were to stay away there wouldn't be a lecture. He lectures just as if there was quite a crowd there.

From the letter of 24 January 1896

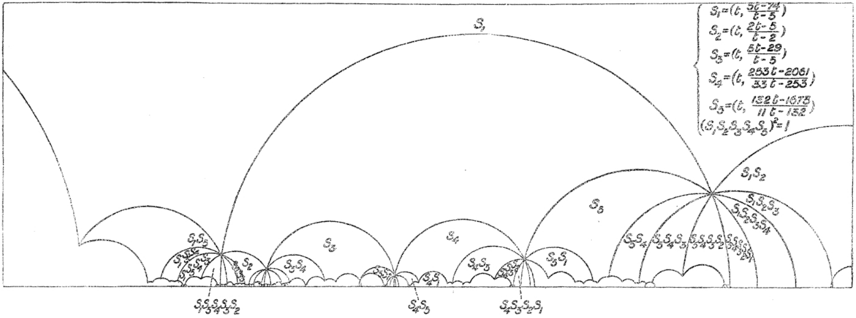
While studying at Cambridge in 1896, Whittaker wrote to his mother with some observations of Cambridge life.



E. W. Hobson

In 1897 Whittaker won the Smith's Prize and a fellowship at Trinity College, Cambridge, for an essay on automorphic functions.

He solved the uniformization problem for the curve $y^2 = x^5 + 1$ of genus 2, using a function with the periodicity of the "Whittaker group," which he pictured as follows.



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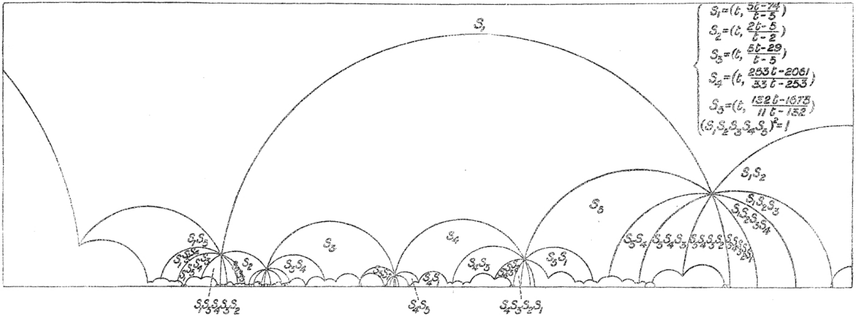

This function is mentioned in Whittaker and Watson, p. 455.

Lester R. Ford (1886–1967)

Ford in the 1940s

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His research included the discovery of Ford circles and their connection with the modular group and continued fractions. Part of his research (on continued fractions for complex numbers) earned him a Harvard Ph.D. when he returned to the US in 1917.

Ford circles became well-known when Ford wrote them up in an article *Fractions* in the *American Mathematical Monthly* of 1938.

Ford in Scotland

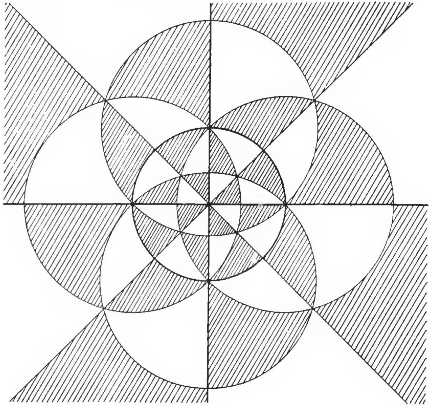
This photo was taken on a later visit to Scotland in 1926.

(Courtesy of Ford's grandaughter Ilisa Kim, obtained by Andrew Ranicki.)

Ford's example of spherical periodicity

Ford begins with automorphic functions on the sphere $\mathbb{C} \cup \{\infty\}$, and the underlying symmetric tessellations. The following pictures are from pp. 60-61 of Ford's 1915 book on automorphic functions.

