The integral affine geometry of Lagrangian bundles

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Daniele Sepe*)

In memory of Angela Iannascolo (1919 - 2001)

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Abstract

In this thesis, a bundle $F \hookrightarrow (M, \omega) \to B$ is said to be Lagrangian if (M, ω) is a $2n$-dimensional symplectic manifold and the fibres are compact and connected Lagrangian submanifolds of (M, ω), *i.e.* $\omega|_F = 0$ for all F. This condition implies that the fibres and the base space are n-dimensional. Such bundles arise naturally in the study of a special class of dynamical systems in Hamiltonian mechanics, namely those called completely integrable Hamiltonian systems. A celebrated theorem due to Liouville [39], Mineur [46] and Arnol'd [2] provides a semi-global (*i.e.* in the neighbourhood of a fibre) symplectic classification of Lagrangian bundles, given by the existence of local action-angle coordinates. A proof of this theorem, due to Markus and Meyer [41] and Duistermaat [20], shows that the fibres and base space of a Lagrangian bundle are naturally integral affine manifolds, *i.e.* they admit atlases whose changes of coordinates can be extended to affine transformations of $\mathbb{R}^n$ which preserve the standard cocompact lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

This thesis studies the problem of constructing Lagrangian bundles from the point of view of affinely flat geometry. The first step to study this question is to construct topological universal Lagrangian bundles using the affine structure on the fibres. These bundles classify Lagrangian bundles topologically in the sense that every such bundle arises as the pullback of one universal bundle. However, not all bundles which are isomorphic to the pullback of a topological universal Lagrangian bundle are Lagrangian, as there exist further smooth and symplectic invariants. Even for bundles which admit local action-angle coordinates (these are classified up to isomorphism by topological universal Lagrangian bundles), there is a cohomological obstruction to the existence of an appropriate symplectic form on the total space, which has been studied by Dazord and Delzant in [18]. Such bundles are called almost Lagrangian. The second half of this thesis constructs the obstruction of Dazord and Delzant using the spectral sequence of a topological universal Lagrangian bundle. Moreover, this obstruction is shown to be related to a cohomological invariant associated to the integral affine geometry of the base space, called the radiance obstruction. In particular, it is shown that the integral affine geometry of the base space of an almost Lagrangian bundle determines whether the bundle is, in fact, Lagrangian. New examples of (almost) Lagrangian bundles are provided to illustrate the theory developed.

Contents

Chapter 1

Introduction

1.1 Guiding questions

This thesis is concerned with the study of Lagrangian bundles, which, in recent years, have been of interest in a number of branches of mathematics and theoretical physics, ranging from symplectic topology to algebraic geometry, from classical mechanics to mirror symmetry. A fibre bundle

$$\pi : (M, \omega) \to B$$

is said to be Lagrangian if the fibres are maximally isotropic submanifolds of the symplectic manifold (M, ω) (cf. Definition 2.1). Such bundles arise naturally in the study of Hamiltonian mechanics and, in particular, in completely integrable Hamiltonian systems (cf. Definition 2.10). Such dynamical systems exhibit a well-understood local dynamical behaviour, as the phase space is fibred by abelian Lie groups along which the motion is quasi-periodic. This is one of the main consequences of the celebrated Liouville-Mineur-Arnol'd theorem (cf. Theorem 2.2), which, in classical mechanics, gives a theoretical approach to constructing the integral of motions by quadratures (cf. [2]). For the purposes of this work, the most important consequence of this theorem is that the fibres and base space of a Lagrangian bundle are *affine* manifolds (cf. Definition 2.12), *i.e.* they admit an atlas whose changes of coordinates are constant on connected components and lie in the group

$$\mathrm{Aff}(\mathbb{R}^n) := \mathrm{GL}(n; \mathbb{R}) \ltimes \mathbb{R}^n$$

of affine diffeomorphisms of $\mathbb{R}^n$. In fact, as pointed out by Duistermaat in [20], it turns out that the fibres and base space of a Lagrangian bundle are *integral* affine manifolds, *i.e.* the coordinate changes in their affine atlases lie in the group

$$\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) := \mathrm{GL}(n; \mathbb{Z}) \ltimes \mathbb{R}^n.$$

The aim of this thesis is the following.

Aim. Study the integral affine geometry of Lagrangian bundles.

In particular, there are two guiding questions throughout.

Guiding Question 1. Does the integral affine structure on the fibres determine the topological classification of Lagrangian bundles?

Guiding Question 2. Is the integral affine geometry on a manifold B related to the problem of constructing Lagrangian bundles over B?

The classification and construction of Lagrangian bundles are problems that have been extensively studied in the last thirty years mainly in relation to completely integrable Hamiltonian systems and mirror symmetry. In both cases, the bundles are allowed to admit some singularities and the interest lies primarily in understanding the nature of these; in one case, the singular fibres arise in concrete classical and quantum integrable systems, such as the spherical pendulum (cf. [15]) or the hydrogen atom in weak electric and magnetic fields (cf. [21]), while in the other they arise in mirror maps between Calabi-Yau manifolds via the Strominger-Yau-Zaslow (SYZ) conjecture (cf. [33, 60]). However, in order to study the nature of singularities, it is necessary to have a deep understanding of the regular part of the bundle, and of how it affects the topology and geometry of singularities. In some cases, there are classification results, both topological (cf. [66, 68]) and symplectic (cf. [50]); moreover, there exist concrete examples of Lagrangian bundles with singularities confirming that the SYZ conjecture holds in some specific cases (cf. [13]).

The following two sections motivate further why the above guiding questions are important and why works in the literature hint at the fact that they can be solved.

1.1.1 Guiding question 1: classification of Lagrangian bundles

The Liouville-Mineur-Arnol'd theorem (cf. Theorem 2.2) gives a complete characterisation of the local behaviour of completely integrable Hamiltonian systems. As such, it can be looked at from various perspectives, *e.g.* dynamically, topologically or symplectically, and there exist several different proofs of the theorem which highlight different aspects of the result (cf. [8, 20, 40]). For the purposes of this thesis, the most important result is the topological and symplectic classification in the neighbourhood of a fibre of a Lagrangian bundle (cf. Theorem 2.2). Such classification is referred to as *semi-global* in various works in the literature, *e.g.* [50, 68]. In particular, this result provides the existence of local *action-angle* coordinates near a fibre of a Lagrangian bundle (cf. Section 4.2), which are Darboux coordinates in this neighbourhood. A natural question to ask is the following.

Question 1.1. Under what constraints does a Lagrangian bundle admit *global* action-angle coordinates?

Failure of the existence of global action-angle coordinates for a Lagrangian bundle yields interesting dynamical behaviour for the underlying completely integrable Hamiltonian system. This phenomenon was first observed with regards to the spherical pendulum (cf. Example 2.11) by Duistermaat in [20] and Cushman in [15]. In this completely integrable Hamiltonian system there is a singularity which generates *monodromy*, *i.e.* the angle coordinates cannot be defined globally for the underlying Lagrangian bundle. However, it is important to observe that the possibility of existence of obstructions to the existence of global action-angle coordinates had also been observed by Nehorošev in [49].

Motivated by Question 1.1, Duistermaat achieved a topological (and partly symplectic) classification of Lagrangian bundles with compact and connected fibres in [20]; under these restrictions, the fibres are diffeomorphic to tori. Duistermaat showed that any such Lagrangian bundle

$$(M, \omega) \to B$$

over an n-dimensional manifold B has two topological invariants, namely:-

i) the *monodromy* representation

$$\chi_* : \pi_1(B; b) \to \mathrm{GL}(n; \mathbb{Z}),$$

where $b \in B$ is a basepoint. This corresponds to the topological monodromy of the bundle (cf. Section 3.2.1);

ii) the *Chern class*

$$c \in \mathrm{H}^2(B; \mathbb{Z}^n_{\chi_*}),$$

where $\mathbb{Z}^n_{\chi_*}$ is the system of local coefficients determined by the above monodromy representation. This cohomology class is the obstruction to the existence of a section $s : B \to M$ (cf. Section 3.2.2).

Remark 1.2 (Dynamical meaning of the Chern class)**.** While monodromy has been observed in several classical and quantum Hamiltonian systems (cf. [10, 15, 21]), there are no examples of completely integrable Hamiltonian systems which exhibit non-trivial Chern class. Thus the dynamical meaning of the latter is still mysterious.

In order to obtain the above topological classification of Lagrangian bundles, [20] provides a proof of the Liouville-Mineur-Arnol‘d theorem which shows that the fibres of a Lagrangian bundle inherit a smoothly varying affine structure, *i.e.* they are affine manifolds whose atlases depend smoothly on B. This result uses a crucial observation which was originally due to Markus and Meyer in [41]. In fact, these affine structures on $\mathbb{T}^n$ are affinely diffeomorphic to the affine structure induced by the standard action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ by translations (cf. Example 2.14.iv), which is integral. Denote this integral affine manifold by $\mathbb{R}^n/\mathbb{Z}^n$.

Moreover, [20] shows that the structure group of the Lagrangian bundle can be reduced to the group

$$\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) := \mathrm{GL}(n; \mathbb{Z}) \ltimes \mathbb{R}^n/\mathbb{Z}^n \tag{1.1}$$

of affine diffeomorphisms of $\mathbb{R}^n/\mathbb{Z}^n$. This observation can be used to construct the topological invariants of Lagrangian bundles by studying the topology of the universal bundle for $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$.

1.1.2 Guiding question 2: construction of Lagrangian bundles

The theory developed in [20] does not suffice to provide a recipe to construct Lagrangian bundles, as the crucial question of the existence of a suitable symplectic form on the total space of the bundle is not studied there. However, the smoothly varying affine structure on the fibres and the reduction of the structure group to $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ are necessary conditions for a $\mathbb{T}^n$-bundle over an n-dimensional manifold to be Lagrangian. Bundles which satisfy these conditions are called *affine* $\mathbb{R}^n/\mathbb{Z}^n$-bundles (cf. [5]).

The proof of the Liouville-Mineur-Arnol‘d theorem in [20] implies that the base space of a Lagrangian bundle is necessarily an integral affine manifold. This is a consequence of the symplectic topology of the bundle. In fact, all integral affine manifolds are the base space of some Lagrangian bundle (cf. Lemma 4.3). Furthermore, the monodromy of a Lagrangian bundle $(M, \omega) \to B$ inducing a given integral affine structure $\mathcal{A}$ on B is determined by the *linear holonomy* $\mathfrak{l}$ of the integral affine manifold $(B, \mathcal{A})$ via

$$\chi_* = \mathfrak{l}^{-T} \tag{1.2}$$

(cf. Section 4.3.1). Hence integral affine geometry is related to the construction of Lagrangian bundles; this suggests that studying Guiding Question 2 might lead to a better understanding of the relation between affine geometry and Lagrangian bundles.

In light of the above results, the problem of constructing Lagrangian bundles becomes the following (cf. Question 4.31).

Question 1.3. Let $(B, \mathcal{A})$ be an n-dimensional integral affine manifold with linear holonomy $\mathfrak{l}$. The cohomology classes in

$$\mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}})$$

classify the isomorphism classes of affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles over B with monodromy $\mathfrak{l}^{-T}$. Which of these bundles can be Lagrangian?

The above problem has been solved from a slightly different point of view by Dazord and Delzant in [18]. This paper proves that there is a homomorphism

$$\mathcal{D}_{(B,\mathcal{A})} : \mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}}) \to \mathrm{H}^3(B; \mathbb{R})$$

whose kernel gives the subgroup of *realisable* Chern classes, *i.e.* Chern classes whose corresponding affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles can be Lagrangian. The terminology comes from the theory of symplectic realisations of Poisson manifolds, which is related to isotropic bundles, a more general family of bundles (cf. [62]).

The idea of the proof of the above result is the following. Let $(B, \mathcal{A})$ be an integral affine manifold with coordinate charts

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n.$$

The manifold $\mathbb{R}^n$ admits an integral affine structure, which comes from the standard atlas. Let $\mathbf{a}$ denote integral affine coordinates on $\mathbb{R}^n$. In light of the above equivalence between integral affine manifolds and base spaces of Lagrangian bundles (cf. Lemma 4.3), there exists a Lagrangian bundle over $\mathbb{R}^n$

$$(\mathrm{T}^*\mathbb{R}^n / P_{\mathbb{R}^n}, \omega_0) \to \mathbb{R}^n, \tag{1.3}$$

where $P \subset (\mathrm{T}^*\mathbb{R}^n, \Omega_{\mathbb{R}^n})$ is the Lagrangian submanifold which covers $\mathbb{R}^n$ with fibre the discrete span

$$\mathbb{Z}\langle \mathrm{d}a^1, \ldots \mathrm{d}a^n \rangle,$$

and ω_0 denotes the induced symplectic form on the quotient. In fact, up to a symplectomorphism which preserves the bundle structure, the bundle of equation (1.3) is the only Lagrangian bundle over $\mathbb{R}^n$ (cf. [49]). Pull back the above bundle by the diffeomorphisms ϕ_α to obtain locally defined Lagrangian bundles

$$(\mathrm{T}^*U_\alpha / P_\alpha, \omega_\alpha) \to U_\alpha;$$

the cohomology class $\mathcal{D}_{(B,\mathcal{A})}(c)$ is the obstruction for the local symplectic forms ω_α to patch together on the total space of the affine $\mathbb{R}^n/\mathbb{Z}^n$-bundle classified by c (cf. Sections 4.4 and 6.3). In particular, this statement holds because there exists a *symplectic reference Lagrangian bundle* associated to the integral affine manifold $(B, \mathcal{A})$ with linear holonomy $\mathfrak{l}$ (cf. Definition 4.25), whose isomorphism class corresponds to

$$0 \in \mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}}).$$

This bundle provides the local symplectic models for affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles over B with monodromy $\mathfrak{l}^{-T}$; these local models are fibrewise symplectomorphic to the local Lagrangian bundles constructed above.

Interestingly, the construction of these local Lagrangian bundles depends entirely on the integral affine geometry of $(B, \mathcal{A})$; in particular, the cohomology class of the symplectic form ω_0 on the total space of the symplectic reference Lagrangian bundle over $(B, \mathcal{A})$ is related to a cohomological invariant of the integral affine structure $\mathcal{A}$ (cf. Theorem 7.5).

Finally, the problem of constructing Lagrangian bundles is interesting also because of the dearth of explicit examples of these bundles with non-trivial topological (and symplectic) invariants. The first examples of Lagrangian bundles with non-trivial Chern classes were provided by Bates in [6]; these examples have been generalised in [55]. In theory, given the classification works [18, 20], it should be possible to carry out a classification of Lagrangian bundles over *any* integral affine manifold $(B, \mathcal{A})$; however, there exists only a classification of such bundles over closed surfaces. This is because, in dimensions higher than two, it is hard to determine whether a given closed manifold is affine (but there are obstructions, cf. Section 1.1.3 and [58, 61]). In dimension two, the only closed manifolds which can be (integral) affine are diffeomorphic to $\mathbb{T}^2$ and the Klein bottle K^2; this is a theorem of Benzecri [9] and Milnor [45]. The classification of Lagrangian bundles over a two-dimensional torus is in [48], while the case of the Klein bottle is dealt with in [53]. Both papers hinge upon very strong results in the theory of affine manifolds due to Fried, Goldman and Hirsch in [27], which allow for a classification of integral affine structures on these manifolds (which should be compared with the corresponding families of affine structures classified by Arrowsmith and Furness in [28, 29]).

1.1.3 Affine geometry

Sections 1.1.1 and 1.1.2 explain why affine geometry permeates the study of Lagrangian bundles. However, the study of affine manifolds precedes the modern interest in the classification of Lagrangian bundles, since the structure of such manifolds has been investigated since 1950 (cf. [4]).

The existence of an affine structure on a closed manifold puts some constraints on the topology of the manifold, as shown by Smillie in [58], where the connected sums of lens spaces are proved not to be affine. This result is obtained by studying the interplay between the topology of an affine manifold and its fundamental group; this is a recurring theme in affine geometry, as illustrated by the conjecture due to Auslander [1], which relates geometric and topological properties of closed affine manifolds to algebraic properties of their fundamental groups. This conjecture is still open, although progress has been made throughout the years (cf. [1]).

The other main outstanding conjecture in affine geometry is due to Markus (cf. [32, 41]); it states that the universal cover of a closed orientable affine manifold $(B, \mathcal{A})$ is $\mathbb{R}^n$ with the standard affine structure if and only if the affine changes of coordinates of $(B, \mathcal{A})$ take values in

$$\mathrm{SL}(n; \mathbb{R}) \ltimes \mathbb{R}^n. \tag{1.4}$$

Note that the study of this conjecture is related to the study of integral affine manifolds, since their coordinate changes lie in the group of equation (1.4). In an attempt to solve this conjecture, Fried, Goldman and Hirsch introduced a topological invariant of affine manifolds, the *radiance obstruction*, which contains important information about the affine structure of the underlying affine manifold (cf. [26] and Chapter 7). For instance, using this cohomology class, the same authors proved that the conjecture of Markus holds when the fundamental group of the manifold is nilpotent (cf. [27]).

1.2 Main results

This thesis provides answers to Guiding Questions 1 and 2, which are outlined in the two sections below.

1.2.1 Answer to Guiding Question 1

The existence of a smoothly varying affine structure on the fibres of Lagrangian bundles with compact and connected fibres determines the topological classification of such bundles. This follows from the fact that the structure group reduces to

$$\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) = \mathrm{GL}(n; \mathbb{Z}) \ltimes \mathbb{R}^n/\mathbb{Z}^n,$$

where the action of $\mathrm{GL}(n; \mathbb{Z})$ on $\mathbb{R}^n/\mathbb{Z}^n$ descends from the standard action on $\mathbb{R}^n$. The splitting

$$\begin{aligned} \sigma : \mathrm{GL}(n; \mathbb{Z}) &\to \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) \\ A &\mapsto (A, \mathbf{0}) \end{aligned}$$

induces a bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{BGL}(n; \mathbb{Z}) \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n), \tag{1.5}$$

via Theorem 3.2. This bundle generates all topological types of Lagrangian bundles in the sense that every such bundle over B is isomorphic to the pull-back

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \chi^*\mathrm{BGL}(n; \mathbb{Z}) \to B,$$

where $\chi : B \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ denotes the classifying map of the associated principal $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$-bundle (cf. Section 3.1.1); this is the content of Theorem 3.6. For this reason, the bundle of equation (1.5) is called the *topological universal Lagrangian bundle*.

As a consequence of Theorem 3.6, the topological invariants of Lagrangian bundles defined in [20] arise as the pull-backs of topological invariants of the topological universal Lagrangian bundle. In particular, there exist a *universal monodromy representation*

$$\mathrm{id}_* : \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \mathrm{GL}(n; \mathbb{Z})$$

and a *universal Chern class*

$$c_U \in \mathrm{H}^2(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); \mathbb{Z}^n_{\mathrm{id}_*}),$$

the latter being the obstruction to the existence of a section for the topological universal Lagrangian bundle. The importance of these universal topological invariants is that the study of properties of the topological invariants of Lagrangian bundles is reduced to the study of these two invariants. In particular, the universal Chern class plays an important role in the Leray-Serre spectral sequence of the topological universal Lagrangian bundle, since it determines some differentials on the E_2 page (cf. Theorem 6.6). Understanding this differential is useful to find an answer to Guiding Question 2.

1.2.2 Answer to Guiding Question 2

Not all pull-backs of the topological universal Lagrangian bundle of equation (1.5) give rise to Lagrangian bundles, as there are necessary conditions on the base space (*e.g.* it needs to be an integral affine manifold whose linear holonomy satisfies the condition of equation (1.2)).

Let $(B, \mathcal{A})$ be an n-dimensional integral affine manifold with linear holonomy $\mathfrak{l}$. The elements of

$$\mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}})$$

classify the isomorphism types of *almost* Lagrangian bundles (cf. Question 1.3 and Definition 4.32). Fix one such

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$$

with Chern class c. The answer to Guiding Question 2 (and the main result of the thesis) is the following.

Main Result. The obstruction for the above almost Lagrangian bundle to be Lagrangian is given by the cohomology class of the cup product

$$c \cdot r_{(B,\mathcal{A})} \in \mathrm{H}^3(B; \mathbb{R}),$$

where $\cdot$ denotes the cup product with twisted coefficients and $r_{(B,\mathcal{A})}$ is the radiance obstruction of the integral affine manifold $(B, \mathcal{A})$.

This result is obtained in four steps.

Step 1 The first explicit examples of *fake* Lagrangian bundles, *i.e.* almost Lagrangian bundles which cannot be Lagrangian, are constructed. This is the content of Theorem 5.1. The underlying integral affine manifold is $\mathbb{R}^3/\mathbb{Z}^3$. While the proof of this result uses very specific tools (in particular, it uses crucially the fact that the integral affine universal cover of $\mathbb{R}^3/\mathbb{Z}^3$ is affinely diffeomorphic to $\mathbb{R}^3$ with standard integral affine structure), it illustrates the topology behind the homomorphism $\mathcal{D}_{\mathbb{R}^3/\mathbb{Z}^3}$ of Dazord and Delzant which computes the obstruction to the existence of an appropriate symplectic form on the total space of an almost Lagrangian bundle over $\mathbb{R}^3/\mathbb{Z}^3$;

Step 2 Some differential $\mathrm{d}^{(2)}$ on the E_2-page of the Leray-Serre spectral sequence with $\mathbb{Z}$-coefficients of the topological universal Lagrangian bundle is proved to be given by taking the cup product with the universal Chern class c_U in Theorem 6.6. In the proof of this theorem, the main ingredient is that the topological universal Lagrangian bundle is the $\mathrm{GL}(n;\mathbb{Z})$-equivariant equivalent of the universal bundle for principal $\mathbb{R}^n/\mathbb{Z}^n$-bundles;

Step 3 Functoriality of the Leray-Serre spectral sequence (cf. [42]) implies that the result of Step 2 applies to the Leray-Serre spectral sequence of any almost Lagrangian bundle. Fix an n-dimensional integral affine manifold $(B, \mathcal{A})$ with linear holonomy $\mathfrak{l}$; the symplectic form on the total space of its symplectic reference Lagrangian bundle defines a cohomology class

$$w_0 \in \mathrm{H}^1(B; \mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n; \mathbb{R})_{\mathfrak{l}}),$$

as shown in Lemma 6.7. The above cohomology group is, in fact, isomorphic to $\mathrm{E}_2^{1,1}$, where $\mathrm{E}_2^{*,*}$ denotes the E_2-page of the Leray-Serre spectral sequence with real coefficients of the almost Lagrangian bundle over $(B, \mathcal{A})$ with Chern class c. Note that the groups $\mathrm{E}_2^{*,*}$ are independent of c. Theorem 6.9 proves that

$$\mathcal{D}_{(B,\mathcal{A})}(c) = -\mathrm{d}_c^{(2)}(w_0),$$

where $\mathrm{d}_c^{(2)} : \mathrm{E}_2^{1,1} \to \mathrm{E}_2^{3,0}$ is the differential. Note that Step 2 implies that $\mathrm{d}_c^{(2)}$ is given by taking cup products with $c^{\mathbb{R}}$, the image of c under the homomorphism

$$\mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}}) \to \mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{H}^2(B; \mathbb{R}^n_{\mathfrak{l}^{-T}})$$

(cf. Corollary 6.8);

Step 4 The cohomology class w_0 is shown to be mapped to the radiance obstruction $r_{(B,\mathcal{A})}$ under a natural isomorphism (cf. Theorem 7.5).

The above Main Result allows to study Lagrangian bundles using integral affine geometry and *vice versa*; an example of the interaction is given in Theorem 7.6, which proves that there exist no closed integral affine manifolds whose radiance obstruction vanishes.

1.3 Structure of thesis

This thesis is structured as follows. Chapter 2 studies the relation between Lagrangian bundles and completely integrable Hamiltonian systems. The Liouville-Mineur-Arnol'd theorem is stated (cf. Theorem 2.2) and a part of it is proved in Theorem 2.3, which implies that the (compact and connected) fibres of a Lagrangian bundle can be naturally endowed with a smoothly varying affine structure affinely diffeomorphic to $\mathbb{R}^n/\mathbb{Z}^n$. This observation, along with the reduction of the structure group to $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$, lays the foundation for the work of Chapter 3, which constructs topological universal Lagrangian bundles (cf. Definition 3.5) starting from the topology of $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$. The topological invariants of Lagrangian bundles, *i.e.* monodromy and Chern class, are constructed starting from their universal counterparts; moreover, these invariants are shown to be *sharp*. Therefore Chapter 3 provides an answer to Guiding Question 1.

On the other hand, Guiding Question 2 is addressed in Chapters 4 to 7. The first of these chapters starts with the observation that not all pull-backs of topological universal Lagrangian bundles are, in fact, Lagrangian; it then proceeds to investigate the symplectic geometry of Lagrangian bundles to obtain further characterisations of such bundles. Existence of local action-angle coordinates is proved in Theorem 4.2, which completes the proof of the Liouville-Mineur-Arnol'd theorem. These canonical coordinates are used to prove that the base space of a Lagrangian bundle is an integral affine manifold (cf. Definition 4.15); any such manifold is also the base of some Lagrangian bundle (cf. Lemma 4.3), which is called the *symplectic reference Lagrangian bundle*. Existence of local action-angle coordinates is, however, not sufficient to construct a suitable symplectic form, as illustrated by introducing the concept of almost Lagrangian bundles (cf. Definition 4.32) and by stating a theorem due to Dazord and Delzant (cf. Theorem 4.4), which gives a necessary and sufficient condition for an almost Lagrangian bundle to be Lagrangian. Chapter 5 studies the problem of differentiating between almost Lagrangian and Lagrangian bundles when the base space is $\mathbb{R}^3/\mathbb{Z}^3$, thus constructing the first explicit examples of fake Lagrangian bundles. Moreover, Section 5.2 gives an example which shows that the homomorphism of Dazord and Delzant does not determine whether the total space of a fake Lagrangian bundle is, in fact, symplectic. Chapter 6 brings topological universal Lagrangian bundles into the problem of constructing Lagrangian bundles over an integral affine manifold $(B, \mathcal{A})$; Theorem 6.9 proves that the homomorphism $\mathcal{D}_{(B,\mathcal{A})}$ of Dazord and Delzant is given by taking twisted cup products with w_0, the cohomology class of the symplectic form of the symplectic reference Lagrangian bundle. This result is obtained by looking at the Leray-Serre spectral sequence of the topological universal Lagrangian bundle: some differential $\mathrm{d}^{(2)}$ on the E_2-page is given by taking cup products with the universal Chern class c_U (cf. Theorem 6.6). Finally, Chapter 7 completes the proof of the Main Result (cf. Section 1.2.2) by noticing that the cohomology class w_0 is, up to isomorphism, equal to the radiance obstruction $r_{(B,\mathcal{A})}$ of the integral affine manifold $(B, \mathcal{A})$. The connection between the symplectic topology of Lagrangian bundles and integral affine geometry is exploited in Theorem 7.6 and Section 7.3; the latter considers some explicit examples which are related to singular Lagrangian bundles.

1.4 (Un)originality claims

Chapters 2 and 4 consist of material that is presented in various other works in the literature, which I have referred the reader to. The exposition is my own and the emphasis on affine geometry is stronger here than it is in most other works in the literature. Chapters 3, 5, 6 and 7 contain original results, unless otherwise stated in the text. Again, it must be said that the construction of the universal radiance obstruction r_U as in Section 7.1 cannot be found elsewhere in the literature; the proved results are well-known and there is no claim of originality there.

Finally, Chapter 3 is largely based on the published article [55], Chapter 5 presents an example that comes from [54], and Chapters 6 and 7 expose results also presented in [56].

Chapter 2

Preliminaries on Lagrangian bundles and affine geometry

This is an expository chapter, introducing the main concepts and methods used in this thesis.

2.1 Definition and examples of Lagrangian bundles

Let (M, ω) be a $2n$-dimensional symplectic manifold.

Definition 2.1 (Lagrangian bundles)**.** A fibre bundle $F \hookrightarrow (M, \omega) \to B$ is said to be *Lagrangian* if the fibres are Lagrangian submanifolds of (M, ω), *i.e.* $\omega|_F = 0$ and $\dim F = \frac{1}{2} \dim M = n$.

Remark 2.2. The above definition implies that $\dim B = n$.

Notation. Throughout this work, M and B denote $2n$-dimensional and n-dimensional manifolds respectively, unless otherwise stated.

Example 2.3 (The cotangent bundle)**.** Let $\pi : \mathrm{T}^*B \to B$ denote the cotangent bundle of a manifold B. It is well-known (*e.g.* [43]) that T^*B admits a symplectic form Ω, called the *canonical* symplectic form. It is defined as follows. Let

$$D\pi : \mathrm{T}\mathrm{T}^*B \to \mathrm{T}B$$

be the differential of the projection map π. For $b \in B$, let $v^* \in \mathrm{T}^*_b B$. Define the canonical 1-form λ to be the linear map

$$v^* \circ D\pi(b, v^*) : \mathrm{T}_{(b,v^*)}\mathrm{T}^*B \to \mathbb{R}$$

at each $(b, v^*) \in \mathrm{T}^*B$. The canonical symplectic form Ω is defined by

$$\Omega = -\mathrm{d}\lambda.$$

If $(a^1, \ldots, a^n)$ are local coordinates on B and $(a^1, \ldots, a^n, p^1, \ldots, p^n)$ are induced local coordinates on T^*B, then a local representative of Ω is given by

$$\sum_{i=1}^{n} \mathrm{d}a^i \wedge \mathrm{d}p^i = -\mathrm{d}\left(\sum_{i=1}^{n} p^i \mathrm{d}a^i \right). \tag{2.1}$$

Equation (2.1) implies that the bundle

$$\mathbb{R}^n \hookrightarrow (\mathrm{T}^*B, \Omega) \to B$$

is Lagrangian. Note further that if μ is a closed 2-form on B, then $(\mathrm{T}^*B, \Omega + \pi^*\mu)$ is a symplectic manifold, and the projection $\pi : (\mathrm{T}^*B, \Omega + \pi^*\mu) \to B$ gives rise to a Lagrangian bundle.

Notation. Throughout this thesis, π denotes projection of a fibre bundle.

Remark 2.4. Lagrangian bundles are henceforth assumed to have compact and connected fibres, unless otherwise stated. However, it is important to bear Example 2.3 in mind, as it is closely related to the local behaviour of Lagrangian bundles in general. Under some restrictions, the classification of Lagrangian bundles with contractible fibres can be carried out (cf. [48]). The constraint in this paper is on the affine structure of the fibres (cf. Section 2.3).

Example 2.5 (Even dimensional tori)**.** Let $\Omega_{\mathbb{R}^{2n}}$ be the canonical symplectic form on $\mathbb{R}^{2n} \cong \mathrm{T}^*\mathbb{R}^n$. Addition of vectors makes $(\mathbb{R}^{2n}, +)$ into a Lie group. Let

$$\Lambda^{2n} \cong \mathbb{Z}^{2n} \subset (\mathbb{R}^{2n}, +) \cong \mathrm{T}^*\mathbb{R}^n$$

be the standard lattice, which, under the projection map (and homomorphism)

$$\pi : (\mathrm{T}^*\mathbb{R}^n, +) \to (\mathbb{R}^n, +), \tag{2.2}$$

maps to the standard lattice $\Lambda^n \subset \mathbb{R}^n$. Translating along the generators of Λ^{2n} yields a $\mathbb{Z}^{2n}$-action on $\mathbb{R}^{2n}$, which descends to a $\mathbb{Z}^n$-action on $\mathbb{R}^n$ under the map of equation (2.2). These $\mathbb{Z}^{2n}$ and $\mathbb{Z}^n$-actions are free and properly discontinuous on $\mathbb{R}^{2n}$ and $\mathbb{R}^n$ respectively; moreover, the $\mathbb{Z}^{2n}$-action leaves $\Omega_{\mathbb{R}^{2n}}$ invariant. Thus there is a Lagrangian bundle

$$\mathbb{T}^n \hookrightarrow (\mathbb{R}^{2n}/\mathbb{Z}^{2n}, \omega) \to \mathbb{R}^n/\mathbb{Z}^n,$$

where $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ and $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ are diffeomorphic to $\mathbb{T}^{2n}$ and $\mathbb{T}^n$ respectively.

Example 2.6 (Kodaira-Thurston manifold)**.** Recall the construction of the Kodaira-Thurston example of a symplectic manifold which is not Kähler (*e.g.* [43]). Let $\Gamma = \mathbb{Z}^2 \ltimes_A \mathbb{Z}^2$, where the group operation is defined by

$$(\mathbf{m}', \mathbf{n}') \cdot (\mathbf{m}, \mathbf{n}) = (\mathbf{m}' + \mathbf{m}, A_{\mathbf{m}'}\mathbf{n} + \mathbf{n}'), \quad A_{\mathbf{m}'} = \begin{pmatrix} 1 & m_2' \\ 0 & 1 \end{pmatrix},$$

where $\mathbf{m}' = (m_1', m_2') \in \mathbb{Z}^2$. Define an action on $\mathbb{R}^4$ by

$$\begin{gathered} \rho : \Gamma \to \mathrm{Diff}(\mathbb{R}^4) \\ \rho_{(\mathbf{m},\mathbf{n})}(\mathbf{a}, \mathbf{p}) = (\mathbf{a} + \mathbf{m}, A_{\mathbf{m}}\mathbf{p} + \mathbf{n}), \end{gathered} \tag{2.3}$$

where $\mathbf{a} = (a^1, a^2),\ \mathbf{p} = (p^1, p^2)$ and $(\mathbf{a}, \mathbf{p}) \in \mathbb{R}^4$. The symplectic form

$$\omega = \mathrm{d}a^2 \wedge \mathrm{d}a^1 + \mathrm{d}p^1 \wedge \mathrm{d}p^2$$

makes the bundle induced by the projection

$$\begin{gathered} \mathbb{R}^4 \to \mathbb{R}^2 \\ (a^1, a^2, p^1, p^2) \mapsto (a^2, p^1) \end{gathered}$$

into a Lagrangian bundle. The orbit $\Gamma \cdot \mathbf{0}$ maps to the standard lattice $\Lambda^2 \subset \mathbb{R}^2$ under the above projection. Note further that the action of Γ on $\mathbb{R}^4$ defined by equation (2.3) preserves the fibres of the above bundle and it leaves ω invariant. Thus ω induces a symplectic form ω' on $\mathbb{R}^4/\Gamma$ which makes the bundle

$$\mathbb{T}^2 \hookrightarrow (\mathbb{R}^4/\Gamma, \omega') \to \mathbb{T}^2$$

Lagrangian. Note that, unlike Examples 2.3 and 2.5, this bundle does not admit a section.

2.2 Relation to completely integrable Hamiltonian systems

The fact that the fibres of the bundles in Examples 2.5 and 2.6 are tori is no coincidence, and it can be explained by investigating the relation between Lagrangian bundles and a special class of dynamical systems arising in Hamiltonian mechanics. In this section, completely integrable Hamiltonian systems are introduced and their relation to Lagrangian bundles is explained.

In order to define a completely integrable Hamiltonian system, it is necessary to introduce the concept of a Poisson bracket, which is central in Hamiltonian dynamics.

Definition 2.7 (Poisson bracket)**.** Let N be a smooth manifold. A *Poisson bracket* on $\mathrm{C}^\infty(N)$ is an anti-symmetric, $\mathbb{R}$-bilinear map $\{.,.\} : \mathrm{C}^\infty(N) \times \mathrm{C}^\infty(N) \to \mathrm{C}^\infty(N)$ which, for all $f, g, h \in \mathrm{C}^\infty(N)$, satisfies

1. Leibniz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$;
2. Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

A *Poisson manifold* is a pair $(N, \{.,.\})$, where $\{.,.\}$ is a Poisson bracket on $\mathrm{C}^\infty(N)$.

Example 2.8 (Symplectic manifolds)**.** Let (M, ω) be a symplectic manifold. For each function $f \in \mathrm{C}^\infty(M)$, define its *Hamiltonian vector field* X_f to be the unique vector field satisfying

$$\iota(X_f)\omega = \mathrm{d}f,$$

where ι denotes interior product. The bilinear map $\{.,.\}$ defined on $\mathrm{C}^\infty(M)$ by setting

$$\{f, g\} := \omega(X_f, X_g) = \mathrm{d}f(X_g) = -\mathrm{d}g(X_f)$$

for all $f, g \in \mathrm{C}^\infty(M)$, yields a Poisson bracket.

Remark 2.9. There exist Poisson manifolds which are not symplectic. For instance, consider $\mathbb{R}^3$ with coordinates (a^1, a^2, a^3) endowed with the bivector

$$\frac{\partial}{\partial a^1} \wedge \frac{\partial}{\partial a^2}.$$

The above bivector induces a Poisson bracket on $\mathrm{C}^\infty(\mathbb{R}^3)$, but $\mathbb{R}^3$ cannot be a symplectic manifold for dimensional reasons.

Definition 2.10 (Completely integrable Hamiltonian systems)**.** Let (M, ω) be a $2n$-dimensional symplectic manifold. A *completely integrable Hamiltonian system* is a map

$$\mathbf{f} = (f_1, \ldots, f_n) : (M, \omega) \to \mathbb{R}^n,$$

whose components satisfy

1. Involutivity: for all $i, j = 1, \ldots, n$,

$$\{f_i, f_j\} = 0,$$

   where $\{.,.\}$ is the Poisson bracket on $\mathrm{C}^\infty(M)$ as defined in Example 2.8;

2. Functional independence: $\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_n \neq 0$ on a dense open subset of M.

Example 2.11 (The spherical pendulum [16, 20])**.** Let $S^2 \subset \mathbb{R}^3$ be the standard embedding, and let $(\mathbf{a}, \mathbf{p})$ be local coordinates on T^*S^2 induced by the inclusion $\mathrm{T}^*S^2 \subset \mathrm{T}^*\mathbb{R}^3$. Consider the function

$$\begin{aligned} H : \mathrm{T}^*S^2 &\to \mathbb{R} \\ (\mathbf{a}, \mathbf{p}) &\mapsto \frac{1}{2}\|\mathbf{p}\|^2 + a^3, \end{aligned}$$

where $\|\mathbf{p}\|^2 = g(\mathbf{p}, \mathbf{p})$, g denotes the metric on T^*S^2 obtained by restricting the pullback of the standard Euclidean metric on $\mathrm{T}^*\mathbb{R}^3 \cong \mathbb{R}^6$, and $\mathbf{a} = (a^1, a^2, a^3)$. Let Ω_{S^2} denote the canonical symplectic form on T^*S^2 and let $\{.,.\}$ be the induced Poisson bracket on $\mathrm{C}^\infty(\mathrm{T}^*S^2)$. The function

$$\begin{aligned} J : \mathrm{T}^*S^2 &\to \mathbb{R} \\ (\mathbf{a}, \mathbf{p}) &\mapsto a^1 v^2 - a^2 v^1 \end{aligned}$$

makes the map $(H, J) : (\mathrm{T}^*S^2, \Omega_{S^2}) \to \mathbb{R}^2$ into a completely integrable Hamiltonian system.

Understanding the dynamics of the spherical pendulum near its equilibrium points was one of the motivations that brought mathematicians to study Lagrangian bundles in further details. In particular, there is interesting dynamics near the unstable equilibrium point of the map (H, J) above, as illustrated in the work of Delos *et al.* [19]. However, it was previous work of Nehorošev in [49], of Duistermaat in [20], and of Cushman in [15] that illustrated how the behaviour near the unstable equilibrium point is reflected in the topology of the map (H, J).

The connection between Lagrangian bundles and completely integrable Hamiltonian systems begins with the following observation.

Observation 2.1 (Duistermaat [20])**.** *Let $F \hookrightarrow (M, \omega) \to B$ be a Lagrangian bundle. Let $U \subset B$ be a coordinate neighbourhood of B and let $\phi : U \to \mathbb{R}^n$ be a coordinate map. Then the composite*

$$\phi \circ \pi : (\pi^{-1}(U), \omega|_{\pi^{-1}(U)}) \to \mathbb{R}^n$$

is a completely integrable Hamiltonian system.

Proof. Set $\phi \circ \pi = (f_1, \ldots, f_n)$. Functional independence of $f_1, \ldots, f_n$ follows from the fact that ϕ is a local diffeomorphism. As for involutivity, recall that if X_i denotes the Hamiltonian vector field associated to f_i, then

$$\{f_i, f_j\} = \omega(X_i, X_j),$$

where, for notational ease, $\omega = \omega|_{\pi^{-1}(U)}$. Since the fibres of the bundle are Lagrangian, the result is proved if each X_i is shown to be tangent to the fibres. It is a standard result that the Hamiltonian vector field X_g is tangent to the level set $\{g = \text{const}\}$ for any function g on a symplectic manifold. In particular, each X_i is tangent to the submanifold defined by

$$\{f_1 = d_1, \ldots, f_n = d_n\} \tag{2.4}$$

where $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{R}^n$ lies in the image of $\phi \circ \pi$. The submanifold of M defined by equation (2.4) is precisely a fibre of the Lagrangian bundle and, thus, the result follows. $\square$

Thus methods from the theory of completely integrable Hamiltonian systems can be used to prove properties of Lagrangian bundles. Moreover, given a completely integrable Hamiltonian system there is a Lagrangian bundle associated to it. This is one of the consequences of the Liouville-Mineur-Arnol'd theorem.

Theorem 2.2 (Liouville [39], Mineur [46], Arnol'd [2])**.** *Let* $\mathbf{f} = (f_1, \ldots, f_n) : (M, \omega) \to \mathbb{R}^n$ *be a completely integrable Hamiltonian system and let* $\mathbf{d} \in \mathbb{R}^n$ *be a regular value of* $\mathbf{f}$*. Suppose that* $F = \mathbf{f}^{-1}(\mathbf{d})$ *is compact and connected. Then*

i) F *is a Lagrangian submanifold of* (M, ω) *and it is diffeomorphic to* $\mathbb{T}^n$*;*

ii) there exists an open neighbourhood $U \subset M$ *of* F *which is symplectomorphic to a neighbourhood* V *of the zero section of* $(\mathrm{T}^*\mathbb{T}^n, \Omega_{\mathbb{T}^n}) \to \mathbb{T}^n$*, as illustrated in the commutative diagram below*

$$\begin{tikzcd}
M & U \arrow[l, hook'] \arrow[r] \arrow[d] & V \arrow[r, hook] \arrow[d] & \mathrm{T}^*\mathbb{T}^n \\
& F \arrow[ul, hook] \arrow[r] & \mathbb{T}^n \arrow[ur, hook] &
\end{tikzcd}$$

The proof of this theorem is split in two halves, namely as the proofs of Theorem 2.3 and Theorem 4.2. This approach is chosen as there are aspects of the proof that have to be studied in detail for the purposes of this thesis. There are several proofs of Theorem 2.2 in the literature, each with a slightly different emphasis (*e.g.* [2, 8, 20]).

It is possible to associate a Lagrangian bundle to a given a completely integrable Hamiltonian system $\mathbf{f} : (M, \omega) \to \mathbb{R}^n$ as follows. Let $\mathcal{R} \subset \mathbb{R}^n$ be the open subset of $\mathbb{R}^n$ consisting of regular values of $\mathbf{f}$. Then

$$\bar{\mathbf{f}} := \mathbf{f}|_{\mathbf{f}^{-1}(\mathcal{R})} : (\mathbf{f}^{-1}(\mathcal{R}), \omega|_{\mathbf{f}^{-1}(\mathcal{R})}) \to \mathcal{R} \subset \mathbb{R}^n \tag{2.5}$$

is a surjective submersion whose level sets are compact and connected; a theorem due to Ehresmann [22] implies that $\bar{\mathbf{f}}$ can be endowed with the structure of a fibre bundle. Moreover, by part (i) of Theorem 2.2, the fibres of $\bar{\mathbf{f}}$ are Lagrangian submanifolds of (M, ω) and thus the fibre bundle of equation (2.5) is Lagrangian.

