Moduli of Bridgeland-Stable Objects

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Ciaran Meachan)

Abstract

In this thesis we investigate wall-crossing phenomena in the stability manifold of an irreducible principally polarized abelian surface for objects with the same invariants as (twists of) ideal sheaves of points. In particular, we construct a sequence of fine moduli spaces which are related by Mukai flops and observe that the stability of these objects is completely determined by the configuration of points. Finally, we use Fourier-Mukai theory to show that these moduli are projective.

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To my wife.

"Beauty in things exists in the mind which contemplates them."

David Hume

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Chapter 1

Introduction

Moduli Problems

Often in algebraic geometry, we would like to understand a particular class of sheaves on a smooth projective variety. Naturally, we can ask if there is a moduli space for our chosen class of sheaves? That is, a space where each closed point corresponds to an isomorphism class of sheaves. It turns out that the moduli space of all coherent sheaves is 'too big'. However, if we define a notion of stability, then the class of stable sheaves (with some fixed numerical invariants) is much more manageable. In particular, they form a bounded family in the sense that there is some scheme of finite-type parametrizing them. These are the moduli spaces we want to study. Remarkably, stable sheaves can be seen as the building blocks for all coherent sheaves via the Harder-Narasimhan and Jordan-Hölder filtrations. More precisely, any coherent sheaf has a unique filtration of semistables and each of these has a filtration of stables whose factors are unique up to permutation. In order to motivate this subject, we provide three good reasons for studying moduli spaces of sheaves. First of all, they provide a natural path to higher dimensional algebraic varieties. In particular, moduli spaces of sheaves on a variety have an incredibly rich and interesting geometry; they are interesting in their own right. Secondly, and rather curiously, understanding moduli spaces often leads to a deeper understanding of the underlying variety. For instance, the moduli space can sometimes provide answers to questions regarding Chow groups, linear systems and intersection numbers. Finally, from an applied point of view, certain moduli spaces have been interpreted as solution spaces to certain differential equations coming from Physics; for example, the Yang-Mills equation which gives rise to the so-called instanton spaces.

Stability Timeline

1960's: The first notion of stability was introduced by Mumford in [Mum62]. He defined a vector bundle E on a smooth curve C to be slope semistable if for all proper subbundles $0 \neq K \subsetneq E$ the degree-rank ratio of K is less than or equal to the degree-rank ratio of E. After developing geometric invariant theory (see [MFK94]), Mumford was able to prove that for any pair of integers (r, d) with $r > 0$, the class of semistable bundles of rank r and degree d on a smooth curve C has a coarse moduli space $\mathcal{M}_C(r, d)$. In particular, he constructed $\mathcal{M}_C(r, d)$ as a projective scheme.

1970's: In trying to generalise Mumford's construction to smooth projective surfaces, Gieseker [Gie77], Maruyama [Mar78] and Takemoto [Tak72] found that two extra ingredients were required:

1. In order to get a compact moduli space, one has to consider torsion-free sheaves which are not locally-free; on a curve, these notions coincide.

2. To define stability of sheaves on a higher dimensional variety X, one must first choose a polarization of X, i.e. a numerical equivalence class of ample line bundles or more simply a specific embedding into projective space.

Their definition of stability was based on the Euler characteristic of torsion-free sheaves and the natural lexicographical ordering of the corresponding degree two polynomials.

1980's: Work on moduli spaces of sheaves on surfaces was significantly stimulated by Donaldson's profound work on four-manifolds. In particular, Donaldson proved that a vector bundle on a complex algebraic surface is slope stable, with respect to the projective embedding, if and only if the corresponding bundle on the underlying four-manifold admits an irreducible Hermitian-Einstein connection; see [Don85]. This result provided an important bridge between algebraic geometry and gauge theory.

1990's: Using the observations of Gieseker, Maruyama and Takemoto mentioned above, it was Simpson who first succeeded in providing a generalised definition of stability in [Sim94]. He defined a pure sheaf E on a polarized variety X to be semistable if for all proper subsheaves $0 \neq K \subsetneq E$ the Hilbert polynomial of K is less than or equal to the Hilbert polynomial of E. The polarization of X is encoded into the Hilbert polynomial and so different ample line bundles will give rise to different notions of stability; see [HL10], Section 4.C. Simpson proved that the class of semistable sheaves with fixed numerical invariants on a projective variety X (with respect to a given polarization) always has a course

moduli space. In particular, he showed via geometric invariant theory that this moduli space is a projective scheme.

2000's: Inspired by Douglas' ideas about 'Π-stability for D-branes' in [Dou01] and [Dou02], Bridgeland introduced the notion of a stability condition on an arbitrary triangulated category $\mathcal{T}$ in [Bri07]; which is essentially an abstraction of the usual properties of slope stability for sheaves on complex projective varieties. More precisely, a stability condition σ is a pair $(Z, \mathcal{A})$ where $\mathcal{A} \subset \mathcal{T}$ is an abelian subcategory (arising as the heart of a bounded t-structure on $\mathcal{T}$) and $Z : K(\mathcal{A}) \to \mathbb{C}$ is a group homomorphism which has the Harder-Narasimhan property. The notion of slope arises naturally as the real-imaginary ratio of the image of an object $E \in \mathcal{A}$ under Z. Then an object $E \in \mathcal{A}$ is defined to be σ-semistable if it is semistable with respect to Z, i.e. the real-imaginary ratio satisfies the usual inequality for all proper subobjects $K \hookrightarrow E$ in $\mathcal{A}$. Remarkably, the space of all stability conditions $\mathrm{Stab}(\mathcal{T})$ comes equipped with a natural topology which makes it into a (possibly infinite-dimensional) complex manifold. Thus we have a geometric invariant naturally associated to a triangulated category $\mathcal{T}$ and ultimately a way of extracting geometry from homological algebra; the triangulated category that we are most interested in is the bounded derived category of coherent sheaves on a smooth projective variety X, denoted $\mathcal{D}(X)$.

Fourier-Mukai Transforms

A Fourier-Mukai transform $\Phi : \mathcal{D}(X) \xrightarrow{\sim} \mathcal{D}(Y)$ is a certain kind of equivalence between the derived category of coherent sheaves on two varieties X and Y. Roughly speaking, applying a Fourier-Mukai transform to an object $E \in \mathcal{D}(X)$ produces a 'frequency spectrum of E' in terms of cohomology sheaves in $\mathcal{D}(Y)$. A simple and yet somewhat powerful observation is that Fourier-Mukai transforms take moduli problems on X isomorphically to moduli problems on Y. Often, the transformed moduli problem is easier to solve and in this way, Fourier-Mukai transforms have proven themselves to be an invaluable tool when studying moduli spaces. These equivalences become particularly interesting when X is not isomorphic to Y and it was Mukai who first constructed such an example in [Muk81]; he showed that the Poincaré bundle $\mathcal{P}$ induces an equivalence between the derived category of an abelian variety A and the derived category of its dual $\hat{A}$ (which in general is not isomorphic to A). This result added significant substance to the Moscow school philosophy that the derived category was, on some deeper level, an invariant of the variety X. Together with Orlov's result [Orl97] which says that all derived equivalences between smooth projective varieties are of Fourier-Mukai-type, it was natural to seek some sort of classification of such transforms:

1. Find the set of Fourier-Mukai partners of a given variety X,

$$\mathrm{FM}(X) = \left\{ Y \mid \mathcal{D}(X) \overset{\sim}{\to} \mathcal{D}(Y) \right\}.$$

2. Find the group of autoequivalences of $\mathcal{D}(X)$,

$$\mathrm{Aut}(\mathcal{D}(X)) = \left\{ F \mid F : \mathcal{D}(X) \overset{\sim}{\to} \mathcal{D}(X) \right\} \Big/ \sim .$$

In [BO01], Bondal and Orlov showed that this classification was rather boring in the case when X (was smooth and projective and) has ample canonical (or anti-canonical) bundle. More precisely, they showed that the only Fourier-Mukai partner of such an X is itself and the group of derived automorphisms is trivial in the sense that it consists solely of automorphisms coming from the variety, twists by line bundles and shifts in the derived category. Moreover, they demonstrated how to reconstruct the variety X from $\mathcal{D}(X)$. Thankfully, if we remove the positivity assumption on the canonical bundle then the theory becomes much more interesting. A particular instance when these questions are both interesting and manageable is when X is Calabi-Yau, i.e. the canonical bundle is trivial. For example, when X is an abelian or K3 surface then Mukai [Muk87b] and Orlov [Orl97] proved that Y is a Fourier-Mukai partner of X if and only if Y is a moduli space of stable sheaves on X. Orlov also proved that the derived automorphism group of an abelian surface sits inside a short exact sequence $0 \to \mathbb{Z} \oplus (X \times \hat{X}) \to \mathrm{Aut}(\mathcal{D}(X)) \to U(X \times \hat{X}) \to 1$ where $U(X \times \hat{X})$ is the group of isometric isomorphisms $f : X \times \hat{X} \overset{\sim}{\to} X \times \hat{X}$ and $\mathbb{Z} \oplus (X \times \hat{X})$ is generated by shifts, translations and twists by line bundles $L \in \mathrm{Pic}^0(X)$; see [Huy06, Section 9.5]. However, the automorphism group for a K3 surface seems to be much more subtle. In this direction, there is a conjectural answer in [Bri08] which is phrased in terms of stability conditions; it is expected to be generated by spherical objects (see [Huy10]).

The underlying connection between this digression on Fourier-Mukai theory and stability conditions is homological mirror symmetry. In [Kon94], Kontsevich proposed a derived equivalence between the category of coherent sheaves on a variety X and the Fukaya category of its mirror $\check{X}$; thus providing a deep connection between the complex geometry of one and the symplectic geometry of the other. We will refrain from discussing any details of this relationship here but just mention that stability conditions were designed, in some sense, to model (what physicists call) 'super conformal field theories' with the hope that the Kähler moduli space associated to this mirror symmetry picture could be realised as

a particular submanifold of the quotient $\text{Stab}(\mathcal{D}(X))/\text{Aut}(\mathcal{D}(X))$; see [Bri09]. From this rather mysterious string-theoretic point of view, stability conditions on smooth projective Calabi-Yau 3-folds are the most interesting but so far, despite many valiant attempts, nobody has managed to construct a single stability condition in this situation. As for dimension one, i.e. elliptic curves, Bridgeland [Bri07] and Macri [Mac07] have shown that there is essentially only one stability condition on $\mathcal{D}(X)$, namely the one with the classical choice of stability function $Z = -\deg + i \cdot \text{rk}$ and $\mathcal{A} = \text{Coh}(X)$. Therefore, this thesis will stick to surfaces, i.e. abelian or K3, where there are plenty of stability conditions.

Preservation of Stability

The main theme of this thesis is preservation of stability. In particular, we scrutinize the following folklore result:

<div align="center">"Stability is preserved under Fourier-Mukai transforms."</div>

Given the many different notions of stability, this statement is quite vague and many people ([BBHR97], [Mac96], [Yos09]) have studied a similar question

<div align="center">When is the image $\Phi(E)$ of a Mumford/Gieseker-stable sheaf E again a Mumford/Gieseker-stable sheaf?</div>

Under suitable conditions, the philosophy holds true but it is not difficult to construct counter-examples on an abelian surface:

Theorem 1.0.1. *Let $(\mathbb{T}, L)$ be a principally polarized abelian surface over $\mathbb{C}$ with $\ell := c_1(L)$ and $\text{Pic}(\mathbb{T}) = \mathbb{Z}[\ell]$. If $\mathcal{M}(r, \ell, -r-1)$ denotes the moduli space of stable sheaves F on $\mathbb{T}$ with $\text{ch}(F) = (r, \ell, -r-1)$ for $r = 0, 1, 2$ then a generic element of $\mathcal{M}(r, \ell, -r-1)$ is a stable sheaf F with stable transform $\hat{F} := \Phi_{\mathcal{P}}(F)$. Moreover, the families of extensions of F by $\hat{F}$ are stable sheaves $E \in \mathcal{M}(2r+1, 2\ell, -(2r+1))$ which, under the standard Fourier-Mukai transform, become unstable.*

Proof See Corollary 2.6.7, Corollary 2.6.9 and Corollary 2.6.11 □

Bridgeland's stability manifold comes with a wall and chamber decomposition in the sense that the set of σ-stable objects (with some fixed numerical invariants) is constant in each chamber and an object of $\mathcal{D}(X)$ can only become stable or unstable by crossing a wall, i.e. a real codimension one submanifold of $\text{Stab}(\mathcal{D}(X))$. Moreover, $\text{Stab}(\mathcal{D}(X))$ has a natural action of the the group of autoequivalences $\text{Aut}(\mathcal{D}(X))$. This allows us to recast the philosophy regarding preservation of

stability under Fourier-Mukai transforms in a much more precise way:

$$E \in \mathcal{D}(X) \text{ is } \sigma\text{-stable} \Leftrightarrow \left\{ \begin{array}{c} \Phi(E) \in \mathcal{D}(X) \text{ is } \Phi(\sigma)\text{-stable} \\ \text{for some } \Phi \in \mathrm{Aut}(\mathcal{D}(X)). \end{array} \right\}$$

Therefore, the following questions become equivalent:

$$\left\{ \begin{array}{c} \text{Does } \Phi \text{ preserve} \\ \text{stability?} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{Are } \sigma \text{ and } \Phi(\sigma) \\ \text{in the same chamber?} \end{array} \right\}$$

To sum up, preservation of stability is a tautologous statement when viewed through the somewhat powerful lens of stability conditions and the examples constructed above must come about because some wall in $\mathrm{Stab}(\mathcal{D}(\mathbb{T}))$ has been crossed. A natural question to ask then is

Can we realise these examples as explicit wall-crossing in $\mathrm{Stab}(\mathcal{D}(\mathbb{T}))$?

Inspired by Arcara and Bertram's excellent paper [ABL07], our method of attack will be to take a one-parameter family of stability conditions $\{\sigma_t\}_{t \in \mathbb{R}_{>0}}$ and observe which walls, if any, are crossed; in a sense, we will go for a 'walk' in $\mathrm{Stab}(\mathcal{D}(\mathbb{T}))$.

Wall-Crossing

Let $\mathcal{M}_X^v(\sigma)$ denote the moduli space of σ-stable objects E on X with Mukai vector $v(E) = v$. Naturally, we can ask:

1. Are $\mathcal{M}_X^v(\sigma)$ projective in general? We saw above that moduli spaces of stable sheaves in the sense of Mumford and Gieseker were manifestly projective but is the same true for Bridgeland-stability?

2. It is expected that wall-crossing corresponds to birational maps between the moduli $\mathcal{M}_X^v(\sigma) \dashrightarrow \mathcal{M}_X^v(\sigma')$. What are they and how do they relate to the underlying geometry of X?