Spherical frame



Its stereographic projection

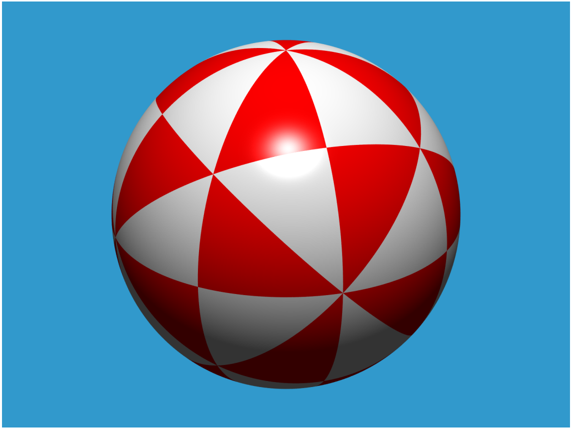
Origin of the spherical model

Alexander Crum Brown

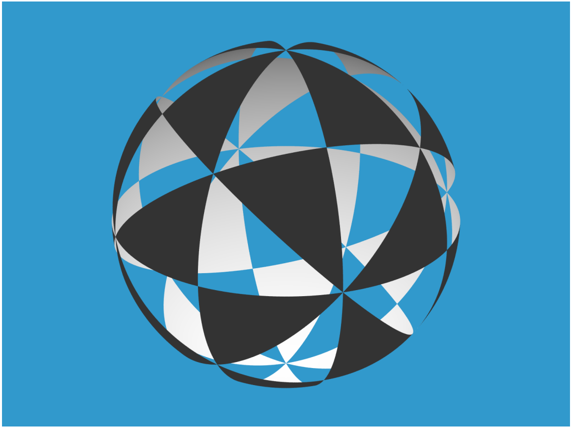
Ford credits the spherical model in the photograph to Professor Crum Brown, who was professor of chemistry at Edinburgh.

(Also known for pioneering contributions to knot theory, working with his brother-in-law Peter Guthrie Tait.)

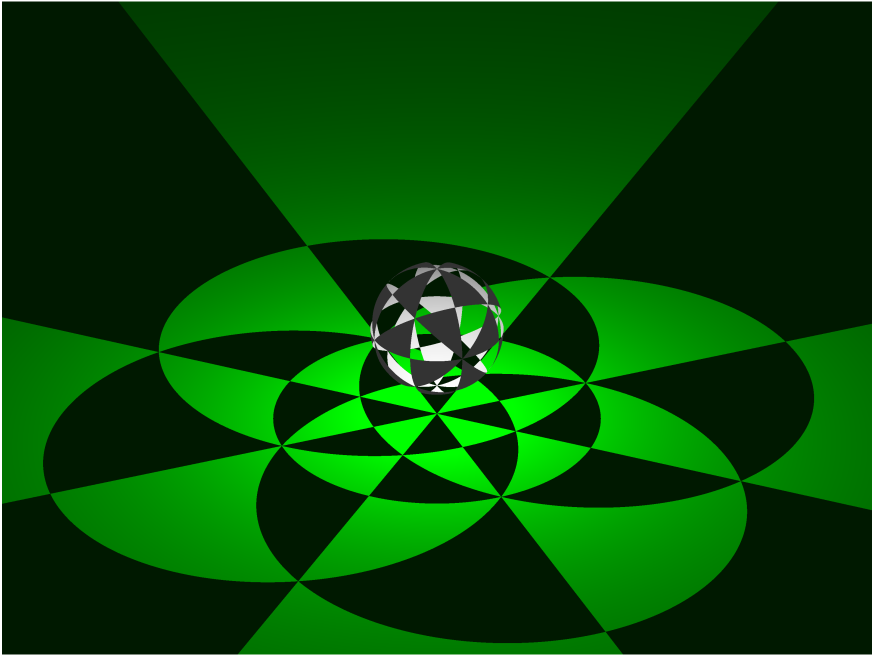
Same object with modern technology (POV-ray)