Theorem 2.2 also provides information regarding the topology of the fibres of Lagrangian bundles. If $F \hookrightarrow (M, \omega) \to B$ is Lagrangian, then Observation 2.1 proves that it is locally given by a completely integrable Hamiltonian system and, thus, that the fibres are diffeomorphic to $\mathbb{T}^n$. In order to use this observation to construct topological invariants of Lagrangian bundles, it is necessary to analyse the proof of the Liouville-Mineur-Arnol'd theorem and, in particular, its connection to affine differential geometry.

2.3 Affine geometry of the fibres of Lagrangian bundles

The essential element to construct topological characteristic classes of Lagrangian bundles is the fact that the fibres are endowed with a natural *affine structure*.

Definition 2.12 (Affine manifolds)**.** An *affine structure* on an n-dimensional manifold B is a choice of atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{R}^n)\}$ whose changes of coordinates

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \to \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$$

are constant on connected components, and are *affine transformations* of $\mathbb{R}^n$, *i.e.* they lie in the group

$$\mathrm{Aff}(\mathbb{R}^n) := \mathrm{GL}(n; \mathbb{R}) \ltimes \mathbb{R}^n,$$

where the action of $\mathrm{GL}(n;\mathbb{R})$ on $\mathbb{R}^n$ is the standard one. An *affine manifold* is a pair $(B, \mathcal{A})$, where B is a manifold and $\mathcal{A}$ is an affine structure on B.

Remark 2.13 (Equivalent definition in terms of linear connections)**.** The original definition of an affine manifold was given in terms of linear connections on tangent bundles. For instance, [4] states that a manifold B is an affine manifold if there exists a flat, torsion-free linear connection ∇ on $\mathrm{T}B$; in the same paper, this condition is shown to be equivalent to Definition 2.12.

Example 2.14 (Affine manifolds)**.**

i) Let $\mathcal{A}$ be the standard atlas of $\mathbb{R}^n$. Then $(\mathbb{R}^n, \mathcal{A})$ is an affine manifold;

ii) Let $N \subset \mathbb{R}^n$ be open. Then the inclusion $N \hookrightarrow (\mathbb{R}^n, \mathcal{A})$ induces an affine structure on N;

iii) More generally, let $D : N \to \mathbb{R}^n$ be a local diffeomorphism. For each $p \in N$ there exists an open neighbourhood $U_p \subset N$ of p such that $D|_{U_p}$ is a diffeomorphism. The collection $\{(U_p, D|_{U_p})\}$ yields a well-defined affine structure on the manifold N;

iv) Let $\Lambda \subset (\mathbb{R}^n, +)$ be a cocompact lattice; any such is isomorphic to $\mathbb{Z}^n$ (cf. [63]). Fix a choice of generators $\lambda_1, \ldots, \lambda_n$ of Λ and define a Λ-action on $\mathbb{R}^n$ by

$$\mathbf{a} \cdot \lambda_i = \mathbf{a} + \mathbf{e}^i,$$

where $\mathbf{a} = (a^1, \ldots, a^n)$ denotes affine coordinates on $\mathbb{R}^n$ defined in (i) above, and $\{\mathbf{e}^1, \ldots, \mathbf{e}^n\}$ is the standard basis of $\mathbb{R}^n$ as a vector space. The quotient $\mathbb{R}^n/\Lambda$ inherits an affine structure $\mathcal{A}_{\boldsymbol{\lambda},\Lambda}$, since the above action is by *affine diffeomorphisms* of $\mathbb{R}^n$, *i.e.* diffeomorphisms which are affine in local affine coordinates. The atlas $\mathcal{A}_{\boldsymbol{\lambda},\Lambda}$ on $\mathbb{T}^n$ depends on the choice of generators λ_i of Λ and on the choice of Λ. However, for any other choice of cocompact lattice Λ' and generators λ_i' of Λ', there exists an affine diffeomorphism

$$(\mathbb{T}^n; \mathcal{A}_{\boldsymbol{\lambda},\Lambda}) \to (\mathbb{T}^n; \mathcal{A}_{\boldsymbol{\lambda}',\Lambda'}),$$

since all cocompact lattices of $(\mathbb{R}^n, +)$ are isomorphic. Thus all such affine structures on $\mathbb{T}^n$ are affinely diffeomorphic to the standard affine structure on $\mathbb{T}^n$ induced by translations along the standard lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. For notational ease, this latter affine manifold is henceforth denoted by $\mathbb{R}^n/\mathbb{Z}^n$;

v) More generally, let $(N, \mathcal{A})$ be an affine manifold and let Γ be a group acting freely and properly discontinuously on the right of N via affine diffeomorphisms of $(N, \mathcal{A})$. Then the manifold N/Γ inherits an affine structure $\mathcal{A}_\Gamma$;

vi) Not all manifolds can be endowed with an affine structure. For instance, a theorem due to Benzecri [9] and Milnor [45] states that the only closed orientable surface that admits an affine structure is diffeomorphic to $\mathbb{T}^2$.

The following theorem relates Lagrangian bundles to affine geometry and is the starting block for the construction of topological invariants of Lagrangian bundles.

Theorem 2.3 (Markus and Meyer [41], Duistermaat [20]). *Let $F \hookrightarrow (M, \omega) \to B$ be a Lagrangian bundle. Then*

i) for each $b \in B$, the fibre $F_b = \pi^{-1}(b)$ is diffeomorphic to $\mathbb{T}^n$;

ii) the fibres are equipped with a smoothly varying affine structure;

iii) the structure group of the fibre bundle reduces to

$$\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) := \mathrm{GL}(n;\mathbb{Z}) \ltimes \mathbb{R}^n/\mathbb{Z}^n,$$

the group of affine diffeomorphisms

$$\mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n,$$

where $\mathbb{R}^n/\mathbb{Z}^n$ denotes the affine manifold of Example 2.14.iv.

Proof. The proof is structured so as to prove assertions (i), (ii) and (iii) sequentially.

Fix $b_0 \in B$, let $U \subset B$ be an open neighbourhood of b_0 which is also a coordinate neighbourhood with coordinate map $\phi : U \to \mathbb{R}^n$. Set $\phi \circ \pi = (f_1, \ldots, f_n)$ and let X_i denote the Hamiltonian vector field of f_i. For any $p \in F_{b_0} = \pi^{-1}(b_0)$,

$$X_1(p), \ldots, X_n(p)$$

form a basis of $\mathrm{T}_p F_{b_0}$ and

$$[X_i, X_j] = X_{\{f_i, f_j\}} = 0 \tag{2.6}$$

for all i, j, since the functions $f_1, \ldots, f_n$ are in involution (cf. Observation 2.1). For each i, let ψ_i^t denote the flow associated to X_i; ψ_i^t preserves F_{b_0} and, since F_{b_0} is compact, the flow is complete. Moreover, equation (2.6) implies that these flows commute, *i.e.*

$$\psi_i^{t^i} \circ \psi_j^{t^j} = \psi_j^{t^j} \circ \psi_i^{t^i}. \tag{2.7}$$

for all $i, j = 1, \ldots, n$, $t^i, t^j \in \mathbb{R}$. Define an $(\mathbb{R}^n, +)$-action on F_{b_0} by

$$\begin{aligned} \Psi : \mathbb{R}^n \times F_{b_0} &\to F_{b_0} \\ (\mathbf{t} = (t^1, \ldots, t^n), p) &\mapsto \Psi(\mathbf{t}, p) = \psi_1^{t^1} \circ \ldots \circ \psi_n^{t^n}(p). \end{aligned} \tag{2.8}$$

Note that it is indeed an $(\mathbb{R}^n, +)$-action by equation (2.7). Since $X_1, \ldots, X_n$ are linearly independent on the connected manifold F, the action is transitive and the isotropy subgroups at points $p, p' \in F$ are conjugate. As $(\mathbb{R}^n, +)$ is abelian, conjugation induces the identity automorphism on isotropy subgroups; for any choice of $p \in F$, denote the isotropy group of the action Ψ by

$$P_{b_0} := \{\mathbf{t} \in \mathbb{R}^n : \Psi(\mathbf{t}, p) = p\}.$$

Thus

$$F_{b_0} \cong \mathbb{R}^n / P_{b_0};$$

since F_{b_0} is compact and n-dimensional, P_{b_0} is an n-dimensional cocompact subgroup of $(\mathbb{R}^n, +)$ and, hence, isomorphic to $\mathbb{Z}^n$. It follows that

$$F_{b_0} \cong \mathbb{T}^n,$$

which proves statement (i).

The main idea to prove (ii) is to show that the above construction induces a smoothly affine structure on the fibres of the bundle. Firstly, note that the above construction shows that

$$F_{b_0} \cong \mathbb{R} \otimes_{\mathbb{Z}} P_{b_0} / P_{b_0}; \tag{2.9}$$

under this identification, F_{b_0} inherits an affine structure as in Example 2.14.iv.

Secondly, this affine structure can be shown to be independent of the choice of maps $f_1, \ldots, f_n$, or equivalently, of the choice of coordinatisation around $b_0 \in B$, as follows. Let $\bar{\phi} : \bar{U} \to \mathbb{R}^n$ be a coordinate map defined on an open neighbourhood of $b_0 \in B$. Set

$$\bar{\phi} \circ \pi = (\bar{f}_1, \ldots, \bar{f}_n)$$

and let $\bar{X}_i$ be the Hamiltonian vector field of the function $\bar{f}_i$. The argument in the proof of Observation 2.1 shows that for any $p \in F_{b_0}$, $\bar{X}_1(p), \ldots, \bar{X}_n(p)$ form a basis of $\mathrm{T}_p F_{b_0}$. Thus there exist constants $m_{ij}(p) \in \mathbb{R}$ such that

$$\bar{X}_i(p) = \sum_{j=1}^{n} m_{ij}(p) X_j;$$

for all i; the matrix $\mathcal{M}(p) = (m_{ij}(p))$ is clearly invertible. Let $\bar{\psi}_i^t$ denote the flow of the vector field $\bar{X}_i$. Commutativity of the flows $\psi_i^{t^i}, \psi_j^{t^j}$ shown in equation (2.7) implies that

$$\bar{\psi}_i^t(p) = \psi_1^{m_{i1}(p)t} \circ \ldots \circ \psi_n^{m_{in}(p)t}(p).$$

If $\bar{\Psi}$ denote the $\mathbb{R}^n$-action on F_{b_0} induced by the flows $\bar{\psi}_1^{t^1}, \ldots, \bar{\psi}_n^{t^n}$ as in equation (2.8), then for all $\mathbf{t} \in \mathbb{R}$ and all $p \in F_{b_0}$

$$\bar{\Psi}(\mathbf{t}, p) = \psi_1^{(\sum_{j=1}^{n} m_{j1}(p)t^j)} \circ \ldots \circ \psi_n^{(\sum_{j=1}^{n} m_{jn}(p)t^j)}(p). \tag{2.10}$$

Let $\bar{P}_{b_0}(p)$ denote the isotropy group of the $(\mathbb{R}^n, +)$-action defined by $\bar{\Phi}$ at p. If $\bar{\mathbf{T}} \in \bar{P}_{b_0}(p)$, then

$$\mathcal{M}^T(p)\bar{\mathbf{T}} \in P_{b_0}(p).$$

Since the $(\mathbb{R}^n, +)$-actions defined by Φ and $\bar{\Phi}$ are abelian, the isotropy subgroups are independent of the choice of $p \in F_{b_0}$. Thus the matrices $\mathcal{M}(p)$ are independent of p and are henceforth denoted simply by $\mathcal{M}$. As

$$\mathbb{R} \otimes_{\mathbb{Z}} P_{b_0}/P_{b_0} \cong F_{b_0} \cong \mathbb{R} \otimes_{\mathbb{Z}} \bar{P}_{b_0}/\bar{P}_{b_0},$$

a change of coordinates around $b_0 \in B$ induces an affine diffeomorphism

$$\mathcal{M}^{-T}(b_0) : \mathbb{R} \otimes_{\mathbb{Z}} P_{b_0}/P_{b_0} \to \mathbb{R} \otimes_{\mathbb{Z}} \bar{P}_{b_0}/\bar{P}_{b_0}. \tag{2.11}$$

In particular, this shows that, up to affine diffeomorphism, the affine structure $\mathcal{A}_{b_0}$ on F_{b_0} is well-defined.

It remains to show that the affine structure $\mathcal{A}_b$ varies smoothly with $b \in U$. To this end, fix a coordinatisation $\phi : U \to \mathbb{R}^n$ around $b_0 \in B$ and consider a local section $s : U \to \pi^{-1}(U)$, which, by shrinking U if needed, exists. Fix a basis $T_0^1, \ldots, T_0^n$ of P_{b_0}. Applying the implicit function theorem to the equation

$$\Psi(\mathbf{t}, s(b)) = s(b),$$

and by shrinking U if needed, it is possible to obtain smooth maps $T^i : U \to \mathbb{R}^n$ such that $T^i(b_0) = T_0^i$ and

$$\Psi(T^i(b), s(b)) = s(b)$$

for all $b \in U$. The maps $T^1, \ldots, T^n$ yield a smoothly varying basis for P_b; by equation (2.9), a choice of P_b determines an affine structure on F_b. Therefore, a smoothly varying basis for P_b yields a smoothly varying affine structure on F_b as claimed. This proves part (ii) of the theorem.

In order to prove (iii), begin by noticing that a choice of local smooth section $s : U \to \pi^{-1}(U)$ yields a trivialisation of the bundle as follows. By shrinking U if needed, assume that it is a coordinate neighbourhood which induces an $(\mathbb{R}^n, +)$-action Ψ on $\pi^{-1}(U)$ as above. Define a map

$$\begin{aligned} \tilde{\Upsilon} : U \times \mathbb{R}^n &\to \pi^{-1}(U) \\ (b, \mathbf{t}) &\mapsto \Psi(\mathbf{t}, s(b)). \end{aligned} \tag{2.12}$$

The above map is smooth as it is the composition of smooth maps. Moreover, if $r_1, \ldots, r_n \in \mathbb{Z}$, then

$$\tilde{\Upsilon}\left(b, \sum_{i=1}^{n} r_i T^i(b)\right) = s(b), \tag{2.13}$$

where, for each i, T^i is the smooth map constructed above. Since each T^i is smooth, the set

$$P := \{(b, \mathbf{t}) \in U \times \mathbb{R}^n : \mathbf{t} \in P_b\} \tag{2.14}$$

is a smooth submanifold of $U \times \mathbb{R}^n$, diffeomorphic to $U \times \mathbb{Z}^n$. Let $U \times \mathbb{T}^n$ denote the quotient of $U \times \mathbb{R}^n$ by P. Equation (2.13) shows that $\tilde{\Upsilon}$ factors through a map

$$\Upsilon : U \times \mathbb{T}^n \to \pi^{-1}(U); \tag{2.15}$$

the arguments used above to prove that each fibre is diffeomorphic to $\mathbb{T}^n$ shows that the map Υ yields a diffeomorphism which makes the following diagram commute

$$\begin{tikzcd} U \times \mathbb{T}^n \arrow[rr, "\Upsilon"] \arrow[dr, "\mathrm{pr}_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U. & \end{tikzcd}$$

In other words, Υ gives a trivialisation of the Lagrangian bundle.

It is important to remark that the above trivialisation depends upon a choice of section. Fix a choice of coordinatisation on U and choose a different section $s' : U \to \pi^{-1}(U)$. Let $\tilde{\Upsilon}' : U \times \mathbb{R}^n \to \pi^{-1}(U)$ be the map defined as in equation (2.12) and denote by Υ' the induced map on $U \times \mathbb{T}^n$. The composite

$$\tilde{\Upsilon}^{-1} \circ \tilde{\Upsilon}' : U \times \mathbb{R}^n \to U \times \mathbb{R}^n$$

is given by

$$(b, \mathbf{t}) \mapsto (b, \mathbf{t} + \mathbf{t}_0(b)), \tag{2.16}$$

where $\mathbf{t}_0 : U \to \mathbb{R}^n$ is a smooth map such that

$$\Psi(\mathbf{t}_0, s(b)) = s'(b).$$

Note that smoothness of $\mathbf{t}_0$ is implied again by the implicit function theorem. This composite descends to a well-defined map

$$\Upsilon^{-1} \circ \Upsilon : U \times \mathbb{T}^n \to U \times \mathbb{T}^n,$$

which is just a translation along the fibres in the affine coordinates of the fibres and, thus, a fibrewise affine diffeomorphism.

Observe also that, for any choice of coordinate neighbourhood $U \subset B$, the fibres of the Lagrangian bundle can be smoothly identified with the affine manifold $\mathbb{R}^n/\mathbb{Z}^n$ of Example 2.14.iv. This can be done by pre-composing any trivialisation $\Upsilon : U \times \mathbb{T}^n \to \pi^{-1}(U)$ with a fibrewise affine diffeomorphism

$$\begin{aligned} U \times \mathbb{R}^n/\mathbb{Z}^n &\to U \times \mathbb{T}^n \\ (b, \boldsymbol{\theta}) &\mapsto (b, A(b)\boldsymbol{\theta}), \end{aligned} \tag{2.17}$$

where $\boldsymbol{\theta} = (\theta^1, \ldots, \theta^n)$ denote affine coordinates on $\mathbb{R}^n/\mathbb{Z}^n$ and

$$A(b) = (T^1(b), \ldots, T^n(b)).$$

Fix $b_0 \in B$, let $U, U' \subset B$ be open coordinate neighbourhoods of b_0, such that $U \cap U'$ is connected, and let $s : U \to \pi^{-1}(U)$, $s' : U' \to \pi^{-1}(U')$ be locally defined sections. The pairs (U, s), (U', s') induce trivialisations

$$\Upsilon : U \times \mathbb{R}^n/\mathbb{Z}^n \to \pi^{-1}(U),\ \Upsilon' : U' \times \mathbb{R}^n/\mathbb{Z}^n \to \pi^{-1}(U').$$

Let

$$\varphi = \Upsilon'^{-1} \circ \Upsilon : (U \cap U') \times \mathbb{R}^n/\mathbb{Z}^n \to (U \cap U') \times \mathbb{R}^n/\mathbb{Z}^n$$

denote the corresponding transition function. Equation (2.11) implies that, up to the choice of s and s', φ acts linearly on each fibre. *A priori* the linear part of φ gives a smooth map

$$\mathcal{M}^{-T} : U \cap U' \to \mathrm{GL}(n; \mathbb{R}),$$

using the notation of the proof of the assertion (ii). Since for each $b \in U \cap U'$, $\mathcal{M}^{-T}(b)$ maps P_b to itself and this has been identified with the standard lattice $\mathbb{Z}^n \subset (\mathbb{R}^n, +)$ via equation (2.17),

$$\mathcal{M}^{-T}(b) \in \mathrm{GL}(n; \mathbb{Z})$$

for all $b \in U \cap U'$. As $U \cap U'$ is connected and the map $\mathcal{M}^{-T}$ is continuous, it is in fact constant. For any given choice of sections s and s', the restriction of φ to each fibre is the composition of $\mathcal{M}^{-T}$ with a translation by equation (2.16). Thus the structure group of the bundle can be restricted to $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ as claimed in (iii). This completes the proof of the theorem. □

Remark 2.15. While the result of the above theorem is well-known, the proof presented above highlights the importance of affine geometry in the structure of Lagrangian bundles; furthermore, it pinpoints where the symplectic structure on M is used to derive the required results. In particular, note that the only place where closure of the symplectic form ω is used in the above proof is equation (2.6), while elsewhere the only essential ingredient is just non-degeneracy. Hence there is a broader class of bundles for which the above theorem holds, namely those bundles $F \hookrightarrow (M, \varpi) \to B$, where ϖ is a non-degenerate 2-form on M, F is a Lagrangian submanifold of (M, ϖ) and $\mathrm{d}\varpi = \pi^*\mu$ for a closed 3-form μ defined on B. The local structure of such bundles has been studied by Fassò and Sansonetto in [25] in relation to a more general form of complete integrability than that given by Definition 2.10.

Remark 2.16. The subgroup of $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ consisting of translations can be identified with the affine manifold $\mathbb{R}^n/\mathbb{Z}^n$ via the map that takes a translation $\mathcal{T}$ to $\mathcal{T}\boldsymbol{\theta}_0$, where $\boldsymbol{\theta}_0 \in \mathbb{R}^n/\mathbb{Z}^n$ is any fixed point. This explains why $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ is denoted as the semidirect product of $\mathrm{GL}(n; \mathbb{Z})$ and $\mathbb{R}^n/\mathbb{Z}^n$.

Notation. A Lagrangian bundle is henceforth denoted by $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ to highlight the importance of the natural affine structure on the fibres.

For a fixed Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$, the proof of Theorem 2.3 implies that at each point $b \in B$, the isomorphism class of the isotropy group P_b of the action Ψ defined in equation (2.8) (with respect to any choice of local coordinates around b) is well-defined.

Definition 2.17 (Period lattice bundle, Duistermaat [20])**.** The isomorphism class of the lattice P_b is called the *period lattice* at $b \in B$. Let P denote the union of period lattices over all points $b \in B$. The natural projection

$$\mathrm{pr} : P \to B$$

yields a $\mathbb{Z}^n$-bundle over B called the *period lattice bundle* associated to the Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$.

Remark 2.18. Under the identification of each fibre of a Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ with the affine manifold $\mathbb{R}^n/\mathbb{Z}^n$, for each $b \in B$, the fibre P_b of the associated period lattice bundle $\mathbb{Z}^n \hookrightarrow P \to B$ can be identified with a choice of n linearly independent vector fields $X_1(b), \ldots, X_n(b)$ tangent to the fibre $F_b \cong \mathbb{R}^n/\mathbb{Z}^n$ whose flows are periodic with period 1. Identifying $X_i(b)$ with the homology of the cycle given by considering the time-1 map of its flow, the period lattice at b can be identified with a choice of basis for $\mathrm{H}_1(F_b; \mathbb{Z}) \cong \mathbb{Z}^n$. This is the original interpretation of the period lattice bundle as given in [20].

Remark 2.19 (Transition functions for the period lattice bundle, [16])**.** The proof of Theorem 2.3 implies that a choice of trivialising cover for a Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ for which the transition functions act on each fibre by elements of $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ yields a trivialising cover for the associated period lattice bundle. Moreover, the transition functions for the latter are given by considering only the linear part of the transition functions of the former.

Historically, the existence of the period lattice bundle associated to a Lagrangian bundle has yielded a topological classification of Lagrangian bundles, as carried out in [20]. The approach taken in this thesis is slightly different, since the existence of the period lattice bundle yields a well-defined affine structure on the fibres of a Lagrangian bundle, which has been used to notice that the structure group of the bundle reduces to $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$.

Chapter 3

Topological classification of Lagrangian bundles

In this chapter, topological universal Lagrangian bundles are constructed using the result of Theorem 2.3. These objects arise from the topology of the group $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$; Theorem 3.6 below proves that they are in fact universal for Lagrangian bundles. Section 3.1 is devoted to studying the topology of the classifying space $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ and to constructing the topological universal Lagrangian bundles. In Section 3.2, the two topological invariants of Lagrangian bundles are constructed using these universal bundles; these characteristic classes have already been constructed in several works in the literature (*e.g.* [18, 20, 40, 68]), but not using the approach taken in this thesis, which emphasises the importance of the smoothly varying affine structure on the fibres of a Lagrangian bundle. This chapter is largely based on the published article [55].

3.1 Topological universal Lagrangian bundles

Theorem 2.3 proves that the structure group of a Lagrangian bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$$

reduces to the group

$$\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) := \mathrm{GL}(n; \mathbb{Z}) \ltimes \mathbb{R}^n/\mathbb{Z}^n,$$

where the action of $\mathrm{GL}(n; \mathbb{Z})$ on $\mathbb{R}^n/\mathbb{Z}^n$ is given by

$$\begin{aligned} \mathrm{GL}(n; \mathbb{Z}) \times \mathbb{R}^n/\mathbb{Z}^n &\to \mathbb{R}^n/\mathbb{Z}^n \\ (A, \boldsymbol{\theta}) &\mapsto A\boldsymbol{\theta}, \end{aligned} \tag{3.1}$$

and $\boldsymbol{\theta} = (\theta^1, \ldots, \theta^n)$ denotes affine coordinates on $\mathbb{R}^n/\mathbb{Z}^n$. Note that the expression $A\boldsymbol{\theta}$ makes sense as $\mathrm{GL}(n; \mathbb{Z})$ stabilises the standard cocompact lattice in $(\mathbb{R}^n, +)$. In this section some bundles, called *topological universal Lagrangian bundles*, are defined using the topology of the group $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$; these are then shown to be universal in the sense that every Lagrangian bundle arises as the pull-back of one of these bundles (cf. Theorem 3.6).

Let 0 and 1 denote the trivial group with additive and multiplicative structures respectively. There is a short exact sequence of groups

$$0 \longrightarrow \mathbb{R}^n/\mathbb{Z}^n \xrightarrow{\tau} \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) \xrightarrow{p} \mathrm{GL}(n;\mathbb{Z}) \longrightarrow 1, \tag{3.2}$$

where the homomorphisms τ, p are defined by

$$\begin{aligned} \tau : \mathbb{R}^n/\mathbb{Z}^n &\to \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) \\ \boldsymbol{\theta} &\mapsto (I, \boldsymbol{\theta}) \end{aligned} \tag{3.3}$$

$$\begin{aligned} p : \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) &\to \mathrm{GL}(n;\mathbb{Z}) \\ (A, \boldsymbol{\theta}) &\mapsto A, \end{aligned} \tag{3.4}$$

and I denotes the identity in $\mathrm{GL}(n;\mathbb{Z})$. The sequence of equation (3.2) is split, *i.e.* there exists a homomorphism $\sigma : \mathrm{GL}(n;\mathbb{Z}) \to \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ which is a right inverse for p. Explicitly, the splitting is defined by

$$\begin{aligned} \sigma : \mathrm{GL}(n;\mathbb{Z}) &\to \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) \\ A &\mapsto (A, \mathbf{0}), \end{aligned} \tag{3.5}$$

where $\mathbf{0}$ denotes the identity in the group $\mathbb{R}^n/\mathbb{Z}^n$. The idea is to construct interesting bundles using the injections σ, τ defined above. In order to do so, it is necessary to introduce the concepts of the *classifying space* of an arbitrary topological group G.

3.1.1 Universal bundles

In this subsection, a few generalities on universal bundles and classifying spaces are recalled to illustrate some of the results needed in the construction of topological universal Lagrangian bundles.

Theorem 3.1 (Universal bundles [44])**.** *Let G be a topological group. There exists a principal G-bundle (defined uniquely up to homotopy), called the* universal G-bundle*,*

$$G \hookrightarrow \mathrm{E}G \to \mathrm{B}G,$$

with the following properties

i $\mathrm{E}G, \mathrm{B}G$ *are CW-complexes;*

ii $\mathrm{E}G$ *is contractible;*

iii if $G \hookrightarrow E \to B$ is any principal G-bundle, there exists a map (defined uniquely up to homotopy)

$$\chi : B \to \mathrm{B}G,$$

called the classifying map*, such that the pull-back bundle $\chi^*\mathrm{E}G \to B$ is isomorphic to the original bundle.*

Notation. Throughout this thesis, all universal bundles, those bundles which are constructed from universal bundles by means of standard constructions, and classifying maps are understood to be defined up to homotopy without further mention.

Definition 3.1 (Classifying space)**.** Let G be a topological group. The base space $\mathrm{B}G$ of its universal bundle is called the *classifying space* of G.

Example 3.2 (Universal bundles)**.**

i) The $\mathbb{Z}$-principal bundle

$$\mathbb{Z} \hookrightarrow \mathbb{R} \xrightarrow{\exp(i\cdot)} S^1 \subset \mathbb{C}$$

is the universal bundle for the group $\mathbb{Z}$;

ii) If G, H are topological groups, then the principal $G \times H$-bundle

$$G \times H \hookrightarrow \mathrm{E}G \times \mathrm{E}H \to \mathrm{B}G \times \mathrm{B}H$$

is the universal $G \times H$-bundle. In particular, the universal $\mathbb{Z}^n$-bundle is given by the bundle

$$\mathbb{Z}^n \hookrightarrow \mathbb{R}^n \to \mathbb{T}^n$$

where the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ is by translations along the standard cocompact lattice in $(\mathbb{R}^n, +)$ (cf. Example 2.14.iv);

iii) For each n, let $S^n \hookrightarrow S^{n+1}$ be the inclusion given by identifying the equator of S^{n+1} with S^n. Taking the direct limit of these inclusions, it is possible to define S^∞, which is a contractible CW-complex, since each inclusion $S^n \hookrightarrow S^{n+1}$ kills $\pi_n(S^n)$. Similarly, the direct limit of the inclusions $\mathbb{C}\mathrm{P}^{n-1} \hookrightarrow \mathbb{C}\mathrm{P}^n$ induced by the standard inclusions $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$, gives rise to a countable CW-complex $\mathbb{C}\mathrm{P}^\infty$. For each n, there is a principal S^1-bundle

$$S^1 \hookrightarrow S^{2n-1} \to \mathbb{C}\mathrm{P}^{n-1},$$

which is well-behaved with respect to the above inclusions, *i.e.* there is a commutative diagram

$$\begin{tikzcd} S^{2n-1} \arrow[r, hook] \arrow[d] & S^{2n+1} \arrow[d] \\ \mathbb{C}\mathrm{P}^{n-1} \arrow[r, hook] & \mathbb{C}\mathrm{P}^n. \end{tikzcd}$$

Thus, in the direct limit, there is a principal S^1-bundle

$$S^1 \hookrightarrow S^\infty \to \mathbb{C}\mathrm{P}^\infty$$

which is the universal bundle for the group S^1.

Remark 3.3. Let $F \hookrightarrow E \to B$ be a fibre bundle with structure group G. Then there exists a principal G-bundle $G \hookrightarrow Q \to B$ such that the original bundle is isomorphic to the bundle $Q \times_G F \to B$, obtained via the Borel construction (cf. [17]). The isomorphism class of the G-principal bundle $Q \to B$ is determined by the isomorphism class of the bundle $E \to B$. This principal G-bundle is said to be *associated* to the original G-bundle. Hence, if $\chi : B \to \mathrm{B}G$ denotes the classifying map for the principal G-bundle $Q \to B$, then χ can also be thought as the classifying map for the G-bundle $E \to B$ with fibre F.

Remark 3.3 implies that, for a given Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$, there is a well-defined notion of a classifying map

$$\chi : B \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n),$$

which is just the classifying map for the principal $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$-bundle associated to $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$.

The following theorems are used in Section 3.1.2; they are stated below without proof (cf. [38, 65]).

Theorem 3.2. *Let G, H be topological groups and let $\iota : H \hookrightarrow G$ be a monomorphism. Then there exists a bundle*

$$G/H \hookrightarrow \mathrm{B}H \to \mathrm{B}G.$$

If, in addition, $\iota(H) \lhd G$, then the above bundle is a G/H-principal bundle.

Theorem 3.3. *An exact sequence of topological groups*

$$1 \to K \to G \to H \to 1$$

induces a fibration (up to homotopy)

$$\mathrm{B}K \to \mathrm{B}G \to \mathrm{B}H.$$

A splitting $H \to G$ induces (up to homotopy) a section $\mathrm{B}H \to \mathrm{B}G$ of the above fibration.

Theorem 3.4. *Let H be a closed subgroup of a topological group G. The structure group of a principal G-bundle $G \hookrightarrow E \to B$ can be reduced to H if and only if there exists a lift $\chi_H : B \to \mathrm{B}H$ of the classifying map $\chi_G : B \to \mathrm{B}G$, i.e. there exists a commutative diagram*

$$\begin{tikzcd} & \mathrm{B}H \arrow[d] \\ B \arrow[ur, "\chi_H"] \arrow[r, "\chi_G"'] & \mathrm{B}G, \end{tikzcd}$$

where the projection $\mathrm{B}H \to \mathrm{B}G$ arises as in Theorem 3.2.

3.1.2 Bundles arising from $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$

In this subsection Theorem 3.2 and 3.3 are used to construct bundles relating the classifying spaces of the groups $\mathbb{R}^n/\mathbb{Z}^n$, $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ and $\mathrm{GL}(n;\mathbb{Z})$. In particular, for each n, the splitting σ defined by equation (3.5) gives rise to the topological universal Lagrangian bundle.

By Theorem 3.3, the short exact sequence

$$0 \longrightarrow \mathbb{R}^n/\mathbb{Z}^n \xrightarrow{\tau} \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) \xrightarrow{p} \mathrm{GL}(n;\mathbb{Z}) \longrightarrow 1,$$

gives rise to a fibration

$$\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n \xhookrightarrow{\tau} \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \xrightarrow{p} \mathrm{BGL}(n;\mathbb{Z}), \tag{3.6}$$

where the induced maps on classifying spaces are denoted with the same symbols as the homomorphisms inducing them. This abuse of notation is justified in light of the fact that the homomorphisms τ and p determine the homomorphisms in homotopy and cohomology of $\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$, $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ and $\mathrm{BGL}(n;\mathbb{Z})$ (cf. [12]).

Lemma 3.5 (Homotopy groups of $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ [55])**.** *The homotopy groups of the classifying space* $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ *are given by*

$$\pi_i(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) = \begin{cases} \mathrm{GL}(n;\mathbb{Z}) & \text{if } i = 1, \\ \mathbb{Z}^n & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Proof. Begin by noticing that $\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$ and $\mathrm{BGL}(n;\mathbb{Z})$ are Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Z}^n,2), \mathrm{K}(\mathrm{GL}(n;\mathbb{Z}),1)$ respectively, *i.e.*

$$\pi_i(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n) = \begin{cases} \mathbb{Z}^n & \text{if } i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\pi_i(\mathrm{BGL}(n;\mathbb{Z})) = \begin{cases} \mathrm{GL}(n;\mathbb{Z}) & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The long exact sequence in homotopy for the fibration of equation (3.6)

$$\ldots \to \pi_n(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n) \to \pi_n(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \pi_n(\mathrm{BGL}(n;\mathbb{Z})) \to \pi_{n-1}(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n) \to \ldots$$

implies that

$$\pi_n(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \cong \pi_n(\mathbb{R}^n/\mathbb{Z}^n) \cong 0$$

for all $n \geq 3$. The remaining part of the long exact sequence collapses to exact sequences

$$\begin{aligned} \pi_3(\mathrm{BGL}(n;\mathbb{Z})) \cong 0 \to \pi_2(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n) \to \pi_2(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \pi_2(\mathrm{BGL}(n;\mathbb{Z})) \cong 0 \\ \pi_1(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n) \cong 0 \to \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \pi_1(\mathrm{BGL}(n;\mathbb{Z})) \to \pi_0(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n) = \{*\}. \end{aligned}$$

Thus $\pi_2(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \cong \pi_2(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n)$ and $\pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \cong \pi_1(\mathrm{BGL}(n;\mathbb{Z}))$, and the result follows. $\square$

Applying Theorem 3.2 to the inclusion $\tau : \mathbb{R}^n/\mathbb{Z}^n \to \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ and to the splitting $\sigma : \mathrm{GL}(n;\mathbb{Z}) \to \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$, obtain two bundles

$$\mathrm{GL}(n;\mathbb{Z}) \hookrightarrow \mathrm{B}(\mathbb{R}^n/\mathbb{Z}^n) \xrightarrow{\tau} \mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) \tag{3.8a}$$

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{BGL}(n;\mathbb{Z}) \xrightarrow{\sigma} \mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) \tag{3.8b}$$

Remark 3.4. Note that since $\sigma(\mathrm{GL}(n;\mathbb{Z}))$ is not a normal subgroup of $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$, the fibres of $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{BGL}(n;\mathbb{Z}) \xrightarrow{\sigma} \mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ are not naturally endowed with the structure of a group.

Definition 3.5 (Topological universal Lagrangian bundles [55, 56])**.** For each n, the bundle of equation (3.8b) is called the *topological universal Lagrangian bundle* of dimension n[1].

Remark 3.6. The fibres of a topological universal Lagrangian bundle are endowed with an affine structure which makes them affinely diffeomorphic to $\mathbb{R}^n/\mathbb{Z}^n$. This is because

[1]Here, the dimension refers to the dimension of the fibre. Throughout this thesis, n is fixed, unless otherwise stated, and the topological universal Lagrangian bundle of dimension n is referred to simply as the topological universal Lagrangian bundle.

the subgroup $\tau(\mathbb{R}^n/\mathbb{Z}^n)$ of $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ can be identified with the affine manifold $\mathbb{R}^n/\mathbb{Z}^n$ (cf. Remark 2.16).

3.1.3 Universality

Theorem 3.6 below proves that the topological universal Lagrangian bundle is truly universal for Lagrangian bundles.

Theorem 3.6 (Universality [56])**.** *Let $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ be a Lagrangian bundle and let $\chi : B \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ denote its classifying map. Then $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ is isomorphic to the pull-back of the topological universal Lagrangian bundle along χ*

$$\begin{tikzcd} \chi^*\mathrm{BGL}(n;\mathbb{Z}) \arrow[r, "\Xi"] \arrow[d] & \mathrm{BGL}(n;\mathbb{Z}) \arrow[d] \\ B \arrow[r, "\chi"'] & \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n). \end{tikzcd}$$

Proof. The universal bundle $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) \hookrightarrow \mathrm{EAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ classifies principal $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$-bundles; thus the associated bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$$

classifies the topological type of $\mathbb{R}^n/\mathbb{Z}^n$-bundles with structure group $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$. For notational ease, set

$$\mathrm{E} = \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n \tag{3.9}$$

throughout the rest of the proof. If $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ is a Lagrangian bundle and $\chi : B \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ is its classifying map, there is an isomorphism of fibre bundles

$$\begin{tikzcd} M \arrow[rr, "\cong"] \arrow[dr] & & \chi^*\mathrm{E} \arrow[dl] \\ & B. & \end{tikzcd}$$

Hence it suffices to show that, up to homotopy, there is an isomorphism of bundles

$$\begin{tikzcd} \mathrm{BGL}(n;\mathbb{Z}) \arrow[rr, "\cong"] \arrow[dr] & & \mathrm{E} \arrow[dl] \\ & \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n). & \end{tikzcd}$$

The group $\mathrm{GL}(n;\mathbb{Z})$ acts freely on the contractible space $\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)$ via σ; thus, up to homotopy, $\mathrm{BGL}(n;\mathbb{Z})$ can be constructed as the quotient $\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)/\mathrm{GL}(n;\mathbb{Z})$. Consider the continuous map

$$\begin{aligned} \varkappa : \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) &\to \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n \\ e &\mapsto (e, \mathbf{0}). \end{aligned}$$

The aim of the proof is to show that $\varkappa$ descends to a map

$$\begin{aligned} \bar{\varkappa} : \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)/\mathrm{GL}(n;\mathbb{Z}) &\to \mathrm{E} \\ [e]_{\mathrm{GL}(n;\mathbb{Z})} &\mapsto [(e,\mathbf{0})]_{\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)}, \end{aligned} \tag{3.10}$$

where E is as in equation (3.9) and $[.]_{\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)}$, $[.]_{\mathrm{GL}(n;\mathbb{Z})}$ denote elements of E and $\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)/\mathrm{GL}(n;\mathbb{Z})$ respectively. The map $\bar{\varkappa}$ is then shown to be the required isomorphism.

Let

$$q : \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n \to \mathrm{E}$$

denote the quotient map. The map

$$\begin{aligned} q \circ \varkappa : \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) &\to \mathrm{E} \\ e &\mapsto [(e, \mathbf{0})]_{\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)} \end{aligned}$$

is continuous; moreover, if, for given e, $e' \in \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)$ there exists $A \in \mathrm{GL}(n;\mathbb{Z})$ such that

$$e' = e \cdot (A, \mathbf{0}),$$

(where the convention is that a group H acts on the right on $\mathrm{E}H$, following [17]), then

$$[(e, \mathbf{0})]_{\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)} = [(e', \mathbf{0})]_{\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)}.$$

In particular, $q \circ \varkappa$ descends to the map $\bar{\varkappa}$, which is therefore well-defined. Reversing the above argument proves that $\bar{\varkappa}$ is injective. Furthermore, note that for all $(e, \mathbf{t}) \in \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n$, there exists an element $(e', \mathbf{0}) \in \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n$ such that

$$[(e, \mathbf{t})]_{\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)} = [(e', \mathbf{0})]_{\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)}.$$

This proves that $\bar{\varkappa}$ is surjective. Thus the map $\bar{\varkappa}$ is continuous and bijective.

It remains to construct an inverse. The continuous map

$$\begin{aligned} \varepsilon : \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n &\to \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \\ (e, \mathbf{t}) &\mapsto e \end{aligned}$$

descends to a map

$$\hat{\varepsilon} : \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n \to \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)/\mathrm{GL}(n;\mathbb{Z}),$$

which satisfies

$$\hat{\varepsilon}(e, \mathbf{t}) = \hat{\varepsilon}(e \cdot (A, \mathbf{t}'), (A, \mathbf{t}')^{-1} \cdot \mathbf{t})$$

for all $(e, \mathbf{t}) \in \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n$ and $(A, \mathbf{t}') \in \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$. Hence, $\hat{\varepsilon}$ descends to a continuous map

$$\bar{\varkappa}^{-1} : \mathrm{E} \to \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)/\mathrm{GL}(n;\mathbb{Z}),$$

which is the inverse of $\bar{\varkappa}$. Therefore $\bar{\varkappa}$ is a homeomorphism.

Finally the map $\bar{\varkappa}$ makes the following diagram commutative

$$\begin{tikzcd} \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)/\mathrm{GL}(n;\mathbb{Z}) \arrow[rr, "\bar{\varkappa}"] \arrow[dr] & & \mathrm{E} \arrow[dl] \\ & \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n), & \end{tikzcd} \tag{3.11}$$

where E is as in equation (3.9). Equation (3.11) proves the result. $\square$

Remark 3.7. As observed in the literature (*e.g.* [8, 20, 40]), the fibres of a Lagrangian bundle are not naturally equipped with a group structure. This can be proved directly using Theorem 3.6 and Remark 3.4.

3.2 Topological invariants

In light of Theorem 3.6, the topological invariants of Lagrangian bundles are pull-backs of *universal* invariants, which characterise the topological universal Lagrangian bundle. In this section, the two topological invariants of Lagrangian bundles are defined in this fashion; these are shown to coincide with different definitions in the literature (*e.g.* [16, 18, 20, 40, 68]) and are proved to be *sharp*, *i.e.* they completely determine the topological type of the bundle.

3.2.1 Monodromy

Fix a Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ with classifying map $\chi : B \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$.

