Homogeneity in supergravity

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. Some of the work in this thesis has been developed in collaboration with José Figueroa-O'Farrill, reported in [1, 2, 3] and with Andree Lischewski, reported in [4]. The work has not been submitted for any other degree or professional qualification.

(*Noel Hustler*)

Lay summary

There are currently two pillars of modern physics that we use to describe the universe around us: the theory of General Relativity which describes the fundamental force of gravity through the curving geometry of spacetime; and the Standard Model, a quantum field theory which describes all the other fundamental forces through the interactions of elementary particles. These two theories are philosophically and physically incompatible with each other; and in the few (but incredibly fundamental) scenarios in which we require both theories to combine in order to model a phenomenon, they either break down, disagree with each other, or more commonly and perversely do both at the same time. As such, we know that they must be but two different approximations of some grander, overarching, and unified theory; our best prospect for such a unified theory is known as string theory.

When we talk about string theory as a potential unified theory, we are actually talking about superstring theory; that is, supersymmetric string theory. Supersymmetry is a hypothetical symmetry between different kinds of elementary particles. If, as many people believe, the Standard Model were supersymmetric, then for each kind of elementary particle we know of, we would expect there to exist another particle known as its superpartner. As an example, we know that electrons exist and so there must be some as yet undetected superpartner particle we will call selectrons. Quarks must mean there exist squarks, photons mean photinos, Higgs bosons mean Higgsinos — you get the idea.

It can be quite difficult to calculate the things that we want to in string theory and so very often we will look at an approximation to string theory in which we ignore some of the more fiddly effects; such an approximation we call a theory of supergravity. String theory and thus supergravity must by necessity contain fundamental aspects of both General Relativity and supersymmetric quantum field theory and so solutions to a theory of supergravity have both the curving geometry of spacetime and fundamental particles with supersymmetry. This thesis draws a deep connection between the amount of supersymmetry that a supergravity solution has and how homogeneous (or simple) the solution's spacetime geometry must be. The theme of this connection is that the more supersymmetry we have, the simpler the spacetime geometry.

Abstract

This thesis is divided into three main parts. In the first of these (comprising chapters 1 and 2) we present the physical context of the research and cover the basic geometric background we will need to use throughout the rest of this thesis.

In the second part (comprising chapters 3 to 5) we motivate and develop the strong homogeneity theorem for supergravity backgrounds. We go on to prove it directly for a number of top-dimensional Poincaré supergravities and furthermore demonstrate how it also generically applies to dimensional reductions of those theories.

In the third part (comprising chapters 6 and 7) we show how further specialising to the case of symmetric backgrounds allows us to compute complete classifications of such backgrounds. We demonstrate this by classifying all symmetric type IIB supergravity backgrounds. Next we apply an algorithm for computing the supersymmetry of symmetric backgrounds and use this to classify all supersymmetric symmetric M-theory backgrounds.

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The stage is too big for the drama.

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Chapter 1

Introduction

1.1 Fundamentals

There are four fundamental forces (or interactions) known to modern physics: Gravitation, strong, weak, and electromagnetic[1]. The first of these is currently described using Einstein's theory of General Relativity and the latter three are currently described using a particular quantum field theory known as the Standard Model.

1.2 General Relativity

In 1915, Albert Einstein published the gravitational field equations of his theory of general relativity [5], giving us a modern framework to describe gravity through the curvature of spacetime (see for example [6]). From its first 'classical' tests: the perihelion precession of Mercury's orbit, gravitational lensing, and the gravitational redshift of light; through to more modern tests such as general relativistic time dilation, frame-dragging, binary pulsars, and most recently the direct detection of gravitational waves [7], we have seen that General Relativity has excellent experimental verification in both strong- and weak-field regimes. However, there are questions pertaining to the nature of spacetime singularities and dark energy, along with the general belief that gravity must be quantised in order to unify it with the three other fundamental forces. Gravity as a quantum field theory appears to be non-renormalisable and for this and other reasons, a consistent quantum theory of gravity is difficult to construct.

1.3 The Standard Model

Quantum field theory (see for example [8]) was developed by the mid twentieth century to unify special relativity and quantum mechanics, and the Standard Model (see for example [9]) is the particular quantum field theory developed during the second half of the twentieth century to

[1] The weak and electromagnetic forces are however unified in the electroweak interaction.

best describe the (gravity-excluded) fundamental phenomena that we observe around us. As a theory of the strong, weak, and electromagnetic forces, the Standard Model splits all elementary particles into either *bosons* or *fermions* depending on whether their spin is integer or half-integer respectively. All matter is contained in the fermionic particle sector of the Standard Model and all interactions are mediated by particles in the bosonic sector. Experimental verification of the Standard Model is excellent in the regimes and energies available to us, and the recent discovery [10, 11] of the Higgs boson at the Large Hadron Collider has now essentially given us observational evidence of the Standard Model's entire *necessary* particle content. However, apart from the obvious issue of not incorporating gravity, there are a number of other problems. Some of the most serious are:

- Free parameters: The Standard Model requires 26 free parameters which is rather a lot more than one would expect or want from a fundamental theory.
- Hierarchy problem: One would expect quantum corrections to make the Higgs mass much, much larger than it is. Either some as-yet unknown mechanism must suppress these corrections or we must have quite an extreme fine-tuning (on the order of 10^{16}) of Standard Model parameters.
- Unification: The strengths of the three fundamental forces in the Standard Model appear to converge at very high energies which would suggest that the forces become unified. However, this unification is not quite exact in the current Standard Model.
- Dark matter and Dark energy: We know from cosmological observations that the Standard Model only describes around 5% of the total energy content of the universe. Of the other 95%, about 27% is dark matter and the rest dark energy [12].
- Neutrino mass: Neutrinos in the Standard Model are massless. However, experimental observation of neutrino oscillation means that there must be mass differences between these neutrinos whence at least two must be massive.
- Matter-antimatter asymmetry: The observable universe is mostly comprised of matter but the Standard Model predicts that matter and antimatter should have been created in almost equal amounts.

1.4 Supersymmetry

One way of extending the symmetries of the Standard Model as a quantum field theory without falling afoul of the Coleman-Mandula no-go theorem[2] [13] is to introduce a symmetry between fermions and bosons; and this is known as *supersymmetry* [14]. Supersymmetry has some tricks up its sleeve – it *can* solve the hierarchy problem, the unification problem, and potentially

[2]With few assumptions, this theorem prohibits the internal and spacetime symmetries of a quantum field theory from being combined in a non-trivial manner.

provide dark matter candidates all in one fell swoop. As one of the very few Coleman-Mandula 'loopholes', this makes it an incredibly attractive proposition. However, supersymmetry requires us to introduce a whole new set of *superpartner* particles, one for each of the fundamental particles we already know of. As of yet, none of these superpartner particles have been experimentally observed.

1.5 Supergravity

Both the Standard Model and General Relativity are based around gauge symmetries, the former around the internal $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ and the latter around the spacetime diffeomorphism group or general coordinate transformations. A natural question to ask then is what happens if we promote supersymmetry from a global (rigid) to a local (gauge) symmetry. It turns out that local supersymmetry implies general coordinate transformations and so automatically implies gravity; such theories are known as *supergravities* [15, 16] and exist in various guises up to a maximum of eleven dimensions. Supergravity theories were developed in the 1970s and 1980s [17, 18, 19] as a potential pathway to unification but it became clear that they were non-renormalisable and thus not suitable candidates.

1.6 String theory

It seems then that in order to construct a unified theory, or even to quantise gravity, something very different is necessary, and the current leading framework for such a quantum theory of gravity is string theory [20, 21, 22]. The basic idea of string theory is that fundamental particles are not point-like but rather very tiny loops of 'string' – starting with this premise, gravity and Yang-Mills gauge theory arise fully-formed out of the deep. Of course, we also receive some necessary extra baggage along for the ride such as supersymmetry and extra dimensions. In fact, only the string theories in ten dimensions were known to be free of gauge and gravitational anomalies [23, 24]. Following the discovery of string dualities [25, 26], the five known ten-dimensional string theories were found to all be aspects of a single theory in eleven dimensions called M-theory [27] – in fact perturbative expansions in different limits of the M-theory parameter space.

The ten-dimensional string theories are difficult to directly analyse non-perturbatively, especially for the case of closed strings, and M-theory itself is still shrouded in significant mystery. However, it turns out that the ten-dimensional supergravities are the low energy effective field theories of the ten-dimensional string theories, and the maximal eleven-dimensional supergravity is thought to be the low energy effective field theory of M-theory. As such, we can learn much about string theory from studying supergravity.

1.7 Back to supergravity

We would like to understand the solution spaces of supergravities and a primary tool is the understanding of supergravity backgrounds – bosonic solutions of the supergravity field equations with fermionic fields set to zero. Now, the role of supersymmetry in string theory and supergravity is pre-eminent and so our interest leans towards those supergravity backgrounds preserving some amount of supersymmetry and in particular, the more supersymmetry, the more the background is in some sense under control. Thus if we are going to study backgrounds, then let us first study those with a large amount of supersymmetry!

In this spirit we wish to tackle the classification of highly supersymmetric supergravity backgrounds and much progress has been made in this endeavour, in a variety of guises. We present two different approaches to this problem: The first being the strong homogeneity theorem for (Poincaré) supergravity backgrounds which links the fraction of supersymmetry of a background to how locally geometrically simple it is. The second through classifying the symmetric backgrounds of $D = 10$ type IIB supergravity and extending the classification of symmetric M-theory backgrounds to include supersymmetry.

Chapter 2

Homogeneity

2.1 Homogeneous spaces

We follow [28] and present some basic material on the topic of homogeneous spaces. We assume that all manifolds are finite-dimensional and connected.

2.1.1 Homogeneity

Let us take a triple $(X, C, \mathcal{G})$ with X a non-empty set in the category C and $\mathcal{G}$ a group. If we have a $\mathcal{G}$-action $\phi : \mathcal{G} \times X \to X$ acting as C-automorphisms, then we call X a *$\mathcal{G}$-space*. If additionally, the action of $\mathcal{G}$ on X is *transitive*, then X is a *homogeneous $\mathcal{G}$-space* or, eliding the particular group, a *homogeneous space*. The action of $\mathcal{G}$ on X is transitive if any of the following equivalent statements are true:

1. There is a single $\mathcal{G}$-orbit;
2. For any two elements $x, y \in X$, $\exists\, a \in \mathcal{G}$ s.t. $\phi(a, x) = y$;
3. For every $x \in X$, the map $\phi_x : \mathcal{G} \twoheadrightarrow X$ is surjective.

From now on, and unless otherwise stated, let us consider C to be the category of smooth manifolds and by homogeneous space we mean a homogeneous space in this category.

2.1.2 Coset manifolds

Let $\mathcal{G}$ be a Lie group with neutral element e, and left and right translations L_a, R_a for $a \in \mathcal{G}$. Taking a closed subgroup $\mathcal{K} \leq \mathcal{G}$, we can construct the set of left cosets $\mathcal{G}/\mathcal{K} = \{a\mathcal{K} : a \in \mathcal{G}\}$ and define the canonical projection map:

$$\begin{aligned} \pi : \mathcal{G} &\to \mathcal{G}/\mathcal{K} \\ a &\mapsto a\mathcal{K}\,, \end{aligned} \tag{2.1}$$

and left translations

$$\begin{array}{rcl} l_a : \mathcal{G}/\mathcal{K} & \to & \mathcal{G}/\mathcal{K} \\ b\mathcal{K} & \mapsto & ab\mathcal{K} \, . \end{array} \tag{2.2}$$

Thus

$$\pi \circ L_a = l_a \circ \pi \, . \tag{2.3}$$

There is a unique way [29] to give $\mathcal{G}/\mathcal{K}$ the structure of a smooth manifold such that π is a submersion, i.e.

$$\mathrm{d}\pi_a : \ \mathrm{T}_a\mathcal{G} \twoheadrightarrow \mathrm{T}_{\pi(a)}(\mathcal{G}/\mathcal{K}) \, , \tag{2.4}$$

and a manifold constructed in this manner we call a *coset manifold*. We have a natural transitive $\mathcal{G}$-action via left translations l_a making $\mathcal{G}/\mathcal{K}$ a homogeneous $\mathcal{G}$-space. Moreover we have a principal $\mathcal{K}$-bundle structure

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathcal{G} \\ & & \big\downarrow{\scriptstyle \pi} \\ & & \mathcal{G}/\mathcal{K} \end{array} \tag{2.5}$$

where the action of $\mathcal{K}$ on $\mathcal{G}$ is via right translations R_a.

Now, let us take *any* smooth manifold $\mathcal{M}$ with ϕ a smooth transitive $\mathcal{G}$-action and $\mathcal{G}$ a Lie group, so $\mathcal{M}$ is a homogeneous $\mathcal{G}$-space. Let us pick a point $m \in \mathcal{M}$ and take the isotropy (stabiliser) group at this point,

$$\mathcal{K} = \mathcal{G}_m = \{a \in \mathcal{G} : \ \phi(a, m) = m\} \, , \tag{2.6}$$

which is a closed subgroup of $\mathcal{G}$. We then have a natural diffeomorphism

$$\begin{array}{rcl} \tau : \mathcal{G}/\mathcal{K} & \to & \mathcal{M} \\ a\mathcal{K} & \mapsto & \phi(a, m) \, , \end{array} \tag{2.7}$$

meaning that $\mathcal{M}$ is diffeomorphic to $\mathcal{G}/\mathcal{K}$. As such we will from now on consider as equivalent and use interchangeably the notions of a coset manifold $\mathcal{G}/\mathcal{K}$ and a homogeneous $\mathcal{G}$-space[1]. In fact, up to isomorphism, all homogeneous spaces are coset manifolds.

Let $o = \pi(e) = \mathcal{K}$ denote the coset neutral element, and $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $\mathcal{G}$ and $\mathcal{K}$ respectively. First, since $\mathcal{K} \leq \mathcal{G}$ we clearly have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \, , \tag{2.8}$$

and from equation (2.4) we see that $\ker \mathrm{d}\pi_e = \mathfrak{k}$. Since $\mathrm{d}\pi$ is surjective, we thus have the isomorphism

$$\mathfrak{g}/\mathfrak{k} \cong \mathrm{T}_o(\mathcal{G}/\mathcal{K}) = \mathrm{T}_o\mathcal{M}. \tag{2.9}$$

[1] Since we are working in the category of smooth manifolds.

We thus see the well known one-to-one correspondence

$$\begin{array}{c} \mathcal{G}\text{-invariant tensor fields} \\ \text{of type } (p,q) \text{ on } \mathcal{G}/\mathcal{K} \end{array} \quad \longleftrightarrow \quad \begin{array}{c} \mathrm{Ad}^{\mathcal{G}/\mathcal{K}}\text{-invariant tensors} \\ \text{of type } (p,q) \text{ on } \mathfrak{g}/\mathfrak{k} \end{array} \tag{2.10}$$

given by evaluation of tensor fields at the origin $o = e\mathcal{K} \in \mathcal{G}/\mathcal{K}$. This shows a highlight of working with homogeneous spaces; geometrical questions about $\mathcal{M}$ may be reformulated in terms of questions about the pair $(\mathcal{G}, \mathcal{K})$ which in turn may be reformulated in terms of questions about the pair $(\mathfrak{g}, \mathfrak{k})$, essentially algebraising many problems.

2.1.3 Reductive homogeneous spaces

If we have a (connected) homogeneous space $\mathcal{M} = \mathcal{G}/\mathcal{K}$ then it is a *reductive homogeneous space* if there exists a subspace $\mathfrak{m} \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \qquad \text{and} \tag{2.11}$$

$$[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m} \,, \tag{2.12}$$

hence as a result of equation (2.9) we have the canonical isomorphism

$$\mathfrak{m} \cong \mathrm{T}_o\mathcal{M} \,, \tag{2.13}$$

making the correspondence in equation (2.10) even more useful.

The *isotropy representation* of (reductive) $\mathcal{G}/\mathcal{K}$ is the homomorphism,

$$\begin{aligned} \mathrm{Ad}^{\mathcal{G}/\mathcal{K}} : \ \mathcal{K} &\to \mathrm{Aut}(\mathfrak{m}) \\ k &\mapsto (\mathrm{d}l_k)_o \,, \end{aligned} \tag{2.14}$$

and it is equivalent to the adjoint representation of $\mathcal{K}$ in $\mathfrak{m}$, i.e. the following diagram commutes

$$\begin{array}{ccc} \mathfrak{m} & \xrightarrow{\mathrm{Ad}^{\mathcal{G}}(k)} & \mathfrak{m} \\ {\scriptstyle \mathrm{d}\pi_e\big|_{\mathfrak{m}}}\downarrow & & \downarrow{\scriptstyle \mathrm{d}\pi_e\big|_{\mathfrak{m}}} \\ \mathrm{T}_o\mathcal{M} & \xrightarrow{(\mathrm{d}l_k)_o} & \mathrm{T}_o\mathcal{M} \end{array} \tag{2.15}$$

where the upper horizontal map is well-defined because when $\mathcal{M}$ is connected, the reductivity property $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ implies $\mathrm{Ad}^{\mathcal{G}}(\mathfrak{m}) \subset \mathfrak{m}$.

As a result of equations (2.13) and (2.14) we can thus identify the tangent bundle of our reductive homogeneous space with the associated bundle of $\mathcal{G}$ via the isotropy representation,

$$\mathrm{T}\mathcal{M} \cong \mathcal{G} \times_{\mathrm{Ad}^{\mathcal{G}/\mathcal{K}}} \mathfrak{m} \,, \tag{2.16}$$

From reductivity we clearly have a horizontal tangent distribution on $\mathcal{G}$ defined by $\mathscr{H}_a = \mathrm{d}L_a(\mathfrak{m})$ (with vertical distribution $\mathscr{V}_a = \mathrm{d}L_a(\mathfrak{k})$) which is invariant under right translations R_a and so defines a connection on the principal $\mathcal{K}$-bundle called the *canonical connection* of the reductive homogeneous space. With the identification in equation (2.16) this connection then induces a canonical connection on $\mathrm{T}\mathcal{M}$. This connection has parallel torsion and curvature, and being induced from the principal $\mathcal{K}$-bundle connection via the isotropy representation, has holonomy $\mathcal{K}$ acting via the isotropy representation. Thus, via the correspondence in equation (2.10), $\mathcal{G}$-invariant vector fields are precisely those vector fields parallel with respect to the canonical connection.

2.1.4 (Pseudo-)Riemannian homogeneous spaces

Let $\mathcal{M} = \mathcal{G}/\mathcal{K}$ be a (not necessarily reductive) homogeneous space and g be a metric on $\mathcal{M}$. Then we say g is $\mathcal{G}$-invariant if the left translations l_a act as isometries with respect to g, meaning we have for all $X, Y \in \mathrm{T}_o\mathcal{M}$ and $a \in \mathcal{G}$,

$$g(X,Y) = g(\mathrm{d}l_a(X), \mathrm{d}l_a(Y)) \ . \tag{2.17}$$

Using the correspondence in equation (2.10) we equivalently have an $\mathrm{Ad}^{\mathcal{G}/\mathcal{K}}$-invariant symmetric bilinear form on $\mathfrak{g}/\mathfrak{k}$. The metric being $\mathcal{G}$-invariant means that the canonical connection for a (pseudo-)Riemannian *reductive* homogeneous space is metric.

2.1.5 Locally homogeneous spaces

We may relax the definition of a homogeneous space somewhat by relaxing the requirement that our G-action be transitive and instead only require that it be *locally* transitive, i.e. for every $m \in U \subset \mathcal{M}$ with U a normal neighbourhood of m, the map $\phi_m : \mathcal{G} \twoheadrightarrow U$ is surjective.

2.2 Symmetric spaces

We very briefly present some basic facts about locally symmetric spaces, which form the underlying geometries of many of the best-studied supergravity backgrounds.

2.2.1 Definition

A pseudo-Riemannian manifold $\mathcal{M}$ is *locally symmetric* if either of the following two equivalent statements is true:

1. For every point $m \in \mathcal{M}$ there exists an involutive *local* isometry ξ_m of which m is an isolated fixed point, i.e. $\xi_m(m) = m$ and $(d\xi_m)_m = -\mathrm{Id}_m$ with Id_m the identity map on $\mathrm{T}_m\mathcal{M}$;

2. $\nabla R = 0$ where R is the curvature tensor of $\mathcal{M}$.

A symmetric space can be given the structure of a reductive homogeneous space $\mathcal{G}/\mathcal{K}$ with direct sum Lie algebra decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ where we also have

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k} \,. \tag{2.18}$$

The canonical connection for a symmetric space has *no* torsion and hence, being metric, is the Levi-Civita connection. Therefore for a symmetric space $\mathcal{G}/\mathcal{K}$, $\mathcal{G}$-invariant tensor fields are parallel with respect to the Levi-Civita connection.

2.2.2 Lorentzian symmetric spaces

The Lorentzian symmetric spaces have been completely classified [30, 31], building on Cartan's classification of Riemannian symmetric spaces [32] via his earlier classification of simple Lie algebras over $\mathbb{R}$.

A Lorentzian (locally) symmetric space $(\mathcal{M}, g)$ is locally isometric to a product

$$\mathcal{M}_0 \times \mathcal{M}_1 \times \ldots \times \mathcal{M}_n \tag{2.19}$$

where $\mathcal{M}_0$ is an *indecomposable* Lorentzian symmetric space and $\mathcal{M}_i$ for $i > 0$ are *irreducible* Riemannian symmetric spaces. In all that follows in classification of symmetric spaces, we will assume we are talking about classification up to local isometry.

Indecomposable Lorentzian symmetric spaces

The indecomposable Lorentzian symmetric spaces are either one-dimensional Minkowski space $\mathbb{R}^{0,1}$ or one of three types: de Sitter, anti-de Sitter, and Cahen-Wallach. The de Sitter and anti-de Sitter spaces are well known but the Cahen-Wallach spaces $\mathrm{CW}_D(\lambda)$ to a lesser degree; these spaces come in $(D-3)$-parameter families but here we will take each family as a single geometry because the family structure will no longer concern us. A detailed overview of Cahen-Wallach spaces in this context can be found in [33].

The three non-trivial types of indecomposable Lorentzian symmetric spaces are listed in table 2.1 along with the ranks of their $\mathfrak{k}$-invariant forms in the form of a Poincaré polynomial:

$$P(t) = \sum_{i=0}^{D} b_i t^i, \qquad \text{where} \quad b_i = \dim_{\mathbb{R}}(\Lambda^i \mathfrak{m})^{\mathfrak{k}} \,. \tag{2.20}$$

[2] Descriptions of $\mathfrak{g}(\lambda)$ and $\mathfrak{k}(\lambda)$ can be found in [33].

Table 2.1: Indecomposable D-dimensional Lorentzian symmetric spaces.

Type	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{k}$-invariant forms
dS_D	$\mathfrak{so}(D,1)$	$\mathfrak{so}(D-1,1)$	$1+t^D$
AdS_D	$\mathfrak{so}(D-1,2)$	$\mathfrak{so}(D-1,1)$	$1+t^D$
$\mathrm{CW}_D(\lambda)$[2]	$\mathfrak{g}(\lambda)$	$\mathfrak{k}(\lambda)$	$1+t(1+t)^{D-2}+t^D$

Irreducible Riemannian symmetric spaces

The irreducible Riemannian symmetric spaces are either of Euclidean, compact, or noncompact type.

The Euclidean type is either $\mathbb{R}$ or S^1 but they are locally isometrically the same.

The compact and noncompact types are subject to a duality such that they come as a pair; a compact space with its dual noncompact space. The complete classification of compact/noncompact pairs includes ten infinite series of pairs coming from the classical Lie groups and twelve exceptional pairs coming from the five exceptional Lie groups. Each space is defined locally by its pair of real Lie algebras $(\mathfrak{g},\mathfrak{k})$ (satisfying equations (2.8), (2.12) and (2.18)) as a homogeneous space $\mathcal{G}/\mathcal{K}$.

In table 2.2 we list all pairs of irreducible Riemannian symmetric spaces of dimension $D \leq 10$ (which is sufficient for us) along with the ranks of their $\mathfrak{k}$-invariant forms and their compact names, of which some of the less familiar ones are described in appendix A.

Statistics of Lorentzian symmetric spaces

Having listed all indecomposable Lorentzian symmetric spaces and all irreducible Riemannian symmetric spaces for $D \leq 10$ we can thus count all the possible (families of) Lorentzian symmetric spaces for $D \leq 11$:

After the application of some basic combinatorics, we list in table 2.3 the number of Riemannian symmetric spaces for $D \leq 10$ and in table 2.4 the number of Lorentzian symmetric spaces for $D \leq 11$.

Table 2.2: Irreducible D-dimensional Riemannian symmetric spaces with $D \leq 10$.

D	$\mathfrak{g}$ (Compact)	$\mathfrak{g}$ (Non-compact)	$\mathfrak{k}$	$\mathfrak{k}$-inv. forms	Compact name
2	$\mathfrak{u}(2)$	$\mathfrak{u}(1,1)$	$\mathfrak{u}(1) \oplus \mathfrak{u}(1)$	$0, 2$	S^2
3	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{su}(2)$	$0, 3$	S^3
4	$\mathfrak{u}(3)$	$\mathfrak{u}(2,1)$	$\mathfrak{u}(2) \oplus \mathfrak{u}(1)$	$0, 2, 4$	$\mathbb{C}\mathrm{P}^2$
4	$\mathfrak{sp}(2)$	$\mathfrak{sp}(1,1)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	$0, 4$	S^4
5	$\mathfrak{su}(3)$	$\mathfrak{sl}(3, \mathbb{R})$	$\mathfrak{so}(3)$	$0, 5$	SLAG_3
5	$\mathfrak{su}(4)$	$\mathfrak{sl}(2, \mathbb{H})$	$\mathfrak{sp}(2)$	$0, 5$	S^5
6	$\mathfrak{u}(4)$	$\mathfrak{u}(3,1)$	$\mathfrak{u}(3) \oplus \mathfrak{u}(1)$	$0, 2, 4, 6$	$\mathbb{C}\mathrm{P}^3$
6	$\mathfrak{sp}(2)$	$\mathfrak{sp}(2, \mathbb{R})$	$\mathfrak{u}(2)$	$0, 2, 4, 6$	$\mathrm{G}^+_{\mathbb{R}}(2, 5)$
6	$\mathfrak{so}(7)$	$\mathfrak{so}(6,1)$	$\mathfrak{so}(6)$	$0, 6$	S^6
7	$\mathfrak{so}(8)$	$\mathfrak{so}(7,1)$	$\mathfrak{so}(7)$	$0, 7$	S^7
8	$\mathfrak{u}(4)$	$\mathfrak{u}(2,2)$	$\mathfrak{u}(2) \oplus \mathfrak{u}(2)$	$0, 2, 4^2, 6, 8$	$\mathrm{G}_{\mathbb{C}}(2, 4)$
8	$\mathfrak{u}(5)$	$\mathfrak{u}(4,1)$	$\mathfrak{u}(4) \oplus \mathfrak{u}(1)$	$0, 2, 4, 6, 8$	$\mathbb{C}\mathrm{P}^4$
8	$\mathfrak{so}(9)$	$\mathfrak{so}(8,1)$	$\mathfrak{so}(8)$	$0, 8$	S^8
8	$\mathfrak{sp}(3)$	$\mathfrak{sp}(2,1)$	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$	$0, 4, 8$	$\mathbb{H}\mathrm{P}^2$
8	$\mathfrak{g}_{2(-14)}$	$\mathfrak{g}_{2(2)}$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	$0, 4, 8$	ASSOC
8	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{sl}(3, \mathbb{C})$	$\mathfrak{su}(3)$	$0, 3, 5, 8$	$\mathrm{SU}(3)$
9	$\mathfrak{su}(4)$	$\mathfrak{sl}(4, \mathbb{R})$	$\mathfrak{so}(4)$	$0, 4, 5, 9$	SLAG_4
9	$\mathfrak{so}(10)$	$\mathfrak{so}(9,1)$	$\mathfrak{so}(9)$	$0, 9$	S^9
10	$\mathfrak{u}(6)$	$\mathfrak{u}(5,1)$	$\mathfrak{u}(5) \oplus \mathfrak{u}(1)$	$0, 2, 4, 6, 8, 10$	$\mathbb{C}\mathrm{P}^5$
10	$\mathfrak{so}(11)$	$\mathfrak{so}(10,1)$	$\mathfrak{so}(10)$	$0, 10$	S^{10}
10	$\mathfrak{so}(7)$	$\mathfrak{so}(5,2)$	$\mathfrak{so}(5) \oplus \mathfrak{so}(2)$	$0, 2, 4, 6, 8, 10$	$\mathrm{G}^+_{\mathbb{R}}(2, 7)$
10	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$	$\mathfrak{sp}(2, \mathbb{C})$	$\mathfrak{sp}(2)$	$0, 3, 7, 10$	$\mathrm{Sp}(2)$

Table 2.3: Number of D-dimensional Riemannian symmetric spaces with $D \leq 10$.

D	1	2	3	4	5	6	7	8	9	10
# of spaces	1	3	5	13	21	47	73	161	253	497

Table 2.4: Number of D-dimensional Lorentzian symmetric spaces with $D \leq 11$.

D	1	2	3	4	5	6	7	8	9	10	11
# of spaces	1	3	8	17	38	77	158	299	580	1067	1978

Chapter 3

The homogeneity theorem

3.1 Introduction

We motivate and develop the framework for the strong homogeneity theorem for (Poincaré) supergravity backgrounds. This chapter is based upon work done in collaboration with José Figueroa-O'Farrill in [1, 2] and builds upon previous work in [34, 35].

Let $(\mathcal{M}, g, \Phi, \$)$ be a supergravity background where $(\mathcal{M}, g)$ is an (oriented) connected finite-dimensional Lorentzian spin manifold with Φ the bosonic field content of the background and $\$$ a real spinor bundle constructed in the usual manner from a (possibly reducible) spin representation $\mathcal{S}$.

3.2 Killing vectors

Killing vector fields are normally defined with respect to a (pseudo-)Riemannian manifold because many of their features are contingent on the existence of a metric tensor, but we will first consider the (tautological) starting case of a smooth manifold so that our extension from the category of (pseudo-)Riemannian manifolds to that of supergravity backgrounds is more natural.

We define a *Killing vector field* on a smooth manifold $\mathcal{M}$ to be a vector field $K \in \Gamma(\mathrm{T}\mathcal{M})$ whose flow is a continuous symmetry of the manifold, i.e. Killing vector fields are the infinitesimal generators of continuous symmetries. In the category of smooth manifolds, we have no geometrical structure and so Killing vector fields are simply vector fields.

Let a geometrical structure on $\mathcal{M}$ be defined by some tensor field $A \in \Gamma(\mathrm{T}^p_q\mathcal{M})$. Then we define a Killing vector field K to be a vector field whose flow leaves A invariant[1], i.e.

$$\mathcal{L}_K A = 0 \,. \tag{3.1}$$

[1] Perhaps up to gauge transformations.

Now, under the Lie bracket of vector fields, for K_1, K_2 Killing vector fields and tensor field A, we have

$$\mathcal{L}_{[K_1,K_2]}A = [\mathcal{L}_{K_1}, \mathcal{L}_{K_2}]A = 0 \ . \tag{3.2}$$

Thus the Killing vector fields form a Lie subalgebra of the Lie algebra of vector fields.

Let us turn to the more familiar case of pseudo-Riemannian manifolds whence the geometrical structure we add is the metric tensor g. A Killing vector field K on a pseudo-Riemannian manifold $(\mathcal{M}, g)$ is a vector field whose flow is isometric, i.e. as in equation (3.1) flowing along K leaves the metric invariant: $\mathcal{L}_K g = 0$. Rewriting this in terms of the Levi-Civita connection ∇ gives us the *Killing equation*,

$$g(\nabla_X K, Y) + g(X, \nabla_Y K) = 0 \ , \tag{3.3}$$

where K is a Killing vector field and X, Y any vector field. This means that the canonical covariant derivative of a Killing vector field on a pseudo-Riemannian manifold is a vector field skew-symmetric relative to the metric, with the converse also true.

3.2.1 Working at a point

Killing vector fields are uniquely determined [36] at a point $m \in \mathcal{M}$ by their value K_m *and* the value of their first derivative $(\nabla K)_m$. In order to parallel transport a Killing vector we must consider it as the parallel section of a connection on some vector bundle. Concretely, Killing vector fields are in one-to-one correspondence [37] with sections of the bundle

$$\mathscr{W} := \mathrm{T}\mathcal{M} \oplus \mathfrak{so}(\mathrm{T}\mathcal{M}) \ , \tag{3.4}$$

parallel with respect to the (*Killing transport*)-associated connection,

$$\begin{array}{rccc} \mathscr{D}_X : & \mathscr{W} & \to & \mathscr{W} \\ & (Y, A) & \mapsto & (\nabla_X Y + A(X), \nabla_X A - R(X, Y)) \ , \end{array} \tag{3.5}$$

where $A \in \mathfrak{so}(\mathrm{T}\mathcal{M})$, and $R \in \Omega^2(\mathfrak{so}(\mathrm{T}\mathcal{M}))$ is the curvature tensor of $(\mathcal{M}, g)$.

We have defined A to be a (metric-relative) skew-symmetric endomorphism of the tangent bundle and so using equation (3.5) we see that a $\mathscr{D}$-parallel section (K, A) satisfies

$$0 = -g(A(X), Y) - g(X, A(Y)) = g(\nabla_X K, Y) + g(X, \nabla_Y K) \ , \tag{3.6}$$

recovering Killing's equation equation (3.3).

We may then construct the Lie bracket for $\mathscr{D}$-parallel sections,

$$[(X, A), (Y, B)] = (A(Y) - B(X), [A, B] + R(X, Y)) \ . \tag{3.7}$$

We can then see that this is a valid Lie bracket because,

$$A(Y) - B(X) = -\nabla_Y X + \nabla_X Y = [X, Y] \,, \tag{3.8}$$

and

$$\begin{aligned} [A, B] + R(X, Y) &= -R(\bullet, X)Y + A(B) + R(\bullet, Y)X - B(A) \\ &= -(\nabla A)Y - A(\nabla Y) + (\nabla B)X + B(\nabla X) \\ &= -\nabla \left(A(Y) - B(X) \right) \\ &= -\nabla [X, Y] \,, \end{aligned} \tag{3.9}$$

and so

$$[(X, A), (Y, B)] = ([X, Y], -\nabla[X, Y]) \,, \tag{3.10}$$

which is the expected extension of the standard Lie bracket on vector fields. We note that this Lie bracket fails to satisfy the Jacobi identity if extended to sections of $\mathscr{W}$ that are not $\mathscr{D}$-parallel.

Thus if we wish to work with Killing vectors at a point $m \in \mathcal{M}$, we can consider Killing vector fields as $\mathscr{D}$-parallel sections of $\mathscr{W}$, with parallel transport on $\mathcal{M}$ by $\mathscr{D}$.

3.2.2 Supergravity Killing vectors

Now, to further specialise, we are interested in Killing vector fields of a supergravity background $(\mathcal{M}, g, \Phi, \$)$ and here we may now consider the collection of bosonic fields Φ to be an additional geometric structure. Hence and again, for the flow of K to be a continuous symmetry of the manifold, now in addition to being an isometry it must also leave Φ invariant: $\mathcal{L}_K \Phi = 0$.

If we have gauge fields, then it is their *field strengths* that we will consider as the relevant objects in Φ and we require invariance only up to gauge transformations. This will always be the implication when claiming $\mathcal{L}_K \Phi = 0$.

We thus define the Lie algebra of Killing vector fields of a supergravity background:

$$\mathfrak{g}_0 := \{(K, -\nabla K) \in \Gamma(\mathscr{W}) : \mathscr{D}_X(K, -\nabla K) = 0 = \mathcal{L}_K \Phi, \ \forall \ X \in \Gamma(\mathrm{T}\mathcal{M})\} \,. \tag{3.11}$$

3.3 Killing spinors

The action of a supergravity theory is invariant under local supersymmetry transformations comprised of the field content of the theory and parametrised by arbitrary sections of the spinor bundle we call the supersymmetry parameter. For a classical supergravity background, which is a solution to the supergravity field equations in which all fermionic fields are set to zero, the supersymmetry transformations of the bosonic fields all disappear automatically and so we are left with the transformations of the fermionic fields which depend upon Φ and the spinorial supersymmetry parameter ε. These transformations must disappear for the background to have

any residual local supersymmetry and this means the existence of *Killing spinor fields* ε that render the transformations trivial.

Thus, the *Killing spinor equations* are the system of equations originating from requiring the disappearance of the supersymmetry transformations of the fermionic content of the theory. The supersymmetry transformation of a gravitino yields a connection $\mathcal{D} = \nabla + \Omega$ on $\$$ (with $\Omega \in \mathrm{T}^*\mathcal{M} \otimes \mathrm{End}(\$)$ depending upon Φ). The supersymmetry transformation of a dilatino or gaugino yields an (algebraic) bundle map on $\$$ we will denote by $\mathcal{P}$ or $\mathcal{Q}$ respectively. Of course, not all theories have dilatinos or gauginos but we always have at least a gravitino and so a connection $\mathcal{D}$. Note that $\mathcal{D}$ is not necessarily a metric connection and is not in general induced from a connection on the tangent bundle; it is a true spinor connection.

Note that Φ is a collection of bosonic fields all living in the exterior algebra bundle and their action on the spinor bundle is via the globalisation of the exterior algebra vector space isomorphism with the Clifford algebra (see appendix B.5), of which the spinor bundle is a bundle of modules.

The connection $\mathcal{D}$, $\mathcal{P}$, and $\mathcal{Q}$ are all linear bundle maps on $\$$ depending upon Φ and the intersection of their kernels,

$$\mathscr{K} := \ker \mathcal{D} \cap \ker \mathcal{P} \cap \ker \mathcal{Q} \subset \$ \tag{3.12}$$

forms the vector subbundle of Killing spinor fields so we define the vector space of Killing spinor fields to be

$$\mathfrak{g}_1 := \Gamma(\mathscr{K}) = \{\varepsilon \in \Gamma(\$) : \mathcal{D}\varepsilon = \mathcal{P}\varepsilon = \mathcal{Q}\varepsilon = 0\} \ . \tag{3.13}$$

3.3.1 Working at a point

If we wish to work with Killing spinors at a point $m \in \mathcal{M}$, we note that $\mathcal{D}$,$\mathcal{P}$, and $\mathcal{Q}$ all contain at most first order derivatives and so a Killing spinor field is uniquely determined by its value at a point, with parallel transport on $\mathcal{M}$ by $\mathcal{D}$.

3.3.2 Supersymmetry

The number of supersymmetries of a background is the dimension of the odd subspace of the supersymmetry superalgebra of the background. We define ν to be the fraction of supersymmetry preserved with respect to the maximum supersymmetry of the theory.

It is clear that the number of supersymmetries of a background is also encoded in the rank of the subbundle of Killing spinor fields of a background. Indeed,

$$\nu = \frac{\mathrm{rank}\,\mathscr{K}}{\mathrm{rank}\,\$} \ . \tag{3.14}$$

3.4 The Killing superalgebra

3.4.1 Lie superalgebras

A superalgebra A is a $\mathbb{Z}_2$-graded $\mathbb{K}$-algebra, i.e. a $\mathbb{K}$-algebra A that decomposes into two subspaces $A = A_0 \oplus A_1$ where the product operator respects the grading $A_i \times A_j \to A_{i+j}$.

A Lie superalgebra [38] $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a superalgebra whose product operator is the supercommutator $[\bullet, \bullet]$ that additionally satisfies, for $x, y, z \in \mathfrak{g}$:

- Super skew-symmetry:

$$[x, y] = -(-1)^{|x||y|}[y, x] \tag{3.15}$$

- Super Jacobi identity:

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0 \tag{3.16}$$

Thus $\mathfrak{g}_0$ is a Lie algebra and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module.

3.4.2 The symmetry superalgebra

We may construct a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ is the Lie algebra of (Φ-preserving) Killing vector fields (equation (3.11)) and $\mathfrak{g}_1$ is the vector space of Killing spinor fields (equation (3.13)). We will call this the *symmetry superalgebra* of a background and in order to show that this may form a Lie superalgebra we first define the supercommutator for each grade combination:

$$[\bullet, \bullet] : \mathfrak{g_0} \otimes \mathfrak{g_0} \to \mathfrak{g_0}$$

We have already seen that $\mathfrak{g}_0$ is a Lie subalgebra of Killing vector fields with respect to Lie bracket defined in equation (3.7) and so we define the even-even bracket to be

$$\begin{array}{rccl} [\bullet, \bullet] : & \mathfrak{g}_0 \otimes \mathfrak{g}_0 & \to & \mathfrak{g}_0 \\ & ((X, A), (Y, B)) & \mapsto & (A(Y) - B(X), [A, B] + R(X, Y)) \ . \end{array} \tag{3.17}$$

$$[\bullet, \bullet] : \mathfrak{g_0} \otimes \mathfrak{g_1} \to \mathfrak{g_1}$$

We define the even-odd bracket using the spinorial Lie derivative [39] (see appendix B.10):

$$\begin{array}{rccl} [\bullet, \bullet] : & \mathfrak{g}_0 \otimes \mathfrak{g}_1 & \to & \mathfrak{g}_1 \\ & ((K, -\nabla K), \varepsilon) & \mapsto & \mathcal{L}_K \varepsilon = \nabla_K \varepsilon + \frac{1}{4} \varkappa_1 (\nabla K) \cdot \varepsilon \ , \end{array} \tag{3.18}$$

where, as defined in appendix B.4.3, $\varkappa_1$ is the sign convention used to define the gamma matrix representation of the Clifford algebra with $\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = 2\varkappa_1 \eta_{ab} \mathbb{1}$.

Now, for a Killing vector field K and any spinor field ε and vector field X, a standard identity of the spinorial Lie derivative is

$$\mathcal{L}_K \nabla_X \varepsilon = \nabla_X \mathcal{L}_K \varepsilon + \nabla_{[K,X]} \varepsilon \ . \tag{3.19}$$

By construction we have $\mathcal{L}_K \Phi = 0$ and so $\mathcal{L}_K \Omega(\Phi) = 0$; as such this identity extends to $\mathcal{D}$, yielding

$$\mathcal{D}_X \mathcal{L}_K \varepsilon = \mathcal{L}_K \mathcal{D}_X \varepsilon - \mathcal{D}_{[K,X]} \varepsilon \ , \tag{3.20}$$

and so for $\mathcal{D}$-parallel ε this tells us that $\mathcal{L}_K \varepsilon$ is also $\mathcal{D}$-parallel. Also, since $\mathcal{P}$ and $\mathcal{Q}$ are linear bundle maps depending only on Φ we have

$$[\mathcal{L}_K, \mathcal{P}] = [\mathcal{L}_K, \mathcal{Q}] = 0 \ . \tag{3.21}$$

Thus ε Killing implies that $\mathcal{L}_K \varepsilon$ is also Killing. As such the even-odd bracket is well-defined when we impose its skew-symmetry.

$[\bullet, \bullet] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$

The brackets so far have been defined in full generality and do not depend on the particulars of the supergravity theory or its spinor bundle $\$$. However, the odd-odd bracket will be different.

Viewing the real spinor bundle as a bundle of Clifford modules, let us presume that we have a (real) pin-invariant spinor inner product $[\![\bullet, \bullet]\!] : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ that naturally globalises with an abuse of notation to $[\![\bullet, \bullet]\!] : \$ \times \$ \to \mathbb{R}$. The details of this inner product will depend upon the signature of the Clifford algebra and the spin representation $\mathcal{S}$ of the spinor bundle but because we are working with a real spinor bundle will either be real symmetric or real symplectic.

Let us denote the Lorentzian inner product on $\mathrm{T}\mathcal{M}$ as $(\!| \bullet, \bullet |\!)$. Then we define the *squaring map* $\Xi : \$ \times \$ \to \mathrm{T}\mathcal{M}$ as the transpose of the Clifford action relative to this spinor inner product and the Lorentzian inner product, i.e. for all $\varepsilon_{1,2} \in \Gamma(\$)$, $X \in \Gamma(\mathrm{T}\mathcal{M})$,

$$(\!| \Xi(\varepsilon_1, \varepsilon_2), X |\!) = [\![\varepsilon_1, X^\flat \cdot \varepsilon_2]\!] \ . \tag{3.22}$$

We will require that the bilinear $[\![\varepsilon_1, X^\flat \cdot \varepsilon_2]\!]$ be symmetric in $\varepsilon_{1,2}$ and so, looking ahead, also the squaring map. Thus, denoting the adjoint of $X^\flat$ with respect to the spinor inner product by $\dot{X}^\flat$ and the symmetry of the spinor inner product by $\varkappa_3$ we have

$$[\![\varepsilon_1, X^\flat \cdot \varepsilon_2]\!] = [\![\dot{X}^\flat \cdot \varepsilon_1, \varepsilon_2]\!] = \varkappa_3 [\![\varepsilon_2, \dot{X}^\flat \cdot \varepsilon_1]\!] \stackrel{!}{=} [\![\varepsilon_2, X^\flat \cdot \varepsilon_1]\!] \ , \tag{3.23}$$

and so in order for the bilinear to be symmetric, we require complementarity of the inner product's symmetry and its 1-form adjoint, i.e. $\dot{X}^\flat = \varkappa_3 X^\flat$. Given this 1-form only carries manifold indices, this means that its adjoint in the Clifford bundle must be its adjoint with

respect to a pinor inner product. The adjoint of a rank-one element of the Clifford algebra with respect to a pinor inner product depends upon which involution induced the inner product. It is self-adjoint under the check involution and anti-self-adjoint under the hat involution. Thus we require either an $\mathbb{R}$-symplectic pinor inner product induced from the hat involution or an $\mathbb{R}$-symmetric pinor inner product induced from the check involution. We note that such a pinor inner product always exists in Lorentzian signature with $d \leq 11$.