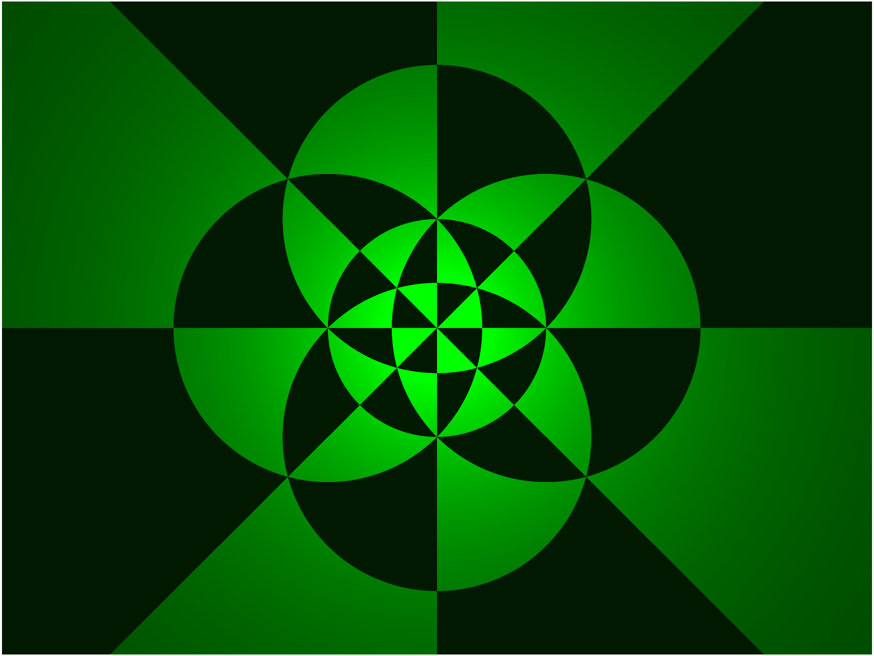


How to make the stereographic projection

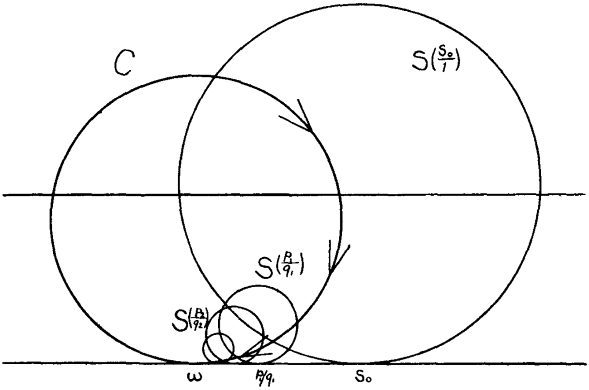


1. First cut out every other triangle in the tessellation of the sphere.
2. Light the sphere from inside at the north pole.
3. Project onto a plane parallel to the tangent plane at the north pole.





The first Ford circles



In a paper of 1917, Ford introduced his circles for the first time.

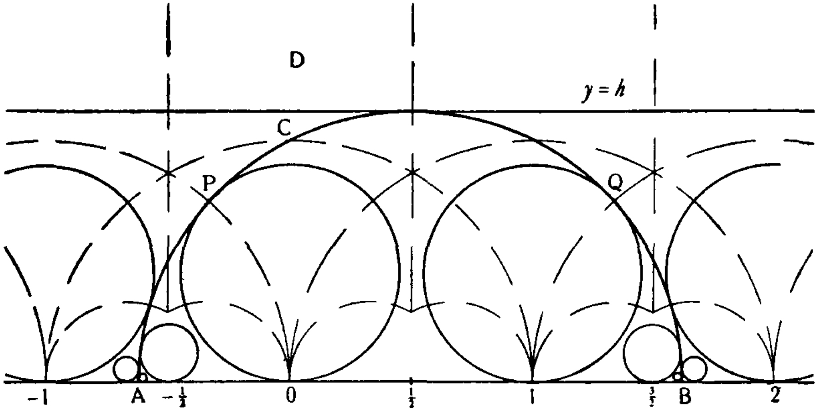
He arrives at them as images of the horizontal line $y = h$ under modular transformations.