Definition 3.8 (Monodromy [55, 56])**.** Let $b_0 \in B$ be a basepoint. The *monodromy* of $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ is defined to be the homomorphism

$$\chi_* : \pi_1(B; b_0) \to \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); \chi(b_0)) \cong \mathrm{GL}(n; \mathbb{Z}).$$

Remark 3.9.

i) There is a notion of *universal* monodromy, which arises from considering the classifying map of the topological universal Lagrangian bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{BGL}(n; \mathbb{Z}).$$

This map is just the identity $\mathrm{id} : \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$. Thus the universal monodromy is simply the identity homomorphism

$$\mathrm{id}_* : \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)),$$

where the dependence upon the basepoint is dropped, as it does not affect the induced homomorphism. Note further that the topological monodromy of $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{BGL}(n; \mathbb{Z})$, *i.e.* the representation

$$\pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \mathrm{Aut}(\pi_1(\mathbb{R}^n/\mathbb{Z}^n)),$$

which arises from considering the action on $\pi_1(\mathbb{R}^n/\mathbb{Z}^n)$ induced by the homotopy equivalences of the fibres obtained by lifting (homotopy classes of) loops in $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$, is given precisely by the identity homomorphism of $\mathrm{GL}(n; \mathbb{Z})$. This follows from the fact that the $\mathrm{GL}(n; \mathbb{Z})$-action on $\mathbb{R}^n/\mathbb{Z}^n$ which defines $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ as a split $\mathbb{R}^n/\mathbb{Z}^n$-extension of $\mathrm{GL}(n; \mathbb{Z})$ is given by the identity homomorphism of $\mathrm{GL}(n; \mathbb{Z})$;

ii) The choice of base point $b_0 \in B$ may affect the image of the homomorphism χ_*, without, however, changing its conjugacy class in $\mathrm{GL}(n; \mathbb{Z})$. The *free monodromy* of a Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ is defined to be the conjugacy class of the image of the monodromy $[\chi_*]$ and is therefore independent of the choice of base point $b \in B$. The free monodromy of a Lagrangian bundle is a genuine invariant of the isomorphism type of the bundle;

iii) The bundle

$$\mathrm{GL}(n;\mathbb{Z}) \hookrightarrow \mathrm{B}(\mathbb{R}^n/\mathbb{Z}^n) \xrightarrow{\tau} \mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$$

of equation (2.3) is the universal covering of $\mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$. Thus if the (free) monodromy χ_* of a Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ is trivial, the classifying map χ admits a lift $\tilde{\chi} : B \to \mathrm{B}(\mathbb{R}^n/\mathbb{Z}^n)$. This implies that the structure group of the Lagrangian bundle can be reduced to the group $\mathbb{R}^n/\mathbb{Z}^n$ (cf. [38]).

The monodromy of a Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ was originally defined to be the homotopy class of the classifying map of the associated period lattice bundle $\mathbb{Z}^n \hookrightarrow P \to B$ (cf. [20]). This can be seen using topological universal Lagrangian bundles.

Definition 3.10 (Universal period lattice bundles [56])**.** The *universal period lattice bundle* associated to the topological universal Lagrangian bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{BGL}(n;\mathbb{Z})$$

is the induced system of local coefficients with fibre $\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$, denoted by

$$\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \hookrightarrow P_n \to \mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n).$$

The pull-back $\chi^* P_n \to B$ is isomorphic to the system of local coefficients obtained by replacing each fibre $\mathbb{R}^n/\mathbb{Z}^n$ of $\chi^*\mathrm{BGL}(n;\mathbb{Z}) \to B$ with $\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$. By Theorem 3.6, $\chi^*\mathrm{BGL}(n;\mathbb{Z}) \to B$ is isomorphic to the Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ whose classifying map is given by χ.

Lemma 3.7. *Let $P \to B$ denote the period lattice bundle associated to the Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ (with classifying map χ) as in Definition 2.17. Then $\chi^* P_n \to B$ is isomorphic to $P \to B$.*

Proof. The $\mathbb{Z}^n$-bundles $P \to B$ and $\chi^* P_n \to B$ are isomorphic if and only if their classifying maps are homotopic. Let

$$\chi_{P_n} : \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{BGL}(n;\mathbb{Z})$$

denote the classifying map of the universal period lattice bundle $\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \hookrightarrow P_n \to \mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$. Then the classifying map of $\chi^* P_n \to B$ is given by $\chi_{P_n} \circ \chi$. Let $\chi_P : B \to \mathrm{BGL}(n;\mathbb{Z})$ denote the classifying map of the period lattice bundle $P \to B$. Since $\mathrm{BGL}(n;\mathbb{Z})$ is a $\mathrm{K}(\mathrm{BGL}(n;\mathbb{Z});1)$ space, the homotopy class of the maps $\chi_P, \chi_{P_n} \circ \chi$ is determined by the induced maps on fundamental groups (up to a choice of basepoint $b \in B$). Fix a basepoint $b \in B$; it suffices to show that $\chi_{P_*} = (\chi_{P_n} \circ \chi)_*$.

Note that the transition functions for $P \to B$ are given by considering the linear part of the transition functions for the associated Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$. Thus, up to homotopy,

$$\chi_P = p \circ \chi,$$

where $p : \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{BGL}(n;\mathbb{Z})$ is the map induced by the homomorphism $p : \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{GL}(n;\mathbb{Z})$ of the long exact sequence of equation (3.2) defining

$\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$. This is because the collection of transition functions of a bundle determines the homotopy class of the classifying map of the bundle (cf. [38]). In particular,

$$\chi_{P_*} = p_* \circ \chi_*.$$

Since $p_* : \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \pi_1(\mathrm{BGL}(n;\mathbb{Z}))$ is the identity (as the homomorphism $p : \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{GL}(n;\mathbb{Z})$ induces the identity map on $\pi_0(\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n))$), then

$$\chi_{P_*} = \chi_*. \tag{3.12}$$

Hence it suffices to prove that the classifying map

$$\chi_{P_n} : \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{BGL}(n;\mathbb{Z})$$

of the universal period lattice bundle induces the identity map on fundamental groups. Note that the homotopy class of χ_{P_n} is determined by the topological monodromy of the topological universal Lagrangian bundle (cf. Remark 3.9.i). In particular, since the latter is given by the identity representation of $\mathrm{GL}(n;\mathbb{Z})$, it follows that the classifying map χ_{P_n} of $\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \hookrightarrow P_n \to \mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ induces the identity on fundamental groups, and the result follows. $\square$

The (free) monodromy χ_* of the Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ determines the homotopy class of the classifying map of $\chi^* P_n \to B$, which is isomorphic to the period lattice bundle $P \to B$ by Lemma 3.7. This proves that the monodromy of Definition 3.8 defined above is equivalent to the notion given in [20].

It is also important to expand upon Remark 3.9.iii, which says that the (free) monodromy can be seen as an obstruction to reducing the structure group of the Lagrangian bundle to $\mathbb{R}^n/\mathbb{Z}^n$. The following lemma, proved using topological universal Lagrangian bundles, shows that the (free) monodromy can be seen as the obstruction for a Lagrangian bundle to be a principal $\mathbb{R}^n/\mathbb{Z}^n$-bundle, as remarked by several authors (cf. [16, 20, 40, 68]).

Lemma 3.8 ([20]). *The (free) monodromy of a Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ is trivial if and only if the bundle is a principal $\mathbb{R}^n/\mathbb{Z}^n$-bundle.*

Proof. If the (free) monodromy is trivial, then the period lattice bundle $P \to B$ is also trivial, *i.e.* $P \cong B \times \mathbb{Z}^n$. A smooth section of this bundle corresponds to a smoothly varying frame $X_1, \ldots, X_n$ of the tangent bundle of the fibre F of the Lagrangian bundle associated to $P \to B$. Fix such a choice.

By considering a simply connected open neighbourhood $U \subset B$, there exist functions

$$\bar{f}_1, \ldots, \bar{f}_n : \pi^{-1}(U) \to \mathbb{R}$$

such that each X_i is the Hamiltonian vector field of $\bar{f}_i$. Note that if Y is a vector field tangent to the fibres, Lagrangianeity implies that, for all i, $\iota(Y)\mathrm{d}\bar{f}_i = 0$, which in turn means that there exists functions

$$f_1, \ldots, f_n : U \to \mathbb{R}$$

such that

$$\mathrm{d}\bar{f}_i = \pi^* \mathrm{d} f_i$$

for all i. The functions $f_1, \dots f_n$ give a local coordinatisation on U and, thus, along with a choice of section $s : U \to \pi^{-1}(U)$, induce a local trivialisation Υ_U of $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$, as proved in Theorem 2.3. Let $\Psi_U : \mathbb{R}^n/\mathbb{Z}^n \times \pi^{-1}(U) \to \pi^{-1}(U)$ denote the locally defined smooth $\mathbb{R}^n/\mathbb{Z}^n$-action given by Theorem 2.3. Note that this action is free, as it is just given by translation in affine coordinates on the fibres. Cover B by such neighbourhoods, and let U' be another such neighbourhood with $U \cap U' \neq \emptyset$. The locally defined smooth actions Ψ_U, $\Psi_{U'}$ depend only on a local choice of frame of the tangent bundle to the fibres of $\pi^{-1}(U \cap U') \to U \cap U'$; having fixed a *globally* defined such choice above, these actions patch together on $U \cup U'$. Thus there is a globally defined free $\mathbb{R}^n/\mathbb{Z}^n$-action Ψ on the fibres of the Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ (which is given by left translations in affine coordinates).

Conversely, if $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ is a principal $\mathbb{R}^n/\mathbb{Z}^n$-bundle, its structure group can be reduced to $\mathbb{R}^n/\mathbb{Z}^n$. If $\chi : B \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ denotes the classifying map, then χ admits a lift $\tilde{\chi} : B \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$ by Theorem 3.4. This implies that the (free) monodromy is trivial and the result follows. □

3.2.2 Chern class

Free monodromy is not the only topological invariant of a Lagrangian bundle. This can be seen by considering the case when the (free) monodromy vanishes and the topological classification reduces to that of principal $\mathbb{R}^n/\mathbb{Z}^n$-bundles. In this case, there is only one other topological invariant associated to the given fibre bundle, namely the obstruction to the existence of a section. In what follows, the corresponding obstruction for any Lagrangian bundle is defined using topological universal Lagrangian bundles; this is then shown to coincide with the definition given in [18, 20, 68].

The approach taken below is obstruction theoretic, which is suitable since all spaces involved are homotopic to CW complexes. For further details regarding the constructions involved, see [17, 65]. The idea is to find the obstruction to the existence of a section of the topological universal Lagrangian bundle. A section of this bundle is a lift of the identity map $\mathrm{id} : \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$, *i.e.* a map (denoted by the dotted arrow) which makes the following diagram commute

$$\begin{tikzcd} & \mathrm{BGL}(n;\mathbb{Z}) \arrow[d, "\sigma"] \\ \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \arrow[ur, dotted] \arrow[r, "\mathrm{id}"] & \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \end{tikzcd}$$

Since $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ is not simply connected, there needs to be extra care to define this obstruction, as the action of the fundamental group of $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ on the homotopy groups of the fibres $\mathbb{R}^n/\mathbb{Z}^n$ needs to be taken into account. To this end, it is important to give another description of the topological monodromy of the topological universal Lagrangian bundle.

Fix a pointed CW decomposition of $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$, *i.e.* a CW decomposition whose 0-skeleton consists of a single point, say b, which exists since $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ is path-connected. Set

$$\varpi = \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); b)$$

for notational ease. Let $\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}$ denote the universal cover of $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ (which, by Remark 3.9.iii, is a $\mathrm{K}(\mathbb{Z}^n;2)$). The given CW decomposition of $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ induces a ϖ-*equivariant* CW decomposition on $\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}$ as follows. Each k-cell e^k_α in the CW decomposition of $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ has preimages $\mathrm{e}^k_{\alpha,\gamma}$ ($\gamma \in \varpi$) under the universal covering map

$$q : \widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)} \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n).$$

Identifying ϖ with the group of deck transformations on $\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}$ acting on the left, the cells $\mathrm{e}^k_{\alpha,\gamma}$ satisfy

$$\gamma' \cdot \mathrm{e}^k_{\alpha,\gamma} = \mathrm{e}^k_{\alpha,\gamma'\gamma}$$

for all $\gamma, \gamma' \in \varpi$. For each $\gamma \in \varpi$, let ς_γ denote the corresponding deck transformation on $\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}$. The commutative diagram

$$\begin{tikzcd} \widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)} \arrow[rr, "\varsigma_\gamma"] \arrow[dr, "q"'] & & \widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)} \arrow[dl, "q"] \\ & \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) & \end{tikzcd}$$

yields a commutative diagram of bundles

$$\begin{tikzcd} q^*\mathrm{BGL}(n;\mathbb{Z}) \arrow[rr, "\Sigma_\gamma"] \arrow[dr] & & (q \circ \varsigma_\gamma)^*\mathrm{BGL}(n;\mathbb{Z}) \arrow[dl] \\ & \widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}, & \end{tikzcd} \tag{3.13}$$

where $q^*\mathrm{BGL}(n;\mathbb{Z}), (q \circ \varsigma_\gamma)^*\mathrm{BGL}(n;\mathbb{Z}) \to \widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}$ denote pull-backs of the topological universal Lagrangian bundle along q and $q \circ \varsigma_\gamma$ respectively. The isomorphism Σ_γ induces an automorphism of each fibre $\mathbb{R}^n/\mathbb{Z}^n$ whose induced map on $\pi_1(\mathbb{R}^n/\mathbb{Z}^n)$ is given by the action of $\gamma \in \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);b)$ on the fundamental group of the fibres of the topological universal Lagrangian bundle. Thus the topological monodromy of the topological universal Lagrangian bundle can be seen as the homomorphism assigning to each $\gamma \in \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);b)$ the homotopy class of the automorphism of $\mathbb{R}^n/\mathbb{Z}^n$ induced by Σ_γ.

With this setup, the problem of finding the obstruction to the existence of a section for the topological universal Lagrangian bundle can be tackled. Since both $\mathrm{BGL}(n;\mathbb{Z})$ and the fibres $\mathbb{R}^n/\mathbb{Z}^n$ are path-connected, there exists a section s_1 defined on the 1-skeleton of $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$. Therefore, if $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)^1$ denotes the 1-skeleton of $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$, there is a commutative diagram

$$\begin{tikzcd} & \mathrm{BGL}(n;\mathbb{Z}) \arrow[d, "\sigma"] \\ \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)^1 \arrow[ur, "s_1"] \arrow[r, hook] & \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n), \end{tikzcd} \tag{3.14}$$

where $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)^1 \hookrightarrow \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ denotes the standard inclusion. Let e^2_α denote the 2-cells in the pointed CW decomposition of $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ fixed above, and let $\mathrm{e}^2_{\alpha,\gamma}$, $\gamma \in \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); b)$ denote the 2-cells in the induced CW decomposition of $\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}$. For each α and γ, let

$$\varrho_\alpha : \partial \mathrm{e}^2_{\alpha,0} \cong S^1 \to \widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}^1$$

denote the attaching map of $\mathrm{e}^2_{\alpha,\gamma}$. The composite

$$\sigma \circ s_1 \circ q \circ \varrho_\alpha : S^1 \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$$

equals $\mathrm{id} \circ q \circ \varrho_\alpha$ by commutativity of the diagram in equation (3.14). Since $\mathrm{id} \circ q$ is defined on the whole of $\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}$, the map $\mathrm{id} \circ q \circ \varrho_\alpha$ can be extended to the interior of the 2-cell $\mathrm{e}^2_{\alpha,\gamma}$ and, thus, it is null-homotopic. By the homotopy lifting property of bundles (cf. [65]), this means that the map $s_1 \circ q \circ \varrho_\alpha : S^1 \to \mathrm{BGL}(n; \mathbb{Z})$ is homotopic to a map $S^1 \to \mathbb{R}^n/\mathbb{Z}^n$. For each 2-cell of $\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}$, the above method yields the homotopy class of a map $h_\gamma : S^1 \to \mathbb{R}^n/\mathbb{Z}^n$ into the fibre. The commutative diagram of equation (3.13) implies that this construction is equivariant, *i.e.*

$$(h_{\gamma'\gamma})_* = \gamma' \cdot (h_\gamma)_* \tag{3.15}$$

for all $\gamma, \gamma' \in \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); b)$, where $\gamma' \cdot$ denotes the action of $\pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); b)$ on $\pi_1(\mathbb{R}^n/\mathbb{Z}^n)$. Thus the maps h_γ give rise to an element of the group

$$\mathrm{Hom}_{\mathbb{Z}[\varpi]}(\mathrm{C}_2(\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}); \mathbb{Z}^n),$$

where $\mathbb{Z}[\varpi]$ denotes the group ring of $\pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); b)$, $\mathrm{C}_2(\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)})$ is the set of 2-cells in the equivariant CW decomposition of $\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}$, and $\mathrm{Hom}_{\mathbb{Z}[\varpi]}$ denotes the set of homomorphisms of $\mathbb{Z}[\pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); b)]$-modules. Both $\mathrm{C}_2(\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)})$ and $\mathbb{Z}^n$ are naturally $\mathbb{Z}[\varpi]$-modules: the former because of the choice of ϖ-equivariant CW decomposition of $\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}$ and the latter via the topological monodromy representation of the topological universal Lagrangian bundle.

Using standard techniques in obstruction theory (cf. [17, 65]), it can be shown that this cochain is in fact a cocycle in the cohomology theory with local coefficients determined by the representation

$$\mathrm{id}_* : \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); b) \to \mathrm{Aut}(\pi_1(\mathbb{R}^n/\mathbb{Z}^n)) \cong \mathrm{GL}(n; \mathbb{Z}) \tag{3.16}$$

given by the universal monodromy representation.

Definition 3.11 (Universal Chern class [56])**.** The cohomology class of the above cocycle

$$c_U \in \mathrm{H}^2(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); \mathbb{Z}^n_{\mathrm{id}_*}),$$

where $\mathbb{Z}^n_{\mathrm{id}_*}$ denotes the system of local coefficients defined by equation (3.16), is called the *universal Chern class.*

Remark 3.12. The above cohomology class represents the obstruction to finding a ϖ-equivariant lift of the universal covering map $q : \widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)} \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$, *i.e.*

a map (denoted by the dotted arrow) which makes the following diagrams commutative

$$\begin{tikzcd} & \mathrm{BGL}(n;\mathbb{Z}) \arrow[d, "\sigma"] \\ \widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)} \arrow[ur, dotted] \arrow[r, "q"] & \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \end{tikzcd}$$

and

$$\begin{tikzcd} \widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)} \arrow[r, dotted] \arrow[d, "\varsigma_\gamma"'] & \mathrm{BGL}(n;\mathbb{Z}) \\ \widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}, \arrow[ur, dotted] & \end{tikzcd}$$

for each $\gamma \in \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); b)$, where ς_γ denotes the deck transformation of the universal cover $\widetilde{\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)}$ induced by γ. This is precisely the obstruction to the existence of a section for the topological universal Lagrangian bundle.

The importance of the universal Chern class c_U is highlighted in Chapter 6 where it is used to study the Leray-Serre spectral sequence associated to the topological universal Lagrangian bundle.

Definition 3.13. Let $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ be a Lagrangian bundle with classifying map $\chi : B \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$. Its *Chern class* is defined to be the pull-back

$$\chi^* c_U \in \mathrm{H}^2(B; \mathbb{Z}^n_{\chi_*}).$$

Remark 3.14.

i) *A priori*, the cohomology class $\chi^* c_U$ is only the *primary* obstruction to the existence of a section for $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$, *i.e.* the vanishing of $\chi^* c_U$ may not mean that there exists a section $s : B \to M$. However, if $\chi^* c_U = 0$, then the argument of Remark 3.12 shows that there exists a section defined on the 2-skeleton of B. Since the fibres $\mathbb{R}^n/\mathbb{Z}^n$ of the bundle are $\mathrm{K}(\mathbb{Z}^n;1)$, it follows from standard arguments in obstruction theory (*e.g.* [65]) that there is no obstruction to extending this section to the other skeleta of B. Thus $\chi^* c_U$ is the only obstruction to the existence of a section;

ii) In [55] the Chern class of a Lagrangian bundle is defined to be the obstruction to the existence of a lift

$$\begin{tikzcd} & \mathrm{BGL}(n;\mathbb{Z}) \arrow[d] \\ B \arrow[ur, dotted] \arrow[r, "\chi"] & \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \end{tikzcd} .$$

This obstruction coincides with the obstruction to the existence of a section $s : B \to \chi^*\mathrm{BGL}(n;\mathbb{Z})$. Since $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ is isomorphic to the pull-back of $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{BGL}(n;\mathbb{Z}) \xrightarrow{\sigma} \mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ along χ, the latter is just $\chi^* c_U$ and the above definition coincides with that of [55];

iii) Definition 3.13 is equivalent to the definition of Chern class given in [6, 18, 20, 68].

3.3 Sharpness

The two topological invariants defined above completely determine the topological type of a Lagrangian bundle, as mentioned in [6, 16, 18, 20, 40, 50, 68]. This can also be seen using arguments in equivariant obstruction theory which are akin to those used to define the universal Chern class. Thus the following sharpness theorem is only stated.

Theorem 3.9 (Sharpness [18, 20, 40, 56]). *Two Lagrangian bundles*

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B \quad , \quad \mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M',\omega') \to B$$

are isomorphic (as fibre bundles) if and only if their free monodromies and Chern classes coincide.

Remark 3.15. It is important to note that the above theorem is merely a topological statement and does not carry any information regarding the symplectic geometry of the bundles. This deeper aspect is investigated in the following chapters.

Chapter 4

Construction of Lagrangian bundles

While the existence of topological universal Lagrangian bundles suffices to solve the classification problem of Lagrangian bundles, it does not address the question of constructing Lagrangian bundles over a given manifold B. In this chapter, the symplectic geometry of Lagrangian bundles is studied in detail in order to provide the complementary tools to tackle the construction problem. Section 4.1 shows that topological universal Lagrangian bundles are, in fact, universal for a broader family of bundles, namely those called affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles (cf. Definition 4.7). The existence of a symplectic form ω which makes the fibres of a Lagrangian bundle into maximally isotropic submanifolds of the total space imposes further restrictions on the topology and geometry of the bundle itself. For instance, Section 4.2 constructs Darboux coordinates for ω in a neighbourhood of a fibre; these coordinates are commonly called *action-angle* coordinates in Hamiltonian mechanics. The existence of such local coordinates implies that the base space of a Lagrangian bundle is an *integral affine manifold* (cf. Definition 4.15); moreover, all integral affine manifolds are the base space of some Lagrangian bundle, as illustrated in Section 4.3, where some affine invariants of the base space are related to topological invariants of the underlying Lagrangian bundle. Finally, Section 4.4 formulates the construction question (cf. Question 4.31) that is studied in Chapters 5, 6 and 7 below.

4.1 Affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles

The topological classification of Lagrangian bundles does not provide a method to determine whether a given $\mathbb{T}^n$-bundle over an n-dimensional manifold B can be endowed with the structure of a Lagrangian bundle. This can be illustrated by studying invariants of $\mathbb{T}^n$-fibrations, which have fewer constraints than the bundles under consideration.

Definition 4.1 ($\mathbb{T}^n$-fibrations)**.** A $\mathbb{T}^n$-fibration is a fibration $M \to B$ whose homotopy fibre is a $\mathrm{K}(\mathbb{Z}^n; 1)$.

Remark 4.2. Any $\mathbb{T}^n$-bundle is a $\mathbb{T}^n$-fibration, but the converse is not true, since the fibres of a $\mathbb{T}^n$-fibration are only homotopy equivalent to an n-torus.

Fix a $\mathbb{T}^n$-fibration $M \to B$ and a basepoint $b \in B$. There is a homomorphism which sends each element $\gamma \in \pi_1(B; b)$ to the homotopy equivalence of the homotopy fibre F_b obtained by lifting γ to M using the homotopy lifting property of fibrations (cf. [17]). Let $\mathrm{Hty}(\mathrm{K}(\mathbb{Z}^n;1))$ denote the group of homotopy classes of self-homotopy equivalences of $\mathrm{K}(\mathbb{Z}^n;1)$ with group operation given by composition of maps (defined up to homotopy). The homomorphism

$$\begin{aligned} \mathrm{Hty}(\mathrm{K}(\mathbb{Z}^n;1)) &\to \mathrm{Aut}(\pi_1(\mathrm{K}(\mathbb{Z}^n;1))) \cong \mathrm{GL}(n;\mathbb{Z}) \\ h &\mapsto h_*, \end{aligned} \tag{4.1}$$

is, in fact, an isomorphism, since it is the restriction of the bijection

$$[\mathrm{K}(\mathbb{Z}^n;1),\mathrm{K}(\mathbb{Z}^n;1)] \to \hom(\mathbb{Z}^n;\mathbb{Z}^n)$$

to the subset of homotopy classes of self-homotopy equivalences of $\mathrm{K}(\mathbb{Z}^n;1)$.

Given a $\mathbb{T}^n$-fibration $M \to B$, there is a homomorphism

$$\chi_* : \pi_1(B;b) \to \mathrm{Hty}(\mathrm{K}(\mathbb{Z}^n;1),$$

defined as follows (cf. [17]). Let $\gamma : [0,1] \to B$ be a loop based at b. Let F_b denote the homotopy fibre of $M \to B$ at b; the inclusion $F_b \hookrightarrow M$ makes the following diagram commutative

$$\begin{tikzcd} F_b \times \{0\} \arrow[r, hook] \arrow[d, hook] & M \arrow[d] \\ F_b \times [0,1] \arrow[r, "H"'] & B, \end{tikzcd}$$

where $H : F_b \times [0,1]\times \to B$ is of the form

$$H(e,t) = \alpha(t)$$

for all $e \in F_b$ and $t \in [0,1]$. By the homotopy lifting property of fibrations (cf. [17]), the map H admits a lift $\tilde{H}$ which makes the following diagram commutative

$$\begin{tikzcd} F_b \times \{0\} \arrow[r, hook] \arrow[d, hook] & M \arrow[d] \\ F_b \times [0,1] \arrow[r, "H"'] \arrow[ur, "\tilde{H}"] & B. \end{tikzcd}$$

Since the composite

$$F_b \xrightarrow{\tilde{H}(-,1)} M \longrightarrow B$$

is the constant map at $b \in B$, it follows that it defines a map

$$F_b \to F_b,$$

the homotopy class of this map does not depend on the homotopy class of α. Therefore, there is a map

$$\chi_* : \pi_1(B;b) \to \mathrm{Hty}(\mathrm{K}(\mathbb{Z}^n;1)) \cong \mathrm{GL}(n;\mathbb{Z})$$

which, in fact, is a homomorphism.

Definition 4.3 (Monodromy of a $\mathbb{T}^n$-fibration [17])**.** The homomorphism χ_* defined above is called the *monodromy* of the fibration $M \to B$.

Remark 4.4. If B is connected, the free monodromy of a $\mathbb{T}^n$-fibration $M \to B$ is the conjugacy class of the image of the monodromy χ_* in $\mathrm{GL}(n;\mathbb{Z})$. This notion is independent of the choice of basepoint in B.

It is also possible to define the obstruction to the existence of a section $s : B \to M$ (up to homotopy) using the same technique as that employed to define the universal Chern class c_U in Section 3.2.2. This yields a cohomology class

$$c \in \mathrm{H}^2(B; \mathbb{Z}^n_{\chi_*}).$$

Definition 4.5 (Chern class of a $\mathbb{T}^n$-fibration [17])**.** The cohomology class c constructed above is called the *Chern class* of the $\mathbb{T}^n$-fibration $M \to B$.

The (free) monodromy and Chern class of a $\mathbb{T}^n$-fibration are the only two topological invariants attached to such fibrations. Thus the topological invariants defined in Chapter 3 are not unique to Lagrangian bundles. The above example motivates the following question.

Question 4.6. Are there topological/smooth/symplectic invariants which determine whether a locally trivial $\mathbb{T}^n$-bundle over an n-dimensional manifold B can be endowed with the structure of a Lagrangian bundle?

One of the most important smooth invariants attached to a Lagrangian bundle is the smoothly varying affine structure on the fibres constructed in the proof of Theorem 2.3. In particular, the fibres of a Lagrangian bundle can be smoothly identified with the affine manifold $\mathbb{R}^n/\mathbb{Z}^n$.

Definition 4.7 (Affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles [5])**.** A locally trivial $\mathbb{T}^n$-bundle is called an *affine* $\mathbb{R}^n/\mathbb{Z}^n$-bundle if its structure group reduces to $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$.

Remark 4.8. The fibres of an affine $\mathbb{R}^n/\mathbb{Z}^n$-bundle can be smoothly identified with the affine manifold $\mathbb{R}^n/\mathbb{Z}^n$. The topological classification of affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles can be carried out as in Section 3.2 above, since their structure group reduces to $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$; in particular, Theorem 3.9 applies to this more general class.

Not all locally trivial $\mathbb{T}^n$-bundles are affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles as illustrated in the next example.

Example 4.9 (Torus bundles over spheres [5, 23])**.** A result due to Farrell and Hsiang in [23] implies that there exist pairs $m, n \geq 3$ such that

$$\pi_{m-1}(\mathrm{Diff}(\mathbb{T}^n)) \neq 0.$$

Fix such a pair, let $f : S^{m-1} \to \mathrm{Diff}(\mathbb{T}^n)$ denote a map whose homotopy class represents a non-trivial element in $\pi_{m-1}(\mathrm{Diff}(\mathbb{T}^n))$, and let $M_f \to S^m$ be the $\mathbb{T}^n$-bundle obtained by using f as the clutching map, *i.e.*

$$M_f := (D^m_+ \times \mathbb{T}^n) \cup_f (D^m_- \times \mathbb{T}^n),$$

where $D^m_\pm$ denote the two hemispheres of S^m and $\cup_f$ denotes the union of the two bundles glued along their boundaries via f, so that

$$(x, \boldsymbol{\theta}) \in \partial(D^m_+ \times \mathbb{T}^n)$$

is glued to

$$(x, f(x)(\boldsymbol{\theta})) \in \partial(D^m_- \times \mathbb{T}^n).$$

Suppose the $\mathbb{T}^n$-bundle $M_f \to B$ is affine. Then its monodromy is trivial since S^m is simply-connected; thus $M_f \to B$ is a principal $\mathbb{T}^n$-bundle by the arguments of Section 3.2. Moreover, since $m \geq 3$, $\mathrm{H}^2(S^m; \mathbb{Z}^n)$ is trivial and hence the bundle $M_f \to B$ admits a section. Therefore it is a trivial $\mathbb{T}^n$-bundle, but this is a contradiction, since its classifying map

$$S^m \to \mathrm{BDiff}(\mathbb{T}^n)$$

induces a non-trivial map on m^{th}-homotopy groups.

Studying the obstruction for a locally trivial $\mathbb{T}^n$-bundle over an n-dimensional manifold B to be an affine $\mathbb{R}^n/\mathbb{Z}^n$-bundle is a problem that is not explored in greater details in the rest of this thesis. However, Example 4.9 above illustrates the importance of the geometry of the fibres of Lagrangian bundles and gives an *a posteriori* reason to study this family of bundles with the methods of the proof of Theorem 2.3.

4.2 Existence of action-angle coordinates

Example 4.9 shows that the problem of constructing Lagrangian bundles over a fixed n-dimensional manifold B depends on the topology of the base space. Consider the following question, which is a weakened version of Question 4.6 above.

Question 4.10. Let $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ be an affine $\mathbb{R}^n/\mathbb{Z}^n$-bundle over an n-dimensional manifold B. Are there smooth/symplectic obstructions to endowing the above with the structure of a Lagrangian bundle?

In general, the answer to the above question is affirmative, as illustrated by the following example.

Example 4.11 (Affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles over S^n for $n \geq 3$ [40, 55])**.** Fix $n \geq 3$. The isomorphisms classes of affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles over S^n are determined by the homotopy classes of maps

$$\chi : S^n \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n).$$

Since both S^n and $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ are CW complexes, these homotopy classes are completely determined by their actions on homotopy groups. Since $n \geq 3$, $\pi_i(S^n) = 0$ for $i = 1, 2$. Since $\pi_k(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) = 0$ for $k \geq 3$ by Lemma 3.5, it follows that any map $\chi : S^n \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ is trivial and, thus, any affine $\mathbb{R}^n/\mathbb{Z}^n$-bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ is also trivial. Suppose that $M \cong S^n \times \mathbb{T}^n$ admits a symplectic form ω. Since M is closed, the cohomology class

$$w = [\omega] \in \mathrm{H}^2(M; \mathbb{R})$$

is non-zero. The Künneth formula implies that

$$\mathrm{H}^*(M; \mathbb{R}) \cong \mathrm{H}^*(S^n; \mathbb{R}) \times \mathrm{H}^*(\mathbb{T}^n; \mathbb{R});$$

in particular, the projection onto second component

$$\mathrm{pr}_2 : S^n \times \mathbb{T}^n \to \mathbb{T}^n \tag{4.2}$$

induces an isomorphism

$$\mathrm{H}^2(\mathbb{T}^n; \mathbb{R}) \cong \mathrm{H}^*(M; \mathbb{R}).$$

Thus there exists a 2-form ω' defined on $\mathbb{T}^n$ such that

$$\omega = \mathrm{pr}_2^* \omega',$$

but this contradicts non-degeneracy of ω. Therefore M cannot be a symplectic manifold and, in particular, no affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles over S^n can be Lagrangian.

Example 4.11 uses the fact that, for $n \geq 3$, the total space of any affine $\mathbb{R}^n/\mathbb{Z}^n$-bundle over S^n is diffeomorphic to $S^n \times \mathbb{T}^n$ which is not a symplectic manifold. This argument does not work for the trivial affine $\mathbb{T}^2$-bundle over S^2, since its total space $S^2 \times \mathbb{T}^2$ is a symplectic manifold, as both S^2 and $\mathbb{T}^2$ are. In this case, the relevant question is whether there exists a symplectic form ω on $S^2 \times \mathbb{T}^2$ which makes the projection

$$S^2 \times \mathbb{T}^2 \to S^2$$

into a Lagrangian bundle. In order to answer this question, it is necessary to investigate the local structure of Lagrangian bundles from a symplectic point of view, as it is done in [8, 20, 40].

4.2.1 Canonical coordinates

This section constructs canonical local symplectic coordinates for Lagrangian bundles, which are commonly referred to in the Hamiltonian mechanics literature as *action-angle* coordinates. This construction can be found in many other works, *e.g.* [8, 20, 40], and is included here for completeness.

The starting point is a slight refinement of the proof of Theorem 2.3 which is due to Duistermaat in [20]. Recall that if $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ is a Lagrangian bundle, for each $b_0 \in B$ there exists an open neighbourhood $U \subset B$ and smooth functions

$$T^i : U \to \mathbb{R}^n,$$

for $i = 1, \ldots, n$, such that $\mathbb{Z}\langle T^1(b), \ldots, T^n(b)\rangle$ yields the period lattice P_b for each $b \in B$ (cf. Definition 2.17). Note that these maps determine the smoothly varying affine structure on the fibres of $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$, since each fibre F_b for $b \in U$ is diffeomorphic to

$$\mathbb{R}\langle T^1(b), \ldots, T^n(b)\rangle / \mathbb{Z}\langle T^1(b), \ldots, T^n(b)\rangle. \tag{4.3}$$

These smooth maps are constructed using a coordinate map $\phi : U \to \mathbb{R}^n$ and by considering the Hamiltonian vector fields $X_1, \ldots, X_n$ of the functions $(f_1, \ldots, f_n) = \phi \circ \pi$, where π denotes the projection $M \to B$.

The following lemma shows that ϕ can always be changed to another coordinate map ϕ' so that the flows of the Hamiltonian vector fields $X'_1, \ldots, X'_n$ of the functions $(f'_1, \ldots, f'_n) = \phi' \circ \pi$ on the fibre F_b are given by

$$t \mapsto \Psi^{t \cdot T^1(b)}, \ldots, t \mapsto \Psi^{t \cdot T^n(b)},$$

where Ψ denotes the action defined by the flows of the Hamiltonian vector fields $X_1, \ldots, X_n$ as in equation (2.8). In other words, B admits an atlas of 'preferred' coordinates, which play a fundamental role in the rest of this thesis. The proof of the lemma is omitted as it can be found in [20], although some of the underlying ideas are discussed in Remark 4.12.

Lemma 4.1 (Duistermaat [20])**.** *Let $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M,\omega) \to B$ be a Lagrangian bundle. For each $b \in B$, there exists an open neighbourhood $U \subset B$ and a diffeomorphism $\phi' : U \to \mathbb{R}^n$ such that the flows of the Hamiltonian vector fields of the functions $(f'_1, \ldots, f'_n) = \pi \circ \phi'$ are periodic with period 1.*

Remark 4.12. The idea underlying Lemma 4.1 is that the smooth maps

$$T^1, \ldots, T^n : U \to \mathbb{R}^n$$

can be thought of as sections of the cotangent bundle $\mathrm{T}^*U \to U$. This can be seen as follows. Let $\phi : U \to \mathbb{R}^n$ be a diffeomorphism or, equivalently, a coordinate map inducing the smooth maps $T^1, \ldots, T^n$. Then ϕ induces a trivialisation

$$\mathrm{T}^*U \cong U \times \mathbb{R}^n.$$

Under this trivialisation, the submanifold

$$P = \{(b, \mathbf{t}) \in U \times \mathbb{R}^n : \mathbf{t} \in P_b\}$$

of equation (2.14) becomes a submanifold of T^*U which is also denoted by P. Furthermore, the smooth maps $T^1, \ldots, T^n$ are identified with sections of the cotangent bundle $\mathrm{T}^*U \to U$. The above trivialisation induces an isomorphism of fibre bundles

$$\mathrm{T}^*U/P \cong U \times \mathbb{T}^n;$$

composing this isomorphism with the trivialisation Υ (which depends on a choice of local section $s : U \to \pi^{-1}(U)$) of equation (2.15), obtain a trivialisation (also denoted by Υ)

$$\Upsilon : \mathrm{T}^*U/P \to \pi^{-1}(U). \tag{4.4}$$

The main idea in the proof of Lemma 4.1 is that, by shrinking U if needed so that it is simply connected, the coordinate map $\phi : U \to \mathbb{R}$ can be chosen so that the sections $T^1, \ldots, T^n$ of $\mathrm{T}^*U \to U$ are *closed*. This is equivalent to the statement of Lemma 4.1, as shown in [20]. Denote this choice of coordinate map by ϕ'.

With this choice, P is a Lagrangian submanifold of $(\mathrm{T}^*U, \Omega_U)$, where Ω_U denotes the canonical symplectic form on T^*U (cf. Example 2.3). Thus T^*U/P inherits a symplectic form ω_0 from the symplectic manifold $(\mathrm{T}^*U, \Omega_U)$. If $a^1, \ldots, a^n$ denote local coordinates on U induced by ϕ', and $(a^1, \ldots, a^n, p^1, \ldots, p^n)$ are the induced local coordinates on T^*U, then

$$\Omega_U = \sum_{i=1}^n \mathrm{d}a^i \wedge \mathrm{d}p^i$$

and

$$\omega_0 = \sum_{i=1}^n \mathrm{d}a^i \wedge \mathrm{d}\theta^i, \tag{4.5}$$

where $\theta^1, \ldots, \theta^n$ are affine coordinates on the fibres $\mathbb{T}^n$ which take values in $\mathbb{R}/\mathbb{Z}$. This is because ϕ' is chosen so that the flows of the vector fields $\partial/\partial\theta^1, \ldots, \partial/\partial\theta^n$ are periodic with period 1. Note that these affine coordinates on the fibres coincide with the affine coordinates on the fibres given by the proof of Theorem 2.3 (cf. equation (2.17)). Finally, by construction, the trivialisation of equation (4.4) satisfies

$$\Upsilon_* \frac{\partial}{\partial \theta^i} = X_i' \tag{4.6}$$

for all i, where $X_1', \ldots, X_n'$ are the Hamiltonian vector fields of the functions

$$(f_1', \ldots, f_n') = \phi' \circ \pi.$$

Lemma 4.1 provides the existence of preferred coordinates $\mathbf{a}$ on the base space B of a Lagrangian bundle; using the ideas in the proof of Theorem 2.3, corresponding affine coordinates $\boldsymbol{\theta}$ on the fibres can be constructed. However, there is freedom in choosing the latter coordinates, as the constraint given by equation (4.6) does not specify affine coordinates uniquely. Recall that the trivialisation Υ constructed in Remark 4.12 depends upon a choice of section

$$s : U \to \pi^{-1}(U).$$

It is natural to demand that the coordinates on the fibres be chosen so that, under Υ, the section s corresponds to a *preferred* Lagrangian section of

$$(\mathrm{T}^*U/P, \omega_0) \to U,$$

namely the zero section. This can always be achieved by translating the affine coordinates along the fibres, as mentioned in [8]. With this preferred choice of coordinates,

$$\Upsilon^* \omega|_{\pi^{-1}(U)} = \omega_0, \tag{4.7}$$

i.e. the map Υ is a symplectomorphism.

Definition 4.13 (Action-angle coordinates)**.** The coordinates

$$(\mathbf{a}, \boldsymbol{\theta}) = (a^1, \ldots, a^n, \theta^1, \ldots, \theta^n)$$

induced by Υ on $\pi^{-1}(U)$ are called local *action-angle coordinates.*

The above discussion proves the following theorem.

Theorem 4.2 (Existence of local action-angle coordinates [2, 20])**.** *Let $F \hookrightarrow (M, \omega) \to B$ be a Lagrangian bundle. There exists an open cover $\mathcal{U} = \{U_\alpha\}$ of B and diffeomorphisms $\phi_\alpha : U_\alpha \to \mathbb{R}$ inducing symplectic trivialisations of the bundle*

$$\Upsilon_\alpha : (\mathrm{T}^*U_\alpha/P_\alpha, \omega_{0,\alpha}) \to (\pi^{-1}(U_\alpha), \omega_\alpha),$$

where $P_\alpha \subset (\mathrm{T}^*U_\alpha, \Omega_\alpha)$ is the Lagrangian submanifold

$$P_\alpha = \{(\mathbf{a}_\alpha, \mathbf{p}_\alpha) \in \mathrm{T}^*U_\alpha : \mathbf{p}_\alpha \in \mathbb{Z}\langle \mathrm{d}a_\alpha^1, \ldots, \mathrm{d}a_\alpha^n \rangle\},$$

$a_\alpha^1, \ldots, a_\alpha^n$ are local coordinates on U_α induced by ϕ_α, and $(\mathbf{a}_\alpha, \mathbf{p}_\alpha)$ are canonical coordinates on T^*U_α. In other words, if $(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha)$ are action-angle coordinates on $\mathrm{T}^*U_\alpha/P_\alpha$, then

$$\omega_{0,\alpha} = \sum_{i=1}^n \mathrm{d}a_\alpha^i \wedge \mathrm{d}\theta_\alpha^i.$$

Remark 4.14. The existence of canonical coordinates in the neighbourhood of a Lagrangian submanifold also follows from Weinstein's Lagrangian neighbourhood theorem [64]. However, the above approach is more revealing from the point of view of affine geometry. For instance, the angle coordinates $\theta^1, \ldots, \theta^n$ on the fibres of a Lagrangian bundle are naturally affine. Moreover, Lemma 4.1 implies that the base space B of a Lagrangian bundle inherits naturally the structure of an affine manifold, as proved in the next section.

4.3 Integral affine geometry of the base space

The existence of local action-angle coordinates imposes constraints on the topology and geometry of Lagrangian bundles. For the purposes of this work, the most important consequence of Lemma 4.1 is that the base space B of a Lagrangian bundle inherits the structure of an *integral affine manifold*.