Now, $\Omega(X)$ acts on spinors via the Clifford action and so the most general form in which we can construct a covariant $\Omega(X)$ is:

$$\Omega(X) = \sum_i a_i W_i^{(n_i)} \cdot X^\flat + b_i X^\flat \cdot W_i^{(n_i)} \ , \tag{3.24}$$

where $W_i^{(n_i)}$ is some Z-valued n_i-form constructed from objects in Φ with $\$$ a bundle of Z-modules, and a_i, b_i are real constants. We will assume that we only have a single term because our argument will distribute over the sum, and so we take

$$\Omega(X) = aW^{(n)} \cdot X^\flat + bX^\flat \cdot W^{(n)} \ . \tag{3.25}$$

The adjoint relative the spinor inner product is then

$$\dot{\Omega}(X) = \varkappa_3 \left(b\dot{W}^{(n)} \cdot X^\flat + aX^\flat \cdot \dot{W}^{(n)} \right) \ . \tag{3.26}$$

Now, for $\varepsilon_{1,2}$ Killing spinors and $X, Y \in \Gamma(\mathrm{T}\mathcal{M})$, using equation (3.22) we have

$$\begin{aligned}
(\!|\nabla_X \Xi(\varepsilon_1, \varepsilon_2), Y|\!) &= X(\!|\Xi(\varepsilon_1, \varepsilon_2), Y|\!) - (\!|\Xi(\varepsilon_1, \varepsilon_2), \nabla_X Y|\!) \\
&= X[\![\varepsilon_1, Y^\flat \cdot \varepsilon_2]\!] - [\![\varepsilon_1, (\nabla_X Y^\flat) \cdot \varepsilon_2]\!] \\
&= [\![\nabla_X \varepsilon_1, Y^\flat \cdot \varepsilon_2]\!] + [\![\varepsilon_1, Y^\flat \cdot \nabla_X \varepsilon_2]\!] \\
&= -[\![\Omega(X) \cdot \varepsilon_1, Y^\flat \cdot \varepsilon_2]\!] - [\![\varepsilon_1, Y^\flat \cdot \Omega(X) \cdot \varepsilon_2]\!] \\
&= -[\![\varepsilon_1, \dot{\Omega}(X) \cdot Y^\flat \cdot \varepsilon_2]\!] - [\![\varepsilon_1, Y^\flat \cdot \Omega(X) \cdot \varepsilon_2]\!] \\
&= -[\![\varepsilon_1, \varkappa_3 \left(b\dot{W}^{(n)} \cdot X^\flat + aX^\flat \cdot \dot{W}^{(n)} \right) \cdot Y^\flat \cdot \varepsilon_2]\!] \\
&\quad -[\![\varepsilon_1, Y^\flat \cdot \left(aW^{(n)} \cdot X^\flat + bX^\flat \cdot W^{(n)} \right) \cdot \varepsilon_2]\!] \\
&= -b[\![\varepsilon_1, \left(Y^\flat \cdot X^\flat \cdot W^{(n)} + \varkappa_3 \dot{W}^{(n)} \cdot X^\flat \cdot Y^\flat \right) \cdot \varepsilon_2]\!] \\
&\quad -a[\![\varepsilon_1, \left(Y^\flat \cdot W^{(n)} \cdot X^\flat + \varkappa_3 X^\flat \cdot \dot{W}^{(n)} \cdot Y^\flat \right) \cdot \varepsilon_2]\!] \ .
\end{aligned} \tag{3.27}$$

Equation (3.3) tells us that in order for $\Xi(\varepsilon_1, \varepsilon_2)$ to be a Killing vector field, $(\!|\nabla_X \Xi(\varepsilon_1, \varepsilon_2), Y|\!)$ must be skew-symmetric in X, Y. This is clearly satisfied if (although not iff) for each term in $\Omega(X)$,

$$\dot{W}^{(n)} = -\varkappa_3 W^{(n)} \ . \tag{3.28}$$

In order for $\Xi(\varepsilon_1, \varepsilon_2)$ to be a supergravity Killing vector field, it must also leave Φ invariant

and to show this will often require us to invoke the field equations of the particular theory.

We will construct the spinor inner product case by case for each supergravity theory and show that it satisfies, for $\varkappa_3$ the symmetry of the pinor inner product:

1. $\dot{X}^\flat = \varkappa_3 X^\flat$, and
2. $\dot{W}^{(n)} = -\varkappa_3 W^{(n)}$, and
3. $\mathcal{L}_{\Xi(\varepsilon_1,\varepsilon_2)}\Phi = 0$.

Let us assume its existence for the rest of this chapter. We can then extend the squaring map naturally (and also call this the squaring map) to

$$\begin{array}{rccl} \chi: & \$ \otimes \$ & \to & \mathscr{W} \\ & (\varepsilon_1, \varepsilon_2) & \mapsto & (\Xi(\varepsilon_1,\varepsilon_2), -\nabla\Xi(\varepsilon_1,\varepsilon_2)) \ , \end{array} \tag{3.29}$$

whose restriction to the subbundle of Killing spinors we will use to define the odd-odd bracket:

$$\begin{array}{rccl} [\bullet,\bullet]: \mathfrak{g}_1 \otimes \mathfrak{g}_1 & \to & \mathfrak{g}_0 \\ (\varepsilon_1, \varepsilon_2) & \mapsto & \chi(\varepsilon_1, \varepsilon_2) \ , \end{array} \tag{3.30}$$

3.4.2.1 Super Jacobi identity

$[\mathfrak{g}_0, \mathfrak{g}_0, \mathfrak{g}_0]$

This component of the super Jacobi identity is nothing more than the standard Jacobi identity of $\mathfrak{g}_0$ as a Lie algebra. We have shown in section 3.2.1 that the Lie bracket of equation (3.7) is equivalent to the standard Lie bracket, whence its Jacobi identity follows from the Jacobi identity of the Lie algebra of vector fields with the standard Lie bracket.

$[\mathfrak{g}_0, \mathfrak{g}_0, \mathfrak{g}_1]$

This identity is, for X, Y, ε Killing,

$$[X,[Y,\varepsilon]] + [Y,[\varepsilon,X]] + [\varepsilon,[X,Y]] \stackrel{!}{=} 0 \ . \tag{3.31}$$

Now, the spinorial Lie derivative satisfies

$$\mathcal{L}_{[X,Y]}\varepsilon = \mathcal{L}_X\mathcal{L}_Y\varepsilon - \mathcal{L}_Y\mathcal{L}_X\varepsilon, \tag{3.32}$$

which is precisely this component of the super Jacobi identity.

$[\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_1]$

This identity is, for $K, \varepsilon_{1,2}$ Killing,

$$[K, [\varepsilon_1, \varepsilon_2]] \stackrel{!}{=} [\varepsilon_1, [K, \varepsilon_2]] + [[K, \varepsilon_1], \varepsilon_2] \,. \tag{3.33}$$

This is equivalent to requiring that, with X any vector field,

$$(\!|\mathcal{L}_K \Xi(\varepsilon_1, \varepsilon_2), X|\!) \stackrel{!}{=} [\![\mathcal{L}_K \varepsilon_1, X^\flat \cdot \varepsilon_2]\!] + [\![\varepsilon, X^\flat \cdot \mathcal{L}_K \varepsilon_2]\!] \,. \tag{3.34}$$

The left hand side is

$$\begin{aligned}(\!|\mathcal{L}_K \Xi(\varepsilon_1, \varepsilon_2), X|\!) &= (\!|\nabla_K \Xi(\varepsilon_1, \varepsilon_2) - \nabla_{\Xi(\varepsilon_1, \varepsilon_2)} K, X|\!) \\ &= [\![\nabla_K \varepsilon_1, X^\flat \cdot \varepsilon_2]\!] + [\![\varepsilon_1, X^\flat \cdot \nabla_K \varepsilon_2]\!] - (\!|\nabla_K \Xi(\varepsilon_1, \varepsilon_2), X|\!) \\ &= [\![\nabla_K \varepsilon_1, X^\flat \cdot \varepsilon_2]\!] + [\![\varepsilon_1, X^\flat \cdot \nabla_K \varepsilon_2]\!] + [\![\varepsilon_1, (\nabla_X K^\flat) \cdot \varepsilon_2]\!] \,.\end{aligned} \tag{3.35}$$

The right hand side is

$$\begin{aligned}[\![\mathcal{L}_K \varepsilon_1, X^\flat \cdot \varepsilon_2]\!] + [\![\varepsilon_1, X^\flat \cdot \mathcal{L}_K \varepsilon_2]\!] &= [\![\nabla_K \varepsilon_1, X^\flat \cdot \varepsilon_2]\!] + [\![\varepsilon_1, X^\flat \cdot \nabla_K \varepsilon_1]\!] \\ &\quad - \tfrac{1}{4} \varkappa_1 [\![\varepsilon_1, \Big((\nabla K) \cdot X^\flat - X^\flat \cdot (\nabla K) \Big) \cdot \varepsilon_2]\!] \\ &= [\![\nabla_K \varepsilon_1, X^\flat \cdot \varepsilon_2]\!] + [\![\varepsilon_1, X^\flat \cdot \nabla_K \varepsilon_1]\!] \\ &\quad - \tfrac{1}{4} \varkappa_1 [\![\varepsilon_1, -4\varkappa_1 (\nabla_X K^\flat) \cdot \varepsilon_2]\!] \\ &= [\![\nabla_K \varepsilon_1, X^\flat \cdot \varepsilon_2]\!] + [\![\varepsilon_1, X^\flat \cdot \nabla_K \varepsilon_2]\!] + [\![\varepsilon_1, (\nabla_X K^\flat) \cdot \varepsilon_2]\!] \,,\end{aligned} \tag{3.36}$$

where we have used the fact that ∇K is anti-self-adjoint with respect to the spinor inner product (which is true for *any* pin-invariant spinor inner product).

Thus, this component of the super Jacobi identity is satisfied.

$[\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1]$

This identity is, for $\varepsilon_{1,2,3}$ Killing,

$$[\varepsilon_1, [\varepsilon_2, \varepsilon_3]] + [\varepsilon_3, [\varepsilon_1, \varepsilon_2]] + [\varepsilon_2, [\varepsilon_3, \varepsilon_1]] \stackrel{!}{=} 0 \,. \tag{3.37}$$

We note that the odd-odd bracket, being symmetric and bilinear, is determined by its value on the diagonal via the polarisation identity:

$$2[\varepsilon_1, \varepsilon_2] = [\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2] - [\varepsilon_1, \varepsilon_1] - [\varepsilon_2, \varepsilon_2] \,, \tag{3.38}$$

and thus equation (3.37) is exactly equivalent to, for ε Killing,

$$[\varepsilon, [\varepsilon, \varepsilon]] \stackrel{!}{=} 0 \ . \tag{3.39}$$

As with the construction of the odd-odd bracket, we will generally have to appeal to the details of the particular supergravity theory to show that this identity holds.

3.4.3 The Killing superalgebra

We have elucidated the construction of the symmetry superalgebra of a supergravity background. However, we are particularly interested in the canonical ideal of this superalgebra that we will call the *Killing superalgebra*:

$$\tilde{\mathfrak{g}} := \mathrm{span}([\mathfrak{g}_1, \mathfrak{g}_1]) \oplus \mathfrak{g}_1 \subset \mathfrak{g} \ . \tag{3.40}$$

This is the sub-superalgebra of the symmetry superalgebra, constructed entirely from the Killing spinors of a background.

3.5 The theorem

3.5.1 Motivation

We have seen that, given we are able to construct a suitable squaring map, Killing spinor fields of a supergravity background square to Killing vector fields of the background and indeed, given a vector space of Killing spinor fields $\mathfrak{g}_1$, we may construct a Lie algebra of Killing vector fields $[\mathfrak{g}_1, \mathfrak{g}_1]$. This fact does not even require the structure of a Lie superalgebra — given we have the squaring map, we see that all super Jacobi identity brackets are satisfied automatically apart from the $[\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1]$ component. For what follows, it will be immaterial to us whether this component of the identity is satisfied. We will call such a structure an *almost Killing superalgebra*, i.e. we have the structure of a Killing superalgebra as described in section 3.4 except that the $[\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1]$ component of the super Jacobi identity is not necessarily satisfied.

Now, supposing we have an almost Killing superalgebra on a supergravity background, we see that supersymmetries of the background generate conventional symmetries of the background. The more conventional symmetries that a background has, the more geometrically simple it is, and there are a number of refinements for describing geometrical simplicity that we may call upon. So a natural question to ask is if there are any thresholds of supersymmetry that saturate the requirements for a particular refinement of geometrical simplicity.

This line of reasoning led to Patrick Meessen's *homogeneity conjecture*, reviewed in [34]:

> All supergravity backgrounds with $\nu > \frac{1}{2}$ are homogeneous.

This is of course the most natural refinement of geometrical simplicity to aim for — homogeneity of a background essentially means that the symmetries of the background saturate the

tangent bundle, or in other words the symmetries make any point in the background look much like any other point in the background, because we can transitively flow along symmetries.

We must however modify this slightly, because in supergravity we only work with local metrics and the background may not in general be complete. As such, the more relevant concept is local homogeneity (see section 2.1.5) and so the conjecture becomes

> All supergravity backgrounds with $\nu > \frac{1}{2}$ are *locally* homogeneous.

We note that there are, of course, no counterexamples to this conjecture, and as we will see, nor can there be.

3.5.2 Working at a point

We have shown that both Killing spinor fields and Killing vector fields, given as sections of the appropriate vector bundles parallel with respect to the appropriate connections, may be defined entirely by their value at a point, with parallel transport around $\mathcal{M}$ via said connections. As such, let us now permanently fix a point $m \in \mathcal{M}$:

The tangent bundle $\mathrm{T}\mathcal{M}$ may be obtained as the vector bundle associated to the spin bundle through the vector representation $\mathcal{V}$ of the correspondent special orthogonal group. As such, we identify the tangent bundle fiber $\mathrm{T}_m\mathcal{M}$ with $\mathcal{V}$.

The spinor bundle $\$$ is the vector bundle associated to the spin bundle through the real (not necessarily irreducible) spinor representation $\mathcal{S}$. As such, we identify the spinor bundle fiber $\$_m$ with $\mathcal{S}$.

The spinor bundle subbundle of Killing spinors then restricts at a point to a subspace of $\mathcal{S}$,

$$\mathcal{W} := \mathscr{K}_m \subset \mathcal{S} \,, \tag{3.41}$$

and the squaring map restricts at a point to the map,

$$\varphi := \Xi_m : \mathcal{S} \times \mathcal{S} \to \mathcal{V} \,. \tag{3.42}$$

3.5.3 Proof

We aim to show that if $\dim \mathcal{W} > \frac{1}{2} \dim \mathcal{S}$ (i.e. $\nu > \frac{1}{2}$), then the restriction of Ξ_m to $\mathcal{W}$ is surjective, i.e.

$$\varphi\big|_{\mathcal{W}} : \mathcal{W} \times \mathcal{W} \twoheadrightarrow \mathcal{V} \,. \tag{3.43}$$

This would mean that $\mathcal{V}$ and so $\mathrm{T}_m\mathcal{M}$ are spanned by the values of supergravity Killing vectors at m and so we can flow along supergravity Killing vector fields to any point in a local neighbourhood U around m. Thus the background is locally homogeneous.

Before continuing however, we will need to impose one more (not unreasonable) requirement on the squaring map: that squaring a single Killing spinor produces a Killing vector that is *causal*,

i.e null or timelike with respect to the Lorentzian inner product:

$$\varkappa_0|\Xi(\epsilon,\epsilon)|^2 = \varkappa_0(\!|\Xi(\epsilon,\epsilon),\Xi(\epsilon,\epsilon)|\!) = \varkappa_0[\![\epsilon,\Xi(\epsilon,\epsilon)^\flat\cdot\epsilon]\!] \leq 0\ , \tag{3.44}$$

where, as defined in appendix C, $\varkappa_0$ is the sign convention of the metric with $\varkappa_0 = +1$ denoting a mostly plus metric and $\varkappa_0 = -1$ a mostly minus metric.

This requirement will be necessary for our proof and we will have to show this for each supergravity theory.

Let $\dim\mathcal{W} > \frac{1}{2}\dim\mathcal{S}$.

Now, we have the Lorentzian inner product on $\mathcal{V}$ and so as $\mathcal{V}$ is thus a semi-inner product space, for any subspace $\mathcal{U}\subset\mathcal{V}$ we have $\dim\mathcal{U}+\dim\mathcal{U}^\perp = \dim\mathcal{V}$ where the perpendicular complement is defined as

$$\mathcal{U}^\perp = \{x\in\mathcal{V} : (\!|x,u|\!) = 0\ \forall\ u\in\mathcal{U}\}\ . \tag{3.45}$$

Thus, on dimensional grounds, the map $\varphi\big|_{\mathcal{W}}$ is surjective iff the perpendicular complement of its image (relative to the Lorentzian inner product on $\mathcal{V}$) is trivial,

$$\{0\} \overset{!}{=} (\mathrm{Im}\,\varphi\big|_{\mathcal{W}})^\perp = \{x\in\mathcal{V} : (\!|x,k|\!) = 0\ \forall\ k\in\mathrm{Im}\,\varphi\big|_{\mathcal{W}}\}\ . \tag{3.46}$$

Using the definition of the squaring map in equation (3.22) this is true iff the only vector $x\in\mathcal{V}$ satisfying

$$[\![w_1, x^\flat\cdot w_2]\!] = 0\ , \tag{3.47}$$

for all $w_{1,2}\in\mathcal{W}$ is the zero vector $x=0$.

Now, let us assume that a vector $x\neq 0$ exists that satisfies equation (3.47).

The spinor inner product on $\mathcal{S}$ also makes $\mathcal{S}$ a semi-inner product space, and so for $\mathcal{W}\subset\mathcal{S}$ we have $\dim\mathcal{W}+\dim\mathcal{W}^\perp = \dim\mathcal{S}$ where the perpendicular complement is defined as

$$\mathcal{W}^\perp = \{\epsilon\in\mathcal{S} : [\![\epsilon,w]\!] = 0\ \forall\ w\in\mathcal{W}\}\ . \tag{3.48}$$

It is clear that such an x is thus necessarily a map

$$x : \mathcal{W}\to\mathcal{W}^\perp\ . \tag{3.49}$$

Now, $\dim\mathcal{W}+\dim\mathcal{W}^\perp = \dim\mathcal{S}$ but $\dim\mathcal{W} > \frac{1}{2}\dim\mathcal{S}$, and so $\dim\mathcal{W}^\perp < \dim\mathcal{W}$. Thus, simply on dimensional grounds, as a map x must have non-trivial kernel. Yet the action of x on $\mathcal{W}$ is the Clifford action and so

$$x^2 = \varkappa_1(\!|x,x|\!)\mathbb{1}\ , \tag{3.50}$$

whence x has non-trivial kernel iff it is null, $(\!|x,x|\!)\overset{!}{=}0$.

Thus, backtracking to equation (3.46), the space $(\mathrm{Im}\,\varphi\big|_{\mathcal{W}})^{\perp}$ must be spanned by null vectors in $\mathcal{V}$ and so is a totally null subspace of $\mathcal{V}$. Any null subspace of a Lorentzian vector space is at most 1-dimensional (see appendix D) and so $\dim(\mathrm{Im}\,\varphi\big|_{\mathcal{W}})^{\perp} \leq 1$. If $\dim(\mathrm{Im}\,\varphi\big|_{\mathcal{W}})^{\perp} = 0$ then $(\mathrm{Im}\,\varphi\big|_{\mathcal{W}})^{\perp}$ is of course trivial and so $\varphi\big|_{\mathcal{W}}$ is surjective as desired. Let us then assume the case $\dim(\mathrm{Im}\,\varphi\big|_{\mathcal{W}})^{\perp} = 1$ whence $(\mathrm{Im}\,\varphi\big|_{\mathcal{W}})^{\perp}$ is spanned by the null vector x.

The perpendicular complement of a null subspace of a Lorentzian vector space contains only itself and spacelike vectors (see appendix D) and so $\mathrm{Im}\,\varphi\big|_{\mathcal{W}}$ is spanned by x and spacelike vectors. However, we earlier imposed that the squaring map acting on a single Killing spinor must only produce *causal* Killing vectors whence they are not spacelike and so they are necessarily collinear with x. Thus

$$\varphi(\epsilon, \epsilon) = \lambda(\epsilon)x \tag{3.51}$$

for some function $\lambda : \mathcal{W} \to \mathbb{R}$.

Now, we consider the squaring map on two different Killing spinors, $\varphi(\epsilon_1, \epsilon_2)$ for $\epsilon_{1,2} \in \mathcal{W}$. By polarisation we have

$$\begin{aligned} 2\varphi(\epsilon_1, \epsilon_2) &= \varphi(\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_2) - \varphi(\epsilon_1, \epsilon_1) - \varphi(\epsilon_2, \epsilon_2) \\ &= \lambda(\epsilon_1 + \epsilon_2)x - \lambda(\epsilon_1)x - \lambda(\epsilon_2)x \\ &= (\lambda(\epsilon_1 + \epsilon_2) - \lambda(\epsilon_1) - \lambda(\epsilon_2))\, x \ , \end{aligned} \tag{3.52}$$

whence $\mathrm{Im}\,\varphi\big|_{\mathcal{W}}$ is collinear with x and so $\dim \mathrm{Im}\,\varphi\big|_{\mathcal{W}} = 1$.

But we already know that $\dim(\mathrm{Im}\,\varphi\big|_{\mathcal{W}})^{\perp} = 1$ and so $\dim \mathcal{V} = \dim \mathrm{Im}\,\varphi\big|_{\mathcal{W}} + \dim(\mathrm{Im}\,\varphi\big|_{\mathcal{W}})^{\perp} = 2$. For $\dim \mathcal{V} = D > 2$ we thus have a contradiction and so $\dim(\mathrm{Im}\,\varphi\big|_{\mathcal{W}})^{\perp} = 0$ whence $\varphi\big|_{\mathcal{W}}$ surjects onto $\mathcal{V}$ and we have shown local homogeneity of the background.

3.5.4 Summary

Let us recap by summarising the necessary requirements for the homogeneity theorem to apply.

We must have a supergravity background $(\mathcal{M}, g, \Phi, \$, [\![\bullet, \bullet]\!])$ where $(\mathcal{M}, g)$ is a connected $(D > 2)$-dimensional Lorentzian spin manifold, Φ the bosonic field content of the background, $\$$ a real spinor bundle constructed in the usual manner from a (possibly reducible) spinor representation $\mathcal{S}$, and $[\![\bullet, \bullet]\!]$ a pin-invariant spinor inner product with symmetry $\varkappa_3$ on $\$$. We construct the squaring map $\Xi : \$ \times \$ \to \mathrm{T}\mathcal{M}$ using equation (3.22). Then we require:

1. For $X \in \Gamma(\mathrm{T}\mathcal{M})$, the squaring map must be symmetric and so we require:

$$\dot{X}^{\flat} = \varkappa_3 X^{\flat} \ , \text{ and} \tag{3.53}$$

2. For $\varepsilon_{1,2} \in \Gamma(\mathscr{K})$ two Killing spinor fields, the vector field $K = \Xi(\varepsilon_1, \varepsilon_2)$ produced by the squaring map must be a *Killing* vector field and so we require that $\mathcal{L}_K g = 0$. Thus, for Ω

taking the form described in equation (3.24), it is sufficient that all $W^{(n)}$ satisfy:

$$\dot{W}^{(n)} = -\varkappa_3 W^{(n)} \ , \text{ and} \tag{3.54}$$

3. For $\varepsilon \in \Gamma(\mathscr{K})$ a single Killing spinor field, the Killing vector field $K = \Xi(\varepsilon, \varepsilon)$ produced by the squaring map must be *causal* and so we require:

$$\varkappa_0(\!|K, K|\!) \leq 0 \ , \text{ and} \tag{3.55}$$

4. For $\varepsilon_{1,2} \in \Gamma(\mathscr{K})$ two Killing spinor fields, the Killing vector field $K = \Xi(\varepsilon_1, \varepsilon_2)$ produced by the squaring map must be a *supergravity* Killing vector field and so we require:

$$\mathcal{L}_K \Phi = 0 \ . \tag{3.56}$$

Satisfying equations (3.53), (3.54) and (3.56) will give us an almost Killing superalgebra and then also satisfying equation (3.55) allows us to deduce local homogeneity of the background using the almost Killing superalgebra. Note that equations (3.53) to (3.55) are all constructed using only the connection $\mathcal{D}$ but showing that equation (3.56) is satisfied will in general require the invocation of $\mathcal{P}$ and $\mathcal{Q}$.

If we wish to construct a full Killing superalgebra we must additionally show that the $[\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1]$ component of the super Jacobi identity is satisfied as described in section 3.4.2.1 which means requiring that for $\varepsilon \in \Gamma(\mathscr{K})$ a single Killing spinor field, the Killing vector field $K = \Xi(\varepsilon, \varepsilon)$ produced by the squaring map must leave it invariant:

$$\mathcal{L}_K \varepsilon = 0 \ . \tag{3.57}$$

Chapter 4

Application of the theorem

4.1 Introduction

Using the results from chapter 3, we run through the application of the strong homogeneity theorem to a number of top-dimensional Poincaré supergravities. In some cases (sections 4.2.7 and 4.3.6), we refer to the original literature for particular results necessary to demonstrate the existence of the Killing superalgebra. This chapter is based upon work done in collaboration with José Figueroa-O'Farrill in [1, 2]

4.2 $D = 11$

4.2.1 Introduction

The Killing superalgebra of $D = 11$ supergravity was described in [35] and the homogeneity theorem in [1]. We briefly review these constructions in our formalism. The homogeneity theorem for $D = 11$ supergravity places a new and firm control on highly supersymmetric $D = 11$ supergravity backgrounds.

4.2.2 Conventions

In the original construction of $D = 11$ supergravity [19], the sign conventions adopted are $(\varkappa_0, \varkappa_1) = (-1, +1)$. Other authors use conventions $(-1, -1)$ [35] and $(+1, +1)$ [40]. Conventions with $\varkappa_0 = \varkappa_1$ have a $\mathrm{C}\ell(1, 10)$-module spinor representation and spinors are real Majorana whereas for $\varkappa_0 = -\varkappa_1$ the spinor representation is a $\mathrm{C}\ell(10, 1)$-module and they are imaginary pseudo-Majorana. We will follow [35] and adopt the conventions $(\varkappa_0, \varkappa_1) = (-1, -1)$.

4.2.3 Definition

The field content of $D = 11$ supergravity (with all fields transforming in varying representations of $\mathrm{SO}(9)$) is:

Symbol	Count	Name	Description	Spin	D.o.f
A	1	Gauge potential	3-form	1	84
g	1	Graviton	Lorentzian metric	2	44
ψ	1	Gravitino	γ-traceless vector-pinor	3/2	128

We construct the field strength,

$$F = \mathrm{d}A \tag{4.1}$$

We are interested in $D = 11$ supergravity backgrounds and so we will set all fermionic field content (ψ) to zero from here on and it will not enter into the discussion. The action and field equations will not concern us in what follows.

Our tangent bundle spin group representation $\mathcal{V}$ is the 11-dimensional real vector representation of $\mathrm{SO}(1,10)$ equipped with the invariant Lorentzian inner product $(\!|\bullet\,,\,\bullet|\!)$.

4.2.4 Spinor representation

Having chosen the convention $(\varkappa_0, \varkappa_1) = (-1,-1)$, the Clifford algebra of relevance is $\mathrm{C}\ell(1,10)$. Let $\mathcal{P}$ denote either one of the two irreducible pinor representations of $\mathrm{C}\ell(1,10)$ which are 32-dimensional and real. These two representations differ by the action of the centre of $\mathrm{C}\ell(1,10)$ and for our purposes it does not matter which one we pick. Let $\mathcal{S}$ be the spinor representation obtained as a restriction of $\mathcal{P}$ to $\mathrm{Spin}(1,10)$. Thus spinors are 32-dimensional and real, and this is the representation with which we construct the spinor bundle $\$$.

The Killing spinor equation obtained from variation of the gravitino is

$$\delta\psi_X = \mathcal{D}_X\varepsilon = \nabla_X\varepsilon + \tfrac{1}{24}\left(3F\cdot X - X\cdot F\right)\cdot\varepsilon \stackrel{!}{=} 0\,, \tag{4.2}$$

or equivalently using equation (B.9),

$$\delta\psi_X = \mathcal{D}_X\varepsilon = \nabla_X\varepsilon + \left(\tfrac{1}{6}\iota_X F + \tfrac{1}{12}X^\flat\wedge F\right)\cdot\varepsilon \stackrel{!}{=} 0\,. \tag{4.3}$$

4.2.5 Spinor inner product

We have a $(\mathrm{Pin}(1,10)$-invariant $\mathbb{R}$-symplectic inner product on $\mathcal{P}$ (so $\varkappa_3 = -1$) induced from the hat involution of $\mathrm{C}\ell(1,10)$ and so equation (3.53) is satisfied.

4.2.6 Almost Killing superalgebra and homogeneity

Viewing F as an endomorphism of $\mathcal{P}$, its adjoint with respect to the spinor inner product is its image under the hat involution on $\mathrm{C}\ell(1,10)$. The exterior algebra isomorphism sends a 4-form to a rank-4 totally antisymmetric element of the Clifford algebra, and so it is invariant under the hat involution. Thus F is self-adjoint as required by equation (3.54) with $\varkappa_3 = -1$.

We construct the squaring map as described in equation (3.22) and, choosing a pseudo-orthonormal basis $\mathfrak{d}_\mu$ for $\mathcal{V}$ and corresponding gamma matrices (see appendix B.11.1) for $\mathcal{P}$, the squaring map takes the concrete form,

$$\Xi(\varepsilon_1, \varepsilon_2) = [\![\varepsilon_1, \Gamma^\mu \varepsilon_2]\!] \mathfrak{d}_\mu = \overline{\varepsilon}_1 \Gamma^\mu \varepsilon_2 \mathfrak{d}_\mu = \varepsilon_1^\dagger \Gamma_0 \Gamma^\mu \varepsilon_2 \mathfrak{d}_\mu \ , \tag{4.4}$$

where we denote the Dirac adjoint $\overline{\varepsilon} := \varepsilon^\dagger \Gamma_0$.

Now, if we look at the $\mathfrak{d}_0$ component of a vector $K = \Xi(\varepsilon, \varepsilon)$ obtained from the squaring map, we see that for non-zero ε,

$$K^0 = \varepsilon^\dagger \Gamma_0 \Gamma^0 \varepsilon = -\varepsilon^\dagger \varepsilon = -\varepsilon^\dagger \varepsilon = -|\varepsilon|^2 < 0 \ , \tag{4.5}$$

This means that a vector field K constructed by squaring a single non-zero spinor field ε is necessarily causal because otherwise we could of course Lorentz-transform to the rest frame where $K^0 = 0$. Thus equation (3.55) is satisfied.

Using equation (4.2) we have for a Killing spinor ε,

$$\nabla_X \varepsilon = -\Omega(X) \cdot \varepsilon = -\left(\tfrac{1}{6} \imath_X F + \tfrac{1}{12} X^\flat \wedge F \right) \cdot \varepsilon \ . \tag{4.6}$$

The adjoint of $\Omega(X)$ with respect to the spinor inner product is its image under the hat involution,

$$\hat{\Omega}(X) = \tfrac{1}{6} \imath_X F - \tfrac{1}{12} X^\flat \wedge F \ . \tag{4.7}$$

Let us define a 2-form θ constructed from two Killing spinor fields $\varepsilon_{1,2}$ via the spinor inner product as

$$\theta_{\mu\nu} = [\![\varepsilon_1, \Gamma_{\mu\nu} \varepsilon_2]\!] \ . \tag{4.8}$$

Then we have

$$\begin{aligned} \nabla_\mu \theta_{\nu\rho} &= \nabla_\mu [\![\varepsilon_1, \Gamma_{\nu\rho} \varepsilon_2]\!] \\ &= [\![\nabla_\mu \varepsilon_1, \Gamma_{\nu\rho} \varepsilon_2]\!] + [\![\varepsilon_1, \Gamma_{\nu\rho} \nabla_\mu \varepsilon_2]\!] \\ &= -[\![\Omega_\mu \varepsilon_1, \Gamma_{\nu\rho} \varepsilon_2]\!] - [\![\varepsilon_1, \Gamma_{\nu\rho} \Omega_\mu \varepsilon_2]\!] \\ &= -[\![\varepsilon_1, \left(\Gamma_{\nu\rho} \Omega_\mu + \hat{\Omega}_\mu \Gamma_{\nu\rho} \right) \varepsilon_2]\!] \ . \end{aligned} \tag{4.9}$$

Now, we have (with liberal use of equation (B.13))

$$\begin{aligned} \Gamma_{\nu\rho}\Omega_\mu + \hat{\Omega}_\mu \Gamma_{\nu\rho} &= \tfrac{1}{12 \cdot 4!} F_{\sigma\tau\kappa\lambda} \left(\Gamma_{\nu\rho} \Gamma^{\sigma\tau\kappa\lambda}{}_\mu - \Gamma^{\sigma\tau\kappa\lambda}{}_\mu \Gamma_{\nu\rho} \right) + \tfrac{2}{3 \cdot 4!} F^{\tau\kappa\lambda}{}_\mu \left(\Gamma_{\nu\rho} \Gamma^{\tau\kappa\lambda} + \Gamma^{\tau\kappa\lambda} \Gamma_{\nu\rho} \right) \\ &= 20 \cdot \tfrac{1}{12 \cdot 4!} F_{\sigma\tau\kappa\lambda} g_{\alpha\mu} \delta^{[\alpha}_{[\nu} \Gamma_{\rho]}{}^{\sigma\tau\kappa\lambda]} + 2 \cdot \tfrac{2}{3 \cdot 4!} F_{\mu\tau\kappa\lambda} \left(\Gamma_{\nu\rho}{}^{\tau\kappa\lambda} - 6 \delta^{[\tau\kappa}_{[\nu\rho]} \Gamma^{\lambda]} \right) \\ &= \tfrac{8}{3 \cdot 4!} F_{\sigma\tau\kappa[\nu} \Gamma_{\rho]\mu}{}^{\sigma\tau\kappa} + \tfrac{2}{3 \cdot 4!} F_{\sigma\tau\kappa\lambda} g_{\mu[\nu} \Gamma_{\rho]}{}^{\sigma\tau\kappa\lambda} + \tfrac{4}{3 \cdot 4!} F_{\mu\tau\kappa\lambda} \Gamma_{\nu\rho}{}^{\tau\kappa\lambda} - \tfrac{8}{4!} F_{\mu\nu\rho\lambda} \Gamma^\lambda \\ &= \left(\tfrac{8}{3 \cdot 4!} F_{\sigma\tau\kappa[\nu} \Gamma_{\rho]\mu}{}^{\sigma\tau\kappa} - \tfrac{4}{3 \cdot 4!} F_{\sigma\tau\kappa\mu} \Gamma_{\nu\rho}{}^{\sigma\tau\kappa} \right) + \tfrac{2}{3 \cdot 4!} F_{\sigma\tau\kappa\lambda} g_{\mu[\nu} \Gamma_{\rho]}{}^{\sigma\tau\kappa\lambda} - \tfrac{8}{4!} F_{\mu\nu\rho\lambda} \Gamma^\lambda \end{aligned} \tag{4.10}$$

If we antisymmetrise this expression over (μ, ν, ρ), the first and second terms disappear and we

are left with

$$(\mathrm{d}\theta)_{\mu\nu\rho} = \nabla_{[\mu}\theta_{\nu\rho]} = \tfrac{1}{3!}F_{\mu\nu\rho\lambda}[\![\varepsilon_1, \Gamma^{\lambda}\varepsilon_2]\!] \ . \tag{4.11}$$

Thus for a Killing vector field produced from two Killing spinors $\varepsilon_{1,2}$ via the squaring map $K = \Xi(\varepsilon_1, \varepsilon_2)$, we have

$$\mathrm{d}\theta = \imath_K F \ , \tag{4.12}$$

and as such, $\mathrm{d}\imath_K F = 0$, and since F is closed we then have

$$\mathcal{L}_K F = \imath_K \mathrm{d}F + \mathrm{d}\imath_K F = 0 \ , \tag{4.13}$$

whence K preserves F and equation (3.56) is satisfied.

We have thus satisfied all the sufficient requirements to have an almost Killing superalgebra *and* for the homogeneity theorem to apply.

4.2.7 Killing superalgebra

The satisfaction of equation (3.57) (the $[\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1]$ super Jacobi identity) is shown in detail in [35] and demonstrates that there is a Killing superalgebra for $D = 11$ supergravity.

4.3 $D = 10$ type IIB

4.3.1 Introduction

The Killing superalgebra of $D = 10$ type IIB supergravity was described in [41] and the homogeneity theorem in [1]. We briefly review these constructions in our formalism. The homogeneity theorem for $D = 11$ supergravity places a new and firm control on highly supersymmetric $D = 10$ type IIB supergravity backgrounds.

4.3.2 Conventions

In the constructing literature of $D = 10$ Type IIB supergravity [42, 43, 44], the sign conventions adopted are $(\varkappa_0, \varkappa_1) = (-1, +1)$. Other authors use the convention $(+1, +1)$ [45, 41]. Conventions with $\varkappa_0 = \varkappa_1$ have a $\mathrm{C}\ell(1, 9)$-module spinor representation whereas for $\varkappa_0 = -\varkappa_1$ the spinor representation is a $\mathrm{C}\ell(9, 1)$-module. Both have real Majorana-Weyl spinors and we will adopt the conventions $(\varkappa_0, \varkappa_1) = (-1, -1)$.

4.3.3 Definition

The field content of type IIB supergravity (with all fields transforming in varying representations of $\mathrm{SO}(8)$) is:

Symbol	Count	Name	Description	Spin	D.o.f
ϕ	1	Dilaton	real scalar	0	1
$C^{(0)}$	1	Axion	0-form (real scalar)	0	1
$C^{(2)}$	1	R-R gauge potential	2-form	1	28
$C^{(4)}$	1	R-R gauge potential	4-form (with self-dual field strength)	1	35
$C^{(2)}$	1	NS-NS gauge potential	2-form	1	28
g	1	Graviton	Lorentzian metric	2	35
${}^{-}\lambda^{i}$	2	Dilatino	negative chirality spinor	½	2×8
${}^{+}\psi^{i}$	2	Gravitino	γ-traceless positive chirality vector-spinor	3/2	2×56

We construct the field strengths,

$$\begin{aligned} H^{(3)} &= \mathrm{d}B^{(2)} \\ G^{(1)} &= \mathrm{d}C^{(0)} \\ G^{(3)} &= \mathrm{d}C^{(2)} - C^{(0)}H^{(3)} \\ G^{(5)} &= \mathrm{d}C^{(4)} - \tfrac{1}{2}\mathrm{d}B^{(2)}\wedge C^{(2)} + \tfrac{1}{2}\mathrm{d}C^{(2)}\wedge B^{(2)} \,. \end{aligned} \tag{4.14}$$

We are interested in type IIB supergravity backgrounds and so we will set all fermionic field content $({}^{-}\lambda^{i}, {}^{+}\psi^{i})$ to zero from here on and it will not enter into the discussion. The action and field equations will not concern us in what follows.

Our tangent bundle spin group representation $\mathcal{V}$ is the 10-dimensional real vector representation of $\mathrm{SO}(1,9)$ equipped with the invariant Lorentzian inner product $(\!|\bullet\,,\,\bullet|\!)$.

4.3.4 Spinor representation

Having chosen the convention $(\varkappa_0, \varkappa_1) = (-1,-1)$, the Clifford algebra of relevance is $\mathrm{C}\ell(1,9)$. Let P denote the irreducible pinor representation of $\mathrm{C}\ell(1,9)$ which is real Majorana and 32-dimensional. Restricting to $\mathrm{Spin}(1,9)$, the spinors of P are real Majorana-Weyl and so reduce to two real 16-dimensional irreducible chiral representations $P = S_+ \oplus S_-$. Now let $\mathcal{P} = P \oplus P$ be two copies of our pinor representation and let $\mathcal{S} = S_+ \oplus S_+$ be the restriction of $\mathcal{P}$ to the positive chiral subspaces of each of the copies of P. $\mathcal{S}$ is then real, 32-dimensional, and is the spinor representation with which we construct the spinor bundle $\$$.

The Killing spinor equations obtained from variation of the gravitino and dilatino are

$$\begin{aligned} \delta\psi_X = \mathcal{D}_X\varepsilon = \nabla_X\varepsilon &+ \tfrac{1}{48}\Big(H^{(3)}\cdot X + X\cdot H^{(3)}\Big)\otimes\theta_3\cdot\varepsilon \\ &+ \tfrac{1}{8}\mathrm{e}^{\phi}\Big(G^{(1)}\cdot X\otimes\theta_2 - G^{(3)}\cdot X\otimes\theta_1 + \tfrac{1}{2}G^{(5)}\cdot X\otimes\theta_2\Big)\cdot\varepsilon \stackrel{!}{=} 0 \end{aligned} \tag{4.15}$$

$$\begin{aligned} \delta\lambda = \mathcal{P}\varepsilon = \mathrm{d}\phi\otimes\mathbb{1}\cdot\varepsilon &+ \tfrac{1}{2}H^{(3)}\otimes\theta_3\cdot\varepsilon \\ &- \mathrm{e}^{\phi}\Big(G^{(1)}\otimes\theta_2 - \tfrac{1}{2}G^{(3)}\otimes\theta_1\Big)\cdot\varepsilon \stackrel{!}{=} 0 \,, \end{aligned} \tag{4.16}$$

where ε is a doublet of positive chirality Majorana-Weyl spinors $\varepsilon = (\varepsilon^1, \varepsilon^2)$ and the 2×2

matrices $(\theta_1, \theta_2, \theta_3) = (\sigma_1, \mathrm{i}\sigma_2, \sigma_3)$ span $\mathfrak{sl}(2, \mathbb{R})$.

4.3.5 Spinor inner product

We have a $(\mathrm{Pin}(1,9)$-invariant $\mathbb{R}$-symplectic inner product on P induced from the hat involution of $\mathrm{C}\ell(1,9)$. We can diagonally extend this to a $\mathbb{R}$-symplectic inner product (so $\varkappa_3 = -1$) on $\mathcal{P}$ and so equation (3.53) is satisfied.

4.3.6 Almost Killing superalgebra and homogeneity

The bosonic fields $H^{(3)}, G^{(1)}, G^{(3)}, G^{(5)}$ all appear in the connection $\mathcal{D}$ as twisted by the 2×2 $\mathfrak{sl}(2, \mathbb{R})$ matrices. The diagonal extension of the $\mathbb{R}$-symplectic inner product implies the trivial inner product on these 2×2 matrices and so the adjoint of a bosonic field twisted by $\mathfrak{sl}(2, \mathbb{R})$ matrix is the image of the field under the hat involution of $\mathrm{C}\ell(1,9)$ composed with transposition of the particular $\mathfrak{sl}(2, \mathbb{R})$ matrix. We cover each term individually:

- $H^{(3)} \otimes \theta_3$: The exterior algebra isomorphism sends a 3-form to a rank-3 totally antisymmetric element of the Clifford algebra, and so $H^{(3)}$ is invariant under the hat involution. θ_3 is symmetric and thus $H^{(3)} \otimes \theta_3$ is self-adjoint.
- $G^{(1)} \otimes \theta_2$: The exterior algebra isomorphism sends a 1-form to a rank-1 element of the Clifford algebra, and so $G^{(1)}$ maps to $-G^{(1)}$ under the hat involution. θ_2 is skew-symmetric and thus $G^{(1)} \otimes \theta_2$ is self-adjoint.
- $G^{(3)} \otimes \theta_1$: The exterior algebra isomorphism sends a 3-form to a rank-3 totally antisymmetric element of the Clifford algebra, and so $G^{(3)}$ is invariant under the hat involution. θ_1 is symmetric and thus $H^{(3)} \otimes \theta_1$ is self-adjoint.
- $G^{(5)} \otimes \theta_2$: The exterior algebra isomorphism sends a 5-form to a rank-5 totally antisymmetric element of the Clifford algebra, and so $G^{(5)}$ maps to $-G^{(5)}$ under the hat involution. θ_2 is skew-symmetric and thus $G^{(5)} \otimes \theta_2$ is self-adjoint.

Thus all requisite items of $\mathcal{D}$ are self-adjoint as required by equation (3.54) with $\varkappa_3 = -1$.

We construct the squaring map as described in equation (3.22) and, choosing a pseudo-orthonormal basis $\mathfrak{d}_\mu$ for $\mathcal{V}$ and corresponding gamma matrices (see appendix B.11.2) for P, the squaring map takes the concrete form where the diagonalisation is made explicit,

$$\Xi(\varepsilon_1, \varepsilon_2) = \sum_i [\![\varepsilon_1^i, \Gamma^\mu \varepsilon_2^i]\!] \mathfrak{d}_\mu = \sum_i \bar{\varepsilon}_1^i \Gamma^\mu \varepsilon_2^i \mathfrak{d}_\mu = \sum_i (\varepsilon_1^i)^\dagger \Gamma_0 \Gamma^\mu \varepsilon_2^i \mathfrak{d}_\mu \ , \tag{4.17}$$

where we denote the Dirac adjoint $\bar{\varepsilon}^i := (\varepsilon^i)^\dagger \Gamma_0$.

Now, if we look at the $\mathfrak{a}_0$ component of a vector $K = \Xi(\varepsilon, \varepsilon)$ obtained from the squaring map, we see that for non-zero ε,

$$K^0 = \sum_i (\varepsilon^i)^\dagger \Gamma_0 \Gamma^0 \varepsilon^i = -\sum_i (\varepsilon^i)^\dagger \varepsilon^i = -\sum_i |\varepsilon^i|^2 < 0 \,. \tag{4.18}$$

This means that a vector field K constructed by squaring a single non-zero spinor field ε is necessarily causal because otherwise we could of course Lorentz-transform to the rest frame where $K^0 = 0$. Thus equation (3.55) is satisfied.

The necessary satisfaction of equations (3.56) and (3.57) are shown in detail in [41] and demonstrate that we have a Killing superalgebra and homogeneity for $D = 10$ type IIB supergravity.

4.4 $D = 10$ type I/heterotic

4.4.1 Introduction

The Killing superalgebra of $D = 10$ type I/heterotic supergravity was described in [41] and the homogeneity theorem in [1]. We briefly review these constructions in our formalism.

We will consider the case of $D = 10$ heterotic supergravity [46] which is $D = 10$ type I supergravity [47] coupled to $\mathcal{N} = 1$ super Yang-Mills [48], in the supergravity limit (i.e. no α' corrections). Of course, $D = 10$ type I supergravity backgrounds are already classified [49] and so the homogeneity theorem in this case tells us nothing new, and we demonstrate it only for completeness.