He finds that the circle $S(\frac{p}{q})$ touching the x-axis at p/q has radius $1/2hq^2$.

Today we take $h = 1$, so that the circles do not overlap.

Ford circles and the modular tessellation

In a second paper of 1917, Ford related his circles to the modular tessellation, in order to prove a theorem of Hurwitz on continued fractions.



Why the connection between fractions and circles?

The key is the fact that a transformation

$$z \mapsto \frac{az+b}{cz+d}, \quad \text{for } a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1,$$

maps circles to circles, $\mathbb{R}$ to itself, and preserves tangency.

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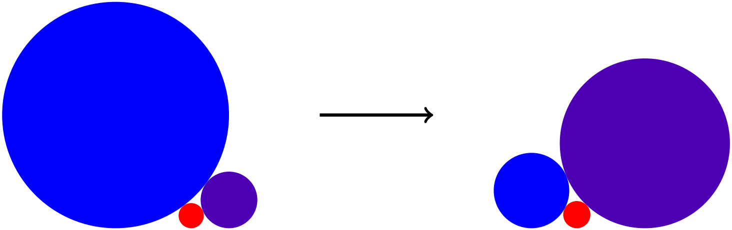
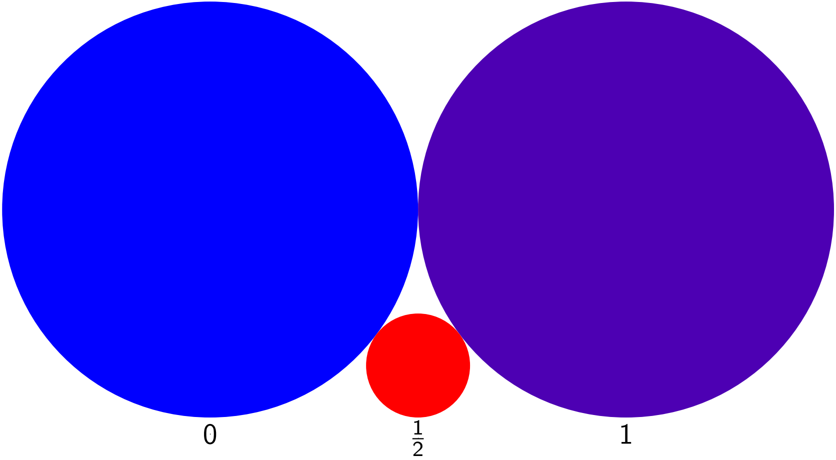
So, if we have some circles that are tangent to each other and the real axis $\mathbb{R}$, the same is true of their images under $z \mapsto \frac{az+b}{cz+d}$.



Example: the modular transformation $z \mapsto z/(z+1)$

This is an example of a *limit rotation* of the hyperbolic plane.
It fixes 0 and maps the circles touching 0 into themselves.
Notice also that it sends $1/n$ to $1/(n+1)$.



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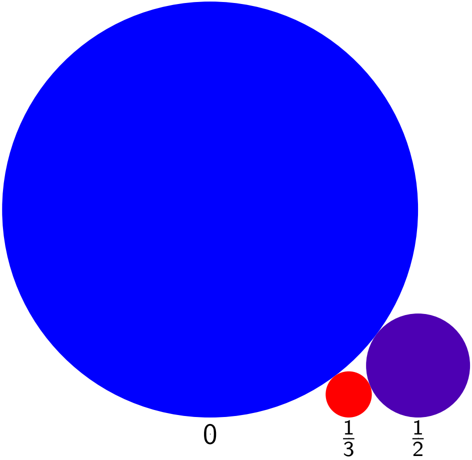

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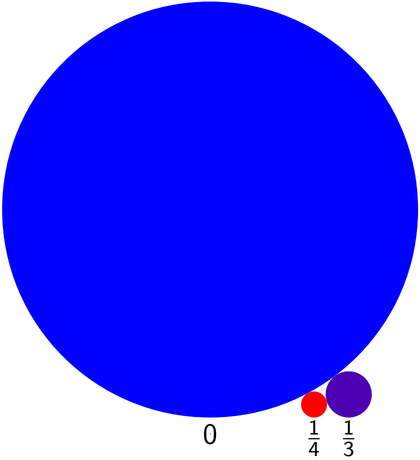