This thesis will address these two questions in the case when X is an irreducible principally polarized abelian surface and our objects have the same invariants as (twists of) ideal sheaves of points. For question one, we actually formally introduce a second parameter s which gives rise to a two-dimensional slice of the three-dimensional stability manifold which we refer to as the (s,t)-plane; each rational point $(s,t) \in \mathbb{Q} \times \mathbb{Q}_{>0}$ gives rise to a stability condition $\sigma_{s,t}$ governed by an abelian category $\mathcal{A}_s$ which is invariant under scaling by t. Then, in a particular region of the (s,t)-plane we can say the following

Theorem 1.0.2. *For any $0 \leq s < 2$ and $t > 0$, the moduli space of $\sigma_{s,t}$-stable objects in $\mathcal{A}_s$ with Chern character $(1, 2\ell, 4 - n)$ is a smooth complex projective variety for each non-negative integer n.*

Proof   See Proposition 3.3.11 and Theorem 3.6.1.   □

The key observation in establishing this result is that, in this particular region, the walls of $\mathrm{Stab}(\mathcal{D}(\mathbb{T}))$ nest; in other words, we cannot have the following behaviour:
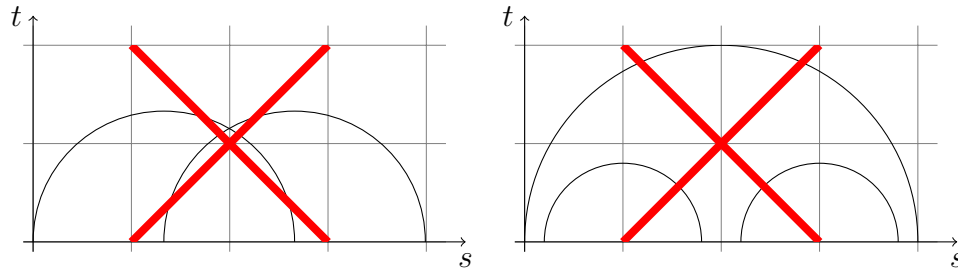
The upshot is that each chamber intersects the the real line and close to the real line we can find a suitable Fourier-Mukai transform which identifies this moduli space with another moduli space which is a priori projective; namely the Hilbert scheme.

In order to tackle question two, we need to borrow another important tool from the forefather of this field. In [Muk84], Mukai proves that the moduli space $\mathcal{M}$ of μ-stable sheaves on an abelian or K3 surface is a symplectic variety. Furthermore, he shows that if $P \subset \mathcal{M}$ is a projective bundle of codimension at least two then there is a birational map $\mathcal{M} \dashrightarrow \mathcal{M}'$ which replaces P by its dual $P^{\vee}$ whilst preserving the symplectic structure; we call this operation a Mukai flop.

To get an idea of wall-crossing phenomena, suppose we are interested in objects with some specific numerical invariants. Let G be our favourite object which exhibits the chosen invariants and suppose it naturally sits in a short exact sequence in $\mathrm{Coh}(X)$ of the form $0 \to F \to E \to G \to 0$. Now, choose an 'appropriate' abelian subcategory $\mathcal{A} \subset \mathcal{D}(X)$ so that we can state the precise set of stable objects with these invariants. In many cases, it will turn out that this sequence is no longer exact in $\mathcal{A}$. Maybe we need to turn the induced triangle once, say, to get

$$0 \to E \to G \to F[1] \to 0.$$

If we imagine the slope as some sort of height function then we can think of these sequences as rotating about G as our formal parameter t varies. More precisely, for some critical value $t_c \in \mathbb{R}_{>0}$ we will have $\sigma_t(E) > \sigma_t(G)$ for all $t < t_c$. In other words, after crossing the wall t_c, we see that G is destabilised by E and we need to replace these extensions by stable ones. On a surface (with trivial canonical

bundle), Serre duality provides a natural pairing between the extension spaces $\mathrm{Ext}^1(F[1], G) \cong \mathrm{Ext}^1(G, F[1])^*$ and thus candidate replacements of the form

$$0 \to F[1] \to G' \to E \to 0.$$

If we can show that the extensions of interest are supported on projective bundles (of codimension at least two) then by Mukai's result, we can cut out the unstable locus and glue in a stable one. We can continue to perform surgeries on our moduli space in this way until we have stable objects for all $t > 0$:

Theorem 1.0.3. *Let $(\mathbb{T}, L)$ be an irreducible principally polarized abelian surface with $\mathrm{Pic}(\mathbb{T}) = \mathbb{Z}[\ell]$ and consider objects $E \in \mathcal{A}_0$ with $\mathrm{ch}(E) = (1, 2\ell, 4 - n)$ where $n \in \mathbb{Z}_{\geq 0}$ and E is σ_t-stable for some $t > 0$. Then we have a set of critical values:*

$$\left\{ t_m = \sqrt{n - 2m - 2} \; : \; 0 \leq m < \frac{n - 2}{2} \right\},$$

away from which, there is a smooth proper moduli space

$$\mathcal{M}_t := \mathcal{M}_t(1, 2\ell, 4 - n)$$

which together with a suitable coherent sheaf $\mathcal{U}_t$ on $\mathbb{T} \times \mathcal{M}_t$ represents the functor: isomorphism classes of flat families of σ_t-stable objects in $\mathcal{A}_0$.

Proof   See Theorem 3.5.8.   □

Using an observation of Maciocia in [Mac11], we can say precisely why these walls exist in terms of the configuration of points with respect to certain curves in $\mathbb{T}$:

Theorem 1.0.4. *The objects $E \in \mathcal{A}_0$ with numerical invariants $\mathrm{ch}(E) = (1, 2\ell, 4 - n)$ that are σ_t-stable for some $t > 0$ are either*

  (a) twisted ideal sheaves of degree four associated to $X \in \mathrm{Hilb}^n(\mathbb{T})$, or

  (b) an extension of a twisted ideal sheaf of degree two by a line bundle supported on a curve, or

  (c) a two-step complex with cohomology consisting of locally-free sheaves which only happens when $n = 5$.

Moreover, an object of type (a) is destabilised by a twisted ideal sheaf of degree two if and only if the associated n points contain a collinear subscheme of colength m; if $n = 5$ then there is a rank two destabiliser if and only if the configuration of X is very specific. Sheaves with sufficiently general configurations of points are σ_t-stable for all $t > 0$.

Proof   See Theorem 3.3.9.   □

The special case of the previous theorem, when $n = 5$, is forced upon us by the (standard) Fourier-Mukai transform. As was stressed above, the transform of a stable object $E \in \mathcal{D}(X)$ is stable with respect to the transformed stability condition. However, in general, the transformed object will have different numerical invariants, i.e. we have an isomorphism of moduli spaces

$$\mathcal{M}_X^v(\sigma) \stackrel{\sim}{\to} \mathcal{M}_X^{\Phi(v)}(\Phi(\sigma)).$$

Since the numerical type fixes the wall and chamber structure we should consider Fourier-Mukai transforms such that $v = \Phi(v)$; which is precisely what happens in our example when $n = 5$ and Φ is Mukai's standard Fourier-Mukai transform.

Understanding the relationship between $\mathcal{M}_X^v(\sigma)$ and $\mathcal{M}_X^v(\Phi(\sigma))$ directly seems to be rather difficult. This is because the geometry of the wall and chamber structure on $\mathrm{Stab}(X)$ is quite complicated. If we suppose that $\mathrm{Stab}(\mathcal{D}(X))$ is connected (not known in general), we could ask (as in [Bri08]) if it is always possible to choose a sequence of adjacent chambers $C_1, \ldots, C_n$ for v with $\sigma_1 = \sigma$, $\sigma_n = \Phi(\sigma)$ and $\sigma_i \in C_i$ for $i = 1, \ldots, n$ so that there is a birational equivalence

$$\mathcal{M}_X^v(\sigma) = \mathcal{M}_{C_1}^v(\sigma_1) \dashrightarrow \cdots \dashrightarrow \mathcal{M}_{C_n}^v(\sigma_n) = \mathcal{M}_X^v(\Phi(\sigma))$$

arising as a sequence of Mukai flops? We cannot answer this question in general but for our special case of $n = 5$ we can indeed construct a chain of such maps; see Section 4.1.6.

Studying the wall and chamber structure for $\mathrm{Stab}(\mathcal{D}(\mathbb{T}))$ with our chosen numerical invariants $(1, 2\ell, 4 - n)$ also allows us to make a connection with our main theme of preservation of stability. More precisely, we can show that all the walls which cross a particular ray $(s = 0)$ in the stability manifold are at least codimension one (Lemma 3.5.2). This gives rise to the following

Theorem 1.0.5. *Let $n \geq 4$ and $X \in \mathrm{Hilb}^n(\mathbb{T})$ be generic. Then the twisted ideal sheaves of degree four associated to X are slope stable (in the sense of Mumford) with slope stable transform.*

Proof   See Corollary 3.5.4.   □

The really interesting part of the thesis comes when we look at the examples for low values of n because they all exhibit such different behaviour; see Section 4.1. For $n = 0$ and 1, we see that our objects are stable in the whole of the (s, t)-plane but things change drastically when $n \geq 2$. In all the examples worked out in the literature so far, there is only ever a finite number of walls and we thought

this was the case with our examples as well until we inspected $n = 2$ and 3 more carefully:

Theorem 1.0.6. *For $n = 2$ and 3 there is an infinite series of walls converging to $2 - \sqrt{n}$.*

Proof   See Corollary 4.1.2 and Corollary 4.1.7.   □

In particular, we find explicit Fourier-Mukai transforms $\Phi \in \mathrm{Aut}(\mathcal{D}(\mathbb{T}))$ which generate these families of walls. For $n = 2$, each wall is a codimension zero wall in the sense that it is effective on every object in $\mathcal{A}_s$ and for $n = 3$, there are actually two infinite families of codimension one and zero walls which alternate all the way down. In some sense, the $n = 3$ case is the most interesting:

Theorem 1.0.7. *For $n = 3$, there is one wall on the line $s = 0$ and thus two moduli spaces M_0 and M_1. Crossing the wall corresponds to a birational transformation $M_0 \dashrightarrow M_1$ which replaces a $\mathbb{P}^1$-fibred codimension one subspace with its dual fibration. The resulting two moduli spaces are isomorphic but this isomorphism is not an extension of the birational map outside the codimension one sublocus.*

Proof   See Theorem 4.1.4 and Remark 4.1.5.   □

For $n = 4$, we find only one wall in the whole (s, t)-plane; in the future, we would like to investigate a possible connection with O'Grady's moduli space [O'Gr03] and Lemma 3.5.7. We have already mentioned that $n = 5$ is a special case because Mukai's standard Fourier-Mukai transform acts on the moduli space $\mathcal{M}_t(1, 2\ell, -1)$ but it also gives rise to moduli spaces of two-step complexes that we can again relate to the geometry of $\mathbb{T}$; see Section 4.1.6. Needless to say, length five is very symmetric and we can draw some pretty pictures of the strata which exist within the moduli space.

Finally, in the last chapter, we return to our original question and successfully identify the walls in $\mathrm{Stab}(\mathcal{D}(\mathbb{T}))$ which realise our examples of non-preservation of stability as explicit wall-crossing.

Understanding the global geometry of the wall and chamber structure on $\mathrm{Stab}(\mathcal{D}(X))$ is a long term goal reaching way beyond the scope of this thesis. However, we feel that, despite concentrating on a particular example, the techniques developed herein will be very useful in understanding the general theory.

Chapter 2

Moduli Spaces, Stability and Fourier-Mukai Transforms

Let X be a smooth projective variety over $\mathbb{C}$ and fix an ample line bundle L.

2.1 Classical Stability

Definition 2.1.1 (Gieseker stability)**.** The Hilbert polynomial $P(E) \in \mathbb{Q}[t]$ of a coherent sheaf E on X is given by

$$n \mapsto \chi(E \otimes L^{\otimes n}) = \sum_{i=0}^{\dim(E)} (-1)^i \dim_{\mathbb{C}} H^i(X, E \otimes L^{\otimes n}).$$

The normalized Hilbert polynomial $p(E)$ is the unique rational multiple of $P(E)$ which is monic. A pure sheaf E on X is said to be semistable if for all proper subsheaves $0 \neq F \subset E$ one has

$$p(F) \leq p(E) \text{ for all } n \gg 0$$

where the polynomials are ordered lexicographically. If the inequality is always strict then E is said to be stable. E is said to be G-twisted semistable, for some pure sheaf G, if for all proper subsheaves $0 \neq F \subset E$ one has

$$p(F \otimes G^{\vee}) \leq p(E \otimes G^{\vee}) \text{ for all } n \gg 0.$$

Proposition 2.1.2. *Let F and G be semistable sheaves.*

 (a) *If $p(F) > p(G)$ then $\mathrm{Hom}(F, G) = 0$.*

 (b) *If $p(F) = p(G)$ and $f : F \to G$ is non-trivial then f is injective if F is stable and surjective if G is stable.*

(c) *If $P(F) = P(G)$ then any non-trivial homomorphism $f : F \to G$ is an isomorphism provided F or G is stable.*

(d) *Any stable sheaf E on X is simple, i.e. $\operatorname{End}(E) = \mathbb{C}$.*

Proof   See [HL10, Proposition 1.2.7 and Corollary 1.2.8].   □

Definition 2.1.3 (Mumford-Takemoto stability). For an ample divisor ω, one defines the slope $\mu_\omega(E)$ of a torsion-free sheaf E on X to be

$$\mu_\omega(E) := \frac{\deg(E)}{\operatorname{rk}(E)} = \frac{c_1(E) \cdot \omega^{\dim(X)-1}}{\operatorname{rk}(E)}$$

where we drop the ω if the context is clear. A torsion-free sheaf E on X is said to be μ_ω-semistable if for all proper subsheaves $0 \neq F \subset E$ one has

$$\mu_\omega(F) \leq \mu_\omega(E).$$

If the inequality is always strict when $\operatorname{rk}(F) < \operatorname{rk}(E)$ then E is said to be μ_ω-stable.

Lemma 2.1.4. *If E is a torsion-free sheaf and ω is the ample divisor corresponding to L, then one has the following chain of implications*

$$E \text{ is } \mu_\omega\text{-stable} \Rightarrow E \text{ is stable} \Rightarrow E \text{ is semistable} \Rightarrow E \text{ is } \mu_\omega\text{-semistable}.$$

If E is a μ_ω-semistable sheaf with rank and degree coprime then E is μ_ω-stable.

Proof   Observe that the coefficient of $t^{\dim(X)-1}$ in $p(E)$ for a torsion-free sheaf E is (a rational multiple of) $\mu_\omega(E)$. See [HL10, Lemma 1.2.13 and Lemma 1.2.14].   □

Theorem 2.1.5.   (a) *Every pure sheaf E has a unique Harder-Narasimhan filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_j = E_j/E_{j-1}$ are semistable sheaves satisfying

$$p_{\max}(E) := p(F_1) > p(F_2) > \cdots > p(F_n) =: p_{\min}(E).$$

In particular, for a torsion-free sheaf E, these factors are μ-semistable with

$$\mu_{\max}(E) := \mu(F_1) \geq \mu(F_2) \geq \cdots \geq \mu(F_n) =: \mu_{\min}(E).$$

(b) Every semistable sheaf has a Jordan-Hölder filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_j = E_j/E_{j-1}$ are stable sheaves satisfying

$$p(F_j) = p(E) \text{ for all } 1 \leq j \leq n.$$

Moreover, these factors are unique up to permuatation. In particular, the associated graded object

$$\mathrm{gr}(E) = \bigoplus_{1 \leq i \leq n} F_i$$

is well-defined and two semistable sheaves E and E' on X are said to be S-equivalent if $\mathrm{gr}(E) \cong \mathrm{gr}(E')$.