4.4.2 Conventions

The defining literature on $D = 10$ type I/heterotic supergravity uses the sign conventions $(\varkappa_0, \varkappa_1) = (-1, +1)$. Conventions with $\varkappa_0 = \varkappa_1$ have a $\mathrm{C}\ell(1, 9)$-module spinor representation whereas for $\varkappa_0 = -\varkappa_1$ the spinor representation is a $\mathrm{C}\ell(9, 1)$-module. Both have real Majorana-Weyl spinors and we will adopt the conventions $(\varkappa_0, \varkappa_1) = (-1, -1)$.

4.4.3 Definition

The field content of heterotic supergravity (with all fields transforming in varying representations of $\mathrm{SO}(8)$) is:

Symbol	Count	Name	Description	Spin	D.o.f
ϕ	1	Dilaton	real scalar	0	1
B	1	NS-NS[1] gauge potential	2-form	1	28
g	1	Graviton	Lorentzian metric	2	35
${}^{-}\lambda$	1	Dilatino	negative chirality spinor	½	8
${}^{+}\psi$	1	Gravitino	γ-traceless positive chirality vector-spinor	⅜	56

However, because $D = 10$ type I/heterotic supergravity is not maximal, we may also have vector supermultiplets with field content:

Symbol	Count	Name	Description	Spin	D.o.f
A	1	Y-M gauge potential	1-form	1	8
${}^{+}\xi$	1	Gaugino	positive chirality spinor	½	8

In principal we may have any number of vector supermultiplets as long as they describe a classical super Yang-Mills gauge theory. But if we wish to consider these theories as classical limits of quantum theories then the choice of gauge group is constrained [50]. However, the choice of gauge group is immaterial to the following argument.

We construct the field strengths,

$$\begin{aligned} H &= \mathrm{d}B \\ F &= \mathrm{d}A + A \wedge A \end{aligned} \tag{4.19}$$

We are interested in $D = 10$ heterotic supergravity backgrounds and so we will set all fermionic field content $({}^{-}\lambda, {}^{+}\psi, {}^{+}\xi)$ to zero from here on and it will not enter into the discussion. The action and field equations will not concern us in what follows.

Our tangent bundle spin group representation $\mathcal{V}$ is the 10-dimensional real vector representation of $\mathrm{SO}(1,9)$ equipped with the invariant Lorentzian inner product $(\!|\bullet, \bullet|\!)$.

4.4.4 Spinor representation

Having chosen the convention $(\varkappa_0, \varkappa_1) = (-1, -1)$, the Clifford algebra of relevance is $\mathrm{C}\ell(1,9)$. Let $\mathcal{P}$ denote the irreducible pinor representation of $\mathrm{C}\ell(1,9)$ which is real Majorana and 32-dimensional. Restricting to $\mathrm{Spin}(1,9)$, the spinors of $\mathcal{P}$ are real Majorana-Weyl and so reduce to two real 16-dimensional irreducible chiral representations $\mathcal{P} = S_+ \oplus S_-$. Let $\mathcal{S} = S_+$ be the restriction of $\mathcal{P}$ to the positive chiral subspace. $\mathcal{S}$ is then real, 16-dimensional, and is the spinor representation with which we construct the spinor bundle $\$$.

The Killing spinor equations obtained from variation of the gravitino, dilatino, and gaugino

[1] If we consider the theory as a classical limit of type I string theory, this is instead a R-R gauge potential.

are

$$\delta\psi_X = \mathcal{D}_X\varepsilon = \nabla_X\varepsilon - \tfrac{1}{12}\left(H\cdot X + X\cdot H\right)\cdot\varepsilon \stackrel{!}{=} 0 \tag{4.20}$$

$$\delta\lambda = \mathcal{P}\varepsilon = (\mathrm{d}\phi)\cdot\varepsilon - \tfrac{1}{2}H\cdot\varepsilon \stackrel{!}{=} 0 \tag{4.21}$$

$$\delta\xi = \mathcal{Q}\varepsilon = F\cdot\varepsilon \stackrel{!}{=} 0\ . \tag{4.22}$$

We can also write the superconnection component (equation (4.20)) as

$$\delta\psi_X = \mathcal{D}_X\varepsilon = \nabla_X\varepsilon - \tfrac{1}{4}\imath_X H\varepsilon \stackrel{!}{=} 0\ . \tag{4.23}$$

4.4.5 Spinor inner product

We have a $(\mathrm{Pin}(1,9)$-invariant $\mathbb{R}$-symplectic inner product on $\mathcal{P}$ induced from the hat involution of $\mathrm{C}\ell(1,9)$ and so equation (3.53) is satisfied.

4.4.6 Almost Killing superalgebra and homogeneity

Viewing H as an endomorphism of $\mathcal{P}$, its adjoint with respect to the spinor inner product is its image under the hat involution on $\mathrm{C}\ell(1,9)$. The exterior algebra isomorphism sends a 3-form to a rank-3 totally antisymmetric element of the Clifford algebra, and so it is invariant under the hat involution. Thus H is self-adjoint as required by equation (3.54) with $\varkappa_3 = -1$. F does not enter into the superconnection and so we do not worry about where it takes values, what it does at the weekend, and why it only calls us up super early on Tuesday mornings.

We construct the squaring map as described in equation (3.22) and, choosing a pseudo-orthonormal basis $\mathfrak{d}_\mu$ for $\mathcal{V}$ and corresponding gamma matrices (see appendix B.11.2) for $\mathcal{P}$, the squaring map takes the concrete form,

$$\Xi(\varepsilon_1,\varepsilon_2) = [\![\varepsilon_1,\Gamma^\mu\varepsilon_2]\!]\mathfrak{d}_\mu = \overline{\varepsilon}_1\Gamma^\mu\varepsilon_2\mathfrak{d}_\mu = \varepsilon_1^\dagger\Gamma_0\Gamma^\mu\varepsilon_2\mathfrak{d}_\mu\ , \tag{4.24}$$

where we denote the Dirac adjoint $\overline{\varepsilon} := \varepsilon^\dagger\Gamma_0$.

Now, if we look at the $\mathfrak{d}_0$ component of a vector $K = \Xi(\varepsilon,\varepsilon)$ obtained from the squaring map, we see that for non-zero ε,

$$K^0 = \varepsilon^\dagger\Gamma_0\Gamma^0\varepsilon = -\varepsilon^\dagger\varepsilon = -\varepsilon^\dagger\varepsilon = -|\varepsilon|^2 < 0\ , \tag{4.25}$$

This means that a vector field K constructed by squaring a single non-zero spinor field ε is necessarily causal because otherwise we could of course Lorentz-transform to the rest frame where $K^0 = 0$. Thus equation (3.55) is satisfied.

Using equation (4.50) we have for a Killing spinor ε,

$$\nabla_X\varepsilon = \tfrac{1}{4}\imath_X H\cdot\varepsilon\ . \tag{4.26}$$

Now for a Killing vector field produced by the squaring map $K^\mu = [\![\varepsilon_1, \Gamma^\mu \varepsilon_2]\!]$ on two Killing spinor fields $\varepsilon_{1,2}$, we have

$$\begin{aligned}
\nabla_\mu K_\nu &= \nabla_\mu [\![\varepsilon_1, \Gamma_\nu \varepsilon_2]\!] \\
&= [\![\nabla_\mu \varepsilon_1, \Gamma_\nu \varepsilon_2]\!] + [\![\varepsilon_1, \Gamma_\nu \nabla_\mu \varepsilon_2]\!] \\
&= \tfrac{1}{8} H_{\mu\rho\sigma} \left([\![\Gamma^{\rho\sigma} \varepsilon_1, \Gamma_\nu \varepsilon_2]\!] + [\![\varepsilon_1, \Gamma_\nu \Gamma^{\rho\sigma} \varepsilon_2]\!] \right) \\
&= -\tfrac{1}{8} H_{\mu\rho\sigma} [\![\varepsilon_1, [\Gamma^{\rho\sigma}, \Gamma_\nu] \varepsilon_2]\!] \\
&= -\tfrac{1}{2} H_{\mu\rho\sigma} [\![\varepsilon_1, \Gamma^\sigma \varepsilon_2]\!] \\
&= -\tfrac{1}{2} K^\sigma H_{\mu\rho\sigma} \ ,
\end{aligned} \tag{4.27}$$

and thus

$$\mathrm{d}K^\flat = -\iota_K H \ . \tag{4.28}$$

Whatsmore, this means that $\mathrm{d}\iota_K H = 0$, and along with the fact that H is closed we thus have,

$$\mathcal{L}_K H = \iota_K \mathrm{d}H + \mathrm{d}\iota_K H = 0 \ , \tag{4.29}$$

whence K preserves H.

Now, using equation (4.21) we have

$$\begin{aligned}
0 = \mathcal{P}\varepsilon &= (\mathrm{d}\phi) \cdot \varepsilon - \tfrac{1}{2} H \cdot \varepsilon \\
&= \overline{\varepsilon} \mathcal{P} \varepsilon = [\![\varepsilon, (\mathrm{d}\phi) \cdot \varepsilon]\!] - \tfrac{1}{2} [\![\varepsilon, H \cdot \varepsilon]\!] \ ,
\end{aligned} \tag{4.30}$$

but the hat involution and the symmetry of the pinor inner product mean that the last term disappears and so we have, for a Killing vector field produced by the squaring map $K^\mu = [\![\varepsilon, \Gamma^\mu \varepsilon]\!]$ on a single Killing spinor field,

$$0 = [\![\varepsilon, (\mathrm{d}\phi) \cdot \varepsilon]\!] = \nabla_\mu \phi [\![\varepsilon, \Gamma^\mu \varepsilon]\!] = K^\mu \nabla_\mu \phi \ , \tag{4.31}$$

whence of course $\mathcal{L}_K \phi = 0$ and so with the squaring map being defined by its diagonal, K leaves ϕ invariant.

Now, using equation (4.22) we have

$$\begin{aligned}
0 = \mathcal{Q}\varepsilon &= F \cdot \varepsilon \\
&= \overline{\varepsilon} \Gamma_\mu \mathcal{Q} \varepsilon = F_{\nu\rho} [\![\varepsilon, \Gamma_\mu \Gamma^{\nu\rho} \varepsilon]\!] \\
&= F_{\nu\rho} [\![\varepsilon, \left(\Gamma_\mu{}^{\nu\rho} - 2\delta_\mu^{[\nu} \Gamma^{\rho]} \right) \varepsilon]\!] \ ,
\end{aligned} \tag{4.32}$$

but the hat involution and the symmetry of the pinor inner product mean that the first term disappears and so for a Killing vector field produced by the squaring map $K^\rho = [\![\varepsilon, \Gamma^\rho \varepsilon]\!]$ on a

single Killing spinor field,

$$\begin{aligned} 0 &= F_{\nu\rho}[\![\varepsilon, \delta_\mu^{[\nu}\Gamma^{\rho]}\varepsilon]\!] \\ &= -2F_{\mu\rho}[\![\varepsilon, \Gamma^\rho\varepsilon]\!] \\ &= -2K^\rho F_{\mu\rho} \ , \end{aligned} \tag{4.33}$$

whence $\iota_K F = 0$. Then

$$\mathcal{L}_K F = \mathrm{d}\iota_K F + \iota_K \mathrm{d}F = \iota_K \mathrm{d}F \tag{4.34}$$

and then using the Bianchi identity $\mathrm{d}F = -[A, F]$ then gives,

$$\mathcal{L}_K F = -[\iota_K A, F] \ , \tag{4.35}$$

which is an infinitesimal gauge transformation with parameter $-\iota_K A$ and so we can always choose a gauge such that $\iota_K A = 0$ whence $\mathcal{L}_K F = 0$ up to gauge transformations.

Thus we have $\mathcal{L}_K H = \mathcal{L}_K \phi = \mathcal{L}_K F = 0$ and equation (3.56) is satisfied.

We have thus satisfied all the sufficient requirements to have an almost Killing superalgebra *and* for the homogeneity theorem to apply.

4.4.7 Killing superalgebra

The Fierz identity for positive chirality spinors with respect to our spinor inner product is:

$$\varepsilon\overline{\varepsilon} = \tfrac{1}{16}\left([\![\varepsilon, \Gamma_\mu\varepsilon]\!]\Gamma^\mu + \tfrac{1}{240}[\![\varepsilon, \Gamma_{\mu\nu\rho\sigma\tau}\varepsilon]\!]\Gamma^{\mu\nu\rho\sigma\tau}\right)\mathbb{P}(-1) \ , \tag{4.36}$$

where $\mathbb{P}(-1) = \frac{1}{2}(\mathbb{1} - \Gamma_{\mathrm{vol}})$ is the projector onto the negative chirality subspace.

For a Killing vector field produced by the squaring map from a single Killing spinor field, $K^\mu = [\![\varepsilon, \Gamma^\mu\varepsilon]\!]$, we have

$$K^\flat \cdot \varepsilon = [\![\varepsilon, \Gamma_\mu\varepsilon, \Gamma]\!]^\mu\varepsilon = \Gamma^\mu\varepsilon\overline{\varepsilon}\Gamma_\mu\varepsilon \tag{4.37}$$

and using equation (4.36),

$$\begin{aligned} K^\flat \cdot \varepsilon &= \tfrac{1}{16}\Gamma^\mu\left([\![\varepsilon, \Gamma_\nu\varepsilon]\!]\Gamma^\nu + \tfrac{1}{240}[\![\varepsilon, \Gamma_{\nu\rho\sigma\tau\kappa}\varepsilon]\!]\Gamma^{\nu\rho\sigma\tau\kappa}\right)\mathbb{P}(-1)\Gamma_\mu\varepsilon \\ &= \tfrac{1}{16}\left([\![\varepsilon, \Gamma_\nu\varepsilon]\!]\Gamma^\mu\Gamma^\nu\Gamma_\mu + \tfrac{1}{240}[\![\varepsilon, \Gamma_{\nu\rho\sigma\tau\kappa}\varepsilon]\!]\Gamma^\mu\Gamma^{\nu\rho\sigma\tau\kappa}\Gamma_\mu\right)\varepsilon \\ &= \tfrac{1}{16}\left([\![\varepsilon, \Gamma_\nu\varepsilon]\!]8\Gamma^\nu\right)\varepsilon = \tfrac{1}{2}[\![\varepsilon, \Gamma_\nu\varepsilon]\!]\Gamma^\nu\varepsilon = \tfrac{1}{2}K^\flat \cdot \varepsilon \ , \end{aligned} \tag{4.38}$$

where we have used the identities valid in ten dimensions:

$$\Gamma^\mu\Gamma^\nu\Gamma_\mu = 8\Gamma^\nu \ , \text{ and} \tag{4.39}$$

$$\Gamma^\mu\Gamma^{\nu\rho\sigma\tau\kappa}\Gamma_\mu = 0 \ . \tag{4.40}$$

Thus we clearly have that

$$K^\flat \cdot \varepsilon = 0 \,. \tag{4.41}$$

Now, for a Killing vector field produced by the squaring map from a single Killing spinor field, $K^\mu = [\![\varepsilon, \Gamma^\mu \varepsilon]\!]$,

$$\mathcal{L}_K \varepsilon = \nabla_K \varepsilon - \tfrac{1}{4}(\nabla K) \cdot \varepsilon \tag{4.42}$$

and using equations (4.26) and (4.28),

$$\begin{aligned} \mathcal{L}_K \varepsilon &= \tfrac{1}{4} \imath_K H \cdot \varepsilon + \tfrac{1}{4} \imath_K H \cdot \varepsilon \\ &= -\tfrac{1}{12} K^\mu H_{\nu\rho\sigma} \left(\Gamma^{\nu\rho\sigma} \Gamma_\mu + \Gamma_\mu \Gamma^{\nu\rho\sigma} \right) \varepsilon \end{aligned} \tag{4.43}$$

and using equation (4.21) on the second term,

$$\mathcal{L}_K \varepsilon = -\tfrac{1}{12} K^\mu H_{\nu\rho\sigma} \Gamma^{\nu\rho\sigma} \Gamma_\mu \varepsilon - \tfrac{1}{6} K^\mu (\nabla_\nu \phi) \Gamma_\mu \Gamma^\nu \varepsilon \tag{4.44}$$

$$\begin{aligned} &= -\tfrac{1}{12} K^\mu H_{\nu\rho\sigma} \Gamma^{\nu\rho\sigma} \Gamma_\mu \varepsilon - \tfrac{1}{6} K^\mu (\nabla_\nu \phi) \left(-\Gamma^\nu \Gamma_\mu - 2\delta^\nu_\mu \mathbb{1} \right) \varepsilon \\ &= -\tfrac{1}{12} K^\mu H_{\nu\rho\sigma} \Gamma^{\nu\rho\sigma} \Gamma_\mu \varepsilon + \tfrac{1}{6} K^\mu (\nabla_\nu \phi) \Gamma^\nu \Gamma_\mu \varepsilon + \tfrac{1}{3} K^\mu (\nabla_\mu \phi) \mathbb{1} \varepsilon \end{aligned} \tag{4.45}$$

and using equation (4.31) on the last term,

$$\mathcal{L}_K \varepsilon = -\tfrac{1}{12} \left(H_{\nu\rho\sigma} \Gamma^{\nu\rho\sigma} - 2(\nabla_\nu \phi) \Gamma^\nu \right) K^\mu \Gamma_\mu \varepsilon \tag{4.46}$$

whence using equation (4.41) finally fields,

$$\mathcal{L}_K \varepsilon = 0 \,. \tag{4.47}$$

As such, equation (3.57) is satisfied and we have a Killing superalgebra.

4.5 $D = 6$, $(1, 0)$

4.5.1 Introduction

The homogeneity theorem and the Killing superalgebra of $D = 6$, $(1, 0)$ supergravity were described in [2] and we briefly review these constructions in our formalism. Of course, supersymmetric $D = 6$, $(1, 0)$ supergravity backgrounds are already classified [51] and either preserve all, half, or none of the maximum supersymmetry and those that are maximally supersymmetric are known to be homogeneous. So, although the construction of the $D = 6$, $(1, 0)$ Killing superalgebra is novel, extending it to the homogeneity theorem in this case tells us nothing new, and we demonstrate it only for completeness.

4.5.2 Conventions

In the reference article for $D = 6$, $(1,0)$ supergravity [52], the sign conventions adopted are $(\varkappa_0, \varkappa_1) = (-1, +1)$ but we will adopt the conventions $(\varkappa_0, \varkappa_1) = (+1, -1)$ as used in [2].

4.5.3 Definition

The field content of $D = 6$, $(1,0)$ supergravity (with all fields transforming in varying representations of $\mathrm{SO}(4)$) is:

Symbol	Count	Name	Description	Spin	D.o.f
B	1	R-R gauge potential	2-form (with anti-self-dual field strength)	1	3
g	1	Graviton	Lorentzian metric	2	9
${}^{+}\psi^{i}$	2	Gravitino	γ-traceless positive chirality vector-spinor	3/2	2×6

We construct the field strength,

$$H = \mathrm{d}B \ . \tag{4.48}$$

We are interested in $D = 6$, $(1,0)$ supergravity backgrounds and so we will set all fermionic field content $({}^{+}\psi^{i})$ to zero from here on and it will not enter into the discussion. The action and field equations will not concern us in what follows.

Our tangent bundle spin group representation $\mathcal{V}$ is the 6-dimensional real vector representation of $\mathrm{SO}(5,1)$ equipped with the invariant Lorentzian inner product $(\!|\bullet, \bullet|\!)$.

4.5.4 Spinor representation

Having chosen the convention $(\varkappa_0, \varkappa_1) = (+1, -1)$, the Clifford algebra of relevance is $\mathrm{C}\ell(5,1)$ and so we have a little work to do. Let P denote the unique irreducible pinor representation of $\mathrm{C}\ell(5,1)$. It is a four-dimensional quaternionic representation that we may view as an eight-dimensional complex representation with an invariant quaternionic structure. Now, let S_1 denote the fundamental representation of the R-symmetry group of $D = 6$, $(1,0)$ supergravity which is $\mathrm{USp}(2) \cong \mathrm{Spin}(3)$, a two-dimensional complex representation also with an invariant quaternionic structure. Taking the tensor product of these two representations $P \otimes S_1$ yields a 16-dimensional complex representation with an invariant real structure and we define $\mathcal{P}$ to be the real subspace of $P \otimes S_1$ with respect to this real structure. Restricting to $\mathrm{Spin}(5,1)$, the spinors of $\mathcal{P}$ are symplectic Majorana-Weyl, real, 16-dimensional, and so reduce to two real 8-dimensional irreducible[2] chiral representations $\mathcal{P} = \mathcal{S}_+ \oplus \mathcal{S}_-$. We note that this decomposition is well-defined in the real subspace because the real structure and the chiral structure commute. We choose the positive chirality representation $\mathcal{S}_+$ as the spinor representation with which to construct the spinor bundle $\$$.

[2] As a representation of $\mathrm{Spin}(5,1) \times \mathrm{USp}(2)$.

The Killing spinor equation obtained from variation of the gravitino is

$$\delta\psi_X = \mathcal{D}_X\varepsilon = \nabla_X\varepsilon + \tfrac{1}{24}\Big(H\cdot X^\flat + X^\flat\cdot H\Big)\cdot\varepsilon \stackrel{!}{=} 0\ , \tag{4.49}$$

or equivalently using equation (B.9),

$$\delta\psi_X = \mathcal{D}_X\varepsilon = \nabla_X\varepsilon + \tfrac{1}{4}\imath_X H\cdot\varepsilon \stackrel{!}{=} 0\ . \tag{4.50}$$

4.5.5 Spinor inner product

We have a $(\mathrm{Pin}(5,1)$-invariant $\mathbb{H}$-hermitian symplectic inner product on P induced from the hat involution of $\mathrm{C}\ell(5,1)$ and a $\mathrm{Spin}(3)$-invariant $\mathbb{H}$-hermitian symmetric inner product on S_1 induced from the hat involution of $\mathrm{C}\ell(3,0)$. They combine to form a $(\mathrm{Pin}(5,1)\times\mathrm{USp}(2))$-invariant $\mathbb{R}$-symplectic inner product on $\mathcal{P}$ (so $\varkappa_3 = -1$) that we will denote $[\![\bullet,\bullet]\!]$. Thus we have an $\mathbb{R}$-symplectic inner product induced from the hat involution (of $\mathrm{C}\ell(5,1)$) and so equation (3.53) is satisfied.

4.5.6 Almost Killing superalgebra and homogeneity

H has no $\mathrm{USp}(2)$ indices and so as an endomorphism of $\mathcal{P}$, its adjoint with respect to the spinor inner product is its image under the hat involution on $\mathrm{C}\ell(5,1)$. The exterior algebra isomorphism sends a 3-form to a rank-3 totally antisymmetric element of the Clifford algebra, and so it is invariant under the hat involution. Thus H is self-adjoint as required by equation (3.54) with $\varkappa_3 = -1$.

We construct the squaring map as described in equation (3.22) and, choosing a pseudo-orthonormal basis $\mathfrak{d}_\mu$ for $\mathcal{V}$ and corresponding gamma matrices (see appendix B.11.3) for $\mathcal{P}$, the squaring map takes the concrete form,

$$\Xi(\varepsilon_1,\varepsilon_2) = [\![\varepsilon_1,\Gamma^\mu\varepsilon_2]\!]\mathfrak{d}_\mu = \bar\varepsilon_1\Gamma^\mu\varepsilon_2\mathfrak{d}_\mu = \varepsilon_1^\dagger\Gamma_{\mathrm{vol}}\Gamma_0\Gamma^\mu\varepsilon_2\mathfrak{d}_\mu\ , \tag{4.51}$$

where we denote the Dirac adjoint $\bar\varepsilon := \varepsilon^\dagger\Gamma_{\mathrm{vol}}\Gamma_0$.

Now, if we look at the $\mathfrak{d}_0$ component of a vector $K = \Xi(\varepsilon,\varepsilon)$ obtained from the squaring map, we see that for non-zero ε,

$$K^0 = \varepsilon^\dagger\Gamma_{\mathrm{vol}}\Gamma_0\Gamma^0\varepsilon = -\varepsilon^\dagger\Gamma_{\mathrm{vol}}\varepsilon = -\varepsilon^\dagger\varepsilon = -|\varepsilon|^2 < 0\ , \tag{4.52}$$

where we have made use of the positive chirality of ε in that $\Gamma_{\mathrm{vol}}\varepsilon = \varepsilon$. This means that a vector field K constructed by squaring a single non-zero spinor field ε is necessarily causal because otherwise we could of course Lorentz-transform to the rest frame where $K^0 = 0$. Thus equation (3.55) is satisfied.

Using equation (4.50) we have for a Killing spinor ε,

$$\nabla_X \varepsilon = -\tfrac{1}{4} \iota_X H \cdot \varepsilon \ . \tag{4.53}$$

Now for a Killing vector field produced by the squaring map $K^\mu = [\![\varepsilon_1, \Gamma^\mu \varepsilon_2]\!]$ on two Killing spinor fields $\varepsilon_{1,2}$, we have

$$\begin{aligned} \nabla_\mu K_\nu &= \nabla_\mu [\![\varepsilon_1, \Gamma_\nu \varepsilon_2]\!] \\ &= [\![\nabla_\mu \varepsilon_1, \Gamma_\nu \varepsilon_2]\!] + [\![\varepsilon_1, \Gamma_\nu \nabla_\mu \varepsilon_2]\!] \\ &= -\tfrac{1}{8} H_{\mu\rho\sigma} \left([\![\Gamma^{\rho\sigma} \varepsilon_1, \Gamma_\nu \varepsilon_2]\!] + [\![\varepsilon_1, \Gamma_\nu \Gamma^{\rho\sigma} \varepsilon_2]\!] \right) \\ &= \tfrac{1}{8} H_{\mu\rho\sigma} [\![\varepsilon_1, [\Gamma^{\rho\sigma}, \Gamma_\nu] \varepsilon_2]\!] \\ &= \tfrac{1}{2} H_{\mu\rho\sigma} [\![\varepsilon_1, \Gamma^\sigma \varepsilon_2]\!] \\ &= \tfrac{1}{2} K^\sigma H_{\mu\rho\sigma} \ , \end{aligned} \tag{4.54}$$

and thus

$$\mathrm{d} K^\flat = \iota_K H \ . \tag{4.55}$$

What is more, this means that $\mathrm{d} \iota_K H = 0$, and along with the fact that H is closed we thus have,

$$\mathcal{L}_K H = \iota_K \mathrm{d} H + \mathrm{d} \iota_K H = 0 \ , \tag{4.56}$$

whence K preserves H and equation (3.56) is satisfied.

We have thus satisfied all the sufficient requirements to have an almost Killing superalgebra *and* for the homogeneity theorem to apply.

4.5.7 Killing superalgebra

For a Killing vector field produced by the squaring map from a single Killing spinor field, $K^\mu = [\![\varepsilon, \Gamma^\mu \varepsilon]\!]$,

$$\mathcal{L}_K \varepsilon = \nabla_K \varepsilon - \tfrac{1}{4} (\nabla K) \cdot \varepsilon \tag{4.57}$$

and using equations (4.53) and (4.55),

$$\begin{aligned} \mathcal{L}_K \varepsilon &= -\tfrac{1}{4} \iota_K H \cdot \varepsilon - \tfrac{1}{4} \iota_K H \cdot \varepsilon \\ &= -\tfrac{1}{2} \iota_K H \cdot \varepsilon \\ &= -\tfrac{1}{12} \left(H \cdot K^\flat + K^\flat \cdot H \right) \cdot \varepsilon \ . \end{aligned} \tag{4.58}$$

Remembering that H is anti-self-dual, we know from appendix B.8 that H annihilates positive chirality spinors and so

$$\mathcal{L}_K\varepsilon = -\tfrac{1}{12}H\cdot K^\flat\cdot\varepsilon\ . \tag{4.59}$$

Now, the Fierz identity for positive chirality spinors with respect to our spinor inner product is:

$$\varepsilon\overline{\varepsilon} = \tfrac{1}{8}\left([\![\varepsilon,\Gamma_\mu\varepsilon]\!]\Gamma^\mu + \tfrac{1}{12}[\![\varepsilon,\Gamma_{\mu\nu\rho}\gamma_i\varepsilon]\!]\Gamma^{\mu\nu\rho}\gamma^i\right)\mathbb{P}(-1)\ , \tag{4.60}$$

where $\mathbb{P}(-1) = \frac{1}{2}(\mathbb{1}-\Gamma_{\rm vol})$ is the projector onto the negative chirality subspace.

Thus

$$\begin{aligned} K^\flat\cdot\varepsilon &= [\![\varepsilon,\Gamma_\mu\varepsilon]\!]\Gamma^\mu\varepsilon \qquad (4.61)\\ &= \overline{\varepsilon}\Gamma_\mu\varepsilon\Gamma^\mu\varepsilon = \Gamma^\mu\varepsilon\overline{\varepsilon}\Gamma_\mu\varepsilon\\ &= \tfrac{1}{8}\Gamma^\mu\left([\![\varepsilon,\Gamma_\nu\varepsilon]\!]\Gamma^\nu + \tfrac{1}{12}[\![\varepsilon,\Gamma_{\nu\rho\sigma}\gamma_i\varepsilon]\!]\Gamma^{\nu\rho\sigma}\gamma^i\right)\mathbb{P}(-1)\Gamma_\mu\varepsilon\\ &= \tfrac{1}{8}\left([\![\varepsilon,\Gamma_\nu\varepsilon]\!]\Gamma^\mu\Gamma^\nu\Gamma_\mu + \tfrac{1}{12}[\![\varepsilon,\Gamma_{\nu\rho\sigma}\gamma_i\varepsilon]\!]\Gamma^\mu\Gamma^{\nu\rho\sigma}\Gamma_\mu\gamma^i\right)\varepsilon\\ &= \tfrac{1}{2}[\![\varepsilon,\Gamma_\nu\varepsilon]\!]\Gamma^\nu\varepsilon\ , \qquad (4.62) \end{aligned}$$

where we have used the identities valid in six dimensions:

$$\Gamma^\mu\Gamma^\nu\Gamma_\mu = 4\Gamma^\nu\ ,\ \text{and} \tag{4.63}$$

$$\Gamma^\mu\Gamma^{\nu\rho\sigma}\Gamma_\mu = 0\ . \tag{4.64}$$

Comparing equation (4.61) and equation (4.62) implies that $K^\flat\cdot\varepsilon = 0$ and so equation (4.59) tells us that $\mathcal{L}_K\varepsilon = 0$. As such, equation (3.57) is satisfied and we have a Killing superalgebra.

4.6 $D = 6$, $(2,0)$

4.6.1 Introduction

The homogeneity theorem and the Killing superalgebra of $D = 6$, $(2,0)$ supergravity were described in [2] and we briefly review these constructions in our formalism. Again, the construction of the $D = 6$, $(2,0)$ Killing superalgebra is novel and, unlike for $D = 6$, $(1,0)$, there is no classification of backgrounds beyond that of the maximally supersymmetric backgrounds in [53] and so the homogeneity theorem tells us something new.

4.6.2 Conventions

For $D = 6$, $(2,0)$ supergravity [54, 55], the sign conventions adopted by [55] is $(\varkappa_0,\varkappa_1) = (-1,+1)$ but we will adopt the conventions $(\varkappa_0,\varkappa_1) = (+1,-1)$ as used in [2].

4.6.3 Definition

The field content of $D = 6$, $(2, 0)$ supergravity (with all fields transforming in varying representations of $\mathrm{SO}(4)$) is:

Symbol	Count	Name	Description	Spin	D.o.f
B^i	5	R-R gauge potential	2-form (with anti-self-dual field strength)	1	5×3
g	1	Graviton	Lorentzian metric	2	9
${}^{+}\psi^j$	4	Gravitino	γ-traceless positive chirality vector-spinor	3/2	4×6

The 2-form gauge potential B^i with $i = 1 \ldots 5$ actually takes values in $\mathbb{V}$, the 5-dimensional vector representation of the R-symmetry group of $D = 6$, $(2, 0)$ supergravity which is $\mathrm{USp}(4) \cong \mathrm{Spin}(5)$. We can represent it as five 2-form gauge fields upon choosing an orthonormal basis for $\mathbb{V}$.

We construct the field strength,

$$H = \mathrm{d}B + B \wedge B \tag{4.65}$$

We are interested in $D = 6$, $(2, 0)$ supergravity backgrounds and so we will set all fermionic field content $({}^{+}\psi^j)$ to zero from here on and it will not enter into the discussion. The action and field equations will not concern us in what follows.

Our tangent bundle spin group representation $\mathcal{V}$ is the 6-dimensional real vector representation of $\mathrm{SO}(5, 1)$ equipped with the invariant Lorentzian inner product that we will denote $(\!|\bullet, \bullet|\!)$.

4.6.4 Spinor representation

Having chosen the convention $(\varkappa_0, \varkappa_1) = (+1, -1)$, the Clifford algebra of relevance is $\mathrm{C}\ell(5, 1)$ and so we again have a little work to do. Let P denote the unique irreducible pinor representation of $\mathrm{C}\ell(5, 1)$. It is a four-dimensional quaternionic representation that we may view as an eight-dimensional complex representation with an invariant quaternionic structure. Now, let S_2 denote the fundamental representation of the R-symmetry group, a four-dimensional complex representation also with an invariant quaternionic structure. Taking the tensor product of these two representations $P \otimes S_2$ yields a 32-dimensional complex representation with an invariant real structure and we define $\mathcal{P}$ to be the real subspace of $P \otimes S_2$ with respect to this real structure. Restricting to $\mathrm{Spin}(5, 1)$, the spinors of $\mathcal{P}$ are symplectic Majorana-Weyl, real, 32-dimensional, and so reduce to two real 16-dimensional irreducible[3] chiral representations $\mathcal{P} = \mathcal{S}_+ \oplus \mathcal{S}_-$. We note that this decomposition is well-defined in the real subspace because the real structure and the chiral structure commute. We choose the positive chirality representation $\mathcal{S}_+$ as the spinor representation with which to construct the spinor bundle $\$$.

[3] As a representation of $\mathrm{Spin}(5, 1) \times \mathrm{USp}(4)$.

The 3-form field strength H is $\mathbb{V}$-valued and the R-symmetry group component of its action on the spinor bundle is via the Clifford action of $\mathrm{C}\ell(\mathbb{V}) \cong \mathrm{C}\ell(0,5)$ on S_2. This appears explicitly in the Killing spinor equation with the gamma matrices γ_i the auxiliary gamma matrices of the real $\mathrm{Spin}(5,1) \times \mathrm{USp}(4)$ representation corresponding to the action of $\mathrm{C}\ell(0,5)$ on S_2.

The Killing spinor equation obtained from variation of the gravitino is

$$\delta\psi_X = \mathcal{D}_X\epsilon = \nabla_X\epsilon + \tfrac{1}{24}\left(H^i \cdot X^\flat + X^\flat \cdot H^i\right) \cdot \gamma_i \cdot \epsilon \stackrel{!}{=} 0 \,, \tag{4.66}$$

or equivalently using equation (B.9),

$$\delta\psi_X = \mathcal{D}_X\varepsilon = \nabla_X\varepsilon + \tfrac{1}{4}\iota_X H^i \cdot \gamma_i \cdot \varepsilon \stackrel{!}{=} 0 \,. \tag{4.67}$$

4.6.5 Spinor inner product

We have a $(\mathrm{Pin}(5,1)$-invariant $\mathbb{H}$-hermitian symplectic inner product on P induced from the hat involution of $\mathrm{C}\ell(5,1)$ and a $\mathrm{Spin}(5)$-invariant $\mathbb{H}$-hermitian symmetric inner product on S_2 induced from the check involution of $\mathrm{C}\ell(0,5)$. They combine to form a $(\mathrm{Pin}(5,1) \times \mathrm{USp}(4))$-invariant $\mathbb{R}$-symplectic inner product on $\mathcal{P}$ (so $\varkappa_3 = -1$) that we will denote $[\![\bullet, \bullet]\!]$. Thus we have an $\mathbb{R}$-symplectic inner product induced from the hat involution (of $\mathrm{C}\ell(5,1)$) and so equation (3.53) is satisfied.

4.6.6 Almost Killing superalgebra and homogeneity

H has $\mathrm{USp}(4)$ indices and so as an endomorphism of $\mathcal{P}$, its adjoint with respect to the spinor inner product is its image under the hat involution on $\mathrm{C}\ell(5,1)$ *and* the check involution on $\mathrm{C}\ell(0,5)$. The exterior algebra isomorphism sends a 3-form to a rank-3 totally antisymmetric element of the Clifford algebra, and so it is invariant under the hat involution of $\mathrm{C}\ell(5,1)$. As an endomorphism of $\mathrm{C}\ell(0,5)$, H is a rank-1 element and so it is also invariant under the check involution of $\mathrm{C}\ell(0,5)$. Thus H is self-adjoint as required by equation (3.54) with $\varkappa_3 = -1$.

We construct the squaring map as described in equation (3.22) and, choosing a pseudo-orthonormal basis $\mathfrak{d}_\mu$ for $\mathcal{V}$ and corresponding gamma matrices (see appendix B.11.4) for $\mathcal{P}$, the squaring map takes the concrete form,

$$\Xi(\varepsilon_1, \varepsilon_2) = [\![\varepsilon_1, \Gamma^\mu\varepsilon_2]\!]\mathfrak{d}_\mu = \bar{\varepsilon}_1\Gamma^\mu\varepsilon_2\mathfrak{d}_\mu = \varepsilon_1^\dagger\Gamma_{\mathrm{vol}}\Gamma_0\Gamma^\mu\varepsilon_2\mathfrak{d}_\mu \,, \tag{4.68}$$

where we denote the Dirac adjoint $\bar{\varepsilon} := \varepsilon^\dagger\Gamma_{\mathrm{vol}}\Gamma_0$.

Now, if we look at the $\mathfrak{d}_0$ component of a vector $K = \Xi(\varepsilon, \varepsilon)$ obtained from the squaring map, we see that for non-zero ε,

$$K^0 = \varepsilon^\dagger\Gamma_{\mathrm{vol}}\Gamma_0\Gamma^0\varepsilon = -\varepsilon^\dagger\Gamma_{\mathrm{vol}}\varepsilon = -\varepsilon^\dagger\varepsilon = -|\varepsilon|^2 < 0 \,, \tag{4.69}$$

where we have made use of the positive chirality of ε in that $\Gamma_{\rm vol}\varepsilon = \varepsilon$. This means that a vector field K constructed by squaring a single non-zero spinor field ε is necessarily causal because otherwise we could of course Lorentz-transform to the rest frame where $K^0 = 0$. Thus equation (3.55) is satisfied.

Using equation (4.67) we have for a Killing spinor ε,

$$\nabla_X \varepsilon = -\tfrac{1}{4}\iota_X H^i \cdot \gamma_i \cdot \varepsilon. \tag{4.70}$$

We define a Killing vector field produced from two Killing spinors $\varepsilon_{1,2}$ via the squaring map,

$$K^\mu = [\![\varepsilon_1, \Gamma^\mu \varepsilon_2]\!] \,, \tag{4.71}$$

and then we have

$$\begin{aligned}
\nabla_\mu K_\nu &= \nabla_\mu [\![\varepsilon_1, \Gamma_\nu \varepsilon_2]\!] \\
&= [\![\nabla_\mu \varepsilon_1, \Gamma_\nu \varepsilon_2]\!] + [\![\varepsilon_1, \Gamma_\nu \nabla_\mu \varepsilon_2]\!] \\
&= -\tfrac{1}{8} H^i_{\mu\rho\sigma} \left([\![\Gamma^{\rho\sigma}\gamma_i \varepsilon_1, \Gamma_\nu \varepsilon_2]\!] + [\![\varepsilon_1, \Gamma_\nu \Gamma^{\rho\sigma}\gamma_i \varepsilon_2]\!] \right) \\
&= \tfrac{1}{8} H^i_{\mu\rho\sigma} [\![\varepsilon_1, [\Gamma^{\rho\sigma}, \Gamma_\nu]\gamma_i \varepsilon_2]\!] \\
&= \tfrac{1}{2} H^i_{\mu\nu\rho} [\![\varepsilon_1, \Gamma^\rho \gamma_i \varepsilon_2]\!]
\end{aligned} \tag{4.72}$$

Then, let us define a $\mathbb{V}$-valued 1-form θ constructed from two Killing spinor fields $\varepsilon_{1,2}$ via the spinor inner product as

$$\theta^i_\mu = [\![\varepsilon_1, \Gamma_\mu \gamma^i \varepsilon_2]\!] \,, \tag{4.73}$$

which gives us[4]

$$\nabla_\mu K_\nu = \tfrac{1}{2} H^i_{\mu\nu\rho} \theta^\rho_i \,. \tag{4.74}$$

Then, pressing on, we have

$$\begin{aligned}
\nabla_\mu \theta^i_\nu &= \nabla_\mu [\![\varepsilon_1, \Gamma_\nu \gamma^i \varepsilon_2]\!] \\
&= [\![\nabla_\mu \varepsilon_1, \Gamma_\nu \gamma^i \varepsilon_2]\!] + [\![\varepsilon_1, \Gamma_\nu \gamma^i \nabla_\mu \varepsilon_2]\!] \\
&= -\tfrac{1}{8} H^j_{\mu\rho\sigma} \left([\![\Gamma^{\rho\sigma}\gamma_j \varepsilon_1, \Gamma_\nu \gamma^i \varepsilon_2]\!] + [\![\varepsilon_1, \Gamma_\nu \gamma^i \Gamma^{\rho\sigma}\gamma_j \varepsilon_2]\!] \right) \\
&= \tfrac{1}{8} H^j_{\mu\rho\sigma} \left([\![\gamma^i \gamma_j \varepsilon_1, \Gamma^{\rho\sigma}\Gamma_\nu \varepsilon_2]\!] - [\![\gamma_j \gamma^i \varepsilon_1, \Gamma_\nu \Gamma^{\rho\sigma} \varepsilon_2]\!] \right) \,.
\end{aligned} \tag{4.75}$$

And using the relation $\gamma_j \gamma^i = \delta^i_j + \gamma_j{}^i$, and then the identity pair $[\Gamma^{\rho\sigma}, \Gamma_\nu] = 4\delta^{[\rho}_\nu \Gamma^{\sigma]}$ and $\Gamma^{\rho\sigma}\Gamma_\nu + \Gamma_\nu \Gamma^{\rho\sigma} = 2\Gamma^{\rho\sigma}{}_\nu$ we have,

$$\begin{aligned}
\nabla_\mu \theta^i_\nu &= \tfrac{1}{8} H^i_{\mu\rho\sigma} [\![\varepsilon_1, [\Gamma^{\rho\sigma}, \Gamma_\nu]\varepsilon_2]\!] - \tfrac{1}{8} H^j_{\mu\rho\sigma} [\![\gamma_j{}^i \varepsilon_1, (\Gamma^{\rho\sigma}\Gamma_\nu + \Gamma_\nu \Gamma^{\rho\sigma})\, \varepsilon_2]\!] \\
&= \tfrac{1}{2} H^i_{\mu\nu\sigma} [\![\varepsilon_1, \Gamma^\sigma \varepsilon_2]\!] + \tfrac{1}{4} H^j_{\mu\rho\sigma} [\![\gamma^i{}_j \varepsilon_1, \Gamma^{\rho\sigma}{}_\nu \varepsilon_2]\!] \,.
\end{aligned} \tag{4.76}$$

[4]With an abuse of notation here, we essentially have $\mathrm{d}K^\flat = \iota_{\theta^\sharp} H$ where contraction includes $\mathrm{USp}(4)$ indices.

Now, using this we can thus compute the exterior derivative of θ,

$$\begin{aligned}(\mathrm{d}\theta)^i_{\mu\nu} &= \nabla_\mu \theta^i_\nu - \nabla_\nu \theta^i_\mu \\ &= H^i_{\mu\nu\sigma}[\![\varepsilon_1, \Gamma^\sigma \varepsilon_2]\!] + \tfrac{1}{4} H^j_{\mu\rho\sigma}[\![\gamma^i{}_j \varepsilon_1, \Gamma^{\rho\sigma}{}_\nu \varepsilon_2]\!] - \tfrac{1}{4} H^j_{\nu\rho\sigma}[\![\gamma^i{}_j \varepsilon_1, \Gamma^{\rho\sigma}{}_\mu \varepsilon_2]\!] \,. \end{aligned} \tag{4.77}$$

We use the identity

$$[\Gamma_{\mu\nu}, \Gamma^{\rho\sigma\tau}] = 12\delta^{[\rho}_{[\mu}\Gamma^{\sigma\tau]}{}_{\nu]} = 6\left(\delta^{[\rho}_\mu \Gamma^{\sigma\tau]}{}_\nu - \delta^{[\rho}_\nu \Gamma^{\sigma\tau]}{}_\mu\right) \,, \tag{4.78}$$

to rewrite the last two terms of equation (4.77) yielding

$$(\mathrm{d}\theta)^i_{\mu\nu} = H^i_{\mu\nu\sigma}[\![\varepsilon_1, \Gamma^\sigma \varepsilon_2]\!] + \tfrac{1}{24} H^j_{\rho\sigma\tau}[\![\gamma^i{}_j \varepsilon_1, [\Gamma_{\mu\nu}, \Gamma^{\rho\sigma\tau}]\varepsilon_2]\!] \,. \tag{4.79}$$

As an endomorphism $\Gamma_{\mu\nu}$ is in the $\mathrm{Spin}(5,1)$ subalgebra of $\mathrm{C}\ell(5,1)$ and so preserves spinor chirality. Thus we see that $H^j_{\rho\sigma\tau}[\Gamma_{\mu\nu}, \Gamma^{\rho\sigma\tau}]\varepsilon_2 = 0$ from the anti-self-duality of H (see appendix B.8) and the positive chirality of ε_2.

We also note that the first term is precisely the contraction of H^i with a Killing vector K constructed out of the two Killing spinors $\varepsilon_{1,2}$ as defined in equation (4.71) and we thus have[5]

$$\mathrm{d}\theta = \iota_K H \,. \tag{4.80}$$

This means that $\mathrm{d}\iota_K H = 0$, and along with the fact that H is closed we thus have,

$$\mathcal{L}_K H = \iota_K \mathrm{d}H + \mathrm{d}\iota_K H = 0 \,, \tag{4.81}$$

whence K preserves H and equation (3.56) is satisfied.

We have thus satisfied all the sufficient requirements to have an almost Killing superalgebra *and* for the homogeneity theorem to apply.