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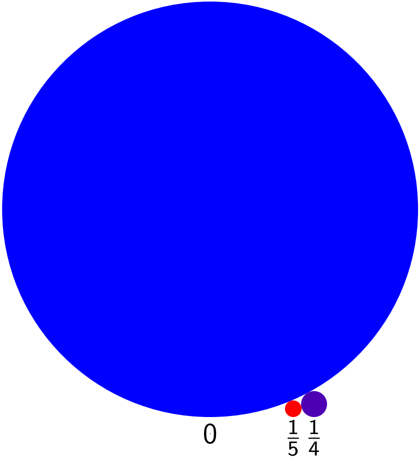

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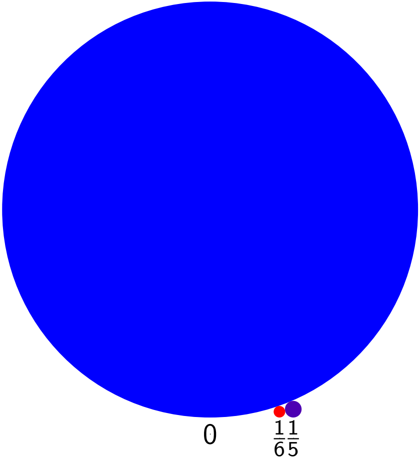


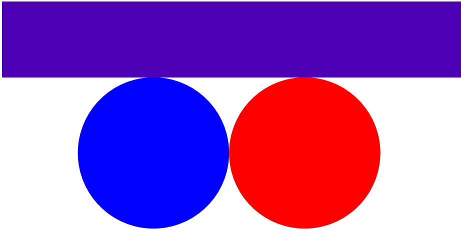
So the circles obtained from unit diameter circles at 0 and 1, by repeatedly filling the gap on the left by a tangential circle, touch at

$$\frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \quad \ldots.$$

The rational positions of the other Ford circles may be explained similarly, by appealing to properties of modular transformations.

The tangential Ford circles are images of the initial tangential circles.

We include the line $\operatorname{Im}(z) = 1$ as an "honorary" circle, touching the real axis at ∞.



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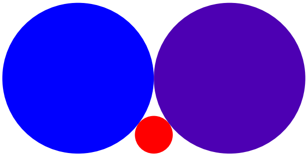

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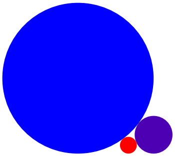

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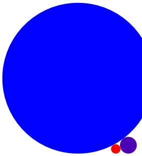
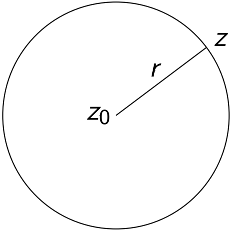


Computing with circles in the complex plane

To obtain several properties of Ford circles at once, we apply the following description of circles in terms of a complex coordinate.



If z lies on a circle with center z_0 and radius r,

then

$$|z - z_0|^2 = r^2, \quad \text{that is,} \quad (z - z_0)\overline{(z - z_0)} = (z - z_0)(\bar{z} - \bar{z}_0) = r^2$$

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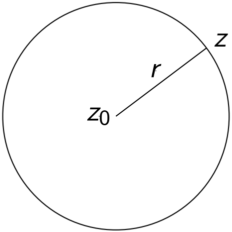

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which gives the equation

$$z\overline{z} - z_0\overline{z} - \overline{z}_0 z + |z_0|^2 - r^2 = 0.$$

Conversely, from such an equation we can read off the center z_0, and then compute the radius r.