Proof   See [HL10, Theorem 1.3.4 and Proposition 1.5.2].   □

Definition 2.1.6. A destabilising sequence for a pure sheaf E on X is a short exact sequence of objects in $\mathrm{Coh}(X)$

$$0 \to K \to E \to Q \to 0 \quad \text{such that} \quad p(K) \geq p(E) \geq p(Q).$$

Using the Harder-Narasimhan and Jordan-Hölder filtrations, we can always choose K or Q to be stable.

2.2   Moduli Spaces of Stable Sheaves

Definition 2.2.1. Let $\mathrm{Fun}(\mathcal{A})$ be the category of contravariant functors (to $\mathfrak{Set}$) for some category $\mathcal{A}$ and consider the functor

$$\mathcal{A} \to \mathrm{Fun}(\mathcal{A}), \quad A \mapsto h_A := \mathrm{Hom}(-, A), \quad f \mapsto h_f := \mathrm{Hom}(-, A) \to \mathrm{Hom}(-, B).$$

A functor $\mathcal{F} \in \mathrm{Fun}(\mathcal{A})$ is said to be corepresented by $A \in \mathcal{A}$ if there is a morphism $\alpha : \mathcal{F} \to h_A$ such that any morphism $\beta : \mathcal{F} \to h_B$ factors through a unique morphism $h_f : h_A \to h_B$, i.e. $\beta = h_f \circ \alpha$. A functor $\mathcal{F} \in \mathrm{Fun}(\mathcal{A})$ is said to be represented by $A \in \mathcal{A}$ if $\mathcal{F} \simeq h_A$. Moreover, the Yoneda lemma says that this functor $\mathcal{A} \to \mathrm{Fun}(\mathcal{A})$ defines an equivalence of $\mathcal{A}$ with the full subcategory of representable functors; see [Huy06, Proposition 1.6].

Now, fix a polynomial $P \in \mathbb{Q}[t]$ and consider the functor (from $\mathfrak{Sch}^{\mathrm{op}}$ to $\mathfrak{Set}$)

$$\mathcal{M}_X^P : S \mapsto \left\{ \mathcal{E} \in \mathrm{Coh}(X \times S) : \begin{array}{c} \mathcal{E} \text{ is } S\text{-flat and } \mathcal{E}_s \text{ is semistable} \\ \text{with } p(\mathcal{E}_s) = P \text{ for all } s \in S \end{array} \right\} \Big/ \sim$$

where S is noetherian of finite-type over $\mathbb{C}$ and $\mathcal{E} \sim \mathcal{E}'$ if there is a line bundle $\mathcal{L}$ on S such that $\mathcal{E} \cong \mathcal{E}' \otimes \pi^* \mathcal{L}$. A scheme M_X^P is called a coarse moduli space of semistable sheaves if it corepresents the functor $\mathcal{M}_X^P$. We call M_X^P a fine moduli space if it represents $\mathcal{M}_X^P$; this is equivalent to the existence of a universal family of sheaves $\mathcal{E}$ on X, i.e. if $\mathcal{E}'$ is a flat family of sheaves on X, parametrized by another scheme S', then there is a unique map $f : S' \to S$ such that $\mathcal{E}' \cong f^* \mathcal{E} := (\mathrm{id}_X \times f)^* \mathcal{E}$.

Theorem 2.2.2. *The class of semistable sheaves on X with Hilbert polynomial P has a coarse moduli space which is projective, i.e. there is a projective scheme M_X^P that corepresents the functor $\mathcal{M}_X^P$. Moreover, the closed points of M_X^P are in bijection with the S-equivalence classes of semistable sheaves.*

Proof   See [Sim94, Theorem 1.21] or [HL10, Theorem 4.3.4].   □

Theorem 2.2.3. *Let $\mathcal{E}$ be a flat family of sheaves on X. If the greatest common divisor of $\{P(0), \ldots, P(\dim(\mathcal{E}))\}$ equals 1 then there is a universal family of semistable sheaves on X with Hilbert polynomial P, i.e. M_X^P is a fine moduli space. In particular, there are no properly semistable sheaves on X.*

Proof   See [HL10, Section 4.6].   □

Corollary 2.2.4. *Let X be a smooth surface and consider the subfunctor $\mathcal{M}_X^P(v) \subset \mathcal{M}_X^P$ of semistable sheaves with a fixed numerical class $v \in K(X)_{\mathrm{num}}$. Let r, c_1, c_2 be the rank and Chern classes corresponding to v. If $\gcd(r, c_1 \cdot \omega, \frac{1}{2} c_1 \cdot (c_1 - K_X) - c_2) = 1$ then there is a universal family of semistable sheaves with Hilbert polynomial P and numerical class v.*

Proof   See [HL10, Corollary 4.6.7].   □

Stability is an open condition in the sense that small deformations of a stable sheaf are again stable. This statement is made precise by the following

Theorem 2.2.5. *Let E be a stable sheaf on X represented by a point $[E] \in M_X^P$. Then*

(a) *the Zariski tangent space of M_X^P at $[E]$ is canonically given by*

$$T_{[E]} M_X^P \cong \mathrm{Ext}^1(E, E).$$

(b) *if E is torsion-free and $\mathrm{Ext}^2(E, E)_0 := \ker(\mathrm{Ext}^2(E, E) \overset{\mathrm{tr}^2}{\to} H^2(\mathcal{O}_X)) = 0$ then M_X^P is smooth at the point $[E]$.*

Proof See [Art89] or [HL10, Corollary 4.5.2 and Theorem 4.5.4]. □

Corollary 2.2.6. *If X is a smooth surface then the dimension of M_X^P at a stable point $[E]$ is bounded below by the expected dimension:*

$$\exp \dim_{[E]} M_X^P := 2\mathrm{rk}(E)c_2(E) - (\mathrm{rk}(E) - 1)c_1(E)^2 - (\mathrm{rk}(E)^2 - 1)\chi(\mathcal{O}_X).$$

Proof See [HL10, p. 114-115]. □

2.3 Fourier-Mukai Transforms

Let X and Y be smooth projective varieties over $\mathbb{C}$.

Definition 2.3.1. The Fourier-Mukai functor corresponding to $\mathcal{P} \in \mathcal{D}(X \times Y)$ (which we often call the kernel) is the integral functor

$$\Phi_{\mathcal{P}} : \mathcal{D}(X) \xrightarrow{\sim} \mathcal{D}(Y) \; ; \; E \mapsto \pi_{2*} \left(\pi_1^*(E) \otimes \mathcal{P} \right)$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the two projection maps which are implicitly understood to be derived; we will always drop the derived signs in this thesis. If this functor is an equivalence of categories, we call it a Fourier-Mukai transform and refer to X and Y Fourier-Mukai partners.

Proposition 2.3.2. *For any $\mathcal{P} \in \mathcal{D}(X \times Y)$ let*

$$\mathcal{P}_L := \mathcal{P}^{\vee} \otimes \pi_2^* \omega_Y [\dim(Y)] \quad \text{and} \quad \mathcal{P}_R := \mathcal{P}^{\vee} \otimes \pi_1^* \omega_X [\dim(X)]$$

where $\mathcal{P}^{\vee}$ is the derived dual with respect to $\mathcal{O}_{X \times Y}$. Then we have the following adjunctions

$$\Phi_{\mathcal{P}_L} \dashv \Phi_{\mathcal{P}} \dashv \Phi_{\mathcal{P}_R}.$$

Proof This follows from Grothendieck-Verdier duality; see [Huy06, Corollary 3.35 & Proposition 5.9] or [Muk81]. □

Proposition 2.3.3. *Let Z be a smooth projective variety and consider $\mathcal{P} \in \mathcal{D}(X \times Y)$ and $\mathcal{Q} \in \mathcal{D}(Y \times Z)$. Then*

$$\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}} \cong \Phi_{\mathcal{R}} : \mathcal{D}(X) \to \mathcal{D}(Z)$$

where

$$\mathcal{R} := \pi_{13*} \left(\pi_{12}^* \mathcal{P} \otimes \pi_{23}^* \mathcal{Q} \right) \in \mathcal{D}(X \times Z)$$

and π_{ij} is the projection from $X \times Y \times Z$ to the ij^{th} factor.

Proof   See [Huy06, Proposition 5.10] or [Muk81, Proposition 1.3].   □

Theorem 2.3.4. *Suppose* $F : \mathcal{D}(X) \xrightarrow{\sim} \mathcal{D}(Y)$ *is an equivalence. Then there exists an object* $\mathcal{P} \in \mathcal{D}(X \times Y)$ *(unique up to isomorphism) such that* $F \simeq \Phi_{\mathcal{P}}$.

Proof   See [Orl97, Theorem 2.18].   □

Remark 2.3.5. If the group of derived autoequivalences $\mathrm{Aut}(\mathcal{D}(X))$ is in some sense a measure of the intrinsic geometry of the variety X then Calabi-Yau varieties turn out to be the most interesting, i.e. those where $\omega_X \simeq \mathcal{O}_X$.

A Fourier-Mukai transform descends in a natural way to a Fourier-Mukai transform at the level of K-groups and cohomology; see [Huy06, Section 5.2]. The way in which the induced transforms interact is given by the infamous

Theorem 2.3.6 (Grothendieck-Riemann-Roch). *Let* $\mathcal{P} \in \mathcal{D}(X \times Y)$. *Then the following diagram is commutative:*

$$
\begin{array}{ccc}
K(X) & \xrightarrow{\Phi^K_{[\mathcal{P}]}} & K(Y) \\
v \downarrow & & \downarrow v \\
H^*(X,\mathbb{Q}) & \xrightarrow{\Phi^H_{v(\mathcal{P})}} & H^*(Y,\mathbb{Q})
\end{array}
$$

where $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$ *for any class* $E \in K(X)$, *i.e.*

$$
\Phi^H_{v(\mathcal{P})}\left(\mathrm{ch}(E)\sqrt{\mathrm{td}(X)}\right) = \mathrm{ch}\left(\Phi^K_{[\mathcal{P}]}(E)\right)\sqrt{\mathrm{td}(Y)}.
$$

Proof   See [Huy06, Corollary 5.29].   □

2.4   Stable Sheaves on Surfaces

Lemma 2.4.1. *Let* X *be a smooth projective surface over* $\mathbb{C}$.

(a) *A sheaf* E *on* X *has pure dimension two if and only if* E *is torsion-free.*

(b) *Any torsion-free sheaf* E *embeds into its reflexive hull* $E^{\vee\vee}$ *such that* $E^{\vee\vee}/E$ *has dimension zero; the support of* $E^{\vee\vee}/E$ *is called the set of singular points of* E. *In other words,* E *is locally-free outside a finite set of points:*

$$
0 \to E \to E^{\vee\vee} \to \mathcal{O}_Z \to 0.
$$

In particular, a torsion-free sheaf of rank one is of the form $L \otimes \mathcal{I}_Z$ *where* L *is a line bundle and* $\mathcal{I}_Z$ *is the ideal sheaf of a codimension two subscheme.*

(c) *If E is torsion-free and $\phi : F \to E$ is any surjection with locally free F then* $\ker(\phi)$ *is also locally-free.*

(d) *If E is torsion-free and $F \subset E$ is locally-free then E/F cannot have torsion supported in dimension zero.*

(e) *A sheaf E on X is locally-free if and only if E is reflexive, i.e. $E \cong E^{\vee\vee}$.*

(f) *The restriction of a locally-free sheaf E on X to any smooth projective curve is again locally-free and the restriction of a torsion-free sheaf E on X to a smooth projective curve avoiding the finitely many singular points of E is locally-free.*

Proof See [HL10, Proposition 1.1.10 and Example 1.1.16]. □

Theorem 2.4.2 (Hodge Index Theorem)**.** *Let H be an ample divisor on a smooth projective surface X and suppose that D is a divisor such that $D \cdot H = 0$. Then $D^2 \leq 0$ with equality if and only if $D \equiv 0$. Moreover, if D is any divisor, then*

$$(D^2)(H^2) \leq (D \cdot H)^2$$

with equality if and only if $D \equiv nH$ for some $n \in \mathbb{Z}$.

Proof. See [Har77, V, Theorem 1.9] for a proof of the first statement. As for the second claim consider $\tilde{D} := (aD + bH)$ where $a, b \in \mathbb{Z}$ are chosen so that $\tilde{D} \cdot H = 0$. Then $\tilde{D}^2 = a^2 D^2 + 2ab D \cdot H + b^2 H^2 \leq 0$ and this quadratic has real roots precisely when the discriminant is non-negative, i.e.

$$(D \cdot H)^2 - (D^2)(H^2) \geq 0.$$

We have equality if and only if $\tilde{D} \equiv 0$, i.e. $D \equiv nH$ for some $n \in \mathbb{Z}$. □

Theorem 2.4.3 (Bogomolov's Inequality)**.** *Let X be a smooth projective surface and H an ample divisor on X. If E is a μ-semistable sheaf on X then*

$$2r(E)\mathrm{ch}_2(E) \leq c_1(E)^2.$$

Proof See [Huy06, Theorem 3.4.1]. □

Definition 2.4.4. Let X be an abelian or K3 surface. The Mukai pairing $\langle -, - \rangle$ is a symmetric bilinear form on the (even part of the) cohomology ring

$$H^{2*}(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$$

defined by the formula

$$\langle (r_1, D_1, s_1), (r_2, D_2, s_2) \rangle = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1.$$

The Mukai vector of an object $E \in \mathcal{D}(X)$ is the element of the sublattice

$$\mathcal{N}(X) = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z} \subset H^*(X, \mathbb{C})$$

defined by the formula

$$v(E) = (\mathrm{rk}(E), c_1(E), s(E)) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)} \in H^*(X, \mathbb{Z})$$

where $s(E) = \mathrm{ch}_2(E) + \epsilon\mathrm{rk}(E)$ and $\epsilon = 0$ or 1 depending on whether X is an abelian or K3 surface. The Riemann-Roch theorem tells us how the Mukai pairing is related to the Euler form. In particular, for any pair of objects $E, F \in \mathcal{D}(X)$ we have

$$\chi(E, F) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_X^i(E, F) = -\langle v(E), v(F) \rangle.$$

See [Huy06, p. 133] for more details.