Definition 4.15 (Integral affine manifold)**.** An n-dimensional affine manifold $(B, \mathcal{A})$ is said to be *integral* if the coordinate changes in its affine structure $\mathcal{A}$ are *integral affine* maps of $\mathbb{R}^n$, *i.e.* to elements of the group

$$\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) := \mathrm{GL}(n;\mathbb{Z}) \ltimes \mathbb{R}^n.$$

Example 4.16 (Integral affine manifolds)**.**

i) The examples of affine manifolds in Example 2.14 are also integral affine;

ii) There exist examples of affine manifolds which are not integral affine, *e.g.* affine structures on $\mathbb{T}^2$ as constructed by Arrowsmith and Furness in [28];

iii) A diffeomorphism
$$(B, \mathcal{A}) \to (B', \mathcal{A}')$$
between (integral) affine manifolds is (integral) affine if it is (integral) affine in local (integral) affine coordinates. There exist examples of affine diffeomorphisms between integral affine manifolds which are not integral affine diffeomorphism. For instance, consider lattices
$$\Lambda_1 = \mathbb{Z},\ \Lambda_2 = 2\mathbb{Z}$$
in $(\mathbb{R}, +)$. The integral affine manifolds $\mathbb{R}/\Lambda_1, \mathbb{R}/\Lambda_2$ are affinely diffeomorphic, but not integrally affinely diffeomorphic.

The following lemma proves that integral affine geometry is deeply related to the study of Lagrangian bundles.

Lemma 4.3 ([20, 40, 55])**.** *A manifold B is the base space of a Lagrangian bundle if and only if it is an integral affine manifold.*

Proof. Let $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ be a Lagrangian bundle. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover as in the statement of Theorem 4.2, *i.e.* there exist diffeomorphisms $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ inducing local coordinates $a^1_\alpha, \ldots, a^n_\alpha$ on U_α whose Hamiltonian vector fields have flows which are periodic with period 1. By shrinking the open sets, it is possible to assume that they are contractible and that each non-empty intersection $U_\alpha \cap U_\beta$ is connected.

Fix α, β such that $U_\alpha \cap U_\beta \neq \emptyset$. Let $a^1_\alpha, \ldots, a^n_\alpha,\ a^1_\beta, \ldots, a^n_\beta$ be coordinates on $U_\alpha \cap U_\beta$ induced by ϕ_α, ϕ_β respectively, and let $X_{1,\alpha}, \ldots, X_{n,\alpha},\ X_{1,\beta}, \ldots, X_{n,\beta}$ be the Hamiltonian vector fields of the functions $a^1_\alpha, \ldots, a^1_\alpha,\ a^1_\beta, \ldots, a^1_\beta$ respectively. Since the flows of the vector fields $X_{i,\beta},\ X_{j,\alpha}$ have period 1 and $U_\alpha \cap U_\beta$ is connected, there exist integers $m^{\beta\alpha}_{ij}$ satisfying

$$X_{i,\beta} = \sum_{j=1}^{n} m^{\beta\alpha}_{ij} X_{j,\alpha} \tag{4.8}$$

for each i (cf. proof of Theorem 2.3). Each $m^{\beta\alpha}_{ij} \in \mathbb{Z}$ since

$$\{X_{1,\alpha}, \ldots, X_{n,\alpha}\} \quad \text{and} \quad \{X_{1,\beta}, \ldots, X_{n,\beta}\}$$

are bases of the period lattice P_b, which has been identified with the standard lattice $\mathbb{Z}^n \subset (\mathbb{R}^n, +)$ via the choice of action coordinates $\mathbf{a}_\alpha$ and $\mathbf{a}_\beta$ (cf. Remark 4.14). As

$$\iota(X_{k,\alpha})\omega = \mathrm{d}a^k_\alpha,\ \iota(X_{k,\beta})\omega = \mathrm{d}a^k_\beta,$$

equation (4.8) implies that

$$\mathrm{d}a^i_\beta = \mathrm{d}\left(\sum_{j=1}^{n} m^{\beta\alpha}_{ij} a^j_\alpha\right).$$

Since $U_\alpha \cap U_\beta$ is connected, this implies that there exist constants $d^{\beta\alpha}_i \in \mathbb{R}$ such that

$$a^i_\beta = \sum_{j=1}^{n} m^{\beta\alpha}_{ij} a^j_\alpha + d^{\beta\alpha}_i \tag{4.9}$$

for all i. The matrix $(m^{\beta\alpha}_{ij})$ is invertible over the integers, as it can be seen by swapping the roles of the indices α and β above. This proves that $\{(U_\alpha, \phi_\alpha)\}$ defines an integral affine structure on B.

Conversely, let $(B, \mathcal{A})$ be an n-dimensional integral manifold, with $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$, local coordinates $a^1_\alpha, \ldots, a^n_\alpha$ and coordinate changes

$$\phi_\beta \circ \phi_\alpha^{-1} = (A_{\beta\alpha}, \mathbf{d}_{\beta\alpha}),$$

where $A_{\beta\alpha} \in \mathrm{GL}(n; \mathbb{Z})$ and $\mathbf{d}_{\beta\alpha} \in \mathbb{R}^n$ are constant on $U_\alpha \cap U_\beta$ for all α, β. Each diffeomorphism

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n$$

induces an isomorphism

$$\mathrm{T}^*U_\alpha \cong \phi_\alpha^* \mathrm{T}^*\mathbb{R}^n;$$

let $(\mathbf{a}_\alpha, \mathbf{p}_\alpha)$ be the induced coordinates on the cotangent bundle T^*U_α. The submanifold

$$P_\alpha := \{(\mathbf{a}_\alpha, \mathbf{p}_\alpha) \in \mathrm{T}^*U_\alpha :\ \mathbf{p}_\alpha \in \mathbb{Z}\langle \mathrm{d}a^1_\alpha, \ldots, \mathrm{d}a^n_\alpha \rangle\}.$$

is a Lagrangian submanifold of $(\mathrm{T}^*U_\alpha, \Omega_\alpha)$, where Ω_α is the canonical symplectic form of Example 2.3. Thus Ω_α descends to a symplectic form $\omega_{0,\alpha}$, which makes the projection

$$\mathrm{T}^*U_\alpha / P_\alpha \to U_\alpha$$

into a Lagrangian bundle. Note that if $(A_{\beta\alpha}, \mathbf{d}_{\beta\alpha}) \in \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$ denotes the change of coordinates defined on $U_\alpha \cap U_\beta$, then the induced transition function of T^*B is given by

$$(\mathbf{a}_\beta, \mathbf{p}_\beta) = (A_{\beta\alpha}\mathbf{a}_\alpha + \mathbf{d}_{\beta\alpha}, A_{\beta\alpha}^{-T}\mathbf{p}_\alpha).$$

In particular, this implies that the locally defined Lagrangian bundles $(\mathrm{T}^*U_\alpha, \omega_{0,\alpha}) \to U_\alpha$ patch together to yield a globally defined Lagrangian bundle

$$(\mathrm{T}^*B/P, \omega_0) \to B, \tag{4.10}$$

and the result follows. $\square$

Remark 4.17 (Properties of $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0) \to B$)**.** The bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0) \to B$ constructed above admits a globally defined section, namely the zero section, since the locally defined $0_\alpha : U_\alpha \to U_\alpha \times \mathbb{R}^n/\mathbb{Z}^n$ patch together. The zero section is also Lagrangian, since each 0_α is.

The submanifold $P \subset \mathrm{T}^*B$ plays an important role in the construction of Lagrangian bundles.

Definition 4.18 (Period lattice bundle of an integral affine manifold)**.** Let $(B, \mathcal{A})$ be an integral affine manifold. The Lagrangian submanifold $P \subset (\mathrm{T}^*B, \Omega)$ constructed in the proof of Lemma 4.3 is called the *period lattice bundle* associated to $(B, \mathcal{A})$.

Remark 4.19. Let $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ be a Lagrangian bundle. Theorem 2.3 and Lemma 4.3 show that B can be covered by integral affine coordinate neighbourhoods U_α over which the bundle is trivial. Let $(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha)$ be action-angle coordinates on $\pi^{-1}(U_\alpha)$, so that the restriction of ω to $\pi^{-1}(U_\alpha)$, denoted by ω_α is equal to

$$\omega_\alpha = \sum_{i=1}^{n} \mathrm{d}a_\alpha^i \wedge \mathrm{d}\theta_\alpha^i.$$

Denote by $\varphi_{\beta\alpha}$ the transition functions of the Lagrangian bundle with respect to the above choice of local trivialisations. Then

i) the restriction of $\varphi_{\beta\alpha}$ to the fibres are affine transformations of $\mathbb{R}^n/\mathbb{Z}^n$ by Theorem 2.3;

ii) the restriction of $\varphi_{\beta\alpha}$ to $U_\alpha \cap U_\beta$ induces a change of integral affine coordinates, denoted by $(A_{\beta\alpha}, \mathbf{d}_{\beta\alpha})$, by Lemma 4.3;

iii) the locally defined symplectic forms $\omega_\alpha, \omega_\beta$ patch together to yield ω and, thus,

$$\varphi_{\beta\alpha}^*\omega_\beta = \omega_\alpha. \tag{4.11}$$

Therefore the transition functions of the Lagrangian bundle with respect to the above choice of locally trivial neighbourhoods are of the form

$$\varphi_{\beta\alpha}(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha) = (A_{\beta\alpha}\mathbf{a}_\alpha + \mathbf{d}_{\beta\alpha}, A_{\beta\alpha}^{-T}\boldsymbol{\theta}_\alpha + \mathbf{g}_{\beta\alpha}(\mathbf{a}_\alpha)), \tag{4.12}$$

where $A_{\beta\alpha}$ and $\mathbf{d}_{\beta\alpha}$ are as in the proof of Lemma 4.3, and $\mathbf{g}_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{R}^n/\mathbb{Z}^n$ is a smooth map which is constrained by the fact that equation (4.11) holds.

4.3.1 Affine holonomy and monodromy

Remark 4.19 allows to relate the monodromy of a Lagrangian bundle over $(B, \mathcal{A})$ to an algebraic invariant of the induced integral affine structure $\mathcal{A}$, called the affine holonomy.

Let $(N, \mathcal{A})$ be an integral affine manifold and $x \in N$ be a basepoint. Define a homomorphism

$$\mathfrak{a} : \pi_1(N; x) \to \mathrm{Aff}(\mathbb{R}^n) \tag{4.13}$$

as follows. Let $\gamma : [0,1] \to N$ be a loop based at x. Cover the image of γ with finitely many integral affine coordinate neighbourhoods $U_1, \ldots, U_n$, labelled so that there exist $0 = t_0 < t_1 < \ldots < t_n = 1$ such that

$$\gamma([t_i, t_{i+1}]) \subset U_{i+1}$$

for each $i = 0, \ldots, n-1$. Let $(A_{i+1,i}, \mathbf{d}_{i+1,i}) \in \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$ denote the changes of coordinates defined on $U_{i+1} \cap U_i$ for $i = 1, \ldots, n$, and set $U_{n+1} = U_1$, so that $(A_{n+1,n}, \mathbf{d}_{n+1,n}) = (A_{1,n}, \mathbf{d}_{1,n})$. Define

$$\mathfrak{a}(\gamma) = \begin{cases} (A_{1,n}, \mathbf{d}_{1,n}) \cdot \ldots \cdot (A_{2,1}, \mathbf{d}_{2,1}) & \text{if } n \geq 2 \\ (I, \mathbf{0}) & \text{otherwise,} \end{cases} \tag{4.14}$$

where $\cdot$ denotes composition in $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$. This construction is independent of the explicit representative γ of a fixed homotopy class in $\pi_1(N; x)$, as shown in [3], and thus it is well-defined and it is a homomorphism, as shown in [4].

Definition 4.20 (Affine and linear holonomy [4, 31])**.** The homomorphism $\mathfrak{a}$ of equation (4.14) is called the *affine holonomy* of the integral affine manifold $(N, \mathcal{A})$. Composing $\mathfrak{a}$ with the projection

$$\mathrm{Lin} : \mathrm{Aff}(\mathbb{R}^n) \to \mathrm{GL}(n; \mathbb{Z}),$$

obtain the *linear holonomy* $\mathfrak{l} : \pi_1(N; x) \to \mathrm{GL}(n; \mathbb{Z})$.

Remark 4.21. The above construction can be carried out for any affine manifold, with the only difference that, in this more general context, the affine (linear) holonomy takes values in $\mathrm{Aff}(\mathbb{R}^n)$ ($\mathrm{GL}(n; \mathbb{R})$ respectively).

Remark 4.22. The affine holonomy is only well-defined up to an explicit choice of affine structure and up to a choice of basepoint. This is because, given any affine structure $\mathcal{A}$ on a manifold N, it is possible to construct an affinely diffeomorphic structure $\mathcal{A}'$ by acting on $\mathbb{R}^n$ with an affine diffeomorphism $(A, \mathbf{b})$. Throughout this work, whenever an affine/linear holonomy homomorphism is considered, it is understood that the basepoint and the explicit integral affine structure are given.

Example 4.23 (Affine holonomy of $\mathbb{R}^n/\mathbb{Z}^n$)**.** Consider the integral affine manifold $\mathbb{R}^n/\mathbb{Z}^n$ obtained by taking the quotient of $\mathbb{R}^n$ by the standard action of $\mathbb{Z}^n$ by translations. Let $\gamma_1, \ldots, \gamma_n$ be generators of $\pi_1(\mathbb{R}^n/\mathbb{Z}^n)$ corresponding to the standard generators $\mathbf{e}^1, \ldots, \mathbf{e}^n$ of $\mathbb{Z}^n$ via the identification of the fundamental group with the group of deck transformations. Then the affine holonomy of $\mathbb{R}^n/\mathbb{Z}^n$ is given by

$$\mathfrak{a}(\gamma_i) = (I, \mathbf{e}^i)$$

for each $i = 1, \ldots, n$.

Recall that the monodromy of a Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ is the topological monodromy of the fibration (cf. Definition 3.8). In particular, it can be calculated in a very similar fashion to the way in which the affine holonomy is defined above. Let $\mathcal{A}$ denote the integral affine structure induced on B by Lemma 4.3, and let $\{U_\alpha\}$ denote an open cover of B by coordinate neighbourhoods over which the bundle can be trivialised, with transition functions given by equation (4.12). Fix a basepoint $b_0 \in B$ and let $\gamma : [0,1] \to B$ be a loop based at b_0. Let $U_1, \ldots, U_n$ be a finite cover of the image of γ defined as above, and let the transition functions be denoted by $\varphi_{i+1,i}$. Then the image of (the homotopy class of) γ under the monodromy homomorphism χ_* is given by

$$\chi_*(\gamma) = \begin{cases} A_{1,n}^{-T} \ldots A_{2,1}^{-T} & \text{if } n \geq 2 \\ I & \text{otherwise.} \end{cases} \tag{4.15}$$

Equations (4.14) and (4.15) imply that

$$\chi_* = \mathfrak{l}^{-T}, \tag{4.16}$$

i.e. the monodromy χ_* of $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ determines the linear holonomy of the induced integral affine structure $\mathcal{A}$ on the base space B.

Remark 4.24. The linear holonomy $\mathfrak{l}$ of an integral affine manifold $(B, \mathcal{A})$ does not determine the isomorphism class of the affine structure $\mathcal{A}$. For instance, the integral affine manifolds $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}/2\mathbb{Z}$ of Example 4.16.iii have trivial linear holonomy (cf. Example 4.23), but they are not diffeomorphic as integral affine manifolds.

4.3.2 Reference bundles for integral affine manifolds

Fix an integral affine manifold $(B, \mathcal{A})$ with linear holonomy $\mathfrak{l}$. The second half of the proof of Lemma 4.3 gives a recipe to construct a Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0) \to B$ with monodromy given by $\mathfrak{l}^{-T}$ and a distinguished Lagrangian section, namely the zero section.

Definition 4.25 (Symplectic reference bundle [55, 68])**.** The bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0) \to B$$

is the *symplectic reference* Lagrangian bundle for the integral affine manifold $(B, \mathcal{A})$.

Definition 4.26 (Topological reference bundle [53])**.** The isomorphism class of

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0) \to B$$

as an affine $\mathbb{R}^n/\mathbb{Z}^n$-bundle is called the *topological reference* Lagrangian bundle for $(B, \mathcal{A})$.

Remark 4.27 (Importance of reference bundles)**.** The existence of reference bundles associated to an integral affine manifold makes the problem of constructing Lagrangian bundles over $(B, \mathcal{A})$ less hard. The more complicated case in which the bundles are allowed to have singularities has been studied extensively, *e.g.* [62, 50, 66, 68]. In particular, [68] remarks that the lack of a reference bundle for manifolds which could be the base space of singular Lagrangian bundles makes the construction problem much harder to study, as illustrated in the fake base space example of the same paper.

Remark 4.28 (Symplectic reference bundle)**.** It is important to notice that the symplectic reference Lagrangian bundle is equipped with a distinguished Lagrangian section. Any other Lagrangian bundle which admits a Lagrangian section is fibrewise symplectomorphic to the symplectic reference Lagrangian bundle, *i.e.* there exists a fibre bundle isomorphism which is also a symplectomorphism of the total spaces. Such a symplectomorphism is sometimes referred to as a *polarisation* in the literature, *e.g.* [48].

Example 4.29 (Symplectic reference Lagrangian bundles [40, 53])**.** Let $\mathbb{R}/\mathbb{Z}, \mathbb{R}/2\mathbb{Z}$ be the integral affine manifolds of Example 4.16.iii. Their linear holonomies are trivial, so that their symplectic reference Lagrangian bundles are principal S^1-bundles with a section, *i.e.* they are globally trivial. Thus their topological reference Lagrangian bundles coincide. However, the total spaces of their symplectic reference Lagrangian bundles are symplectomorphic to

$$(\mathbb{R}^2/\mathbb{Z}^2, \omega_1),\ (\mathbb{R}^2/2\mathbb{Z} \oplus \mathbb{Z}, \omega_2)$$

respectively, where ω_1, ω_2 are symplectic forms which descend from the standard symplectic form Ω obtained by considering $\mathbb{R}^2 \cong \mathrm{T}^*\mathbb{R}$, cf. [40]. These two symplectic manifolds cannot be symplectomorphic since the total spaces have different volumes. This example can been refined to work in the case when the total spaces have the same volume, cf. [53].

The idea that lies at the heart of Example 4.29 is that the symplectic reference Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0) \to B$ associated to an integral affine manifold $(B, \mathcal{A})$ contains information about the integral affine structure $\mathcal{A}$. This idea is further explored in Chapter 7.

4.4 Almost Lagrangian bundles

In light of Lemma 4.3, the base space of a Lagrangian bundle is necessarily an integral affine manifold. However, not all $\mathbb{T}^n$-bundles over an n-dimensional integral affine manifold are Lagrangian as the next example shows (cf. Example 4.9).

Example 4.30 (A torus bundle over an integral affine manifold)**.** A result of Farrell and Hsiang [23] implies that

$$\pi_4(\mathrm{Diff}(\mathbb{T}^{67})) \otimes \mathbb{Q} \cong \mathbb{Q}^{67}; \tag{4.17}$$

in particular, the above homotopy group is non-trivial. Let $f : S^4 \to \mathrm{Diff}(\mathbb{T}^{67})$ represent a non-trivial homotopy class in $\pi_4(\mathrm{Diff}(\mathbb{T}^{67}))$. Consider the open subset $B = \mathbb{R}^{67} \setminus V \subset \mathbb{R}^{67}$ obtained by removing the closed subspace

$$V = \{\mathbf{x} \in \mathbb{R}^{67} : x^1 = 0, \ldots, x^6 = 0\} \cong \mathbb{R}^{61}.$$

Note that B is homotopy equivalent to S^5 and that it inherits an integral affine structure $\mathcal{A}$ from $\mathbb{R}^{67}$, as in Example 2.14.ii. The linear holonomy $\mathfrak{l}$ of $(B, \mathcal{A})$ is trivial, since $\pi_1(B)$ is trivial. Let $M_f \to B$ be the isomorphism class of the $\mathbb{T}^{67}$-bundle classified by f, as in Example 4.9. This bundle has trivial monodromy and so the condition of equation (4.16) is satisfied. However, it is not a Lagrangian bundle, as otherwise it would be trivial, since it would be a principal $\mathbb{T}^{67}$-bundle with a section. Note that equation

(4.17) implies that there exist countably many pairwise non-isomorphic such examples over B.

Note that what fails in the above example is the lack of a smoothly varying affine structure on the fibres, as in Example 4.9. The question that is considered in the rest of the thesis is stated below.

Question 4.31. Let $(B, \mathcal{A})$ be an n-dimensional integral affine manifold with linear holonomy $\mathfrak{l}$, and let $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ be an affine $\mathbb{R}^n/\mathbb{Z}^n$-bundle classified by the homotopy class of a map

$$\chi : B \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n),$$

which satisfies the condition of equation (4.16). What is the obstruction to endowing M with a symplectic form ω which makes the bundle Lagrangian?

Definition 4.32 (Almost Lagrangian bundles)**.** For a fixed n-dimensional integral affine manifold $(B, \mathcal{A})$ with linear holonomy $\mathfrak{l}$, the affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles over B whose monodromy satisfies the condition of equation (4.16) are called *almost Lagrangian.*

Remark 4.33 (Classification of almost Lagrangian bundles)**.** The isomorphism classes of almost Lagrangian bundles over an n-dimensional integral affine manifold $(B, \mathcal{A})$ with linear holonomy $\mathfrak{l}$ are classified by elements of the group $\mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}})$. This is because they are affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles with monodromy $\mathfrak{l}^{-T}$ (cf. Remark 4.8). The almost Lagrangian bundle corresponding to $0 \in \mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}})$ is isomorphic to the topological reference Lagrangian bundle of Definition 4.26. The proof of Lemma 4.3 shows that the total space of this bundle can be endowed with a symplectic form ω which makes the bundle Lagrangian. In particular, if ω is chosen so that the bundle admits a Lagrangian section, the resulting Lagrangian bundle is fibrewise symplectomorphic to the symplectic reference Lagrangian bundle (cf. [53]).

Remark 4.34 (Local trivialisations of almost Lagrangian bundles)**.** Fix an almost Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ over $(B, \mathcal{A})$ and denote by P its period lattice bundle

$$\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n; \mathbb{Z}) \hookrightarrow P \to B. \tag{4.18}$$

The isomorphism class of the covering projection of equation (4.18) is classified by a homomorphism

$$\pi_1(B) \to \mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}(n; \mathbb{Z}),$$

which, up to a choice of basepoint, equals the inverse transposed $\mathfrak{l}^{-T}$ of the linear holonomy of $(B, \mathcal{A})$ by definition. If $P_{(B,\mathcal{A})} \to B$ denotes the period lattice bundle associated to $(B, \mathcal{A})$ (cf. Definition 4.18), then there is an isomorphism

$$P \cong P_{(B,\mathcal{A})}. \tag{4.19}$$

Let $\mathcal{U} = \{U_\alpha\}$ be a good open cover of B by integral affine coordinate neighbourhoods. The restriction

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \pi^{-1}(U_\alpha) \to U_\alpha \tag{4.20}$$

is an almost Lagrangian bundle. Since U_α is contractible, the above bundle is trivial and therefore there exists a section $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$. Fix such a section. Using the ideas of the proof of Theorem 2.3, each fibre $\mathbb{R}^n/\mathbb{Z}^n$ of the bundle of equation (4.20) can be identified with the integral affine manifold

$$\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n; \mathbb{R})/\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n; \mathbb{Z})$$

by defining a free and transitive action of $\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})$ on the fibre $\mathbb{R}^n/\mathbb{Z}^n$ whose isotropy group is $\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$ (cf. [68]). Since $P_\alpha = P|_{U_\alpha} \to U_\alpha$ is trivial, the above identification can be extended to a trivialisation

$$\pi^{-1}(U_\alpha) \cong (P_\alpha \otimes_{\mathbb{Z}} \mathbb{R})/P_\alpha$$

using the section s_α. The isomorphism of equation (4.19) extends to yield an isomorphism

$$P \otimes_{\mathbb{Z}} \mathbb{R} \cong P_{(B,\mathcal{A})} \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{T}^*B,$$

where the second isomorphism follows from the definition of $P_{(B,\mathcal{A})}$. In particular, the restriction of this isomorphism to $\pi^{-1}(U_\alpha)$ defines a trivialisation

$$\Upsilon_\alpha : \pi^{-1}(U_\alpha) \to \mathrm{T}^*U_\alpha/P_{(B,\mathcal{A})}|_{U_\alpha} \tag{4.21}$$

which maps the section s_α to the zero section of $\mathrm{T}^*U_\alpha/P_{(B,\mathcal{A})}|_{U_\alpha} \to U_\alpha$. Note that if ω_0 denotes the symplectic form making the bundle

$$\mathrm{T}^*B/P_{(B,\mathcal{A})} \to B$$

into the symplectic reference Lagrangian bundle associated to $(B,\mathcal{A})$, and $\omega_{0,\alpha}$ is its restriction to $\mathrm{T}^*U_\alpha/P_{(B,\mathcal{A})}|_{U_\alpha}$, then

$$\Upsilon_\alpha^*\omega_{0,\alpha}$$

makes the bundle of equation (4.20) Lagrangian. Furthermore, the section s_α is also Lagrangian.

If $(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha)$ denote action-angle coordinates on $\mathrm{T}^*U_\alpha/P_{(B,\mathcal{A})}|_{U_\alpha}$, then they are action-angle coordinates on $\pi^{-1}(U_\alpha) \to U_\alpha$ via the trivialisation Υ_α. Thus almost Lagrangian bundles admit local action-angle coordinates. Furthermore, the transition functions

$$\varphi_{\beta\alpha} = \Upsilon_\beta \circ \Upsilon_\alpha^{-1} : \mathrm{T}^*(U_\alpha \cap U_\beta)/P_{(B,\mathcal{A})}|_{U_\alpha\cap U_\beta} \to \mathrm{T}^*(U_\alpha \cap U_\beta)/P_{(B,\mathcal{A})}|_{U_\alpha\cap U_\beta}$$

take the familiar form (cf. equation (4.12))

$$\varphi_{\beta\alpha}(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha) = (A_{\beta\alpha}\mathbf{a}_\alpha + \mathbf{d}_{\beta\alpha}, A_{\beta\alpha}^{-T}\boldsymbol{\theta}_\alpha + \mathbf{g}_{\beta\alpha}(\mathbf{a}_\alpha)), \tag{4.22}$$

where the first component comes from an affine change of coordinates on $U_\alpha \cap U_\beta$, and the linear part of the second component is determined by the transition functions for $P_{(B,\mathcal{A})} \to B$ by construction. Note that the maps $\varphi_{\beta\alpha}$ are not necessarily fibrewise symplectomorphisms as the locally defined forms $\Upsilon_\alpha^*\omega_{0,\alpha}$ do not necessarily patch together.

Remark 4.35 (Equivalent definition of almost Lagrangian bundles [18]). Fix an n-dimensional integral affine manifold $(B,\mathcal{A})$ with linear holonomy $\mathfrak{l}$, and let $P_{(B,\mathcal{A})} \subset \mathrm{T}^*B$ denote its associated period lattice bundle. Denote by $\mathcal{P}$ the sheaf of smooth sections of the projection $P \to B$. Note that P is isomorphic to the period lattice bundle of any Lagrangian bundle over B whose monodromy equals $\mathfrak{l}^{-T}$, as proved in [20]. Thus there is an isomorphism of cohomology groups

$$\mathrm{H}^i(B;\mathcal{P}) \cong \mathrm{H}^i(B;\mathbb{Z}^n_{\mathfrak{l}^{-T}})$$

for all i. The sheaf $\mathcal{P}$ fits into a short exact sequence (cf. [20])

$$0 \to \mathcal{P} \to \mathcal{C}^\infty(\mathrm{T}^*B) \to \mathcal{C}^\infty(\mathrm{T}^*B/P) \to 0, \tag{4.23}$$

where $\mathcal{C}^\infty(\mathrm{T}^*B)$ and $\mathcal{C}^\infty(\mathrm{T}^*B/P)$ are the sheaves of smooth sections of the cotangent bundle $\mathrm{T}^*B \to B$ and of the topological reference Lagrangian bundle $\mathrm{T}^*B/P \to B$ respectively. The long exact sequence in cohomology induced by equation (4.23) collapses to isomorphisms

$$\mathrm{H}^i(B;\mathcal{C}^\infty(\mathrm{T}^*B/P)) \cong \mathrm{H}^{i+1}(B;\mathcal{P})$$

for $i \geq 1$, since $\mathcal{C}^\infty(\mathrm{T}^*B)$ is a fine sheaf (cf. [37]). In particular,

$$\mathrm{H}^1(B;\mathcal{C}^\infty(\mathrm{T}^*B/P)) \cong \mathrm{H}^2(B;\mathcal{P}).$$

Hence

$$\mathrm{H}^1(B;\mathcal{C}^\infty(\mathrm{T}^*B/P)) \cong \mathrm{H}^2(B;\mathbb{Z}^n_{\mathfrak{l}^{-T}}). \tag{4.24}$$

The group $\mathrm{H}^1(B;\mathcal{C}^\infty(\mathrm{T}^*B/P))$ classifies the isomorphism classes of bundles over B which are locally isomorphic to $\mathrm{T}^*B/P \to B$ and have structure sheaf $\mathcal{C}^\infty(\mathrm{T}^*B/P)$ (cf. [18, 35]). These are bundles which admit local trivialisations which are isomorphic to the trivialisations for $\mathrm{T}^*B/P \to B$ and with transition functions given by locally defined sections of $\mathrm{T}^*B/P \to B$. In light of Remark 4.33 and equation (4.24), an almost Lagrangian bundle satisfies this condition and, conversely, any bundle over B with the above properties is almost Lagrangian. This is the point of view taken in [18], and, more generally, in other works in the literature, *e.g.* [20, 40, 68].

Remark 4.36 (Relation to integrable systems [25])**.** Almost Lagrangian bundles are the correct geometric setting for studying the type of generalised Liouville integrability that Fassò and Sansonetto consider in [25]. It can be shown that the total space of an almost Lagrangian bundle admits an appropriate non-degenerate 2-form with respect to which the fibres are maximally isotropic submanifolds. This work will appear in [52].

Not all almost Lagrangian bundles are Lagrangian. Fix an n-dimensional integral affine manifold $(B,\mathcal{A})$ with linear holonomy $\mathfrak{l}$, and let $P \subset \mathrm{T}^*B$ be the period lattice bundle associated to $(B,\mathcal{A})$ (cf. Definition 4.18). The sheaf of smooth sections $\mathcal{P}$ of $P \to B$ fits in a short exact sequence

$$0 \to \mathcal{P} \to \mathcal{Z}(\mathrm{T}^*B) \to \mathcal{Z}(\mathrm{T}^*B/P) \to 0, \tag{4.25}$$

where $\mathcal{Z}(\mathrm{T}^*B)$ and $\mathcal{Z}(\mathrm{T}^*B/P)$ are the sheaves of closed sections of $\mathrm{T}^*B \to B$ and $\mathrm{T}^*B/P \to B$ respectively. Note that Lagrangianeity of P is necessary to construct the above exact sequence (cf. Lemma 4.1). The induced long exact sequence in cohomology groups yields a homomorphism

$$\mathcal{D}_{(B,\mathcal{A})} : \mathrm{H}^2(B;\mathcal{P}) \to \mathrm{H}^2(B;\mathcal{Z}(\mathrm{T}^*B)),$$

where the subscript denotes the dependence upon the integral affine structure of the manifold B. This homomorphism depends on $\mathcal{A}$ as the latter determines P as a Lagrangian submanifold of (T^*B,Ω) (cf. Definition 4.18). By Poincaré Lemma (cf. [64]),

$$\mathrm{H}^2(B;\mathcal{Z}(\mathrm{T}^*B)) \cong \mathrm{H}^3(B;\mathbb{R}),$$

where $\mathrm{H}^3(B;\mathbb{R})$ denotes cohomology with real coefficients, and the isomorphism between this cohomology theory and the cohomology with coefficients in the constant sheaf $\mathcal{R}$ over B with $\mathbb{R}$ coefficients is used tacitly. Hence there is a homomorphism

$$\mathcal{D}_{(B,\mathcal{A})} : \mathrm{H}^2(B;\mathcal{P}) \to \mathrm{H}^3(B;\mathbb{R}). \tag{4.26}$$

Theorem 4.4 (Dazord and Delzant [18])**.** *Let $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ be an almost Lagrangian bundle over $(B,\mathcal{A})$ with linear holonomy $\mathfrak{l}$, and let*

$$c \in \mathrm{H}^2(B;\mathbb{Z}^n_{\mathfrak{l}^{-T}}) \cong \mathrm{H}^2(B;\mathcal{P})$$

be its Chern class. Then $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ is Lagrangian if and only if

$$\mathcal{D}_{(B,\mathcal{A})}c = 0. \tag{4.27}$$

The remaining chapters of this thesis are devoted to proving that the homomorphism $\mathcal{D}_{(B,\mathcal{A})}$ is related to a differential on the E_2-page of the Leray-Serre spectral sequence of the topological universal Lagrangian bundles constructed in Chapter 3. This observation allows to give a different proof of Theorem 4.4, which highlights the importance of integral affine geometry. The aim is to prove that $\mathcal{D}_{(B,\mathcal{A})}$ is determined by the universal Chern class c_U and the *radiance obstruction* $r_{(B,\mathcal{A})}$, a cohomological invariant associated to an (integral) affine manifold $(B,\mathcal{A})$. The next chapter computes the map $\mathcal{D}_{(B,\mathcal{A})}$ in the case $(B,\mathcal{A}) = \mathbb{R}^3/\mathbb{Z}^3$, thus providing the first explicit examples of almost Lagrangian bundles which are not Lagrangian. Such bundles are called *fake* Lagrangian. These examples should serve as motivation for Chapters 6 and 7.

Chapter 5

Fake Lagrangian bundles over $\mathbb{R}^3/\mathbb{Z}^3$

In this chapter, the problem of determining which almost Lagrangian bundles are Lagrangian is solved when the base space is $\mathbb{R}^3/\mathbb{Z}^3$, *i.e.* the integral affine manifold obtained by considering the standard affine action of $\mathbb{Z}^3$ on $\mathbb{R}^3$, as in Example 2.14.iv. Section 5.1 provides a necessary and sufficient condition for an almost Lagrangian bundle over $\mathbb{R}^3/\mathbb{Z}^3$ to be Lagrangian (cf. Theorem 5.1). As a result, the first known examples of fake Lagrangian bundles are constructed. In Section 5.2 a specific example is considered to illustrate that the total space of a fake Lagrangian bundle can, in fact, also be the total space of a Lagrangian bundle over a different integral affine manifold.

Remark 5.1. Throughout this chapter, the isomorphism between singular cohomology with real coefficients and de Rham cohomology is used tacitly.

5.1 Almost Lagrangian bundles over $\mathbb{R}^3/\mathbb{Z}^3$

Let a^1, a^2, a^3 denote local integral affine coordinates on $\mathbb{R}^3/\mathbb{Z}^3$, which are the standard mod 1 coordinates on the 3-torus. The linear holonomy of $\mathbb{R}^3/\mathbb{Z}^3$ is trivial (cf. Example 4.23); thus any almost Lagrangian bundle over $\mathbb{R}^3/\mathbb{Z}^3$ is a principal $\mathbb{R}^3/\mathbb{Z}^3$-bundle (cf. Section 3.2.1). The symplectic reference Lagrangian bundle for $\mathbb{R}^3/\mathbb{Z}^3$ is

$$(\mathrm{T}^*(\mathbb{R}^3/\mathbb{Z}^3)/P, \omega_0) \to \mathbb{R}^3/\mathbb{Z}^3,$$

where ω_0 is a symplectic form which descends from the canonical symplectic form $\Omega_{\mathbb{R}^3/\mathbb{Z}^3}$ on $\mathrm{T}^*(\mathbb{R}^3/\mathbb{Z}^3)$. The Lagrangian submanifold $P \subset (\mathrm{T}^*(\mathbb{R}^3/\mathbb{Z}^3), \Omega_{\mathbb{R}^3/\mathbb{Z}^3})$ is spanned by the differentials of the local integral affine coordinates on $\mathbb{R}^3/\mathbb{Z}^3$. If $(\mathbf{a}, \mathbf{p})$ denote local coordinates on $\mathrm{T}^*(\mathbb{R}^3/\mathbb{Z}^3)$,

$$P = \{(\mathbf{a}, \mathbf{p}) \in \mathrm{T}^*\mathbb{R}^3/\mathbb{Z}^3 : \mathbf{p} \in \mathbb{Z}\langle \mathrm{d}a^1, \mathrm{d}a^2, \mathrm{d}a^3 \rangle\}. \tag{5.1}$$

If $\mathcal{P}$ denotes the sheaf of sections of the covering map $P \to B$, then the isomorphism classes of almost Lagrangian bundles over $\mathbb{R}^3/\mathbb{Z}^3$ are classified by the cohomology group

$$\mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathcal{P}) \cong \mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}^3),$$

(cf. Remark 4.35). In particular, the coefficient system $\mathbb{Z}^3$ can be identified with the integral span

$$\mathbb{Z}\langle \mathrm{d}a^1, \mathrm{d}a^2, \mathrm{d}a^3 \rangle,$$

since the differential forms $\mathrm{d}a^1, \mathrm{d}a^2, \mathrm{d}a^3$ are defined globally. The universal coefficient theorem (cf. [59]) implies that

$$\mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}\langle \mathrm{d}a^1, \mathrm{d}a^2, \mathrm{d}a^3\rangle) \cong \mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}) \otimes \mathbb{Z}\langle \mathrm{d}a^1, \mathrm{d}a^2, \mathrm{d}a^3\rangle.$$

Choose an ordered basis η_1, η_2, η_3 of $\mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z})$, such that

$$\begin{aligned} \eta_1 &\mapsto \mathrm{d}a^1 \wedge \mathrm{d}a^2 \\ \eta_2 &\mapsto \mathrm{d}a^2 \wedge \mathrm{d}a^3 \\ \eta_3 &\mapsto \mathrm{d}a^3 \wedge \mathrm{d}a^1 \end{aligned} \tag{5.2}$$

under the monomorphism

$$\mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}) \hookrightarrow \mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{R}) \cong \mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Fix $\mathrm{d}a^3, \mathrm{d}a^1, \mathrm{d}a^2$ as an ordered basis of the above coefficient system $\mathcal{P}$. Let

$$c = [c_{lm}]$$

be an element of $\mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathcal{P})$, where $[c_{lm}] \in \mathrm{M}(3; \mathbb{Z})$ is the 3×3 integer-valued matrix representing c with respect to the above choices of ordered bases. Let $F \hookrightarrow M \to \mathbb{R}^3/\mathbb{Z}^3$ be the isomorphism class of the almost Lagrangian bundle over $\mathbb{R}^3/\mathbb{Z}^3$ whose Chern class is given by c. The following theorem gives a necessary and sufficient condition for this bundle to be Lagrangian.

Theorem 5.1. *The bundle $F \hookrightarrow M \to \mathbb{R}^3/\mathbb{Z}^3$ is Lagrangian if and only if*

$$\mathrm{Tr}[c_{lm}] = 0. \tag{5.3}$$

The strategy of the proof is as follows. The long exact sequence in homotopy for the almost Lagrangian bundle $F \hookrightarrow M \to \mathbb{R}^3/\mathbb{Z}^3$ (cf. [17]) is trivial except for

$$0 \to \pi_1(F) \to \pi_1(M) \to \pi_1(\mathbb{R}^3/\mathbb{Z}^3) \to 0,$$

since $F \cong \mathbb{R}^3/\mathbb{Z}^3$. Thus, as an abstract group, $\pi_1(M)$ is a $\mathbb{Z}^3$-extension of $\mathbb{Z}^3$; since the monodromy of the bundle is trivial, the only invariant classifying the above extension is a cohomology class in $\mathrm{H}^2(\mathbb{Z}^3; \mathbb{Z}^3)$, which is isomorphic to

$$\mathrm{H}^2(\mathrm{B}\mathbb{Z}^3; \mathbb{Z}^3) \cong \mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}^3).$$

In fact, the Chern class c of the almost Lagrangian bundle $F \hookrightarrow M \to \mathbb{R}^3/\mathbb{Z}^3$ classifies this extension under the above isomorphism. Henceforth, denote $\pi_1(M) = \Gamma_c$ to highlight the dependence of the fundamental group of M on the Chern class of the bundle.

The following lemma shows that the total space of any almost Lagrangian bundle over $\mathbb{R}^3/\mathbb{Z}^3$ is diffeomorphic to a quotient of $\mathrm{T}^*\mathbb{R}^3$ by some action of Γ_c.

Lemma 5.2. *There exists a representation $\Gamma_c \to \mathrm{Diff}(\mathrm{T}^*\mathbb{R}^3)$ which makes M diffeomorphic to $\mathrm{T}^*\mathbb{R}^3/\Gamma_c$.*

Proof. Let $q : \mathbb{R}^3 \to \mathbb{R}^3/\mathbb{Z}^3$ denote the universal covering. The induced integral affine structure on $\mathbb{R}^3$ is just the standard structure (cf. Example 2.14.i). The pullback bundle

$$\begin{tikzcd} q^*M \arrow[r, "Q"] \arrow[d] & M \arrow[d] \\ \mathbb{R}^3 \arrow[r, "q"'] & \mathbb{R}^3/\mathbb{Z}^3 \end{tikzcd}$$

is almost Lagrangian over $\mathbb{R}^3$, by naturality of the definition of almost Lagrangian bundles (cf. Definition 4.32). Since $\mathbb{R}^3$ is 2-connected, *i.e.* $\pi_i(\mathbb{R}^3)$ is trivial for $i = 1, 2$, the isomorphism type of $q^*M \to \mathbb{R}^3$ is trivial by Theorem 3.9. In particular, there is an isomorphism of almost Lagrangian bundles

$$\begin{tikzcd} \mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3} \arrow[rr, "\cong"] \arrow[dr] & & q^*M \arrow[dl] \\ & \mathbb{R}^3, & \end{tikzcd}$$

where $P_{\mathbb{R}^3} = q^*P$, and P is defined as in equation (5.1). Thus there is a commutative diagram

$$\begin{tikzcd} \mathrm{T}^*\mathbb{R}^3 \arrow[r, "Q'"] \arrow[dr] & \mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3} \arrow[r, "Q"] \arrow[d] & M \arrow[d] \\ & \mathbb{R}^3 \arrow[r, "q"'] & \mathbb{R}^3/\mathbb{Z}^3, \end{tikzcd} \tag{5.4}$$

where $Q' : \mathrm{T}^*\mathbb{R}^3 \to \mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}$ denotes the quotient map. Note that both Q' and Q are covering maps, induced by smooth $\mathbb{Z}^3$-actions on $\mathrm{T}^*\mathbb{R}^3$ (translation along the submanifold $P_{\mathbb{R}^3}$) and on $\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}$ (a lift of the $\mathbb{Z}^3$-action on $\mathbb{R}^3$) respectively. Thus the composite

$$Q \circ Q' : \mathrm{T}^*\mathbb{R}^3 \to M$$

is a fibration with unique path lifting property and discrete fibres; since both M and $\mathrm{T}^*\mathbb{R}^3$ are manifolds, this implies that $Q \circ Q'$ is a covering map (cf. [59]). Moreover, as $\mathrm{T}^*\mathbb{R}^3$ is simply connected, it is a universal cover of M and thus M is homeomorphic to $\mathrm{T}^*\mathbb{R}^3/\Gamma_c$. However, the action of Γ_c is obtained by composing the aforementioned smooth $\mathbb{Z}^3$-actions and, therefore, it is smooth. Hence, the result follows. □

Remark 5.2. The submanifold $P_{\mathbb{R}^3} = q^*P \subset \mathrm{T}^*\mathbb{R}^3$ is the period lattice bundle associated to the integral affine manifold $\mathbb{R}^3$ with standard integral affine structure. In particular, if a^1, a^2, a^3 are integral affine coordinates on $\mathbb{R}^3$ induced by pulling back the integral affine structure on $\mathbb{R}^3/\mathbb{Z}^3$, and $(\mathbf{a}, \mathbf{p})$ are induced coordinates on $\mathrm{T}^*\mathbb{R}^3$, then

$$P_{\mathbb{R}^3} = \{(\mathbf{a}, \mathbf{p}) \in \mathrm{T}^*\mathbb{R}^3/\mathbb{Z}^3 : \mathbf{p} \in \mathbb{Z}\langle \mathrm{d}a^1, \mathrm{d}a^2, \mathrm{d}a^3\rangle\}$$

(cf. equation (5.1)).