4.6.7 Killing superalgebra

For a Killing vector field produced by the squaring map from a single Killing spinor field, $K^\mu = [\![\varepsilon, \Gamma^\mu \varepsilon]\!]$,

$$\mathcal{L}_K \varepsilon = \nabla_K \varepsilon - \tfrac{1}{4}(\nabla K)\cdot\varepsilon \tag{4.82}$$

and using equations (4.70) and (4.74),

$$\begin{aligned}\mathcal{L}_K \varepsilon &= -\tfrac{1}{8} K^\mu H^i_{\mu\nu\rho}\Gamma^{\nu\rho}\gamma_i \varepsilon - \tfrac{1}{8}\theta^\mu_i H^i_{\mu\nu\rho}\Gamma^{\nu\rho}\varepsilon \\ &= \tfrac{1}{48} H^i_{\mu\nu\rho}\left(\Gamma^{\mu\nu\rho}\Gamma_\sigma + \Gamma_\sigma \Gamma^{\mu\nu\rho}\right)\left(K^\sigma \gamma_i + \theta^\sigma_i\right)\varepsilon \,. \end{aligned} \tag{4.83}$$

[5] Compare this and equation (4.74).

Remembering that H is anti-self-dual, we know from appendix B.8 that H annihilates positive chirality spinors and so

$$\begin{aligned}\mathcal{L}_K\varepsilon &= \tfrac{1}{48}H^i_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_\sigma\left(K^\sigma\gamma_i+\theta_i^\sigma\right)\varepsilon &(4.84)\\ &= \tfrac{1}{48}H^i_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_\sigma\left([\![\varepsilon,\Gamma^\sigma\varepsilon]\!]\gamma_i+[\![\varepsilon,\Gamma^\sigma\gamma_i\varepsilon]\!]\right)\varepsilon &(4.85)\\ &= \tfrac{1}{48}H^i_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_\sigma\left(\varepsilon\overline{\varepsilon}\Gamma^\sigma\varepsilon\gamma_i+\varepsilon\overline{\varepsilon}\Gamma^\sigma\gamma_i\varepsilon\right) &(4.86)\\ &= \tfrac{1}{48}H^i_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_\sigma\left(\gamma_i\varepsilon\overline{\varepsilon}+\varepsilon\overline{\varepsilon}\gamma_i\right)\Gamma^\sigma\varepsilon &(4.87)\end{aligned}$$

Now, the Fierz identity for positive chirality spinors with respect to our spinor inner product is:

$$\varepsilon\overline{\varepsilon}=\tfrac{1}{16}\left([\![\varepsilon,\Gamma_\mu\varepsilon]\!]\Gamma^\mu-[\![\varepsilon,\Gamma_\mu\gamma_i\varepsilon]\!]\Gamma^\mu\gamma^i+\tfrac{1}{24}[\![\varepsilon,\Gamma_{\mu\nu\rho}\gamma_{ij}\varepsilon]\!]\Gamma^{\mu\nu\rho}\gamma^{ij}\right)\mathbb{P}(-1)\,, \tag{4.88}$$

where $\mathbb{P}(-1)=\frac{1}{2}(\mathbb{1}-\Gamma_{\rm vol})$ is the projector onto the negative chirality subspace.

Thus, using again equations (4.63) and (4.64) we have

$$\Gamma_\sigma\varepsilon\overline{\varepsilon}\Gamma^\sigma\varepsilon=\tfrac{1}{4}\left([\![\varepsilon,\Gamma_\mu\varepsilon]\!]\Gamma^\mu-[\![\varepsilon,\Gamma_\mu\gamma_j\varepsilon]\!]\Gamma^\mu\gamma^j\right)\varepsilon\,, \tag{4.89}$$

and so

$$\begin{aligned}\Gamma_\sigma\left(\gamma_i\varepsilon\overline{\varepsilon}+\varepsilon\overline{\varepsilon}\gamma_i\right)\Gamma^\sigma\varepsilon &= \tfrac{1}{4}\left(2[\![\varepsilon,\Gamma_\mu\varepsilon]\!]\Gamma^\mu\gamma_i-[\![\varepsilon,\Gamma_\mu\gamma_j\varepsilon]\!]\Gamma^\mu(\gamma_i\gamma^j+\gamma^j\gamma_i)\right)\varepsilon\\ &= \tfrac{1}{2}\Gamma_\mu\left([\![\varepsilon,\Gamma^\mu\varepsilon]\!]\gamma_i+[\![\varepsilon,\Gamma^\mu\gamma_i\varepsilon]\!]\right)\varepsilon\,.\end{aligned} \tag{4.90}$$

Substituting equation (4.90) into equation (4.87) then yields

$$\mathcal{L}_K\varepsilon=\tfrac{1}{96}H^i_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_\sigma\left([\![\varepsilon,\Gamma^\sigma\varepsilon]\!]\gamma_i+[\![\varepsilon,\Gamma^\sigma\gamma_i\varepsilon]\!]\right)\varepsilon\,. \tag{4.91}$$

Finally, comparing equation (4.85) and equation (4.91) implies that $\mathcal{L}_K\varepsilon=0$ and so equation (3.57) is satisfied and we have a Killing superalgebra.

4.7 $D=4,\mathcal{N}=1$

4.7.1 Introduction

We construct the Killing superalgebra of $D=4$, $\mathcal{N}=1$ supergravity [56, 57, 58, 59]. This will turn out to be trivial because the lack of bosonic field content in addition to the metric means that the superconnection is just the lift of the Levi-Civita connection and so Killing spinors are parallel spinors whence Killing vector fields produced using the squaring map are parallel vectors. Of course, supersymmetric $D=4$, $\mathcal{N}=1$ supergravity backgrounds are already completely classified [60] and so the homogeneity theorem in this case tells us nothing new, and we demonstrate it only for completeness.

4.7.2 Conventions

In much of the original literature describing $D = 4$, $\mathcal{N} = 1$ supergravity, the sign conventions adopted are $(\varkappa_0, \varkappa_1) = (-1, +1)$ but we will adopt the conventions $(\varkappa_0, \varkappa_1) = (-1, -1)$. However, the lack of bosonic field content means this will be immaterial to the discussion.

4.7.3 Definition

The field content of $D = 4$, $\mathcal{N} = 1$ supergravity (with all fields transforming in varying representations of $\mathrm{SO}(2)$) is:

Symbol	Count	Name	Description	Spin	D.o.f
g	1	Graviton	Lorentzian metric	2	2
ψ	1	Gravitino	γ-traceless vector-spinor	3/2	2

We are interested in $D = 4$, $\mathcal{N} = 1$ supergravity backgrounds and so we will set all fermionic field content (ψ) to zero from here on and it will not enter into the discussion. The action and field equations will not concern us in what follows.

Our tangent bundle spin group representation $\mathcal{V}$ is the four-dimensional real vector representation of $\mathrm{SO}(1,3)$ equipped with the invariant Lorentzian inner product $(\!|\bullet, \bullet|\!)$.

4.7.4 Spinor representation

Having chosen the convention $(\varkappa_0, \varkappa_1) = (-1, -1)$, the Clifford algebra of relevance is $\mathrm{C}\ell(1,3)$. Let P denote the unique irreducible pinor representation of $\mathrm{C}\ell(1,3)$ which is 4-dimensional and real. Restricting to $\mathrm{Spin}(1,3)$, the spinor representation reduces to chiral representations but the chiral structure does not commute with the real structure and so the chiral representations are complex. We thus have a choice to work with real Majorana spinors or complex Weyl spinors and we of course opt for the former. Thus, let S be the real Majorana spinor representation obtained as a restriction of P to $\mathrm{Spin}(1,3)$. Thus spinors are 4-dimensional and real, and this is the representation with which we construct the spinor bundle $\$$.

There are no bosonic fields other than the metric and so the Killing spinor equation obtained from variation of the gravitino is

$$\delta\psi_X = \mathcal{D}_X\varepsilon = \nabla_X\varepsilon \stackrel{!}{=} 0\,. \tag{4.92}$$

We see that Killing spinors are just parallel spinors.

4.7.5 Spinor inner product

We have a $(\mathrm{Pin}(1,3)$-invariant $\mathbb{R}$-symplectic inner product on P induced from the hat involution of $\mathrm{C}\ell(1,3)$ and so equation (3.53) is satisfied.

4.7.6 Almost Killing superalgebra and homogeneity

We have no bosonic fields in addition to the metric and so equations (3.54) and (3.56) are automatically satisfied.

We construct the squaring map as described in equation (3.22) and, choosing a pseudo-orthonormal basis $\mathfrak{d}_\mu$ for $\mathcal{V}$ and corresponding gamma matrices (see appendix B.11.5) for $\mathcal{P}$, the squaring map takes the concrete form,

$$\Xi(\varepsilon_1, \varepsilon_2) = [\![\varepsilon_1, \Gamma^\mu \varepsilon_2]\!] \mathfrak{d}_\mu = \overline{\varepsilon}_1 \Gamma^\mu \varepsilon_2 \mathfrak{d}_\mu = \varepsilon_1^\dagger \Gamma_0 \Gamma^\mu \varepsilon_2 \mathfrak{d}_\mu \ , \tag{4.93}$$

where we denote the Dirac adjoint $\overline{\varepsilon} := \varepsilon^\dagger \Gamma_0$.

Now, if we look at the $\mathfrak{d}_0$ component of a vector $K = \Xi(\varepsilon, \varepsilon)$ obtained from the squaring map, we see that for non-zero ε,

$$K^0 = \varepsilon^\dagger \Gamma_0 \Gamma^0 \varepsilon = -\varepsilon^\dagger \varepsilon = -|\varepsilon|^2 < 0 \ , \tag{4.94}$$

This means that a vector field K constructed by squaring a single non-zero spinor field ε is necessarily causal because otherwise we could of course Lorentz-transform to the rest frame where $K^0 = 0$. Thus equation (3.55) is satisfied.

We have thus satisfied all the sufficient requirements to have an almost Killing superalgebra *and* for the homogeneity theorem to apply.

4.7.7 Killing superalgebra

For a Killing vector field produced by the squaring map $K^\mu = [\![\varepsilon_1, \Gamma^\mu \varepsilon_2]\!]$ on two Killing spinor fields $\varepsilon_{1,2}$, we have

$$\begin{aligned} \nabla_\mu K_\nu &= \nabla_\mu [\![\varepsilon_1, \Gamma_\nu \varepsilon_2]\!] \\ &= [\![\nabla_\mu \varepsilon_1, \Gamma_\nu \varepsilon_2]\!] + [\![\varepsilon_1, \Gamma_\nu \nabla_\mu \varepsilon_2]\!] \end{aligned} \tag{4.95}$$

and so using equation (4.92),

$$\nabla_\mu K_\nu = 0 \ . \tag{4.96}$$

Thus for a Killing vector field produced by the squaring map from a single Killing spinor field, $K^\mu = [\![\varepsilon, \Gamma^\mu \varepsilon]\!]$ we have

$$\mathcal{L}_K \varepsilon = \nabla_K \varepsilon - \tfrac{1}{4} (\nabla K) \cdot \varepsilon = 0 \ . \tag{4.97}$$

Thus equation (3.57) is satisfied and we have a Killing superalgebra.

Chapter 5

Dimensional reduction of the theorem

5.1 Introduction

In chapter 4 we have shown that the strong homogeneity theorem of chapter 3 applies to a number of a top-dimensional Poincaré supergravity theories. However, although the number of Poincaré supergravity theories is limited, we do not care very much for demonstrating in detail a proof of the theorem for each animal in the zoo, especially because they become more and more unruly as the number of fields increase with dimensional reduction. Therefore we will instead demonstrate that if we have a *theory* to which the strong homogeneity theorem applies, then the strong homogeneity theorem also applies to any dimensional reduction[1] of said *theory*.

We will begin by defining what we mean by a dimensional reduction and follow that up by detailing the decomposition of the massless field content of a theory. Then we will discuss the well-known fact that given a supersymmetric supergravity background, its oxidation must preserve *at least* the same amount of supersymmetry. As examples, we will detail the dimensional reductions of the Killing spinor equations of $D = 11$ and $D = 6$, $(1, 0)$ supergravities. Finally we will show that given a supergravity *theory* satisfying the strong homogeneity theorem, the supergravity *theory* obtained via dimensional reduction also satisfies the strong homogeneity theorem.

To be clear, we will be looking at a general background that upon oxidation preserves greater than half the maximum supersymmetry and is a solution to a supergravity theory known to satisfy the homogeneity theorem. The oxidised background is thus homogeneous, and we will then show that this homogeneity persists after reversing the oxidation via dimensional reduction back to the original background. The key requirement here is that the oxidation be homogeneous as a result of the homogeneity theorem.

[1] As defined in section 5.2.

5.2 Dimensional reduction

A supergravity *compactification* [61, 62] is a generalisation of Kaluza-Klein theory whereby we reduce a supergravity background in $D = p + q$ dimensions to one in p dimensions. We start with a supergravity background $(\mathcal{M}, \hat{g}, \hat{\Phi})$ with $(\mathcal{M}, \hat{g})$ a D-dimensional Lorentzian manifold and $\hat{\Phi}$ the field content of the theory. If possible, we identify an q-parameter group K of spacelike isometries of the background, i.e. spacelike Killing vector fields ξ_i with $\mathcal{L}_{\xi_i}\hat{\Phi} = 0$, $i = 1 \dots q$. We additionally require that the action of this group is free and that the norms of ξ_i are nowhere zero. This guarantees for us that the space $\mathcal{N} = \mathcal{M}/\mathcal{K}$ inherits a Lorentzian metric and via the slice theorem is a smooth manifold. We can then view $\mathcal{M}$ as a principal $\mathcal{K}$-bundle

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathcal{M} \\ & & \big\downarrow \pi \\ & & \mathcal{N} \end{array} \tag{5.1}$$

whose local trivialization is the product $\mathcal{N} \times \mathcal{K}$.

At a point $m \in \mathcal{M}$ the tangent space $\mathrm{T}_m\mathcal{M}$ decomposes into a horizontal and vertical subspace $\mathrm{T}_m\mathcal{M} = \mathscr{H}_m \oplus \mathscr{V}_m$ where $\mathscr{V}_m = \ker \mathrm{d}\pi$ and $\mathscr{H}_m = \mathscr{V}_m^{\perp}$ with respect to $\hat{g}$. This induces a decomposition of the frame bundle structure group from $\mathrm{SO}(1, D-1)$ to $\mathrm{SO}(1, p-1) \times \mathrm{SO}(q)$ and so we proceed to decompose any fields in our theory under $\mathrm{SO}(1, p-1) \times \mathrm{SO}(q)$ and expand in terms of eigenfunctions of the new field-relevant differential operator $\mathscr{D}_{\text{int}}$ on $\mathcal{K}$, yielding an expanded (and possibly infinite) new set of fields. We find that the eigenvalues of these eigenfunctions determine the mass of each of our new fields and are inversely proportional to the length scale R of $\mathcal{K}$. In the limit $R \to 0$ it is thus only the *zero modes* of the expansion that describe the massless fields.

As supergravity is a low-energy effective field theory and R is presumed small, we are only interested in the dynamics of massless fields and so our next step is the *truncation* of each expanded field to just its zero modes. However, at this point we must be careful; such a truncation may well not (and indeed, usually does not) satisfy our original equations of motion because our massless modes may appear in source terms for massive modes. A *consistent truncation* [63] is a compactification with truncation where any solution to the compactified, truncated theory is also a solution of the original theory; i.e. compactification with truncation and variation of the Lagrangian commute.

Let us consider the case where our internal manifold is a flat torus T^q. The eigenfunctions in our field expansions are then simply the irreducible representations of $\mathrm{U}(1)^q$ and we have one singlet zero mode with the other modes all doublets under at least one $\mathrm{U}(1)$ component. If we keep only the zero mode in our truncation then we are guaranteed it will be consistent because singlets alone cannot act as sources for doublets[2]. One can also see in this specific case that truncation is an equivalent prescription to requiring that solutions of the original theory do not

[2]The Noether charges of the $\mathrm{U}(1)^q$ symmetry must be conserved.

depend on the coordinates of the torus. We call such a compactification where solutions have no dependence on the coordinates of the torus a *dimensional reduction*.

The converse operation to dimensional reduction is *oxidation*. We take any solution of a dimensionally reduced supergravity theory in p dimensions with spacetime $\mathcal{M}_p$ and form a trivial fiber bundle over $\mathcal{M}_p$ with fiber T^q, recomposing fields accordingly to transform in the relevant representations of $\mathrm{SO}(1,D-1)$. After oxidation, the new solution[3] to the D-dimensional supergravity theory has no dependence on the fiber coordinates.

We draw attention to the fact that the process of dimensional reduction is thus simply a rewriting of equations and repackaging of degrees of freedom with oxidation the inverse process. A small technicality is that dimensional reduction of a D-dimensional pure supergravity background does not always result in an n-dimensional pure supergravity background. The reduced theory may contain extra matter multiplets. However, a p-dimensional pure supergravity background can always be oxidised to D-dimensions.

In order to simplify the discussion, we will assume $q = 1$ in all that follows; i.e. dimensional reduction and oxidation are with respect to S^1 only. Thus we may adapt coordinates such that the manifold fiber coordinate is ξ and so the spacelike vector field whose flow generates $\mathcal{K}$ we denote $\hat{Z} = \partial_\xi$. Hatted objects and capitalised indices are explicitly in D dimensions, and unhatted objects and lower-case indices explicitly in $(D-1)$ dimensions. Manifold indices are Greek and flat frame indices are Latin. The frame fiber coordinate is z.

5.3 Decomposition of massless fields

Massless fields of $\mathrm{SO}(1,D-1)$ transform as induced representations of the little group $\mathrm{SO}(D-2)$.

Metric

A metric $\hat{g}$ is a rank two symmetric traceless tensor field and so transforms as the $\frac{1}{2}\mathbf{D}(\mathbf{D}-\mathbf{3})$ of $\mathrm{SO}(D-2)$. It decomposes under $\mathrm{SO}(D-3)$ into $\frac{1}{2}(\mathbf{D}-\mathbf{1})(\mathbf{D}-\mathbf{4}) \oplus (\mathbf{D}-\mathbf{3}) \oplus \mathbf{1}$, a metric g, gauge field 1-form A, and scalar ϕ respectively.

Without loss of generality, the decomposition may be written in the Kaluza-Klein ansatz form as

$$\hat{g} = \mathrm{e}^{\pi^*\phi}\left(\pi^* g + \hat{\Theta}^2\right) \ , \tag{5.2}$$

where

$$\hat{\Theta} = \hat{Z}^\flat + \hat{A} \ , \tag{5.3}$$

and

$$\mathrm{d}\hat{A} = \mathrm{d}\hat{\Theta} = \pi^* F = \pi^*(\mathrm{d}A) \ , \tag{5.4}$$

for some 2-form F on $\mathcal{N}$.

[3] It is of course a solution because dimensional reduction is a consistent truncation.

Flux

For gauge fields, we are interest in invariance up to gauge transformations, and so it is the gauge field strength that is the relevant object. A (massless) flux or field strength $\hat{G}$ is an n-form or rank n totally antisymmetric tensor field and so transforms as the $\binom{\mathbf{D-2}}{\mathbf{n}}$ of $\mathrm{SO}(D-2)$. It decomposes under $\mathrm{SO}(D-3)$ into $\binom{\mathbf{D-3}}{\mathbf{n}} \oplus \binom{\mathbf{D-3}}{\mathbf{n-1}}$, an n-form J and $(n-1)$-form H respectively.

Without loss of generality, the decomposition may be written as

$$\hat{G} = \hat{J} + \hat{\Theta} \wedge \iota_{\hat{Z}} \hat{G} \,, \tag{5.5}$$

with $\hat{J}$ a *horizontal* n-form on $\mathcal{M}$, i.e.

$$\iota_{\hat{Z}} \hat{J} = 0 \,. \tag{5.6}$$

Now, as a dimensional reduction we know that $\hat{G}$ is $\mathcal{K}$-invariant by construction and so

$$\begin{aligned} 0 = \mathcal{L}_{\hat{Z}} \hat{G} &= \iota_{\hat{Z}} \mathrm{d}\hat{G} + \mathrm{d}\iota_{\hat{Z}} \hat{G} \\ &= \mathcal{L}_{\hat{Z}} \hat{J} + \mathcal{L}_{\hat{Z}} \iota_{\hat{Z}} \hat{G} = \iota_{\hat{Z}} \mathrm{d}\hat{J} + \hat{\Theta} \wedge \iota_{\hat{Z}} \mathrm{d}\iota_{\hat{Z}} \hat{G} \,, \end{aligned} \tag{5.7}$$

whence

$$\iota_{\hat{Z}} \mathrm{d}\hat{J} = \iota_{\hat{Z}} \mathrm{d}\iota_{\hat{Z}} \hat{G} = 0 \,. \tag{5.8}$$

But then $\mathcal{L}_{\hat{Z}} \hat{J} = \mathcal{L}_{\hat{Z}} \iota_{\hat{Z}} \hat{G} = 0$ and so $\hat{J}$ and $\iota_{\hat{Z}} \hat{G}$ are both horizontal and invariant whence they are both *basic* and thus

$$\hat{J} = \pi^* J \,, \text{ and} \tag{5.9}$$

$$\iota_{\hat{Z}} \hat{G} = \pi^* H \,, \tag{5.10}$$

for J and H an n-form and $(n-1)$-form respectively on $\mathcal{N}$.

Thus we have the decomposition,

$$\hat{G} = \pi^* J + \hat{\Theta} \wedge \pi^* H \,. \tag{5.11}$$

In the case where we have an (anti-)self-dual n-form, it decomposes under $\mathrm{SO}(D-3)$ into just $\binom{\mathbf{D-3}}{\mathbf{n}}$, an n-form. In terms of equation (5.11) this is because H and J are Hodge-dual (up to an appropriate sign) in $D-1$ dimensions.

Vector-spinor

A (massless) gravitino is a γ-traceless vector-spinor field and so transforms as the $(\mathbf{D-3})\mathbf{2}^{\lfloor \mathbf{D/2} \rfloor - \mathbf{1}}$ of $\mathrm{SO}(D-2)$. It decomposes under $\mathrm{SO}(D-3)$ into $(\mathbf{D-4})\mathbf{2}^{\lfloor (\mathbf{D-1})/\mathbf{2} \rfloor - \mathbf{1}} \oplus \mathbf{2}^{\lfloor (\mathbf{D-1})/\mathbf{2} \rfloor - \mathbf{1}}$, a vector-spinor and spinor respectively. These fields will further decompose into chiral eigenstates if D is odd.

Spinor

A (massless) spinor field transforms as the $\mathbf{2}^{\lfloor \mathbf{D}/\mathbf{2} \rfloor - \mathbf{1}}$ of $\mathrm{SO}(D-2)$. It decomposes under $\mathrm{SO}(D-3)$ into $\mathbf{2}^{\lfloor (\mathbf{D}-\mathbf{1})/\mathbf{2} \rfloor - \mathbf{1}}$, a spinor. This field will further decompose into chiral eigenstates if D is odd.

5.4 Decomposition of Killing spinor equations

The Killing spinor equations of a supergravity theory originate from the disappearance of the supersymmetry variations of the fermionic field content: gravitinos, dilatinos, and gauginos. Variation of the gravitino gives rise to a differential equation and variation of dilatinos and gauginos gives rise to algebraic equations, all on the spinor bundle. These equations are all linear and the kernel of this system of equations is the subbundle of Killing spinors. Because we only work locally, we may always identify the representation of the spin group associated to the spinor bundle of a $(D-1)$-dimensional manifold as a possibly decomposable sub-representation of that of the D-dimensional manifold and so the rank of the spinor bundle does not change under dimensional reduction. As such, decomposition of the Killing spinor equations under dimensional reduction or recomposition under oxidation does not change the rank of the kernel of this system and so the number of supersymmetries is preserved. We demonstrate two concrete examples of dimensional reduction of the Killing spinor equations.

5.4.1 $D = 11$ Killing spinor equations reduction

For $D = 11$ supergravity (see section 4.2.3) we will label our field content $(\hat{g}, \hat{F}, \hat{\psi})$ with $\hat{g}$ the metric, $\hat{F}$ a 4-form field strength, and $\hat{\psi}$ a Majorana vector-spinor field (gravitino). Under dimensional reduction, a $D = 11$ background reduces to a pure $D = 10$ type IIA background.

The field content of $D = 10$ type IIA supergravity (with all fields transforming in varying representations of $\mathrm{SO}(8)$) is:

Symbol	Count	Name	Description	Spin	D.o.f
ϕ	1	Dilaton	real scalar	0	1
$A^{(1)}$	1	R-R gauge potential	1-form	1	8
$B^{(2)}$	1	NS-NS gauge potential	2-form	1	28
$C^{(3)}$	1	R-R gauge potential	3-form	1	56
g	1	Graviton	Lorentzian metric	2	35
λ	1	Dilatino	spinor	½	16
ψ	1	Gravitino	γ-traceless vector-spinor	3/2	112

We construct the field strengths,

$$\begin{aligned} F^{(2)} &= \mathrm{d}A^{(1)} \\ H^{(3)} &= \mathrm{d}B^{(2)} \\ G^{(4)} &= \mathrm{d}C^{(3)} - H^{(3)} \wedge A^{(1)} \ . \end{aligned} \tag{5.12}$$

The spinor representation is Majorana-Weyl and so the gravitino and dilatino can be further chirally decomposed but we will not do so.

Note that in this section, we vary our practice and instead follow the conventions of [64] and use $(\varkappa_0, \varkappa_1) = (+1, +1)$. The fields decompose concretely as

$$\hat{g} = \begin{pmatrix} \mathrm{e}^{-2\phi/3} g_{\mu\nu} + \mathrm{e}^{4\phi/3} A_\mu A_\nu & \mathrm{e}^{4\phi/3} A_\mu \\ \mathrm{e}^{4\phi/3} A_\nu & \mathrm{e}^{4\phi/3} \end{pmatrix} \tag{5.13}$$

$$\begin{aligned} \hat{F}_{\mu\nu\rho\sigma} &= G_{\mu\nu\rho\sigma} - 4A_{[\mu} H_{\nu\rho\sigma]} \\ \hat{F}_{\mu\nu\rho\xi} &= H_{\mu\nu\rho} \end{aligned} \tag{5.14}$$

$$\begin{aligned} \hat{\psi}_a &= \mathrm{e}^{\phi/6} \left(\psi_a - \tfrac{1}{6} \hat{\Gamma}_a \lambda \right) \\ \hat{\psi}_z &= \tfrac{1}{3} \mathrm{e}^{\phi/6} \hat{\Gamma}_z \lambda . \end{aligned} \tag{5.15}$$

The supersymmetry parameter decomposes as

$$\hat{\varepsilon} = \mathrm{e}^{\phi/6} \varepsilon . \tag{5.16}$$

The gamma matrices decompose from $\mathrm{C}\ell(1,10)$ to $\mathrm{C}\ell(1,9)$ as

$$\begin{aligned} \hat{\Gamma}^a &= \Gamma^a \\ \hat{\Gamma}_z &= \Gamma_{11} := \Gamma_0 \ldots \Gamma_9 . \end{aligned} \tag{5.17}$$

Then we have the decomposed spin connection,

$$\begin{aligned} \hat{\omega}_{aBC} \hat{\Gamma}^{BC} &= \mathrm{e}^{\phi/3} \omega_{abc} \hat{\Gamma}^{bc} - \tfrac{2}{3} \mathrm{e}^{\phi/3} \partial^b \phi \hat{\Gamma}_{ab} + \mathrm{e}^{4\phi/3} F_{ab} \hat{\Gamma}^b \hat{\Gamma}_z \\ \hat{\omega}_{zBC} \hat{\Gamma}^{BC} &= -\tfrac{4}{3} \mathrm{e}^{\phi/3} \partial^b \phi \hat{\Gamma}_b \hat{\Gamma}_z - \tfrac{1}{2} \mathrm{e}^{4\phi/3} F_{bc} \hat{\Gamma}^{bc} . \end{aligned} \tag{5.18}$$

We can write the $D = 11$ gravitino variation (superconnection, equation (4.2)) in the form

$$\delta \hat{\phi}_A = \hat{\mathcal{D}}_A \hat{\varepsilon} = \hat{\nabla}_A \hat{\varepsilon} + \tfrac{1}{576} \hat{F}_{BCDE} \left(3 \hat{\Gamma}^{BCDE} \hat{\Gamma}_A - \hat{\Gamma}_A \hat{\Gamma}^{BCDE} \right) \hat{\varepsilon} , \tag{5.19}$$

which decomposes as

$$\begin{aligned}\delta\hat{\psi}_a &= \mathrm{e}^{\phi/2}\Big(\nabla_a - \tfrac{1}{6}\partial^b\phi\hat{\Gamma}_{ab} + \tfrac{1}{4}\mathrm{e}^{\phi}F_{ab}\hat{\Gamma}^b\hat{\Gamma}_z - \tfrac{1}{144}H_{bcd}(3\hat{\Gamma}^{bcd}\hat{\Gamma}_a + \hat{\Gamma}_a\hat{\Gamma}^{bcd})\hat{\Gamma}_z \\ &\qquad + \tfrac{1}{576}G_{bcde}(3\hat{\Gamma}^{bcde}\hat{\Gamma}_a - \hat{\Gamma}_a\hat{\Gamma}^{bcde})\Big)\varepsilon \\ \delta\hat{\psi}_z &= \mathrm{e}^{\phi/2}\left(-\tfrac{1}{3}\partial^b\phi\hat{\Gamma}_b\hat{\Gamma}_z - \tfrac{1}{8}\mathrm{e}^{\phi}F_{bc}\hat{\Gamma}^{bc} + \tfrac{1}{36}H_{bcd}\hat{\Gamma}^{bcd} + \tfrac{1}{288}\mathrm{e}^{\phi}G_{bcde}\hat{\Gamma}^{bcde}\hat{\Gamma}_z\right)\varepsilon.\end{aligned} \tag{5.20}$$

Then, using the gravitino and gamma matrix decompositions, we can further recast this decomposition into the type IIA dilatino and gravitino variations,

$$\begin{aligned}\delta\psi_a &= \left(\nabla_a - \tfrac{1}{8}H_{abc}\Gamma^{bc}\Gamma_{11} - \tfrac{1}{16}\mathrm{e}^{\phi}F_{bc}\Gamma^{bc}\Gamma_a\Gamma_{11} + \tfrac{1}{192}\mathrm{e}^{\phi}G_{bcde}\Gamma^{bcde}\Gamma_a\right)\varepsilon \\ \delta\lambda &= \left(\partial_b\phi\Gamma^b - \tfrac{1}{12}H_{bcd}\Gamma^{bcd}\Gamma_{11} - \tfrac{3}{8}\mathrm{e}^{\phi}F_{bc}\Gamma^{bc}\Gamma_{11} + \tfrac{1}{96}\mathrm{e}^{\phi}G_{bcde}\Gamma^{bcde}\right)\varepsilon.\end{aligned} \tag{5.21}$$

The gravitino and dilatino variations above are the Killing spinor equations of $D = 10$ type IIA supergravity. We see that they are a direct recasting of the $D = 11$ supergravity Killing spinor equation with respect to the decomposition of fields induced by a spacelike symmetry of the background. Thus a $D = 10$ type IIA Killing spinor oxidises to a $D = 11$ Killing spinor.

5.4.2 $D = 6$, $(1, 0)$ Killing spinor equations reduction

For $D = 6$, $(1, 0)$ supergravity (see section 4.5) we will label our field content $(\hat{g}, \hat{H}, \hat{\psi})$ with $\hat{g}$ the $6d$ metric, $\hat{H}$ an anti-self-dual 3-form field strength, and $\hat{\psi}$ a positive chirality symplectic Majorana-Weyl vector-spinor field (gravitino). Under dimensional reduction, a $D = 6$, $(1, 0)$ background reduces to a $D = 5$, $\mathcal{N} = 2$ background. However, it reduces not to pure $D = 5$, $\mathcal{N} = 2$ supergravity but to $D = 5$, $\mathcal{N} = 2$ supergravity coupled to a vector multiplet. Be that as it may, we will see that suitable $D = 6$, $(1, 0)$ backgrounds can still be consistently truncated to pure $D = 5$, $\mathcal{N} = 2$ supergravity whence pure $D = 5$, $\mathcal{N} = 2$ backgrounds always oxidise to $D = 6$, $(1, 0)$ backgrounds.

The field content of pure $D = 5$, $\mathcal{N} = 2$ supergravity (with all fields transforming in varying representations of $\mathrm{SO}(3)$) is:

Symbol	Count	Name	Description	Spin	D.o.f
B	1	R-R gauge potential	1-form	1	3
g	1	Graviton	Lorentzian metric	2	5
ψ	2	Gravitino	γ-traceless vector-spinor	$3/2$	2×4

The field content of a $D = 5$, $\mathcal{N} = 2$ vector multiplet (with all fields transforming in varying representations of $\mathrm{SO}(3)$) is:

Symbol	Count	Name	Description	Spin	D.o.f
ϕ	1	Dilaton	real scalar	0	1
A	1	Gauge potential	1-form	1	3
ξ	1	Dilatino	spinor	½	4

We construct the field strengths,

$$\begin{aligned} H &= \mathrm{d}B \\ F &= \mathrm{d}A\ . \end{aligned} \tag{5.22}$$

We invert the conventions of [65] and stick to $(\varkappa_0, \varkappa_1) = (+1, -1)$ (resulting in some sign differences). The fields decompose concretely as

$$\hat{g} = \begin{pmatrix} \mathrm{e}^{-\phi} g_{\mu\nu} + \mathrm{e}^{2\phi} A_\mu A_\nu & \mathrm{e}^{2\phi} A_\mu \\ \mathrm{e}^{2\phi} A_\nu & \mathrm{e}^{2\phi} \end{pmatrix} \tag{5.23}$$

$$\hat{H}_{\mu\nu\xi} = \tfrac{1}{\sqrt{2}} H_{\mu\nu} \tag{5.24}$$

$$\begin{aligned} \hat{\psi}_a &= \psi_a \\ \hat{\psi}_z &= \lambda. \end{aligned} \tag{5.25}$$

However, in order to reduce to a pure background we must impose some constraints and redefine some fields, in order to perform a consistent truncation to the bosonic sector of pure $D = 5$, $\mathcal{N} = 2$ supergravity. Further details can be found in [66]. First, we may consistently set $\phi = 0$ such that the $D = 6$ equations of motion are satisfied. Then the $D = 6$ Einstein equation imposes that $|H|^2 = 2|F|^2$ and so, satisfying this, we make the field identification and definition

$$G_{\mu\nu} = 2\partial_{[\mu} V_{\nu]} := \sqrt{\tfrac{3}{2}} H_{\mu\nu} = \sqrt{3} F_{\mu\nu} = \sqrt{12} \partial_{[\mu} A_{\nu]}. \tag{5.26}$$

The field decompositions then become

$$\hat{g} = \begin{pmatrix} g_{\mu\nu} + \frac{1}{3} V_\mu V_\nu & \frac{1}{\sqrt{3}} V_\mu \\ \frac{1}{\sqrt{3}} V_\nu & 1 \end{pmatrix} \tag{5.27}$$

$$\hat{H}_{\mu\nu\xi} = \tfrac{1}{\sqrt{3}} G_{\mu\nu}, \tag{5.28}$$

The supersymmetry parameter decomposes as

$$\hat{\varepsilon} = \varepsilon. \tag{5.29}$$

The gamma matrices decompose from $\mathrm{C}\ell(5,1)$ to $\mathrm{C}\ell(4,1)$ as

$$\begin{aligned} \hat{\Gamma}_a &= \Gamma_a \otimes \sigma_1 \\ \hat{\Gamma}_z &= -\mathbb{1}_4 \otimes i\sigma_2 \\ \hat{\Gamma}_7 &= \Gamma_0 \ldots \Gamma_5 = \mathbb{1}_4 \otimes \sigma_3. \end{aligned} \tag{5.30}$$

We have the decomposed spin connection

$$\begin{aligned}\hat{\omega}_{aBC}\hat{\Gamma}^{BC} &= \omega_{abc}\hat{\Gamma}^{bc} + \tfrac{1}{\sqrt{3}}G_{ab}\hat{\Gamma}^{b}\hat{\Gamma}_{z}\\ \hat{\omega}_{zBC}\hat{\Gamma}^{BC} &= -\tfrac{1}{2\sqrt{3}}G_{bc}\hat{\Gamma}^{bc}.\end{aligned} \tag{5.31}$$

We have the $D = 6$, $(1, 0)$ gravitino variation (superconnection)

$$\delta\hat{\psi}_A = \hat{\mathcal{D}}_A\hat{\varepsilon} = \hat{\nabla}_A\hat{\varepsilon} + \tfrac{1}{48}\hat{H}_{BCD}\hat{\Gamma}^{BCD}\hat{\Gamma}_A\hat{\varepsilon}, \tag{5.32}$$

which decomposes as

$$\begin{aligned}\delta\hat{\psi}_a &= \left(\nabla_a + \tfrac{1}{8\sqrt{3}}(\hat{\Gamma}_{bc}\hat{\Gamma}_a - 2\hat{\Gamma}^b\delta^c_a)G_{bc}\hat{\Gamma}_z\right)\hat{\varepsilon}\\ \delta\hat{\psi}_z &= 0.\end{aligned} \tag{5.33}$$

With our identifications, the z-component of the gravitino variation is automatically zero. Now, we note that, using the positive chirality of $\hat{\varepsilon}$,

$$\begin{aligned}\hat{\Gamma}_a\hat{\Gamma}_b\hat{\varepsilon} &= ((\Gamma_a\Gamma_b)\otimes\mathbb{1}_2)\hat{\varepsilon}\\ \hat{\Gamma}_a\hat{\Gamma}_z\hat{\varepsilon} &= (\Gamma_a\otimes\sigma_3)\hat{\varepsilon} = (\Gamma_a\otimes\mathbb{1}_2)\hat{\Gamma}_7\hat{\varepsilon} = (\Gamma_a\otimes\mathbb{1}_2)\hat{\varepsilon}.\end{aligned} \tag{5.34}$$

We thus see directly in the restriction to $\mathrm{Spin}(4, 1)$ how we reduce from a positive chirality symplectic Majorana-Weyl supersymmetry parameter to a symplectic Majorana supersymmetry parameter. This finally reduces the $D = 6$, $(1, 0)$ gravitino variation to that of $D = 5$, $\mathcal{N} = 2$,

$$\delta\psi_a = \left(\nabla_a + \tfrac{1}{8\sqrt{3}}(\Gamma_{bc}\Gamma_a - 2\Gamma^b\delta^c_a)G_{bc}\right)\varepsilon. \tag{5.35}$$

The gravitino variation above is the Killing spinor equation of pure $D = 5, \mathcal{N} = 2$ supergravity. We see that it is a direct recasting of the $D = 6$, $(1, 0)$ supergravity Killing spinor equation with respect to the decomposition of fields induced by a spacelike symmetry of the background along with a consistent truncation of the $D = 5$ vector multiplet in the bosonic sector. We thus have a consistent procedure to take any pure $D = 5$, $\mathcal{N} = 2$ background and oxidise it to a $D = 6$, $(1, 0)$ background. As part of this procedure, the dilatino component of the $D = 5$ Killing spinor equations is identically zero and thus a $D = 5$, $\mathcal{N} = 2$ Killing spinor oxidises to a $D = 6$, $(1, 0)$ Killing spinor.

5.5 The homogeneity theorem for dimensional reductions

Let us start with a $(D - 1)$-dimensional supergravity background $\mathcal{X} = (\mathcal{N}, g, \Phi)$ that oxidises to a D-dimensional supergravity background $\mathcal{Y} = (\mathcal{M}, \hat{g}, \hat{\Phi})$ with principal bundle structure $\pi : \mathcal{M} \to \mathcal{N}$. We have seen that if the kernel of the system of Killing spinor equations of $\mathcal{X}$ has dimension k then the kernel of the system of Killing spinor equations of $\mathcal{Y}$ has at least dimension

k. Now, let us suppose that k is larger than half the rank of the spinor bundle of $\mathcal{Y}$ and that $\mathcal{Y}$ is a background of a supergravity theory that satisfies the strong homogeneity theorem (sections 4.2, 4.3, 4.4.1 and 4.5 to 4.7). The strong homogeneity theorem then tells us that the oxidised background $\mathcal{Y}$ is locally homogeneous. This means that for every point $m \in \mathcal{M}$ we can construct a frame for the tangent space made entirely out of Killing vector fields and these Killing vector fields preserve $\hat{\Phi}$. Moreover, these Killing vector fields are all constructed from the Killing spinor fields via the squaring map $\hat{\Xi}$.

Now, the Killing spinors of $\mathcal{Y}$ are the oxidised Killing spinors of $\mathcal{X}$ and so the Killing spinors of $\mathcal{Y}$ must be constant along the oxidation fiber because a dimensional reduction is a consistent truncation. So for a $\mathcal{Y}$-Killing spinor $\hat{\varepsilon}$,

$$\hat{\mathcal{L}}_{\hat{Z}}\hat{\varepsilon} = 0. \tag{5.36}$$

The Killing vector fields of $\mathcal{Y}$ are constructed via the squaring map $\hat{\Xi}$ from the Killing spinors of $\mathcal{Y}$ and thus must also be constant along the oxidation fiber, so for $\mathcal{Y}$-Killing vector $\hat{K}$ and $\mathcal{Y}$-Killing spinors $\hat{\varepsilon}_1, \hat{\varepsilon}_2$,

$$\hat{\mathcal{L}}_{\hat{Z}}\hat{K} = \hat{\mathcal{L}}_{\hat{Z}}\hat{\Xi}(\hat{\varepsilon}_1, \hat{\varepsilon}_2) = 0. \tag{5.37}$$

That $\hat{K}$ is invariant is precisely the condition that $\hat{K}$ is *projectable* meaning it is the horizontal lift of a $K \in \Gamma(\mathrm{T}\mathcal{N})$ such that $\pi_*\hat{K} = K$.

Moreover this means that any such $\hat{K}$ commutes with $\hat{Z}$ and so, taking $\hat{\mathfrak{g}}_0$ to be the Lie algebra of Killing vector fields on $\mathcal{M}$, and $h \subset \hat{\mathfrak{g}}_0$ the line generated by $\hat{Z}$ in $\hat{\mathfrak{g}}_0$ where $N(h)$ is the normaliser of h in $\hat{\mathfrak{g}}_0$, we have a $(D-1)$-dimensional horizontal subalgebra of projectable Killing vector fields,

$$\hat{\mathfrak{h}}_0 = N(h)/h\ . \tag{5.38}$$

What's more, being projectable and horizontal, $\hat{\mathfrak{h}}_0$ is basic. The restriction of the push down π_* to basic vector fields is injective whence we may push down $\hat{\mathfrak{h}}_0$ to a $(D-1)$-dimensional algebra of vector fields $\mathfrak{h}_0$ on $\mathcal{N}$. As $\mathfrak{h}_0$ is $(D-1)$-dimensional, at any point $n \in \mathcal{N}$ it spans the tangent space $\mathrm{T}_n\mathcal{N}$.

Now, we wish to show that each of the decomposed objects on $\mathcal{N}$ is left invariant by the vector fields in $\mathfrak{h}_0$. We have g, A, ϕ from the metric $\hat{g}$ and J, H from a flux $\hat{G}$. However, A being a gauge field must be invariant up to gauge transformations and so we instead require invariance of its field strength whence F is the relevant object. Now, since $\mathfrak{h}_0$ is just the push down of a basic Lie algebra of vector fields on $\mathcal{M}$ and the Lie derivative commutes with pullbacks via *naturality* we may equivalently show that the pullbacks of these objects $(\pi^*g, \pi^*F, \pi^*\phi, \pi^*J, \pi^*H)$ are $\hat{\mathfrak{h}}_0$-invariant.

Invariance of the metric decomposition

Now, from equation (5.2),

$$\hat{g}(\hat{Z},\hat{Z}) = \mathrm{e}^{\pi^*\phi}\,, \tag{5.39}$$

and for a Killing vector field $\hat{K} \in \hat{\mathfrak{h}}_0$,

$$\mathcal{L}_{\hat{K}}\left(\hat{g}(\hat{Z},\hat{Z})\right) = \left(\mathcal{L}_{\hat{K}}\hat{g}\right)(\hat{Z},\hat{Z}) + 2\hat{g}\left(\mathcal{L}_{\hat{K}}\hat{Z},\hat{Z}\right) = 0 + 0 = 0\,, \tag{5.40}$$

because $\mathcal{L}_{\hat{K}}\hat{Z} = 0$ from equation (5.37) and $\mathcal{L}_{\hat{K}}\hat{g} = 0$ because $\hat{K}$ is Killing, whence

$$\mathcal{L}_{\hat{K}}\pi^*\phi = 0\,. \tag{5.41}$$

Again from equation (5.2),

$$\hat{g}(\hat{Z}) = \mathrm{e}^{\pi^*\phi}\hat{\Theta}\,, \tag{5.42}$$

and for a Killing vector field $\hat{K} \in \hat{\mathfrak{h}}_0$,

$$\mathcal{L}_{\hat{K}}\left(\hat{g}(\hat{Z})\right) = \left(\mathcal{L}_{\hat{K}}\hat{g}\right)(\hat{Z}) + \hat{g}\left(\mathcal{L}_{\hat{K}}\hat{Z}\right) = 0 + 0 = 0\,, \tag{5.43}$$

where again we have used $\mathcal{L}_{\hat{K}}\hat{Z} = 0$ from equation (5.37) and $\mathcal{L}_{\hat{K}}\hat{g} = 0$ because $\hat{K}$ is Killing. But

$$\begin{aligned}\mathcal{L}_{\hat{K}}\left(\hat{g}(\hat{Z})\right) &= \mathcal{L}_{\hat{K}}\left(\mathrm{e}^{\pi^*\phi}\hat{\Theta}\right)\\ &= \left(\mathcal{L}_{\hat{K}}\mathrm{e}^{\pi^*\phi}\right)\hat{\Theta} + \mathrm{e}^{\pi^*\phi}\left(\mathcal{L}_{\hat{K}}\hat{\Theta}\right)\\ &= \mathrm{e}^{\pi^*\phi}\left(\mathcal{L}_{\hat{K}}\hat{\Theta}\right)\,,\end{aligned} \tag{5.44}$$

where we have used equation (5.41). Thus

$$\mathcal{L}_{\hat{K}}\hat{\Theta} = 0\,, \tag{5.45}$$

whence $\mathcal{L}_{\hat{K}}\mathrm{d}\hat{\Theta} = 0$ and so

$$\mathcal{L}_{\hat{K}}\pi^*F = 0\,. \tag{5.46}$$

Once more from equation (5.2), using equations (5.41) and (5.45),

$$\begin{aligned}\mathcal{L}_{\hat{K}}\hat{g} &= \mathcal{L}_{\hat{K}}\left(\mathrm{e}^{\pi^*\phi}\left(\pi^*g + \hat{\Theta}^2\right)\right)\\ &= \mathrm{e}^{\pi^*\phi}\mathcal{L}_{\hat{K}}\left(\pi^*g + \hat{\Theta}^2\right)\\ &= \mathrm{e}^{\pi^*\phi}\mathcal{L}_{\hat{K}}\pi^*g\,,\end{aligned} \tag{5.47}$$

and so using $\mathcal{L}_{\hat{K}}\hat{g} = 0$ we have

$$\mathcal{L}_{\hat{K}}\pi^*g = 0\,. \tag{5.48}$$

Thus all components of the metric $\hat{g}$ are $\hat{K}$-invariant.