Lemma 2.4.5. *Let (X, L) be a polarized abelian or K3 surface with $\ell := c_1(L)$.*

(a) *A pure dimension two sheaf E on X is semistable if for all proper subsheaves $0 \neq F \subset E$ one has $\mu(F) \leq \mu(E)$; if $\mu(F) = \mu(E)$ then one has*

$$\frac{\chi(F)}{\mathrm{rk}(F)} \leq \frac{\chi(E)}{\mathrm{rk}(E)}.$$

(b) *A pure dimension one sheaf E on X is semistable if for all proper subsheaves $0 \neq F \subset E$ one has*

$$\frac{\chi(F)}{\mathrm{deg}(F)} \leq \frac{\chi(E)}{\mathrm{deg}(E)}.$$

(c) *A pure dimension zero sheaf E on X is just a sheaf supported at a finite set of points. Any such sheaf is semistable. The only stable pure dimension zero sheaves are the structure sheaves of single (closed) points.*

Proof If $v(E) = (\mathrm{rk}(E), c_1(E), \chi(E) - \epsilon\mathrm{rk}(E))$ then the statements follow immediately from the Hilbert polynomial:

$$
\begin{aligned}
P(E, n) &= \chi\left(E \otimes L^n\right) = -\left\langle v\left(L^{-n}\right), v(E)\right\rangle \\
&= -\left\langle\left(1, -n\ell, \frac{1}{2}(n\ell)^2 + \epsilon\right), (\mathrm{rk}(E), c_1(E), \chi(E) - \epsilon\mathrm{rk}(E))\right\rangle \\
&= \left(\frac{\mathrm{rk}(E)\ell^2}{2}\right) n^2 + (c_1(E) \cdot \ell)\, n + \chi(E).
\end{aligned}
$$

(a) If $\mathrm{rk}(E) \neq 0$ then

$$
p(E, n) = n^2 + \frac{2\mu(E)}{\ell^2} n + \frac{2\chi(E)}{\ell^2 \mathrm{rk}(E)}.
$$

(b) If $\mathrm{rk}(E) = 0$ and $c_1(E) \neq 0$ then

$$
p(E, n) = n + \frac{\chi(E)}{\deg(E)}.
$$

(c) If $\mathrm{rk}(E) = 0$, $c_1(E) = 0$ and $\chi(E) \neq 0$ then

$$
p(E, n) = 1.
$$

In particular, the Hilbert polynomial is the constant polynomial and any zero-dimensional sheaf is semistable. Moreover, it can only be stable if it has no proper subsheaves at all.

$\square$

Lemma 2.4.6. *Let X be an abelian or K3 surface. If $E \in \mathcal{D}(X)$ is simple then*

$$
\dim_{\mathbb{C}}\mathrm{Ext}^1_X(E, E) = 2 + v(E)^2 \geq 0,
$$

with equality precisely when E is spherical.

Proof See [Muk87b, Corollary 2.5] and [Bri08, Lemma 5.1]. $\square$

We say that an object $E \in \mathcal{D}(X)$ is rigid if $\dim_{\mathbb{C}} \mathrm{Ext}^1_X(E, E) = 0$. If E is simple then being rigid is equivalent to $v(E)^2 = -2$. Similarly, we say that a simple object E is semi-rigid if $\dim \mathrm{Ext}^1_X(E, E) = 2 \Leftrightarrow v(E)^2 = 0$. Semi-rigid objects give rise to Fourier-Mukai transforms:

Theorem 2.4.7. *Let X be an abelian or K3 surface such that $\mathrm{NS}(X) = \mathbb{Z}[\ell]$. If* $\mathrm{FM}(X)$ *denotes the set of Fourier-Mukai partners of X and $\mathcal{M}(v)$ denotes the moduli space of μ-stable sheaves with Mukai vector v, then*

$$\mathrm{FM}(X) = \{\mathcal{M}(r, c\ell, \chi) \mid 2r\chi = c^2\ell^2,\ (r, \chi) = 1,\ r \geq \chi\}.$$

Proof   This result is a slight refinement of Mukai's result; see [Muk87b] and [Muk78, Remark 7.13]. The statement above appears as [HLOY03, Theorem 2.1]. □

2.5   Abelian Surfaces

All the results in this section are quoted in the context of surfaces but many of them hold more generally for abelian varieties of dimension g.

Definition 2.5.1. Let $\mathbb{T}$ be an abelian surface over $\mathbb{C}$ and let $\hat{\mathbb{T}}$ denote the dual abelian variety, i.e. the smooth projective surface that represents the Picard functor

$$\mathrm{Pic}^0_{\mathbb{T}} : S \mapsto \mathrm{Pic}^0(S \times \mathbb{T}) := \{L \in \mathrm{Pic}(S \times \mathbb{T})\ :\ c_1(L_s) = 0 \text{ for all } s \in S\} / \sim .$$

In particular, $\hat{\mathbb{T}}$ is a fine moduli space and so $\mathbb{T} \times \hat{\mathbb{T}}$ carries a universal line bundle $\mathcal{P}$. This is called the Poincaré bundle and is uniquely determined by two conditions:

- If $\hat{x} \in \hat{\mathbb{T}}$ corresponds to a line bundle $L \in \mathrm{Pic}(\mathbb{T})$ then $\mathcal{P}_{\mathbb{T} \times \{\hat{x}\}} \cong L$.

- The fibre over the identity $e \in \mathbb{T}$ is trivial, i.e. $\mathcal{P}_{\{e\} \times \hat{\mathbb{T}}} \cong \mathcal{O}_{\hat{\mathbb{T}}}$.

Theorem 2.5.2. *Let $\mathcal{P} \in \mathcal{D}(\mathbb{T} \times \hat{\mathbb{T}})$ be the Poincaré bundle on $\mathbb{T} \times \hat{\mathbb{T}}$. Then $\Phi_{\mathcal{P}} : \mathcal{D}(\mathbb{T}) \xrightarrow{\sim} \mathcal{D}(\hat{\mathbb{T}})$ is an equivalence and*

$$\hat{\Phi}_{\mathcal{P}} \circ \Phi_{\mathcal{P}} \simeq (-1_{\mathbb{T}})^*[-2]$$

where $\hat{\Phi}_{\mathcal{P}} : \mathcal{D}(\hat{\mathbb{T}}) \xrightarrow{\sim} \mathcal{D}(\mathbb{T})$ and $(-1_{\mathbb{T}})$ is the morphism of group inversion. We call $\Phi_{\mathcal{P}}$ the standard Fourier-Mukai transform.

Proof   See [Muk81, Theorem 2.2]. □

Corollary 2.5.3. *Let F be a coherent sheaf on $\mathbb{T}$. Then there is a spectral sequence*

$$E_2^{p,q} = \hat{\Phi}_{\mathcal{P}}^p\left(\Phi_{\mathcal{P}}^q(F)\right) \Rightarrow \left(\hat{\Phi}_{\mathcal{P}} \circ \Phi_{\mathcal{P}}\right)^{p+q}(F) = \begin{cases} (-1)_{\mathbb{T}}^* F & \text{if } p+q = 2 \\ 0 & \text{o/w.} \end{cases}$$

with the following properties:

$$\hat{\Phi}_{\mathcal{P}}^{p}\left(\Phi_{\mathcal{P}}^{0}(F)\right) = 0 \quad \text{if } p = 0, 1 \qquad \hat{\Phi}_{\mathcal{P}}^{p}\left(\Phi_{\mathcal{P}}^{2}(F)\right) = 0 \quad \text{if } p = 1, 2$$

$$\hat{\Phi}_{\mathcal{P}}^{0}\left(\Phi_{\mathcal{P}}^{1}(F)\right) \hookrightarrow \hat{\Phi}_{\mathcal{P}}^{2}\left(\Phi_{\mathcal{P}}^{0}(F)\right) \qquad \hat{\Phi}_{\mathcal{P}}^{0}\left(\Phi_{\mathcal{P}}^{2}(F)\right) \twoheadrightarrow \hat{\Phi}_{\mathcal{P}}^{2}\left(\Phi_{\mathcal{P}}^{1}(F)\right).$$

We call this the Mukai spectral sequence.

Definition 2.5.4. A coherent sheaf E on X is said to be WIT_i with respect to a Fourier-Mukai transform $\Phi_{\mathcal{P}} : \mathcal{D}(X) \xrightarrow{\sim} \mathcal{D}(Y)$ if

$$\Phi_{\mathcal{P}}^{j}(E) := R^{j}\Phi_{\mathcal{P}}(E) = 0 \text{ for all } j \neq i.$$

We say that E is WIT if it is WIT_i for some i and denote the transform $\Phi_{\mathcal{P}}^{i}(E)$ of E by $\hat{E}$; embedding $\text{Coh}(X) \hookrightarrow \mathcal{D}(X)$ as complexes concentrated in degree zero, we see that $\Phi_{\mathcal{P}}(E) \cong \hat{E}[-i]$ if E is WIT_i. Furthermore, E is said to be IT_i if for all $y \in \hat{\mathbb{T}}$,

$$H^{j}(X, E \otimes \mathcal{P}_y) = 0 \text{ for all } j \neq i.$$

Since $\Phi_{\mathcal{P}}^{j}(E)_y \cong H^{j}(E \otimes \mathcal{P}_y)$, we see by base change ([Har77, III, Theorem 12.11]) that if E is IT_i then E is WIT_i and $\hat{E}$ is locally-free.

Corollary 2.5.5. *Let E be a WIT_i sheaf on $\mathbb{T}$. Then $\hat{E}$ is a WIT_{2-i} sheaf on $\hat{\mathbb{T}}$ and $\hat{\hat{E}} \cong (-1_{\mathbb{T}})^{*}E$. Moreover, if E is WIT_2 then $\hat{E}$ is locally-free.*

Proof See [Muk81, Corollary 2.4]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Example 2.5.6. Let $\mathcal{O}_x$ be the one dimensional skyscraper sheaf supported at $x \in \mathbb{T}$. Since $H^{i}(\mathbb{T}, \mathcal{O}_x \otimes \mathcal{P}_{\hat{y}}) = 0$ for every $i > 0$ and $\hat{y} \in \hat{\mathbb{T}}$, we see that $\mathcal{O}_x$ is IT_0 and $\widehat{\mathcal{O}_x} \cong \mathcal{P}_x$. Hence by Corollary 2.5.5, $\mathcal{P}_x$ is WIT_2 and $\widehat{\mathcal{P}_x} \cong \mathcal{O}_{-x}$. Note that $\mathcal{P}_x$ is not IT. See [Muk81, Example 2.6].

Theorem 2.5.7 (Parseval's Theorem)**.** *Suppose $\Phi_{\mathcal{P}} : \mathcal{D}(X) \xrightarrow{\sim} \mathcal{D}(Y)$ is an equivalence. Then for any $E, F \in \mathcal{D}(X)$, we have*

$$\text{Ext}_{X}^{i}(E, F) \cong \text{Ext}_{Y}^{i}(\Phi_{\mathcal{P}}(E), \Phi_{\mathcal{P}}(F)).$$

In particular, if E and F are WIT_j and WIT_k respectively, then

$$\text{Ext}_{X}^{i}(E, F) \cong \text{Ext}_{Y}^{i+j-k}(\hat{E}, \hat{F}) \quad \text{for all } i \in \mathbb{Z}.$$

Proof We have the following natural isomorphisms

$$
\begin{aligned}
\mathrm{Ext}^i_X(E, F) \quad &\cong \quad \mathrm{Hom}_{\mathcal{D}(X)}(E, F[i]) \\
&\cong \quad \mathrm{Hom}_{\mathcal{D}(Y)}(\Phi_{\mathcal{P}}(E), \Phi_{\mathcal{P}}(F)[i]) \\
&\cong \quad \mathrm{Hom}_{\mathcal{D}(Y)}(\hat{E}[-j], \hat{F}[i-k]) \\
&\cong \quad \mathrm{Ext}^{i+j-k}_Y(\hat{E}, \hat{F}).
\end{aligned}
$$

See [Muk81, Corollary 2.5]. □

Lemma 2.5.8. *Let* $\mathcal{P} \in \mathcal{D}(\mathbb{T} \times \hat{\mathbb{T}})$ *be the Poincaré bundle on* $\mathbb{T} \times \hat{\mathbb{T}}$. *Poincaré duality and the cohomological Fourier-Mukai transform compare via*

$$
\Phi^H_{\mathcal{P}} = (-1)^{\frac{n(n+1)}{2}} \cdot \mathrm{PD}_n : H^n(\mathbb{T}, \mathbb{Q}) \xrightarrow{\sim} H^{4-n}(\hat{\mathbb{T}}, \mathbb{Q}) = H^{4-n}(\mathbb{T}, \mathbb{Q})^*.
$$

In particular, the cohomological Fourier-Mukai transform defines an isomorphism of integral(!) cohomology

$$
\Phi^H_{\mathcal{P}} : H^*(\mathbb{T}, \mathbb{Z}) \xrightarrow{\sim} H^*(\hat{\mathbb{T}}, \mathbb{Z}) \; ; \; \alpha \mapsto \pi_{2*}\left(\pi_1^*(\alpha).\mathrm{ch}(\mathcal{P})\right) \quad \text{and} \quad \hat{\Phi}^H_{\mathcal{P}} \circ \Phi^H_{\mathcal{P}} = (-1)^n.
$$

Proof See [Huy06, Lemma 9.23 and Corollary 9.24]. □

Corollary 2.5.9. *If* E *is a coherent sheaf on* $\mathbb{T}$ *then*

$$
\mathrm{PD}\left(\mathrm{ch}_{2-i}(E)\right) = \sum_j (-1)^{i+j}\mathrm{ch}_i\left(\Phi^j_{\mathcal{P}}(E)\right).
$$

In particular, if E *is* WIT_k *then*

$$
\mathrm{ch}_i(\hat{E}) = (-1)^{i+k}\mathrm{PD}\left(\mathrm{ch}_{2-i}(E)\right)
$$

Proof Follows immediately from Theorem 2.3.6 and Lemma 2.5.8. □

Lemma 2.5.10. *Let* $\tau_x : \mathbb{T} \to \mathbb{T}$; $y \mapsto x + y$ *be the translation morphism. Then for any* $x \in \mathbb{T}$ *and* $\hat{x} \in \hat{\mathbb{T}}$ *we have the following isomorphisms of functors*

$$
\begin{aligned}
\Phi_{\mathcal{P}} \circ \tau_x^* \quad &\simeq \quad (- \otimes \mathcal{P}_{-x}) \circ \Phi_{\mathcal{P}} \\
\Phi_{\mathcal{P}} \circ (- \otimes \mathcal{P}_{\hat{x}}) \quad &\simeq \quad \tau_{\hat{x}}^* \circ \Phi_{\mathcal{P}}.
\end{aligned}
$$

Proof See [Muk81, Section 3.1]. □

Proposition 2.5.11. *Suppose* $\mathrm{Pic}(\mathbb{T}) \cong \mathbb{Z}$ *and let* $\mathcal{E} \in \mathcal{D}(\mathbb{T} \times \hat{\mathbb{T}})$ *be an object such that* $\Phi_{\mathcal{E}} : \mathcal{D}(\mathbb{T}) \xrightarrow{\sim} \mathcal{D}(\hat{\mathbb{T}})$ *is an equivalence. Then, up to a shift,* $\mathcal{E}$ *is isomorphic to a sheaf. Moreover, every such* $\mathcal{E}$ *(except for* $\mathcal{O}_{\Delta}$*) is locally-free and we can index the set of such* $\mathcal{E}$*'s (up to equivalence) by their slope* $\mu(\mathcal{E}_x) \in \mathbb{Q} \cup \infty$.

Proof See [Orl02, Proposition 3.2]. For the second part observe that any $\mathcal{E}_x$ must be semi-homogeneous. That is, if $\mathrm{ch}(\mathcal{E}_x) = (r, c, \chi)$ then we must have $c^2 = r\chi$. In [Muk87b], Mukai shows that the corresponding moduli space is fine if and only if $\gcd(r, c, \chi) = 1$. This forces us to have $r = a^2$, $\chi = b^2$ and $c = \pm ab$ for coprime integers $a > 0$ and b which allows us to associate a unique rational number to $\mathcal{E}$. $\qquad\square$

2.6 Preservation of Stability

General philosophy asserts that

> "Stability is preserved under Fourier-Mukai transforms."