Lemma 5.2 illustrates also how Γ_c acts on $\mathrm{T}^*\mathbb{R}^3$. The restriction of the action to the normal subgroup of Γ_c corresponding to $\pi_1(F)$ acts on the fibres of the cotangent bundle $\mathrm{T}^*\mathbb{R}^3 \to \mathbb{R}^3$ inducing the covering map $Q' : \mathrm{T}^*\mathbb{R}^3 \to \mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}$. The quotient $\Gamma_c/\pi_1(F) \cong \pi_1(\mathbb{R}^3/\mathbb{Z}^3)$ acts on $\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}$; this action lifts the action of $\pi_1(\mathbb{R}^3/\mathbb{Z}^3)$ on $\mathbb{R}^3$ by deck transformations.

Suppose that the almost Lagrangian bundle $F \hookrightarrow M \to \mathbb{R}^3/\mathbb{Z}^3$ is, in fact, Lagrangian, so that the total space M admits an appropriate symplectic form ω. The following lemma is the equivalent of Lemma 5.2 in the category of Lagrangian bundles.

Lemma 5.3. *The symplectic manifold (M, ω) is symplectomorphic to the quotient $(\mathrm{T}^*\mathbb{R}^3/\Gamma_c, \omega')$, where ω' descends from the canonical symplectic form $\Omega_{\mathbb{R}^3}$ on $\mathrm{T}^*\mathbb{R}^3$ (cf. Example 2.3).*

Proof. It suffices to prove that Γ_c acts by symplectomorphisms on $(\mathrm{T}^*\mathbb{R}^3, \Omega_{\mathbb{R}^3})$. Fix the notation as in the proof of Lemma 5.2. The pullback bundle $q^*M \to \mathbb{R}^3$ is Lagrangian, since the symplectic form ω on M pulls back to a closed 2-form $Q^*\omega$ which is non-degenerate as Q is a local diffeomorphism. Since

$$q^*M \cong \mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3},$$

$Q^*\omega$ defines a symplectic form which makes the bundle

$$\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3} \to \mathbb{R}^3$$

Lagrangian. Remark 5.2 implies that the canonical symplectic form $\Omega_{\mathbb{R}^3}$ on $\mathrm{T}^*\mathbb{R}^3$ descends to a symplectic form ω_0 on $\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}$.

The Lagrangian bundles

$$(\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}, \omega_0) \to \mathbb{R}^3 \quad \text{and} \quad (\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}, Q^*\omega) \to \mathbb{R}^3$$

are fibrewise symplectomorphic. This can be seen as follows. The bundle

$$p : \mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3} \to \mathbb{R}^3$$

admits a section s which is Lagrangian with respect to the symplectic form ω_0. Let $\beta = s^*Q^*\omega$ and note that translation by s along the fibres induces a fibrewise symplectomorphism

$$(\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}, Q^*\omega) \cong (\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}, \omega_0 + p^*\beta)$$

(cf. [53]). Since $\mathbb{R}^3$ is 2-connected, $\beta = \mathrm{d}\beta'$ for some 1-form β' which is unique up to addition of closed 1-forms. Thus $(\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}, Q^*\omega)$ is fibrewise symplectomorphic to $(\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}, \omega_0 + \mathrm{d}(p^*\beta'))$. The 1-form β' is a section of the cotangent bundle $\mathrm{T}^*\mathbb{R}^3 \to \mathbb{R}^3$, thus

$$Q' \circ \beta' : \mathbb{R}^3 \to \mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}$$

is a section of the bundle

$$\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3} \to \mathbb{R}^3,$$

where $Q' : \mathrm{T}^*\mathbb{R}^3 \to \mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}$ is the quotient map in the proof of Lemma 5.2. The isomorphism

$$\begin{aligned} \mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3} &\to \mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3} \\ (\mathbf{a}, \boldsymbol{\theta}) &\mapsto (\mathbf{a}, \boldsymbol{\theta} - Q' \circ \beta'(\mathbf{a})), \end{aligned} \tag{5.5}$$

gives a symplectomorphism

$$(\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}, \omega_0) \cong (\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}, \omega_0 + p^*\mathrm{d}\beta').$$

The composition of the various fibrewise symplectomorphisms together yields a fibrewise symplectomorphism

$$(q^*M, Q^*\omega) \cong (\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}, \omega_0).$$

By virtue of the commutative diagram of equation (5.4), the action of Γ_c on $\mathrm{T}^*\mathbb{R}^3$ is by symplectomorphisms, since the normal group $\pi_1(F)$ acts by symplectomorphisms (translations along $P_{\mathbb{R}^3}$) of $\Omega_{\mathbb{R}^3}$ and the $\mathbb{Z}^3$-action on $(q^*M, Q^*\omega)$ is also by symplectomorphisms by construction (as the form $Q^*\omega$ descends to define ω). This proves the result. □

In light of Lemmata 5.2 and 5.3, it suffices to study whether there exist symplectic actions of Γ_c on $(\mathrm{T}^*\mathbb{R}^3, \Omega_{\mathbb{R}^3})$, such that

i the restriction of the action to the normal subgroup $\pi_1(F)$ induces the quotient map
$$Q' : (\mathrm{T}^*\mathbb{R}^3, \Omega_{\mathbb{R}^3}) \to (\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}, \omega_0);$$

ii the action of $\Gamma_c/\pi_1(F)$ lifts the integral affine action of $\mathbb{Z}^3$ on $\mathbb{R}^3$ by translations along the standard lattice $\mathbb{Z}^3 \subset (\mathbb{R}^3, +)$.

Proof of Theorem 5.1. Fix an element $c \in \mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}^3)$ and suppose that the associated almost Lagrangian bundle $F \hookrightarrow M \to \mathbb{R}^3/\mathbb{Z}^3$ with trivial monodromy is Lagrangian. Set $\Gamma_c = \pi_1(M)$. By abuse of notation, let a^1, a^2, a^3 be integral affine coordinates on $\mathbb{R}^3$ inducing coordinates $(\mathbf{a}, \mathbf{p})$ on $\mathrm{T}^*\mathbb{R}^3$, so that the canonical symplectic form $\Omega_{\mathbb{R}^3}$ is given by

$$\Omega_{\mathbb{R}^3} = \sum_{i=1}^{3} \mathrm{d}a^i \wedge \mathrm{d}p^i.$$

As mentioned above, the action of Γ_c on $\mathrm{T}^*\mathbb{R}^3$ is constructed by combining the action of $\pi_1(F)$ on $\mathrm{T}^*\mathbb{R}^3$ given by translations along $P_{\mathbb{R}^3}$, with the lift of the action of $\pi_1(\mathbb{R}^3/\mathbb{Z}^3)$ on $\mathbb{R}^3$ to $\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}$. In order to study such an action, a presentation of Γ_c is given in terms of generators of $\pi_1(F)$ and $\pi_1(\mathbb{R}^3/\mathbb{Z}^3)$.

Choose the standard generators $\gamma_1, \gamma_2, \gamma_3$ of $\pi_1(\mathbb{R}^3/\mathbb{Z}^3)$ which are given by the homotopy classes of loops of the form

$$\begin{aligned}
\gamma_1 &: t \mapsto (ta^1, 0, 0) \\
\gamma_2 &: t \mapsto (0, ta^2, 0) \\
\gamma_3 &: t \mapsto (0, 0, ta^3),
\end{aligned}$$

where $t \in [0,1]$ and $\mathbf{a}$ are integral affine coordinates on $\mathbb{R}^3/\mathbb{Z}^3$. The integral affine structure on $\mathbb{R}^3/\mathbb{Z}^3$ is such that the deck transformation on $\mathbb{R}^3$ corresponding to γ_i is given by

$$\mathbf{a} \mapsto \mathbf{a} + \mathbf{e}^i, \tag{5.6}$$

for each i, where $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ is the standard basis of $\mathbb{R}^3$. Let $(\mathbf{a}, \boldsymbol{\theta})$ be action-angle coordinates on $\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}$, so that

$$\omega_0 = \sum_{i=1}^{3} \mathrm{d}a^i \wedge \mathrm{d}\theta^i.$$

The lift of the $\pi_1(\mathbb{R}^3/\mathbb{Z}^3)$-action on $\mathbb{R}^3$ to $\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}$ is by symplectomorphisms; equation (5.6) implies that the group acts by translations on $\mathbb{R}^3$, so the lifted action of the generators $\gamma_1, \gamma_2, \gamma_3$ is of the form

$$\begin{aligned} \gamma_1 &: (\mathbf{a}, \boldsymbol{\theta}) \mapsto (\mathbf{a} + \mathbf{e}^1, \boldsymbol{\theta} + \mathbf{f}) \\ \gamma_2 &: (\mathbf{a}, \boldsymbol{\theta}) \mapsto (\mathbf{a} + \mathbf{e}^2, \boldsymbol{\theta} + \mathbf{g}) \\ \gamma_3 &: (\mathbf{a}, \boldsymbol{\theta}) \mapsto (\mathbf{a} + \mathbf{e}^3, \boldsymbol{\theta} + \mathbf{h}), \end{aligned} \tag{5.7}$$

where $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathrm{C}^\infty(\mathbb{R}^3; \mathbb{R}^3/\mathbb{Z}^3)$ are constrained by the fact that, for each i,

$$\gamma_i^* \omega_0 = \omega_0,$$

and by commutativity of the generators.

A choice of action-angle coordinates $(\mathbf{a}, \boldsymbol{\theta})$ induces a choice of generators $\sigma_1, \sigma_2, \sigma_3$ of the fundamental group of the fibre of the Lagrangian bundle $(\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}, \omega_0) \to \mathbb{R}^3$ by setting σ_i to be the loop defined by following the flow of the Hamiltonian vector field of the function a^i from time 0 to time 1 (cf. [20]). Explicitly, these are given by the homotopy classes of the following loops in $\mathrm{T}^*\mathbb{R}^3/P_{\mathbb{R}^3}$

$$\begin{aligned} \sigma_1 &: t \mapsto (0, 0, 0, t\theta^1, 0, 0) \\ \sigma_2 &: t \mapsto (0, 0, 0, 0, t\theta^2, 0) \\ \sigma_3 &: t \mapsto (0, 0, 0, 0, 0, t\theta^3), \end{aligned}$$

where it is important to remember that the angle coordinates $\boldsymbol{\theta}$ are coordinates mod 1 by Lemma 4.1. Note that under the action defined by equation (5.7), these loops are mapped to homotopic loops and, thus, $\sigma_1, \sigma_2, \sigma_3$ descend to generators of $\pi_1(F)$. Hence,

$$\gamma_1, \gamma_2, \gamma_3, \sigma_1, \sigma_2, \sigma_3$$

are generators for Γ_c. It remains to describe the relations among these generators. To this end, recall that there is a short exact sequence of groups

$$0 \to \pi_1(F) \to \Gamma_c \to \pi_1(\mathbb{R}^3/\mathbb{Z}^3) \to 0$$

arising from the long exact sequence in homotopy for the Lagrangian bundle $F \hookrightarrow (M, \omega) \to \mathbb{R}^3/\mathbb{Z}^3$. Since the bundle has trivial monodromy and $\pi_1(F)$ injects into Γ_c, for each i, σ_i commutes with all other generators of Γ_c. The obstruction for the generators $\gamma_1, \gamma_2, \gamma_3$ to commute in Γ_c is given by c (identified as a cohomology class in $\mathrm{H}^2(\mathbb{Z}^3; \mathbb{Z}^3)$ via the isomorphism

$$\mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}^3) \cong \mathrm{H}^2(\mathbb{Z}^3; \mathbb{Z}^3)$$

(cf. [12])). Fix ordered bases of $\mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z})$ and $\mathbb{Z}^3$ so that $c = [c_{kl}]$ as in the statement of the theorem. There is an isomorphism

$$\mathbb{Z}\langle \mathrm{d}a^1, \mathrm{d}a^2, \mathrm{d}a^3 \rangle \cong \pi_1(F)$$

defined by

$$\begin{aligned} \mathrm{d}a^3 &\mapsto \sigma_3 \\ \mathrm{d}a^1 &\mapsto \sigma_1 \\ \mathrm{d}a^2 &\mapsto \sigma_2, \end{aligned} \tag{5.8}$$

i.e. arising from associating to each generator of the period lattice bundle P the homotopy class of the loop traced by the flow of the corresponding Hamiltonian vector field. The generators $\eta_1, \eta_2, \eta_3 \in \mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z})$ correspond to three 2-cells in the standard CW-decomposition of $\mathbb{R}^3/\mathbb{Z}^3$ (cf. [36]), denoted by $\mathrm{e}_1^2, \mathrm{e}_2^2, \mathrm{e}_3^2$ respectively. The attaching maps of these cells define relations amongst the generators $\gamma_1, \gamma_2, \gamma_3$ in $\pi_1(\mathbb{R}^3/\mathbb{Z}^3)$; given the choices of generators of $\mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z})$ and $\pi_1(\mathbb{R}^3/\mathbb{Z}^3)$, the relations are given by the words

$$\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1},\ \gamma_2\gamma_3\gamma_2^{-1}\gamma_3^{-1},\ \gamma_3\gamma_1\gamma_3^{-1}\gamma_1^{-1}.$$

In light of the isomorphism of equation (5.8), obstruction theoretic arguments imply that the lift of the above relations to Γ_c is given by

$$\begin{aligned} \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} &= \sigma_3^{c_{11}}\sigma_1^{c_{12}}\sigma_2^{c_{13}} \\ \gamma_2\gamma_3\gamma_2^{-1}\gamma_3^{-1} &= \sigma_3^{c_{21}}\sigma_1^{c_{22}}\sigma_2^{c_{23}} \\ \gamma_3\gamma_1\gamma_3^{-1}\gamma_1^{-1} &= \sigma_3^{c_{21}}\sigma_1^{c_{22}}\sigma_2^{c_{23}}. \end{aligned} \tag{5.9}$$

Any Γ_c-action $(\mathrm{T}^*\mathbb{R}^3, \Omega_{\mathbb{R}^3})$ acting by symplectomorphisms is defined on the above generators by

$$\begin{aligned} \sigma_1 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a}, \mathbf{p} + \mathbf{e}^1) \\ \sigma_2 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a}, \mathbf{p} + \mathbf{e}^2) \\ \sigma_3 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a}, \mathbf{p} + \mathbf{e}^3) \\ \gamma_1 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a} + \mathbf{e}^1, \mathbf{p} + \tilde{\mathbf{f}}) \\ \gamma_2 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a} + \mathbf{e}^2, \mathbf{p} + \tilde{\mathbf{g}}) \\ \gamma_3 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a} + \mathbf{e}^3, \mathbf{p} + \tilde{\mathbf{h}}), \end{aligned} \tag{5.10}$$

where $(\mathbf{a}, \mathbf{p})$ are canonical coordinates on $\mathrm{T}^*\mathbb{R}^3$, and

$$\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}} \in \mathrm{C}^\infty(\mathbb{R}^3; \mathbb{R}^3)$$

are lifts of the maps

$$\mathbf{f},\ \mathbf{g},\ \mathbf{h} \in \mathrm{C}^\infty(\mathbb{R}^3; \mathbb{R}^3/\mathbb{Z}^3)$$

of equation (5.7). There are further constraints on the above action, namely that

i) the deck transformation corresponding to each γ_i is a symplectomorphism of $(\mathrm{T}^*\mathbb{R}^3, \Omega_{\mathbb{R}^3})$;

ii) the action need satisfy the relations of equation (5.9).

For instance, if $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$, then

$$\gamma_1^*\Omega_{\mathbb{R}^3} = \Omega_{\mathbb{R}^3}$$

implies that

$$\frac{\partial \tilde{f}_i}{\partial a^j} = \frac{\partial \tilde{f}_j}{\partial a^i} \tag{5.11}$$

for all $i \neq j$ and throughout $\mathbb{R}^3$; on the other hand, the first relation in equation (5.9) implies that

$$\begin{aligned}
\tilde{g}_3(\mathbf{a}-\mathbf{e}^1-\mathbf{e}^2) - \tilde{g}_3(\mathbf{a}-\mathbf{e}^2) + \tilde{f}_3(\mathbf{a}-\mathbf{e}^1) - \tilde{f}_3(\mathbf{a}-\mathbf{e}^1-\mathbf{e}^2) &= c_{11}\\
\tilde{g}_1(\mathbf{a}-\mathbf{e}^1-\mathbf{e}^2) - \tilde{g}_1(\mathbf{a}-\mathbf{e}^2) + \tilde{f}_1(\mathbf{a}-\mathbf{e}^1) - \tilde{f}_1(\mathbf{a}-\mathbf{e}^1-\mathbf{e}^2) &= c_{12}\\
\tilde{g}_2(\mathbf{a}-\mathbf{e}^1-\mathbf{e}^2) - \tilde{g}_2(\mathbf{a}-\mathbf{e}^2) + \tilde{f}_2(\mathbf{a}-\mathbf{e}^1) - \tilde{f}_2(\mathbf{a}-\mathbf{e}^1-\mathbf{e}^2) &= c_{13}.
\end{aligned} \tag{5.12}$$

Set

$$\frac{\partial \tilde{f}_3}{\partial a^2} = F(\mathbf{a}) = \frac{\partial \tilde{f}_2}{\partial a^3}, \qquad \frac{\partial \tilde{g}_3}{\partial a^1} = G(\mathbf{a}) = \frac{\partial \tilde{g}_1}{\partial a^3}, \qquad \frac{\partial \tilde{h}_2}{\partial a^1} = H(\mathbf{a}) = \frac{\partial \tilde{h}_1}{\partial a^2}.$$

Clearly $F, G, H \in \mathrm{C}^\infty(\mathbb{R}^3)$. Write

$$\tilde{f}_3(\mathbf{a}) = \int^{a^2} F(a^1, \tilde{a}^2, a^3)\mathrm{d}\tilde{a}^2 + A(a^1, a^3),$$

etc. The relations of equation (5.12) imply the following equalities

$$\begin{aligned}
\int_{a^2-1}^{a^2} F(a^1-1, \tilde{a}^2, a^3)\mathrm{d}\tilde{a}^2 - \int_{a^1-1}^{a^1} G(\tilde{a}^1, a^2-1, a^3)\mathrm{d}\tilde{a}^1 &= c_{11},\\
\int_{a^3-1}^{a^3} G(a^1, a^2-1, \tilde{a}^3)\mathrm{d}\tilde{a}^3 - \int_{a^2-1}^{a^2} H(a^1, \tilde{a}^2, a^3-1)\mathrm{d}\tilde{a}^2 &= c_{22},\\
\int_{a^1-1}^{a^1} H(\tilde{a}^1, a^2, a^3-1)\mathrm{d}\tilde{a}^1 - \int_{a^3-1}^{a^3} F(a^1-1, a^2, \tilde{a}^3)\mathrm{d}\tilde{a}^3 &= c_{33},
\end{aligned} \tag{5.13}$$

which hold for all $\mathbf{a} \in \mathbb{R}^3$. Integrating the first equality in equation (5.13) over $[a^3 - 1, a^3]$, obtain that

$$\int_{a^3-1}^{a^3}\int_{a^2-1}^{a^2} F(a^1-1, \tilde{a}^2, \tilde{a}^3)\mathrm{d}\tilde{a}^2\mathrm{d}\tilde{a}^3 - \int_{a^3-1}^{a^3}\int_{a^1-1}^{a^1} G(\tilde{a}^1, a^2-1, \tilde{a}^3)\mathrm{d}\tilde{a}^1\mathrm{d}\tilde{a}^3 = c_{11}$$

for all $\mathbf{a} \in \mathbb{R}^3$. The functions F and G are smooth and these integrals are over compact subsets of $\mathbb{R}^3$ (for some fixed a^1, a^2, a^3); by Fubini's theorem, the order of integration can be switched, so that

$$\int_{a^2-1}^{a^2}\int_{a^3-1}^{a^3} F(a^1-1, \tilde{a}^2, \tilde{a}^3)\mathrm{d}\tilde{a}^3\mathrm{d}\tilde{a}^2 - \int_{a^1-1}^{a^1}\int_{a^3-1}^{a^3} G(\tilde{a}^1, a^2-1, \tilde{a}^3)\mathrm{d}\tilde{a}^3\mathrm{d}\tilde{a}^1 = c_{11}. \tag{5.14}$$

The second and third equations in (5.13) can now be used to derive the necessary condition. Indeed the a^3-integrals of equation (5.14) are given by

$$\begin{aligned}\int_{a^3-1}^{a^3} G(a^1, a^2-1, \tilde{a}^3)\mathrm{d}\tilde{a}^3 &= \int_{a^2-1}^{a^2} H(a^1, \tilde{a}^2, a^3-1)\mathrm{d}\tilde{a}^2 + c_{22}, \\ \int_{a^3-1}^{a^3} F(a^1-1, a^2, \tilde{a}^3)\mathrm{d}\tilde{a}^3 &= \int_{a^1-1}^{a^1} H(\tilde{a}^1, a^2, a^3-1)\mathrm{d}\tilde{a}^1 - c_{33},\end{aligned} \tag{5.15}$$

and so equation (5.14) becomes

$$\begin{aligned}\int_{a^2-1}^{a^2}\int_{a^1-1}^{a^1} H(\tilde{a}^1, \tilde{a}^2, a^3-1)\mathrm{d}\tilde{a}^1\mathrm{d}\tilde{a}^2 - \int_{a^1-1}^{a^1}\int_{a^2-1}^{a^2} H(\tilde{a}^1, \tilde{a}^2, a^3-1)\mathrm{d}\tilde{a}^2\mathrm{d}\tilde{a}^1 \\ = c_{11} + c_{22} + c_{33}.\end{aligned} \tag{5.16}$$

Using Fubini's theorem again, the left hand side of equation (5.16) vanishes and so the necessary condition follows.

It remains to show that the condition is sufficient. Fix a cohomology class $c \in \mathrm{H}^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}^3)$ represented by an integral matrix $[c_{lm}]$ with $c_{11}+c_{22}+c_{33} = 0$. Consider the Γ_c-action on $\mathrm{T}^*\mathbb{R}^3$ defined by the following vector valued functions $\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}}$

$$\begin{aligned}\tilde{\mathbf{f}}(a^1, a^2, a^3) &= (c_{12}a^2 - c_{32}a^3, c_{12}a^1 + c_{11}a^3, -c_{32}a^1 + c_{11}a^2), \\ \tilde{\mathbf{g}}(a^1, a^2, a^3) &= (-c_{13}a^2, c_{23}a^3 - c_{13}a^1, c_{23}a^2), \\ \tilde{\mathbf{h}}(a^1, a^2, a^3) &= (c_{31}a^3 - c_{22}a^2, -c_{22}a^1 - c_{21}a^3, c_{31}a^1 - c_{21}a^2).\end{aligned}$$

This action is symplectic and the bundle obtained in the quotient is a Lagrangian bundle over $\mathbb{R}^3/\mathbb{Z}^3$ with trivial monodromy and Chern class given by c. This shows that the condition is sufficient and therefore completes the proof of the theorem. □

Remark 5.3 (Fake Lagrangian bundles)**.** The almost Lagrangian bundles over $\mathbb{R}^3/\mathbb{Z}^3$ whose Chern classes do not satisfy the condition of Theorem 5.1 are the first explicit examples of fake Lagrangian bundles.

5.2 Relation to symplectic topology

Fake Lagrangian and Lagrangian bundles are topologically indistinguishable as they are both examples of affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles (cf. Section 4.1). Theorem 5.1 characterises the difference between these two notions when the base space is the integral affine manifold $\mathbb{R}^3/\mathbb{Z}^3$. It is also possible to consider this result from a different point of view. Let M be the total space of a principal $\mathbb{R}^3/\mathbb{Z}^3$-bundle over $\mathbb{R}^3/\mathbb{Z}^3$ with Chern class c; Theorem 5.1 gives a necessary and sufficient condition on the topology of M (which depends on c) for it to admit a symplectic form which makes the given principal $\mathbb{R}^3/\mathbb{Z}^3$-bundle Lagrangian. From this point of view, the following is a natural question to investigate.

Question 5.4. Is the condition of Theorem 5.1 necessary for M to be a symplectic manifold?

This question studies the extent to which the obstruction to the existence of a suitable symplectic form on the total space of an almost Lagrangian bundle is, in fact, an obstruction to the existence of a symplectic structure. In this section a negative answer is provided to the above question, using, as the starting block, an examples of a fake Lagrangian bundle over $\mathbb{R}^3/\mathbb{Z}^3$.

The example under consideration has trivial monodromy and Chern class given by the integral matrix

$$c = \operatorname{diag}(1, 0, 0),$$

where the notation and conventions followed are those of Section 5.1. Throughout the rest of this section, let M denote (a representative of the diffeomorphism type of) the total space of this fake Lagrangian bundle and set $\Gamma = \pi_1(M)$. This bundle is denoted by $F \hookrightarrow M \to \mathbb{R}^3/\mathbb{Z}^3$. In what follows, not only is it shown that M admits a symplectic form, but also that this symplectic form makes M the total space of a Lagrangian bundle over a manifold which is diffeomorphic to $\mathbb{T}^3$.

As in Section 5.1, let $\gamma_1, \gamma_2, \gamma_3$ and $\sigma_1, \sigma_2, \sigma_3$ be generators of $\pi_1(\mathbb{R}^3/\mathbb{Z}^3)$ and $\pi_1(F)$ respectively. The group Γ is generated by $\gamma_1, \gamma_2, \gamma_3, \sigma_1, \sigma_2, \gamma_3$ and the only relation which is not commutation of two generators is given by

$$\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1} = \sigma_3.$$

The manifold M can be constructed smoothly as the quotient $\mathrm{T}^*\mathbb{R}^3/\Gamma$, where the (integral affine action) of Γ (identified as a group of deck transformations on $\mathrm{T}^*\mathbb{R}^3$) is given by

$$\begin{aligned}
\sigma_1 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a}, \mathbf{p} + \mathbf{e}_1) \\
\sigma_2 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a}, \mathbf{p} + \mathbf{e}_2) \\
\gamma_3 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a}, \mathbf{p} + \mathbf{e}_3) \\
\gamma_1 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a} + \mathbf{e}_1, \mathbf{p} + (0, 0, a^2)) \\
\gamma_2 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a} + \mathbf{e}_2, \mathbf{p}) \\
\gamma_3 &: (\mathbf{a}, \mathbf{p}) \mapsto (\mathbf{a} + \mathbf{e}_3, \mathbf{p}),
\end{aligned} \tag{5.17}$$

and $\mathbf{a} = (a^1, a^2, a^3), \mathbf{p} = (p^1, p^2, p^3)$ are canonical coordinates on $\mathrm{T}^*\mathbb{R}^3$ as in Section 5.1. The projection $\pi : M \to \mathbb{R}^3/\mathbb{Z}^3$ arises from the projection $\tilde{\pi} : \mathrm{T}^*\mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\tilde{\pi} : (a^1, a^2, a^3, p^1, p^2, p^3) \mapsto (a^1, a^2, a^3).$$

Identify $\mathrm{T}^*\mathbb{R}^3 \cong \mathbb{R}^6$ and consider the projection

$$\begin{aligned}
\tilde{\pi}_L &: \mathbb{R}^6 \to \mathbb{R}^3 \\
(a^1, a^2, a^3, p^1, p^2, p^3) &\mapsto (a^1, p^2, a^3).
\end{aligned}$$

The symplectic form

$$\tilde{\eta} = \mathrm{d}a^1 \wedge \mathrm{d}p^3 - a^1 \mathrm{d}a^1 \wedge \mathrm{d}a^2 + \mathrm{d}p^2 \wedge \mathrm{d}a^2 + \mathrm{d}a^3 \wedge \mathrm{d}p^1$$

makes the vector bundle $\tilde{\pi}_L : \mathbb{R}^6 \to \mathbb{R}^3$ into a Lagrangian bundle with fibre $\mathbb{R}^3$. Moreover, $\tilde{\eta}$ is invariant under the action of Γ and therefore descends to a well-defined symplectic form η on M. Denote by N the normal subgroup of Γ generated by $\sigma_1, \gamma_2, \sigma_3$;

$N \cong \mathbb{Z}\langle \sigma_1, \gamma_2, \sigma_3 \rangle$ acts trivially on $\mathbb{R}^3 = \tilde{\pi}_L(\mathbb{R}^6)$. Therefore there exists a commutative diagram of bundles

$$\begin{tikzcd}
(\mathbb{R}^6, \tilde{\eta}) \arrow[r, "N"] \arrow[dr, "\tilde{\pi}_L"'] & (\mathbb{R}^3 \times \mathbb{R}^3/\mathbb{Z}^3, \bar{\eta}) \arrow[r, "\Gamma/N"] \arrow[d] & (M, \eta) \arrow[d, "\pi_L"] \\
& \mathbb{R}^3 \arrow[r, "\Gamma/N"] & \mathbb{R}^3/(\Gamma/N)
\end{tikzcd}$$

where the horizontal arrows denote quotient maps induced by the action of N on $\mathbb{R}^6$, and Γ/N on $\mathbb{R}^3 \times \mathbb{R}^3/\mathbb{Z}^3$ and $\mathbb{R}^3$. In particular, note that $\mathbb{R}^3/(\Gamma/N)$ is diffeomorphic to $\mathbb{T}^3$ and that the fibre of $\pi_L : M \to \mathbb{T}^3$ is diffeomorphic to a 3-torus, which is Lagrangian with respect to the symplectic form η. Thus the bundle $\pi_L : M \to \mathbb{R}^3/\mathbb{Z}^3$ is Lagrangian. Note that the vector bundle $\tilde{\pi}_L : \mathbb{R}^6 \to \mathbb{R}^3$ admits a section

$$s : (a^1, p^2, a^3) \mapsto (a^1, p^2, a^3, 0, 0, 0)$$

which is invariant under the action of Γ. In particular, $\pi_L : M \to \mathbb{R}^3/\mathbb{Z}^3$ admits a global section and so the bundle has zero Chern class. On the other hand, since $\Gamma \ncong \mathbb{Z}^6$, M is not diffeomorphic to $\mathbb{T}^6$ and so the bundle $\pi_L : M \to \mathbb{R}^3/\mathbb{Z}^3$ has non-trivial monodromy. Therefore M is the total space of a Lagrangian bundle over $\mathbb{T}^3$ with non-trivial monodromy and zero Chern class.

Remark 5.5. It is interesting to notice that the integral affine structure $\mathcal{A}$ that the Lagrangian bundle constructed above induces on the base space $\mathbb{T}^3$ is *not* affinely diffeomorphic to the the standard one. This follows from the fact that the monodromy of the Lagrangian bundle is not trivial and, thus, the linear holonomy of the integral affine structure on $\mathbb{T}^3$ is not trivial in light of equation (4.16). Any integral affine manifold diffeomorphic to $\mathbb{T}^3$ which is also (integrally) affinely diffeomorphic to $\mathbb{R}^3/\mathbb{Z}^3$ must have trivial linear holonomy. Therefore $(\mathbb{T}^3; \mathcal{A})$ is not affinely diffeomorphic to $\mathbb{R}^3/\mathbb{Z}^3$ and the result follows.

Remark 5.6. The idea for the above construction comes from the classification of $\mathbb{T}^2$-bundles over $\mathbb{T}^2$ carried out in [51]. In particular, a manifold diffeomorphic to the example due to Kodaira and Thurston (cf. Example 2.6) is the total space of inequivalent Lagrangian bundles over $\mathbb{T}^2$, one inducing the standard integral affine structure on the base space but with non-trivial Chern class and the other admitting a section, but with non-trivial monodromy (cf. [30]). The example above can be considered as a generalisation, with the added result that one bundle is Lagrangian while the other is fake.

Chapter 6

The spectral sequence of topological universal Lagrangian bundles

This chapter studies the difference between the notion of almost Lagrangian and Lagrangian bundles over a fixed integral affine manifold $(B, \mathcal{A})$. The main result is Theorem 6.9 which proves that this obstruction is given by the cup product (in the appropriate cohomology theory) of the Chern class of the bundle with the cohomology class of the symplectic form on the symplectic reference bundle associated to $(B, \mathcal{A})$. The proof of Theorem 6.9 also provides a proof to Theorem 4.4 (cf. [18]). Section 6.1 proves that the problem of studying the topology of the topological universal Lagrangian bundle is simply the problem of studying the *equivariant* topology of the universal $\mathbb{R}^n/\mathbb{Z}^n$-bundle. Lemma 6.1 allows to study the Leray-Serre spectral sequence of the topological universal Lagrangian bundle starting from that of the universal $\mathbb{R}^n/\mathbb{Z}^n$-bundle. Since the differentials on the Leray-Serre spectral sequence of the latter are well-understood (cf. Lemma 6.4), it is possible to explicitly compute some differentials on the E_2-page of the spectral sequence of the former, which depend on the universal Chern class c_U (cf. Theorem 6.6). This is the content of Section 6.2. Finally, Section 6.3 relates these results to the study of almost Lagrangian bundles first by highlighting the importance of the symplectic reference Lagrangian bundle, which encapsulates information about the integral affine structure of $(B, \mathcal{A})$, and then by proving Theorem 6.9.

6.1 A preparatory lemma

In this section, the topological universal Lagrangian bundle is identified as the equivariant equivalent of the universal bundle for principal $\mathbb{R}^n/\mathbb{Z}^n$-bundles. The arguments at the end of Section 3.2.1 show that the (free) monodromy of a Lagrangian bundle is the obstruction to be a principal $\mathbb{R}^n/\mathbb{Z}^n$-bundle. Lemma 6.1 uses topological methods to prove the equivalent result for all bundles which are classified by the topological universal Lagrangian bundle, *e.g.* almost Lagrangian bundles, affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles.

Recall that the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R}^n/\mathbb{Z}^n \arrow[r, "\tau"] & \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) \arrow[r, shift left, "p"] & \mathrm{GL}(n;\mathbb{Z}) \arrow[l, shift left, "\sigma"] \arrow[r] & 1, \end{tikzcd}$$

induces bundles

$$\tau : \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$$
$$\sigma : \mathrm{BGL}(n;\mathbb{Z}) \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$$

(cf. equations (3.8a) and (3.8b)). The following lemma proves that the pull-back of the topological universal Lagrangian bundle along τ is the universal bundle for principal $\mathbb{R}^n/\mathbb{Z}^n$-bundles.

Lemma 6.1. *The pull-back bundle*

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \tau^*\mathrm{BGL}(n;\mathbb{Z}) \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$$

is a universal bundle for principal $\mathbb{R}^n/\mathbb{Z}^n$-bundles.

Proof. The group $\mathbb{R}^n/\mathbb{Z}^n$ acts freely on $\mathrm{EAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ via the inclusion τ. Thus there is a commutative diagram

$$\begin{tikzcd} & \mathrm{EAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) \arrow[dl, "\mathrm{pr}_{\mathbb{R}^n/\mathbb{Z}^n}"'] \arrow[dr, "\mathrm{pr}_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)}"] & \\ \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n \arrow[rr, "\tau"'] & & \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n), \end{tikzcd} \tag{6.1}$$

where the vertical maps are the quotient maps induced by the $\mathbb{R}^n/\mathbb{Z}^n$ and $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ actions on $\mathrm{EAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ respectively. The group $\mathbb{R}^n/\mathbb{Z}^n$ acts on itself by left translations; tautologically, the associated bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{EAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathbb{R}^n/\mathbb{Z}^n} \mathbb{R}^n/\mathbb{Z}^n \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n,$$

where the action of $\mathbb{R}^n/\mathbb{Z}^n$ on itself is by left translations, is a universal principal $\mathbb{R}^n/\mathbb{Z}^n$-bundle. Thus it suffices to show that there exists a bundle isomorphism

$$\mathrm{EAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathbb{R}^n/\mathbb{Z}^n} \mathbb{R}^n/\mathbb{Z}^n \cong \tau^*\mathrm{BGL}(n;\mathbb{Z}).$$

Recall from the proof of Theorem 3.6 that the topological universal Lagrangian bundle is isomorphic to the associated bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{EAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n),$$

where the left action of $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ on $\mathbb{R}^n/\mathbb{Z}^n$ is given by

$$\begin{aligned} \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n &\to \mathbb{R}^n/\mathbb{Z}^n \\ ((A, \mathbf{s}), \mathbf{t}) &\mapsto A\mathbf{t} + \mathbf{s}. \end{aligned}$$

Hence it suffices to prove that there exists a bundle isomorphism

$$\mathrm{EAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathbb{R}^n/\mathbb{Z}^n} \mathbb{R}^n/\mathbb{Z}^n \cong \tau^*(\mathrm{EAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n).$$

Equation (6.1) implies that

$$\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n = \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)/(\mathbb{R}^n/\mathbb{Z}^n);$$

if $x \in \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)$, its equivalence class in $\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$ is denoted by $[x]_{\mathbb{R}^n/\mathbb{Z}^n}$. Consider the map

$$\begin{aligned} \varsigma : \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n &\to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n \times \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n \\ (x, \mathbf{t}) &\mapsto ([x]_{\mathbb{R}^n/\mathbb{Z}^n}, (x, \mathbf{t})). \end{aligned} \tag{6.2}$$

It is continuous as it arises from the composition of continuous maps; thus, the map $\bar{\varsigma} = q \circ \varsigma$, where

$$q : \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n \times \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n \times \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n$$

is the quotient map, is also continuous. Note that, by construction,

$$\bar{\varsigma}(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n) \subset \tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n).$$

In fact, $\bar{\varsigma}$ is onto $\tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n)$, for if

$$([x]_{\mathbb{R}^n/\mathbb{Z}^n}, [y, \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)}) \in \tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n),$$

then the equivalence classes of $x, y \in \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)$ under the action of $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ are equal. In particular, there exists an element $a \in \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ such that $y = x \cdot a$; thus

$$[y, \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} = [x \cdot a, \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} = [x, a \cdot \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)}.$$

Hence,

$$\bar{\varsigma}(x, a \cdot \mathbf{t}) = [y, \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)},$$

and so

$$\bar{\varsigma}(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n) = \tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n).$$

Let $\mathbf{s} \in \mathbb{R}^n/\mathbb{Z}^n$ and identify it with an element of $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ via the injection τ. Then, since

$$[x \cdot \mathbf{s}, \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} = [x, \mathbf{s} \cdot \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)}$$

for all $x \in \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)$, $\mathbf{t} \in \mathbb{R}^n/\mathbb{Z}^n$, it follows that

$$\bar{\varsigma}(x \cdot \mathbf{s}, \mathbf{t}) = \bar{\varsigma}(x, \mathbf{s} \cdot \mathbf{t}). \tag{6.3}$$

Equation (6.3) implies that $\bar{\varsigma}$ descends to a well-defined, continuous, surjective map

$$\hat{\varsigma} : \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathbb{R}^n/\mathbb{Z}^n} \mathbb{R}^n/\mathbb{Z}^n \to \tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n),$$

which is going to yield the required isomorphism. Note that $\hat{\varsigma}$ is also injective, since if

$$\hat{\varsigma}([x, \mathbf{t}]_{\mathbb{R}^n/\mathbb{Z}^n}) = \hat{\varsigma}([x', \mathbf{t}']_{\mathbb{R}^n/\mathbb{Z}^n}),$$

for some $x, x' \in \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)$, $\mathbf{t}, \mathbf{t}' \in \mathbb{R}^n/\mathbb{Z}^n$, then there exists $\mathbf{s} \in \mathbb{R}^n/\mathbb{Z}^n$ (again identified as an element of $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ via τ) such that

$$x' = x \cdot \mathbf{s} \quad \text{and} \quad \mathbf{t}' = \mathbf{s}^{-1} \cdot \mathbf{t}.$$

Thus

$$[x', \mathbf{t}']_{\mathbb{R}^n/\mathbb{Z}^n} = [x \cdot \mathbf{s}, \mathbf{s}^{-1} \cdot \mathbf{t}]_{\mathbb{R}^n/\mathbb{Z}^n} = [x', \mathbf{t}']_{\mathbb{R}^n/\mathbb{Z}^n},$$

and the map $\hat{\varsigma}$ is injective.

It remains to exhibit a continuous inverse. Let

$$\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$$

be the bundle obtained from the universal bundle for $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ and considering the cartesian product of the total space $\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)$ with $\mathbb{R}^n/\mathbb{Z}^n$. There is a continuous map

$$\begin{aligned} \mathrm{pr}_2 : \tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n) &\to \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n \\ ([x]_{\mathbb{R}^n/\mathbb{Z}^n}, (y, \mathbf{t})) &\mapsto (y, \mathbf{t}), \end{aligned}$$

which is just projection onto the second component. The composite

$$\vartheta : \tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathbb{R}^n/\mathbb{Z}^n} \mathbb{R}^n/\mathbb{Z}^n$$

is therefore also continuous. Note that, by definition,

$$([x]_{\mathbb{R}^n/\mathbb{Z}^n}, (y, \mathbf{t})) \in \tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n)$$

if and only if there exists $a(x, y) \in \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ such that $y = x \cdot a(x, y)$. When such element exists, it is unique since $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ acts freely on $\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)$. In particular,

$$\vartheta([x]_{\mathbb{R}^n/\mathbb{Z}^n}, (y, \mathbf{t})) = [y, \mathbf{t}]_{\mathbb{R}^n/\mathbb{Z}^n} = [x, a(x, y) \cdot \mathbf{t}]_{\mathbb{R}^n/\mathbb{Z}^n}.$$

Note that for any $b \in \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ and any $([x]_{\mathbb{R}^n/\mathbb{Z}^n}, (y, \mathbf{t})) \in \tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n/\mathbb{Z}^n)$,

$$\vartheta([x]_{\mathbb{R}^n/\mathbb{Z}^n}, (y \cdot b, \mathbf{t})) = \vartheta([x]_{\mathbb{R}^n/\mathbb{Z}^n}, (y, b \cdot \mathbf{t})),$$

since the action of $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ on $\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n)$ is free. Therefore ϑ descends to a continuous map

$$\begin{aligned} \hat{\vartheta} : \tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n) &\to \mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathbb{R}^n/\mathbb{Z}^n} \mathbb{R}^n/\mathbb{Z}^n \\ ([x]_{\mathbb{R}^n/\mathbb{Z}^n}, [y, \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)}) &\mapsto [x, a(x, y)\mathbf{t}]_{\mathbb{R}^n/\mathbb{Z}^n}. \end{aligned} \tag{6.4}$$

Note that for all $([x]_{\mathbb{R}^n/\mathbb{Z}^n}, [y, \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)}) \in \tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n)$,

$$\begin{aligned} \hat{\varsigma} \circ \hat{\vartheta}([x]_{\mathbb{R}^n/\mathbb{Z}^n}, [y, \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)}) &= ([x]_{\mathbb{R}^n/\mathbb{Z}^n}, [x, a(x, y) \cdot \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)}) \\ &= ([x]_{\mathbb{R}^n/\mathbb{Z}^n}, [x \cdot a(x, y), \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)}) \\ &= ([x]_{\mathbb{R}^n/\mathbb{Z}^n}, [y, \mathbf{t}]_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)}), \end{aligned}$$

so that

$$\hat{\varsigma} \circ \hat{\vartheta} = \mathrm{id}_{\tau^*(\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)} \mathbb{R}^n/\mathbb{Z}^n)}.$$

It is also clear that

$$\hat{\vartheta} \circ \hat{\varsigma} = \mathrm{id}_{\mathrm{EAff}(\mathbb{R}^n/\mathbb{Z}^n) \times_{\mathbb{R}^n/\mathbb{Z}^n} \mathbb{R}^n/\mathbb{Z}^n};$$

hence $\hat{\varsigma}$ is a homeomorphism. Moreover, $\hat{\varsigma}, \hat{\vartheta}$ are morphisms of fibre bundles and, in particular, isomorphisms. This proves the lemma. □

Remark 6.1. Lemma 6.1 proves that an affine $\mathbb{R}^n/\mathbb{Z}^n$-bundle with trivial monodromy is, in fact, a principal $\mathbb{R}^n/\mathbb{Z}^n$-bundle. This is because the classifying map $\chi : B \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ factors through a map $\tilde{\chi} : B \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$ by Theorem 3.2. Hence, the bundle is isomorphic to the pull-back of the bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \tau^*\mathrm{BGL}(n;\mathbb{Z}) \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n \tag{6.5}$$

along $\tilde{\chi}$. Since Lemma 6.1 shows that (6.5) is a model for the universal bundle for principal $\mathbb{R}^n/\mathbb{Z}^n$-bundles, the result follows.