Invariance of the flux decomposition

For a Killing vector field $\hat{K} \in \hat{\mathfrak{h}}_0$ we have

$$\mathcal{L}_{\hat{K}} \hat{G} = 0 \,, \tag{5.49}$$

but then we have that

$$\mathcal{L}_{\hat{K}} \iota_{\hat{Z}} \hat{G} = \iota_{[\hat{K},\hat{Z}]} \hat{G} = 0 \,, \tag{5.50}$$

whence from equation (5.10),

$$\mathcal{L}_{\hat{K}} \pi^* H = 0 \,. \tag{5.51}$$

Now, rewriting equation (5.11) we have

$$\pi^* J = \hat{G} - \hat{\Theta} \wedge \pi^* H \,, \tag{5.52}$$

and so

$$\mathcal{L}_{\hat{K}} \pi^* J = \mathcal{L}_{\hat{K}} \left(\hat{G} - \hat{\Theta} \wedge \pi^* H \right) = 0 \,, \tag{5.53}$$

which is clear from equations (5.45), (5.49) and (5.51).

Thus all components of the flux $\hat{G}$ are $\hat{K}$-invariant.

We see that $\mathfrak{h}_0$ is a Lie algebra of supergravity Killing vector fields that span the tangent space of $\mathcal{N}$ and so the background $\mathcal{X}$ is locally homogeneous.

5.6 Conclusion

We have shown that if we have a Poincaré supergravity theory in D dimensions that has been shown to satisfy the strong homogeneity theorem, then a theory that may be constructed as a dimensional reduction (as defined in section 5.2) of this theory will also satisfy the strong homogeneity theorem. This is because any background in $D-1$ dimensions can be oxidised to a background in D dimensions where the strong homogeneity theorem applies whence if the background preserves greater than half the maximum supersymmetry, it is locally homogeneous. Local homogeneity of the oxidised background has then been shown to imply local homogeneity of the dimensionally reduced background. This process can be iterated and so we have shown the strong homogeneity theorem to apply to all dimensional reductions of $D = 11, 10$ and $D = 6$, $\mathcal{N} = 2, 4$ Poincaré supergravities along with $D = 4$ minimal supergravity. This leaves $\mathcal{N} = 3, 5, 6$ supergravities where the strong homogeneity theorem has yet to be demonstrated. The spectrum and coverage of Poincaré supergravity theories is shown in table 5.1.

We note that this does *not* mean that the dimensional reduction of *any* homogeneous *background* is again homogeneous. It also does not mean that *any* homogeneous *background* is necessarily homogeneous upon oxidation.

Table 5.1: Poincaré supergravity theories and their dimensional reductions.

D\SUSY	32	24	20	16	12	8	4
11	M						
10	IIA, IIB			I			
9	$\mathcal{N}=2$			$\mathcal{N}=1$			
8	$\mathcal{N}=2$			$\mathcal{N}=1$			
7	$\mathcal{N}=4$			$\mathcal{N}=2$			
6	$(2,2)$, $(3,1)$, $(4,0)$	$(2,1)$, $(3,0)$		$(1,1)$, $(2,0)$		$(1,0)$	
5	$\mathcal{N}=8$	$\mathcal{N}=6$		$\mathcal{N}=4$		$\mathcal{N}=2$	
4	$\mathcal{N}=8$	$\mathcal{N}=6$	$\mathcal{N}=5$	$\mathcal{N}=4$	$\mathcal{N}=3$	$\mathcal{N}=2$	$\mathcal{N}=1$

- Strong homogeneity theorem proved directly
- Strong homogeneity theorem applies as a dimensional reduction
- The theory has no metric in its spectrum
- The theory has not yet been explicitly constructed
- Strong homogeneity theorem not yet demonstrated

Chapter 6

Symmetric type IIB supergravity backgrounds

6.1 Introduction

The homogeneity theorem greatly simplifies the programme of classification of supergravity backgrounds preserving many supersymmetries. It means that in order to classify highly supersymmetric backgrounds, we may instead classify locally homogeneous backgrounds.

However, although this is a significant improvement, we are left with the still daunting task of classifying homogeneous backgrounds. It would seem that this will require first classifying homogeneous Lorentzian manifolds. Progress has been achieved relatively recently in the three- [67] and four-[68] dimensional cases and a programme in general dimensions [69, 70, 71] has been under way for a period of time. In general dimensions, the classification of manifolds $\mathcal{M} = \mathcal{G}/\mathcal{H}$ with semisimple $\mathcal{G}$ has been reduced to the case of compact stabilizer $\mathcal{H}$ but an approach to classifying manifolds with non-semisimple $\mathcal{G}$ remains out of reach.

As such, we may be forgiven for heading straight for the low-hanging fruit: the symmetric supergravity backgrounds. The symmetric M-theory backgrounds have been classified in [33] and in this spirit we look to classify the symmetric backgrounds of type IIB supergravity. This chapter is based upon work done in collaboration with José Figueroa-O'Farrill in [3].

6.2 $D = 10$ type IIB supergravity

$D = 10$ type IIB supergravity [42, 43, 44] is the low energy effective field theory of type IIB string theory [72]. It is the unique $\mathcal{N} = (2, 0)$ (chiral) supergravity theory in ten dimensions and unlike $D = 10$ type IIA supergravity [73, 74, 75], cannot be constructed as a dimensional reduction of $D = 11$ maximal supergravity [19]. However, type IIB supergravity can be related to type IIA supergravity through T-duality [76, 77, 78, 79] via the their common dimensional reduction to $D = 9$. It has been the subject of extensive study as a tool for understanding

type IIB string theory and furthermore as a limiting case of M-theory through the unification of ten-dimensional string theories [27]. More recently, the $\mathrm{AdS}\,/\,\mathrm{CFT}$ correspondence [80] implied that type IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ is equivalent to $\mathcal{N} = 4$ super Yang-Mills theory on the boundary with gauge group $\mathrm{SU}(N)$, with $N \gg 1$ and in the strong coupling regime.

6.2.1 Definition

We continue on from our earlier definition of $D = 10$ type IIB supergravity (see section 4.3.3). Now we are interested in the field equations of the theory and so we begin with the action.

Because of the self-duality of $G^{(5)}$, it is not possible to write a covariant action for type IIB supergravity. However, we can write a non-self-dual action that when varied yields the correct field equations as long as we add in the self-duality condition by hand as an additional field equation. This (bosonic) action is (in the string frame)

$$\begin{aligned} S = \int \left\{ \mathrm{e}^{-2\phi} \left(\mathrm{R} + 4|\mathrm{d}\phi|^2 - \tfrac{1}{2}|H^{(3)}|^2 \right) - \tfrac{1}{2} \left(|G^{(1)}|^2 + |G^{(3)}|^2 + \tfrac{1}{2}|G^{(5)}|^2 \right) \right\} \mathrm{dvol} \\ - \tfrac{1}{2} \int C^{(4)} \wedge H^{(3)} \wedge \mathrm{d}C^{(2)} \,, \end{aligned} \tag{6.1}$$

where R is the Ricci scalar curvature of g and dvol is the signed volume element. We define the inner product on differential forms $\langle X, Y \rangle \,\mathrm{dvol} = X \wedge *Y$ and the norm $|X|^2 = \langle X, X \rangle$.

Varying the action with respect to each of the fields and supplementing with the $G^{(5)}$ self-duality condition yields the following (bosonic) equations of motion,

$$\begin{aligned} \triangle \phi &= \tfrac{1}{16}|H^{(3)}|^2 - \tfrac{1}{16}\mathrm{e}^{2\phi}|G^{(3)}|^2 - \tfrac{1}{8}\mathrm{e}^{2\phi}|G^{(1)}|^2 \\ \mathrm{d} \star G^{(1)} &= -H^{(3)} \wedge \star G^{(3)} \\ \mathrm{d} \star G^{(3)} &= -H^{(3)} \wedge G^{(5)} \\ \mathrm{d} \star H^{(3)} &= \mathrm{e}^{2\phi} G^{(3)} \wedge G^{(5)} \\ \mathrm{d} \star G^{(5)} &= H^{(3)} \wedge G^{(3)} \\ G^{(5)} &= \star G^{(5)} \\ \mathrm{Ric}(X, Y) &= -4(X\phi)(Y\phi) + \tfrac{1}{2}\mathrm{e}^{2\phi} G^{(1)}(X) G^{(1)}(Y) + \tfrac{1}{2}\langle \iota_X H^{(3)}, \iota_Y H^{(3)} \rangle \\ &\quad + \tfrac{1}{2}\mathrm{e}^{2\phi}\langle \iota_X G^{(3)}, \iota_Y G^{(3)} \rangle + \tfrac{1}{4}\mathrm{e}^{2\phi}\langle \iota_X G^{(5)}, \iota_Y G^{(5)} \rangle \\ &\quad - \tfrac{1}{8}|H^{(3)}|^2 g(X, Y) - \tfrac{1}{8}\mathrm{e}^{2\phi}|G^{(3)}|^2 g(X, Y) \,, \end{aligned} \tag{6.2}$$

where Ric stands for the Ricci tensor.

6.2.2 Symmetries

$D = 10$ type IIB supergravity exhibits two symmetries that we will make use of in our analysis:

$\mathbf{SL(2,\mathbb{R})}$ symmetry

The action has a global $\mathrm{SL}(2,\mathbb{R})$ symmetry [43]. This symmetry (in the Einstein frame[1]) acts upon g and $C^{(4)}$ inertly, on $C^{(2)}$ with $B^{(2)}$ as a doublet, and on the axi-dilaton $\tau = C^{(0)} + \mathrm{i}\mathrm{e}^{-\phi}$ via Möbius transformations. Taking a group element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \ , \tag{6.3}$$

these transformation are

$$\begin{pmatrix} (B^{(2)})' \\ (C^{(2)})' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B^{(2)} \\ C^{(2)} \end{pmatrix} \qquad \text{and} \qquad \tau' = \frac{a\tau + b}{c\tau + d} \ . \tag{6.4}$$

The type IIB string theory S-duality is the preservation of the $\mathrm{SL}(2,\mathbb{Z})$ subgroup of this symmetry [79, 81].

Homothety symmetry

A feature common to ungauged and massless supergravities is a homothety symmetry known as the $\mathbb{R}^+$ or trombone symmetry [82].

The field equation (6.2) are invariant under homothetic transformations of the fields, the weight of the transformation corresponding to the number of Lorentz indices[2] of the field. In this particular case the transformation is, taking an element $t \in \mathbb{R}$

$$(g, \phi, C^{(0)}, C^{(2)}, C^{(4)}, B^{(2)}) \mapsto (\mathrm{e}^{2t} g, \phi, C^{(0)}, \mathrm{e}^{2t} C^{(2)}, \mathrm{e}^{4t} C^{(4)}, \mathrm{e}^{2t} B^{(2)}) \ . \tag{6.5}$$

The Lagrangian is thus not invariant but transforms as $\mathcal{L} \to \mathrm{e}^{8t}\mathcal{L}$ giving this symmetry its trombone moniker.

6.3 $D = 10$ Type IIB symmetric backgrounds

A $D = 10$ Type IIB background is $(\mathcal{M}, g, \phi, H^{(3)}, G^{(1)}, G^{(3)}, G^{(5)})$. This background is a homogeneous background when the underlying geometry is homogeneous, $\mathcal{M} = \mathcal{G}/\mathcal{K}$ with $\mathcal{G} \subseteq \mathrm{I}(\mathcal{M})$ and the field content is $\mathcal{G}$-invariant up to gauge transformations. We can realise this $\mathcal{G}$-invariance up to gauge transformations very simply by requiring the $\mathcal{G}$-invariance of the gauge-invariant field strengths. Now, since the (scalar) dilaton ϕ is not a field strength, this means it is necessarily constant and as such we can eliminate it from the field equations by defining the new fields $F^{(i)} := \mathrm{e}^{\phi} G^{(i)}$, $(i = 1, 2, 3)$. The field equation (6.2) simplify to

[1] $ds^2_{\mathrm{E}} = \mathrm{e}^{-\phi/2} ds^2_{\mathrm{S}}$

[2] Or for spinorial fields, the number of Lorentz indices minus a half.

$$
\begin{aligned}
|H^{(3)}|^2 &= |F^{(3)}|^2 + 2|F^{(1)}|^2 \\
\mathrm{d}\star F^{(1)} &= \star F^{(3)} \wedge H^{(3)} \\
\mathrm{d}\star F^{(3)} &= F^{(5)} \wedge H^{(3)} \\
\mathrm{d}\star H^{(3)} &= F^{(3)} \wedge F^{(5)} \\
\mathrm{d}\star F^{(5)} &= H^{(3)} \wedge F^{(3)} \\
F^{(5)} &= \star F^{(5)} \\
\mathrm{Ric}(X,Y) &= \tfrac{1}{2}F^{(1)}(X)F^{(1)}(Y) + \tfrac{1}{2}\langle \iota_X H^{(3)}, \iota_Y H^{(3)}\rangle + \tfrac{1}{2}\langle \iota_X F^{(3)}, \iota_Y F^{(3)}\rangle \\
&\quad + \tfrac{1}{4}\langle \iota_X F^{(5)}, \iota_Y F^{(5)}\rangle - \tfrac{1}{8}|H^{(3)}|^2 g(X,Y) - \tfrac{1}{8}|F^{(3)}|^2 g(X,Y)\ .
\end{aligned}
\tag{6.6}
$$

Now, let us further specialise to the case where the underlying geometry is not only homogeneous but symmetric in which case we say that we have a symmetric background. For a symmetric space we have that any invariant n-form $A^{(n)}$ is parallel with respect to the canonical connection (see section 2.2.1) which is the Levi-Civita connection ($\nabla A^{(n)} = 0$) and so its dual $\star A^{(n)}$ is also parallel ($\nabla \star A^{(n)} = 0$) whence it is both closed ($\mathrm{d}A^{(n)} = 0$) and co-closed ($\mathrm{d}\star A^{(n)} = 0$). The field equations then further simplify to

$$
|H^{(3)}|^2 = |F^{(3)}|^2 + 2|F^{(1)}|^2 \tag{6.7}
$$

$$
0 = \star F^{(3)} \wedge H^{(3)} \tag{6.8}
$$

$$
0 = F^{(5)} \wedge H^{(3)} \tag{6.9}
$$

$$
0 = F^{(3)} \wedge F^{(5)} \tag{6.10}
$$

$$
0 = H^{(3)} \wedge F^{(3)} \tag{6.11}
$$

$$
F^{(5)} = \star F^{(5)} \tag{6.12}
$$

$$
\begin{aligned}
\mathrm{Ric}(X,Y) &= \tfrac{1}{2}F^{(1)}(X)F^{(1)}(Y) + \tfrac{1}{8}\left(4\langle \iota_X H^{(3)}, \iota_Y H^{(3)}\rangle - |H^{(3)}|^2 g(X,Y)\right) \\
&\quad + \tfrac{1}{8}\left(4\langle \iota_X F^{(3)}, \iota_Y F^{(3)}\rangle - |F^{(3)}|^2 g(X,Y)\right) + \tfrac{1}{4}\langle \iota_X F^{(5)}, \iota_Y F^{(5)}\rangle\ .
\end{aligned}
\tag{6.13}
$$

Now, using the classification and enumeration of Lorentzian symmetric spaces described in section 2.2.2, we can take a particular Lorentzian symmetric space $\mathcal{M} = \mathcal{M}_0 \times \mathcal{M}_1 \times \ldots \times \mathcal{M}_n$, where each factor determined by its Lie algebra pair $(\mathfrak{k}, \mathfrak{m})_i$, along with the set of $\mathfrak{k}$-invariant forms for each factor.

Using the correspondence in equation (2.10) we then algebraise the field equations (6.7) to (6.13). For each field strength n-form (evaluated at the origin), we construct the most general possible (real-)parametrised ansatz $\mathfrak{k}$-invariant n-form out of the the set of available $\mathfrak{k}$-invariant forms on our factors, with $F^{(1)} \in \mathfrak{m}^{\mathfrak{k}}$, $F^{(3)}, H^{(3)} \in (\Lambda^3\mathfrak{m})^{\mathfrak{k}}$, and $F^{(5)} \in (\Lambda^5_+\mathfrak{m})^{\mathfrak{k}}$. Substituting in these ansatz field strengths to the algebraised field equations yields a system of polynomial equations in the parameters of the ansätze whose solution space is the moduli space of the

background.

6.4 Classification of Type IIB backgrounds

6.4.1 Organisation

In section 6.4.2 we make some general observations that will aid us in understanding and computing moduli spaces. In section 6.4.3 we lay out the notation we will use unless otherwise stated. We then make a start in section 6.4.4 by ruling out whole classes of backgrounds based on general arguments. In section 6.4.5 we list geometries that have been individually ruled out and explain or give reference for the arguments used. Then in sections 6.4.6 to 6.4.9 we list those geometries for which we have found solutions[3]. Finally in section 6.4.10 we summarise our results.

6.4.2 Observations

Interchangeability of invariant forms

When computing the existence and moduli space of backgrounds, the spaces in each of the pairs $(\mathrm{S}^5, \mathrm{SLAG}_3)$, $(\mathbb{C}\mathrm{P}^3, \mathrm{G}^+_{\mathbb{R}}(2,5))$, and $(\mathbb{H}\mathrm{P}^2, \mathrm{ASSOC})$ are interchangeable because they have the same invariant forms which can also be identically normalised.

The existence of a background with a S^n factor implies the existence of backgrounds with the S^n factor replaced by any other n-dimensional possibly reducible compact (non-flat) Riemannian factor. Similarly the existence of a background with an AdS_n factor implies the existence of backgrounds with the AdS_n factor replaced by $\mathrm{AdS}_p \times \mathcal{M}^{n-p}$ where $\mathcal{M}^{n-p}$ is a possibly reducible non-compact (non-flat) Riemannian factor. This is due to the fact that invariant forms for S^n and AdS_n factors are nothing but multiples of their volume forms and so can be substituted by the volume forms of the possibly reducible factors that replace them with no effect. Note that this argument clearly does not work in reverse!

Residual SO(2) symmetry

When $F^{(1)} = 0$ we have that $C^{(0)}$ is constant and then so is the axi-dilaton $\tau = C^{(0)} + \mathrm{i}\mathrm{e}^{-\phi}$. We may thus use the $\mathrm{SL}(2,\mathbb{R})$ symmetry of the field equations described in section 6.2.2 to transform the axi-dilaton to $\tau = \mathrm{i}$. The subgroup of $\mathrm{SL}(2,\mathbb{R})$ that stabilises $\tau = \mathrm{i}$ is $\mathrm{SO}(2)$ and so any background that has non-zero $H^{(3)}$ or $F^{(3)}$ and $\tau = \mathrm{i}$ will correspond to an $\mathrm{SO}(2)$ orbit of backgrounds. So, when we have a backgrounds with $F^{(1)} = 0$ and non-zero $H^{(3)}$ and $F^{(3)}$, we may use this residual $\mathrm{SO}(2)$ symmetry acting as rotations on the ($H^{(3)}$,$F^{(3)}$) plane to simplify

[3] Some of those solutions will have flatness forced on one or more of the factors and thus will not be a solution for the original geometry. An example is $\mathrm{AdS}_3 \times \mathrm{S}^4 \times \mathrm{S}^3$.

the background as a representative of the $\mathrm{SL}(2,\mathbb{R})$ orbit. A demonstrative example where we do this can be found in section 6.4.8.

Limit solutions

If we have an exactly solved moduli space for a particular background, we may see that in certain limits [36] we recover a different background, with one or more factors of the original geometry becoming flat. In this way we can see that certain backgrounds must exist even if we cannot compute their entire moduli space.

Polynomial systems

Parametrisation

We are working with geometries that are products of a number of factors. When we construct the most general invariant n-form for a particular product geometry, it is a parametrised sum of all the possible independent n-forms constructed from the invariant forms of the factors. When we substitute this most general form into the field equations it reduces to a system of polynomial equations in said parameters. We then add further constraints into the system based on geometrical considerations, such as requiring that Ricci-flat and non-Ricci-flat have vanishing and non-vanishing Ricci tensors respectively through the Einstein equation.

A given geometry will have spaces of invariant 1-forms, 3-forms, and self-dual 5-forms of dimensions $m_1 = \dim \mathfrak{m}^{\mathfrak{k}}$, $m_3 = \dim \left(\Lambda^3 \mathfrak{m}\right)^{\mathfrak{k}}$ and $m_5^+ = \dim \left(\Lambda^5_+ \mathfrak{m}\right)^{\mathfrak{k}}$, respectively. This gives us a total of $m_1 + 2m_3 + m_5^+$ parameters which are then constrained by the field equations to form the moduli space.

Reduction via symmetries

If $F^{(1)} = 0$, we may use the $\mathrm{SO}(2)$ stabiliser subgroup of the $\mathrm{SL}(2,\mathbb{R})$ symmetry to eliminate *one* of the 3-form parameters (the parameters that we later call α_i, β_j) to give a representative background of an $\mathrm{SL}(2,\mathbb{R})$ orbit of backgrounds as described in section 6.4.2. We might then also use the homothety invariance described in section 6.2.2 to eliminate one parameter if it helps us to simplify things. Solving this final system gives us the moduli space of the background.

Numerical solutions

We desire in all cases to solve analytically for the moduli space and in most cases it is possible to solve this polynomial system over the reals. However, in some cases the system becomes unwieldy and an analytical solution is no longer computationally possible [83]. In such cases, we resort to a search for numerical solutions.

Our technique is blunt: we take the sum of the squares of our normalised polynomial system $F = \sum_i f_i^2$ and then use a low discrepancy quasi-random sampling of the homothetically

compactified solution space of our system as seeds for standard numerical minimisation routines applied to F. We accept local minima as valid solutions as long as $|F| < 10^{-30}$. Note that calculations were carried out with a working precision of 10^{-60}. Checks using the application of this technique to the polynomial systems that were analytically solvable were encouraging. However, a pinch of salt is prescribed.

We applied this technique in two ways to help us with difficult polynomial systems. First, to trawl the solution space of a system to hint at whether solutions may exist and if so, to indicate the (non-)compactness of factor geometries. Second, and when solutions are suggested to exist, to present potential ansätze for finding exact solutions.

In some cases, where the moduli space of a background is too complicated to solve exactly, we may have already seen the background as a limit of the moduli space of another background as explained in section 6.4.2. In these cases, we may not even look for numerical solutions because there is nothing further to gain. In particular, by considering the balance of curvatures between factors, we know that we do not miss any non-compact factor geometries by doing this.

6.4.3 Notation

We will use the following notation throughout this section:

Symbol	Description
$\underline{A}^{(n)}$	Any invariant n-form composed of components only from the Lorentzian symmetric space factor. The rank of the form will be omitted if it is clear from the context. The form may be trivial.
$\overline{A}^{(n)}$	Any invariant n-form composed of components only from the Riemannian symmetric space factor. The rank of the form will be omitted if it is clear from the context. The form may be trivial.
ν_0	The volume form on the Lorentzian symmetric space factor
ν_i	The volume form on the ith Riemannian symmetric space factor
$\mathrm{d}\vartheta^i$	The volume form on the ith Riemannian flat factor. The index is suppressed when there is only one flat factor.
$\mathrm{d}\vartheta^{123\ldots}$	A composite Riemannian flat factor volume form, $\mathrm{d}\vartheta^1 \wedge \mathrm{d}\vartheta^2 \wedge \mathrm{d}\vartheta^3 \wedge \ldots$
$\underline{X}, \underline{Y}, \ldots$	Vector fields tangent to the Lorentzian symmetric space factor
$\overline{X}, \overline{Y}, \ldots$	Vector fields tangent to the Riemannian symmetric space factor
g_0	The metric on the Lorentzian symmetric space factor
g_i	The metric on the ith Riemannian symmetric space factor
$\overline{g}$	The metric on the Riemannian symmetric space factor

$\lambda, \alpha, \beta, \kappa$ Real parameters with (generally) λ parametrising $F^{(1)}$, α parametrising $H^{(3)}$, β parametrising $F^{(3)}$, and κ parametrising $F^{(5)}$

R_i The Ricci scalar curvature on the ith factor ($i = 0$ Lorentzian, $i > 0$ Riemannian)

6.4.4 Special Cases

de Sitter backgrounds

We analyse backgrounds of the form $\mathrm{dS}_d \times \mathcal{M}^{10-d}$ for $d \geq 2$. The most general 5-form we can construct takes the form

$$F^{(5)} = \nu_0^{(d)} \wedge \overline{\tau}_5^{(5-d)} + \overline{F}^{(5)} \ , \tag{6.14}$$

whence

$$\langle \iota_{\underline{X}} F^{(5)}, \iota_{\underline{Y}} F^{(5)} \rangle = \langle \iota_{\underline{X}} \nu_0^{(d)}, \iota_{\underline{Y}} \nu_0^{(d)} \rangle |\overline{\tau}_5^{(5-d)}|^2 = -|\overline{\tau}_5^{(5-d)}|^2 g(\underline{X}, \underline{Y}) \ . \tag{6.15}$$

The most general 3-form we can construct (K standing for $H^{(3)}$ or $F^{(3)}$) takes the form

$$K^{(3)} = \nu_0^{(d)} \wedge \overline{\tau}_K^{(3-d)} + \overline{K}^{(3)} \ , \tag{6.16}$$

whence

$$\langle \iota_{\underline{X}} K^{(3)}, \iota_{\underline{Y}} K^{(3)} \rangle = \langle \iota_{\underline{X}} \nu_0^{(d)}, \iota_{\underline{Y}} \nu_0^{(d)} \rangle |\overline{\tau}_K^{(5-d)}|^2 = -|\overline{\tau}_K^{(5-d)}|^2 g(\underline{X}, \underline{Y}) \text{ , and} \tag{6.17}$$

$$|K^{(3)}|^2 = |\nu_0^{(d)}|^2 |\overline{\tau}_K^{(3-d)}|^2 + |\overline{K}^{(3)}|^2 = -|\overline{\tau}_K^{(3-d)}|^2 + |\overline{K}^{(3)}|^2 \ , \tag{6.18}$$

so

$$4\langle \iota_{\underline{X}} K^{(3)}, \iota_{\underline{Y}} K^{(3)} \rangle - |K^{(3)}|^2 g(\underline{X}, \underline{Y}) = -\left(3|\overline{\tau}_K^{(5-d)}|^2 + |\overline{K}^{(3)}|^2\right) g(\underline{X}, \underline{Y}) \ . \tag{6.19}$$

Finally, since dS_d has no invariant 1-forms,

$$F^{(1)}(\underline{X}) F^{(1)}(\underline{Y}) = 0 \ . \tag{6.20}$$

Therefore, the Einstein equation equation (6.13) on the de Sitter factor is

$$\mathrm{Ric}(\underline{X}, \underline{Y}) = -\tfrac{1}{8}\left(3|\overline{\tau}_H^{(3-d)}|^2 + 3|\overline{\tau}_F^{(3-d)}|^2 + 2|\overline{\tau}_5^{(5-d)}|^2 + |\overline{H}^{(3)}|^2 + |\overline{F}^{(3)}|^2\right) g(\underline{X}, \underline{Y}) \ . \tag{6.21}$$

We notice that this implies a negative-semidefinite curvature contradicting the fact that dS_d has positive curvature. As such, this rules out all backgrounds with a de Sitter factor. Since we only used the Einstein equation (which was not simplified from equation (6.6) to equation (6.13)) this is true in general for homogeneous backgrounds and not only symmetric ones.

Backgrounds with a 1-dimensional Lorentzian factor

We analyse backgrounds of the form $\mathbb{R}^{0,1}\times\mathcal{M}^9$ for $d\geq 2$, i.e. with metric of the form $g=-\mathrm{d}t^2+\overline{g}$, and define T to be a vector field tangent to the $\mathbb{R}^{0,1}$ factor.

The most general 5-form we can construct takes the form

$$F^{(5)}=\nu_0\wedge\overline{\tau}^{(4)}_5+\overline{F}^{(5)}\ , \tag{6.22}$$

whence

$$\langle\iota_T F^{(5)},\iota_T F^{(5)}\rangle=-|\overline{\tau}^{(4)}_5|^2 g(T,T)=|\overline{\tau}^{(4)}_5|^2\ . \tag{6.23}$$

The most general 3-form we can construct (K standing for $H^{(3)}$ or $F^{(3)}$) takes the form

$$K^{(3)}=\nu_0\wedge\overline{\tau}^{(2)}_K+\overline{K}^{(3)}\ , \tag{6.24}$$

whence

$$\langle\iota_T K^{(3)},\iota_T K^{(3)}\rangle=-|\overline{\tau}^{(2)}_K|^2 g(T,T)=|\overline{\tau}^{(2)}_K|^2\text{ , and} \tag{6.25}$$

$$|K^{(3)}|^2=|\overline{K}^{(3)}|^2-|\overline{\tau}^{(3)}_K|^2\ , \tag{6.26}$$

so

$$4\langle\iota_T K^{(3)},\iota_T K^{(3)}\rangle-|K^{(3)}|^2 g(T,T)=3|\overline{\tau}^{(2)}_K|^2+|\overline{K}^{(3)}|^2\ . \tag{6.27}$$

For $F^{(1)}$, we have the most general form

$$F^{(1)}=\alpha\mathrm{d}t+\overline{F}^{(1)}\ , \tag{6.28}$$

whence

$$F^{(1)}(T)F^{(1)}(T)=\alpha^2 \tag{6.29}$$

Therefore, the Einstein equation equation (6.13) on the $-\mathrm{d}t^2$ factor is

$$\mathrm{Ric}(T,T)=\tfrac{1}{8}\left(4\alpha^2+3|\overline{\tau}^{(2)}_H|^2+3|\overline{\tau}^{(2)}_F|^2+2|\overline{\tau}^{(4)}_5|^2+|\overline{H}^{(3)}|^2+|\overline{F}^{(3)}|^2\right)\ . \tag{6.30}$$

This factor is flat and so the Ricci curvature on the factor must be identically zero. All the terms are positive-semidefinite being either a squared real parameter or the squared norms of purely Riemannian forms. Thus $F^{(1)}$, $H^{(3)}$, and $F^{(3)}$ are all zero and we are left with $F^{(5)}=\overline{F}^{(5)}$ since this component does not contribute to the curvature. However, $F^{(5)}$ is self-dual and a self-dual form has zero norm so

$$0\stackrel{!}{=}|F^{(5)}|^2=|\overline{F}^{(5)}|^2\ , \tag{6.31}$$

whence $F^{(5)}$ is also forced to be zero. Therefore we have $F^{(1)}=H^{(3)}=F^{(3)}=F^{(5)}=0$. This means that $\overline{g}$ is Ricci-flat and hence flat and so locally isometric to the Minkowski vacuum. Again,

since we only used the Einstein equation this is true in general for homogeneous backgrounds and not only symmetric ones.

Cahen-Wallach backgrounds

We analyse backgrounds with metric of the form $\mathrm{CW}_d(\lambda)\times\mathcal{M}^{10-d}$. The most general n-form we can construct takes the form

$$W^{(n)}=\nu_0^{(d)}\wedge\overline{W}_1^{(n-d)}+\overline{W}_2^{(n)}+\sum_i\tau_i^{(s_i)}\wedge\overline{Z}_i^{(n-s_i)}\ , \tag{6.32}$$

where the first term is the wedge product of the Cahen-Wallach volume form with an invariant form on the Riemannian factors, the second term is an invariant form on the Riemannian factors, and the third term is a sum of wedge products of Cahen-Wallach non-volume invariant forms and invariant forms on the Riemannian factors. Using the fact that all Cahen-Wallach non-volume invariant forms are null and so $|\tau_i^{(s_i)}|^2=0$, we have

$$\langle\iota_{\overline{X}}W^{(n)},\iota_{\overline{Y}}W^{(n)}\rangle=-\langle\iota_{\overline{X}}\overline{W}_1^{(n-d)},\iota_{\overline{Y}}\overline{W}_1^{(n-d)}\rangle+\langle\iota_{\overline{X}}\overline{W}_2^{(n)},\iota_{\overline{Y}}\overline{W}_2^{(n)}\rangle\ , \tag{6.33}$$

$$\langle\iota_{\underline{X}}W^{(n)},\iota_{\underline{Y}}W^{(n)}\rangle=-|\overline{W}_1^{(n-d)}|^2g(\underline{X},\underline{Y})\ \text{, and} \tag{6.34}$$

$$|W^{(n)}|^2=-|\overline{W}_1^{(n-d)}|^2+|\overline{W}_2^{(n)}|^2\ , \tag{6.35}$$

so

$$4\langle\iota_{\underline{X}}W^{(n)},\iota_{\underline{Y}}W^{(n)}\rangle-|W^{(n)}|^2g(\underline{X},\underline{Y})=-\left(3|\overline{W}_1^{(n-d)}|^2+|\overline{W}_2^{(n)}|^2\right)g(\underline{X},\underline{Y})\ . \tag{6.36}$$

Therefore, the Einstein equation equation (6.13) on the Cahen-Wallach factor is

$$\mathrm{Ric}(\underline{X},\underline{Y})=-\tfrac{1}{8}\left(3|\overline{H}_1^{(3-d)}|^2+3|\overline{F}_1^{(3-d)}|^2+2|\overline{F}_1^{(5-d)}|^2+|\overline{H}_2^{(3)}|^2+|\overline{F}_2^{(3)}|^2\right)g(\underline{X},\underline{Y})\ . \tag{6.37}$$

Cahen-Wallach spaces are Ricci-null and so the Ricci curvature on this factor must be identically zero. All the terms are negative-semidefinite and so each term is forced to zero. In particular, this means that $|H^{(3)}|^2=|F^{(3)}|^2=0$ whence equation (6.7) tells us that $|F^{(1)}|^2=0$. Since there can be no $\overline{F}_1^{(1-d)}$ component on dimensional grounds, this means that $|\overline{F}_2^{(1)}|^2=0$ and so $\overline{F}_2^{(1)}=0$. From self-duality we have $|F^{(5)}|^2=0$ and since $|\overline{F}_1^{(5-d)}|^2=0$, this means that $|\overline{F}_2^{(5-d)}|^2=0$ and so $\overline{F}_2^{(5-d)}=0$. Thus $H^{(3)}$, $F^{(1)}$, $F^{(3)}$, and $F^{(5)}$ all take the form of equation (6.32) where the first two terms are absent. But then, equation (6.33) tells us that they do not contribute to the Ricci tensor of the Riemannian factor either, which is then forced to be Ricci-flat and so flat. We are thus left with only $\mathrm{CW}_d(\lambda)\times\mathbb{R}^{10-d}$.

High-dimensional factors

9-dimensional irreducible Riemannian factors

These are the cases already dealt with in section 6.4.4.

8-dimensional irreducible Riemannian factors

- $(\mathfrak{su}(\mathbf{3}) \oplus \mathfrak{su}(\mathbf{3}), \mathfrak{su}(\mathbf{3}))$ and $(\mathfrak{sl}(\mathbf{3}, \mathbb{C}), \mathfrak{su}(\mathbf{3}))$:
  The only available complementary space is AdS_2. We have a single invariant 3-form and there are no available invariant 1-forms so the field equations (6.7) to (6.9) dictate $F^{(1)} = H^{(3)} = F^{(3)} = 0$. The most general self-dual 5-form is

$$F^{(5)} = \kappa(\nu_0 \wedge \tau^{(3)} - \tau^{(5)}) \ . \tag{6.38}$$

  With the equations of motion thus satisfied, the Einstein equation then yields

$$\mathrm{R}_0 = -\tfrac{1}{4}\kappa^2 g_0 \qquad \text{and} \qquad \mathrm{R}_1 = \tfrac{1}{16}\kappa^2 g_1 \ , \tag{6.39}$$

  which gives a solution for the compact case $\mathrm{AdS}_2 \times \mathrm{SU}(3)$.

- Other 8-dimensional Riemannian factors:
  The only available complementary space is AdS_2. As such there are no invariant 3-forms or 5-forms and so Ricci-flatness is forced. However, these spaces are not Ricci-flat and so we have a contradiction with the negative curvature of AdS_2 and thus these spaces are ruled out.

7-dimensional irreducible Riemannian factors

The only 7-dimensional Riemannian factor is S^7. We have no invariant 1-forms or 5-forms in the Riemannian part and only the invariant 3-form of the Lorentzian volume. Thus the field equations (6.7) to (6.9) dictate $H^{(3)} = F^{(3)} = F^{(1)} = 0$ which forces Ricci-flatness, again contradicting the curvature of AdS_3 and thus these spaces are ruled out.

$\mathrm{AdS}_{d>7}$ backgrounds

In this case there are no invariant 3-forms or invariant 5-forms and so the Einstein equation forces Ricci-flatness, again contradicting the curvature of $\mathrm{AdS}_{d>7}$ and thus these spaces are ruled out.

6.4.5 Individually inadmissible geometries

Here we list the geometries not already ruled out by general arguments but which we have shown to not admit any solutions. Although we list the geometries by using the compact version of the Riemannian symmetric spaces, their non-compact duals are similarly ruled out.

- $\mathrm{AdS}_7 \times \mathrm{S}^3$
- $\mathrm{AdS}_7 \times \mathrm{S}^2 \times \mathrm{T}^1$
- $\mathrm{AdS}_7 \times \mathrm{T}^3$
- $\mathrm{AdS}_6 \times \mathrm{S}^2 \times \mathrm{T}^2$
- $\mathrm{AdS}_5 \times \mathrm{S}^4 \times \mathrm{T}^1$
- $\mathrm{AdS}_5 \times \mathrm{S}^3 \times \mathrm{T}^2$
- $\mathrm{AdS}_5 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{T}^1$
- $\mathrm{AdS}_5 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^1$
- $\mathrm{AdS}_5 \times \mathrm{S}^2 \times \mathrm{T}^3$
- $\mathrm{AdS}_4 \times \mathrm{S}^5 \times \mathrm{T}^1$
- $\mathrm{AdS}_4 \times \mathrm{SLAG}_3 \times \mathrm{T}^1$
- $\mathrm{AdS}_4 \times \mathrm{S}^4 \times \mathrm{T}^2$
- $\mathrm{AdS}_4 \times \mathrm{S}^3 \times \mathrm{S}^3$
- $\mathrm{AdS}_4 \times \mathrm{S}^3 \times \mathrm{T}^3$
- $\mathrm{AdS}_4 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{T}^2$
- $\mathrm{AdS}_4 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^2$
- $\mathrm{AdS}_4 \times \mathrm{S}^2 \times \mathrm{T}^4$
- $\mathrm{AdS}_3 \times \mathrm{S}^5 \times \mathrm{T}^2$
- $\mathrm{AdS}_3 \times \mathrm{SLAG}_3 \times \mathrm{T}^2$
- $\mathrm{AdS}_3 \times \mathrm{S}^4 \times \mathrm{S}^3$
- $\mathrm{AdS}_3 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{S}^3$
- $\mathrm{AdS}_3 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^1$
- $\mathrm{AdS}_3 \times \mathrm{G}^+_{\mathbb{R}}(2,5) \times \mathrm{T}^1$
- $\mathrm{AdS}_3 \times \mathbb{C}\mathrm{P}^3 \times \mathrm{T}^1$
- $\mathrm{AdS}_3 \times \mathrm{S}^4 \times \mathrm{T}^3$
- $\mathrm{AdS}_3 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{S}^2 \times \mathrm{T}^1$
- $\mathrm{AdS}_3 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{T}^3$
- $\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^1$
- $\mathrm{AdS}_2 \times \mathrm{S}^6 \times \mathrm{T}^2$
- $\mathrm{AdS}_2 \times \mathrm{S}^5 \times \mathrm{T}^3$
- $\mathrm{AdS}_2 \times \mathrm{SLAG}_3 \times \mathrm{T}^3$

In most of these cases we rule the geometry out most easily by analysing the Einstein equation along the flat directions whence more often than not their flatness forces all parameters to zero, which then contradicts the fact that the geometry is not Ricci-flat. Three of the geometries require other arguments. The first being $\mathrm{AdS}_4 \times \mathrm{S}^3 \times \mathrm{S}^3$, where we see that with the two available 3-forms belonging to different Riemannian factors, we cannot simultaneously satisfy the field equations (6.7) to (6.8) and so all parameters are set to zero, which then contradicts the fact that the geometry is not Ricci-flat. The other two are $\mathrm{AdS}_3 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{T}^3$, and the equivalent pair $\mathrm{AdS}_2 \times \mathrm{S}^5 \times \mathrm{T}^3$ and $\mathrm{AdS}_2 \times \mathrm{SLAG}_3 \times \mathrm{T}^3$. These require subtler arguments which we omit here for brevity's sake, but can be found in [33].

6.4.6 AdS_5 backgrounds

$\mathbf{AdS_5 \times S^5}$ and $\mathbf{AdS_5 \times SLAG_3}$

The field equations admit the following solution:

$$\begin{aligned} F^{(1)} &= F^{(3)} = H^{(3)} = 0 \\ F^{(5)} &= \kappa(\nu_0 - \nu_1) \ . \end{aligned} \tag{6.40}$$

The Einstein equation yields

$$\mathrm{R}_0 = -\tfrac{1}{4}\kappa^2 g_0 \ , \qquad \text{and} \qquad \mathrm{R}_1 = \tfrac{1}{4}\kappa^2 g_1 \ , \tag{6.41}$$

giving a solution for the compact cases $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathrm{AdS}_5 \times \mathrm{SLAG}_3$.

$\mathbf{AdS_5 \times S^3 \times S^2}$

The field equations admit the following solution:

$$\begin{aligned} F^{(1)} &= F^{(3)} = H^{(3)} = 0 \\ F^{(5)} &= \kappa(\nu_0 - \nu_1 \wedge \nu_2) \ . \end{aligned} \tag{6.42}$$

The Einstein equation yields

$$\mathrm{R}_0 = -\tfrac{1}{4}\kappa^2 g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{4}\kappa^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = \tfrac{1}{4}\kappa^2 g_2 \ , \tag{6.43}$$

giving a solution for the compact case $\mathrm{AdS}_5 \times \mathrm{S}^3 \times \mathrm{S}^2$. The existence of this background follows from the existence of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background via the argument in section 6.4.2.

6.4.7 $\mathbf{AdS_4}$ backgrounds

$\mathbf{AdS_4 \times S^3 \times S^2 \times T^1}$

The field equations admit the following solution, with $\xi_i = \pm 1$:

$$\begin{aligned} F^{(1)} &= \lambda \mathrm{d}\vartheta \\ F^{(3)} &= 0 \\ H^{(3)} &= \xi_1 \sqrt{2} \lambda \nu_2 \wedge \mathrm{d}\vartheta \\ F^{(5)} &= \xi_2 \sqrt{5} \lambda (\nu_0 \wedge \mathrm{d}\vartheta + \nu_1 \wedge \nu_2) \ . \end{aligned} \tag{6.44}$$

The Einstein equation yields

$$\mathrm{R}_0 = -\tfrac{3}{2}\lambda^2 g_0 \ , \qquad \mathrm{R}_1 = \lambda^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = 2\lambda^2 g_2 \ , \tag{6.45}$$

giving a solution for $\mathrm{AdS}_4 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{T}^1$.

6.4.8 $\mathbf{AdS_3}$ backgrounds

For AdS_3 and AdS_2 the moduli spaces become increasingly difficult to compute as the number of 3-form components increases, and even when computed may be difficult to interpret.

$\mathbf{AdS_3 \times S^5 \times S^2}$ and $\mathbf{AdS_3 \times SLAG_3 \times S^2}$

The field equations admit the following solution:

$$\begin{aligned} F^{(1)} &= F^{(3)} = H^{(3)} = 0 \\ F^{(5)} &= \kappa(\nu_1 - \nu_0 \wedge \nu_2) \ . \end{aligned} \tag{6.46}$$

The Einstein equation yields:

$$\mathrm{R}_0 = -\tfrac{1}{4}\kappa^2 g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{4}\kappa^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = -\tfrac{1}{4}\kappa^2 g_2 \ , \tag{6.47}$$

giving a solution for the cases $\mathrm{AdS}_3 \times \mathrm{S}^5 \times \mathrm{H}^2$ and $\mathrm{AdS}_3 \times \mathrm{SLAG}_3 \times \mathrm{H}^2$. We could have deduced the existence of such a background from that of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background via the argument in section 6.4.2.

$\mathbf{AdS_3 \times S^4 \times S^3}$

The field equations admit the following solution with $\xi = \pm 1$:

$$\begin{aligned} F^{(1)} &= F^{(5)} = 0 \\ F^{(3)} &= \beta(\nu_0 + \xi\nu_2) \\ H^{(3)} &= \alpha(\nu_0 + \xi\nu_2) \ . \end{aligned} \tag{6.48}$$

As explained in section 6.4.2, there is a residual $\mathrm{SO}(2)$ subgroup of the $\mathrm{SL}(2,\mathbb{R})$ symmetry group which we may use to simplify the solution further. This subgroup acts by rotations on $(F^{(3)}, H^{(3)})$ or, equivalently here, on (α, β). Thus we will use this to set $\beta = 0$, whence the simplified solution as a representative of the $\mathrm{SL}(2,\mathbb{R})$ orbit is:

$$\begin{aligned} F^{(1)} &= F^{(3)} = F^{(5)} = 0 \\ H^{(3)} &= \alpha(\nu_0 \pm \nu_2) \ . \end{aligned} \tag{6.49}$$

The Einstein equation yields:

$$\mathrm{R}_0 = -\tfrac{1}{2}\alpha^2 g_0 \ , \qquad \mathrm{R}_1 = 0, \qquad \text{and} \qquad \mathrm{R}_2 = \tfrac{1}{2}\alpha^2 g_2 \ . \tag{6.50}$$

As $\mathrm{R}_1 = 0$ is forced, this geometry is ruled out and what we actually obtain is a background with underlying geometry $\mathrm{AdS}_3 \times \mathrm{T}^4 \times \mathrm{S}^3$.