Given the many different notions of stability, this statement is quite vague and several people have studied the following question:

> When is the transform $\Phi_{\mathcal{P}}(E)$ of a μ-stable sheaf E again a μ-stable sheaf?

Under 'suitable' conditions, the philosophy holds true but it is not difficult to construct counter-examples, i.e. μ-stable sheaves which become unstable after applying a Fourier-Mukai transform.

Definition 2.6.1. Let $(\mathbb{T}, L)$ be an irreducible principally polarized abelian surface over $\mathbb{C}$ with $\ell := c_1(L)$ and $\mathrm{Pic}(\mathbb{T}) = \mathbb{Z}[\ell]$. In other words, L is an ample line bundle with $\chi(L) = 1$ and $\phi_L : \mathbb{T} \xrightarrow{\sim} \hat{\mathbb{T}}$; $x \mapsto \tau_x^* L \otimes L^*$. This identification allows us to view the standard Fourier-Mukai transform as an autoequivalence of $\mathcal{D}(\mathbb{T})$. To be more precise, let Φ be the Fourier-Mukai transform with kernel $(\mathrm{id}_{\mathbb{T}} \times \phi_L)^* \mathcal{P}$ or, equivalently

$$\Phi := \phi_L^* \circ \Phi_{\mathcal{P}} : \mathcal{D}(\mathbb{T}) \xrightarrow{\sim} \mathcal{D}(\mathbb{T}).$$

Applying Riemann-Roch to the principal polarization we see that $1 = \chi(L) = \ell^2/2$, i.e. $\ell^2 = 2$ and $\deg(E) := c_1(E) \cdot \ell \in 2\mathbb{Z}$ for any sheaf E on $\mathbb{T}$. Let D_L denote the zero set of the unique holomorphic section of L, i.e. $L = \mathcal{O}(D_L)$. Translations of D_L by $x \in \mathbb{T}$ are given by $D_x := \tau_x D_L$ and we make a note of the fact that $D_x \in |\tau_{-x}^* L| \cong |L\mathcal{P}_{-\hat{x}}|$. Let $\mathrm{Hilb}^n(\mathbb{T})$ be the Hilbert scheme of length

n subschemes of $\mathbb{T}$. If $X \subset \mathbb{T}$ is a finite subscheme of length n, we shall abuse notation and denote the corresponding point in $\mathrm{Hilb}^n(\mathbb{T})$ by X as well. Following [Mac11], we make the following definition

$$X \in \mathrm{Hilb}^n(\mathbb{T}) \text{ is collinear if } X \subset D_x \text{ for some } x \in \mathbb{T}.$$

Our convention will be to use the letters P, Q, Y, Z, W, X to denote zero-dimensional subschemes of length $1, 2, 3, 4, 5, n$ respectively. Lastly, if F and G are sheaves then $E = F \ltimes G$ will denote a representative of the equivalence class of non-split extensions of F by G, i.e. a short exact sequence of the form

$$0 \to G \to E \to F \to 0.$$

Also, for convenience, we shall often drop the tensor product sign between sheaves.

Lemma 2.6.2. *Let E be semistable sheaf on $\mathbb{T}$ with positive degree. Then for all $\hat{x} \in \hat{\mathbb{T}}$ we have $H^2(\mathbb{T}, E\mathcal{P}_{\hat{x}}) = 0$ and so $\Phi^2(E) = 0$.*

Proof By Serre duality and Proposition 2.1.2 (a), we have

$$H^2(\mathbb{T}, E\mathcal{P}_{\hat{x}}) \cong \mathrm{Ext}^2_{\mathbb{T}}(\mathcal{O}_{\mathbb{T}}, E\mathcal{P}_{\hat{x}}) \cong \mathrm{Hom}(E, \mathcal{P}_{-\hat{x}}) = 0.$$

The second statement follows from base change ([Har77, III, Theorem 12.11]); that is, $\Phi^2(E)_{\hat{x}} \cong H^2(\mathbb{T}, E\mathcal{P}_{\hat{x}})$. $\square$

Lemma 2.6.3. *Let $0 \to A \to E \to B \to 0$ be a μ-destabilising sequence for a sheaf $E = F \ltimes G$ where F and G are μ-stable sheaves with $\mu(G) < \mu(E) < \mu(F)$. Then we have the following chain of inequalities*

$$\mu(G) < \mu(B) \leq \mu(E) \leq \mu(A) < \mu(F).$$

Proof By definition of a μ-destabilising sequence for E, we have $\mu(B) \leq \mu(E) \leq \mu(A)$ and so it remains to show that $\mu(A) < \mu(F)$ and $\mu(G) < \mu(B)$. Replacing A by one of the sheaves in the associated graded object of a factor in its Harder-Narasimhan filtration, we can assume A to be stable. If $\mu(A) > \mu(F)$ then $\mathrm{Hom}(A, F) = 0 = \mathrm{Hom}(A, G)$ by Proposition 2.1.2(a) and so $\mathrm{Hom}(A, E) = 0$; contradiction. Similarly, if $\mu(A) = \mu(F)$ then $A \cong F$ by Proposition 2.1.2(b) which would provide a splitting of the extension contradicting our assumption. Therefore, $\mu(A) < \mu(F)$. For the last part, define the following additive function

$$Z : \mathrm{Coh}(\mathbb{T}) \to \mathbb{C} \; ; \; E \mapsto -\deg(E) + i\,\mathrm{rk}(E) \in \exp(i\pi\phi(E)) \cdot \mathbb{R}_{\geq 0}$$

for some $\phi(E) \in (0, 1]$ which we call the phase of E. That is, Z sends a short exact sequence $0 \to G \to E \to F \to 0$ to a parallelogram. Observe that F is μ-stable if and only if $\phi(F') < \phi(F)$ for all proper subsheaves $0 \neq F' \subset F$. Indeed, $Z(F)/\mathrm{rk}(F) = -\mu(F) + i$ and $\mu(F') < \mu(F) \Leftrightarrow -\mu(F') > -\mu(F) \Leftrightarrow \phi(F') < \phi(F)$. Let $f : A \to F$ be the composite map and set $K := \ker(f)$ and $I := \mathrm{Im}(f)$. Then $K \hookrightarrow A$ lifts to an injection $K \hookrightarrow G$ with quotient J (say), i.e. $\phi(K) < \phi(G)$ and K is confined by two parallelograms ensuring the desired inequality, $\mu(G) < \mu(B)$.

$\square$
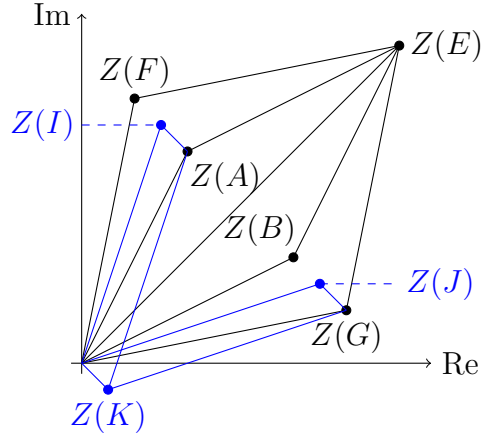
Lemma 2.6.4. *Let $(\mathbb{T}, L)$ be an irreducible principally polarized abelian surface and $r \in \mathbb{Z}_{>0}$. Let F be a μ-stable, WIT_1 sheaf with $\mathrm{ch}(F) = (r, \ell, -(r+1))$ and μ-stable transform. Then $E = F \ltimes \hat{F}$ is a μ-stable sheaf which, under the standard Fourier-Mukai transform, becomes unstable.*

Proof. The WIT_1 condition tells us that $v(\hat{F}) = (r+1, \ell, -r)$. Observe that there are indeed non-trivial extensions of F by $\hat{F}$ since

$$
\begin{aligned}
\chi(F, \hat{F}) &= -\langle v(F), v(\hat{F}) \rangle \\
&= -\langle (r, \ell, -(r+1)), (r+1, \ell, -r) \rangle \\
&= -2 - r^2 - (r+1)^2 < 0 \quad \Rightarrow \quad \dim_{\mathbb{C}} \mathrm{Ext}^1(F, \hat{F}) \neq 0.
\end{aligned}
$$

Since F (and hence $\hat{F}$) is WIT_1 we have that $E = F \ltimes \hat{F}$ is WIT_1 also.

Suppose $0 \to A \to E \to B \to 0$ is a μ-destabilising sequence for E. Then, by Lemma 2.6.3, we have the following inequality

$$
\mu(E) \leq \mu(A) < \mu(F) \quad \Leftrightarrow \quad 0 < \frac{4 \cdot \mathrm{rk}(A)}{2r+1} \leq \deg(A) < \frac{2 \cdot \mathrm{rk}(A)}{r}.
$$

Since A is a subobject of E we have $1 \leq \mathrm{rk}(A) \leq 2r$. For $\mathrm{rk}(A) \leq r$, the upper bound is at most 2 and so $0 < \deg(A) < 2$; contradiction. However, when $r + 1 \leq \mathrm{rk}(A) \leq 2r$, the lower bound is strictly greater than 2 and the upper bound is at most 4, i.e. $2 < \deg(A) < 4$; contradiction. Therefore, destabilising objects for E cannot exist and so we deduce that it is μ-stable.

Explicitly, we have an extension $\hat{F} \to E \to F$ with Chern characters

$$(r + 1, \ell, -r) \to (2r + 1, 2\ell, -(2r + 1)) \to (r, \ell, -(r + 1))$$

and increasing slopes

$$\mu(\hat{F}) = \frac{2}{r + 1} \to \mu(E) = \frac{4}{2r + 1} \to \mu(F) = \frac{2}{r} \quad (\nearrow).$$

Applying the standard Fourier-Mukai transform gives rise to the flipped extension $F \to \hat{E} \to \hat{F}$ where the Chern characters have been reversed and $\hat{E}$ is destabilised by F. $\qquad \square$

Remark 2.6.5. Experiments with Maple suggest that the previous Lemma should be true when $v(F) = (r, d\ell, -(r + 1))$ and $d|r$ or $d|(r + 1)$. In fact, if $v(F) = (r, d\ell, -k)$, we expect a generic element of the space of extensions of F by $\hat{F}$ to be a μ-stable sheaf with unstable transform.

The main issue with Lemma 2.6.4 is whether we can find a coherent sheaf F which satisfies all the hypotheses. For the case when $r = 1$, we can appeal to the following

Theorem 2.6.6. *Let $X \in \mathrm{Hilb}^n(\mathbb{T})$ for $n \geq 2$. Then $L\mathcal{I}_X$ is a μ-stable WIT_1 sheaf with μ-stable transform.*

Proof See [Muk87a, Theorem 0.3] or [Mac11, Theorem 11.1]. $\qquad \square$

Corollary 2.6.7. *There exist μ-stable sheaves E with $\mathrm{ch}(E) = (3, 2\ell, -3)$ which, under the standard Fourier-Mukai transform, become unstable.*

Proof Let $Y \in \mathrm{Hilb}^3(\mathbb{T})$ and apply Lemma 2.6.4 to $F := L\mathcal{I}_Y$. $\qquad \square$

For $r = 2$, we can utilise the following

Lemma 2.6.8. *Let $\mathcal{M}(2, \ell, -3)$ denote the moduli space of μ-stable sheaves F on $\mathbb{T}$ with $\mathrm{ch}(F) = (2, \ell, -3)$. Then it is always possible to choose an IT_1 sheaf $F \in \mathcal{M}(2, \ell, -3)$ with μ-stable transform.*

Proof By [Yos01, Theorem 0.1] we know that this moduli space is non-empty. Since $\deg(F) = 2 > 0$ we have $H^2(F\mathcal{P}_{\hat{x}}) = 0$ for all $\hat{x} \in \hat{\mathbb{T}}$ by Lemma 2.6.2. Suppose F is not IT$_1$ then $H^0(F\mathcal{P}_{-\hat{x}}) \neq 0$ for some $\hat{x} \in \hat{\mathbb{T}}$, i.e. there is a non-trivial map $\mathcal{P}_{\hat{x}} \to F$ with torsion-free quotient Q. Indeed, if Q had a torsion subsheaf T supported on a curve then $\ker(F \to Q/T) \cong L$ by Hilbert's syzygy theorem which would contradict the stability of F. Similarly, if T was supported on points then the length of the subscheme must be zero. Therefore, by Lemma 2.4.1 (b), $Q \cong L\mathcal{I}_Z\mathcal{P}_{\hat{y}}$ for some $Z \in \mathrm{Hilb}^4(\mathbb{T})$. Next consider the family of extensions $0 \to \mathcal{P}_{\hat{x}} \to F \to L\mathcal{I}_Z\mathcal{P}_{\hat{y}} \to 0$ and observe that

$$
\begin{aligned}
\dim \mathbb{P}\mathrm{Ext}^1(L\mathcal{I}_Z\mathcal{P}_{\hat{y}}, \mathcal{P}_{\hat{x}}) &= -\chi(L\mathcal{I}_Z\mathcal{P}_{\hat{y}}, \mathcal{P}_{\hat{x}}) + \dim \mathrm{Ext}^2(L\mathcal{I}_Z\mathcal{P}_{\hat{y}}, \mathcal{P}_{\hat{x}}) - 1 \\
&= \langle v(L\mathcal{I}_Z\mathcal{P}_{\hat{y}}), v(\mathcal{P}_{\hat{x}})\rangle + \dim \mathrm{Hom}(\mathcal{P}_{\hat{x}}, L\mathcal{I}_Z\mathcal{P}_{\hat{y}}) - 1 \\
&= \begin{cases} 2 & \text{if } Z \text{ is generic} \\ 3 & \text{if } Z \text{ is collinear.} \end{cases}
\end{aligned}
$$

Since $\dim \mathcal{M}(2, \ell, -3) = 2 + v(F)^2 = 16$ and $\mathcal{P}_{\hat{x}}$ moves in a 2-dimensional family whilst $L\mathcal{I}_Z$ moves in a 10-dimensional family, we see that the space of all such extensions is at least codimension 1 inside $\mathcal{M}(2, \ell, -3)$. In other words, a generic element of $\mathcal{M}(2, \ell, -3)$ will be IT$_1$.