Generation of the Ford Circles

All Ford circles are images of the line $\mathrm{Im}(z) = 1$, that is $z - \overline{z} = 2i$, under transformations in the modular group:

$$z \mapsto w = \frac{az + b}{cz + d}, \quad \text{where } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1.$$

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which we recognise as the equation of the circle with center $z_0 = \frac{a}{c} + \frac{i}{2c^2}$ and for which we easily find radius $r = \frac{1}{2c^2}$.

Basic properties of the Ford circles

These follow immediately from properties of modular transformations.

1. The circle touching the real axis at the reduced fraction a/c has radius $1/2c^2$.
   Such a circle is the image of the line $\text{Im}(z) = 1$ under $z \mapsto \frac{az+b}{cz+d}$. This also explains why the circles for reduced fractions a/c and a'/c have the same radius.

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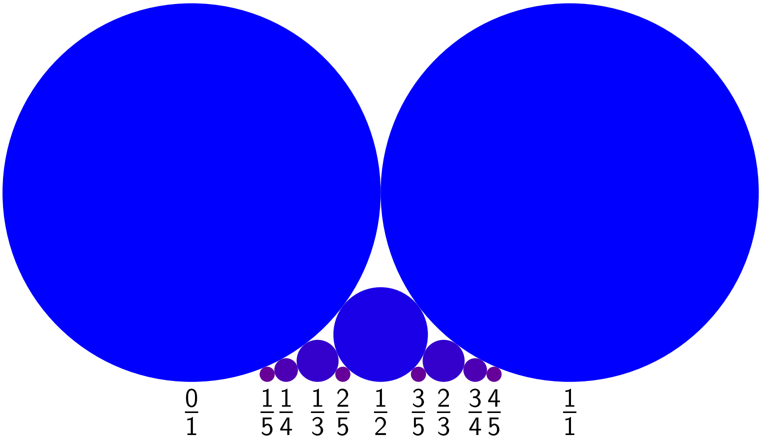
3. The circle between these tangential circles touches at $\frac{a+b}{c+d}$.
   Because the latter circle is the image of the circle between the circles touching $z = 0$ and $z = \infty$, namely the circle touching $z = 1$, under $z \mapsto \frac{az+b}{cz+d}$.

The mediant property

The last property implies that in any Farey series, such as



the term between $\frac{a}{c}$ and the next term but one, $\frac{b}{d}$, equals $\frac{a+b}{c+d}$.

Back to Farey

This property was discovered by Farey in 1816 (without proof), and published in *Philosophical Magazine* 47 (1816) pp. 385–386.

LXXIX. *On a curious Property of vulgar Fractions. By Mr. J. Farey, Sen.*

To Mr. Tilloch.

Sir, — On examining lately, some very curious and elaborate Tables of "Complete decimal Quotients," calculated by Henry Goodwyn, Esq. of Blackheath, of which he has printed a copious specimen, for private circulation among curious and practical calculators, preparatory to the printing of the whole of these useful Tables, if sufficient encouragement, either public or individual, should appear to warrant such a step: I was fortunate while so doing, to deduce from them the following general property; viz.

If all the possible vulgar fractions of different values, whose greatest denominator (when in their lowest terms) does not exceed any given number, be arranged in the order of their values, or quotients; then if both the numerator and the denominator of any fraction therein, be added to the numerator and the denominator, respectively, of the fraction next but one to it (on either side), the sums will give the fraction next to it; although, perhaps, not in its lowest terms.

I am not acquainted, whether this curious property of vulgar fractions has been before pointed out?; or whether it may admit of any easy or general demonstration?; which are points on which I should be glad to learn the sentiments of some of your mathematical readers; and am

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Your obedient humble servant,

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Nor did Cauchy, who saw Farey's question reprinted in a French journal, supplied a proof, and attributed the discovery to Farey.

More on the modular tessellation and Ford circles

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A Motif of Mathematics, Docent Press 2011

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The following picture, from Francis Bonahon's web site

http://www-bcf.usc.edu/~fbonahon/STML49/FareyFord.html

shows blue Ford circles on a multicolored modular tessellation. Also see his book *Low-Dimensional Geometry*, AMS 2009.