$\mathbf{AdS_3 \times \mathbb{C}P^2 \times S^3}$

The field equations admit the following solution with ω the Kähler form in the Hermitian symmetric space $\mathbb{C}\mathrm{P}^2$, $\xi = \pm 1$, and where we have used the residual $\mathrm{SO}(2)$ symmetry to set $F^{(3)} = 0$:

$$\begin{aligned} F^{(1)} &= F^{(3)} = 0 \\ H^{(3)} &= \alpha(\nu_0 + \xi\nu_2) \\ F^{(5)} &= \tfrac{1}{2}(1+\xi)\kappa(\nu_0 + \xi\nu_2) \wedge \omega \ . \end{aligned} \tag{6.51}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\tfrac{1}{2}(\alpha^2+\kappa^2)g_0\ , \qquad \mathrm{R}_1 = 0, \qquad \text{and} \qquad \mathrm{R}_2 = \tfrac{1}{2}(\alpha^2+\kappa^2)g_2\ . \tag{6.52}$$

As $\mathrm{R}_1 = 0$ is forced, this geometry is ruled out and what we actually obtain is a background with underlying geometry $\mathrm{AdS}_3 \times \mathrm{T}^4 \times \mathrm{S}^3$.

$\mathbf{AdS_3 \times S^3 \times S^2 \times S^2}$

The field equations admit the following two branches of solutions with $\xi = \pm 1$:

1. $$\begin{aligned} F^{(1)} &= F^{(3)} = H^{(3)} = 0 \\ F^{(5)} &= \kappa_1(\nu_0 \wedge \nu_2 - \nu_1 \wedge \nu_3) + \kappa_2(\nu_0 \wedge \nu_3 - \nu_1 \wedge \nu_2)\ . \end{aligned} \tag{6.53}$$

   The Einstein equation then yields:

   $$\begin{aligned} \mathrm{R}_0 &= -\tfrac{1}{4}(\kappa_1^2+\kappa_2^2)g_0\ , & \mathrm{R}_1 &= \tfrac{1}{4}(\kappa_1^2+\kappa_2^2)g_1\ , \\ \mathrm{R}_2 &= -\tfrac{1}{4}(\kappa_1^2-\kappa_2^2)g_2\ , \qquad \text{and} & \mathrm{R}_3 &= \tfrac{1}{4}(\kappa_1^2-\kappa_2^2)g_3\ , \end{aligned} \tag{6.54}$$

   which gives a solution for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{H}^2$. This solution degenerates to a solution for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ when $\kappa_1^2 = \kappa_2^2$.

   The existence of the special case $\kappa_1 = 0$ of this background follows from the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background via the argument in section 6.4.2.

2. $$\begin{aligned} F^{(1)} &= F^{(3)} = 0 \\ H^{(3)} &= \alpha(\nu_0 + \xi\nu_1) \\ F^{(5)} &= \kappa(\nu_0 - \nu_1) \wedge (\nu_2 - \xi\nu_3)\ . \end{aligned} \tag{6.55}$$

   The Einstein equation then yields:

   $$\mathrm{R}_0 = -\tfrac{1}{2}(\alpha^2+\kappa^2)g_0\ , \qquad \mathrm{R}_1 = \tfrac{1}{2}(\alpha^2+\kappa^2)g_1\ , \qquad \text{and} \qquad \mathrm{R}_2 = \mathrm{R}_3 = 0\ . \tag{6.56}$$

   As $\mathrm{R}_2 = \mathrm{R}_3 = 0$ is forced, what we actually obtain is a solution for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$.

$\mathbf{AdS_3 \times S^3 \times S^3 \times T^1}$

The field equations admit the following solution:

$$\begin{aligned} F^{(1)} &= F^{(5)} = 0 \\ F^{(3)} &= \beta_1\nu_0 + \beta_2\nu_1 + \beta_3\nu_2 \\ H^{(3)} &= \alpha_1\nu_0 + \alpha_2\nu_1 + \alpha_3\nu_2\ , \end{aligned} \tag{6.57}$$

with

$$\begin{gathered}\alpha_1^2=\alpha_2^2+\alpha_3^2,\qquad \beta_1^2=\beta_2^2+\beta_3^2,\qquad \alpha_1\beta_2=\beta_1\alpha_2,\\ \alpha_1\beta_3=\beta_1\alpha_3,\qquad \text{and}\qquad \alpha_2\beta_3=\beta_2\alpha_3\ .\end{gathered}\tag{6.58}$$

The last three equations say that the vectors $(\alpha_1,\alpha_2,\alpha_3)$ and $(\beta_1,\beta_2,\beta_3)$ are collinear, so that $F^{(3)}$ and $H^{(3)}$ point in the same direction. In this case we can then use the residual $\mathrm{SO}(2)$ symmetry to set the $\beta_i=0$, whence we arrive at the simplified solution:

$$\begin{aligned}F^{(1)}&=F^{(3)}=F^{(5)}=0\\ H^{(3)}&=\alpha_1\nu_0+\alpha_2\nu_1+\alpha_3\nu_2\ ,\end{aligned}\tag{6.59}$$

with

$$\alpha_1^2=\alpha_2^2+\alpha_3^2\ .\tag{6.60}$$

The Einstein equation then yields:

$$\mathrm{R}_0=-\tfrac{1}{2}\alpha_1^2 g_0\ ,\qquad \mathrm{R}_1=\tfrac{1}{2}\alpha_2^2 g_1\ ,\qquad \text{and}\qquad \mathrm{R}_2=\tfrac{1}{2}\alpha_3^2 g_2\ ,\tag{6.61}$$

giving a solution for $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{T}^1$. This solution degenerates to one for $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ whenever $\alpha_2=0$ or $\alpha_3=0$.

$\mathbf{AdS_3\times S^3\times S^2\times T^2}$

The field equations admit the following two branches of solutions with $\xi_i=\pm1$:

1. $$\begin{aligned}F^{(1)}&=0\\ F^{(3)}&=\xi_1\sqrt{\kappa_2^2-\kappa_1^2}\,\nu_2\wedge\mathrm{d}\vartheta^2\\ H^{(3)}&=\xi_2\sqrt{\kappa_2^2-\kappa_1^2}\,\nu_2\wedge\mathrm{d}\vartheta^1\\ F^{(5)}&=(\kappa_1\nu_0+\kappa_2\nu_1)\wedge\nu_2-(\kappa_2\nu_0+\kappa_1\nu_1)\wedge\mathrm{d}\vartheta^1\wedge\mathrm{d}\vartheta^2\ .\end{aligned}\tag{6.62}$$

   The Einstein equation then yields:

   $$\mathrm{R}_0=-\tfrac{1}{2}\kappa_2^2 g_0\ ,\qquad \mathrm{R}_1=\tfrac{1}{2}\kappa_1^2 g_1\ ,\qquad \text{and}\qquad \mathrm{R}_2=(\kappa_2^2-\kappa_1^2)g_2\ ,\tag{6.63}$$

   giving a solution for the compact case $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{S}^2\times\mathrm{T}^2$. This solution degenerates to one for $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ if $\kappa_1^2=\kappa_2^2$, and to one for $\mathrm{AdS}_3\times\mathrm{S}^2\times\mathrm{T}^5$ if $\kappa_1^2=0$.

2. Using the residual $\mathrm{SO}(2)$ symmetry, we can write a second solution as:

   $$\begin{aligned}F^{(1)}&=F^{(3)}=0\\ H^{(3)}&=\alpha(\nu_0+\xi_1\nu_1)\\ F^{(5)}&=\kappa(\nu_0+\xi_1\nu_1)\wedge(\nu_2-\xi_1\mathrm{d}\vartheta^{12})\ .\end{aligned}\tag{6.64}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\tfrac{1}{2}(\alpha^2+\kappa^2)g_0\ , \qquad \mathrm{R}_1 = \tfrac{1}{2}(\alpha^2+\kappa^2)g_1\ , \qquad \text{and} \qquad \mathrm{R}_2 = 0\ . \tag{6.65}$$

As $\mathrm{R}_2 = 0$ is forced, what we actually obtain is a solution for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$.

$\mathbf{AdS_3 \times S^3 \times T^4}$

The field equations admit the following solution, where we have used the residual $\mathrm{SO}(2)$ symmetry transformation to set $F^{(3)} = 0$ and $\xi = \pm 1$:

$$\begin{aligned} F^{(1)} &= F^{(3)} = 0 \\ H^{(3)} &= \alpha(\nu_0 + \xi\nu_1) \\ F^{(5)} &= \kappa(\nu_0 + \xi\nu_1) \wedge (\mathrm{d}\vartheta^{12} - \xi\mathrm{d}\vartheta^{34})\ . \end{aligned} \tag{6.66}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\tfrac{1}{2}(\alpha^2+\kappa^2)g_0 \qquad \text{and} \qquad \mathrm{R}_1 = \frac{1}{2}(\alpha^2+\kappa^2)g_1\ , \tag{6.67}$$

which is a solution for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$.

$\mathbf{AdS_3 \times S^2 \times S^2 \times T^3}$

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (3, 8, 5)$ and so the most general forms here have a total of 24 parameters. The resultant system of polynomials does not lend itself to symbolic solution, although we can exhibit an exact solution of the following form where $\xi = \pm 1$:

$$\begin{aligned} F^{(1)} &= 0 \\ H^{(3)} &= \nu_1 \wedge (\alpha_1\mathrm{d}\vartheta^2 + \alpha_2\mathrm{d}\vartheta^3) + \nu_2 \wedge (\alpha_3\mathrm{d}\vartheta^2 + \alpha_4\mathrm{d}\vartheta^3) \\ F^{(3)} &= \xi\left(\nu_1 \wedge (\alpha_2\mathrm{d}\vartheta^2 - \alpha_1\mathrm{d}\vartheta^3) - \nu_2 \wedge (\alpha_4\mathrm{d}\vartheta^2 - \alpha_3\mathrm{d}\vartheta^3)\right) \\ F^{(5)} &= \sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2}(\nu_0 \wedge \mathrm{d}\vartheta^{23} - \nu_1 \wedge \nu_2 \wedge \mathrm{d}\vartheta^1)\ . \end{aligned} \tag{6.68}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\tfrac{1}{2}(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2)g_0\ , \qquad \mathrm{R}_1 = (\alpha_1^2+\alpha_2^2)g_1\ , \qquad \text{and} \qquad \mathrm{R}_2 = (\alpha_3^2+\alpha_4^2)g_2\ , \tag{6.69}$$

giving a solution for $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^3$.

$\mathbf{AdS_3 \times S^2 \times T^5}$

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (5, 16, 11)$ and so the most general forms here have a total of 48 parameters. The resultant system of polynomials does not lend

itself to symbolic solution. However we have seen that such backgrounds exist as limits of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{T}^2$.

6.4.9 $\mathbf{AdS_2}$ backgrounds

For AdS_2 backgrounds the number of components increases yet again and the larger number of parameters means that we will tend to have only partial results. This means that in many cases we have been unable to determine the moduli space fully but we have found and exhibited some exact solutions where possible to demonstrate existence.

$\mathbf{AdS_2 \times S^5 \times S^3}$ and $\mathbf{AdS_2 \times SLAG_3 \times S^3}$

The field equations admit the following solution:

$$\begin{aligned} F^{(1)} &= F^{(3)} = H^{(3)} = 0 \\ F^{(5)} &= \kappa(\nu_1 + \nu_0 \wedge \nu_2) \ . \end{aligned} \tag{6.70}$$

The Einstein equation yields:

$$\mathrm{R}_0 = -\tfrac{1}{4}\kappa^2 g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{4}\kappa^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = -\tfrac{1}{4}\kappa^2 g_2 \ , \tag{6.71}$$

which yields a solution for $\mathrm{AdS}_2 \times \mathrm{S}^5 \times \mathrm{H}^3$ and $\mathrm{AdS}_2 \times \mathrm{SLAG}_3 \times \mathrm{H}^3$. The existence of this background follows from that of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathrm{AdS}_4 \times \mathrm{SLAG}_3$ backgrounds via the argument in section 6.4.2.

$\mathbf{AdS_2 \times S^4 \times S^3 \times T^1}$

The field equations admit the following two branches of solutions with $\xi_i = \pm 1$:

1. $$\begin{aligned} F^{(1)} &= \lambda \mathrm{d}\vartheta \\ F^{(3)} &= F^{(5)} = 0 \\ H^{(3)} &= \frac{\xi_1}{\sqrt{2}} \lambda \left(\nu_0 \wedge \mathrm{d}\vartheta + \xi_2 \sqrt{5} \nu_2 \right) \ . \end{aligned} \tag{6.72}$$

   The Einstein equation yields:

   $$\mathrm{R}_0 = -\tfrac{1}{2}\lambda^2 g_0 \ , \qquad \mathrm{R}_1 = -\tfrac{1}{4}\lambda^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = \lambda^2 g_2 \ , \tag{6.73}$$

   giving a solution for $\mathrm{AdS}_2 \times \mathrm{H}^4 \times \mathrm{S}^3 \times \mathrm{T}^1$.

2.

$$\begin{aligned} F^{(1)} &= \alpha \mathrm{d}\vartheta \\ F^{(3)} &= \xi_1 \sqrt{2} \alpha \nu_0 \wedge \mathrm{d}\vartheta \\ H^{(3)} &= 0 \\ F^{(5)} &= \xi_2 \alpha (\nu_0 \wedge \nu_2 - \nu_1 \wedge \mathrm{d}\vartheta) \, . \end{aligned} \tag{6.74}$$

The Einstein equation yields:

$$\mathrm{R}_0 = -\alpha^2 g_0 \, , \qquad \mathrm{R}_1 = \tfrac{1}{2}\alpha^2 g_1 \, , \qquad \text{and} \qquad \mathrm{R}_2 = 0 \, . \tag{6.75}$$

As $\mathrm{R}_2 = 0$ is forced, what we actually obtain is a solution for $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{T}^4$.

$\mathbf{AdS_2 \times S^5 \times S^2 \times T^1}$ and $\mathbf{AdS_2 \times SLAG_3 \times S^2 \times T^1}$

The field equations admit the following solution with $\xi_i = \pm 1$:

$$\begin{aligned} F^{(1)} &= \sqrt{3} \lambda \mathrm{d}\vartheta \\ F^{(3)} &= \sqrt{5} \left(\xi_1 \sqrt{\alpha^2 + \lambda^2} \nu_0 \wedge \mathrm{d}\vartheta + \xi_2 \alpha \nu_2 \wedge \mathrm{d}\vartheta \right) \\ H^{(3)} &= \alpha \nu_0 \wedge \mathrm{d}\vartheta + \xi_1 \xi_2 \sqrt{\alpha^2 + \lambda^2} \nu_2 \wedge \mathrm{d}\vartheta \\ F^{(5)} &= 0 \, . \end{aligned} \tag{6.76}$$

The Einstein equation yields:

$$\begin{aligned} \mathrm{R}_0 &= -(3\alpha^2 + 2\lambda^2) g_0 \, , & \mathrm{R}_1 &= \tfrac{1}{2}\lambda^2 g_1 \, , \\ \mathrm{R}_2 &= (3\alpha^2 + \lambda^2) g_2 \, , \qquad \text{and} & \mathrm{R}_3 &= 0 \, , \end{aligned} \tag{6.77}$$

giving a solution for $\mathrm{AdS}_2 \times \mathrm{S}^5 \times \mathrm{S}^2 \times \mathrm{T}^1$ and $\mathrm{AdS}_2 \times \mathrm{SLAG}_3 \times \mathrm{S}^2 \times \mathrm{T}^1$. This solution degenerates to one for $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{T}^6$ when $\lambda = 0$.

$\mathbf{AdS_2 \times S^3 \times S^3 \times S^2}$

The field equations admit the following solution:

$$\begin{aligned} F^{(1)} &= F^{(3)} = H^{(3)} = 0 \\ F^{(5)} &= \nu_0 \wedge (\kappa_1 \nu_1 + \kappa_2 \nu_2) + (\kappa_2 \nu_1 - \kappa_1 \nu_2) \wedge \nu_3 \, . \end{aligned} \tag{6.78}$$

The Einstein equation yields:

$$\begin{aligned} \mathrm{R}_0 &= -\tfrac{1}{4}(\kappa_1^2 + \kappa_2^2) g_0 \, , & \mathrm{R}_1 &= -\tfrac{1}{4}(\kappa_1^2 - \kappa_2^2) g_1 \, , \\ \mathrm{R}_2 &= \tfrac{1}{4}(\kappa_1^2 - \kappa_2^2) g_2 \, , \qquad \text{and} & \mathrm{R}_3 &= \tfrac{1}{4}(\kappa_1^2 + \kappa_2^2) g_3 \, , \end{aligned} \tag{6.79}$$

giving a solution for $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{H}^3 \times \mathrm{S}^2$. This solution degenerates to $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{T}^6$ whenever $\kappa_1^2 = \kappa_2^2$.

$\mathbf{AdS_2 \times \mathbb{C}P^2 \times S^3 \times T^1}$

The field equations admit the following five branches of solutions, with ω the Kähler form in the Hermitian symmetric space $\mathbb{C}\mathrm{P}^2$ and $\xi_i = \pm 1$:

1.
$$\begin{aligned} F^{(1)} &= F^{(3)} = F^{(5)} = 0 \\ H^{(3)} &= \alpha(\xi_1\sqrt{2}\nu_0 + \omega) \wedge \mathrm{d}\vartheta \ , \end{aligned} \tag{6.80}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\alpha^2 g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{2}\alpha^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = 0 \ . \tag{6.81}$$

Since $\mathrm{R}_2 = 0$, what we actually obtain is a solution for $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{T}^4$.

2.
$$\begin{aligned} F^{(1)} &= \xi_1\sqrt{6(\alpha^2 - \beta^2)}\mathrm{d}\vartheta \\ F^{(3)} &= \xi_2\sqrt{10}\alpha\nu_0 \wedge \mathrm{d}\vartheta + \frac{\beta}{\sqrt{5}}\omega \wedge \mathrm{d}\vartheta \\ H^{(3)} &= \xi_2\sqrt{2}\beta\nu_0 \wedge \mathrm{d}\vartheta + \alpha\omega \wedge \mathrm{d}\vartheta \\ F^{(5)} &= 0 \ . \end{aligned} \tag{6.82}$$

The Einstein equation yields:

$$\mathrm{R}_0 = -2(2\alpha^2 + \beta^2)g_0 \ , \qquad \mathrm{R}_1 = \tfrac{3}{2}(\alpha^2 + \beta^2)g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = (\alpha^2 - \beta^2)g_2 \ , \tag{6.83}$$

giving a solution for $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{S}^3 \times \mathrm{T}^1$.

3.
$$\begin{aligned} F^{(1)} &= F^{(3)} = H^{(3)} = 0 \\ F^{(5)} &= \kappa(\omega + \xi\sqrt{2}\nu_0) \wedge (\omega - \xi\sqrt{2}\nu_2) \ . \end{aligned} \tag{6.84}$$

The Einstein equation yields:

$$\mathrm{R}_0 = -2\kappa^2 g_0 \ , \qquad \mathrm{R}_1 = \kappa^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = 0 \ . \tag{6.85}$$

Since $\mathrm{R}_2 = 0$, what we actually obtain is a solution for $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{T}^4$.

4.
$$\begin{aligned} F^{(1)} &= \xi_1\sqrt{2}\alpha_1\mathrm{d}\vartheta \\ F^{(3)} &= F^{(5)} = 0 \\ H^{(3)} &= \frac{1}{\sqrt{5}}\alpha_1\nu_2 + (\xi_2\sqrt{\alpha_1^2 + 2\alpha_2^2}\nu_0 + \alpha_2\omega) \wedge \mathrm{d}\vartheta \ . \end{aligned} \tag{6.86}$$

The Einstein equation yields:

$$\mathrm{R}_0 = -(\alpha_1^2 + \alpha_2^2)g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{2}(\alpha_2^2 - \alpha_1^2)g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = 2\alpha_1^2 g_2 \ , \tag{6.87}$$

giving a solution for $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{S}^3 \times \mathrm{T}^1$ and $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{H}^2 \times \mathrm{S}^3 \times \mathrm{T}^1$, depending on the sign of $\alpha_2^2 - \alpha_1^2$. When $\alpha_1^2 = \alpha_2^2$ we obtain a solution for $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{T}^5$.

5.
$$\begin{aligned} F^{(1)} &= \xi_1 \beta \mathrm{d}\vartheta \\ F^{(3)} &= \frac{1}{\sqrt{2}} \beta \nu_0 \wedge \mathrm{d}\vartheta \\ H^{(3)} &= 0 \\ F^{(5)} &= \xi_2 \beta (\nu_1 \wedge \mathrm{d}\vartheta - \nu_0 \wedge \nu_2) \ . \end{aligned} \tag{6.88}$$

The Einstein equation yields:

$$\mathrm{R}_0 = -\beta^2 g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{2}\beta^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = 0 \ . \tag{6.89}$$

Since $\mathrm{R}_2 = 0$, what we actually obtain is a solution for $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{T}^4$.

$\mathbf{AdS_2 \times S^3 \times S^3 \times T^2}$

The field equations admit the following solution with $\xi_i = \pm 1$:

$$\begin{aligned} F^{(1)} &= 0 \\ F^{(3)} &= \xi_1 \sqrt{\kappa_1^2 + \kappa_2^2}\, \nu_0 \wedge \mathrm{d}\vartheta^1 \\ H^{(3)} &= \xi_2 \sqrt{\kappa_1^2 + \kappa_2^2}\, \nu_0 \wedge \mathrm{d}\vartheta^2 \\ F^{(5)} &= \nu_0 \wedge (\kappa_2 \nu_1 + \kappa_1 \nu_2) + (\kappa_1 \nu_1 - \kappa_2 \nu_2) \wedge \mathrm{d}\vartheta^{12} \ . \end{aligned} \tag{6.90}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -(\kappa_1^2 + \kappa_2^2) g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{2}\kappa_1^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = \tfrac{1}{2}\kappa_2^2 g_2 \ , \tag{6.91}$$

which gives a solution for $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{T}^2$. This solution degenerates to one for $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{T}^5$ whenever $\kappa_1 = 0$ or $\kappa_2 = 0$.

$\mathbf{AdS_2 \times} G^+_{\mathbb{R}}\mathbf{(2,5) \times T^2}$ **and** $\mathbf{AdS_2 \times \mathbb{C}P^3 \times T^2}$

The field equations admit the following six solutions with ω the Kähler form in the relevant Hermitian symmetric space $G^+_{\mathbb{R}}(2,5)$ or $\mathbb{C}\mathrm{P}^3$, and $\xi_i = \pm 1$:

1.
$$\begin{aligned} F^{(1)} &= F^{(3)} = F^{(5)} = 0 \\ H^{(3)} &= \alpha(\xi_1 \sqrt{3} \nu_0 + \omega) \wedge \mathrm{d}\vartheta^1 \ . \end{aligned} \tag{6.92}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\tfrac{1}{2}\alpha^2 g_0 \qquad \text{and} \qquad \mathrm{R}_1 = \tfrac{1}{6}\alpha^2 g_1 \ . \tag{6.93}$$

2.

$$\begin{aligned} F^{(1)} &= \lambda \mathrm{d}\vartheta^1 \\ H^{(3)} &= \frac{\lambda}{\sqrt{2}}(\xi_2\sqrt{3}\nu_0 \wedge \mathrm{d}\vartheta^1 + \xi_3 \omega \wedge \mathrm{d}\vartheta^2) \\ F^{(3)} &= \xi_1\sqrt{2}\lambda\nu_0 \wedge \mathrm{d}\vartheta^2 \\ F^{(5)} &= 0 \ , \end{aligned} \tag{6.94}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\tfrac{3}{2}\lambda^2 g_0 \qquad \text{and} \qquad \mathrm{R}_1 = \tfrac{1}{2}\lambda^2 g_1 \ . \tag{6.95}$$

3.

$$\begin{aligned} F^{(1)} &= \lambda \mathrm{d}\vartheta^1 \\ H^{(3)} &= \frac{\lambda}{\sqrt{2}}(\xi_2\nu_0 \wedge \mathrm{d}\vartheta^2 + \xi_3 \frac{1}{\sqrt{3}}\omega \wedge \mathrm{d}\vartheta^1) \\ F^{(3)} &= \xi_1\sqrt{2}\lambda\nu_0 \wedge \mathrm{d}\vartheta^1 \\ F^{(5)} &= 0 \ , \end{aligned} \tag{6.96}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\lambda^2 g_0 \qquad \text{and} \qquad \mathrm{R}_1 = \tfrac{1}{3}\lambda^2 g_1 \ . \tag{6.97}$$

4.

$$\begin{aligned} F^{(1)} &= \lambda \mathrm{d}\vartheta^1 \\ H^{(3)} &= 0 \\ F^{(3)} &= \xi_1\sqrt{2}\lambda\nu_0 \wedge \mathrm{d}\vartheta^2 \\ F^{(5)} &= \xi_2\lambda(\nu_0 \wedge \omega \wedge \mathrm{d}\vartheta^1 - \nu_1 \wedge \mathrm{d}\vartheta^2) \ , \end{aligned} \tag{6.98}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\tfrac{3}{2}\lambda^2 g_0 \qquad \text{and} \qquad \mathrm{R}_1 = \tfrac{1}{2}\lambda^2 g_1 \ . \tag{6.99}$$

5.

$$\begin{aligned} F^{(1)} &= \lambda \mathrm{d}\vartheta^1 \\ H^{(3)} &= 0 \\ F^{(3)} &= \xi_1\sqrt{2}\lambda\nu_0 \wedge \mathrm{d}\vartheta^1 \\ F^{(5)} &= \xi_2\frac{\lambda}{\sqrt{3}}(\nu_0 \wedge \omega \wedge \mathrm{d}\vartheta^2 + \nu_1 \wedge \mathrm{d}\vartheta^1) \ , \end{aligned} \tag{6.100}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\lambda^2 g_0 \qquad \text{and} \qquad \mathrm{R}_1 = \tfrac{1}{3}\lambda^2 g_1 \ . \tag{6.101}$$

All six are solutions for $\mathrm{AdS}_2 \times G^+_{\mathbb{R}}(2,5) \times \mathrm{T}^2$ and $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^3 \times \mathrm{T}^2$.

$\mathbf{AdS_2 \times S^4 \times S^2 \times T^2}$

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (2,4,2)$ and so the most general forms here have a total of 12 parameters. The field equations admit the following ? solutions with $\xi_i = \pm 1$:

1.
$$\begin{aligned} F^{(1)} &= \alpha \mathrm{d}\vartheta^2 \\ F^{(3)} &= \left(\beta \nu_2 + \xi_1 \sqrt{2\alpha^2 + \beta^2}\nu_0\right) \wedge \mathrm{d}\vartheta^2 \\ H^{(3)} &= 0 \\ F^{(5)} &= \xi_2 \alpha \left(\nu_0 \wedge \nu_2 \wedge \mathrm{d}\vartheta^1 - \nu_1 \wedge \mathrm{d}\vartheta^2\right) \ . \end{aligned} \tag{6.102}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\tfrac{1}{2}(2\alpha^2 + \beta^2) g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{2}\alpha^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = \tfrac{1}{2}\beta^2 g_2 \ , \tag{6.103}$$

which gives a solution for $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^2$ in the generic case, or $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{T}^6$ if $\alpha = 0$ and $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{T}^4$ if $\beta = 0$. Using the argument in section 6.4.2 we obtain solutions for $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^2$ and $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^4$.

2.
$$\begin{aligned} F^{(1)} &= \alpha \mathrm{d}\vartheta^1 \\ F^{(3)} &= \left(\beta \nu_2 + \xi_1 \sqrt{2\alpha^2 + \beta^2}\nu_0\right) \wedge \mathrm{d}\vartheta^2 \\ H^{(3)} &= 0 \\ F^{(5)} &= \xi_2 \sqrt{3}\alpha \left(\nu_0 \wedge \nu_2 \wedge \mathrm{d}\vartheta^1 - \nu_1 \wedge \mathrm{d}\vartheta^2\right) \ . \end{aligned} \tag{6.104}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\tfrac{1}{2}(3\alpha^2 + \beta^2) g_0 \ , \qquad \mathrm{R}_1 = \alpha^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = \tfrac{1}{2}(\beta^2 - \alpha^2) g_2 \ , \tag{6.105}$$

which gives a solution for $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^2$ for $\beta^2 > \alpha^2$, $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{T}^4$ for $\beta^2 = \alpha^2$ and $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{H}^2 \times \mathrm{T}^2$ for $\beta^2 < \alpha^2$. Again this also gives solutions for $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^2$, $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^4$, and $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{H}^2 \times \mathrm{T}^2$.

3.
$$\begin{aligned} F^{(1)} &= \alpha \mathrm{d}\vartheta^1 \\ F^{(3)} &= \xi_3 \sqrt{\beta^2 + 2\alpha^2}\nu_0 \wedge \mathrm{d}\vartheta^2 + \beta \nu_2 \wedge \mathrm{d}\vartheta^1 \\ H^{(3)} &= \sqrt{\beta^2 + \tfrac{3}{2}\alpha^2}(\xi_1 \nu_0 \wedge \mathrm{d}\vartheta^1 + \xi_2 \nu_2 \wedge \mathrm{d}\vartheta^2) \\ F^{(5)} &= 0 \ . \end{aligned} \tag{6.106}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -(\tfrac{3}{2}\alpha^2 + \beta^2)g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{4}\alpha^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = (\alpha^2 + \beta^2)g_2 \ , \tag{6.107}$$

which gives a solution for $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^2$.

4.

$$\begin{aligned} F^{(1)} &= \alpha \mathrm{d}\vartheta^1 \\ F^{(3)} &= \xi_3 \sqrt{\beta^2 + 2\alpha^2}\nu_0 \wedge \mathrm{d}\vartheta^1 + \beta\nu_2 \wedge \mathrm{d}\vartheta^2 \\ H^{(3)} &= \sqrt{\beta^2 + \tfrac{1}{2}\alpha^2}(\xi_1\nu_0 \wedge \mathrm{d}\vartheta^2 + \xi_2\nu_2 \wedge \mathrm{d}\vartheta^1) \\ F^{(5)} &= 0 \ . \end{aligned} \tag{6.108}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -(\alpha^2 + \beta^2)g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{4}\alpha^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = (\tfrac{1}{2}\alpha^2 + \beta^2)g_2 \ , \tag{6.109}$$

which gives a solution for $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^2$.

5.

$$\begin{aligned} F^{(1)} &= \alpha \mathrm{d}\vartheta^1 \\ F^{(3)} &= \beta(\nu_2 \wedge \mathrm{d}\vartheta^1 - \xi_2\xi_3\nu_0 \wedge \mathrm{d}\vartheta^2) + \sqrt{2}\xi_1\xi_3\alpha\nu_0 \wedge \mathrm{d}\vartheta^1 \\ H^{(3)} &= \xi_1\beta(\nu_0 \wedge \mathrm{d}\vartheta^1 - \xi_2\xi_3\nu_2 \wedge \mathrm{d}\vartheta^2) + \tfrac{1}{\sqrt{2}}\alpha(\xi_3\nu_2 \wedge \mathrm{d}\vartheta^1 + \xi_2\nu_0 \wedge \mathrm{d}\vartheta^2) \\ F^{(5)} &= 0 \ . \end{aligned} \tag{6.110}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -(\alpha^2 + \beta^2)g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{4}\alpha^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = (\tfrac{1}{2}\alpha^2 + \beta^2)g_2 \ , \tag{6.111}$$

which gives a solution for $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^2$.

6.

$$\begin{aligned} F^{(1)} &= \alpha \mathrm{d}\vartheta^1 \\ F^{(3)} &= \beta(\nu_2 \wedge \mathrm{d}\vartheta^2 - 3\xi_2\xi_3\nu_0 \wedge \mathrm{d}\vartheta^1) + \xi_1\xi_3\sqrt{2(\alpha^2 - 4\beta^2)}\nu_0 \wedge \mathrm{d}\vartheta^2 \\ H^{(3)} &= \xi_1\sqrt{\tfrac{3}{2}(\alpha^2 - 4\beta^2)}(\nu_0 \wedge \mathrm{d}\vartheta^1 - \xi_2\xi_3\nu_2 \wedge \mathrm{d}\vartheta^2) + \sqrt{3}\beta(\xi_2\nu_0 \wedge \mathrm{d}\vartheta^2 + \xi_3\nu_2 \wedge \mathrm{d}\vartheta^1) \\ F^{(5)} &= 0 \ . \end{aligned} \tag{6.112}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -(\tfrac{3}{2}\alpha^2 - \beta^2)g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{4}\alpha^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = (\alpha^2 - \beta^2)g_2 \ , \tag{6.113}$$

which gives a solution for $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^2$.

We have omitted certain branches of the solutions where one of the spheres is forced to be flat. There is an additional branch which does not seem to be explicitly parametrisable, in the sense that the equations are not solvable in terms of radicals.

$\mathbf{AdS_2 \times S^3 \times S^2 \times S^2 \times T^1}$

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (1, 4, 3)$ and so the most general forms here have a total of 12 parameters. The resultant system of polynomials does not lend itself to symbolic solution but an exact background with geometry $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{H}^2 \times \mathrm{T}^1$ can be written down as a limit of the background found for $\mathrm{AdS}_4 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{T}^1$. In addition, we can exhibit an exact solution of the following form with $\xi_i = \pm 1$:

$$\begin{aligned} F^{(1)} &= \sqrt{2}\lambda \mathrm{d}\vartheta \\ H^{(3)} &= \xi_1 \sqrt{5}\lambda \nu_1 + \left(\xi_2 \sqrt{\alpha^2 + \beta^2 + \lambda^2}\nu_0 + \alpha\nu_2 + \beta\nu_3\right) \wedge \mathrm{d}\vartheta \\ F^{(3)} &= F^{(5)} = 0 \ . \end{aligned} \tag{6.114}$$

The Einstein equation then yields:

$$\begin{aligned} \mathrm{R}_0 &= -\tfrac{1}{2}(2\lambda^2 + \alpha^2 + \beta^2) g_0 \ , & \mathrm{R}_1 &= 2\lambda^2 g_1 \ , \\ \mathrm{R}_2 &= \tfrac{1}{2}(\alpha^2 - \lambda^2) g_2 \ , \qquad \text{and} & \mathrm{R}_3 &= \tfrac{1}{2}(\beta^2 - \lambda^2) g_3 \ , \end{aligned} \tag{6.115}$$

giving solutions for $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^1$, $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{H}^2 \times \mathrm{S}^2 \times \mathrm{T}^1$, and $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{H}^2 \times \mathrm{H}^2 \times \mathrm{T}^1$. This solution degenerates to one for $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^4$ whenever $\lambda = 0$, $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{T}^6$ whenever $\lambda = \alpha = 0$ or $\lambda = \beta = 0$, and $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{H}^2 \times \mathrm{T}^2$ whenever $\alpha^2 = \lambda^2$ with $\beta^2 < \lambda^2$ or $\beta^2 = \lambda^2$ with $\alpha^2 < \lambda^2$.

$\mathbf{AdS_2 \times \mathbb{C}P^2 \times S^2 \times T^2}$

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (2, 6, 5)$ and so the most general forms here have a total of 19 parameters. The resultant system of polynomials does not lend itself to symbolic solution. Numerical optimization suggests that solutions exist for both $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{S}^2 \times \mathrm{T}^2$ and $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{H}^2 \times \mathrm{T}^2$. We can exhibit an exact solution of the following form:

$$\begin{aligned} F^{(1)} &= \lambda(\mathrm{d}\vartheta^1 + \mathrm{d}\vartheta^2) \\ H^{(3)} &= \tfrac{\sqrt{7}}{2}\lambda(\nu_0 \wedge \mathrm{d}\vartheta^1 + \nu_2 \wedge \mathrm{d}\vartheta^2) \\ F^{(3)} &= \tfrac{1}{2}\lambda(\nu_0 \wedge (\mathrm{d}\vartheta^1 + 4\mathrm{d}\vartheta^2) + \nu_2 \wedge \mathrm{d}\vartheta^2) \\ F^{(5)} &= 0 \ . \end{aligned} \tag{6.116}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\tfrac{5}{2}\lambda^2 g_0 \ , \qquad \mathrm{R}_1 = \tfrac{1}{2}\lambda^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = \tfrac{3}{2}\lambda^2 g_2 \ , \tag{6.117}$$

giving a solution for $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{S}^2 \times \mathrm{T}^2$. The solution does not depend on any of the invariant forms of $\mathbb{C}\mathrm{P}^2$, whence it also gives a solution for $\mathrm{AdS}_2 \times X^4 \times \mathrm{S}^2 \times \mathrm{T}^2$, where X is any compact (since the curvature must be positive) four-dimensional Riemannian symmetric space: S^4, $\mathbb{C}\mathrm{P}^2$ or $\mathrm{S}^2 \times \mathrm{S}^2$. In particular, this solution belongs to the branch of $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^2$ with $F^{(5)} = 0$ and $F^{(3)} \neq 0$ whose general solution cannot be expressed in terms of radicals.

We can also exhibit an exact solution of the following form with $\xi_i = \pm 1$:

$$\begin{aligned} F^{(1)} &= H^{(3)} = 0 \\ F^{(3)} &= \xi_1 \nu_0 \wedge (\kappa_1 \mathrm{d}\vartheta^1 - \kappa_2 \mathrm{d}\vartheta^2) + \tfrac{1}{\sqrt{2}} \xi_2 \nu_1 \wedge (\kappa_2 \mathrm{d}\vartheta^1 + \kappa_1 \mathrm{d}\vartheta^2) \\ F^{(5)} &= \sqrt{2}(\nu_1 \wedge (\kappa_1 \mathrm{d}\vartheta^1 - \kappa_2 \mathrm{d}\vartheta^2) + \nu_0 \wedge \nu_2 \wedge (\kappa_2 \mathrm{d}\vartheta^1 + \kappa_1 \mathrm{d}\vartheta^2)) \ . \end{aligned} \tag{6.118}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -(\kappa_1^2 + \kappa_2^2) g_0 \ , \qquad \mathrm{R}_1 = \tfrac{3}{4}(\kappa_1^2 + \kappa_2^2) g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = -\tfrac{1}{2}(\kappa_1^2 + \kappa_2^2) g_2 \ , \tag{6.119}$$

giving a solution for $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{H}^2 \times \mathrm{T}^2$. Although we do use the volume form of $\mathbb{C}\mathrm{P}^2$ in this case, the previous argument again applies via the argument in section 6.4.2.

$\mathbf{AdS_2 \times S^4 \times T^4}$

We have already found such backgrounds when studying the geometries $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^3 \times \mathrm{T}^1$ and $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^2$ but, now surprisingly, we can in fact solve the moduli space exactly and we find an additional branch.

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (4, 8, 4)$ and so the most general forms here have a total of 24 parameters. Just for this geometry we change notation and let $\alpha, \beta, \beta', \gamma, \gamma', \delta$ denote invariant 1-forms on T^4, so each has four parameters ($\alpha_{1,2,3,4}$ for example). Also, let X, Y denote vector fields tangent to T^4. The general forms are then given by:

$$\begin{aligned} F^{(1)} &= \alpha \\ F^{(3)} &= \nu_0 \wedge \beta + \star\gamma \\ H^{(3)} &= \nu_0 \wedge \beta' + \star\gamma' \\ F^{(5)} &= \nu_0 \wedge \star\delta + \nu_1 \wedge \delta \ . \end{aligned} \tag{6.120}$$

The field equations (6.7) to (6.12) become

$$\begin{aligned} -|\beta'|^2 + |\gamma'|^2 &= -|\beta|^2 + |\gamma|^2 + 2|\alpha|^2 \\ 0 &= -\langle\beta,\beta'\rangle + \langle\gamma,\gamma'\rangle \\ 0 &= \beta'\wedge\delta \\ 0 &= \langle\delta,\gamma'\rangle \\ 0 &= \beta\wedge\delta \\ 0 &= \langle\delta,\gamma\rangle \\ 0 &= \langle\beta,\gamma'\rangle - \langle\beta',\gamma\rangle\ , \end{aligned} \tag{6.121}$$

and the T^4 components of the Einstein equation equation (6.13) become

$$\begin{aligned} 0 = \tfrac{1}{2}\langle\iota_X\alpha,\iota_Y\alpha\rangle - \tfrac{1}{2}\langle\iota_X\beta,\iota_Y\beta\rangle - \tfrac{1}{2}\langle\iota_X\beta',\iota_Y\beta'\rangle - \tfrac{1}{2}\langle\iota_X\gamma,\iota_Y\gamma\rangle - \tfrac{1}{2}\langle\iota_X\gamma',\iota_Y\gamma'\rangle \\ +\tfrac{1}{2}\langle\iota_X\delta,\iota_Y\delta\rangle + \tfrac{1}{8}\left(|\beta|^2+|\beta'|^2+3|\gamma|^2+3|\gamma'|^2-2|\delta|^2\right)g(X,Y)\ . \end{aligned} \tag{6.122}$$

We first show that $\delta\neq 0$. Indeed, tracing the above equation we see that

$$\tfrac{1}{2}|\delta|^2 = \tfrac{1}{2}|\alpha|^2 + |\gamma|^2 + |\gamma'|^2\ , \tag{6.123}$$

whence if $\delta=0$, so are α,γ,γ'. Two of the remaining equations for β and β' are then $|\beta|^2=|\beta'|^2$ and $\langle\beta,\beta'\rangle=0$. Using the $\mathrm{SO}(4)$ symmetry of T^4 we can choose $\beta=\beta_1\mathrm{d}\vartheta^1$ and $\beta'=\beta'_2\mathrm{d}\vartheta^2$ with $\beta_1^2=(\beta'_2)^2$. Then the (33) component of equation equation (6.122) says that $|\beta|^2+|\beta'|^2=0$, whence $\beta=\beta'=0$, contradicting the fact that the geometry is not Ricci-flat. Therefore $\delta\neq 0$.

Using the $\mathrm{SO}(4)$ symmetry we may set $\delta=\delta_1\mathrm{d}\vartheta^1$, with $\delta_1\neq 0$, and since $\beta\wedge\delta=0=\beta'\wedge\delta$, also $\beta=\beta_1\mathrm{d}\vartheta^1$ and $\beta'=\beta'_1\mathrm{d}\vartheta^1$. Since γ and γ' are perpendicular to δ, we can use the stabilising $\mathrm{SO}(3)$ to set $\gamma=\gamma_2\mathrm{d}\vartheta^2$ and then the stabilising $\mathrm{SO}(2)$ to set $\gamma'=\gamma'_2\mathrm{d}\vartheta^2+\gamma'_3\mathrm{d}\vartheta^3$, whereas α remains arbitrary. The (14), (24) and (34) components of equation equation (6.122) give

$$\alpha_1\alpha_4=\alpha_2\alpha_4=\alpha_3\alpha_4=0\ , \tag{6.124}$$

whence we have two branches to consider:

1. First branch: $\alpha_4\neq 0$, whence $\alpha_1=\alpha_2=\alpha_3=0$. Equations equation (6.121) become simply

$$\gamma_2\gamma'_2=\beta_1\beta'_1 \qquad\text{and}\qquad -(\beta'_1)^2+(\gamma'_2)^2+(\gamma'_3)^2=-\beta_1^2+\gamma_2^2+2\alpha_4^2\ , \tag{6.125}$$

whereas equations equation (6.122) now become

$$\begin{aligned}
0 &= \gamma_2'\gamma_3' \\
0 &= \beta_1^2 + (\beta_1')^2 + 3\gamma_2^2 + 3(\gamma_2')^2 - (\gamma_3')^2 - 2\delta_1^2 \\
0 &= \beta_1^2 + (\beta_1')^2 - \gamma_2^2 - (\gamma_2')^2 + 3(\gamma_3')^2 - 2\delta_1^2 \\
0 &= \beta_1^2 + (\beta_1')^2 - \gamma_2^2 - (\gamma_2')^2 - (\gamma_3')^2 - \tfrac{2}{3}\delta_1^2 \\
0 &= 4\alpha_4^2 + \beta_1^2 + (\beta_1')^2 + 3\gamma_2^2 + 3(\gamma_2')^2 + 3(\gamma_3')^2 - 2\delta_1^2 \ .
\end{aligned} \tag{6.126}$$

Subtracting the second of the above equations from the last, we find that $\alpha_4 = \gamma_3' = 0$. Subtracting the third from the last we now find $\gamma_2 = \gamma_2' = 0$. Finally subtracting the next to last equation from the last equation gives that $\delta_1 = 0$, which is a contradiction.

2. Second branch: $\alpha_4 = 0$. Then the (44) component of equation equation (6.122) says that the term multiplying $g(X, Y)$ vanishes separately, whence the resulting equations are now

$$\begin{aligned}
0 &= \alpha_1\alpha_2 \\
0 &= \alpha_1\alpha_3 \\
0 &= \alpha_2\alpha_3 - \gamma_2'\gamma_3' \\
0 &= \gamma_2\gamma_2' - \beta_1\beta_1' \\
0 &= (\beta_1')^2 - (\gamma_2')^2 - (\gamma_3')^2 - \beta_1^2 + \gamma_2^2 + 2\alpha_1^2 + 2\alpha_2^2 + 2\alpha_3^2 \\
0 &= 4\alpha_3^2 + \beta_1^2 + (\beta_1')^2 + 3\gamma_2^2 + 3(\gamma_2')^2 - (\gamma_3')^2 - 2\delta_1^2 \\
0 &= 4\alpha_2^2 + \beta_1^2 + (\beta_1')^2 - \gamma_2^2 - (\gamma_2')^2 + 3(\gamma_3')^2 - 2\delta_1^2 \\
0 &= 4\alpha_1^2 - 3\beta_1^2 - 3(\beta_1')^2 + 3\gamma_2^2 + 3(\gamma_2')^2 + 3(\gamma_3')^2 + 2\delta_1^2 \\
0 &= \beta_1^2 + (\beta_1')^2 + 3\gamma_2^2 + 3(\gamma_2')^2 + 3(\gamma_3')^2 - 2\delta_1^2 \ .
\end{aligned} \tag{6.127}$$

There are two branches of solutions. In both of them $\alpha_3 = \beta_1' = \gamma_2' = \gamma_3' = 0$.