Choose $F \in \mathcal{M}(2, \ell, -3)$ to be IT$_1$ and suppose that $\hat{F}$ is not μ-stable. Then there is a short exact sequence $0 \to A \to \hat{F} \to B \to 0$ with $\mu(A) \geq \mu(\hat{F}) \geq \mu(B)$. Thus, $\deg(A) \geq 2\mathrm{rk}(A)/3 > 0$ which implies $\Phi^2(A) = 0$ by Lemma 2.6.2, i.e. A is WIT$_1$. Replacing B by one of the sheaves in the associated graded object of a factor in its Harder-Narasimhan filtration, we can assume B to be stable with $\deg(B) \leq 0$. Therefore, by a similar argument to Lemma 2.6.2, we see that $H^0(B\mathcal{P}_{-\hat{x}}) = 0$ for all $\hat{x} \in \hat{\mathbb{T}}$ since if it were non-zero then $B \cong \mathcal{P}_{\hat{x}}$ would contradict the fact that $\Phi^2(B) = 0$, i.e. B is WIT$_1$ as well. Thus, applying Φ to the destabilising sequence gives rise to the following short exact sequence

$$0 \to \hat{A} \to F \to \hat{B} \to 0.$$

But F is μ-stable which implies $\mu(\hat{A}) < \mu(F) = 1 \Leftrightarrow \deg(\hat{A}) < \mathrm{rk}(\hat{A})$. However, $\deg(\hat{A}) = \deg(A) \geq 2$ and so $\mathrm{rk}(\hat{A}) > 2$; contradiction. $\qquad\square$

Corollary 2.6.9. *There exist μ-stable sheaves E with $\mathrm{ch}(E) = (5, 2\ell, -5)$ which, under the standard Fourier-Mukai transform, become unstable.*

Proof By Lemma 2.6.8, we can choose a μ-stable WIT$_1$ sheaf F with $\mathrm{ch}(F) = (2, \ell, -3)$ and μ-stable transform. Take such an F and apply Lemma 2.6.4. $\qquad\square$

If we replace μ-stable by stable in the hypothesis of Lemma 2.6.4 then we can also consider the case when $r = 0$.

Lemma 2.6.10. *Let $P \in \mathrm{Hilb}^1(\mathbb{T})$. Then $L\mathcal{I}_P$ is WIT_1 with $\widehat{L\mathcal{I}_P} \cong \mathcal{O}_{\hat{D}_{-p}}\mathcal{P}_p$ a line bundle of degree zero supported on the divisor $D_{-p} := \tau_{-p}D_L$.*

Proof See [Mac11, Section 5]. Applying Φ to the twisted structure sequence of $P \in \mathbb{T}$ yields

$$0 \to \Phi^0(L\mathcal{I}_P) \to \hat{L}^{-1} \to \mathcal{P}_p \to \Phi^1(L\mathcal{I}_P) \to 0.$$

Since $\hat{L}^{-1} \to \mathcal{P}_p$ is a non-zero map from a rank one torsion-free sheaf to a torsion-free sheaf, we see that it must be an injection and $\Phi^0(L\mathcal{I}_P) = 0$. Therefore, $L\mathcal{I}_P$ is WIT_1 and by Lemma 2.4.1 (d) & (f) we have $\widehat{L\mathcal{I}_P} \cong \mathcal{P}_p/\hat{L}^{-1}$ is a locally-free sheaf supported on $D_{-p} \in |\hat{L}\mathcal{P}_p| \cong |\tau_p^*\hat{L}|$. Since $\mathrm{ch}(\mathcal{P}_p/\hat{L}^{-1}) = (0, \ell, -1)$ we can use Riemann-Roch to conclude that $\widehat{L\mathcal{I}_P}$ has rank one and degree zero. $\square$

Corollary 2.6.11. *There exist stable sheaves E with $\mathrm{ch}(E) = (1, 2\ell, -1)$ which, under the standard Fourier-Mukai transform, become unstable.*

Proof Apply Lemma 2.6.4 (with μ-stable replaced by stable) to $F := \mathcal{O}_{D_x}$. In particular, observe that $E = F \ltimes \hat{F}$ is rank one and torsion-free, i.e. E is μ-stable and hence stable. The transform is destabilised by its torsion. $\square$

Corollary 2.6.12. *$\mathcal{O}_{D_z}(-n)$ is WIT_1 with $\widehat{\mathcal{O}_{D_z}(-n)}$ torsion-free for all $n \in \mathbb{Z}_{>0}$.*

Proof Choose $X \in \mathrm{Hilb}^n(\mathbb{T})$ such that $X \subset D_z$. Then applying Φ to the structure sequence $0 \to \mathcal{O}_{D_z}(-n) \to \mathcal{O}_{D_z} \to \mathcal{O}_X \to 0$ yields

$$0 \to \mathcal{H}_X \to \Phi^1(\mathcal{O}_{D_z}(-n)) \to \hat{L}\mathcal{P}_z\mathcal{I}_{-\hat{z}} \to 0$$

since sheaves supported in dimension zero are WIT_0 and we see that $\widehat{\mathcal{O}_{D_z}(-n)}$ cannot have torsion since $\hat{L}\mathcal{P}_z\mathcal{I}_{-\hat{z}}$ and $\mathcal{H}_X$ are both torsion-free. $\square$

Corollary 2.6.13. *$\mathcal{O}_{D_z}(1)$ is WIT_1 with $\widehat{\mathcal{O}_{D_z}(1)} \cong \mathcal{O}_{\hat{D}_{x-z}}(1)$ for some $x \in \mathbb{T}$.*

Proof Applying the transform to $0 \to \mathcal{O}_{D_z} \to \mathcal{O}_{D_z}(1) \to \mathcal{O}_x \to 0$ yields

$$0 \to \Phi^0(\mathcal{O}_{D_z}(1)) \to \mathcal{P}_x \to \hat{L}\mathcal{P}_z\mathcal{I}_{-\hat{z}} \to \Phi^1(\mathcal{O}_{D_z}(1)) \to 0.$$

Since $\mathcal{P}_x \to \Phi^1(\mathcal{O}_{D_z})$ is a non-zero[1] map from a rank one torsion-free sheaf to a torsion-free sheaf, we see that it must be an injection and $\Phi^0(\mathcal{O}_{D_z}(1)) = 0$. Therefore, $\mathcal{O}_{D_z}(1)$ is WIT$_1$ and by Lemma 2.4.1 (d) & (f) we have $\widehat{\mathcal{O}_{D_z}(1)} \cong L\mathcal{P}_z\mathcal{I}_{-\hat{z}}/\mathcal{P}_x$ is a locally-free sheaf supported on $D_{x-z} \in |\hat{L}\mathcal{P}_{z-x}|$. Since $\mathrm{ch}(L\mathcal{P}_z\mathcal{I}_{-\hat{z}}/\mathcal{P}_x) = (0, \ell, 0)$ we can use Riemann-Roch to conclude that $\widehat{\mathcal{O}_{D_z}(1)}$ has rank one and degree -1. $\qquad\square$

---

[1]If it were zero it would contradict Parseval's relation on the connecting homomorphism $\mathcal{O}_x \to \mathcal{O}_{D_z}[1]$ in $\mathcal{D}(\mathbb{T})$ which is non-zero for non-split extensions by definition.

Chapter 3

Stability Conditions on Smooth Projective Surfaces

3.1 Basic Construction of Stability Conditions

We begin this Chapter with a brief summary of the theory of stability conditions on triangulated categories; see [Bri07] and [Bri08] for more details. Throughout this section, let X be an abelian or K3 surface.

Definition 3.1.1. A torsion pair in an abelian category $\mathcal{A}$ is a pair of full subcategories $\mathcal{T}, \mathcal{F} \subset \mathcal{A}$ which satisfy $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and such that every object $E \in \mathcal{A}$ fits into a short exact sequence

$$0 \to T \to E \to F \to 0$$

for some $T \in \mathcal{T}$ and $F \in \mathcal{F}$. The objects of $\mathcal{T}$ and $\mathcal{F}$ are called torsion and torsion-free respectively.

Theorem 3.1.2. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. If $\mathcal{A}$ is the heart of a bounded t-structure on a triangulated category $\mathcal{D}$, then the full subcategory*

$$\mathcal{A}^{\sharp} = \{E \in \mathcal{D} \mid H^{-1}(E) \in \mathcal{F},\ H^0(E) \in \mathcal{T},\ H^i(E) = 0\ \text{for}\ i \neq -1, 0\}$$

is the heart of another t-structure on $\mathcal{D}$. In particular, $\mathcal{A}^{\sharp}$ is an abelian category.

Proof See [HRS96, Proposition 2.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Thus, any object $E \in \mathcal{A}^{\sharp}$ is isomorphic to a complex of the form

$$E^{-1} \xrightarrow{\ f\ } E^0$$

with $\mathrm{coker}(f) \in \mathcal{T}$ and $\ker(f) \in \mathcal{F}$. That is, we have a short exact sequence of

objects in $\mathcal{A}^{\sharp}$

$$0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0.$$

One says that $\mathcal{A}^{\sharp}$ is obtained from the category $\mathcal{A}$ by tilting with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$. Note that the pair $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in $\mathcal{A}^{\sharp}$ and that tilting with respect to this pair gives you back the original category $\mathcal{A}$ with a shift, namely $\mathcal{A}[1]$. A good way to think about tilting is illustrated by Bridgeland's 'filmstrip' picture:
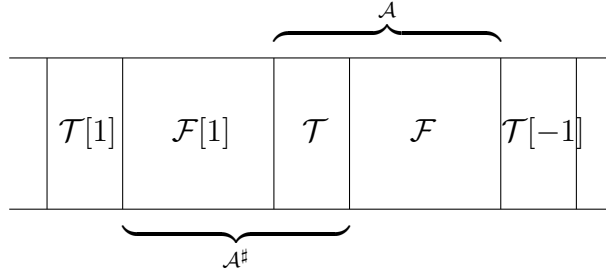
Remark 3.1.3. A short exact sequence $0 \to K \to E \to Q \to 0$ of objects in $\mathcal{A}^{\sharp}$ gives rise to a long exact sequence of objects of $\mathcal{A}$:

$$0 \to H^{-1}(K) \to H^{-1}(E) \to H^{-1}(Q) \to H^0(K) \to H^0(E) \to H^0(Q) \to 0$$

where $H^{-1}(K), H^{-1}(E), H^{-1}(Q) \in \mathcal{F}$ and $H^0(K), H^0(E), H^0(Q) \in \mathcal{T}$.

Definition 3.1.4. A stability function on an abelian category $\mathcal{A}$ (which we will implicitly assume is the the heart of a bounded t-structure) is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that

$$0 \neq E \in \mathcal{A} \implies Z(E) \in \mathbb{R}_{>0}\exp(i\pi\phi(E)) \text{ with } 0 < \phi(E) \leq 1.$$

The real number $\phi(E) \in (0, 1]$ is called the phase of the object E. The notion of 'slope' can naturally be defined as

$$\mu_Z(E) := -\frac{\mathrm{Re}(Z(E))}{\mathrm{Im}(Z(E))}.$$

A nonzero object $E \in \mathcal{A}$ is said to be semistable with respect to a stability function Z if for all proper subobjects $0 \neq K \subset E$ in $\mathcal{A}$ one has

$$\mu_Z(K) \leq \mu_Z(E).$$

As usual, if the inequality is always strict then E is said to be stable. The stability function is said to have the Harder-Narasimhan property if every nonzero object

$E \in \mathcal{A}$ has a finite filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_j = E_j/E_{j-1}$ are semistable objects of $\mathcal{A}$ with

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$$

A pair $\sigma = (Z, \mathcal{A})$ is called a Bridgeland stability condition if Z is a stability function on $\mathcal{A}$ which has the Harder-Narasimhan property; see [Bri07, Proposition 5.3].
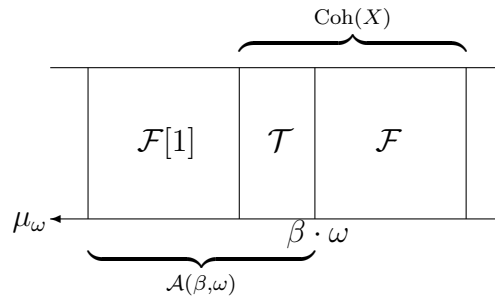
Lemma 3.1.5. *For any pair $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbb{R}$ with $\omega \in \mathrm{Amp}(X)$ there is a unique torsion pair $(\mathcal{T}, \mathcal{F})$ on the category $\mathrm{Coh}(X)$ such that*

$$\mathcal{T} \; := \; \{E \in \mathrm{Coh}(X) \mid (\mu_\omega)_{\min}(E/\mathrm{tors}(E)) > \beta \cdot \omega\} \cup \{\text{torsion sheaves}\}, \quad \text{and}$$
$$\mathcal{F} \; := \; \{E \in \mathrm{Coh}(X) \mid E \text{ is torsion-free and } (\mu_\omega)_{\max}(E) \le \beta \cdot \omega\}.$$

Proof   See [Bri08, Lemma 6.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Tilting with respect to this torsion pair gives a bounded t-structure on $\mathcal{D}(X)$ with heart

$$\mathcal{A}(\beta, \omega) = \{E \in \mathcal{D}(X) \mid H^{-1}(E) \in \mathcal{F}, \; H^0(E) \in \mathcal{T}, \; H^i(E) = 0 \text{ for } i \ne -1, 0\}.$$

Remark 3.1.6. $\mathcal{A}(\beta, \omega)$ does not really depend on β, only on $\beta \cdot \omega$.

Lemma 3.1.7. *Take a pair $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbb{Q}$ with $\omega \in \mathrm{Amp}(X)$. Then the group homomorphism*

$$Z : K(X) \to \mathbb{C} \; ; \; v(E) \mapsto \langle \exp(\beta + i\omega), v(E) \rangle$$

is a stability function on the abelian category $\mathcal{A}(\beta, \omega)$ (with the Harder-Narasimhan property) providing β and ω are chosen so that for all spherical sheaves E on X one has $Z(E) \notin \mathbb{R}_{\le 0}$. This holds in particular whenever $\omega^2 > 2$.

Proof   See [Bri08, Lemma 6.2 & Proposition 7.1] or [ABL07, Corollary 2.1].   □

Remarks 3.1.8. In fact, Bridgeland shows that we can drop the rationality assumption on β and ω and still get stability functions with the Harder-Narasimhan property; see [Bri08, Section 11]. However, rational classes will be enough for this thesis. Also notice that the condition on spherical objects is vacuously satisfied on abelian surfaces because there are none; see [Bri08, Lemma 15.1].

The main stability function used in [ABL07] is different in the sense that it ignores all contributions from the Todd class. More specifically, for most of the paper, they use $Z = \langle \exp(\beta + i\omega), \mathrm{ch}(E) \rangle$ rather than the one above; which actually improves their results (see [ABL07, Section 6]). Given that the Todd class is trivial on an abelian surface, these stability functions coincide and so we make no distinction between the two in this thesis.

To help us understand $\mathcal{A}(\beta, \omega)$ a little better, we have

Proposition 3.1.9. *The minimal objects in the abelian category* $\mathcal{A}(\beta, \omega)$, *i.e. those with no proper subobjects, are precisely the objects*

- $\mathcal{O}_x$, *where* $x \in X$ *is a closed point and*

- $E[1]$, *where* E *is a* μ_ω-*stable locally free sheaf with* $\mu_\omega(E) = \beta \cdot \omega$.