Furthermore, by naturality of the Chern class, the obstruction to the existence of a section of the bundle (6.5) is given by

$$\tau^* c_U \in \mathrm{H}^2(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n; \mathbb{Z}^n_{(\tau\circ\mathrm{id})_*}),$$

where c_U denotes the universal Chern class of Definition 3.11. Note that $\pi_1(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n)$ is trivial, so that the above coefficient system is just the usual constant $\mathbb{Z}^n$-system of coefficients on $\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$. Since (6.5) is a model for the universal bundle for principal $\mathbb{T}^n$-bundles,

$$\tau^* c_U = c_{\mathbb{R}^n/\mathbb{Z}^n}, \tag{6.6}$$

where $c_{\mathbb{R}^n/\mathbb{Z}^n}$ is the obstruction to the existence of a section for the bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{E}\mathbb{R}^n/\mathbb{Z}^n \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n.$$

6.2 The spectral sequence of a topological universal Lagrangian bundle

In this section the methods of [14] are adapted to prove that some differential on the E_2-page of the Leray-Serre spectral sequence of the topological universal Lagrangian bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{BGL}(n;\mathbb{Z}) \xrightarrow{\sigma} \mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$$

is given (up to some isomorphisms) by taking cup products with the universal Chern class

$$c_U \in \mathrm{H}^2(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); \mathbb{Z}^n_{\mathrm{id}_*}).$$

Aside from its intrinsic value, this result is important when studying the problem of distinguishing fake Lagrangian and Lagrangian bundles over a fixed integral affine manifold, as illustrated in Section 6.3.

Given a spectral sequence, the main idea in [14] is to use *auxiliary* spectral sequences to reduce the problem of determining differentials on the E_2-page of the original spectral sequence to a simpler one. While [14] deals with the cohomology of group extensions and, in particular, with abelian extensions, the case of the topological universal Lagrangian bundle and, more generally, of affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles can be thought of as a natural generalisation. This is because the interesting part of the long exact sequence in homotopy for an affine $\mathbb{R}^n/\mathbb{Z}^n$-bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ is

$$0 \to \pi_2 M \to \pi_2 B \to \pi_1 \mathbb{R}^n/\mathbb{Z}^n \to \pi_1 M \to \pi_1 B \to 1.$$

In many known cases, the connecting homomomorphism $\pi_2 B \to \pi_1 \mathbb{R}^n/\mathbb{Z}^n$ vanishes identically (say, when B has contractible universal cover) and the results of [14] can be applied directly. However, this is not the case in general, as shown by the examples due to Bates [6].

Remark 6.2. The reason why it is possible to adapt the methods of [14] to the present case is that the fibres of affine $\mathbb{R}^n/\mathbb{Z}^n$-bundles have very simple topology which is completely determined by the fundamental group.

6.2.1 Auxiliary spectral sequences

The E_2-page of the Leray-Serre spectral sequence[1] for the topological universal Lagrangian bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{BGL}(n;\mathbb{Z}) \xrightarrow{\sigma} \mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$$

with $\mathbb{Z}$ coefficients is denoted by

$$\mathrm{E}_2^{p,q} \cong \mathrm{H}^p(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); \mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})_{\rho_q}).$$

The homomorphism

$$\rho_q : \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \mathrm{Aut}(\mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}))$$

classifies the local coefficient system defined by replacing each fibre $\mathbb{R}^n/\mathbb{Z}^n$ of the topological universal Lagrangian bundle with its q-th cohomology group with integer coefficients $\mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$. Henceforth, fix a basepoint in $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$, so that the above homomorphisms are also fixed. Denote by

$$\breve{\rho}_1 : \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \mathrm{Aut}(\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}))$$

the homomorphism classifying the local coefficient system with fibre $\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$ over $\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$. Following [14], introduce *auxiliary* spectral sequences, whose E_2-pages are given by

$$\begin{aligned} \bar{\mathrm{E}}_2^{p,q} &\cong \mathrm{H}^p(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); \mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n; \mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}))_{\bar{\rho}_q}), \\ \hat{\mathrm{E}}_2^{p,q} &\cong \mathrm{H}^p(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n); \mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n; \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}))_{\hat{\rho}_q}), \end{aligned} \tag{6.7}$$

where the above local coefficient systems are given by

$$\begin{aligned} \bar{\rho}_q &= \rho_q \otimes \rho_1 : \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \mathrm{Aut}(\mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \otimes \mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})), \\ \hat{\rho}_q &= \rho_q \otimes \breve{\rho}_1 : \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \mathrm{Aut}(\mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \otimes \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})), \end{aligned}$$

via the isomorphisms

$$\begin{aligned} \mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n; \mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})) &\cong \mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \otimes \mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}), \\ \mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n; \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})) &\cong \mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \otimes \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}), \end{aligned}$$

induced by the universal coefficient theorem. These spectral sequences are henceforth referred to as the Leray-Serre spectral sequences with $\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$ and $\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$ coefficients respectively.

There is a pairing

$$\bar{\mathrm{E}}_2^{p,q} \otimes_{\mathbb{Z}} \hat{\mathrm{E}}_2^{p',q'} \to \mathrm{E}_2^{p+p',q+q'} \tag{6.8}$$

[1] For generalities on the Leray-Serre spectral sequence construction see [17, 42].

induced by taking cup products and by the standard duality

$$\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \to \mathbb{Z}.$$

Definition 6.3 (Auxiliary pairing)**.** The pairing of equation (6.8) is called the *auxiliary pairing* associated to the topological universal Lagrangian bundle.

If $\mathrm{d}^{(2)}$, $\bar{\mathrm{d}}^{(2)}$, $\hat{\mathrm{d}}^{(2)}$ denote the differentials on the E_2-page of the spectral sequences $\mathrm{E}_2^{*,*}$, $\bar{\mathrm{E}}_2^{*,*}$, $\hat{\mathrm{E}}_2^{*,*}$ respectively, there is the multiplicative formula

$$\mathrm{d}^{(2)}(x \cdot y) = \bar{\mathrm{d}}^{(2)}(x) \cdot y + (-1)^{p+q} x \cdot \hat{\mathrm{d}}^{(2)}(y), \tag{6.9}$$

where $x \in \hat{\mathrm{E}}_2^{p,q}$, $y \in \bar{\mathrm{E}}_2^{p',q'}$ and $x \cdot y$ denotes the auxiliary pairing between x and y. Equation (6.9) follows from the fact that the Leray-Serre cohomology sequence for a fibration preserves the cup product structures on both the base and the fibre. In particular, any differential in the spectral sequence is a *derivation* (cf. [17, 42]).

As shown in [14], there is an isomorphism

$$\theta : \mathrm{E}_2^{p,1} \to \bar{\mathrm{E}}_2^{p,0} \tag{6.10}$$

induced by the isomorphism

$$\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \to \mathrm{H}^0(\mathbb{R}^n/\mathbb{Z}^n;\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})).$$

The identity map

$$\mathrm{id} : \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \to \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$$

defines an element $g^1 \in \mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}))$, since

$$\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})) \cong \mathrm{Hom}(\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z});\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})).$$

The element g^1, in turn, defines an element

$$f^1 \in \hat{\mathrm{E}}_2^{0,1} \cong \mathrm{H}^0(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}))_{\hat{\rho}_1}),$$

since g^1 is fixed by the action of $\pi_1\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ defined by $\hat{\rho}_1$. The following two propositions are presented below without proof.

Proposition 6.2 (Proposition 2.1 [14])**.** *Let* $x \in \mathrm{E}_2^{p,1}$. *Then*

$$x = \theta(x) \cdot f^1, \tag{6.11}$$

where $\cdot$ *denotes the auxiliary pairing defined above.*

Proposition 6.3 (Proposition 2.2 [14])**.** *Let* $x \in \mathrm{E}_2^{p,1}$. *Then*

$$\mathrm{d}^{(2)}(x) = (-1)^{p+1}\theta(x) \cdot \hat{\mathrm{d}}^{(2)}(f^1).$$

Proposition 6.3 reduces the problem of determining $\mathrm{d}^{(2)}$ to that of determining $\hat{\mathrm{d}}^{(2)}(f^1)$, which depends on the universal Chern class c_U: this is the content of Theorem 6.6, which provides an explicit expression for this differential in terms of the universal Chern class c_U.

Before computing $\hat{\mathrm{d}}^{(2)}(f^1)$, it is useful to compute the corresponding differential for the bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{E}\mathbb{R}^n/\mathbb{Z}^n \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n, \tag{6.12}$$

which, in light of Lemma 6.1, is isomorphic to the pull-back of the topological universal Lagrangian bundle along the universal covering $\tau : \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$. Let

$$\hat{\mathrm{E}}^{p,q}_{2,\mathbb{R}^n/\mathbb{Z}^n} \cong \mathrm{H}^p(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n; \mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n; \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})))$$

denote the E_2-page of the Leray-Serre spectral sequence for the bundle (6.12) with $\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$ coefficients, and let $\hat{\mathrm{d}}^{(2)}_{\mathbb{R}^n/\mathbb{Z}^n}$ denote the corresponding differential. Let $f^1_{\mathbb{R}^n/\mathbb{Z}^n} \in \hat{\mathrm{E}}^{0,1}_{2,\mathbb{R}^n/\mathbb{Z}^n}$ be the element arising from the identity map

$$\mathrm{id} : \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \to \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$$

as above. In fact, in light of Lemma 6.1, the relation between f^1 and $f^1_{\mathbb{R}^n/\mathbb{Z}^n}$ is given by

$$f^1_{\mathbb{R}^n/\mathbb{Z}^n} = \tau^* f^1,$$

where $\tau : \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ is the universal covering. Let

$$\psi_{\mathbb{R}^n/\mathbb{Z}^n} : \mathrm{H}^2(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n; \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})) \to \hat{\mathrm{E}}^{2,0}_{2,\mathbb{R}^n/\mathbb{Z}^n} \tag{6.13}$$

be the isomorphism arising via an identification

$$\mathrm{H}^0(\mathbb{R}^n/\mathbb{Z}^n; \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})) \cong \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}).$$

The following lemma computes the image of $f^1_{\mathbb{R}^n/\mathbb{Z}^n}$ under the differential $\hat{\mathrm{d}}^{(2)}_{\mathbb{R}^n/\mathbb{Z}^n}$. This is a well-known result (*e.g.* [30]), but a proof is included for completeness.

Lemma 6.4.

$$\hat{\mathrm{d}}^{(2)}_{\mathbb{R}^n/\mathbb{Z}^n}(f^1_{\mathbb{R}^n/\mathbb{Z}^n}) = \psi(c_{\mathbb{R}^n/\mathbb{Z}^n}),$$

where $c_{\mathbb{R}^n/\mathbb{Z}^n} \in \mathrm{H}^2(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n; \mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}))$ *is the Chern class of the universal bundle* $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{E}\mathbb{R}^n/\mathbb{Z}^n \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$*, as in equation (6.6).*

Proof. By naturality of the Chern class, it is enough to prove the result when $n = 1$. In this case, the universal bundle is isomorphic (up to homotopy) to

$$S^1 \hookrightarrow S^\infty \to \mathbb{C}\mathrm{P}^\infty. \tag{6.14}$$

Since S^∞ is contractible, the differential

$$\hat{\mathrm{d}}^{(2)}_{S^1} : \hat{\mathrm{E}}^{0,1}_{2,S^1} \to \hat{\mathrm{E}}^{2,0}_{2,S^1}$$

is an isomorphism. In particular, since $f^1_{S^1} \in \hat{\mathrm{E}}^{0,1}_{2,S^1}$ is a generator, $\hat{\mathrm{d}}^{(2)}_{S^1}(f^1_{S^1})$ is a generator of $\hat{\mathrm{E}}^{2,0}_{2,S^1}$, and thus equal to $\pm\psi_{S^1}(c_{S^1})$. In fact, the normalisation axiom for the Chern classes (cf. [43]) implies that

$$\hat{\mathrm{d}}^{(2)}_{S^1}(f^1_{S^1}) = \psi_{S^1}(c_{S^1}).$$

□

Let $\mathrm{E}^{*,*}_{2,\mathbb{R}^n/\mathbb{Z}^n}$, $\bar{\mathrm{E}}^{*,*}_{2,\mathbb{R}^n/\mathbb{Z}^n}$ denote the E_2-pages of the Leray-Serre spectral sequences of

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{E}\mathbb{R}^n/\mathbb{Z}^n \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$$

with $\mathbb{Z}$ and $\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$ coefficients respectively, and denote by $\mathrm{d}^{(2)}_{\mathbb{R}^n/\mathbb{Z}^n}$, $\hat{\mathrm{d}}^{(2)}_{\mathbb{R}^n/\mathbb{Z}^n}$ their respective differentials. Lemma 6.4 and Proposition 6.3 can be combined to prove the following corollary, which is a version of Theorem 6.6 for principal $\mathbb{R}^n/\mathbb{Z}^n$-bundles.

Corollary 6.5. *Let $x \in \mathrm{E}^{p,1}_{2,\mathbb{R}^n/\mathbb{Z}^n}$. Its image $\mathrm{d}^{(2)}_{\mathbb{R}^n/\mathbb{Z}^n}(x) \in \mathrm{E}^{p+2,0}_{2,\mathbb{R}^n/\mathbb{Z}^n}$ is given by*

$$\mathrm{d}^{(2)}_{\mathbb{R}^n/\mathbb{Z}^n}(x) = (-1)^{p+1}\theta_{\mathbb{R}^n/\mathbb{Z}^n}(x) \cdot_{\mathbb{R}^n/\mathbb{Z}^n} \psi_{\mathbb{R}^n/\mathbb{Z}^n}(c_{\mathbb{R}^n/\mathbb{Z}^n}),$$

where $\theta_{\mathbb{R}^n/\mathbb{Z}^n}$, $\cdot_{\mathbb{R}^n/\mathbb{Z}^n}$ are the analogues of the isomorphism of equation (6.10) and the auxiliary pairing of equation (6.8) for the bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{E}\mathbb{R}^n/\mathbb{Z}^n \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$.

6.2.2 The differential $\mathrm{d}^{(2)}$

One of the aims of this chapter is to prove the analogue of Corollary 6.5 for the topological universal Lagrangian bundle. This is the content of the following theorem, which states that, up to some isomorphisms, the differential

$$\mathrm{d}^{(2)} : \mathrm{E}^{p,1}_2 \to \mathrm{E}^{p+2,0}_2$$

is given by taking the cup product with the universal Chern class c_U. The notion of taking cup products makes sense since the local coefficient systems involved are dual to one another, the duality arising from the standard duality between $\mathrm{H}^*(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$ and $\mathrm{H}_*(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$ (cf. Remark 6.2).

Theorem 6.6. *Let $x \in \mathrm{E}^{p,1}_2$. Its image $\mathrm{d}^{(2)}(x) \in \mathrm{E}^{p+2,0}_2 \cong \mathrm{H}^{p+2}(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);\mathbb{Z})$ is given by*

$$\mathrm{d}^{(2)}(x) = (-1)^{p+1}\theta(x) \cdot \psi(c_U), \tag{6.15}$$

where $\psi : \mathrm{H}^2(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);\mathbb{Z}^n_{\mathrm{id}_*}) \to \hat{\mathrm{E}}^{2,0}_2$ is the isomorphism induced by the identification

$$\mathrm{H}^0(\mathbb{R}^n/\mathbb{Z}^n;\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})) \cong \mathbb{Z}^n.$$

Proof. In light of Proposition 6.3, it suffices to show that

$$\hat{\mathrm{d}}^{(2)}(f^1) = \psi(c_U), \tag{6.16}$$

which is just the equivariant version of the result of Lemma 6.4. The idea of the proof is to use Lemma 6.4 and functoriality of the Leray-Serre spectral sequence to deduce the result.

By Lemma 6.1, the pull-back bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \tau^*\mathrm{BGL}(n;\mathbb{Z}) \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$$

is a universal bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{E}\mathbb{R}^n/\mathbb{Z}^n \to \mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$. In particular, there is a commutative diagram

$$\begin{tikzcd}
\hat{\mathrm{E}}_2^{0,1} \arrow[r, "\tau^*"] \arrow[d, "\hat{\mathrm{d}}^{(2)}"'] & \hat{\mathrm{E}}_{2,\mathbb{R}^n/\mathbb{Z}^n}^{0,1} \arrow[d, "\hat{\mathrm{d}}^{(2)}_{\mathbb{R}^n/\mathbb{Z}^n}"] \\
\hat{\mathrm{E}}_2^{2,0} \arrow[r, "\tau^*"'] & \hat{\mathrm{E}}_{2,\mathbb{R}^n/\mathbb{Z}^n}^{2,0}
\end{tikzcd} \tag{6.17}$$

arising from functoriality of the Leray-Serre spectral sequence (cf. [42]). By Lemma 6.4,

$$\hat{\mathrm{d}}^{(2)}_{\mathbb{R}^n/\mathbb{Z}^n} \circ \tau^*(f^1) = \psi_{\mathbb{R}^n/\mathbb{Z}^n}(c_{\mathbb{R}^n/\mathbb{Z}^n}), \tag{6.18}$$

since $f^1_{\mathbb{R}^n/\mathbb{Z}^n} = \tau^* f^1$. Equation (6.18) and the commutativity of the diagram in equation (6.17) imply that

$$\tau^* \circ \hat{\mathrm{d}}^{(2)}(f^1) = \psi_{\mathbb{R}^n/\mathbb{Z}^n}(c_{\mathbb{R}^n/\mathbb{Z}^n}). \tag{6.19}$$

Note that, by definition, the isomorphism ψ is the *equivariant* version of $\psi_{\mathbb{R}^n/\mathbb{Z}^n}$, *i.e.* there is a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);\mathbb{Z}^n_{\mathrm{id}_*}) \arrow[r, "\psi"] \arrow[d, "\tau^*"'] & \hat{\mathrm{E}}_2^{2,0} \arrow[d, "\tau^*"] \\
\mathrm{H}^2(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}^n) \arrow[r, "\psi_{\mathbb{R}^n/\mathbb{Z}^n}"'] & \hat{\mathrm{E}}_{2,\mathbb{R}^n/\mathbb{Z}^n}^{2,0}.
\end{tikzcd} \tag{6.20}$$

By equation (6.6), the commutative diagram (6.20) implies that

$$\tau^* \circ \psi(c_U) = \psi_{\mathbb{R}^n/\mathbb{Z}^n}(c_{\mathbb{R}^n/\mathbb{Z}^n}). \tag{6.21}$$

In particular, combining equations (6.19) and (6.21), obtain that

$$\hat{\mathrm{d}}^{(2)}(f^1) - \psi(c_U) = \mu \in \ker \tau^*. \tag{6.22}$$

It therefore remains to show that $\mu = 0$.

This is achieved in the following two steps.

Step 1 Prove that $\psi^{-1}(\mu)$ lies in the image of the homomorphism

$$p^* : \mathrm{H}^2(\mathrm{BGL}(n;\mathbb{Z});\mathbb{Z}^n_{\mathrm{id}^G_*}) \to \mathrm{H}^2(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);\mathbb{Z}^n_{\mathrm{id}_*})$$

induced by the fibration

$$p : \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{BGL}(n;\mathbb{Z});$$

Step 2 Prove that $\psi^{-1}(\mu)$ lies in the kernel of the homomorphism

$$\sigma^* : \mathrm{H}^2(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);\mathbb{Z}^n_{\mathrm{id}_*}) \to \mathrm{H}^2(\mathrm{BGL}(n;\mathbb{Z});\mathbb{Z}^n_{\mathrm{id}^G_*})$$

induced by the topological universal Lagrangian bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{BGL}(n;\mathbb{Z}) \xrightarrow{\sigma} \mathrm{BAff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n).$$

Step 1

Recall that there is a fibration

$$\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n \overset{\tau}{\hookrightarrow} \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \xrightarrow{p} \mathrm{BGL}(n;\mathbb{Z}) \tag{6.23}$$

arising from the group $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n) = \mathrm{GL}(n;\mathbb{Z}) \ltimes \mathbb{R}^n/\mathbb{Z}^n$ (cf. Theorem 3.3 and equation (3.6)), and consider the bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \sigma^*\mathrm{BGL}(n;\mathbb{Z}) \to \mathrm{BGL}(n;\mathbb{Z}) \tag{6.24}$$

obtained by pulling back the topological universal Lagrangian bundle along the map $\sigma : \mathrm{BGL}(n;\mathbb{Z}) \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ induced by the section $\sigma : \mathrm{GL}(n;\mathbb{Z}) \to \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$. There is an associated system of local coefficients

$$\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \hookrightarrow \sigma^*P_n \to \mathrm{BGL}(n;\mathbb{Z}), \tag{6.25}$$

which is just the pull-back of the universal period lattice bundle $P_n \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$ along σ. The pull-back

$$\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \hookrightarrow p^*\sigma^*P_n \to \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)$$

is classified by the (conjugacy class of the) homomorphism

$$\sigma_* \circ p_* : \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)) \to \pi_1(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n)), \tag{6.26}$$

which is the identity, as the composition of homomorphisms

$$\sigma \circ p : \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) \to \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$$

preserves path-connected components. Hence, the following cohomology rings are isomorphic

$$\mathrm{H}^*(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);\mathbb{Z}^n_{\mathrm{id}_*}) \cong \mathrm{H}^*(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})_{(\sigma\circ p)_*}). \tag{6.27}$$

There is a twisted version of the Leray-Serre spectral sequence, constructed by Siegel in [57], which allows to calculate the cohomology ring

$$\mathrm{H}^*(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})_{(\sigma\circ p)_*})$$

using the fibration

$$\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n \overset{\tau}{\hookrightarrow} \mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n) \xrightarrow{p} \mathrm{BGL}(n;\mathbb{Z})$$

and the system of local coefficients on $\mathrm{BGL}(n;\mathbb{Z})$ of equation (6.25). Let $\breve{\mathrm{E}}_2^{p,q}$ denote the E_2-page of this spectral sequence; since $\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n$ is simply connected, there is an exact sequence

$$0 \longrightarrow \breve{\mathrm{E}}_2^{2,0} \xrightarrow{p^*} \mathrm{H}^2(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);\mathbb{Z}^n_{\mathrm{id}_*}) \xrightarrow{\tau^*} \breve{\mathrm{E}}_2^{0,2}, \tag{6.28}$$

where the isomorphism of equation (6.27) is used tacitly. There are isomorphisms

$$\begin{aligned}\check{\mathrm{E}}_2^{2,0} &\cong \mathrm{H}^2(\mathrm{BGL}(n;\mathbb{Z});\mathbb{Z}^n_{\mathrm{id}_*^G}),\\ \check{\mathrm{E}}_2^{0,2} &\cong [\mathrm{H}^2(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}^n)]^G \subset \mathrm{H}^2(\mathrm{B}\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}^n),\end{aligned}$$

where $\mathrm{id}_*^G : \pi_1\mathrm{BGL}(n;\mathbb{Z}) \cong \mathrm{GL}(n;\mathbb{Z}) \to \mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}(n;\mathbb{Z})$ is the identity and $[\,.\,]^G$ denotes the group of $\mathrm{GL}(n;\mathbb{Z})$-invariant elements. Since $\ker\tau^* = \operatorname{im} p^*$, it follows that $\psi^{-1}(\mu) \in \operatorname{im} p^*$. This completes the proof of Step 1.

Step 2

Consider the bundle of equation (6.24). This bundle admits a section, which is induced by the identity map $\mathrm{BGL}(n;\mathbb{Z}) \to \mathrm{BGL}(n;\mathbb{Z})$. Thus

$$\sigma^*(c_U) = 0 \in \mathrm{H}^2(\mathrm{BGL}(n;\mathbb{Z});\mathbb{Z}^n_{\mathrm{id}_*^G}). \tag{6.29}$$

Moreover, if $\hat{\mathrm{E}}_{2,G}^{*,*}$ denotes the Leray-Serre spectral sequence for

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \sigma^*\mathrm{BGL}(n;\mathbb{Z}) \to \mathrm{BGL}(n;\mathbb{Z})$$

with $\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})$ coefficients, it follows from [14] that the differential

$$\hat{\mathrm{d}}_G^{(2)} : \hat{\mathrm{E}}_{2,G}^{0,1} \to \hat{\mathrm{E}}_{2,G}^{2,0}$$

vanishes identically, since the map pr^* induced in cohomology by the projection $\mathrm{pr} : \sigma^*\mathrm{BGL}(n;\mathbb{Z}) \to \mathrm{BGL}(n;\mathbb{Z})$ is injective as the bundle admits a section. Thus there is the following commutative diagram arising from functoriality of spectral sequences

$$\begin{array}{ccc} \hat{\mathrm{E}}_2^{0,1} & \xrightarrow{\sigma^*} & \hat{\mathrm{E}}_{2,G}^{0,1} \\ {\scriptstyle \hat{\mathrm{d}}^{(2)}}\downarrow & & \downarrow{\scriptstyle \hat{\mathrm{d}}_G^{(2)}\equiv 0} \\ \hat{\mathrm{E}}_2^{2,0} & \xrightarrow[\sigma^*]{} & \hat{\mathrm{E}}_{2,G}^{2,0}, \end{array}$$

which implies that

$$\sigma^*\hat{\mathrm{d}}^{(2)}(f^1) = 0. \tag{6.30}$$

If $\psi_G : \mathrm{H}^2(\mathrm{BGL}(n;\mathbb{Z});\mathbb{Z}^n_{\mathrm{id}_*^G}) \to \hat{\mathrm{E}}_{2,G}^{2,0}$ is the isomorphism arising from the identification

$$\mathrm{H}^0(\mathbb{R}^n/\mathbb{Z}^n;\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})) \cong \mathbb{Z}^n,$$

there is a commutative diagram (cf. equation (6.20))

$$\begin{array}{ccc} \mathrm{H}^2(\mathrm{BAff}(\mathbb{R}^n/\mathbb{Z}^n);\mathbb{Z}^n_{\mathrm{id}_*}) & \xrightarrow{\psi} & \hat{\mathrm{E}}_2^{2,0} \\ {\scriptstyle \sigma^*}\downarrow & & \downarrow{\scriptstyle \sigma^*} \\ \mathrm{H}^2(\mathrm{BGL}(n;\mathbb{Z});\mathbb{Z}^n_{\mathrm{id}_*^G}) & \xrightarrow[\psi_G]{} & \hat{\mathrm{E}}_{2,G}^{2,0}. \end{array} \tag{6.31}$$

Equation (6.29) and the diagram (6.31) imply that

$$\sigma^*\psi^*(c_U) = 0. \tag{6.32}$$

Applying σ^* to both sides of equation (6.22), and using equations (6.30) and (6.32), obtain that

$$\sigma^* \mu = 0,$$

which proves Step 2.

By Step 1, there exists

$$\nu \in \mathrm{H}^2(\mathrm{BGL}(n;\mathbb{Z}); \mathbb{Z}^n_{\mathrm{id}^G_*})$$

such that $\psi^{-1}(\mu) = p^*\nu$. , then

$$\sigma^* \circ \psi \circ p^*(\nu) = \sigma^* \mu = 0, \tag{6.33}$$

since $\mu \in \ker \sigma^*$ by Step 2. Commutativity of the diagram in equation (6.31) implies that

$$\sigma^* \circ \psi \circ p^*(\nu) = \psi \circ \sigma^* \circ p^*(\nu).$$

Since ψ is an isomorphism, equation (6.33) implies that

$$\sigma^* \circ p^*(\nu) = 0;$$

as $\sigma^* \circ p^*$ is the identity on $\mathrm{H}^2(\mathrm{BGL}(n;\mathbb{Z}); \mathbb{Z}^n_{\mathrm{id}^G_*})$, it follows that $\nu = 0$. Therefore,

$$\mu = 0$$

as required. □

6.3 Relation to almost Lagrangian bundles

Theorem 6.6 provides information on some differentials on the E_2-page of the Leray-Serre spectral sequence of almost Lagrangian bundles by functoriality of the Leray-Serre spectral sequence. Throughout this section, fix an integral affine manifold $(B, \mathcal{A})$ with linear holonomy $\mathfrak{l}$ and an almost Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ with Chern class

$$c \in \mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}}).$$

The aim of this section is to use Theorem 6.6 to compute the obstruction for the above bundle to be Lagrangian; Theorem 6.9 proves that this obstruction is given by the cohomology class of the cup product of the Chern class of the bundle with the cohomology class of the symplectic form on the total space of the symplectic reference Lagrangian bundle associated to $(B, \mathcal{A})$ (cf. Lemma 6.7 below).

6.3.1 Symplectic reference Lagrangian bundles

The symplectic reference Lagrangian bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0) \to B$$

associated to the integral affine manifold $(B, \mathcal{A})$ (cf. Definition 4.25) plays an important role in determining whether almost Lagrangian bundles over $(B, \mathcal{A})$ are, in fact, Lagrangian. The symplectic form ω_0 defines a cohomological invariant of $(B, \mathcal{A})$, as shown in the next lemma.

Lemma 6.7. *The 2-form ω_0 defines a cohomology class*

$$w_0 \in \mathrm{H}^1(B; \mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n; \mathbb{R})_{\mathfrak{l}}),$$

where $\mathfrak{l} : \pi_1(B) \to \mathrm{GL}(n; \mathbb{Z}) \subset \mathrm{GL}(n; \mathbb{R})$.

Proof. The cohomology theory used throughout this proof is Čech-de Rham (cf. [11]). The cohomology class of a closed differential form can be represented as the obstruction to finding a globally defined potential. Let $\mathcal{U} = \{U_\alpha\}$ be a good open cover by integral affine coordinate neighbourhoods of $(B, \mathcal{A})$. The proof of Lemma 4.3 shows that there exist local action-angle coordinates $(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha)$ on $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^n/\mathbb{Z}^n$, so that

$$\omega_\alpha = \omega_0|_{\pi^{-1}(U_\alpha)} = \sum_i \mathrm{d}a^i_\alpha \wedge \mathrm{d}\theta^i_\alpha = \mathrm{d}\left(\sum_i a^i_\alpha \mathrm{d}\theta^i_\alpha\right), \tag{6.34}$$

The transition functions $\varphi_{\beta\alpha}$ for this choice of local trivialisations of $\mathrm{T}^*B/P_{(B,\mathcal{A})} \to B$ are given by

$$\varphi_{\beta\alpha}(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha) = (A_{\beta\alpha}\mathbf{a}_\alpha + \mathbf{d}_{\beta\alpha}, A^{-T}_{\beta\alpha}\boldsymbol{\theta}_\alpha), \tag{6.35}$$

where $A_{\beta\alpha} \in \mathrm{GL}(n; \mathbb{Z})$ and $\mathbf{d}_{\beta\alpha} \in \mathbb{R}^n$ is constant (cf. Lemma 4.3). For each α, set

$$\nu_\alpha = \sum_i a^i_\alpha \mathrm{d}\theta^i_\alpha.$$

The cohomology class of ω_0 (as a differential form on $\mathrm{T}^*B/P_{(B,\mathcal{A})}$) is given in Čech cohomology by the cocycle

$$\tau_{\beta\alpha} = \varphi^*_{\beta\alpha}\nu_\beta - \nu_\alpha = \sum_{i,k} c^i_{\beta\alpha}\mathrm{d}\theta^i_\beta.$$

Since $\mathcal{U}$ is a good cover for B, $\tau = \{\tau_{\beta\alpha}\}$ defines a one dimensional cohomology class w_0 on B with coefficients in the local coefficient system

$$\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n; \mathbb{R}) \hookrightarrow P^*_{\mathbb{R}} \to B,$$

whose monodromy is given by $\mathfrak{l} : \pi_1(B) \to \mathrm{GL}(n; \mathbb{Z}) \subset \mathrm{GL}(n; \mathbb{R})$, since this equals the inverse transposed of the monodromy of the period lattice bundle $P_{(B,\mathcal{A})}$. This proves the result. □

Remark 6.4. Let ω' be any other symplectic form on $\mathrm{T}^*B/P_{(B,\mathcal{A})}$ making the topological reference Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \mathrm{T}^*B/P_{(B,\mathcal{A})} \to B$ Lagrangian. This bundle admits a section s (cf. Remark 4.17). Let

$$\mu = s^*\omega'$$

denote the pull-back of the symplectic form ω' to B. If $\pi : \mathrm{T}^*B/P_{(B,\mathcal{A})} \to B$ denotes the projection map, then $\pi^*\mu$ is a closed 2-form on $\mathrm{T}^*B/P_{(B,\mathcal{A})}$ and the 2-form

$$\omega_0 + \pi^*\mu$$

is symplectic, since ω_0 is non-degenerate. Moreover, the fibres of π are Lagrangian submanifolds of $\omega_0 + \pi^*\mu$. Thus

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0 + \pi^*\mu) \to B$$

is a Lagrangian bundle. Translation along s induces a fibrewise symplectomorphism

$$(\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega') \to (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0 + \pi^*\mu),$$

as in the proof of Lemma 5.3 and [53]. Let $\mathcal{U} = \{U_\alpha\}$ be the good open cover of B of the proof of Lemma 6.7, let $(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha)$ be action-angle coordinates for the restriction of the symplectic reference Lagrangian bundle to each $\pi^{-1}(U_\alpha)$, and denote by $\pi^*\mu_\alpha$ the restriction of the above 2-form. In light of the above fibrewise symplectomorphism,

$$\omega'_\alpha = \omega'|_{\pi^{-1}(U_\alpha)} = \sum_{i=1}^{n} \mathrm{d}a^i_\alpha \wedge \mathrm{d}\theta^i_\alpha + \pi^*\mu_\alpha;$$

set

$$\pi^*\mu_\alpha = \mathrm{d}\pi^*\chi_\alpha$$

for some 1-form χ_α, which exists as U_α is simply connected. Then

$$\omega'_\alpha = \mathrm{d}\left(\sum_{i=1}^{n} a^i_\alpha \mathrm{d}\theta^i_\alpha + \pi^*\chi_\alpha\right). \tag{6.36}$$

Set

$$\bar{\nu}_\alpha = \nu_\alpha + \pi^*\chi_\alpha = \sum_{i=1}^{n} a^i_\alpha \mathrm{d}\theta^i_\alpha + \pi^*\chi_\alpha,$$

and consider the cocycle

$$\bar{\tau}_{\beta\alpha} = \varphi^*_{\beta\alpha}\bar{\nu}_\beta - \bar{\nu}_\alpha = \tau_{\beta\alpha} + \varphi^*_{\beta\alpha}\pi^*\chi_\beta - \pi^*\chi_\alpha.$$

If $\phi_{\beta\alpha}$ denotes the change in integral affine coordinates on $(B, \mathcal{A})$, then $\pi \circ \varphi_{\beta\alpha} = \phi_{\beta\alpha} \circ \pi$ by equation (6.35). In particular,

$$\varphi^*_{\beta\alpha}\pi^*\chi_\beta - \pi^*\chi_\alpha = \pi^*(\phi^*_{\beta\alpha}\chi_\beta - \chi_\alpha) =: \pi^*\eta_{\beta\alpha}.$$

The cocycle $\eta = \{\eta_{\beta\alpha}\}$ represents the cohomology class of β in Čech-de Rham cohomology and, thus, $\pi^*\eta$ represents the cohomology class of $\pi^*\beta$ in $\mathrm{H}^2(\mathrm{T}^*B/P_{(B,\mathcal{A})}; \mathbb{R})$. Thus

$$\omega' - \pi^*\beta = \omega' - \pi^*s^*\omega'$$

also defines the cohomology class w_0 of Lemma 6.7. This construction is independent of the choice of section s (cf. [11]).

Remark 6.5 (Alternative definition of w_0)**.** Fix an n-dimensional integral affine manifold $(B, \mathcal{A})$ with linear holonomy $\mathfrak{l}$. Let

$$\mathbb{R}^n \xhookrightarrow{\iota} \mathrm{T}^*B/P_{(B,\mathcal{A})} \xrightarrow{\pi} B$$

be (a representative of the isomorphism class of) the topological reference Lagrangian bundle, so that $P_{(B,\mathcal{A})} \subset \mathrm{T}^*B$ is classified by $\mathfrak{l}^{-T}$. Let ω be a symplectic form on $\mathrm{T}^*B/P_{(B,\mathcal{A})}$ making the above bundle Lagrangian. Denote its cohomology class by

$$w \in \mathrm{H}^2(\mathrm{T}^*B/P_{(B,\mathcal{A})}; \mathbb{R});$$

since $\mathbb{R}$ is a field, there is an isomorphism

$$\mathrm{H}^2(\mathrm{T}^*B/P_{(B,\mathcal{A})};\mathbb{R}) \cong \mathrm{E}_\infty^{0,2} \oplus \mathrm{E}_\infty^{1,1} \oplus \mathrm{E}_\infty^{2,0},$$

where $\mathrm{E}_*^{*,*}$ denotes the Leray-Serre spectral sequence with real coefficients for the above bundle. Recall that the projection

$$\mathrm{H}^2(\mathrm{T}^*B/P_{(B,\mathcal{A})};\mathbb{R}) \to \mathrm{E}_\infty^{0,2} \subset \mathrm{E}_2^{0,2} \subset \mathrm{H}^2(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})$$

corresponds to the map

$$\iota^* : \mathrm{H}^2(\mathrm{T}^*B/P_{(B,\mathcal{A})};\mathbb{R}) \to \mathrm{H}^2(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})$$

(cf. [59]). Since $\iota^*\omega = 0$, it follows that the $\mathrm{E}_\infty^{0,2}$ component of w is zero. As the Chern class of the bundle vanishes, Corollary 6.8 implies that

$$\mathrm{E}_\infty^{1,1} = \mathrm{E}_2^{1,1};$$

thus the $\mathrm{E}_\infty^{1,1}$ component of w defines a class in

$$\mathrm{E}_2^{1,1} \cong \mathrm{H}^1(B;\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})_{\mathfrak{l}}).$$

Remark 6.4 above implies that there exists a closed 2-form β on B such that

$$\omega = \omega_0 + \pi^*\beta;$$

since

$$\pi^* : \mathrm{H}^2(B;\mathbb{R}) \to \mathrm{H}^2(\mathrm{T}^*B/P_{(B,\mathcal{A})};\mathbb{R})$$

corresponds to the homomorphism

$$\mathrm{E}_2^{2,0} \to \mathrm{E}_\infty^{2,0} \hookrightarrow \mathrm{H}^2(\mathrm{T}^*B/P_{(B,\mathcal{A})};\mathbb{R})$$

(cf. [42]), the $\mathrm{E}_\infty^{1,1}$ component of the cohomology class of ω equals the $\mathrm{E}_\infty^{1,1}$ component the cohomology class of ω_0. The proof of Lemma 6.7 shows that the cohomology class of ω_0 is

$$w_0 \in \mathrm{H}^1(B;\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})_{\mathfrak{l}}) = \mathrm{E}_2^{1,1} = \mathrm{E}_\infty^{1,1}.$$

In particular, this shows that w_0 is the $\mathrm{E}_\infty^{1,1}$ component of the cohomology class of *any* symplectic form ω on $\mathrm{T}^*B/P_{(B,\mathcal{A})}$ making the bundle

$$\mathbb{R}^n \xhookrightarrow{\iota} \mathrm{T}^*B/P_{(B,\mathcal{A})} \xrightarrow{\pi} B$$

Lagrangian.

Example 6.6 (Symplectic forms on a topological reference Lagrangian bundle)**.** The integral affine manifold $\mathbb{R}^2/\mathbb{Z}^2$ has trivial linear holonomy. The corresponding topological reference Lagrangian bundle is isomorphic to

$$\mathbb{R}^2/\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2.$$

Let $\boldsymbol{\theta}_B, \boldsymbol{\theta}_F$ denote integral affine coordinates on the base and fibre respectively. The symplectic form ω_0 for which the zero section is Lagrangian is given by

$$\omega_0 = \mathrm{d}\theta_B^1 \wedge \mathrm{d}\theta_F^1 + \mathrm{d}\theta_B^2 \wedge \mathrm{d}\theta_F^2.$$

Consider the section

$$\begin{aligned} s : \mathbb{R}^2/\mathbb{Z}^2 &\to \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^2/\mathbb{Z}^2 \\ (\theta_B^1, \theta_B^2) &\mapsto (\theta_B^1, \theta_B^2, \theta_B^2, 0); \end{aligned} \tag{6.37}$$

the pull-back

$$s^*\omega_0 = \mathrm{d}\theta_B^1 \wedge \mathrm{d}\theta_B^2$$

defines a generator of $\mathrm{H}^2(B;\mathbb{R})$. Denote the projection in the above bundle by π. The symplectic form

$$\omega = \omega_0 - \pi^* s^* \omega_0$$

makes the above bundle Lagrangian with a Lagrangian section, given by s. Moreover, translation by s gives a fibrewise symplectomorphism

$$(\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^2/\mathbb{Z}^2, \omega_0) \cong (\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^2/\mathbb{Z}^2, \omega).$$

Note, however, that the cohomology class of ω is not equal to the cohomology class of ω_0; however, their $\mathrm{E}_\infty^{1,1}$ components coincide, where

$$\mathrm{E}_\infty^{1,1} = \mathrm{H}^1(\mathbb{R}^2/\mathbb{Z}^2; \mathrm{H}^1(\mathbb{R}^2/\mathbb{Z}^2;\mathbb{R})) \cong \mathrm{H}^1(\mathbb{R}^2/\mathbb{Z}^2;\mathbb{R}) \otimes_{\mathbb{R}} \mathrm{H}^1(\mathbb{R}^2/\mathbb{Z}^2;\mathbb{R}).$$

The cohomology class w_0 does not depend solely on the linear holonomy $\mathfrak{l}$ of $(B, \mathcal{A})$, as the next example illustrates (cf. Example 4.29).