(a) Letting $\xi_i = \pm 1$, the first branch is given by

$$\beta_1 = \xi_1\sqrt{3}\alpha_2, \qquad \gamma_2 = \xi_2\alpha_2, \qquad \text{and} \qquad \delta_1 = \xi_3\sqrt{3}\alpha_2 \ . \tag{6.128}$$

(b) Letting $\xi_i = \pm 1$, the second branch is given by

$$\beta_1 = \xi_1\sqrt{2}\alpha_1, \qquad \gamma_2 = 0, \qquad \text{and} \qquad \delta_1 = \xi_2\alpha_1 \ . \tag{6.129}$$

In summary, we have two kinds of backgrounds with this geometry with $\xi_i = \pm 1$:

1.

$$\begin{aligned} F^{(1)} &= \alpha_2 \mathrm{d}\vartheta^2 \\ F^{(3)} &= \alpha_2 \left(\xi_1 \sqrt{3} \nu_0 \wedge \mathrm{d}\vartheta^1 - \xi_2 \mathrm{d}\vartheta^{134}\right) \\ H^{(3)} &= 0 \\ F^{(5)} &= \xi_3 \sqrt{3} \alpha_2 \left(\nu_0 \wedge \mathrm{d}\vartheta^{234} + \nu_1 \wedge \mathrm{d}\vartheta^1\right) \ . \end{aligned} \tag{6.130}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -2\alpha_2^2 g_0 \qquad \text{and} \qquad \mathrm{R}_1 = \tfrac{3}{4}\alpha_2^2 g_1 \ . \tag{6.131}$$

2.

$$\begin{aligned} F^{(1)} &= \alpha_1 \mathrm{d}\vartheta^1 \\ F^{(3)} &= \xi_1 \sqrt{2} \alpha_1 \nu_0 \wedge \mathrm{d}\vartheta^1 \\ F^{(5)} &= \xi_2 \alpha_1 \left(\nu_0 \wedge \mathrm{d}\vartheta^{234} + \nu_1 \wedge \mathrm{d}\vartheta^1\right) \ . \end{aligned} \tag{6.132}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\alpha_1^2 g_0 \qquad \text{and} \qquad \mathrm{R}_1 = \tfrac{1}{2}\alpha_1^2 g_1 \ . \tag{6.133}$$

This latter branch is precisely (up to relabelling) the one we found earlier when looking for backgrounds with geometries $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^3 \times \mathrm{T}^1$ and $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^2$.

Either of these two branches gives solutions for $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^4$.

$\mathbf{AdS_2 \times S^3 \times S^2 \times T^3}$

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (3, 8, 5)$ and so the most general forms here have a total of 24 parameters. The resultant system of polynomials does not lend itself to symbolic solution. However, we can exhibit two exact solutions with $\xi_i = \pm 1$:

1.

$$\begin{aligned} F^{(1)} &= \lambda \mathrm{d}\vartheta^1 \\ H^{(3)} &= \xi_1 \sqrt{\tfrac{5}{2}} \lambda \nu_1 + \xi_2 \sqrt{\alpha^2 + \lambda^2} \nu_0 \wedge \mathrm{d}\vartheta^1 + \alpha \nu_2 \wedge \mathrm{d}\vartheta^1 + \xi_3 \tfrac{1}{\sqrt{2}} \mathrm{d}\vartheta^{123} \\ F^{(3)} &= F^{(5)} = 0 \ . \end{aligned} \tag{6.134}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\tfrac{1}{4}(2\alpha^2 + 3\lambda^2) g_0 \ , \qquad \mathrm{R}_1 = \lambda^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = \tfrac{1}{4}(2\alpha^2 - \lambda^2) g_2 \ , \tag{6.135}$$

giving solutions for $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{T}^3$, $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{T}^5$, and $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{H}^2 \times \mathrm{T}^3$.

2. And the rather uglier,

$$\begin{aligned}
F^{(1)} &= \lambda \mathrm{d}\vartheta^3 \\
H^{(3)} &= \xi_1 \sqrt{\frac{43-\sqrt{57}}{28}}\lambda\nu_1 + \xi_2\sqrt{\frac{9+5\sqrt{57}}{56}}\lambda\nu_0\wedge\mathrm{d}\vartheta^3 + \xi_3\frac{4\sqrt{3}}{3+\sqrt{57}}\lambda\nu_2\wedge\mathrm{d}\vartheta^3 \\
&\quad - \xi_1\xi_2\xi_3\sqrt{\frac{\sqrt{57}-3}{4}} \\
F^{(3)} &= 0 \\
F^{(5)} &= \xi_4\sqrt{\frac{2(6+\sqrt{57})}{7}}\lambda(\nu_1\wedge\mathrm{d}\vartheta^{12} + \nu_0\wedge\nu_2\wedge\mathrm{d}\vartheta^3) + \xi_1\xi_2\xi_4\tfrac{1}{\sqrt{2}}\lambda(\nu_0\wedge\mathrm{d}\vartheta^{123} + \nu_1\wedge\nu_2) \\
&\quad + \xi_1\xi_3\xi_4\sqrt{2}\lambda(\nu_0\wedge\nu_1 - \nu_2\wedge\mathrm{d}\vartheta^{123}) \ .
\end{aligned} \tag{6.136}$$

The Einstein equation then yields:

$$\mathrm{R}_0 = -\frac{29+\sqrt{57}}{16}\lambda^2 g_0 \ , \qquad \mathrm{R}_1 = \lambda^2 g_1 \ , \qquad \text{and} \qquad \mathrm{R}_2 = \frac{13+\sqrt{57}}{16}\lambda^2 g_2 \ , \tag{6.137}$$

giving a solution for $\mathrm{AdS}_2\times\mathrm{S}^3\times\mathrm{S}^2\times\mathrm{T}^3$.

$\mathbf{AdS_2\times S^2\times S^2\times S^2\times T^2}$

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (2, 8, 6)$ and so the most general forms here have a total of 24 parameters. The resultant system of polynomials does not lend itself to symbolic solution. However, we know that solutions exist for $\mathrm{AdS}_2\times\mathrm{S}^2\times\mathrm{S}^2\times\mathrm{S}^2\times\mathrm{T}^2$ and $\mathrm{AdS}_2\times\mathrm{S}^2\times\mathrm{S}^2\times\mathrm{H}^2\times\mathrm{T}^2$ as limits of the solutions for $\mathrm{AdS}_2\times\mathrm{S}^4\times\mathrm{S}^2\times\mathrm{T}^2$ and $\mathrm{AdS}_2\times\mathrm{S}^4\times\mathrm{H}^2\times\mathrm{T}^2$.

$\mathbf{AdS_2\times\mathbb{C}P^2\times T^4}$

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (4, 12, 10)$ and so the most general forms here have a total of 38 parameters. The resultant system of polynomials does not lend itself to symbolic solution. However, we know that solutions exist as a limit of $\mathrm{AdS}_2\times\mathbb{C}\mathrm{P}^2\times\mathrm{S}^3\times\mathrm{T}^1$.

$\mathbf{AdS_2\times S^3\times T^5}$

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (5, 16, 11)$ and so the most general forms here have a total of 48 parameters. The resultant system of polynomials does not lend itself to symbolic solution. However, we know that solutions exist as limits of $\mathrm{AdS}_2\times\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{T}^2$, $\mathrm{AdS}_2\times\mathbb{C}\mathrm{P}^2\times\mathrm{S}^3\times\mathrm{T}^1$ and $\mathrm{AdS}_2\times\mathrm{S}^3\times\mathrm{S}^2\times\mathrm{T}^3$.

$\mathbf{AdS_2\times S^2\times S^2\times T^4}$

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (4, 16, 12)$ and so the most general forms here have a total of 48 parameters. The resultant system of polynomials does not lend

itself to symbolic solution. However, we know that solutions exist for both $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^4$ and $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{H}^2 \times \mathrm{T}^4$ as a limits of $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^1$ and $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{H}^2 \times \mathrm{T}^1$ respectively.

$\mathbf{AdS_2 \times S^2 \times T^6}$

In the notation of section 6.4.2 we have $(m_1, m_3, m_5^+) = (6, 32, 26)$ and so the most general forms here have a total of 96 parameters. The resultant system of polynomials does not lend itself to symbolic solution. However, we know that solutions exist as a limits of $\mathrm{AdS}_2 \times \mathrm{S}^5 \times \mathrm{S}^2 \times \mathrm{T}^1$, $\mathrm{AdS}_2 \times \mathrm{SLAG}_3 \times \mathrm{S}^2 \times \mathrm{T}^1$, $\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^2$, and $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{H}^3 \times \mathrm{S}^2$.

6.4.10 Summary

We have identified (up to local isometry) all symmetric type IIB supergravity backgrounds; that is, type IIB supergravity backgrounds whose underlying geometry is a symmetric space $\mathcal{G}/\mathcal{H}$ and with all field content $\mathcal{G}$-invariant up to gauge transformations. For approximately two thirds of all these backgrounds we have determined the full moduli space and for the rest we have shown existence either as limits of other geometries or as exact solutions.

There are two classes of solutions, with underlying geometry either:

- Cahen-Wallach spaces (possible degenerate): $\mathrm{CW}_d(\lambda) \times \mathbb{R}^{10-d}$, or
- Backgrounds with an Anti-de Sitter factor: $\mathrm{AdS}_d \times \mathcal{M}^{10-d}$ with $2 \leq d \leq 5$.

The Anti-de Sitter class we summarise in tables 6.2 and 6.3, distinguished by whether or not we have determined the exact moduli space. In table 6.2 we list the backgrounds for which we have determined the exact moduli space. There are three numbers associated to each background that correspond to the three types of moduli:

- *Geometric*:
  These moduli correspond to the number of free parameters of a background. One of these moduli always corresponds to the homothety symmetry (see section 6.2.2) of the background and so this is always ≥ 1.
- *Duality*:
  These moduli correspond to the $\mathrm{SL}(2,\mathbb{R})$ orbit of a background (see section 6.2.2) and can be one[4] of:
  - 0: this corresponds to backgrounds where $F^{(1)} \neq 0$,
  - 2: this corresponds to backgrounds where $F^{(1)} = H^{(3)} = F^{(3)} = 0$, so that the $\mathrm{SL}(2,\mathbb{R})$ orbit is parametrised by the axi-dilaton τ , and
  - 3: this corresponds to backgrounds where $F^{(1)} = 0$ but $H^{(3)}$ or $F^{(3)}$ are non-zero.

[4]Geometries with multiple solution branches may have different duality moduli for each branch. The only examples are: $\mathrm{AdS}_2 \times \mathrm{G}^+_{\mathbb{R}}(2,5) \times \mathrm{T}^2$ and $\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^3 \times \mathrm{T}^2$.

- *Other*:

  These moduli correspond to the the dimension of an orbit of the action of $\mathrm{SO}(n)$ on backgrounds with a T^n factor. The only non-zero example of this where we have determined the full moduli space is $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ where, when $\gamma \neq 0$, the moduli parametrises the orbit of $\mathrm{SO}(4)$ acting on the flat component of $F^{(5)}$ that is an (anti-)self-dual 2-form which in $\mathbb{R}^4$.

In table 6.3 we list the backgrounds for which we have *not* determined the full moduli space. Each background is either of the first status or one or more of the latter two:

- *Some exact solutions*:

  We have not determined the full moduli space of the background, but have demonstrated some exact solutions in order to show existence. There may be no other solutions, but we cannot claim to know.

- $\exists$ *as limit of*:

  The background was found as a limit of one or more other backgrounds when the radius of curvature of one of their Riemannian factors goes to infinity (see section 6.4.2). We thus have a family of solutions from the limit(s) but they may describe only a portion of the full moduli space.

- $\exists$ *from*:

  This can mean either:

  1. The background was found by looking at a geometry with fewer flat directions, but where the field equations forced one or more of the Riemannian factors to be flat. We thus have a family of solutions from the limit(s) but they may describe only a portion of the full moduli space.
  2. The background can be shown to exist via the argument in section 6.4.2. We thus have a family of solutions but they may describe only a portion of the full moduli space.

The list of backgrounds is not particularly surprising, with only a few backgrounds that are not AdS-sphere-flat products. We note here that symmetric supergravity backgrounds do not necessarily oxidise to symmetric supergravity backgrounds; although the curvature of the oxidation connection may be invariant, the oxidation connection itself may well not be invariant.

It is left to determine which of these backgrounds is supersymmetric. The spirit of this is address in chapter 7 although we actually determine supersymmetry for symmetric M-theory backgrounds. The same technique can be applied for symmetric type IIB backgrounds although the dilaton component of the Killing spinor equations makes things slightly more difficult.

After the proof of Patrick Meessen's homogeneity conjecture [34, 1, 2, 84] we were reminded of his follow-up conjecture that all supergravity backgrounds preserving more than $3/4$ of the maximum supersymmetry are not only homogeneous but moreover symmetric. As far as we

know there are no counterexamples – we note that all M-theory backgrounds with more than 29 supercharges are already known to be maximally supersymmetric [85] and so symmetric [86], and all $D = 10$ type IIB backgrounds with more than 28 supercharges are known to be maximally supersymmetric [87] and so symmetric [86]. However, we have not yet found a way to directly connect supersymmetry to symmetric geometry in particular. Our current feeling is that this may require a "theorem"[5] and not a "Theorem". As ammunition for this position, we note that there is a symmetric M-theory background with 26 supercharges (and so more than $3/4$ of the maximum supersymmetry) where the Killing spinors do not generate the full symmetric algebra on their own [88]. However, we also know [89, 90] that we should be able to 'dial up' the symmetry algebra of a background, and since we know that this background is symmetric, the full symmetric algebra presumably could still be encoded in the supersymmetries.

Table 6.2: AdS backgrounds with known moduli space

Geometry	Moduli		
	geometric	duality	other
$\mathrm{AdS}_5 \times \mathrm{S}^5$	1	2	0
$\mathrm{AdS}_5 \times \mathrm{SLAG}_3$	1	2	0
$\mathrm{AdS}_5 \times \mathrm{S}^3 \times \mathrm{S}^2$	1	2	0
$\mathrm{AdS}_4 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{T}^1$	1	0	0
$\mathrm{AdS}_3 \times \mathrm{S}^5 \times \mathrm{H}^2$	1	2	0
$\mathrm{AdS}_3 \times \mathrm{SLAG}_3 \times \mathrm{H}^2$	1	2	0
$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{H}^2$	2	2	0
$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{T}^1$	2	3	0
$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{T}^2$	2	3	0
$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$	2	3	2
$\mathrm{AdS}_2 \times \mathrm{SU}(3)$	1	2	0
$\mathrm{AdS}_2 \times \mathrm{G}^+_{\mathbb{R}}(2,5) \times \mathrm{T}^2$	1	0/3	0
$\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^3 \times \mathrm{T}^2$	1	0/3	0
$\mathrm{AdS}_2 \times \mathrm{S}^5 \times \mathrm{H}^3$	1	2	0
$\mathrm{AdS}_2 \times \mathrm{SLAG}_3 \times \mathrm{H}^3$	1	2	0
$\mathrm{AdS}_2 \times \mathrm{H}^4 \times \mathrm{S}^3 \times \mathrm{T}^1$	1	0	0
$\mathrm{AdS}_2 \times \mathrm{S}^5 \times \mathrm{S}^2 \times \mathrm{T}^1$	2	0	0
$\mathrm{AdS}_2 \times \mathrm{SLAG}_3 \times \mathrm{S}^2 \times \mathrm{T}^1$	2	0	0
$\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{H}^3 \times \mathrm{S}^2$	2	2	0
$\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{T}^2$	2	3	0
$\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^2 \times \mathrm{S}^3 \times \mathrm{T}^1$	2	0	0
$\mathrm{AdS}_2 \times \mathbb{C}\mathrm{H}^2 \times \mathrm{S}^3 \times \mathrm{T}^1$	2	0	0
$\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^2 \times \mathrm{T}^2$	2	0	0
$\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{H}^2 \times \mathrm{T}^2$	2	0	0
$\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{T}^4$	2	0	0

[5] Exhaustive proof in the nomenclature of Victor Kac.

Table 6.3: AdS backgrounds with unknown moduli space

Geometry	Status
$AdS_3 \times S^2 \times S^2 \times T^3$	Some exact solutions
$AdS_3 \times S^2 \times T^5$	$\exists$ as limit of $AdS_3 \times S^3 \times S^2 \times T^2$
$AdS_2 \times S^3 \times S^2 \times S^2 \times T^1$	Some exact solutions
$AdS_2 \times S^3 \times H^2 \times S^2 \times T^1$	Some exact solutions
$AdS_2 \times S^3 \times H^2 \times H^2 \times T^1$	Some exact solutions
$AdS_2 \times \mathbb{C}P^2 \times S^2 \times T^2$	Some exact solutions
$AdS_2 \times \mathbb{C}P^2 \times H^2 \times T^2$	Some exact solutions
$AdS_2 \times S^3 \times S^2 \times T^3$	Some exact solutions
$AdS_2 \times S^3 \times H^2 \times T^3$	Some exact solutions
$AdS_2 \times S^2 \times S^2 \times S^2 \times T^2$	$\exists$ from $AdS_2 \times S^4 \times S^2 \times T^2$
$AdS_2 \times S^2 \times S^2 \times H^2 \times T^2$	$\exists$ from $AdS_2 \times S^4 \times H^2 \times T^2$
$AdS_2 \times \mathbb{C}P^2 \times T^4$	$\exists$ from $AdS_2 \times \mathbb{C}P^2 \times S^3 \times T^1$
$AdS_2 \times S^3 \times T^5$	$\exists$ as limit of $AdS_2 \times S^3 \times S^3 \times T^2$, $AdS_2 \times \mathbb{C}P^2 \times S^3 \times T^1$, $AdS_2 \times S^3 \times S^2 \times T^3$ and $AdS_2 \times S^3 \times H^2 \times T^3$
$AdS_2 \times S^2 \times S^2 \times T^4$	$\exists$ from $AdS_2 \times S^3 \times S^2 \times S^2 \times T^1$ and $AdS_2 \times S^4 \times T^4$
$AdS_2 \times S^2 \times H^2 \times T^4$	$\exists$ from $AdS_2 \times S^3 \times S^2 \times H^2 \times T^1$
$AdS_2 \times S^2 \times T^6$	$\exists$ from $AdS_2 \times S^5 \times S^2 \times T^1$, $AdS_2 \times SLAG_3 \times S^2 \times T^1$ and $AdS_2 \times S^4 \times S^2 \times T^2$; $\exists$ as limit of $AdS_2 \times S^3 \times H^3 \times S^2$

Chapter 7

Supersymmetric symmetric M-theory backgrounds

7.1 Introduction

A classification of all symmetric M-theory backgrounds was made in [33] (and additionally for symmetric Type IIB backgrounds in [3]) using Élie Cartan's classification of irreducible Riemannian symmetric spaces (e.g. [32]) and Cahen and Wallach's classification of indecomposable Lorentzian symmetric spaces [30, 31]. However this classification dealt with backgrounds at the bosonic level and said nothing about which backgrounds are supersymmetric. In this chapter we build on this result by determining which symmetric M-theory backgrounds are supersymmetric and if so what fraction of the maximum supersymmetry is preserved. This represents the next step in classifying homogeneous M-theory backgrounds and concludes the classification of supersymmetric symmetric M-theory backgrounds.

The general idea is as follows: The number of supersymmetries preserved by an M-theory background $(\mathcal{M}, g, F)$ is equal to the dimension of the kernel of the M-theory superconnection $\mathcal{D}$ (see equation (4.2)). We use a general algorithm (derived in detail in [4]) for computing algebraically and concretely the holonomy algebra of *any* invariant spinor connection on a reductive homogeneous space to compute the holonomy algebra of $\mathcal{D}$. We then identify the dimension of the kernel of the algebra and thus the number of supersymmetries preserved by the background. We apply this to all symmetric M-theory backgrounds listed in [33] and so identify all supersymmetric symmetric M-theory backgrounds.

This chapter is based upon work done in collaboration with Andree Lischewski in [4].

7.2 Invariant spin connections on symmetric spaces

We give a *very* brief framing of the algorithm we will use to compute the supersymmetry of symmetric M-theory backgrounds. Much more detail can be found in [4].

Let $(\mathcal{M}, g)$ be a pseudo-Riemannian reductive homogeneous spin manifold with signature (s,t) and homogeneous decomposition $\mathcal{M} = \mathcal{G}/\mathcal{K}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. The isotropy representation is the homomorphism $\mathrm{Ad}^{\mathcal{G}/\mathcal{K}} : \mathcal{K} \to \mathrm{SO}^+(\mathfrak{m})$ and we denote the g-corresponding $\mathrm{Ad}^{\mathcal{G}/\mathcal{K}}$-invariant bilinear form on $\mathfrak{m}$ as $q_{\mathfrak{m}}$.

The homogeneous oriented frame bundle is

$$\mathscr{P} = \mathcal{G} \times_{\mathrm{Ad}^{\mathcal{G}/\mathcal{K}}} \mathrm{SO}^+(\mathfrak{m}) \,, \tag{7.1}$$

and by picking a lift of the isotropy representation to $\mathrm{Spin}^+(\mathfrak{m})$ through choice of map $\widetilde{\mathrm{Ad}}^{\mathcal{G}/\mathcal{K}}$ such that the diagram

$$\begin{array}{ccc} & & \mathrm{Spin}^+(\mathfrak{m}) \\ & \nearrow^{\widetilde{\mathrm{Ad}}^{\mathcal{G}/\mathcal{K}}} & \downarrow{\scriptstyle \lambda} \\ \mathcal{K} & \xrightarrow{\mathrm{Ad}^{\mathcal{G}/\mathcal{K}}} & \mathrm{SO}^+(\mathfrak{m}) \end{array} \tag{7.2}$$

commutes (where λ is the double covering map), we may fix a homogeneous spin structure $(\mathscr{Q}, f)$ with

$$\mathscr{Q} = \mathcal{G} \times_{\widetilde{\mathrm{Ad}}^{\mathcal{G}/\mathcal{K}}} \mathrm{Spin}^+(\mathfrak{m}) \,, \tag{7.3}$$

and $f : \mathscr{Q} \to \mathscr{P}$ simply the double covering map λ. We can then construct the homogeneous spinor bundle

$$\$ = \mathscr{Q} \times_{\mathrm{Spin}^+} \mathcal{S}_{\mathfrak{m}} \cong \mathcal{G} \times_{\widetilde{\mathrm{Ad}}^{\mathcal{G}/\mathcal{K}}} \mathcal{S}_{\mathfrak{m}} \,. \tag{7.4}$$

In this context, let $\nabla : \Gamma(\$) \to \Gamma(\mathrm{T}^*\mathcal{M} \otimes \$)$ denote the spinor covariant derivative induced by the lift of the Levi-Civita connection and let there be a connection on the spinor bundle

$$\mathcal{D} = \nabla + \Omega : \Gamma(\$) \to \Gamma(\mathrm{T}^*\mathcal{M} \otimes \$) \tag{7.5}$$

where $\Omega : \mathrm{T}\mathcal{M} \to \mathrm{C}\ell(\mathrm{T}\mathcal{M}, g) \cong \Lambda^*(\mathrm{T}\mathcal{M})$ is a vector bundle homomorphism and is left-invariant (using the exterior algebra isomorphism, see appendix B.5): $\forall X \in \mathrm{T}\mathcal{M}, a \in \mathcal{G}$,

$$l^*_{a^{-1}} \Omega(\mathrm{d}l_a(X)) = \Omega(X) \,. \tag{7.6}$$

From Theorem 3.2 and Corollary 3.3 in [4] we see that $\mathcal{D}$ is naturally induced by a $\mathcal{G}$-invariant connection $\tilde{\mathcal{D}}$ on the homogeneous principal bundle

$$\begin{array}{ccc} \mathrm{C}\ell^\times(\mathfrak{m}, q_{\mathfrak{m}}) & \longrightarrow & \mathcal{G} \times_{\widetilde{\mathrm{Ad}}_{\mathcal{G}/\mathcal{K}}} \mathrm{C}\ell^\times(\mathfrak{m}, q_{\mathfrak{m}}) \\ & & \downarrow \\ & & \mathcal{G}/\mathcal{K} \end{array} \tag{7.7}$$

Using the known theory [91, 92, 93] of invariant connections on reductive homogeneous spaces we may thus find the parallel sections of $\mathcal{D}$ by finding the kernel of the holonomy algebra of a

particular map $\alpha : \mathfrak{g} \to \mathrm{C}\ell(\mathfrak{m}, q_{\mathfrak{m}})$ associated to $\tilde{\mathcal{D}}$. The map α decomposes with $\mathcal{D} = \nabla + \Omega$ into $\alpha^{\mathfrak{k}} + \alpha^{\mathfrak{m}}$ of which

$$\alpha^{\mathfrak{k}} = \widetilde{\mathrm{ad}}^{\mathcal{G}/\mathcal{K}} : \mathfrak{k} \to \mathfrak{spin}(\mathfrak{m}, q_{\mathfrak{m}}) \subset \mathrm{C}\ell(\mathfrak{m}, q_{\mathfrak{m}}) \ , \tag{7.8}$$

and

$$\alpha^{\mathfrak{m}} = \Omega_o : \mathfrak{m} \to \mathrm{C}\ell(\mathfrak{m}, q_{\mathfrak{m}}) \ . \tag{7.9}$$

We also have the curvature map κ which measures the failure of α to be a Lie algebra homomorphism,

$$\begin{array}{rccl} \kappa : & \mathfrak{g} \times \mathfrak{g} & \to & \mathrm{C}\ell(\mathfrak{m}, q_{\mathfrak{m}}) \\ & (X_1, X_2) & \mapsto & [\alpha(X_1), \alpha(X_2)] - \alpha([X_1, X_2]) \ , \end{array} \tag{7.10}$$

Then, with $\widehat{\mathrm{Im}}(\kappa) := \mathrm{span}(\mathrm{Im}(\kappa))$, $\mathfrak{hol}(\alpha)$ is the $\mathfrak{g}$-module generated by $\widehat{\mathrm{Im}}(\kappa)$, i.e.

$$\mathfrak{hol}(\alpha) = \widehat{\mathrm{Im}}(\kappa) + [\alpha(\mathfrak{g}), \widehat{\mathrm{Im}}(\kappa)] + [\alpha(\mathfrak{g}), [\alpha(\mathfrak{g}), \widehat{\mathrm{Im}}(\kappa)]] + \ldots \subset \mathrm{C}\ell(\mathfrak{m}, q_{\mathfrak{m}}) \ , \tag{7.11}$$

and for the case of a symmetric space,

$$\mathfrak{hol}(\alpha) = \widehat{\mathrm{Im}}(\kappa) + [\alpha(\mathfrak{m}), \widehat{\mathrm{Im}}(\kappa)] + [\alpha(\mathfrak{m}), [\alpha(\mathfrak{m}), \widehat{\mathrm{Im}}(\kappa)]] + \ldots \subset \mathrm{C}\ell(\mathfrak{m}, q_{\mathfrak{m}}) \ . \tag{7.12}$$

Finally, for ease of computation, we wish to work in the algebra $\mathrm{C}\ell(s,t)$ instead of $\mathrm{C}\ell(\mathfrak{m}, q_{\mathfrak{m}})$ and so we fix an orientation-preserving isometry $w : \mathbb{R}^{s,t} \to \mathfrak{m}$ and instead use the map $\alpha_w := w^*\alpha : \mathfrak{g} \to \mathrm{C}\ell(s,t)$ (and concomitant $\kappa_w : \mathfrak{g} \times \mathfrak{g} \to \mathrm{C}\ell(s,t)$) in all calculations.

7.3 Supersymmetric symmetric M-theory backgrounds

We continue on from our earlier definition of $D = 11$ supergravity (see section 4.2.3). Now we are interested in the field equations of the theory and so we begin with the action. The $D = 11$ (bosonic) action is (in the string frame)

$$S = \tfrac{1}{2} \int \left\{ \left(\mathrm{R} - \tfrac{1}{2}|F|^2 \right) \mathrm{dvol} - \tfrac{1}{6} F \wedge F \wedge A \right\} \ , \tag{7.13}$$

where R is the Ricci scalar curvature of g and dvol is the signed volume element. We define the inner product on differential forms $\langle X, Y \rangle \, \mathrm{dvol} = X \wedge *Y$ and the norm $|X|^2 = \langle X, X \rangle$.

Varying the action with respect to each of the fields yields the following (bosonic) equations of motion

$$\begin{aligned} \mathrm{d} \star F &= \tfrac{1}{2} F \wedge F \\ \mathrm{Ric}(X, Y) &= \tfrac{1}{2} \langle \iota_X F, \iota_Y F \rangle - \tfrac{1}{6} g(X, Y) |F|^2 \ . \end{aligned} \tag{7.14}$$

In the case of a symmetric space, the 4-form F is parallel with respect to the canonical connection (see section 2.2.1) which is the Levi-Civita connection ($\nabla F = 0$) and so its dual $\star F$ is also parallel ($\nabla \star F = 0$) whence it is both closed ($\mathrm{d}F = 0$) and co-closed ($\mathrm{d} \star F = 0$). The

field equations then simplify to

$$0 = F \wedge F \tag{7.15}$$

$$\mathrm{Ric}(X,Y) = \tfrac{1}{2}\langle \iota_X F, \iota_Y F\rangle - \tfrac{1}{6} g(X,Y)|F|^2 \ . \tag{7.16}$$

We recall the Killing spinor equation of $D = 11$ supergravity (equation (4.2)) takes the form $\mathcal{D}_X = \nabla_X + \Omega(X)$ where

$$\Omega(X) = \tfrac{1}{24}\left(3F \cdot X - X \cdot F\right) \ . \tag{7.17}$$

Killing spinors are (non-trivial) spinors parallel to this connection and the number of linearly independent Killing spinors is equal to the number of supersymmetries of a background. It is clear that the algorithm of section 7.2 may be applied to this connection in order to determine the rank of the subbundle of Killing spinors and so the fraction of the maximum supersymmetry preserved by the background.

Symmetric M-theory backgrounds have been classified in [33] and we follow that article's conventions (which coincide with those in section 4.2.3) and reference particular backgrounds directly from it. Background geometries are of the form $\mathcal{M} = \mathcal{M}_0 \times \mathcal{M}_1 \times \ldots \times \mathcal{M}_n$ with $\mathcal{M}_0$ an indecomposable Lorentzian symmetric space and $\mathcal{M}_i$ for $i > 0$ irreducible Riemannian symmetric spaces, with each factor determined by its Lie algebra pair $(\mathfrak{k}, \mathfrak{m})_i$. From here forwards in this chapter we will elide pullbacks and conflate F with its evaluation at the origin o.

7.4 Exclusion of backgrounds

We note that to exclude a background, it is only necessary to identify a single element in the holonomy algebra that has trivial kernel. This element alone then ensures that the algebra has trivial kernel. As such, we do not in general have to produce the whole holonomy algebra $\mathfrak{hol}(\alpha)$ by stabilising the dimension of successive commutators of $\alpha(\mathfrak{m})$ with $\widehat{\mathrm{Im}}(\kappa)$ in order to rule out a background.

7.4.1 Special cases

7.4.1.1 No spin structure

We ignore all backgrounds from [33] which involve a $\mathbb{C}\mathrm{P}^2$ or $\mathbb{C}\mathrm{H}^2$ factor. They are solutions to the field equations but $\mathbb{C}\mathrm{P}^2$ does not admit a spin structure, and so it makes no sense to speak about supersymmetry in this case. The non-compact dual $\mathbb{C}\mathrm{H}^2$ does not admit a homogeneous spin structure as its isotropy representation is equivalent to that of $\mathbb{C}\mathrm{P}^2$.

7.4.1.2 $n \geq 3$ and F two-factor homogeneous

Let us assume that $n \geq 3$ and either:

$$F = F_1 \ , \qquad \text{or} \qquad F = F_1 \wedge H_2 \wedge H_3 \ , \qquad \text{or} \qquad F = H_2 \wedge H_3 \ , \tag{7.18}$$

with $F_1 \in (\Lambda \mathfrak{m}_1)^{\mathfrak{k}}$ and $H_i \in (\Lambda_{\rm hom} \mathfrak{m}_i)^{\mathfrak{k}}$, i.e. H_i are homogeneous forms.

Now, taking $X_2 \in \mathfrak{m}_2$, $X_3 \in \mathfrak{m}_3$ such that either $\iota_{X_i} H_i \neq 0$ or $\iota_{X_i} H_i = 0$, we then have that $[X_2, X_3]_{\mathfrak{k}} = 0$ and $\alpha(X_i) \propto X_i \cdot F$, so

$$\kappa(X_2, X_3) \propto [X_2 \cdot F, X_3 \cdot F] \propto (X_2 \cdot X_3) \cdot (F \cdot F) \in \mathrm{C}\ell(1, 10) \ . \tag{7.19}$$

Thus in this case F must have a kernel as considered as a spinor endomorphism or there are no $\mathcal{D}$-parallel spinors. Homogeneous elements act invertibly and so the last case is ruled out and the first two cases are reduced to F_1 having kernel. In particular, this means that F_1 cannot be homogeneous.

An immediate application of this is when $n \geq 3$ and F is proportional to the volume form on any (possibly composite) factor, then there are no $\mathcal{D}$-parallel spinors.

Note that this does not apply to Freund-Rubin backgrounds because $n = 2$.

7.4.1.3 $n \geq 3$, F two-factor homogeneous, and F_1 inhomogeneous

Here we look at geometries where section 7.4.1.2 applies but the F_1 factor is inhomogeneous and so we must do a little more work.

- (4.6.1), (4.7.5): $F_1 = F \propto \omega \wedge \omega$ with ω the Kähler form of a 6-dimensional factor
  The action of ω on the (complex) spinor module is well known [94] and using these results it is straightforward to see that F considered as an endomorphism on the spinor module has no zero eigenvalue if $\mathcal{M}_1$ is 6-dimensional. This is easily verified with a concrete realisation of F.

- (4.7.6): $F_1 \in (\Lambda^4 \mathfrak{m}_0)^{\mathfrak{k}} \oplus (\Lambda^4 \mathfrak{m}_1)^{\mathfrak{k}}$ is decomposable and has trivial kernel on both factors; so F_1 has trivial kernel.

- (4.7.7): $F_1 \in (\Lambda^4 \mathfrak{m}_1)^{\mathfrak{k}} \oplus (\Lambda^4 \mathfrak{m}_2)^{\mathfrak{k}}$ is decomposable and has trivial kernel on both factors; so F_1 has trivial kernel.

7.4.1.4 A background that isn't a background

We suggest that the classical background (4.7.4): $\mathrm{AdS}_2 \times \mathrm{G}_{\mathbb{C}}(2,4) \times \mathrm{T}^1$ does not exist. There are two polynomial equations resulting from the field equation $F \wedge F = 0$, the second of which is not found in [33]. In the notation of [33] this is $f_1^2 + f_2^2 = 0$ and it means that no background with such an underlying geometry exists.

7.4.1.5 Summary

We summarise the backgrounds excluded by the previous arguments in table 7.1.

Table 7.1: Backgrounds ruled out through general arguments

Ref. [33]	Background	F	Reason
4.2	$\mathrm{AdS}_7 \times \mathrm{S}^2 \times \mathrm{S}^2$	$f\nu_1 \wedge \nu_2$	section 7.4.1.2
4.4.2	$\mathrm{AdS}_5 \times \mathrm{H}^2 \times \mathrm{S}^4$	$f\nu_2$	section 7.4.1.2
4.5.1	$\mathrm{AdS}_4 \times \mathrm{S}^5 \times \mathrm{S}^2$, $\mathrm{AdS}_4 \times \mathrm{SLAG}_3 \times \mathrm{S}^2$	$f\nu_0$	section 7.4.1.2
4.5.2	$\mathrm{AdS}_4 \times \mathrm{S}^4 \times \mathrm{S}^3$, $\mathrm{AdS}_4 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^3$	$f\nu_0$	section 7.4.1.2
4.5.2	$\mathrm{AdS}_4 \times \mathrm{S}^4 \times \mathrm{H}^3$	$f\nu_1$	section 7.4.1.2
4.5.2	$\mathrm{AdS}_4 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{H}^3$	$f\nu_1 \wedge \nu_2$	section 7.4.1.2
4.6.1	$\mathrm{AdS}_3 \times \mathbb{C}\mathrm{P}^3 \times \mathrm{H}^2$, $\mathrm{AdS}_3 \times \mathrm{G}^+_{\mathbb{R}}(2,5) \times \mathrm{H}^2$	$f\omega \wedge \omega$	section 7.4.1.3
4.6.2	$\mathrm{AdS}_3 \times \mathrm{S}^4 \times \mathrm{H}^4$, $\mathrm{AdS}_3 \times \mathrm{S}^4 \times \mathrm{H}^2 \times \mathrm{H}^2$	$f\nu_1$	section 7.4.1.2
4.6.2	$\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{H}^4$	$f\nu_1 \wedge \nu_2$	section 7.4.1.2
4.7.2	$\mathrm{AdS}_2 \times \mathrm{H}^2 \times \mathrm{S}^7$	$f\nu \wedge \sigma$	section 7.4.1.2
4.7.3	$\mathrm{AdS}_2 \times \mathrm{H}^5 \times \mathrm{S}^4$, $\mathrm{AdS}_2 \times (\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)) \times \mathrm{S}^4$	$f\nu_2$	section 7.4.1.2
4.7.5	$\mathrm{AdS}_2 \times \mathrm{H}^3 \times \mathbb{C}\mathrm{P}^3$ $\mathrm{AdS}_2 \times \mathrm{H}^3 \times \mathrm{G}^+_{\mathbb{R}}(2,5)$	$f\omega \wedge \omega$	section 7.4.1.3
4.7.6	$\mathrm{AdS}_2 \times \mathbb{C}\mathrm{P}^3 \times \mathrm{T}^3$, $\mathrm{AdS}_2 \times \mathrm{G}^+_{\mathbb{R}}(2,5) \times \mathrm{H}^3$	$f(\omega \pm \sqrt{3}\nu_0) \wedge \mathrm{d}\vartheta^{12}$	section 7.4.1.3
4.7.7	$\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{T}^3$	$f(\nu_1 \pm \nu_2) \wedge \mathrm{d}\vartheta^3$	section 7.4.1.3
4.7.8	$\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{S}^3 \times \mathrm{H}^2$	$f\nu_0 \wedge \nu_3$	section 7.4.1.2
4.7.8	$\mathrm{AdS}_2 \times \mathrm{S}^4 \times \mathrm{H}^3 \times \mathrm{H}^2$	$f\nu_1$	section 7.4.1.2

7.4.2 Computed geometries

After this analysis, there are a number of backgrounds for which the existence of supersymmetries is still undecided. In all these cases, the 4-form F depends on various parameters which are subject to additional algebraic equations or inequalities. The resulting equations for the parameters f_i appearing in F are much more involved and so in order to complete our analysis we turn to the computer.

We take the symmetric space data for a geometry and reduce it in the same fashion as described in chapter 6 but as an M-theory background. This gives us a polynomial system the solution space of which is the moduli space of the background. We then construct a concrete $\mathrm{C}\ell(1,10)$ representation and will sometimes adapt it to the product structure of the space (see appendix B.11.6 as an example) by requiring that the volume form on each component acts diagonally. In these cases, our representation will no longer be real but eventual eigenvalue computation is greatly simplified. However, in some cases, reality will be useful because we can show an absence of real eigenvalues. Using this representation we construct a concrete

realisation of the map κ and then, following the spirit of the previous analysis, we choose one or more convenient pairs $X, Y \in \mathfrak{m}$ and compute the eigenvalues of $\kappa(X, Y)$. These eigenvalues are parametrised by the background moduli space parameters and their solution space is the supersymmetry moduli space. We rule out the existence of any zero eigenvalues either directly (general eigenvalue arguments without appeal to background moduli space) or by seeing that the background moduli space and supersymmetry moduli space do not intersect. Such backgrounds are show in table 7.2.

Table 7.2: Backgrounds ruled out through direct computation (referencing [33])

- 4.4.1: $\mathrm{AdS}_5 \times \mathbb{C}\mathrm{P}^3 / \mathrm{G}^+_{\mathbb{R}}(2,5)$
  We take a generator constructed from one vector from each factor. We see that there can be no zero eigenvalues.
- 4.4.3: $\mathrm{AdS}_5 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2$
  We take three generators; one constructed from two vectors on the AdS_5 factor, one constructed from two vectors on an S^2 factor, and one constructed from one vector from the AdS_5 and one from (the same) S^2. We see that there can be no simultaneous zero eigenvalues in the F-moduli space.
- 4.6.5: $\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2$
  Taking a generator constructed from two vectors on an S^2 factor we see that the generator is skew-symmetric and has constant complex eigenvectors even though we use a concretely real representation.
- 4.7.1: $\mathrm{AdS}_2 \times \mathrm{SLAG}_4$
  Taking a generator constructed from two vectors in the AdS_2 factor we see that there are no zero eigenvalues.
- 4.7.11: $\mathrm{AdS}_2 \times \mathrm{S}^5 \,/\, \mathrm{SLAG}_3 \times \mathrm{T}^4$
  Taking a generator constructed from two vectors on the AdS_2 factor, we see that there can be no zero eigenvalues.
- 4.7.12: $\mathrm{AdS}_2 \times \mathrm{S}^5 \,/\, \mathrm{SLAG}_3 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2$
  Taking a generator constructed from two vectors on the AdS_2 factor we see that the generator has no zero eigenvalues in the F-moduli space.
- 4.7.12: $\mathrm{AdS}_2 \times \mathrm{H}^5 \,/ (\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)) \times \mathrm{S}^2 \times \mathrm{S}^2$
  Taking a generator constructed from two vectors on the AdS_2 factor we see that the generator has no zero eigenvalues in the F-moduli space.
- 4.7.14: $\mathrm{AdS}_2 \times \mathrm{S}^3 \,/\, \mathrm{H}^3 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{S}^2 \times \mathrm{S}^2$
  Taking a generator constructed from two vectors on the AdS_2 factor we see that the generator has no zero eigenvalues in the F-moduli space via non-flatness of *all* $\mathrm{S}^2 \,/\, \mathrm{H}^2$ factors.
- 4.7.14: $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{H}^2 \times \mathrm{H}^2 \times \mathrm{S}^2$
  Taking a generator constructed from two vectors on the AdS_2 factor we see that the generator has no zero eigenvalues in the F-moduli space via non-flatness of *all* $\mathrm{S}^2 \,/\, \mathrm{H}^2$ factors.
- 4.7.16: $\mathrm{AdS}_2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^1$
  Taking a generator constructed from two vectors on the AdS_2 factor we see that the generator has no zero eigenvalues in the F-moduli space.

In all of the above cases we find that all eigenvalues are necessarily nonzero within the F-moduli space of the bosonic field equations. As such, none of the above backgrounds admit supersymmetries.

7.4.3 Limit geometries

Let us consider a geometry $\mathcal{M} \times \mathcal{N}$ ($\mathcal{N}$ not flat) and take the geometric limit [36] in which the curvature of $\mathcal{N}$ goes to zero yielding the limit geometry $\mathcal{M} \times \mathrm{T}^n$. For the geometry $\mathcal{M} \times \mathcal{N}$ the most general ansatz 4-form F is a sum of available invariant 4-forms whose parameterisation, then constrained by the field equations, forms the solution moduli space. In this geometric limit, we generally have access to extra invariant forms due to the triviality of the flat component and so must a priori consider extra terms in our 4-form ansatz. However, we now also have to impose a flatness condition coming from the Einstein equation.

Let us specialise to consider the case where this flatness condition forces all parameters of these extra invariant 4-forms to zero. In this case the moduli space of the limit geometry is a subspace of the moduli space of our original geometry and the holonomy algebra of the limit geometry is a subalgebra of the holonomy algebra of the original geometry. As such, we may deduce the absence of supersymmetry of such a limit geometry from the absence of supersymmetry of the original geometry as demonstrated – as long as we do not use the relevant non-flatness conditions – via a realised generator common to both holonomy algebras, i.e. constructed from vectors on the $\mathcal{M}$ factor. In this way, we may also rule out supersymmetry for the backgrounds in table 7.3.

Table 7.3: Backgrounds ruled out as limit geometries

Ref. [33]	Background	As a limit of
4.4.3	$\mathrm{AdS}_5 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^2$	$\mathrm{AdS}_5 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2$
4.6.5	$\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{T}^2$	$\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{S}^2$
4.7.12	$\mathrm{AdS}_2 \times \mathrm{S}^5 \,/\, \mathrm{SLAG}_3 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{T}^2$	$\mathrm{AdS}_2 \times \mathrm{S}^5 \,/\, \mathrm{SLAG}_3 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{S}^2$
4.7.14	$\mathrm{AdS}_2 \times \mathrm{S}^3 \,/\, \mathrm{H}^3 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^2$	$\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{H}^2 \times \mathrm{S}^2 \times \mathrm{S}^2$
4.7.17	$\mathrm{AdS}_2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{S}^2 \times \mathrm{T}^3$	$\mathrm{AdS}_2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^1$ The orthogonal symmetry of the T^3 component means this limit is the full F-moduli space.

7.5 Supersymmetric backgrounds

Table 7.4 exhausts all symmetric M-theory backgrounds $(\mathcal{M}, g, F)$ (i.e. $(\mathcal{M}, g)$ a symmetric space and F an invariant closed 4-form subject to the field equations) preserving some supersymmetry. However, we have *not* computed the full supersymmetry moduli space, so there may be additional supersymmetric solutions for these geometries.