Proof   See [Huy08, Proposition 2.2] and compare with [Bri08, Lemma 6.3 & Lemma 10.1].   □

Huybrechts uses this observation to prove the following

Theorem 3.1.10. *Let* $\Phi_{\mathcal{E}[1]} : \mathcal{D}(X') \xrightarrow{\sim} \mathcal{D}(X)$ *be an exact equivalence such that* $\mu_\omega(\mathcal{E}_y) = \beta \cdot \omega$ *for all closed points* $y \in X'$. *Then it descends to an equivalence of hearts*

$$\Phi^{-1} = \Phi_{\mathcal{E}^\vee[1]} \; : \; \mathcal{A}_X(\beta, \omega) \xrightarrow{\sim} \mathcal{A}_{X'}(\beta', \omega')$$

for some $\beta', \omega' \in \mathrm{NS}(X')$ *where* $\Phi^H_{\mathcal{E}^\vee[1]} \exp(\beta + i\omega) = \lambda \exp(\beta' + i\omega')$ *and* $\lambda \in \mathbb{C}^*$.

Proof   See [Huy08, Corollary 1.3 & Corollary 5.3].   □

Remark 3.1.11. Notice that if either β or ω were irrational, then the only simple objects in $\mathcal{A}(\beta, \omega)$ are the skyscraper sheaves.

3.2   Principally Polarized Abelian Surfaces

For the rest of the chapter, let $(\mathbb{T}, L)$ be an irreducible principally polarized abelian surface over $\mathbb{C}$ with $\ell := c_1(L)$ and $\mathrm{Pic}(\mathbb{T}) = \mathbb{Z}[\ell]$; see Definition 2.6.1. Following the notation above, we introduce formal parameters by setting $\beta := s\ell$ for $s \in \mathbb{Q}$ and $\omega := t\ell$ for $t \in \mathbb{Q}_{>0}$. For clarity, we restate Lemma 3.1.5 as a

Definition 3.2.1. For each $s \in \mathbb{Q}$, define a torsion pair $(\mathcal{T}_s, \mathcal{F}_s)$ in $\mathrm{Coh}(\mathbb{T})$ such that

$$
\begin{aligned}
\mathcal{T}_s &:= \{E \in \mathrm{Coh}(\mathbb{T}) \ : \ \mu_{\min}(E/\mathrm{tors}(E)) > 2s\} \cup \{\text{torsion sheaves}\}, \\
\mathcal{F}_s &:= \{E \in \mathrm{Coh}(\mathbb{T}) \ : \ E \text{ is torsion-free and } \mu_{\max}(E) \leq 2s\}.
\end{aligned}
$$

Notice that the categories $\mathcal{T}_s$ and $\mathcal{F}_s$ are invariant under rescaling ω. Tilting with respect to this torsion pair produces abelian subcategories of $\mathcal{D}(\mathbb{T})$

$$
\mathcal{A}_s := \mathcal{A}(s\ell, t\ell) = \{E \in \mathcal{D}(\mathbb{T}) \ : \ H^{-1}(E) \in \mathcal{F}_s, H^0(E) \in \mathcal{T}_s, H^i(E) = 0 \text{ for } i \neq -1, 0\}.
$$

which are also independent of t. It is the stability conditions governed by $\mathcal{A}_s$ that will be the main focus of this thesis; see [ABL07] and [AB09] for more details.

The stability function is given by

$$
\begin{aligned}
Z_{s,t}(E) &:= \langle \exp(s\ell + it\ell), v(E) \rangle \\
&= -\chi(E) + s\deg(E) - \mathrm{rk}(E)\left(s^2 - t^2\right) + it\left(\deg(E) - 2s\mathrm{rk}(E)\right)
\end{aligned}
$$

where $\mathrm{ch}(E) = (\mathrm{rk}(E), c_1(E), \chi(E))$. Consequently, the slope function becomes

$$
\mu_{s,t}(E) := -\frac{\mathrm{Re}(Z_{s,t}(E))}{\mathrm{Im}(Z_{s,t}(E))} = \frac{\chi(E) - s\deg(E) + \mathrm{rk}(E)\left(s^2 - t^2\right)}{t\left(\deg(E) - 2s\mathrm{rk}(E)\right)}.
$$

Notice that the objects of $\mathcal{A}_s$ which have infinite $\mu_{s,t}$-slope are either:

a) Torsion sheaves supported in dimension zero, or

b) Shifts $E[1]$ of μ-stable vector bundles with $c_1(E) = s\mathrm{rk}(E)\ell$.

As before, an object $E \in \mathcal{A}_s$ is $\mu_{s,t}$-stable if it is stable with respect to the central charge $Z_{s,t}$, i.e.

$$
\mu_{s,t}(K) < \mu_{s,t}(E) \text{ for all subobjects } K \subset E \text{ in } \mathcal{A}_s.
$$

In particular, K destabilises E if $\mu_{s,t}(K) \geq \mu_{s,t}(E)$ and the case of equality gives rise to critical values of s and t which we call walls. Since the destabilising condition is a quadratic in s, we see that the walls are all semicircles with centre

on the s-axis. More precisely, manipulating the expression for $\mu_{s,t}(K) \geq \mu_{s,t}(E)$ yields $a(s^2 + t^2) - 2bs + c \geq 0$ where $a := \mathrm{rk}(E)\deg(K) - \mathrm{rk}(K)\deg(E)$, $b := \mathrm{rk}(E)\chi(K) - \mathrm{rk}(K)\chi(E)$ and $c := \deg(E)\chi(K) - \deg(K)\chi(E)$. Therefore, if $\mu(K) \neq \mu(E)$ then this this gives rise to a semicircle with centre $(b/a, 0)$ and radius $\sqrt{(b/a)^2 - (c/a)}$; otherwise, we get a straight line at $s = c/2b$ which we think of as a semicircle with an infinite radius.

The Chern character of E determines a set of walls and each connected component of the complement of this set in the (s, t)-plane is called a chamber. In [Bri08, Section 9], Bridgeland proves that the set of $\mu_{s,t}$-stable objects is constant in each chamber. In the case when the value of s is understood we will simplify $\mu_{s,t}$ to just μ_t; the $s = 0$ ray will be of considerable importance.

Our expression for the stability function suggests how to generalize the rank and degree of an object $E \in \mathrm{Coh}(\mathbb{T})$ to an object $E \in \mathcal{A}_s$. More precisely, we have

Definition 3.2.2. The rank is an integer-valued linear function:

$$\mathrm{rk} : K(\mathcal{D}(\mathbb{T})) \to \mathbb{Z}$$

on the Grothendieck group of the derived category of coherent sheaves, with the property that $\mathrm{rk}(E) \geq 0$ for all coherent sheaves E on $\mathbb{T}$. We can define an analogous rank function for each $s \in \mathbb{Q}$ to be the imaginary part of the stability function:

$$r_s : K(\mathcal{D}(\mathbb{T})) \to \mathbb{Q} \; ; \; r_s(E) = \deg(E) - 2s \cdot \mathrm{rk}(E)$$

which has the property that $r_s(E) \geq 0$ for all $E \in \mathcal{A}_s$ and $r_s(T) > 0$ for all coherent sheaves in $\mathcal{T}_s$ supported in codimension ≤ 1.

Similarly, the degree is an integer-valued linear function:

$$\deg : K(\mathcal{D}(\mathbb{T})) \to \mathbb{Z} \; ; \; \deg(E) = c_1(E) \cdot \ell$$

with the property that for all coherent sheaves E:

$$\mathrm{rk}(E) = 0 \Rightarrow (\deg(E) \geq 0 \text{ and } \deg(E) = 0 \Leftrightarrow E \text{ is supported in codim} \geq 2).$$

There is an analogous two-parameter family of degree functions given by the real part of the stabilty function:

$$d_{s,t} : K(\mathcal{D}(\mathbb{T})) \to \mathbb{Q} \; ; \; d_{s,t}(E) = \chi(E) - s\deg(E) + \mathrm{rk}(E)(s^2 - t^2),$$

i.e. a ray of degree functions for each rank r_s. Suppose $E \in \mathcal{A}_s$ with $r_s(E) =$

$\deg(E) - 2s \cdot \mathrm{rk}(E) = 0$ then E fits into a unique exact sequence of objects in $\mathcal{A}_s$:

$$0 \to F[1] \to E \to T \to 0$$

where T is a torsion sheaf supported in codimension 2, and F is a μ-semistable sheaf with $\mu(F) = 2s$. To see this, just consider the short exact sequence $0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0$ and use the fact that r_s is additive.

Proposition 3.2.3. *Suppose $r_s(E) = 0$ for an object $E \in \mathcal{A}_s$. Then for all $t > 0$,*

$$d_{s,t}(E) \geq 0 \ \text{ and } \ d_{s,t}(E) = 0 \Leftrightarrow E = 0.$$

Proof Follows from the fact that r_s and $d_{s,t}$ are given by the imaginary and real parts of the stability function respectively.   □

Corollary 3.2.4. $Z_{s,t}(E) = -d_{s,t}(E) + itr_s(E)$ *and each 'slope' function:*

$$\mu_{s,t} = \frac{d_{s,t}}{tr_s} = -\frac{\mathrm{Re}(Z_{s,t}(E))}{\mathrm{Im}(Z_{s,t}(E))}$$

has the usual properties of a slope function on the objects of $\mathcal{A}_s$. That is, given an exact sequence of objects $0 \to K \to E \to Q \to 0$ in $\mathcal{A}_s$ then

$$\mu_{s,t}(K) < \mu_{s,t}(E) \Leftrightarrow \mu_{s,t}(E) < \mu_{s,t}(Q) \quad \& \quad \mu_{s,t}(K) = \mu_{s,t}(E) \Leftrightarrow \mu_{s,t}(E) = \mu_{s,t}(Q).$$

Proof Follows from the definitions of r_s and $d_{s,t}$.   □

Corollary 3.2.5. *Suppose $E \in \mathcal{A}_s$ is a μ-stable sheaf, i.e. $E \in \mathcal{A}_s \cap \mathrm{Coh}(\mathbb{T})$ and as a member of $\mathrm{Coh}(\mathbb{T})$ is μ-stable with respect to the polarization. Then*

$$K \hookrightarrow E \ \text{ in } \ \mathcal{A}_s \quad \Rightarrow \quad \mu(K) < \mu(E).$$

Proof Let $Q \in \mathcal{A}_s$ denote the quotient of E by K. Taking cohomology of the short exact sequence $0 \to K \to E \to Q \to 0$ of objects in $\mathcal{A}_s$ gives rise to a long exact sequence of objects in $\mathrm{Coh}(\mathbb{T})$:

$$0 \to H^{-1}(Q) \to K \to E \to H^0(Q) \to 0.$$

Since $H^{-1}(Q) \in \mathcal{F}_s$ and E is torsion-free we see that $K \in \mathcal{T}_s$ must be torsion-free as well with $\mu(H^{-1}(Q)) < \mu(K)$. Splitting the sequence via $K/H^{-1}(Q)$ gives two short exact sequences in $\mathrm{Coh}(\mathbb{T})$:

$$0 \to H^{-1}(Q) \to K \to K/H^{-1}(Q) \to 0, \quad 0 \to K/H^{-1}(Q) \to E \to H^0(Q) \to 0.$$

Therefore, $\mu(K) < \mu(K/H^{-1}(Q))$ and $K/H^{-1}(Q) \subset E \Rightarrow \mu(K/H^{-1}(Q)) < \mu(E)$, i.e. $\mu(K) < \mu(E)$. If $H^{-1}(Q) = 0$ then $K \subset E$ and so the inequality follows immediately. $\square$

Definition 3.2.6. Let $\mathcal{E} \in \mathcal{D}(\mathbb{T} \times \hat{\mathbb{T}})$ be an object such that $\Phi_{\mathcal{E}} : \mathcal{D}(\mathbb{T}) \xrightarrow{\sim} \mathcal{D}(\hat{\mathbb{T}})$ is an equivalence. By Theorem 2.4.7, we have $\chi(\mathcal{E}_x, \mathcal{E}_x) = 0$ and hence $\mathrm{ch}(\mathcal{E}_x) = (a^2, -ab\ell, b^2)$ for two coprime integers $a > 0$ and b. By Theorem 2.5.11, we know that (up to a shift) $\mathcal{E}$ is uniquely determined by the rational number $s = b/a$. Now, by Theorem 3.1.10, we see that $\Phi_{\mathcal{E}^\vee[1]} : \mathcal{D}(\hat{\mathbb{T}}) \xrightarrow{\sim} \mathcal{D}(\mathbb{T})$ descends to an equivalence

$$\Phi_{\mathcal{E}[1]} : \mathcal{A}_{\mathbb{T}}(s\ell, t\ell) \xrightarrow{\sim} \mathcal{A}_{\hat{\mathbb{T}}}(s'\ell, t'\ell)$$

where $\Phi^H_{\mathcal{E}[1]} \exp(s\ell + it\ell) = \lambda \exp(s'\ell + it'\ell)$ for some $s', t', \lambda \in \mathbb{Q}$. Therefore, we can use the identification $\phi_L : \mathbb{T} \xrightarrow{\sim} \hat{\mathbb{T}}$; $x \mapsto \tau_x^* L \otimes L^*$ to define an autoequivalence

$$\Phi_s := \Phi_{\mathcal{E}^\vee[1]} \circ \phi_L : \mathcal{D}(\mathbb{T}) \xrightarrow{\sim} \mathcal{D}(\mathbb{T})$$

which descends to an equivalence $\Phi_{-s} : \mathcal{A}_s \xrightarrow{\sim} \mathcal{A}_{s'}$. In particular, an object $E \in \mathcal{A}_s$ is $\mu_{s,t}$-stable if and only if $\Phi_{-s}(E) \in \mathcal{A}_{s'}$ is $\mu_{s',t'}$-stable. As usual, let $\hat{\Phi}_s$ denote the inverse transform. Notice that the standard Fourier-Mukai transform with kernel the Poincaré line bundle $\mathcal{P}$ is just Φ_0 in this notation.

Lemma 3.2.7. *Suppose* $\Phi_{-s} : \mathcal{A}(s\ell, t\ell) \xrightarrow{\sim} \mathcal{A}(s'\ell, t'\ell)$ *is the abelian equivalence associated to* $\mathcal{E} \in \mathcal{D}(\mathbb{T} \times \hat{\mathbb{T}})$ *with* $\mathrm{ch}(\mathcal{E}_x) = (a^2, -ab\ell, b^2)$ *and* $\mathrm{ch}(\mathcal{E}_{\hat{x}}) = (a^2, ac\ell, c^2)$. *Then* $s' = c/a$, $t' = 1/(a^2 t)$ *and* $\lambda = a^2 t^2$.