Example 6.7. Let $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}/2\mathbb{Z}$ be the integral affine manifolds of Example 4.16.iii and 4.29. These affine manifolds have trivial linear holonomy. Let ω_1 and ω_2 be symplectic forms on their respective symplectic reference Lagrangian bundles. These bundles are isomorphic as affine $\mathbb{R}/\mathbb{Z}$-bundles and their total spaces can be identified with $S^1 \times S^1$. The cohomology classes

$$w_{0,1}, w_{0,2} \in \mathrm{H}^1(S^1; \mathrm{H}^1(\mathbb{R}/\mathbb{Z};\mathbb{R})) \hookrightarrow \mathrm{H}^2(S^1 \times S^1;\mathbb{R})$$

defined from ω_1 and ω_2 as in Lemma 6.7 satisfy

$$w_{0,2} = 2w_{0,1}.$$

In particular, the above example hints at the fact that w_0 is an *integral affine invariant* of the manifold $(B, \mathcal{A})$. This is evident from the cocycle $\tau = \{\tau_{\beta\alpha}\}$ representing w_0 in the proof of Lemma 6.7, since it depends on the translational components of the changes of integral affine coordinates of $(B, \mathcal{A})$.

Remark 6.8. It is important to notice that the differential 1-forms $\mathrm{d}\theta_\alpha^1, \ldots, \mathrm{d}\theta_\alpha^n$ represent, in fact, integral cohomology classes in

$$\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R}) \cong \mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

for all indices α. This is because these forms are dual to the flows of the vector fields $\partial/\partial\theta_\alpha^1, \ldots, \partial/\partial\theta_\alpha^n$ from time 0 to time 1. These curves define a basis of the integral homology groups

$$\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}),$$

(cf. Remark 4.12). Thus the reason why real coefficients are used throughout is that the translational components of the changes of integral affine coordinates of $(B, \mathcal{A})$ are not necessarily integral. This is further studied in Remark 7.18.

6.3.2 Realisability theorem

This section is devoted to proving Theorem 6.9, which shows that the homomorphism $\mathcal{D}_{(B,\mathcal{A})}$ of Dazord and Delzant (cf. Theorem 4.4) is, in fact, determined by a differential $\mathrm{d}^{(2)}$ on E_2-page of the Leray-Serre spectral sequence of the topological universal Lagrangian bundle and by the cohomology class w_0 of Lemma 6.7. Let $\mathrm{E}^{*,*}_{2,B}$ denote the E_2-page of the Leray-Serre spectral sequence with integer coefficients of the almost Lagrangian bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$$

fixed above. Theorem 6.6 and functoriality of the Leray-Serre spectral sequence imply the following corollary.

Corollary 6.8. *Let $x \in \mathrm{E}^{p,1}_{2,B}$. Its image under the differential $\mathrm{d}^{(2)}_B$ is given by*

$$\mathrm{d}^{(2)}_B(x) = (-1)^{p+1}\theta_B(x) \cdot_B \psi_B(c),$$

where θ_B, ψ_B, $\cdot_B$ are the pull-backs of θ, ψ, $\cdot$ defined for the topological universal Lagrangian bundle.

The main idea of Theorem 6.9 is to study the differential

$$\mathrm{d}^{(2)}_{B,\mathbb{R}} : \mathrm{E}^{1,1}_{2,B,\mathbb{R}} \to \mathrm{E}^{3,0}_{2,B,\mathbb{R}}$$

on the E_2-page of the Leray-Serre spectral sequence with real coefficients (hence the subscript $\mathbb{R}$) associated to the above almost Lagrangian bundle. Corollary 6.8 can be used to show that if $x \in \mathrm{E}^{p,1}_{2,B,\mathbb{R}}$, then

$$\mathrm{d}^{(2)}_{B,\mathbb{R}}(x) = (-1)^{p+1}\theta_{B,\mathbb{R}}(x) \cdot_{B,\mathbb{R}} \psi_{B,\mathbb{R}}(c^{\mathbb{R}}),$$

where $c^{\mathbb{R}} \in \mathrm{H}^2(B; \mathbb{R}^n_{\chi_*})$ is the image of c under the homomorphism

$$\mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}-T}) \to \mathrm{H}^2(B; \mathbb{R}^n_{\mathfrak{l}-T}) \tag{6.38}$$

induced by the standard inclusion

$$\mathbb{Z}^n \hookrightarrow \mathbb{R}^n \cong \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R},$$

and $\theta_{B,\mathbb{R}}$, $\psi_{B,\mathbb{R}}$ are the appropriate isomorphisms. Henceforth, the subscripts B, $\mathbb{R}$ are omitted in order to simplify notation.

Remark 6.9 (Realisability theorem)**.** Theorem 6.9 may be called a *realisability* theorem, since it provides a way to determine which cohomology classes in $\mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}-T})$ can be realised as the Chern class of some Lagrangian bundle over $(B, \mathcal{A})$. The terminology comes from the theory of symplectic realisations of Poisson manifolds (cf. [62]); these are symplectic manifolds which fibre over a Poisson manifold such that the projection map is a Poisson morphism. Such manifolds arise in the study of isotropic bundles, which are a natural generalisation of Lagrangian bundles and are associated to a more general notion of integrability due to Miščenko and Fomenko (cf. [47]).

Firstly, it is shown that

$$w_0 \in \mathrm{E}_2^{1,1}.$$

The homomorphisms

$$\rho_q : \pi_1(B) \to \mathrm{Aut}(\mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R}))$$

are completely determined by ρ_1, since

$$\mathrm{H}^q(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R}) \cong \underbrace{\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R}) \wedge \ldots \wedge \mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})}_{q}.$$

Recall that the monodromy $\mathfrak{l}^{-T}$ classifies the local coefficient system

$$\mathrm{H}_1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \hookrightarrow P_{(B,\mathcal{A})} \to B;$$

thus

$$\rho_1 = \mathfrak{l} : \pi_1(B) \to \mathrm{GL}(n;\mathbb{Z}) \cong \mathrm{Aut}(\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z})) \hookrightarrow \mathrm{Aut}(\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})),$$

where the last inclusion follows from the injection

$$\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \hookrightarrow \mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})$$

induced by the universal coefficient theorem. Thus

$$w_0 \in \mathrm{E}_2^{1,1},$$

where w_0 is the cohomology class defined in Lemma 6.7.

Secondly, using local action-angle coordinates, it is possible to give an explicit cocycle representing the form w_0 in Čech-de Rham cohomology. Let $\mathcal{U} = \{U_\alpha\}$ be a good open cover by integral affine coordinate neighbourhoods of $(B,\mathcal{A})$. Remark 4.34 shows that there exist local trivialisations

$$\Upsilon_\alpha : \pi^{-1}(U_\alpha) \to \mathrm{T}^*U_\alpha/P_{(B,\mathcal{A})}|_{U_\alpha}$$

inducing action-angle coordinates $(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha)$ on $\pi^{-1}(U_\alpha)$; the corresponding transition functions $\varphi_{\beta\alpha}$ are of the form

$$\varphi_{\beta\alpha}(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha) = (A_{\beta\alpha}\mathbf{a}_\alpha + \mathbf{d}_{\beta\alpha}, A_{\beta\alpha}^{-T}\boldsymbol{\theta}_\alpha + \mathbf{g}_{\beta\alpha}(\mathbf{a}_\alpha)),$$

where the first component corresponds to a change in integral affine coordinates on $(B,\mathcal{A})$. The family of 1-forms

$$\bar{\tau}_{\beta\alpha} = \sum_{i=1}^{n} d^i_{\beta\alpha}\mathrm{d}\theta^i_\beta$$

define an element in $\mathrm{H}^1(B;\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})_{\rho_1})$, since the forms $\mathrm{d}\theta^1_\beta, \ldots, \mathrm{d}\theta^n_\beta$ are closed and the family of cohomology classes of $\bar{\tau}_{\beta\alpha}$ form a Čech cocycle. This last statement holds since

$$(\delta\bar{\tau})_{\alpha_1\alpha_2\alpha_3} = \mathrm{d}\left(\sum_{i=1}^{n} d^i_{\alpha_2\alpha_3} g^i_{\alpha_2\alpha_1}\right)$$

and the right hand side is an exact form. Since the angle coordinates $\boldsymbol{\theta}_\alpha$ are pulled back from angle coordinates on the symplectic reference Lagrangian bundle via the local trivialisations Υ_α, it follows that the cohomology class that $\{\bar{\tau}_{\beta\alpha}\}$ defines equals the cohomology class w_0, since for all indices α, β

$$\tau_{\beta\alpha} = \bar{\tau}_{\beta\alpha}$$

(cf. proof of Lemma 6.7).

With the above constructions in place, it is possible to prove the main theorem of this chapter.

Theorem 6.9. *Let $(B, \mathcal{A})$ denote an integral affine manifold with linear holonomy $\mathfrak{l}$. An almost Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ over $(B, \mathcal{A})$ is a Lagrangian bundle if and only if*

$$\mathrm{d}^{(2)}(w_0) = 0, \tag{6.39}$$

where $\mathrm{d}^{(2)} : \mathrm{E}_2^{1,1} \to \mathrm{E}_2^{3,0}$ is the differential on the E^2-page of the Leray-Serre spectral sequence for $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ with real coefficients.

Proof. Čech-de Rham cohomology and the corresponding interpretation of the Leray-Serre spectral sequence with real coefficients are used throughout this proof (cf. [11]). Firstly, suppose that the bundle is, in fact, Lagrangian. Let ω denote a symplectic form on M making $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ Lagrangian. Theorem 4.2 and Remark 4.19 imply that there exists a good open cover $\mathcal{U} = \{U_\alpha\}$ of B with local action-angle coordinates $(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha)$ on each $\pi^{-1}(U_\alpha)$ and transition functions

$$\varphi_{\alpha_2\alpha_1}(\mathbf{a}_{\alpha_1}, \boldsymbol{\theta}_{\alpha_1}) = (A_{\alpha_2\alpha_1}\mathbf{a}_{\alpha_1} + \mathbf{d}_{\alpha_2\alpha_1}, A^{-T}_{\alpha_2\alpha_1}\boldsymbol{\theta}_{\alpha_1} + \mathbf{g}_{\alpha_2\alpha_1}(\mathbf{a}_{\alpha_1})) \tag{6.40}$$

(cf. Equation (4.12)), with the functions $\mathbf{g}_{\alpha_2\alpha_1}$ constrained by

$$\varphi^*_{\alpha_2\alpha_1}\omega_{\alpha_2} = \omega_{\alpha_1}, \tag{6.41}$$

where ω_{α_i} denotes the restriction of ω to $\pi^{-1}(U_{\alpha_i})$ for $i = 1, 2$. On each intersection, equation (6.41) is equivalent to

$$\sum_{i,j=1}^{n} A^{ij}_{\alpha_2\alpha_1}\mathrm{d}a^j_{\alpha_1} \wedge \mathrm{d}g^i_{\alpha_2\alpha_1} = 0; \tag{6.42}$$

since each $U_{\alpha_1} \cap U_{\alpha_2}$ is simply-connected (in fact, contractible), the forms $\mathrm{d}a^i_{\alpha_1}$ are exact and so the above equation implies that

$$\mathrm{d}\left(\sum_{i,j=1}^{n} A^{ij}_{\alpha_2\alpha_1} a^j_{\alpha_1} \mathrm{d}g^i_{\alpha_2\alpha_1}\right) = 0. \tag{6.43}$$

As in the proof of Lemma 6.7, let

$$\sigma_\alpha = \sum_{i=1}^{n} a^i_\alpha \mathrm{d}\theta^i_\alpha$$

be a locally defined potential for ω. Since equation (6.41) holds,

$$\kappa_{\alpha_2\alpha_1} = \varphi^*_{\alpha_2\alpha_1}\sigma_{\alpha_2} - \sigma_{\alpha_1}$$

is a closed 1-form. The Čech cocycle $\{\kappa_{\alpha_2\alpha_1}\}$ represents the cohomology class of ω in $\mathrm{H}^2(M;\mathbb{R})$. Take

$$\underbrace{\sum_{i,j=1}^{n} A^{ij}_{\alpha_2\alpha_1} a^j_{\alpha_1} \mathrm{d}g^i_{\alpha_2\alpha_1}}_{:=\zeta_{\alpha_2\alpha_1}} + \underbrace{\sum_{i=1}^{n} d^i_{\alpha_2\alpha_1} \mathrm{d}\theta^i_{\alpha_2}}_{=\tau_{\alpha_2\alpha_1}}$$

as a representative of $\kappa_{\alpha_2\alpha_1}$. Note that $\{\tau_{\alpha_2\alpha_1}\}$ is a representative of the cohomology class $w_0 \in \mathrm{E}_2^{1,1}$; if δ denotes the Čech-boundary map, then

$$(\delta\tau)_{\alpha_1\alpha_2\alpha_3} = \mathrm{d}\left(\sum_{i=1}^{n} d^i_{\alpha_2\alpha_3} g^i_{\alpha_2\alpha_1}\right) := \mathrm{d}\eta_{\alpha_1\alpha_2\alpha_3}.$$

The local potentials $\eta_{\alpha_1\alpha_2\alpha_3}$ are defined up to a choice of constants. The functions $\{-(\delta\eta)_{\alpha_1\alpha_2\alpha_3\alpha_4}\}$ are a Čech-de Rham cocycle whose corresponding cohomology class in $\mathrm{E}_2^{3,0}$ is $\mathrm{d}^{(2)}(w_0)$ since the cover $\mathcal{U}$ is good (cf. [11]).

The Čech boundary of $\{\zeta_{\alpha_2\alpha_1}\}$ is equal to

$$(\delta\zeta)_{\alpha_1\alpha_2\alpha_3} = \mathrm{d}\left(\sum_{i=1}^{n} (d^i_{\alpha_1\alpha_3} - d^i_{\alpha_1\alpha_2}) g^i_{\alpha_1\alpha_2}\right) := \mathrm{d}\xi_{\alpha_1\alpha_2\alpha_3}. \tag{6.44}$$

It can be checked that for all indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

$$(\delta\xi)_{\alpha_1\alpha_2\alpha_3\alpha_4} = -(\delta\eta)_{\alpha_1\alpha_2\alpha_3\alpha_4}; \tag{6.45}$$

the proof of this equality is postponed to Appendix A. In particular, $\delta\xi$ is a Čech-de Rham cocycle representing the cohomology class $\mathrm{d}^{(2)}(w_0)$. In order to prove that this class vanishes, it suffices to show that $\xi_{\alpha_1\alpha_2\alpha_3}$ can be chosen so that

$$(\delta\xi)_{\alpha_1\alpha_2\alpha_3\alpha_4} = 0$$

for all indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

Since equation (6.42) holds and the cover $\mathcal{U}$ is good, the 1-forms $\zeta_{\alpha_2\alpha_1}$ are closed and, hence, exact. Set

$$\zeta_{\alpha_2\alpha_1} = \mathrm{d}\epsilon_{\alpha_2\alpha_1}$$

for each pair of indices α_1, α_2. Then

$$(\delta\zeta)_{\alpha_1\alpha_2\alpha_3} = \mathrm{d}(\epsilon_{\alpha_2\alpha_3} - \epsilon_{\alpha_1\alpha_3} + \epsilon_{\alpha_3\alpha_2}) = \mathrm{d}(\delta\epsilon)_{\alpha_1\alpha_2\alpha_3}. \tag{6.46}$$

Equations (6.44) and (6.46) imply that

$$\xi_{\alpha_1\alpha_2\alpha_3} = (\delta\epsilon)_{\alpha_1\alpha_2\alpha_3} + C_{\alpha_1\alpha_2\alpha_3}$$

for some constants $C_{\alpha_1\alpha_2\alpha_3}$. By substituting $\xi_{\alpha_1\alpha_2\alpha_3}$ with $\xi_{\alpha_1\alpha_2\alpha_3} - C_{\alpha_1\alpha_2\alpha_3}$, it may be assumed that

$$\xi_{\alpha_1\alpha_2\alpha_1} = (\delta\epsilon)_{\alpha_1\alpha_2\alpha_3},$$

which, in turn, implies that for all indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

$$(\delta\xi)_{\alpha_1\alpha_2\alpha_3\alpha_4} = (\delta^2\epsilon)_{\alpha_1\alpha_2\alpha_3\alpha_4} = 0.$$

This proves that $\mathrm{d}^{(2)}(w_0) = 0$.

Conversely, suppose that $\mathrm{d}^{(2)}(w_0) = 0$ for the almost Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$. Let $\mathcal{U} = \{U_\alpha\}$ be the good cover of B given by Remark 4.34, *i.e.* there exist local action-angle coordinates $(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha)$ on $\pi^{-1}(U_\alpha)$ and the transition functions are of the form

$$\varphi_{\alpha_2\alpha_1}(\mathbf{a}_{\alpha_1}, \boldsymbol{\theta}_{\alpha_1}) = (A_{\alpha_2\alpha_1}\mathbf{a}_{\alpha_1} + \mathbf{d}_{\alpha_2\alpha_1}, A^{-T}_{\alpha_2\alpha_1}\boldsymbol{\theta}_{\alpha_1} + \mathbf{g}_{\alpha_2\alpha_1}(\mathbf{a}_{\alpha_1})),$$

without the constraint on the functions $\mathbf{g}_{\alpha_2\alpha_1}$ given by equation (6.41). Recall that the form

$$\omega_\alpha = \sum_{i=1}^n \mathrm{d}a^i_\alpha \wedge \mathrm{d}\theta^i_\alpha$$

makes the bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow \pi^{-1}(U_\alpha) \to U_\alpha$ Lagrangian. The obstruction to patching these forms together to yield a globally defined symplectic form ω on M which makes the bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ Lagrangian is given by the Čech cocycle

$$\varphi^*_{\alpha_2\alpha_1}\omega_{\alpha_2} - \omega_{\alpha_1} = \sum_{i,j=1}^n A^{ij}_{\alpha_2\alpha_1}\mathrm{d}a^j_{\alpha_1} \wedge \mathrm{d}g^i_{\alpha_2\alpha_1}. \tag{6.47}$$

Since the cover $\mathcal{U}$ is good, this cocycle represents a cohomology class in $\mathrm{H}^1(B; \mathcal{Z}^2(\mathrm{T}^*B))$, where $\mathcal{Z}^2(\mathrm{T}^*B)$ denotes the sheaf of closed sections of the bundle $\mathrm{T}^*B \wedge \mathrm{T}^*B \to B$. In light of the isomorphism

$$\mathrm{H}^1(B; \mathcal{Z}^2(\mathrm{T}^*B)) \cong \mathrm{H}^3(B; \mathbb{R})$$

(cf. [11, 18]), the above cocycle defines a cohomology class

$$\upsilon \in \mathrm{H}^3(B; \mathbb{R}).$$

Using the notation of the first half of the proof, a cocycle representing υ in Čech cohomology is given by $-\delta\xi$ (this simply unravels the above isomorphism). The equality of equation (6.45) still holds, since it is not necessary to have that the transition functions $\varphi_{\alpha_2\alpha_1}$ are symplectomorphisms in order to prove it (cf. Appendix A). Thus

$$-\delta\xi = \delta\eta.$$

Note that the Čech-de Rham cocycle $-\delta\eta$ is a representative of $\mathrm{d}^{(2)}(w_0)$; by assumption, this vanishes and

$$\upsilon = 0.$$

Therefore

$$\sum_{i,j=1}^n A^{ij}_{\alpha_2\alpha_1}\mathrm{d}a^j_{\alpha_1} \wedge \mathrm{d}g^i_{\alpha_2\alpha_1}$$

is a coboundary; thus there exists closed 2-forms k_α defined on each U_α such that

$$(\delta k)_{\alpha_2\alpha_1} = \sum_{i,j=1}^n A^{ij}_{\alpha_2\alpha_1}\mathrm{d}a^j_{\alpha_1} \wedge \mathrm{d}g^i_{\alpha_2\alpha_1} \tag{6.48}$$

for all indices α_1, α_2.

The forms

$$\omega_\alpha - \pi^* k_\alpha$$

defined on each $\pi^{-1}(U_\alpha)$ patch together by virtue of equations (6.47) and (6.48). Denote the resulting 2-form on M by ω. It is closed and non-degenerate since each $\omega_\alpha - \pi^* k_\alpha$ is; moreover, the fibres of the bundle are Lagrangian submanifolds of (M, ω), since they are Lagrangian submanifolds of the relevant symplectic manifold $(\pi^{-1}(U_\alpha), \omega_\alpha - \pi^* k_\alpha)$ and, hence, the result follows. □

Remark 6.10.

i) Let $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ be an almost Lagrangian bundle over the integral affine manifold $(B, \mathcal{A})$ with Chern class c, and let w_0 be the cohomology class of Lemma 6.7. If

$$\mathrm{d}^{(2)} : \mathrm{E}_2^{1,1} \to \mathrm{E}_2^{3,0}$$

denotes the differential on the E^2-page of the Leray-Serre spectral sequence of $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ with real coefficients, Theorem 6.9 proves that

$$\mathrm{d}^{(2)}(w_0) = -\mathcal{D}_{(B,\mathcal{A})}(c),$$

where $\mathcal{D}_{(B,\mathcal{A})}$ denotes the homomorphism of Dazord and Delzant (cf. Theorem 4.4). This is because the cohomology class $\mathcal{D}_{(B,\mathcal{A})}(c)$ is given by the Čech-de Rham cocycle

$$\{\sum_{i,j=1}^{n} A^{ij}_{\beta\alpha} \mathrm{d}a^j_\alpha \wedge \mathrm{d}g^i_{\beta\alpha}\},$$

whose cohomology class can also be represented by the Čech cocycle $-\delta\xi$ defined above. The relation of equation (6.45) proves the claim;

ii) The reason why Čech-de Rham cohomology is used throughout the above proofs is to highlight the importance of the role of local action-angle coordinates in the construction of Lagrangian bundles.

Chapter 7

Relation to integral affine geometry

Chapter 6 proves that the obstruction for an almost Lagrangian bundle over an integral affine manifold $(B, \mathcal{A})$ to be Lagrangian is given (up to isomorphism) by taking the cup product of the Chern class of the bundle with the cohomology class w_0 of the symplectic form on the symplectic reference Lagrangian bundle. This chapter proves that the cohomology class w_0 is an integral affine invariant of the base space $(B, \mathcal{A})$, namely its radiance obstruction $r_{(B,\mathcal{A})}$. Section 7.1 constructs the universal radiance obstruction r_U (cf. Definition 7.1), which arises from the topology of the group $\mathrm{Aff}(\mathbb{R}^n)$; the radiance obstruction $r_{(B,\mathcal{A})}$ is the pull-back of r_U along the classifying map of the affine tangent bundle

$$\mathrm{T}^{\mathrm{Aff}} B \to B$$

associated to the integral affine manifold $(B, \mathcal{A})$. The relation between $r_{(B,\mathcal{A})}$ and w_0 is studied in Section 7.2, which proves that, up to isomorphism, these two cohomology classes coincide (cf. Theorem 7.5). In light of Theorem 6.9, this fact implies that the homomorphism $\mathcal{D}_{(B,\mathcal{A})}$ is determined by the integral affine structure $\mathcal{A}$; moreover, Theorem 7.5 allows to study integral affine geometry using techniques from Lagrangian bundles and *vice versa*. This interaction is exploited in two examples, namely in Theorem 7.6 and in Section 7.3; in the latter, some examples connected to the study of singular Lagrangian bundles are studied.

7.1 The radiance obstruction of an affine manifold

In this section the radiance obstruction of an affine manifold is introduced, following the work of Goldman and Hirsch in [31]. This obstruction was first introduced by Smillie in [58], and further investigated in [27, 31, 32]. It is an important cohomological invariant of an affine manifold, as it encapsulates many essential properties of the affine structure.

7.1.1 Universal radiance obstruction

In this subsection, the radiance obstruction is defined in group theoretic terms, starting from the structure of the group

$$\mathrm{Aff}(\mathbb{R}^n) = \mathrm{GL}(n; \mathbb{R}) \ltimes \mathbb{R}^n.$$

There is an exact sequence

$$0 \longrightarrow \mathbb{R}^n \xrightarrow{\iota} \mathrm{Aff}(\mathbb{R}^n) \xrightarrow{\mathrm{Lin}} \mathrm{GL}(n;\mathbb{R}) \longrightarrow 1, \tag{7.1}$$

where the action of $\mathrm{GL}(n;\mathbb{R})$ on $\mathbb{R}^n$ is given by

$$\begin{aligned} \mathrm{GL}(n;\mathbb{R}) \times \mathbb{R}^n &\to \mathbb{R}^n \\ (A, \mathbf{b}) &\mapsto A\mathbf{b}, \end{aligned}$$

and the map $\mathrm{Lin} : \mathrm{Aff}(\mathbb{R}^n) \to \mathrm{GL}(n;\mathbb{R})$ is simply projection onto the first component. Define the map

$$\begin{aligned} \mathrm{Trans} : &\mathrm{Aff}(\mathbb{R}^n) \to \mathbb{R}^n \\ &(A, \mathbf{b}) \mapsto \mathbf{b}, \end{aligned} \tag{7.2}$$

which satisfies

$$\mathrm{Trans}((A, \mathbf{b}) \cdot (A', \mathbf{b}')) = \mathrm{Trans}(A, \mathbf{b}) + \mathrm{Lin}(A, \mathbf{b})\mathrm{Trans}(A', \mathbf{b}'),$$

for all $(A, \mathbf{b}), (A', \mathbf{b}') \in \mathrm{Aff}(\mathbb{R}^n)$. Thus Trans is a *crossed homomorphism* (cf. [12]) and it defines an element

$$r_U \in \mathrm{H}^1(\mathrm{Aff}(\mathbb{R}^n); \mathbb{R}^n_{\mathrm{Lin}}),$$

where $\mathbb{R}^n_{\mathrm{Lin}}$ denotes $\mathbb{R}^n$ as an $\mathrm{Aff}(\mathbb{R}^n)$-module via the homomorphism Lin.

Definition 7.1 (Universal radiance obstruction [31, 32])**.** The cohomology class r_U is called the *universal radiance obstruction.*

Remark 7.2 (A characteristic class)**.** A cohomology class in $\mathrm{H}^1(\mathrm{Aff}(\mathbb{R}^n); \mathbb{R}^n_{\mathrm{Lin}})$ corresponds to a choice of (conjugacy class of) splitting of the split extension of $\mathrm{Aff}(\mathbb{R}^n)$ by $\mathbb{R}^n$, where the action of the former on the latter is given by the homomorphism Lin. The splitting corresponding to r_U can be constructed as follows. Consider the pull-back

$$\begin{tikzcd} \mathrm{Lin}^*(\mathrm{Aff}(\mathbb{R}^n)) \arrow[r, "L"] \arrow[d] & \mathrm{Aff}(\mathbb{R}^n) \arrow[d, "\mathrm{Lin}"] \\ \mathrm{Aff}(\mathbb{R}^n) \arrow[r, "\mathrm{Lin}"'] & \mathrm{GL}(n;\mathbb{R}), \end{tikzcd}$$

where

$$\mathrm{Lin}^*(\mathrm{Aff}(\mathbb{R}^n)) := \{((A, \mathbf{b}), (A', \mathbf{b}')) \in \mathrm{Aff}(\mathbb{R}^n) \times \mathrm{Aff}(\mathbb{R}^n) : \mathrm{Lin}(A, \mathbf{b}) = \mathrm{Lin}(A', \mathbf{b}')\}.$$

Note that $\mathrm{Lin}^*(\mathrm{Aff}(\mathbb{R}^n))$ is a subgroup of $\mathrm{Aff}(\mathbb{R}^n) \times \mathrm{Aff}(\mathbb{R}^n)$ and that the natural homomorphism

$$\mathrm{Lin}^*(\mathrm{Aff}(\mathbb{R}^n)) \to \mathrm{Aff}(\mathbb{R}^n)$$

given by the projection onto the first component has kernel isomorphic to $\mathbb{R}^n$. The graph of the identity map

$$\mathrm{id} : \mathrm{Aff}(\mathbb{R}^n) \to \mathrm{Aff}(\mathbb{R}^n)$$

is a splitting of the exact sequence

$$0 \to \mathbb{R}^n \to \mathrm{Lin}^*(\mathrm{Aff}(\mathbb{R}^n)) \to \mathrm{Aff}(\mathbb{R}^n) \to 1, \tag{7.3}$$

inducing the crossed homomorphism Trans : $\mathrm{Aff}(\mathbb{R}^n) \to \mathbb{R}^n$ defined in equation (7.2). The cohomology class defined by Trans classifies the (conjugacy class of the) above splitting. On the other hand, the cohomology class of the crossed homomorphism Trans is the universal radiance obstruction by definition. Hence, r_U classifies the (conjugacy class of the) above splitting.

Let $(B, \mathcal{A})$ be an affine manifold with affine holonomy representation

$$\mathfrak{a} : \pi_1(B) \to \mathrm{Aff}(\mathbb{R}^n)$$

(cf. Definition 4.20). Let $\mathfrak{l} = \mathrm{Lin} \circ \mathfrak{a}$ denote its linear holonomy and let $\Gamma = \pi_1(B)$ throughout.

Definition 7.3 (Group theoretic definition of radiance obstruction [31])**.** The cohomology class

$$r_{(B,\mathcal{A})} = \mathfrak{a}^* r_U \in \mathrm{H}^1(\Gamma; \mathbb{R}^n_{\mathfrak{l}})$$

is called the *radiance obstruction* of the affine manifold $(B, \mathcal{A})$.

Remark 7.4 (Well-definedness [31])**.** The class $r_{(B,\mathcal{A})}$ depends on the choice of affine holonomy representation $\mathfrak{a}$ (cf. Remark 4.22), but its vanishing does not. Henceforth, whenever the radiance obstruction of an affine manifold is mentioned, it is understood that a choice of affine holonomy representation has been fixed.

Definition 7.5 (Radiant manifolds [27, 31])**.** An affine manifold $(B, \mathcal{A})$ whose radiance obstruction $r_{(B,\mathcal{A})}$ vanishes is called a *radiant* manifold.

Remark 7.6 (Characterisation of radiant manifolds [31])**.** An affine manifold $(B, \mathcal{A})$ is radiant if and only if its affine holonomy representation $\mathfrak{a}$ can be chosen so that its image lies entirely in $\mathrm{GL}(n; \mathbb{R}) \subset \mathrm{Aff}(\mathbb{R}^n)$. Equivalently, an affine manifold $(B, \mathcal{A})$ is radiant if and only if there exists an affine structure $\mathcal{A}'$ which is affinely diffeomorphic to the given one and whose changes of coordinates are constant linear transformations of $\mathbb{R}^n$.

7.1.2 The topology of the universal radiance obstruction

The universal radiance obstruction r_U can also be described as a topological obstruction, exploiting the isomorphism

$$\mathrm{H}^*(G; K_\rho) \cong \mathrm{H}^*(\mathrm{K}(G; 1); K_\rho),$$

where G is any topological group and K_ρ is a G-module via the representation $\rho : G \to \mathrm{Aut}(K)$.

Let $\mathfrak{a} : \Gamma \to \mathrm{Aff}(\mathbb{R}^n)$ be a representation of a discrete group Γ. The composition $\mathrm{Trans} \circ \mathfrak{a}$ defines a crossed homomorphism which represents a cohomology class

$$r_\Gamma \in \mathrm{H}^1(\Gamma; \mathbb{R}^n_{\mathrm{Lin}\circ\mathfrak{a}}).$$

Remark 7.7. If $(B, \mathcal{A})$ is an affine manifold whose fundamental group is isomorphic to Γ and whose affine holonomy is given by the above representation $\mathfrak{a}$, then

$$r_{(B,\mathcal{A})} = r_\Gamma,$$

where $r_{(B,\mathcal{A})}$ is the radiance obstruction of Definition 7.3.

Lemma 7.1. *The cohomology class r_Γ vanishes if and only if the representation $\mathfrak{a}$ is conjugate in* $\mathrm{Aff}(\mathbb{R}^n)$ *to a representation whose image lies entirely in* $\sigma(\mathrm{GL}(n;\mathbb{R}))$*, where*

$$\begin{aligned} \sigma : \mathrm{GL}(n;\mathbb{R}) &\to \mathrm{Aff}(\mathbb{R}^n) \\ A &\mapsto (A, \mathbf{0}) \end{aligned} \tag{7.4}$$

is a splitting for the exact sequence of equation (7.1) (cf. equation (3.5)).

Proof. For $\gamma \in \Gamma$, set

$$\mathfrak{a}(\gamma) = (\mathfrak{l}(\gamma), \mathrm{Trans}(\mathfrak{a}(\gamma))),$$

where $\mathfrak{l} = \mathrm{Lin} \circ \mathfrak{a}$ and the injective homomorphisms ι and σ have been used tacitly. Suppose that r_Γ vanishes. By definition (cf. [12]), there exists $\mathbf{b} \in \mathbb{R}^n$ such that

$$\mathrm{Trans}(\mathfrak{a}(\gamma)) = \mathfrak{l}(\gamma)\mathbf{b} - \mathbf{b}$$

for all $\gamma \in \Gamma$. Then

$$(I, \mathbf{b}) \cdot (\mathfrak{l}(\gamma), \mathrm{Trans}(\mathfrak{a}(\gamma))) \cdot (I, -\mathbf{b}) = (\mathfrak{l}(\gamma), \mathbf{0}) \tag{7.5}$$

for all $\gamma \in \Gamma$. Thus the representation $\mathfrak{a}$ can be conjugated by an element in $\iota(\mathbb{R}^n)$ to lie in $\sigma(\mathrm{GL}(n;\mathbb{R}))$.

Conversely, suppose there exists an element $(A, \mathbf{b}) \in \mathrm{Aff}(\mathbb{R}^n)$ such that

$$(A, \mathbf{b}) \cdot (\mathfrak{l}(\gamma), \mathrm{Trans}(\mathfrak{a}(\gamma))) \cdot (A^{-1}, -A^{-1}\mathbf{b}) \in \sigma(\mathrm{GL}(n;\mathbb{R}))$$

for all $\gamma \in \Gamma$. Then

$$\mathrm{Trans}(\mathfrak{a}(\gamma)) = \mathfrak{l}(\gamma)A^{-1}\mathbf{b} - A^{-1}\mathbf{b}$$

for all $\gamma \in \Gamma$; thus $r_\Gamma = 0$ and the result follows. □

For any topological group G, let G^δ denote the group endowed with the discrete topology. In light of equation (7.1), there is a split short exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^\delta \xrightarrow{\ \iota\ } \mathrm{Aff}(\mathbb{R}^n)^\delta \underset{\sigma}{\overset{\mathrm{Lin}}{\rightleftarrows}} \mathrm{GL}(n;\mathbb{R})^\delta \longrightarrow 1.$$

Applying Theorem 3.2 to the splitting σ, obtain a fibration

$$(\mathbb{R}^n)^\delta \hookrightarrow \mathrm{BGL}(n;\mathbb{R})^\delta \to \mathrm{BAff}(\mathbb{R}^n)^\delta,$$

where

$$\mathrm{BGL}(n;\mathbb{R})^\delta \simeq \mathrm{EAff}(\mathbb{R}^n)^\delta / \mathrm{GL}(n;\mathbb{R})^\delta.$$

Note that $\mathrm{BAff}(\mathbb{R}^n)^\delta$ and $\mathrm{BGL}(n;\mathbb{R})^\delta$ are $\mathrm{K}(\mathrm{Aff}(\mathbb{R}^n);1)$ and $\mathrm{K}(\mathrm{GL}(n;\mathbb{R});1)$ respectively. Using the ideas of the proof of Theorem 3.6 and Lemma 6.1, the following lemma can be proved.

Lemma 7.2. *There exists a bundle isomorphism*

$$\begin{tikzcd} \mathrm{EAff}(\mathbb{R}^n)^\delta/\mathrm{GL}(n;\mathbb{R})^\delta \arrow[rr, "\cong"] \arrow[dr] & & \mathrm{EAff}(\mathbb{R}^n)^\delta \times_{\mathrm{Aff}(\mathbb{R}^n)^\delta} (\mathbb{R}^n)^\delta \arrow[dl] \\ & \mathrm{BAff}(\mathbb{R}^n)^\delta. & \end{tikzcd}$$

Let Γ be a discrete topological group. Any representation

$$\mathfrak{a} : \Gamma \to \mathrm{Aff}(\mathbb{R}^n)$$

factors through $\mathrm{Aff}(\mathbb{R}^n)^\delta$, since Γ is discrete. The homomorphism $\mathfrak{a}$ induces a map (defined up to homotopy)

$$\bar{\mathfrak{a}} : \mathrm{B}\Gamma \to \mathrm{BAff}(\mathbb{R}^n)^\delta$$

whose induced map on fundamental groups coincides with $\mathfrak{a}$ (up to a choice of base-points). A lift

$$\begin{tikzcd} & \mathrm{BGL}(n;\mathbb{R})^\delta \arrow[d, "\sigma"] \\ \mathrm{B}\Gamma \arrow[r, "\bar{\mathfrak{a}}"'] \arrow[ur, dotted] & \mathrm{BAff}(\mathbb{R}^n)^\delta \end{tikzcd}$$

exists if and only if

$$\bar{\mathfrak{a}}_*(\Gamma) \subset \sigma_*(\pi_1(\mathrm{BGL}(n;\mathbb{R})^\delta)),$$

since the fibre of the projection $\sigma : \mathrm{BGL}(n;\mathbb{R})^\delta \to \mathrm{BAff}(\mathbb{R}^n)^\delta$ is discrete (cf. [65]). Note that the induced map

$$\sigma_* : \pi_1(\mathrm{BGL}(n;\mathbb{R})^\delta) \cong \mathrm{GL}(n;\mathbb{R}) \to \pi_1(\mathrm{BAff}(\mathbb{R}^n)^\delta) \cong \mathrm{Aff}(\mathbb{R}^n)$$

equals the homomorphism σ of equation (7.4) by construction. Hence, the map $\bar{\mathfrak{a}}$ admits a lift if and only if the representation

$$\bar{\mathfrak{a}}_* : \pi_1(\mathrm{B}\Gamma) \cong \Gamma \to \pi_1(\mathrm{BAff}(\mathbb{R}^n)^\delta) \cong \mathrm{Aff}(\mathbb{R}^n)$$

lies entirely within the image of σ_*. The latter statement is true if, up to conjugation, the image of the representation $\mathfrak{a}$ lies entirely in $\sigma(\mathrm{GL}(n;\mathbb{R}))$, which is true if and only if $r_\Gamma = 0$. In particular, letting $\Gamma = \mathrm{Aff}(\mathbb{R}^n)^\delta$ and $\mathfrak{a} : \mathrm{Aff}(\mathbb{R}^n)^\delta \to \mathrm{Aff}(\mathbb{R}^n)$ be the identity homomorphism, the above discussion proves the following theorem.

Theorem 7.3. *The universal radiance obstruction r_U is the obstruction to the existence of a section for the fibration*

$$(\mathbb{R}^n)^\delta \hookrightarrow \mathrm{BGL}(n;\mathbb{R})^\delta \to \mathrm{BAff}(\mathbb{R}^n)^\delta.$$

7.1.3 A geometric interpretation

It is possible to give a geometric interpretation to the radiance obstruction of an affine manifold, which, thus far, is simply a cohomological invariant of the fundamental group of the manifold.

Firstly, note that the tangent bundle of an affine manifold can be endowed with the structure of a *flat* affine bundle.

Definition 7.8 (Flat affine bundle [31])**.** Let $(F, \mathcal{B}) \hookrightarrow E \to N$ be an affine bundle, so that $(F, \mathcal{B})$ is an affine manifold and its structure group is $\mathrm{Aff}(F, \mathcal{B})$. It is said to be *flat* if it admits locally constant transition functions.

Example 7.9 (Tangent bundles of affine manifolds)**.** The tangent bundle of an affine manifold $(N, \mathcal{A})$ (thought of as a vector bundle) is naturally a flat linear bundle, since the standard transition functions are locally constant. This is because the changes of affine coordinates on the manifold are locally constant too. The inclusion

$$\mathrm{GL}(n; \mathbb{R}) \hookrightarrow \mathrm{Aff}(\mathbb{R}^n)$$

makes the tangent bundle $\mathrm{T}N \to N$ a flat affine bundle.

There is, however, a different choice of flat affine bundle structure that can be chosen on the tangent bundle of an affine manifold $(B, \mathcal{A})$, which is called *natural* in [31]. Let $(B, \mathcal{A})$ be an n-dimensional affine manifold and let

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n$$

denote a local coordinate map. Define affine trivialisations

$$\begin{aligned} t_\alpha : \mathrm{T}U_\alpha &\to U_\alpha \times \mathbb{R}^n \\ (x, \mathbf{v}) &\mapsto (x, (D\phi_\alpha(x))(\mathbf{v}) + \phi_\alpha(x)), \end{aligned} \tag{7.6}$$

where D denotes derivative. Note that the map t_α is truly a trivialisation as ϕ_α is a diffeomorphism. Denote the locally constant changes in affine coordinates by

$$\phi_\beta \circ \phi_\alpha^{-1}(\mathbf{a}_\alpha) = A_{\beta\alpha}\mathbf{a}_\alpha + \mathbf{d}_{\beta\alpha}, \tag{7.7}$$

where $\mathbf{a} \in \mathbb{R}^n$. The transition functions for the local affine trivialisations of equation (7.6) are given by

$$\begin{aligned} f_{\beta\alpha} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n &\mapsto (U_\alpha \cap U_\beta) \times \mathbb{R}^n \\ (x, \mathbf{y}) &\mapsto (x, A_{\beta\alpha}\mathbf{y} + \mathbf{d}_{\beta\alpha}). \end{aligned} \tag{7.8}$$

Since the changes of coordinates of equation (7.7) are locally constant, it follows that the trivialisations of equation (7.6) induce a flat affine bundle structure on $\mathrm{T}B$, which is henceforth denoted by $\mathrm{T}^{\mathrm{Aff}}B$.

Definition 7.10 (Affine tangent bundle [31])**.** For an affine manifold $(B, \mathcal{A})$, the affine bundle $\mathrm{T}^{\mathrm{Aff}}B \to B$ constructed above is called the *affine* tangent bundle.

Remark 7.11 (Dependence on affine structure)**.** For a given manifold B there can be inequivalent affine structures on B which induce inequivalent flat affine structures on the tangent bundle $\mathrm{T}B \to B$ (cf. Example 7.16).

Fix an n-dimensional manifold $(B, \mathcal{A})$ and let $\mathrm{T}^{\mathrm{Aff}}B \to B$ be its affine tangent bundle. Since its fibres are contractible, there exists a section to this bundle, *e.g.* the zero section. However, it is not necessarily true that the affine tangent bundle admits a *flat* section.

Definition 7.12 (Flat section of a flat affine bundle [31])**.** Let $(F, \mathcal{B}) \hookrightarrow E \to N$ be a flat affine bundle. A section $s : N \to E$ is *flat* if, for any local trivialisation

$\pi^{-1}(U_\alpha) \to U_\alpha \times F$, the composite

$$U_\alpha \xrightarrow{s_\alpha} \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F \xrightarrow{\mathrm{pr}_2} F$$

is locally constant.