Table 7.4: Supersymmetric M-theory backgrounds (referencing [33])

- 3.3: $\mathbb{R}^{10,1}$, The Minkowski vacuum.
  Maximally supersymmetric.
- 3.5: $\mathrm{CW}(\lambda)_{d>2} \times \mathbb{R}^{11-d}$
  These backgrounds are known to be supersymmetric with at least 16 supersymmetries.
- 4.2 and 4.5.1: $\mathrm{AdS}_7 \times \mathrm{S}^4$ and $\mathrm{AdS}_4 \times \mathrm{S}^7$
  The well-known maximally supersymmetric Freund-Rubin backgrounds.
- 4.6.3: $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{T}^2$ and $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^5$
  Their dimensional reductions $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ and $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ along S^1 to $D = 10$ type IIA supergravity are known to admit 16 supersymmetries. Thus, the $D = 11$ geometries admit at least 16 supersymmetries. On the other hand, running the algorithm for these geometries shows directly that there are at most 16 linearly independent spinors annihilated by $\kappa \subset \mathfrak{hol}$.
- 4.6.4 and 4.7.9: $\mathrm{AdS}_{2,3} \times \mathrm{S}^{3,2} \times \mathrm{T}^6$
  These backgrounds are known to admit supersymmetries for $F = f\nu \wedge \omega$, where ν is the volume form of the 2-dimensional factor and ω denotes the Kähler form on T^6. The dimensional reduction $\mathrm{AdS}_{2,3} \times S^{3,2}$ to $D = 5$ supergravity is known to admit 8 supersymmetries. With the same argument as above, also the $D = 11$ geometries admit 8 supersymmetries.
- 4.6.4: $\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^4$
  This background is known to admit 8 supersymmetries for $F = f_0(\nu_1 \wedge \nu_2 + \sqrt{f_1}\nu_1 \wedge \chi_\mathrm{i} + \sqrt{1-f_1}\nu_2 \wedge \chi_\mathrm{r})$ where $\nu_{1,2}$ are the two sphere volume forms and $\chi_{\mathrm{r,i}}$ the real and imaginary parts of the holomorphic 2-form of the T^4.
- 4.7.14: $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{S}^2 \,/\, \mathrm{H}^2 \times \mathrm{T}^4$
  These backgrounds are known to admit 8 supersymmetries for $F = f_0(\nu_0 \wedge \nu_2 + \sqrt{f_1}\nu_2 \wedge \chi_\mathrm{r} + \sqrt{1+f_1}\nu_0 \wedge \chi_\mathrm{i})$ where ν_0 is the AdS volume form, ν_2 the $\mathrm{S}^2 \,/\, \mathrm{H}^2$ volume form, and $\chi_{\mathrm{r,i}}$ the real and imaginary parts of the holomorphic 2-form of the T^4
- 4.7.17: $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{T}^5$
  This background is known to admit 8 supersymmetries for $F = f_0(\nu_0 \wedge (f_2 \mathrm{d}\vartheta^{34} + f_3 \mathrm{d}\vartheta^{35} - \mathrm{d}\vartheta^{12}) - \sqrt{f_1}\nu_1 \wedge (f_2 \mathrm{d}\vartheta^{24} + f_3 \mathrm{d}\vartheta^{25} + \mathrm{d}\vartheta^{13}) - \sqrt{1-f_1}\nu_2 \wedge (f_2 \mathrm{d}\vartheta^{14} + f_3 \mathrm{d}\vartheta^{15} - \mathrm{d}\vartheta^{23}))$ where ν_0 is the AdS volume form and $\nu_{1,2}$ the two sphere volume forms.
- 4.7.17: $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{T}^7$
  This background is known to admit 8 supersymmetries for $F = f_0(\nu_0 \wedge \chi_\mathrm{r} \pm \nu_1 \wedge \chi_\mathrm{i}) + f_1(\nu_0 \wedge \chi_\mathrm{i} \mp \nu_1 \wedge \chi_\mathrm{r})$ where ν_0 is the AdS volume form, ν_1 the sphere volume form, and $\chi_{\mathrm{r,i}}$ the real and imaginary parts of the holomorphic 2-form of a T^4 inside the T^7.

7.6 Summary

It is not particularly surprising to see that all symmetric M-theory background geometries admitting supersymmetry are already known and they are all anti-de Sitter, sphere, flat factor products or Cahen-Wallach pp-waves. We have not computed the supersymmetry moduli space for a number of these geometries due to purely computational complexity but a further analysis could reveal their full supersymmetry moduli spaces.

This technique can be applied to the symmetric type IIB backgrounds of chapter 6 although the dilaton component of the Killing spinor equations (equation (4.16)) will additionally need to be dealt with; but this simply becomes another algebraic constraint to satisfy.

Chapter 8

Conclusion

We have seen the strong homogeneity theorem constructed and applied to the majority of Poincaré supergravity theories. In fact, we have so far shown that it applies to all Poincaré supergravities apart from those with $N = 3, 5, 6$ although we presume it will apply in general. This means that any background of such a theory that preserves more than half the maximum amount of supersymmetry is necessarily locally homogeneous and so knowledge of the background at a point determines the background since supergravity backgrounds are not in general complete. This reduces the problem of classifying highly supersymmetric supergravity backgrounds to one of classifying highly supersymmetric *homogeneous* backgrounds – essentially we may use homogeneity to aid in reducing the classification problem.

There are two clear natural refinements for describing geometrical simplicity that we may further apply, that of local reductive homogeneity and local symmetry. There are non-locally-symmetric backgrounds with $1/2 < \nu < 3/4$ but a possible threshold for local reductive homogeneity is perhaps less clear. Patrick Meessen's symmetry conjecture, that backgrounds preserving more than $3/4$ of the maximum supersymmetry are locally symmetric is open and although the symmetric algebra does not in general appear to be *explicitly* generated by the Killing spinors, a less direct mechanism may be at play.

We have seen that using the classification of Lorentzian symmetric spaces, we may classify symmetric supergravity backgrounds and determine their moduli spaces entirely apart from cases where the computational complexity of the resultant system of polynomials stumps us. This was done previously for M-theory backgrounds, and we have here completed a classification for $D = 10$ type IIB backgrounds. We have also seen that there is a purely algebraic algorithm for computing the fraction of supersymmetry that such backgrounds preserve and using this we have classified all supersymmetric symmetric M-theory backgrounds. We were not able to compute the full supersymmetry moduli spaces in general, again due to computational complexity but determined which backgrounds were supersymmetric and which were not. This algorithm can easily be applied with minimal modification to our classification of $D = 10$ type IIB backgrounds.

Appendices

Appendix A

Nomenclature

Symbol	Description
$\mathfrak{e}_n$	A (pseudo-)orthonormal basis
e^x	An exponential
e	The identity element of a group
o	The neutral element of a coset manifold. For $\mathcal{M} = \mathcal{G}/\mathcal{K}$ with neutral element $e \in \mathcal{G}$, then $o = e\mathcal{K} = \mathcal{K}$
$\hookrightarrow$	Injection
$\twoheadrightarrow$	Surjection
$P^{(n)}$	An n-form
ι_X	Interior product
$X^\flat$	The musical isomorphism $\flat : \ V \to V^*$
$\varkappa_i$	A sign convention, $\varkappa_i = \pm 1$
$A \cdot \psi$	Clifford multiplication of A with ψ
Γ_a	Gamma matrix of a Clifford algebra representation
γ_i	Auxiliary gamma matrix of a symplectic Majorana Clifford algebra representation
σ_i	Pauli matrices with $i = 0 \ldots 3$ and $\sigma_0 = \mathbb{1}_2$
$\mathbb{P}(\pm 1)$	Chiral spinor projector
∇	The Levi-Civita connection
R	Ricci scalar curvature
Ric	Ricci tensor
dvol	Signed volume element
$\star A$	Hodge dual of A
$\triangle$	Laplacian
$[\![\bullet, \bullet]\!]$	Pinor inner product
$(\!(\bullet, \bullet)\!)$	Lorentzian inner product
$\langle\!\langle \bullet, \bullet \rangle\!\rangle_{\mathfrak{m}}$	An Ad-invariant bilinear form on $\mathfrak{m}$

$\langle A, B\rangle$	Inner product on differential forms, $\langle A, B\rangle \operatorname{dvol} = A \wedge *B$
$[\bullet, \bullet]$	Commutator or supercommutator
$\mathrm{I}(\mathcal{M})$	The isometry group of a manifold $\mathcal{M}$
$\mathfrak{g}, \mathfrak{k} \ldots$	The Lie algebra of a Lie group $\mathcal{G}, \mathcal{K} \ldots$
$\mathrm{G}^+_{\mathbb{R}}(p, n)$	The Grassmannian of oriented real p-planes in $\mathbb{R}^n$
$\mathrm{G}_{\mathbb{C}}(p, n)$	The Grassmannian of complex p-planes in $\mathbb{C}^n$
SLAG_n	The Grassmannian of special Lagrangian planes in $\mathbb{C}^n$
ASSOC	The Grassmannian of associative 3-planes in $\mathbb{R}^7$
$\mathcal{G}^+$	The identity component of the Lie group $\mathcal{G}$
$\mathrm{C}\ell(s, t)$	The Clifford algebra of signature (s, t)
$\mathrm{C}\ell(V, q)$	The Clifford algebra with real vector space V and symmetric bilinear form q
$\mathrm{C}\ell^\times$	The Clifford group

Appendix B

Clifford algebra and spinors

We require an understanding of Clifford algebras, pinors, spinors, and the Majorana and symplectic Majorana constraints and so we will briefly cover some of the necessary definitions and results that we will need. Useful references on this topic are [95, 96, 97].

B.1 Definition

Given a real vector space V with a symmetric bilinear form[1] η, we may define the (real) Clifford algebra[95] $\mathrm{C}\ell(V,\eta)$. The bilinear form has signature (s,t) (with $d = s + t$, and s and t the number of positive and negative eigenvalues respectively of η). We call the resulting Clifford algebra $\mathrm{C}\ell(s,t)$ and it is defined as the associative unital algebra generated by x, y subject to the following relations (note the sign):

$$x \cdot y + y \cdot x = -2\eta(x,y)\ . \tag{B.1}$$

B.2 Automorphisms

We denote by $x \to \tilde{x}$ the canonical automorphism of the Clifford algebra induced by the isometry $x \mapsto -x$ on V. This automorphism decomposes the Clifford algebra into the even and odd subalgebras $\mathrm{C}\ell(s,t) = \mathrm{C}\ell(s,t)_{\mathrm{Even}} \oplus \mathrm{C}\ell(s,t)_{\mathrm{Odd}}$. This means that Clifford algebras are superalgebras. The canonical automorphism's action on an element of the Clifford algebra x of rank n is clearly $x_n \mapsto (-1)^n x_n$.

We denote by $x \to \check{x}$ the anti-automorphism of the Clifford algebra induced by the anti-automorphism of $\otimes V$ defined by reversing the order of a simple product. This is called the check involution and its action on an element of the Clifford algebra x of rank n is $x_n \mapsto (-1)^{n(n-1)/2} x_n$.

[1] Or equivalently a quadratic form.

We denote by $x \to \hat{x}$ the anti-automorphism of the Clifford algebra induced by the composition of $x \to \tilde{x}$ and $x \to \check{x}$. This is called the hat involution and its action on an element of the Clifford algebra x of rank n is $x_n \mapsto (-1)^{n(n+1)/2} x_n$.

B.3 Groups

The Clifford algebra contains three groups which are of interest: We start with the group of invertible elements (the Clifford group) $\mathrm{C}\ell^{\times}(s,t)$ which must of course contain all other groups. Then the Clifford group has two subgroups of more direct consequence:

- The Pin group (unital simple products):

$$\mathrm{Pin}(s,t) = \{y \in \mathrm{C}\ell(s,t) : \ y = \prod_n x_n \text{ with } x_n^2 = \pm 1\} \tag{B.2}$$

- The Spin group (unital simple products stabilised by the canonical automorphism):

$$\begin{aligned} \mathrm{Spin}(s,t) &= \{y \in \mathrm{C}\ell(s,t) : \ y = \prod_n x_n \text{ with } x_n^2 = \pm 1 \text{ and } n \bmod 2 = 0\} \\ &= \mathrm{Pin}(s,t) \cap \mathrm{C}\ell(s,t)_{\mathrm{Even}} \end{aligned} \tag{B.3}$$

B.4 Representations

The Clifford algebra (as we have defined it) is a real associative algebra but we are interested in *complex* representations and so a representation[2] of a Clifford algebra $\mathrm{C}\ell(s,t)$ on a complex vector space $\mathcal{S}$ is a homomorphism

$$\rho : \mathrm{C}\ell(s,t) \to \mathrm{End}(\mathcal{S}) \ . \tag{B.4}$$

We also then naturally have representations of the Pin and Spin subgroups of the Clifford algebra, elements of which are *pinors* and *spinors* respectively.

B.4.0.1 Matrix ring isomorphisms

Clifford algebras themselves are (non-canonically) algebra-isomorphic to matrix rings or direct sums of matrix rings over a field $\mathbb{K}$ as an algebra over $\mathbb{R}$. The matrix dimension and field $\mathbb{K}$ depend upon the signature mod 8 via Bott periodicity [98]. For a Clifford algebra $\mathrm{C}\ell(s,t)$ and $d = s + t$ we have the algebra isomorphisms [96],

[2] A Clifford module

$(s-t) \bmod 8$	$\mathrm{C}\ell(s,t) \cong$ Matrix Ring	n
$0, 6$	$\mathrm{Mat}_n(\mathbb{R})$	$2^{d/2}$
$2, 4$	$\mathrm{Mat}_n(\mathbb{H})$	$2^{(d-2)/2}$
$1, 5$	$\mathrm{Mat}_n(\mathbb{C})$	$2^{(d-1)/2}$
3	$\mathrm{Mat}_n(\mathbb{H}) \oplus \mathrm{Mat}_n(\mathbb{H})$	$2^{(d-3)/2}$
7	$\mathrm{Mat}_n(\mathbb{R}) \oplus \mathrm{Mat}_n(\mathbb{R})$	$2^{(d-1)/2}$

Each matrix ring $\mathrm{Mat}_n(\mathbb{K})$ as an $\mathbb{R}$-algebra has an irreducible representation $\rho : \mathrm{Mat}_n(\mathbb{K}) \to \mathrm{End}(\mathbb{K}^n)$ and as they are simple, it is unique. As such we have either one or two[3] (in even and odd dimensions respectively) unique irreducible representation(s) of a Clifford algebra $\mathrm{C}\ell(s,t)$ corresponding to the particular matrix ring algebra isomorphism. Given a representation of $\mathrm{C}\ell(s,t)$ we automatically get a representation of $\mathrm{Pin}(s,t)$ and so these representations are also pinor representations. We also automatically get a representation of $\mathrm{Spin}(s,t)$ although the representation may not remain irreducible under the spin group.

Turning now to spinors and so to the even subalgebra, we have the isomorphisms [96],

$$\begin{aligned} \mathrm{C}\ell(s+1,t)_{\mathrm{Even}} &\cong \mathrm{C}\ell(s,t) \\ \mathrm{C}\ell(s,t+1)_{\mathrm{Even}} &\cong \mathrm{C}\ell(t,s) \ , \end{aligned} \tag{B.5}$$

which lead to the induced matrix ring algebra isomorphisms for the even Clifford subalgebras,

$(s-t) \bmod 8$	$\mathrm{C}\ell(s,t)_{\mathbf{Even}} \cong$ Matrix Ring	n
$1, 7$	$\mathrm{Mat}_n(\mathbb{R})$	$2^{(d-1)/2}$
$3, 5$	$\mathrm{Mat}_n(\mathbb{H})$	$2^{(d-3)/2}$
$2, 6$	$\mathrm{Mat}_n(\mathbb{C})$	$2^{(d-2)/2}$
4	$\mathrm{Mat}_n(\mathbb{H}) \oplus \mathrm{Mat}_n(\mathbb{H})$	$2^{(d-4)/2}$
0	$\mathrm{Mat}_n(\mathbb{R}) \oplus \mathrm{Mat}_n(\mathbb{R})$	$2^{(d-2)/2}$

As such we have either one or two (in odd and even dimensions respectively) unique irreducible representation(s) of an even Clifford subalgebra $\mathrm{C}\ell(s,t)_{\mathrm{Even}}$ corresponding to the particular matrix ring algebra isomorphism. Thus we have also have either one or two unique irreducible spinor representation(s) of $\mathrm{Spin}(s,t)$.

B.4.1 Real and quaternionic structures

We have chosen to work with complex representations but we see that the representations may naturally be real or quaternionic. This is manifested by either a real or quaternionic structure on the complex representation:

Let V be a complex vector space. Then $\varphi : V \to V$ is a *real* structure iff:

- φ is antilinear: $\forall \lambda \in \mathbb{C}$ and $v \in V$, $\varphi(\lambda v) = \lambda^* \varphi(v)$

[3] For $\mathrm{Mat}_n(\mathbb{C})$ we also have the inequivalent conjugate representation.

- $\varphi^2 = \mathbb{1}$.

As an involution, a real structure has two eigenvalues ± 1 and so decomposes V into two eigenspaces $V = V_+ \oplus V_-$ and these eigenspaces are real because the structure map is not complex linear but antilinear. A real structure φ is thus just an abstraction of complex conjugation.

Now, let V again be a complex vector space. Then $\varphi : V \to V$ is a *quaternionic* structure iff:

- φ is antilinear: $\forall \lambda \in \mathbb{C}$ and $v \in V$, $\varphi(\lambda v) = \lambda^* \varphi(v)$
- $\varphi^2 = -\mathbb{1}$.

Since $\varphi^2 = -\mathbb{1}$ and $\varphi(iv) = -i\varphi(v)$, the quaternionic structure defines a left quaternionic j-action on V and so a left $\mathbb{H}$-action on V: If $\mathbb{H} \ni q = a + bj$ for $a, b \in \mathbb{C}$, we have $qv = av + b\varphi(v)$.

Of course, we are interested in pinor and spinor representations and so these structures must be pin- or spin-invariant. For example, in signature $(s - t) \bmod 8 = 1$ we only have a spin-invariant real structure and so only the spinor representation is real.

B.4.2 Majorana and symplectic Majorana spinors

With a spin-invariant real structure on our representation we can always choose a basis in which this structure acts either as the identity (in which case we have concretely real *Majorana* spinors) or as the imaginary identity (in which case we have purely imaginary *pseudo-Majorana* spinors). For example, $D = 11$ supergravity may equally well be formulated with a spinor bundle modelled upon $\mathrm{C}\ell(10, 1)$ [19] in which case we have (purely imaginary) pseudo-Majorana spinors or on $\mathrm{C}\ell(1, 10)$ [35, 40] in which case we have (concretely real) Majorana spinors.

Unfortunately, in signatures $(s - t) \bmod 8 = 3, 4, 5$ we have no spin-invariant real structure to work with. However, in these signatures we do always have spin-invariant quaternionic structures. Let us start with a representation V of $\mathrm{C}\ell(s, t)$ with a quaternionic structure φ_1 and take some other auxiliary representation W of a group G that also has a quaternionic structure φ_2. We may construct the tensor product of representations $V \otimes W$ which is a representation of $\mathrm{C}\ell(s, t) \times G$. We then have a new structure $\varphi_1 \otimes \varphi_2$ satisfying:

- $\varphi_1 \otimes \varphi_2$ is antilinear: $\forall \lambda \in \mathbb{C}$, $v \in V$, $w \in W$,

  $(\varphi_1 \otimes \varphi_2)(\lambda v \otimes w) = (\varphi_1(\lambda v) \otimes \varphi_2(w)) = (\lambda^* \varphi_1(v) \otimes \varphi_2(w)) = \lambda^* (\varphi_1 \otimes \varphi_2)(v \otimes w)$
- $(\varphi_1 \otimes \varphi_2)^2 = (\varphi_1^2) \otimes (\varphi_2^2) = (-\mathbb{1}_V) \otimes (-\mathbb{1}_W) = \mathbb{1}_{V \otimes W}$.

As such, the tensor product of two representations each with quaternionic structures is a new representation with a real structure.

Using this construction, if we have a spinor representation with no real structure, then we may combine it with an auxiliary quaternionic representation to create a real structure on a new reducible representation. Given a choice of basis in which the new real structure acts as either the identity or the imaginary identity, we then have concretely real *symplectic Majorana* spinors

or purely imaginary *symplectic pseudo-Majorana* spinors respectively. See appendices B.11.3 and B.11.4 for concrete examples of reducible symplectic Majorana representations.

B.4.3 Gamma matrices

A matrix representation of the Clifford algebra is concretely furnished by the gamma matrices Γ_n,

$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2\varkappa_1\eta_{ab}\mathbb{1} \ , \tag{B.6}$$

where $\varkappa_1 = \pm 1$ is a sign convention typically (but not strictly) chosen to be $+1$ in the physics literature and -1 in the mathematics literature including [95]. Note that this does not change our definition of the Clifford algebra equation (B.1) but merely enables us to make contact with the conventions in the literature.

We define totally antisymmetric products of gamma matrices as

$$\Gamma_{a_1\ldots a_n} := \Gamma_{[a_1}\ldots\Gamma_{a_n]} \ . \tag{B.7}$$

B.5 Exterior algebra isomorphism

We have the vector space isomorphism $\Lambda(V) \cong \mathrm{C}\ell(V)$ concretely defined by

$$\begin{aligned} \Lambda(V) &\to \mathrm{C}\ell(V) \\ \frac{1}{n!}X^{a_1\ldots a_n}\mathfrak{e}_{a_1}\wedge\ldots\wedge\mathfrak{e}_{a_n} &\mapsto X^{a_1\ldots a_n}\Gamma_{a_1\ldots a_n} \ . \end{aligned} \tag{B.8}$$

This isomorphism globalises to a vector bundle isomorphism between the exterior bundle and the Clifford bundle. A spinor bundle is a bundle of Clifford modules by construction and so we have a natural action of sections of the exterior bundle on sections of the spinor bundle through this vector bundle isomorphism and the natural Clifford action on the spinor bundle.

Using this isomorphism, we have the following relations between the interior and exterior products in the exterior algebra and the Clifford product in the Clifford algebra. So, for a vector $X \in V^*$ and n-form $P \in \Lambda^n(V)$ we have,

$$\iota_X P = \tfrac{1}{2}\varkappa_1\left(X^\flat\cdot P - (-1)^n P\cdot X^\flat\right) \tag{B.9}$$

$$X^\flat\wedge P = \tfrac{1}{2}\left(X^\flat\cdot P + (-1)^n P\cdot X^\flat\right) \ . \tag{B.10}$$

B.6 Inner products

The two involutions of the Clifford algebra induce two pinor inner products on a pinor representation $\mathcal{P}$ such that

$$[\![x \cdot \epsilon_1, \epsilon_2]\!] = [\![\epsilon_1, \dot{x} \cdot \epsilon_2]\!] \,, \tag{B.11}$$

where $\epsilon_{1,2} \in \mathcal{P}$ and $\dot{x}$ denotes either the hat or the check involution of $x \in \mathrm{C}\ell(V) \cong \mathrm{End}(\mathcal{P})$.

The inner products can take the form of any of the eight types of inner product on $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ depending on the matrix ring algebra isomorphism of the Clifford algebra:

- $\mathbb{R}$: symmetric or symplectic
- $\mathbb{C}$: symmetric, symplectic, hermitian symmetric, or hermitian symplectic
- $\mathbb{H}$: hermitian symmetric or hermitian symplectic

Also, if there are two inequivalent pinor representations, an inner product may intertwine them and thus only be defined on their direct sum.

B.7 Identities

B.7.1 Product identities

Rearranging the Clifford algebra definition on gamma matrices equation (B.6) yields

$$\Gamma_a \Gamma_b = 2\varkappa_1 \eta(a,b) \mathbb{1} - \Gamma_b \Gamma_a . \tag{B.12}$$

This rearrangement can be iterated to yield the general product formula,

$$\Gamma^{a_1 \ldots a_n} \Gamma_{b_1 \ldots b_m} = \sum_{t=0}^{\min(n,m)} t! \binom{n}{t} \binom{m}{t} \varkappa_1{}^t (-1)^{t(2n+t+1)/2} \delta^{[a_1}_{[b_1} \ldots \delta^{a_t}_{b_t} \Gamma^{a_{t+1} \ldots a_n]}{}_{b_{t+1} \ldots b_m]} \,. \tag{B.13}$$

B.7.2 Volume identities

We define the Clifford volume element of signature $d = s + t$ to be

$$\Gamma_{\mathrm{vol}} := \Gamma^{1 \ldots d} . \tag{B.14}$$

As a consequence, we have

$$\Gamma_{a_1 \ldots a_n} \Gamma_{\mathrm{vol}} = (-1)^{n(d-1)} \Gamma_{\mathrm{vol}} \Gamma_{a_1 \ldots a_n} \,. \tag{B.15}$$

Upon manipulation of equation (B.13), noting that only the $t = n$ term is non-zero, we have

the more general identity

$$\Gamma_{a_1\ldots a_n}\Gamma_{\rm vol} = \varkappa_1{}^n(-1)^{n(n-1)/2}\tfrac{1}{(d-n)!}\epsilon_{a_1\ldots a_n,b_1\ldots b_{d-n}}\Gamma^{b_1\ldots b_{d-n}} \ . \tag{B.16}$$

Using the index formulation of the Hodge star operator on an n-form A,

$$(*A)_{b_1\ldots b_{d-n}} = \tfrac{1}{(d-n)!}A^{a_1\ldots a_n}\epsilon_{a_1\ldots a_n,b_1\ldots b_{d-n}} \ , \tag{B.17}$$

we arrive at the identity,

$$A_{a_1\ldots a_n}\Gamma^{a_1\ldots a_n}\Gamma_{\rm vol} = \varkappa_1{}^n(-1)^{n(n-1)/2}(*A)_{b_1\ldots b_{d-n}}\Gamma^{b_1\ldots b_{d-n}} \ . \tag{B.18}$$

B.8 (Anti-)self-dual forms

In dimensions $(s-t) \bmod 4 = 0$ we may construct (anti-)self-dual forms $\star A^{\varkappa_2} = \varkappa_2 A^{\varkappa_2}$. In these dimensions the eigenvalues of $\Gamma_{\rm vol}$ are real and chiral spinors satisfy the relation

$$\Gamma_{\rm vol}\epsilon^{\varkappa_3} = \varkappa_3\epsilon^{\varkappa_3} \ . \tag{B.19}$$

We thus have the chirality projectors

$$\mathbb{P}(\varkappa_3) = \tfrac{1}{2}\left(\Gamma_{\rm vol} + \varkappa_3\mathbb{1}\right) \ , \tag{B.20}$$

where $\mathbb{P}(\varkappa_3)$ projects the $\varkappa_3$ chirality subspace.

Then, for an (anti-)-self-dual n-form, equation (B.18) becomes

$$A^{\varkappa_2}_{a_1\ldots a_n}\Gamma^{a_1\ldots a_n}\Gamma_{\rm vol} = \varkappa_2\varkappa_1{}^n(-1)^{n(n-1)/2}A^{\varkappa_2}_{a_1\ldots a_n}\Gamma^{a_1\ldots a_n} \ , {}^{\varkappa_2}A_{a_1\ldots a_n} \tag{B.21}$$

and so

$$A^{\varkappa_2}_{a_1\ldots a_n}\Gamma^{a_1\ldots a_n}\mathbb{P}(\varkappa_2\varkappa_1{}^n(-1)^{n(n-1)/2+1}) = 0 \ . \tag{B.22}$$

We see that an (anti-)self-dual n-form thus annihilates $\varkappa_2\varkappa_1{}^n(-1)^{n(n-1)/2+1}$ chirality spinors. Two relevant examples in Lorentzian signature:

- $(s,t) = (5,1)$: we have (anti-)self-dual 3-forms and, if using Clifford algebra sign $\varkappa_1 = -1$ we have

  $$\varkappa_2\varkappa_1{}^n(-1)^{n(n-1)/2+1} = \varkappa_2\cdot(-1)^3\cdot(-1)^{3(3-1)/2+1} = -\varkappa_2 \ , \tag{B.23}$$

  meaning that in this case self-dual 3-forms annihilate negative chirality spinors and anti-self-dual 3-forms annihilate positive chirality spinors.

- $(s,t) = (1,9)$: we have self-dual 5-forms and, if using Clifford algebra sign $\varkappa_1 = -1$ we

have

$$\varkappa_2 {\varkappa_1}^n (-1)^{n(n-1)/2+1} = \varkappa_2 \cdot (-1)^5 \cdot (-1)^{5(5-1)/2+1} = \varkappa_2 \ , \tag{B.24}$$

meaning that in this case self-dual 5-forms annihilate positive chirality spinors and anti-self-dual 5-forms annihilate negative chirality spinors.

B.9 Covariant spinor derivative

We derive the form of the covariant spinor derivative so as to make clear how the sign adopted in the definition of the Clifford algebra $\varkappa_1$ enters the definition.

We first suppose the covariant spinor derivative along a vector X takes the form, for unknown θ in the spin subalgebra,

$$\nabla_X \epsilon = \partial_X \epsilon + \theta(X) \cdot \epsilon \ . \tag{B.25}$$

We then require the Leibniz condition be compatible with the Clifford action and so,

$$\nabla_X (W \cdot \epsilon) = (\nabla_X W) \cdot \epsilon + W \cdot \nabla_X \epsilon \ . \tag{B.26}$$

Substituting in our supposed form yields the condition, for any vectors W,

$$\theta(X) \cdot W - W \cdot \theta(X) \stackrel{!}{=} \nabla_X W - \partial_X W \ . \tag{B.27}$$

We may then explicitly compute θ,

$$\begin{aligned} \theta(X)_{ab} W_c (\Gamma^{ab}\Gamma^c - \Gamma^c \Gamma^{ab}) &= \theta(X)_{ab} W_c (-4\varkappa_1 \eta^{c[a}\Gamma^{b]}) \\ &= -4\varkappa_1 \theta(X)_{ab} W^a \Gamma^b \\ &\stackrel{!}{=} X^c (\partial_c W_a \Gamma^a + {\omega_c}^{ab} W_b \Gamma_a - \partial_c W_a \Gamma^a) \\ &= X^c {\omega_c}^{ab} W_b \Gamma_a \ , \end{aligned} \tag{B.28}$$

where we have used the definition of the covariant derivative of a vector W in terms of the spin connection ω,

$$\nabla_\mu W^a = \partial_c W^a + {\omega_c}^{ab} W_b \ . \tag{B.29}$$

Thus we have

$$\theta(X)^{ab} = \tfrac{1}{4}\varkappa_1 X^c {\omega_c}^{ab} \ , \tag{B.30}$$

and so

$$\nabla_X \epsilon = \partial_X \epsilon + \tfrac{1}{4}\varkappa_1 \omega_X \cdot \epsilon \ . \tag{B.31}$$

B.10 Spinorial Lie derivative

We derive the form of the spinorial Lie derivative [39] so as to make clear how the sign adopted in the definition of the Clifford algebra $\varkappa_1$ enters the definition.

We first suppose the spinorial Lie derivative along a Killing vector K takes the form, for unknown θ in the Spin subalgebra,

$$\mathcal{L}_K\epsilon = \nabla_K\epsilon + \theta(K)\cdot\epsilon\ . \tag{B.32}$$

We then require the Leibniz condition be compatible with the Clifford action and so,

$$\mathcal{L}_K(W\cdot\epsilon) = [K,W]\cdot\epsilon + W\cdot\mathcal{L}_K\epsilon\ . \tag{B.33}$$

Substituting in our supposed form yields the condition, for any vector W,

$$\theta(K)\cdot W - W\cdot\theta(K) \stackrel{!}{=} -\nabla_W K\ . \tag{B.34}$$

We may then explicitly compute θ,

$$\begin{aligned} -W^a\Gamma^b\nabla_a K_b &\stackrel{!}{=} \theta(K)_{ab}W_c(\Gamma^{ab}\Gamma^c - \Gamma^c\Gamma^{ab}) \\ &= \theta(K)_{ab}W_c(-4\varkappa_1\eta^{c[a}\Gamma^{b]}) \\ &= -4\varkappa_1\theta(K)_{ab}W^a\Gamma^b\ , \end{aligned} \tag{B.35}$$

whence

$$\theta(K)_{ab} = \tfrac{1}{4}\varkappa_1\nabla_a K_b\ , \tag{B.36}$$

and so

$$\mathcal{L}_K\epsilon = \nabla_K\epsilon + \tfrac{1}{4}\varkappa_1(\nabla K)\cdot\epsilon\ . \tag{B.37}$$

B.11 Explicit realisations of pinor representations

B.11.1 $D = 11$

In $D = 11$ we use sign conventions (see appendices B.4.3 and C) $(\varkappa_0,\varkappa_1) = (-1,-1)$ or $(\varkappa_0,\varkappa_1) = (+1,+1)$ meaning we work with $\mathrm{C}\ell(1,10)$. We use a Majorana representation constructed as

$\Gamma_0 = \mathrm{i}\sigma_1\otimes\sigma_0\otimes\sigma_0\otimes\sigma_0\otimes\sigma_2$	$\Gamma_1 = \sigma_1\otimes\sigma_0\otimes\sigma_0\otimes\sigma_3\otimes\sigma_3$	$\Gamma_2 = \sigma_1\otimes\sigma_0\otimes\sigma_0\otimes\sigma_1\otimes\sigma_3$
$\Gamma_3 = \sigma_2\otimes\sigma_3\otimes\sigma_2\otimes\sigma_0\otimes\sigma_0$	$\Gamma_4 = \sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_0\otimes\sigma_0$	$\Gamma_5 = \sigma_2\otimes\sigma_2\otimes\sigma_0\otimes\sigma_0\otimes\sigma_0$
$\Gamma_6 = \sigma_1\otimes\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_3$	$\Gamma_7 = \sigma_1\otimes\sigma_0\otimes\sigma_2\otimes\sigma_2\otimes\sigma_3$	$\Gamma_8 = \sigma_1\otimes\sigma_2\otimes\sigma_3\otimes\sigma_2\otimes\sigma_3$
$\Gamma_9 = \sigma_1\otimes\sigma_0\otimes\sigma_0\otimes\sigma_0\otimes\sigma_1$	$\Gamma_{10} = \sigma_3\otimes\sigma_0\otimes\sigma_0\otimes\sigma_0\otimes\sigma_0$.	

The gamma matrices are all real and

$$\Gamma_{\text{vol}} := \Gamma^{0\ldots10} = \mathbb{1} \ . \tag{B.38}$$

B.11.2 $D = 10$

In $D = 10$ we use sign conventions $(\varkappa_0, \varkappa_1) = (-1, -1)$ meaning we work with $\mathrm{C}\ell(1, 9)$. We use a Majorana-Weyl representation constructed as

$$\begin{array}{lll}
\Gamma_0 = \mathrm{i}\sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_2 & \Gamma_1 = \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_3 \otimes \sigma_3 & \Gamma_2 = \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_1 \otimes \sigma_3 \\
\Gamma_3 = \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes \sigma_0 & \Gamma_4 = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_0 \otimes \sigma_0 & \Gamma_5 = \sigma_2 \otimes \sigma_2 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \\
\Gamma_6 = \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_3 & \Gamma_7 = \sigma_1 \otimes \sigma_0 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_3 & \Gamma_8 = \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_3 \\
\Gamma_9 = \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_1 \ . &&
\end{array}$$

The gamma matrices are all real and

$$\Gamma_{\text{vol}} := \Gamma^{0\ldots9} = \sigma_3 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \ . \tag{B.39}$$

B.11.3 $D = 6$, $(1, 0)$

In $D = 6$, $(1, 0)$ we use sign conventions $(\varkappa_0, \varkappa_1) = (+1, -1)$ meaning we work with $\mathrm{C}\ell(5, 1)$. We use a symplectic Majorana-Weyl representation constructed as

$$\begin{array}{lll}
\Gamma_0 = \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_1 & \Gamma_1 = \mathrm{i}\sigma_1 \otimes \sigma_0 \otimes \sigma_2 \otimes \sigma_3 & \Gamma_2 = \mathrm{i}\sigma_2 \otimes \sigma_3 \otimes \sigma_0 \otimes \sigma_0 \\
\Gamma_3 = \mathrm{i}\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_0 & \Gamma_4 = \mathrm{i}\sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_2 & \Gamma_5 = \mathrm{i}\sigma_2 \otimes \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \ .
\end{array}$$

The gamma matrices are all real and

$$\Gamma_{\text{vol}} := \Gamma^{0\ldots5} = \sigma_3 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \ . \tag{B.40}$$

The auxiliary representation of $\mathrm{C}\ell(3, 0)$ is then

$$\begin{array}{lll}
\gamma_1 = \mathrm{i}\sigma_0 \otimes \sigma_0 \otimes \sigma_2 \otimes \sigma_0 & \gamma_2 = \mathrm{i}\sigma_3 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_3 & \gamma_3 = \mathrm{i}\sigma_3 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \ .
\end{array}$$

The auxiliary gamma matrices are all real and

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}\mathbb{1} \ , \tag{B.41}$$

$$\gamma_i = \gamma^i \ , \tag{B.42}$$

$$\gamma_{\text{vol}} := \gamma^{1\ldots3} = \mathbb{1} \ , \tag{B.43}$$

$$[\Gamma_a, \gamma_i] = 0 \ . \tag{B.44}$$

B.11.4 $D = 6$, $(2, 0)$

In $D = 6$, $(2, 0)$ we use sign conventions $(\varkappa_0, \varkappa_1) = (+1, -1)$ meaning we work with $\mathrm{C}\ell(5, 1)$. We use a symplectic Majorana-Weyl representation constructed as

$$\Gamma_0 = \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_1 \quad \Gamma_1 = \mathrm{i}\sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_2 \otimes \sigma_3 \quad \Gamma_2 = \mathrm{i}\sigma_2 \otimes \sigma_0 \otimes \sigma_3 \otimes \sigma_0 \otimes \sigma_0$$
$$\Gamma_3 = \mathrm{i}\sigma_2 \otimes \sigma_0 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_0 \quad \Gamma_4 = \mathrm{i}\sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_2 \quad \Gamma_5 = \mathrm{i}\sigma_2 \otimes \sigma_0 \otimes \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \ .$$

The gamma matrices are all real and

$$\Gamma_{\mathrm{vol}} := \Gamma^{0\ldots5} = \sigma_3 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \ . \tag{B.45}$$

The auxiliary representation of $\mathrm{C}\ell(0,5)$ is then

$$\gamma_1 = \sigma_0 \otimes \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \quad \gamma_2 = \sigma_0 \otimes \sigma_3 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \quad \gamma_3 = \sigma_0 \otimes \sigma_2 \otimes \sigma_0 \otimes \sigma_2 \otimes \sigma_0$$
$$\gamma_4 = \sigma_3 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_3 \quad \gamma_5 = \sigma_3 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \ .$$

The auxiliary gamma matrices are all real and

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}\mathbb{1} \ , \tag{B.46}$$
$$\gamma_i = -\gamma^i \ , \tag{B.47}$$
$$\gamma_{\mathrm{vol}} := \gamma^{1\ldots5} = \mathbb{1} \ , \tag{B.48}$$
$$[\Gamma_a, \gamma_i] = 0 \ . \tag{B.49}$$

B.11.5 $D = 4, \mathcal{N} = 1$

In $D = 4$, $\mathcal{N} = 1$ we use sign conventions $(\varkappa_0, \varkappa_1) = (+1, +1)$ meaning we work with $\mathrm{C}\ell(1,3)$. We use a Majorana representation constructed as

$$\Gamma_0 = \mathrm{i}\sigma_0 \otimes \sigma_2 \quad \Gamma_1 = \sigma_0 \otimes \sigma_1 \quad \Gamma_2 = \sigma_3 \otimes \sigma_3$$
$$\Gamma_3 = \sigma_1 \otimes \sigma_3 \ .$$

The gamma matrices are all real and

$$\Gamma_{\mathrm{vol}} := \Gamma^{0\ldots3} = \mathrm{i}\sigma_2 \otimes \sigma_3 \ . \tag{B.50}$$

B.11.6 $D = 11$ adapted to $\mathrm{AdS}_2 \times \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2$

Again we use sign conventions $(\varkappa_0, \varkappa_1) = (-1, -1)$ or $(\varkappa_0, \varkappa_1) = (+1, +1)$ meaning we work with $\mathrm{C}\ell(1,10)$. We use a (not concretely real) Majorana representation constructed as

$$\Gamma_0 = \mathrm{i}\sigma_3 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_1 \otimes \sigma_0 \quad \Gamma_1 = \sigma_0 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_0 \quad \Gamma_2 = \sigma_0 \otimes \sigma_3 \otimes \sigma_0 \otimes \sigma_3 \otimes \sigma_1$$
$$\Gamma_3 = \sigma_0 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \quad \Gamma_4 = \sigma_0 \otimes \sigma_0 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \quad \Gamma_5 = \sigma_0 \otimes \sigma_0 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1$$
$$\Gamma_6 = \sigma_0 \otimes \sigma_0 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \quad \Gamma_7 = \sigma_0 \otimes \sigma_2 \otimes \sigma_0 \otimes \sigma_2 \otimes \sigma_3 \quad \Gamma_8 = \sigma_0 \otimes \sigma_1 \otimes \sigma_0 \otimes \sigma_2 \otimes \sigma_3$$
$$\Gamma_9 = \sigma_2 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_1 \otimes \sigma_0 \quad \Gamma_{10} = \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_1 \otimes \sigma_0 \ .$$

All the volume forms act diagonally but the gamma matrices are now *not* all real.

Appendix C

Sign conventions

We have seen in appendix B.4.3 that we have a choice of sign $\varkappa_1 = \pm 1$ to make when constructing a gamma matrix representation of the Clifford algebra. There is yet another choice of sign available in the physics literature and this is the sign convention of the Lorentzian metric tensor. We can either pick it to be *mostly plus* or *mostly minus* corresponding to the sign of the majority of the eigenvalues of the metric tensor as a matrix. This is often historically called picking an east-coast or west-coast sign and there is discussion [99] over whether one or the other is preferable. We will denote this sign choice by $\varkappa_0 = \pm 1$ where $\varkappa_0 = +1$ denotes a mostly plus metric and $\varkappa_0 = -1$ mostly minus.

In most applications the choice of metric sign $\varkappa_0$ is irrelevant [100, 101] but the combination of $(\varkappa_0, \varkappa_1)$ imposes a choice of pinor representation [102] and this can have consequences because although change of metric sign is an isomorphism of spinor representations, the pinor representations are *not* isomorphic.

We will not delve into this any further, but will specify our sign conventions where necessary and in some cases try to present sign-agnostic results.

Appendix D

Lorentzian vector spaces

We recite a few simple facts about Lorentzian vector spaces.

D.1 Causal character

Taking a Lorentzian vector space V, let us denote its Lorentzian inner product by $(\!|\bullet, \bullet|\!)$ and choose a pseudo-orthonormal basis $\mathfrak{e}$ such that $-(\!|\mathfrak{e}_0, \mathfrak{e}_0|\!) = (\!|\mathfrak{e}_i, \mathfrak{e}_i|\!) = \varkappa_0$ and $(\!|\mathfrak{e}_0, \mathfrak{e}_i|\!) = (\!|\mathfrak{e}_i, \mathfrak{e}_j|\!) = 0$ for $i > 0 \neq j > 0$.

A vector $v \in V$ may be characterised as:

$$\begin{aligned} &\bullet \text{ Timelike:} && \varkappa_0(\!|v, v|\!) < 0 \\ &\bullet \text{ Null:} && \varkappa_0(\!|v, v|\!) = 0 \\ &\bullet \text{ Spacelike:} && \varkappa_0(\!|v, v|\!) < 0\ , \end{aligned} \tag{D.1}$$

and we add two further descriptions for utility:

$$\begin{aligned} &\bullet \text{ Causal:} && \varkappa_0(\!|v, v|\!) \leq 0 \\ &\bullet \text{ Anti-causal:} && \varkappa_0(\!|v, v|\!) \geq 0\ . \end{aligned} \tag{D.2}$$

D.2 Some results on null subspaces

Inner product of linearly independent null vectors

Let us take two linearly independent null vectors v_i with $i = 1, 2$. Without loss of generality, with respect to a pseudo-orthonormal basis we have $v_i = a_i \mathfrak{e}_0 + u_i$ with a_i real non-zero constants, and u_i spacelike vectors.

Let us suppose that $(\!|v_1, v_2|\!) = 0$. Thus $0 = (\!|v_1, v_2|\!) = \varkappa_0 \left(-a_1 a_2 + (\!|u_1, u_2|\!)\right)$ and so $(\!|u_1, u_2|\!) = a_1 a_2$. Also $0 = (\!|v_i, v_i|\!) = \varkappa_0 \left(-a_i^2 + (\!|u_i, u_i|\!)\right)$ and so $(\!|u_i, u_i|\!) = a_i^2$. But then we have Cauchy-Schwarz equality $|(\!|u_1, u_2|\!)|^2 = (\!|u_1, u_1|\!)(\!|u_2, u_2|\!)$ meaning that $u_2 = \lambda u_1$ for some non-zero λ.

Then $a_1 a_2 = (\!|u_1, u_2|\!) = \lambda(\!|u_1, u_1|\!) = \lambda a_1^2$ and so $a_2 = \lambda a_1$. But then $v_2 = \lambda a_1 \mathfrak{d}_0 + \lambda u_1 = \lambda v_1$ contradicting linear independence. Thus $(\!|v_1, v_2|\!) \neq 0$.

A totally null subspace is at most one-dimensional

Let us assume we have a totally null subspace with dimension greater than one. Then we have at least two linearly independent null vectors v_1 and v_2. From polarisation we have that $(\!|v_1 + v_2, v_1 + v_2|\!) = 2(\!|v_1, v_2|\!)$ but from the inner product of linearly independent null vectors we know that $(\!|v_1, v_2|\!) \neq 0$ and so $(\!|v_1 + v_2, v_1 + v_2|\!) \neq 0$ meaning $v_1 + v_2$ is not null and thus the subspace is not totally null. Thus a totally null subspace is at most one-dimensional.

The perpendicular complement of a totally null subspace contains only itself and spacelike vectors

The perpendicular complement of a subspace $W \subset V$ with respect to the Lorentzian inner product is defined as

$$\mathcal{W}^\perp = \{v \in V : (\!|v, w|\!) = 0 \; \forall \; w \in W\} \; , \tag{D.3}$$

but a totally null subspace W is at most one-dimensional and so if non-trivial is spanned by a single null vector w. Its perpendicular complement is thus

$$\mathcal{W}^\perp = \{v \in V : (\!|v, w|\!) = 0\} \; . \tag{D.4}$$

Now, $(\!|w, w|\!) = 0$ and so $w \in W^\perp$ but we have seen that for v_n a null vector linearly independent of w, $(\!|v_n, w|\!) \neq 0$ and so $v_n \notin W^\perp$.

Let us now take our null vector w and any timelike vector v_t. Without loss of generality, with respect to a pseudo-orthonormal basis we can write these as $w = a\mathfrak{d}_0 + w_s$ and $v_t = b\mathfrak{d}_0$, for a, b real non-zero constants. Then $(\!|v_t, w|\!) = \varkappa_0 ab \neq 0$ and so also $v_t \notin W^\perp$.

Thus we see that the perpendicular complement of a totally null subspace contains only itself and spacelike vectors.

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