Example 7.13.

i) Let $(B, \mathcal{A})$ be an affine manifold, let $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ be an affine coordinate chart and consider the affine tangent bundle $\mathrm{T}^{\mathrm{Aff}} U_\alpha = \mathrm{T}^{\mathrm{Aff}} B|_{U_\alpha}$ with trivialisation given by equation (7.6). The section

$$\begin{aligned} s_\alpha : U_\alpha &\to \mathrm{T}^{\mathrm{Aff}} U_\alpha \\ x &\mapsto (x, -(D\phi_\alpha(x))^{-1}(\phi_\alpha(x))) \end{aligned} \tag{7.9}$$

is flat;

ii) On the other hand, the zero section 0_α is not flat, since the composite

$$U_\alpha \xrightarrow{0_\alpha} \mathrm{T}^{\mathrm{Aff}} U_\alpha \longrightarrow U_\alpha \times \mathbb{R}^n \xrightarrow{\mathrm{pr}_2} \mathbb{R}^n$$

is the map

$$x \mapsto \phi_\alpha(x),$$

which is a diffeomorphism by definition.

Fix an affine manifold $(B, \mathcal{A})$ and let $\mathrm{T}^{\mathrm{Aff}} B \to B$ denote the corresponding affine tangent bundle with trivialisations given by equation (7.6) and transition functions $f_{\beta\alpha}$ as in equation (7.8). The map

$$f_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Aff}(\mathbb{R}^n)$$

is locally constant; thus it factors through

$$\bar{f}_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Aff}(\mathbb{R}^n)^\delta.$$

For each α, endow the fibres of $\mathrm{T}^{\mathrm{Aff}} U_\alpha \to U_\alpha$ with the discrete topology and denote the resulting space by $(\mathrm{T}^{\mathrm{Aff}} U_\alpha)^\delta$. The trivialisation t_α of equation (7.6) induces a trivialisation

$$\bar{t}_\alpha : (\mathrm{T}^{\mathrm{Aff}} U_\alpha)^\delta \to U_\alpha \times (\mathbb{R}^n)^\delta.$$

These trivialisations, along with the maps $\bar{f}_{\beta\alpha}$, define a bundle

$$(\mathbb{R}^n)^\delta \hookrightarrow (\mathrm{T}^{\mathrm{Aff}} B)^\delta \to B, \tag{7.10}$$

which is classified by the homotopy class of a map

$$\chi_{\mathrm{Aff}} : B \to \mathrm{BAff}(\mathbb{R}^n)^\delta.$$

Moreover, in light of Lemma 7.2, the bundle of equation (7.10) is isomorphic to the pull-back bundle

$$\chi_{\mathrm{Aff}}^* \mathrm{BGL}(n; \mathbb{R})^\delta \to B.$$

Therefore the cohomology class

$$\chi_{\mathrm{Aff}}^* r_U$$

is the obstruction to the existence of a section for the bundle of equation (7.10).

The homotopy class of the map χ_{Aff} is determined by its induced map on fundamental groups, since $\mathrm{BAff}(\mathbb{R}^n)^\delta$ is a $\mathrm{K}(\mathrm{Aff}(\mathbb{R}^n);1)$. By construction, this map equals (up to conjugation) the affine holonomy

$$\mathfrak{a} : \pi_1(B) = \Gamma \to \mathrm{Aff}(\mathbb{R}^n)$$

of the affine manifold $(B, \mathcal{A})$ (cf. [4]). Let

$$\bar{\mathfrak{a}} : \mathrm{B}\Gamma \to \mathrm{Aff}(\mathbb{R}^n)^\delta$$

be the map (defined up to homotopy) induced by the affine holonomy and let $\chi_{\tilde{B}} : B \to \mathrm{B}\Gamma$ denote the classifying map for the universal covering $\tilde{B} \to B$. Then the following diagram commutes (up to homotopy)

$$\begin{tikzcd} B \arrow[rr, "\chi_{\mathrm{Aff}}"] \arrow[dr, "\chi_{\tilde{B}}"'] & & \mathrm{BAff}(\mathbb{R}^n)^\delta \\ & \mathrm{B}\Gamma, \arrow[ur, "\bar{\mathfrak{a}}"'] & \end{tikzcd} \tag{7.11}$$

as

$$(\mathrm{T}^{\mathrm{Aff}} B)^\delta \cong \tilde{B} \times_\Gamma (\mathbb{R}^n)^\delta,$$

where the action of Γ on $(\mathbb{R}^n)^\delta$ is given by the representation $\mathfrak{a}$. The latter defines an action on $(\mathbb{R}^n)^\delta$ since the transition functions are locally constant. In particular,

$$\chi^*_{\mathrm{Aff}} r_U = \chi^*_{\tilde{B}} \circ \bar{\mathfrak{a}}^* r_U = \chi^*_{\tilde{B}} r_{(B,\mathcal{A})},$$

where the second equality follows from the definition of the radiance obstruction of $(B, \mathcal{A})$. The map $\chi^*_{\tilde{B}}$ is an isomorphism on one dimensional cohomology with any coefficient system (cf. [31]) and thus

$$r_{(B,\mathcal{A})} = (\chi^*_{\tilde{B}})^{-1} \circ \chi^*_{\mathrm{Aff}} r_U. \tag{7.12}$$

Remark 7.14. By abuse of nomenclature and notation, the class $\chi^*_{\mathrm{Aff}} r_U$ is henceforth also referred to as the radiance obstruction of $(B, \mathcal{A})$ and denoted by $r_{(B,\mathcal{A})}$.

With this identification, the radiance obstruction $r_{(B,\mathcal{A})}$ is the obstruction to the existence of a section to the bundle of equation (7.10). By construction, a section for the aforementioned bundle exists if and only if $\mathrm{T}^{\mathrm{Aff}} B \to B$ admits a flat section. Therefore the following theorem holds.

Theorem 7.4 (Goldman and Hirsch [31])**.** *Let $(B, \mathcal{A})$ be an n-dimensional affine manifold with linear holonomy $\mathfrak{l}$. The radiance obstruction*

$$r_{(B,\mathcal{A})} \in \mathrm{H}^1(B; \mathbb{R}^n_{\mathfrak{l}})$$

is the obstruction to the existence of a flat section for the flat affine bundle $\mathrm{T}^{\mathrm{Aff}} B \to B$.

Remark 7.15. There are other ways to define the radiance obstruction $r_{(B,\mathcal{A})}$ of an affine manifold $(B, \mathcal{A})$, as shown in [31, 32, 34]; these different definitions can be used to prove different properties of this characteristic class.

Example 7.16 (Flat affine structures on $\mathrm{T}(S^1 \times \mathbb{R})$)**.** The manifold $S^1 \times \mathbb{R}$ can be endowed with several affine structures, as shown in [7]. In this example, two inequivalent affine structures on this manifold are constructed to illustrate that there exist inequivalent flat affine structures on its tangent bundle. Firstly, consider the $\mathbb{Z}$-action on $\mathbb{R}^2$ given by translations in a fixed direction $\mathbf{b}_0 \neq 0$; this action is free, proper and by affine transformations on $\mathbb{R}^2$. Thus the manifold $\mathbb{R}^2/\mathbb{Z}$ is affine (cf. Example 2.14.v) and its affine holonomy

$$\mathfrak{a}_{\mathbb{R}^2/\mathbb{Z}} : \Gamma_1(\mathbb{R}^2/\mathbb{Z}) \to \mathrm{Aff}(\mathbb{R}^2)$$

is defined on a generator γ as

$$\mathfrak{a}_{\mathbb{R}^2/\mathbb{Z}}(\gamma) = (I, \mathbf{b}_0), \tag{7.13}$$

and extended by linearity. The crossed homomorphism $\mathrm{Trans} \circ \mathfrak{a}_{\mathbb{R}^2/\mathbb{Z}}$ defines a non-zero cohomology class

$$r_{\mathbb{R}^2/\mathbb{Z}} \in \mathrm{H}^1(\mathbb{R}^2/\mathbb{Z}; \mathbb{R}^2),$$

where the local coefficient system is trivial since the linear holonomy $\mathfrak{l}_{\mathbb{R}^2/\mathbb{Z}} = \mathrm{Lin} \circ \mathfrak{a}_{\mathbb{R}^2/\mathbb{Z}}$ is trivial. The non-vanishing of $r_{\mathbb{R}^2/\mathbb{Z}}$ follows from the fact that for any $(A, \mathbf{d}) \in \mathrm{Aff}(\mathbb{R}^2)$,

$$(A, \mathbf{d}) \cdot \mathfrak{a}_{\mathbb{R}^2/\mathbb{Z}} \cdot (A^{-1}, -A^{-1}\mathbf{d})$$

has a non-trivial translational component since $\mathbf{b}_0 \neq \mathbf{0}$; thus, Lemma 7.1 implies that $r_{\mathbb{R}^2/\mathbb{Z}} \neq 0$. In light of Theorem 7.4, the flat affine bundle $\mathrm{T}^{\mathrm{Aff}}(\mathbb{R}^2/\mathbb{Z}) \to \mathbb{R}^2/\mathbb{Z}$ does not admit a flat section. On the other hand, the inclusion

$$\mathbb{R}^2 \setminus \{\mathbf{0}\} \hookrightarrow \mathbb{R}^2$$

induces an affine structure on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ which has trivial affine holonomy. Therefore the affine tangent bundle

$$\mathrm{T}^{\mathrm{Aff}}(\mathbb{R}^2 \setminus \{\mathbf{0}\}) \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$$

admits a flat section. These two bundles are isomorphic as vector bundles, but are not as flat affine bundles.

7.2 Relation to Lagrangian bundles

In this section, the radiance obstruction $r_{(B,\mathcal{A})}$ is related to the problem of constructing Lagrangian bundles over B inducing the integral affine structure $\mathcal{A}$. In particular, the radiance obstruction $r_{(B,\mathcal{A})}$ is identified with the cohomology class w_0 of Lemma 6.7. Throughout the following, fix an integral affine manifold $(B, \mathcal{A})$ whose linear holonomy is denoted by $\mathfrak{l}$.

Remark 7.17. The importance of the radiance obstruction in the study of Lagrangian bundles has also been observed by Gross and Siebert in [34], where mirror symmetry is studied from the point of view of Lagrangian bundles.

Recall that w_0 is the cohomology class of the symplectic form ω_0 of the symplectic reference Lagrangian bundle associated to $(B, \mathcal{A})$

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0) \to B$$

into a Lagrangian bundle for which the zero section is Lagrangian. Let $\mathcal{C}^\infty(\mathrm{T}^{\mathrm{Aff}}B)$ and $\mathcal{C}^\infty(\mathrm{T}^*\mathbb{R}^n/\mathbb{Z}^n)$ denote the sheaves of sections of the affine tangent bundle $\mathrm{T}^{\mathrm{Aff}}B \to B$ and of 1-forms on the fibres of the symplectic reference Lagrangian bundle respectively. The symplectic form ω_0 defines an isomorphism of sheaves

$$\mathcal{C}^\infty(\mathrm{T}^*\mathbb{R}^n/\mathbb{Z}^n) \cong \mathcal{C}^\infty(\mathrm{T}B). \tag{7.14}$$

In local action-angle coordinates $(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha)$, the above isomorphism is given by

$$\sum_{i=1}^n h^i_\alpha \mathrm{d}\theta^i_\alpha \mapsto \sum_{i=1}^n h^i_\alpha \frac{\partial}{\partial a^i_\alpha}, \tag{7.15}$$

where $h^1_\alpha, \ldots, h^n_\alpha$ are smooth functions. Equation (7.15) follows from the fact that ω_0 is given by

$$\sum_{i=1}^n \mathrm{d}a^i_\alpha \wedge \mathrm{d}\theta^i_\alpha$$

in local action-angle coordinates $(\mathbf{a}_\alpha, \theta_\alpha)$. By equation (7.15), the isomorphism of equation (7.14) restricted to the subsheaf $\mathcal{C}^\infty_{\mathrm{flat}}(\mathrm{T}^*\mathbb{R}^n/\mathbb{Z}^n)$ of *locally constant* sections descends to an isomorphism of sheaves

$$\mathcal{C}^\infty_{\mathrm{flat}}(\mathrm{T}^*\mathbb{R}^n/\mathbb{Z}^n) \cong \mathcal{C}^\infty_{\mathrm{flat}}(\mathrm{T}^{\mathrm{Aff}}B).$$

Since $\mathrm{T}^*\mathbb{R}^n/\mathbb{Z}^n$ admits a frame of closed forms, a locally constant section of $M^* \to B$ is given in local action-angle coordinates by

$$\sum_{i=1}^n r^i_\alpha \mathrm{d}\theta^i_\alpha,$$

where $r^1_\alpha, \ldots, r^n_\alpha \in \mathbb{R}$ are constant. Such sections are in 1-1 correspondence with cohomology classes in $\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})$, since the forms $\mathrm{d}\theta^1_\alpha, \ldots, \mathrm{d}\theta^n_\alpha$ induce a basis of $\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})$. Hence, the symplectic form ω_0 induces an isomorphism of sheaves

$$\mathcal{P}^* \cong \mathcal{C}^\infty_{\mathrm{flat}}(\mathrm{T}^{\mathrm{Aff}}B), \tag{7.16}$$

where $\mathcal{P}^*$ denotes the sheaves of sections of the local coefficient system

$$\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R}) \hookrightarrow P^* \to B$$

associated to the symplectic reference Lagrangian bundle. This isomorphism induces an isomorphism of cohomology groups

$$\Phi : \mathrm{H}^*(B;\mathcal{P}^*) \to \mathrm{H}^*(B;\mathcal{C}^\infty_{\mathrm{flat}}(\mathrm{T}^{\mathrm{Aff}}B)). \tag{7.17}$$

Since both $\mathcal{P}^*$ and $\mathcal{C}^\infty_{\mathrm{flat}}(\mathrm{T}^{\mathrm{Aff}}B)$ are locally constant sheaves, the above isomorphism induces an isomorphism

$$\Phi : \mathrm{H}^*(B;\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})_{\mathfrak{l}}) \to \mathrm{H}^1(B;\mathbb{R}^n_{\mathfrak{l}}).$$

Theorem 7.5. *The cohomology class $w_0 \in \mathrm{H}^1(B;\mathrm{H}^1(\mathbb{R}^n/\mathbb{Z}^n;\mathbb{R})_{\mathfrak{l}})$ defined by the symplectic form ω_0 as in Lemma 6.7, maps to the radiance obstruction $r_{(B,\mathcal{A})} \in \mathrm{H}^1(B;\mathbb{R}^n_{\mathfrak{l}})$ via the isomorphism Φ induced by the symplectic form ω_0 of equation (7.17).*

Proof. Throughout the proof Čech cohomology is used. First, by Theorem 7.4, the radiance obstruction $r_{(B,\mathcal{A})}$ is the obstruction to the existence of a flat section to $\mathrm{T}^{\mathrm{Aff}}B \to B$. Let $\mathcal{U} = \{U_\alpha\}$ be a good open cover by integral affine coordinate neighbourhoods of $(B,\mathcal{A})$ and let

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n$$

denote the coordinate map. The section

$$\begin{aligned} s_\alpha : U_\alpha &\to \mathrm{T}^{\mathrm{Aff}}U_\alpha \\ x &\mapsto (x, -(D\phi_\alpha(x))^{-1}(\phi_\alpha(x))) \end{aligned} \tag{7.18}$$

is flat (cf. Example 7.13.i). The collection

$$\hat{\tau} = \{\hat{\tau}_{\beta\alpha}\} := \{s_\beta - s_\alpha\}$$

is a Čech cocycle; moreover, each $\hat{\tau}_{\beta\alpha}$ yields a flat section of

$$\mathrm{T}^{\mathrm{Aff}}(U_\alpha \cap U_\beta) \to U_\alpha \cap U_\beta.$$

Thus $\hat{\tau}$ defines a cohomology class in $\mathrm{H}^1(B;\mathbb{R}^n_{\mathfrak{l}})$ which equals $r_{(B,\mathcal{A})}$ (cf. [31]).

Let $\mathbf{a}_\alpha$ denote affine coordinates on U_α induced by ϕ_α and, as usual, set

$$\phi_\beta \circ \phi_\alpha^{-1}(\mathbf{a}_\alpha) = A_{\beta\alpha}\mathbf{a}_\alpha + \mathbf{d}_{\beta\alpha} \in \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n).$$

Using the affine trivialisations

$$t_\alpha : \mathrm{T}^{\mathrm{Aff}}U_\alpha \to U_\alpha \times \mathbb{R}^n$$

of equation (7.6), the difference $s_\beta - s_\alpha$ is given by

$$\hat{\tau}_{\beta\alpha} = \sum_{i=1}^n d^i_{\beta\alpha}\frac{\partial}{\partial a^i_\beta}. \tag{7.19}$$

In light of equation (7.15), the preimage of $\hat{\tau}_{\beta\alpha}$ under the isomorphism of equation (7.16) is given by

$$\sum_{i=1}^n d^i_{\beta\alpha}\mathrm{d}\theta^i_\beta. \tag{7.20}$$

The cocycle of equation (7.20) corresponds to the cocycle τ defining the cohomology class w_0 in the proof of Lemma 6.7 and, thus, the result follows. □

The radiance obstruction of an integral affine manifold $(B,\mathcal{A})$ corresponds to the cohomology class of the symplectic form ω_0 on the symplectic reference Lagrangian bundle over $(B,\mathcal{A})$. Theorem 7.5 allows to use tools from affine geometry to study problems in the symplectic geometry of Lagrangian bundles and *vice versa*. For instance, the following theorem holds.

Theorem 7.6. *There exist no closed radiant integral affine manifolds.*

Proof. Suppose the contrary. Let $(B,\mathcal{A})$ be a closed radiant integral affine manifold and let $P_{(B,\mathcal{A})}$ denote the period lattice bundle associated to it (cf. Definition 4.18).

Consider the symplectic reference Lagrangian bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0) \to B; \tag{7.21}$$

since B and $\mathbb{R}^n/\mathbb{Z}^n$ are closed, so is $\mathrm{T}^*B/P_{(B,\mathcal{A})}$. Therefore the cohomology class

$$[\omega_0] \in \mathrm{H}^2(\mathrm{T}^*B/P_{(B,\mathcal{A})}; \mathbb{R})$$

is non-zero. Lemma 6.7 proves that $[\omega_0]$ vanishes if and only if w_0 vanishes. However, Theorem 7.5 implies that

$$w_0 = \Phi^{-1}(r_{(B,\mathcal{A})}) = 0,$$

where the second equality follows by assumption. Therefore $[\omega_0] = 0$, but this is a contradiction. □

Remark 7.18. It is important to notice the difference between the bundle $P^* \to B$ and the period lattice bundle $P \to B$ associated to an integral affine manifold $(B, \mathcal{A})$. The former is an *affine* invariant of B (via the symplectic form ω_0), while the latter is only a *linear* invariant, since it is the pull-back of a universal lattice defined over $\mathrm{BGL}(n; \mathbb{Z})$. The period lattice bundle can be endowed with the structure of an *affine lattice* of $(B, \mathcal{A})$ if and only if the radiance obstruction $r_{(B,\mathcal{A})}$ is an *integral* form, which, in turn, is true if and only if the coordinate changes of the atlas $\mathcal{A}$ can be chosen to lie in the group of affine transformations of $\mathbb{Z}^n$

$$\mathrm{Aff}(\mathbb{Z}^n) := \mathrm{GL}(n; \mathbb{Z}) \ltimes \mathbb{Z}^n.$$

Such manifolds are henceforth called *strongly integral affine manifolds*, although it must be noticed that this terminology is not standard (cf. [34]). In view of Theorem 7.5, the symplectic form ω_0 on the symplectic reference Lagrangian bundle of a strongly integral affine manifold $(B, \mathcal{A})$ is itself integral.

The following corollary is obtained by combining the above theorem with Theorem 6.9.

Corollary 7.7. *Let $(B, \mathcal{A})$ denote an integral affine manifold with linear holonomy $\mathfrak{l}$. An almost Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ over $(B, \mathcal{A})$ is Lagrangian if and only if its Chern class $c \in \mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}})$ satisfies*

$$\theta(\Phi^{-1}(r_{(B,\mathcal{A})})) \cdot \psi(c^{\mathbb{R}}) = 0, \tag{7.22}$$

where the notation is the same as in Theorem 6.9, and $r_{(B,\mathcal{A})}$ is the radiance obstruction of the integral affine manifold $(B, \mathcal{A})$.

This corollary proves that the homomorphism $\mathcal{D}_{(B,\mathcal{A})}$ of Dazord and Delzant [18] is completely determined by the integral affine structure on the base of an almost Lagrangian bundle and by the universal Chern class c_U.

Remark 7.19. If $(B, \mathcal{A})$ is a strongly integral affine manifold with linear holonomy $\mathfrak{l}$, Corollary 7.7 can be strengthened to say that an almost Lagrangian bundle over $(B, \mathcal{A})$ is Lagrangian if and only if its Chern class $c \in \mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}})$ satisfies

$$\theta(\Phi^{-1}(r_{(B,\mathcal{A})})) \cdot \psi(c) = 0. \tag{7.23}$$

In particular, if $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ is a Lagrangian bundle over a strongly integral affine manifold $(B, \mathcal{A})$ (*i.e.* it induces the affine structure $\mathcal{A}$ on B), then ω can always be chosen to be integral. This should be compared with Remark 1.2 of [34]. It is not true that for a fixed integral affine manifold $(B, \mathcal{A})$ there is a strongly integral affine manifold $(B', \mathcal{A}')$ in the same integral affine diffeomorphism class. This can be seen by considering an integral affine two-torus with trivial linear holonomy and translational components which are not integral (cf. [48]).

The following corollary is a special case of Corollary 7.7.

Corollary 7.8. *If $(B, \mathcal{A})$ is a radiant affine manifold, then $\mathcal{D}_{(B,\mathcal{A})} = 0$.*

Remark 7.20. Corollary 7.8 should be compared with what is known in the literature regarding exactness of the symplectic form on the total space of a Lagrangian (or isotropic) bundle, *e.g.* [20].

7.3 Some examples

In this section a manifold is endowed with various radiant integral affine structures to illustrate how the classification of Lagrangian bundles depends on the integral affine geometry of the base space.

Let $B = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be endowed with three integral affine structures $\mathcal{A}_1,\ \mathcal{A}_2,\ \mathcal{A}_3$ defined as follows. The manifold B inherits an integral affine structure from $\mathbb{R}^2$ via the natural inclusion

$$B \hookrightarrow \mathbb{R}^2.$$

Denote this integral affine structure by $\mathcal{A}_0$. Its universal cover $\tilde{B}$ can also be endowed with an integral affine structure $\tilde{\mathcal{A}}_0$; an explicit description of the affine structure on $\tilde{B}$ can be found in [7]. It is important to notice that this affine structure on $\tilde{B}$ is not affinely isomorphic to the standard affine structure on $\mathbb{R}^2$. For any matrix $A \in \mathrm{GL}(2;\mathbb{Z})$, it is possible to define a $\mathbb{Z}$-action on $(\tilde{B}, \tilde{\mathcal{A}}_0)$ which induces an integral affine structure on B whose affine holonomy is given by the representation defined on the generator γ of $\pi_1(B)$ by

$$\gamma \mapsto (A, \mathbf{0}).$$

For $A_1,\ A_2,\ A_3 \in \mathrm{SL}(2;\mathbb{Z})$, let $\mathcal{A}_1,\ \mathcal{A}_2,\ \mathcal{A}_3$ be the corresponding radiant integral affine structures on B. Consider the integral affine manifold

$$(Y, \mathcal{A}_{Y_3}) = (B, \mathcal{A}_1) \times (B, \mathcal{A}_2) \times (B, \mathcal{A}_3). \tag{7.24}$$

This affine manifold is radiant, as it can be seen by looking at its affine holonomy. Thus $\mathcal{D}_{(Y,\mathcal{A}_Y)} = 0$ by Corollary 7.8. Note that Y has the homotopy type of a three-torus and so it has $\mathrm{H}^3(Y;\mathbb{R}) \cong \mathbb{R}$. This integral affine manifold therefore provides an example of trivial homomorphism $\mathcal{D}_{(Y,\mathcal{A}_Y)}$ of Dazord and Delzant even though its range is non-trivial.

Let $\mathfrak{l}_Y$ denote the linear holonomy of $(Y, \mathcal{A}_Y)$. The twisted cohomology group

$$\mathrm{H}^2(Y; \mathbb{Z}^6_{\mathfrak{l}_Y^{-T}})$$

is not trivial if and only if at least one of the A_i is *unipotent.* If this condition is satisfied, then $(Y, \mathcal{A}_Y)$ provides the first example of a manifold whose associated homomorphism $\mathcal{D}_{(Y,\mathcal{A}_Y)}$ is trivial notwithstanding the fact that both its domain and range are not trivial. More generally, by taking the product of k radiant integral affine manifolds of the form described above, it is possible to construct such examples in any even dimension greater than or equal to 6.

Consider the product

$$(Z_{\mathbf{n}}, \mathcal{A}_{Z_{\mathbf{n}}}) = (B, \mathcal{A}_{n_1}) \times \ldots (B, \mathcal{A}_{n_k}),$$

where $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k_+$, each $n_j \neq 0$ and the radiant integral affine structure $\mathcal{A}_{n_j}$ on B has linear holonomy generated by the matrix

$$\begin{pmatrix} 1 & 0 \\ -n_j & 1 \end{pmatrix}. \tag{7.25}$$

Let $\mathfrak{l}_{Z_{\mathbf{n}}}$ be the linear holonomy of $(Z_{\mathbf{n}}, \mathcal{A}_{Z_{\mathbf{n}}})$. All elements of the cohomology group

$$\mathrm{H}^2(Z_{\mathbf{n}}; \mathbb{Z}^{2k}_{\mathfrak{l}^{-T}_{Z_{\mathbf{n}}}})$$

(which, by the above remark, is non trivial) can be realised as the Chern class of some regular Lagrangian bundle over $(Z_{\mathbf{n}}, \mathcal{A}_{Z_{\mathbf{n}}})$. This example is interesting because each $(B, \mathcal{A}_{n_j})$ is the affine model in the neighbourhood of a focus-focus singularity of a completely integrable Hamiltonian system, which is homeomorphic to a two torus pinched n_j times, as shown in [7, 67]. Thus $(Z_{\mathbf{n}}, \mathcal{A}_{Z_{\mathbf{n}}})$ is a local affine model for a product of focus-focus singularities. Such products occur naturally amongst non-degenerate singularities of Lagrangian bundles, which have been classified topologically by Zung in [66].

The above examples show that there are no obstructions to constructing Lagrangian bundles with non-trivial Chern classes over radiant integral affine manifolds (cf. Corollary 7.8). In order to construct a singular Lagrangian bundle $(M, \omega) \to B$, such that (M, ω) is a smooth symplectic manifold, it is necessary that its regular part $(M_{\mathrm{reg}}, \omega_{\mathrm{reg}}) \to B_{\mathrm{reg}}$ also be Lagrangian. Suppose that B_{reg} is a smooth manifold, so that Lemma 4.3 implies that it is integral affine. Let $\mathcal{A}_{\mathrm{reg}}$ be the induced integral affine structure. Corollary 7.8 shows that if $(B_{\mathrm{reg}}, \mathcal{A}_{\mathrm{reg}})$ is a radiant integral affine manifold, there is no symplectic obstruction to constructing regular Lagrangian bundles over $(B_{\mathrm{reg}}, \mathcal{A}_{\mathrm{reg}})$. In particular, all elements of $\mathrm{H}^2(B_{\mathrm{reg}}; \mathbb{Z}^n_{\mathfrak{l}_{\mathrm{reg}}})$ can be realised as Chern classes of some Lagrangian bundle over $(B_{\mathrm{reg}}, \mathcal{A}_{\mathrm{reg}})$. However, it still remains to understand how the integral affine structure $\mathcal{A}_{\mathrm{reg}}$ affects the topology and symplectic geometry near the singular fibres of $(M, \omega) \to B$. For instance, the linear holonomy in the affine model near a focus-focus singularity (cf. [7]) determines the topology of the singular set.

Chapter 8

Conclusion

This thesis has established the deep link between integral affine geometry and the topology and symplectic geometry of Lagrangian bundles. In light of Theorem 7.5, it is possible to use the methods of symplectic topology and Lagrangian bundles to study the geometry of integral affine manifolds and *vice versa*. There are a few questions that naturally stem from the work in this thesis, which are stated and briefly discussed below; these address problems which can be solved by exploiting the above relation.

Generalisation to isotropic bundles

Lagrangian bundles are a special case of *isotropic* bundles. These are of the form

$$F \hookrightarrow (M, \omega) \to B,$$

where M is $2n$-dimensional, and the fibres F are isotropic submanifolds of (M, ω), *i.e.* $\omega|_F = 0$ for all F. These bundles are related to a more general notion of integrability than Liouville integrability (cf. Definition 2.10), known as *non-commutative* or *super*-integrability, which was first formulated by Fomenko and Miščenko in [47]. Arguments akin to those used to prove the Liouville-Mineur-Arnol'd theorem (cf. 2.2) imply that the fibres of isotropic bundles are k-dimensional tori ($k \leq n$) and that the manifold B inherits the structure of a *regular Poisson manifold*, *i.e.* a manifold endowed with a Poisson structure (cf. Definition 2.7) whose rank is constant (cf. [24]). In fact, the rank of the induced Poisson structure is precisely $2n - 2k$; this implies that the manifold B is foliated by symplectic manifolds of dimension $2n - 2k$ (cf. [62]).

The work of Dazord and Delzant in [18] addresses the issue of constructing isotropic bundles when the regular Poisson manifold B is fixed. A natural question to ask is the following.

Question 8.1. Can the methods of this thesis be extended to the isotropic case? In particular, does there exist a link between integral affine geometry and the construction problem for isotropic bundles?

A good reason to study this problem is that the current theory of classification of non-degenerate singularities of completely integrable Hamiltonian systems is based on the study of isotropic bundles in [18]. Thus finding an answer to Question 8.1 might shed some light towards understanding the role of integral affine geometry in the classification of singularities of Lagrangian fibrations.

Symplectic topology of total spaces of Lagrangian bundles

Symplectic topology is normally regarded as being *flabby*, while integral affine geometry is *rigid*. Total spaces of Lagrangian bundles are examples of symplectic manifolds whose symplectic topology depends on the integral affine geometry of the base space. For instance, any integral affine diffeomorphism of $(B, \mathcal{A})$ can be lifted to a fibrewise symplectomorphism of its associated symplectic reference Lagrangian bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0) \to B.$$

The group of integral affine diffeomorphisms $\mathrm{Aff}_{\mathbb{Z}}(B, \mathcal{A})$ often has multiple connected components, as it is a subgroup of $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$.

Question 8.2. Does the injection

$$\mathrm{Aff}_{\mathbb{Z}}(B, \mathcal{A}) \hookrightarrow \mathrm{Symp}(M, \omega)$$

give information about the topology of $\mathrm{Symp}(M, \omega)$? For instance, do distinct connected components of $\mathrm{Aff}_{\mathbb{Z}}(B, \mathcal{A})$ map to distinct components of $\mathrm{Symp}(M, \omega)$?

The above question is interesting to study because it illustrates the interplay between the flabbiness of symplectic topology and the rigidity of integral affine geometry in the context of Lagrangian bundles. A first step towards answering Question 8.2 would be to study which integral affine diffeomorphisms of the base space $(B, \mathcal{A})$ lift to fibrewise symplectomorphisms of a Lagrangian bundle $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (M, \omega) \to B$ with non-trivial Chern class.

Integral affine geometry via Lagrangian bundles

Theorem 7.6 gives an example of an application of the methods of symplectic topology to the study of integral affine manifolds. The link connecting the two subjects is the radiance obstruction $r_{(B,\mathcal{A})}$ of an integral affine manifold $(B, \mathcal{A})$. The importance of this cohomological invariant has been remarked by several authors in both fields (cf. [31, 32, 34]). In light of the importance of Lagrangian bundles in Hamiltonian mechanics and theoretical physics, a study of the topological properties of integral affine manifolds is needed.

As a starting point, it should be noted that the radiance obstruction $r_{(B,\mathcal{A})}$ provides information about the cohomology group

$$\mathrm{H}^1(B; \mathbb{R}^n_{\mathfrak{l}}),$$

where $\mathfrak{l}$ denotes the linear holonomy of $(B, \mathcal{A})$. For instance, the following question can be asked.

Question 8.3. Suppose $(B, \mathcal{A})$ is an integral affine manifold with linear holonomy $\mathfrak{l}$. What are the elements of $\mathrm{H}^1(B; \mathbb{R}^n_{\mathfrak{l}})$ which can arise as the radiance obstruction of an integral affine structure $\mathcal{A}'$ on B whose linear holonomy is $\mathfrak{l}$?

Question 8.3 is closely related to the following question, which explores the difference between integral and strongly integral affine manifolds (cf. Remark 7.19).

Question 8.4. Let $(B, \mathcal{A})$ be an integral affine manifold with linear holonomy $\mathfrak{l}$. Does there exist a strongly integral affine structure $\mathcal{A}'$ on B with linear holonomy $\mathfrak{l}$?

In particular, an affirmative answer to Question 8.4 would imply that integral elements of $\mathrm{H}^1(B;\mathbb{R}^n_l)$ arise as the radiance obstruction of some integral affine structure on B. The existence of strongly integral affine manifolds with given linear holonomy is related to the study of singular Lagrangian bundles in mirror symmetry (cf. [34]). Moreover, answers to the above questions would provide some insight in the topology of integral affine manifolds, which could then be used to attempt to tackle longstanding conjectures in affine geometry, *e.g.* Markus' and Auslander's conjectures (cf. Section 1.1.3), for integral affine manifolds.

Classification of singularities of Lagrangian fibrations

A *Lagrangian fibration* $(M,\omega) \to B$ is a singular Lagrangian bundle. As mentioned in the Introduction to this thesis, the interest in Lagrangian fibrations stemming from Hamiltonian mechanics and mirror symmetry concentrates on the topology and symplectic geometry of the singularities.

There exist several works in the literature which offer some results on the classification of singular Lagrangian fibrations. For instance, Zung has completed the topological classification of *non-degenerate* singularities, first by showing that they are locally given by products of lower dimensional singularities in [66], then by producing a Dazord-Delzant type of theory for the construction of non-degenerate singular bundles in [68]. It would be interesting to study the classification and construction problem of Lagrangian fibrations using as a starting point integral affine geometry. The works of Castaño-Bernard and Matessi in [13] and of Gross and Siebert in [34] are along this line.

The construction of Lagrangian fibrations with non-trivial Chern class (in the sense of Zung's paper [68]) would be a good first step towards understanding the role of integral affine geometry in Lagrangian fibrations. The following question arises from Section 7.3.

Question 8.5. Is it possible to construct Lagrangian fibrations whose regular parts are isomorphic to some Lagrangian bundle with non-trivial Chern class over $(Z_{\mathbf{n}}, \mathcal{A}_{Z_{\mathbf{n}}})$? What about the Lagrangian bundles with non-trivial Chern class constructed by Bates in [6]?

Appendix A

Proof of equation (6.45)

Throughout the appendix fix the notation as in the proof of Theorem 6.9. Let $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ be an almost Lagrangian bundle over $(B, \mathcal{A})$ and let $\mathcal{U} = \{U_\alpha\}$ be the good cover of B given by Remark 4.34, *i.e.* there exist local action-angle coordinates $(\mathbf{a}_\alpha, \boldsymbol{\theta}_\alpha)$ on $\pi^{-1}(U_\alpha)$ and the transition functions are of the form

$$\varphi_{\alpha_2\alpha_1}(\mathbf{a}_{\alpha_1}, \boldsymbol{\theta}_{\alpha_1}) = (A_{\alpha_2\alpha_1}\mathbf{a}_{\alpha_1} + \mathbf{d}_{\alpha_2\alpha_1}, A^{-T}_{\alpha_2\alpha_1}\boldsymbol{\theta}_{\alpha_1} + \mathbf{g}_{\alpha_2\alpha_1}(\mathbf{a}_{\alpha_1})), \tag{A.1}$$

where the first component corresponds to a change in integral affine coordinates on the base space $(B, \mathcal{A})$. The cocycle condition for the transition functions of equation (A.1) imply the following equalities

$$\mathbf{d}_{\alpha_1\alpha_2} - \mathbf{d}_{\alpha_1\alpha_3} = A_{\alpha_1\alpha_3}\mathbf{d}_{\alpha_3\alpha_2} \tag{A.2a}$$

$$\mathbf{g}_{\alpha_1\alpha_2} - \mathbf{g}_{\alpha_1\alpha_3} = A^{-T}_{\alpha_1\alpha_3}\mathbf{g}_{\alpha_3\alpha_2} \tag{A.2b}$$

for all indices $\alpha_1, \alpha_2, \alpha_3$. Let

$$\mathrm{d}^{(2)} : \mathrm{E}_2^{1,1} \to \mathrm{E}_2^{3,0}$$

denote a differential on the E_2-page of the Leray-Serre spectral sequence of $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow M \to B$ with real coefficients. Let w_0 be the cohomology class of the symplectic form of the symplectic reference Lagrangian bundle

$$\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow (\mathrm{T}^*B/P_{(B,\mathcal{A})}, \omega_0) \to B$$

associated to $(B, \mathcal{A})$. Using the Čech-de Rham interpretation of this spectral sequence (cf. [11]), a cocycle representing $w_0 \in \mathrm{E}_2^{1,1}$ is given by

$$\tau_{\alpha_2\alpha_1} = \sum_{i=1}^{n} d^i_{\alpha_2\alpha_1}\mathrm{d}\theta^i_{\alpha_2}.$$

Define

$$\eta_{\alpha_1\alpha_2\alpha_3} := \sum_{i=1}^{n} d^i_{\alpha_2\alpha_3} g^i_{\alpha_2\alpha_1}$$

as in the proof of Theorem 6.9.

The obstruction to the existence of an appropriate symplectic form ω on M which makes the above bundle Lagrangian is the cohomology class υ represented by the Čech cocycle

$$\sum_{i,j=1}^{n} A^{ij}_{\alpha_2\alpha_1} \mathrm{d}a^j_{\alpha_1} \wedge \mathrm{d}g^i_{\alpha_2\alpha_1}.$$

Set

$$\xi_{\alpha_1\alpha_2\alpha_3} := \sum_{i=1}^{n} (d^i_{\alpha_1\alpha_3} - d^i_{\alpha_1\alpha_2}) g^i_{\alpha_1\alpha_2}$$

as in the proof of Theorem 6.9. The cocycle $-\delta\xi$ is another Čech representative of υ (cf. [11]). The following lemma proves the equality of equation (6.45).

Lemma A.1. *For all indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$,*

$$(\delta\xi)_{\alpha_1\alpha_2\alpha_3\alpha_4} = -(\delta\eta)_{\alpha_1\alpha_2\alpha_3\alpha_4}.$$

Proof. On the one hand,

$$\begin{aligned} (\delta\eta)_{\alpha_1\alpha_2\alpha_3\alpha_4} &= \eta_{\alpha_2\alpha_3\alpha_4} - \eta_{\alpha_1\alpha_3\alpha_4} + \eta_{\alpha_1\alpha_2\alpha_4} - \eta_{\alpha_1\alpha_2\alpha_3} \\ &= \sum_{i=1}^{n} d^i_{\alpha_3\alpha_4} g^i_{\alpha_3\alpha_2} - \sum_{i=1}^{n} d^i_{\alpha_3\alpha_4} g^i_{\alpha_3\alpha_1} + \sum_{i=1}^{n} d^i_{\alpha_2\alpha_4} g^i_{\alpha_2\alpha_1} - \sum_{i=1}^{n} d^i_{\alpha_2\alpha_3} g^i_{\alpha_2\alpha_1}. \end{aligned} \tag{A.3}$$

On the other,

$$\begin{aligned} (\delta\xi)_{\alpha_1\alpha_2\alpha_3\alpha_4} &= \xi_{\alpha_2\alpha_3\alpha_4} - \xi_{\alpha_1\alpha_3\alpha_4} + \xi_{\alpha_1\alpha_2\alpha_4} - \xi_{\alpha_1\alpha_2\alpha_3} \\ &= \sum_{i=1}^{n} (d^i_{\alpha_2\alpha_4} - d^i_{\alpha_2\alpha_3}) g^i_{\alpha_2\alpha_3} - \sum_{i=1}^{n} (d^i_{\alpha_1\alpha_4} - d^i_{\alpha_1\alpha_3}) g^i_{\alpha_1\alpha_3} \\ &\quad + \sum_{i=1}^{n} (d^i_{\alpha_1\alpha_4} - d^i_{\alpha_1\alpha_2}) g^i_{\alpha_1\alpha_2} - \sum_{i=1}^{n} (d^i_{\alpha_1\alpha_3} - d^i_{\alpha_1\alpha_2}) g^i_{\alpha_1\alpha_2} \\ &= \sum_{i=1}^{n} ((d^i_{\alpha_2\alpha_4} - d^i_{\alpha_2\alpha_3}) g^i_{\alpha_2\alpha_3} - (d^i_{\alpha_1\alpha_4} - d^i_{\alpha_1\alpha_3}) g^i_{\alpha_1\alpha_3} \\ &\quad + (d^i_{\alpha_1\alpha_4} - d^i_{\alpha_1\alpha_3}) g^i_{\alpha_1\alpha_2}). \end{aligned} \tag{A.4}$$

Equation (A.2b) with the first and third indices being equal yields

$$\mathbf{g}_{\alpha\beta} = -A^{-T}_{\alpha\beta} \mathbf{g}_{\beta\alpha} \tag{A.5}$$

for all indices α, β. Applying equation (A.5) to the last line of equation (A.4) and using equation (A.2a) repeatedly, the following equality is obtained

$$\begin{aligned} (\delta\xi)_{\alpha_1\alpha_2\alpha_3\alpha_4} = - \sum_{i,j,k=1}^{n} \Big(& \underbrace{(d^i_{\alpha_2\alpha_4} - d^i_{\alpha_2\alpha_3})}_{=\sum_{k=1}^{n} A^{ik}_{\alpha_2\alpha_3} d^k_{\alpha_3\alpha_4}} (A^{-T}_{\alpha_2\alpha_3})^{ij} g^j_{\alpha_3\alpha_2} - \underbrace{(d^i_{\alpha_1\alpha_4} - d^i_{\alpha_1\alpha_3})}_{=\sum_{k=1}^{n} A^{ik}_{\alpha_1\alpha_3} d^k_{\alpha_3\alpha_4}} (A^{-T}_{\alpha_1\alpha_3})^{ij} g^j_{\alpha_3\alpha_1} \\ & + \underbrace{(d^i_{\alpha_1\alpha_4} - d^i_{\alpha_1\alpha_3})}_{=\sum_{k=1}^{n} A^{ik}_{\alpha_1\alpha_3} d^k_{\alpha_3\alpha_4}} (A^{-T}_{\alpha_1\alpha_2})^{ij} g^j_{\alpha_2\alpha_1} \Big) \end{aligned}$$

$$= -\left(\sum_{j=1}^{n} d^{j}_{\alpha_3\alpha_4} g^{j}_{\alpha_3\alpha_2} - \sum_{j=1}^{n} d^{k}_{\alpha_3\alpha_4} g^{j}_{\alpha_3\alpha_1} + \sum_{j=1}^{n} \underbrace{\sum_{k=1}^{n} A^{jk}_{\alpha_2\alpha_3} d^{k}_{\alpha_3\alpha_4}}_{= d^{j}_{\alpha_2\alpha_4} - d^{j}_{\alpha_2\alpha_3}} g^{j}_{\alpha_2\alpha_1} \right).$$

Therefore, equation (A.3) implies that

$$(\delta\xi)_{\alpha_1\alpha_2\alpha_3\alpha_4} = -(\delta\eta)_{\alpha_1\alpha_2\alpha_3\alpha_4},$$

and the result follows. $\square$

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