Proof By [Mac97, Lemma 1.4], the cohomological transform is given by

$$\Phi^H_{\mathcal{E}^\vee} = \begin{pmatrix} c^2 & -2ac & a^2 \\ cd & -(ad+bc) & ab \\ d^2 & -2bd & b^2 \end{pmatrix} \quad \Rightarrow \quad \Phi^H_{-s} = -\begin{pmatrix} b^2 & -2ab & a^2 \\ bd & -(ad+bc) & ac \\ d^2 & -2cd & c^2 \end{pmatrix}$$

where $d \in \mathbb{Z}$ is such that $ad - bc = 1$. Therefore, $\Phi^H_{-s} \exp(s\ell + it\ell) = \lambda \exp(s'\ell + it'\ell)$ equates to

$$-\begin{pmatrix} b^2 & -2ab & a^2 \\ bd & -(ad+bc) & ac \\ d^2 & -2cd & c^2 \end{pmatrix} \begin{pmatrix} 1 \\ b/a + it \\ b^2/a^2 - t^2 + i(2b/a)t \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ s' + it' \\ s'^2 - t'^2 + i2s't' \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} a^2 t^2 \\ act^2 + it \\ c^2 t^2 - 1/a^2 + i(2c/a)t \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ s' + it' \\ s'^2 - t'^2 + i2s't' \end{pmatrix}$$

from which we can read off the values of s', t' and λ. $\square$

The main goal of this chapter is to find smooth projective varieties which represent the moduli functors (from $\mathfrak{Sch}^{\mathrm{op}}$ to $\mathfrak{Set}$):

$$\mathcal{M}_{s,t}(r, c\ell, \chi) : \Sigma \mapsto \left\{ E \in \mathcal{D}(\mathbb{T} \times \Sigma) : \begin{array}{c} E_\sigma \in \mathcal{A}_s \text{ is } \mu_{s,t}\text{-stable and} \\ \mathrm{ch}(E_\sigma) = (r, c\ell, \chi) \\ \text{for all } \sigma \in \Sigma \end{array} \right\} \Big/ \sim,$$

where Σ is a noetherian scheme of finite-type over $\mathbb{C}$, $i_\sigma : \mathbb{T} \times \{\sigma\} \hookrightarrow \mathbb{T} \times \Sigma$, $E_\sigma = i_\sigma^* E$ and $E \sim E'$ if there is a line bundle $\mathcal{L}$ on Σ such that $E \cong E' \otimes \pi_\Sigma^* \mathcal{L}$. Therefore, by Lemma 3.2.7

$$\Phi_{-s} : \mathcal{M}_{s,t}(\mathrm{ch}(E)) \xrightarrow{\sim} \mathcal{M}_{s',t'}(\mathrm{ch}(\Phi_{-s}(E)))$$

where we implicitly understand that we have pulled back the transform to include Σ. More precisely, if $M_{s,t}$ together with a universal object $\mathcal{E}$ represents $\mathcal{M}_{s,t}$ then $\Phi_{-s}(M_{s,t})$ together with $\Phi_{-s}(\mathcal{E})$ represents $\mathcal{M}_{s',t'}$. Since the numerical invariants determine the wall and chamber structure of $\mathrm{Stab}(\mathcal{D}(\mathbb{T}))$, we expect to see special behaviour when $\mathrm{ch}(E) = \mathrm{ch}(\Phi_{-s}(E))$.

Proposition 3.2.8. *Suppose $E \in \mathcal{D}(\mathbb{T})$ with $\mathrm{ch}(E) = (r, c\ell, \chi)$ satisfies*

$$r > 0 \quad \text{and} \quad c - sr > 0.$$

Then $E \in \mathcal{A}_s$ is $\mu_{s,t}$-semistable for all $t \gg 0$ iff E is a shift of a e^β-twisted semistable sheaf on $\mathbb{T}$ where $\beta := s\ell$. (See Definition 2.1.1 for the definition of twisted-stability.)

Proof   See [Bri08, Proposition 14.2]. $\square$

In other words, for t very large we see that the functor $\mathcal{M}_{s,t\gg 0}(\mathrm{ch}(E))$ is corepresented by the moduli space of twisted-semistable sheaves. That is, we have a GIT construction and the moduli space is necessarily projective. Since t' is always a multiple of $1/t$, we can use Φ_{-s} to get (for free) an analogous result for small t:

Corollary 3.2.9. $\Phi_{-s} : \mathcal{M}_{s,t\gg 1}(\mathrm{ch}(E)) \xrightarrow{\sim} \mathcal{M}_{s',t\ll 1}(\mathrm{ch}(\Phi_{-s}(E)))$. *In particular,*

$$\mathcal{M}_{s,t\gg 1}(\mathrm{ch}(E)) \text{ projective} \quad \Rightarrow \quad \mathcal{M}_{s',t\ll 1}(\mathrm{ch}(\Phi_{-s}(E))) \text{ projective}.$$

Proof   Follows from Theorem 2.3.6, Lemma 3.2.7 and Proposition 3.2.8. $\square$

We conclude this section with a couple of useful observations regarding $s = 0$:

Lemma 3.2.10. *Suppose E is a coherent sheaf on $\mathbb{T}$. Then*

i) $E \in \mathcal{T}_0$ *and* $\deg(E) = 2$ *implies* $E/\mathrm{tors}(E)$ *is* μ-*semistable*,

ii) $E \in \mathcal{F}_0$ *and* $\deg(E) = 0$ *implies* E *is* μ-*semistable*.

Proof i) Suppose there exists a μ-destabilising sequence $0 \to A \to E/\mathrm{tors}(E) \to B \to 0$ with $\mu(A) > \mu(E) > \mu(B)$. Then $\deg(B) < 2\mathrm{rk}(B)/\mathrm{rk}(E) < 2$ which implies $\deg(E) \leq 0$ but this is a contradiction since $E \in \mathcal{T}_0 \Rightarrow \mu_{\min}(E) > 0 \Leftrightarrow \deg_{\min}(E) > 0$.

ii) From definition of $\mathcal{F}_0$ we see that E cannot have any subsheaves of positive slope. $\qquad\square$

3.3 Stable Objects

Some examples of $\mu_{s,t}$-stable objects are provided by the following

Lemma 3.3.1. *Let $X \in \mathrm{Hilb}^n(\mathbb{T})$ and $\mathcal{O}_{D_z}(\alpha)$ denote a line bundle of degree α supported on the divisor $D_z := \tau_z D_L$. Then*

(i) $L^m \mathcal{P}_{\hat{x}}$ *is* $\mu_{s,t}$-*stable in* $\mathcal{A}_s$ *for all* $m \in \mathbb{Z}_{>0}$, $t > 0$ *and* $s < m$,

(ii) $L^m[1]\mathcal{P}_{\hat{x}}$ *is* $\mu_{s,t}$-*stable in* $\mathcal{A}_s$ *for all* $m \in \mathbb{Z}_{\leq 0}$, $t > 0$ *and* $s \geq m$,

(iii) $\mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}}$ *is* μ_t-*stable in* $\mathcal{A}_0$ *for all* $t > 0$,

(iv) $L\mathcal{I}_X \mathcal{P}_{\hat{x}}$ *is* $\mu_{s,t}$-*stable in* $\mathcal{A}_s$ *for all* $t > 0$ *and* $0 \leq s < 1$,

Proof Without loss of generality, let $\hat{x} = \hat{e} = \hat{y} \in \hat{\mathbb{T}}$.

(i). Suppose $0 \to K \to L^m \to Q \to 0$ is a destabilising sequence for L^m in $\mathcal{A}_s$ with $\mathrm{ch}(K) = (r, c\ell, \chi)$. By Corollary 3.2.5, we see that $K \in \mathcal{T}_s$ is a torsion-free sheaf with $\mu(K) < \mu(L^m) \Leftrightarrow c < mr$. Replacing K by one of its Harder-Narasimhan factors, we can assume K is μ-semistable and so $r\chi \leq c^2$ by Bogomolov (Theorem 2.4.3). Thus, we have

$$\mu_{s,t}(K) \geq \mu_{s,t}(L^m) \Leftrightarrow 0 < (rm - c)t^2 \;\leq\; (c - mr)s^2 + \left(rm^2 - \chi\right)s + m\chi - cm^2$$
$$= (m - s)(msr - mc + \chi - cs)$$

where $m - s > 0$ since $L^m \in \mathcal{T}_s$. Now observe that

$$r(msr - mc + \chi - cs) = (r\chi - c^2) + (c - rs)(c - rm)$$

where the first term is non-positive by Bogomolov and the second term is negative since $K \in \mathcal{T}_s \Rightarrow c - rs > 0$. Therefore, every factor K' of the Harder-Narasimhan filtration of K has $\mu_{s,t}(K') < \mu_{s,t}(L^m)$ and so K cannot destabilise L^m. (For an alternative proof, see [AB09, Proposition 3.6].)

(ii). Follows in exactly the same way as i).

(iii). Suppose $0 \to K \to \mathcal{O}_{D_z}(\alpha) \to Q \to 0$ is a destabilising sequence for $\mathcal{O}_{D_z}(\alpha)$ in $\mathcal{A}_0$ with $\mathrm{ch}(K) = (r, c\ell, \chi)$. Taking cohomology, we get a long exact sequence of objects in $\mathrm{Coh}(\mathbb{T})$

$$0 \to H^{-1}(Q) \to K \to \mathcal{O}_{D_z}(\alpha) \to H^0(Q) \to 0.$$

If we factor the map $K \to \mathcal{O}_{D_z}(\alpha)$ through its image $K/H^{-1}(Q)$ then we see that $\mathrm{rk}(K/H^{-1}(Q)) = 0$ which implies $H^0(Q)$ is a torsion sheaf. If we suppose that $H^0(Q)$ is supported on a curve then, in terms of Chern characters, the sequence reads

$$(r, (c - 1 + d)\ell, *) \to (r, c\ell, \chi) \to (0, \ell, \alpha - 1) \to (0, d\ell, *).$$

Then $H^{-1}(Q) \in \mathcal{F}_0$ and $K = H^0(K) \in \mathcal{T}_0$ imply $d = 0$ and $c = 1$. Therefore, $H^0(Q)$ is supported on points and $\deg(Q) = \deg(H^0(Q)) - \deg(H^{-1}(Q)) = 0$, i.e. Q has infinite μ_t-slope and no object can destabilise $\mathcal{O}_{D_z}(\alpha)$.

(iv). A similar argument as for iii) proves the statement for $s = 0$. Thus, it remains to consider $0 < s < 1$. Observe that each wall is a semicircle with centre on the s-axis and since $L\mathcal{I}_X$ is μ_t-stable in $\mathcal{A}_0$ for all $t > 0$, no wall can intersect the line $s = 0$. As above, if K is a μ_t-destabiliser for some $t > 0$ then by Corollary 3.2.5 we see that $K \in \mathcal{T}_s$ must be a torsion-free sheaf with $\mu(K) < \mu(L\mathcal{I}_X) \Leftrightarrow c < r$. Thus, we have

$$\mu_{s,t}(K) \geq \mu_{s,t}(L\mathcal{I}_X) \Leftrightarrow 0 < (r - c)t^2 \leq (c - r)s^2 + \chi(L\mathcal{I}_X)(rs - c) + \chi(1 - s).$$

Completing the square shows us that the centre of the semicircle is

$$s = -\frac{1}{2} \frac{\chi - r\chi(L\mathcal{I}_X)}{r - c}, \qquad t = 0.$$

Since $s > 0$, we have $\chi < r\chi(L\mathcal{I}_X)$ and the destabilising condition above reduces to

$$0 < (c - r)\left(s^2 - \chi(L\mathcal{I}_X)\right) \Leftrightarrow 0 < s^2 < \chi(L\mathcal{I}_X) = 1 - n \Leftrightarrow n = 0,$$

i.e. we are in case i) which has already been proven. $\qquad \square$

For the rest of this section, we fix our Chern character to be $(1, 2\ell, 4 - n)$ for some non-negative integer n and focus on the case when $s = 0$. That is, we aim to give a precise set of μ_t-stable objects $E \in \mathcal{A}_0$ with these invariants. Before

going on, let us make several observations. First of all, if E is a torsion-free sheaf then by Lemma 2.4.1(b) $E \cong L^2 \mathcal{I}_X \mathcal{P}_{\hat{x}}$ for some $X \in \mathrm{Hilb}^n(\mathbb{T})$ and we have the following useful

Proposition 3.3.2. *Let* $X \in \mathrm{Hilb}^n(\mathbb{T})$ *for* $n \geq 4$. *Then*

$$L^2 \mathcal{I}_X \text{ is } \mathrm{WIT}_1 \Leftrightarrow X \text{ is not collinear.}$$

Proof See [Mac11, Corollary 7.2]. □

By Proposition 3.2.8 and Theorem 2.2.4, we know that $\mathcal{M}_{t \gg 1}(1, 2\ell, 4 - n)$ is represented by the moduli space of stable sheaves which is projective. Thus, by Proposition 3.3.2, we see that

$$\Phi_0 : \mathcal{M}_{t \gg 1}(1, 2\ell, 4 - n) \xrightarrow{\sim} \mathcal{M}_{t \ll 1}(n - 4, 2\ell, -1)$$

provides a fine projective moduli space for $\mathcal{M}_{t \ll 1}(n - 4, 2\ell, -1)$ when $n \geq 4$.

Lemma 3.3.3. *Suppose* $E \in \mathcal{A}_0$ *with* $\mathrm{ch}(E) = (1, 2\ell, 4 - n)$ *is* μ_t*-stable for some* $t > 0$ *and* $H^{-1}(E) \neq 0$. *Then* $H^{-1}(E)$ *is locally-free and if* $H^0(E)$ *has torsion, it is supported in dimension zero.*

Proof Suppose, for a contradiction, that $H^{-1}(E)$ is not locally-free. We know it is torsion-free since $H^{-1}(E) \in \mathcal{F}_0$ and so the torsion sequence gives us the following short exact sequence in $\mathcal{A}_0$

$$\mathcal{O}_Z \to H^{-1}(E)[1] \to H^{-1}(E)^{**}[1].$$

Thus, we have an injection $\mathcal{O}_Z \hookrightarrow E$ in $\mathcal{A}_0$ which destabilises E for all $t > 0$; contradiction.

Suppose $H^0(E)$ has torsion and define $E' := \ker(E \to H^0(E)/\mathrm{tors}(H^0(E)))$. Then E' fits into the following diagram

$$
\begin{array}{ccccc}
H^{-1}(E)[1] & \longrightarrow & E' & \longrightarrow & \mathrm{tors}(H^0(E)) \\
\| & & \downarrow & & \downarrow \\
H^{-1}(E)[1] & \longrightarrow & E & \longrightarrow & H^0(E) \\
& & \downarrow & & \downarrow \\
& & H^0(E)/\mathrm{tors}(H^0(E)) & = \!\!= & H^0(E)/\mathrm{tors}(H^0(E))
\end{array}
$$

That is, $H^{-1}(E') = H^{-1}(E)$ is locally-free and $H^0(E') = \mathrm{tors}(H^0(E))$ can only be supported in dimension zero. Indeed, the fact that $H^{-1}(E) \in \mathcal{F}_0$ and $H^0(E) \in \mathcal{T}_0$ forces $\deg(H^0(E)) = 2$; if $\deg(H^0(E)) = 4$ then $\deg(H^{-1}(E)) = 0$ and $H^{-1}(E)[1]$

would destabilise E for all $t > 0$. Now, if $H^0(E)$ had torsion supported on a curve then $\deg(\mathrm{tors}(H^0(E))) \geq 2$ which implies $\deg(H^0(E)/\mathrm{tors}(H^0(E))) \leq 0$ contradicting the fact that $H^0(E) \in \mathcal{T}_0$; unless $H^0(E)$ is torsion but then $\mathrm{rk}(E) = \mathrm{rk}(H^0(E)) - \mathrm{rk}(H^{-1}(E)) < 0$ giving another contradiction. $\qquad\square$

Proposition 3.3.4. *Suppose $E \in \mathcal{A}_0$ with $\mathrm{ch}(E) = (1, 2\ell, 4 - n)$ is μ_t-stable for some $t > 0$. Then, either*

1. *E is a torsion-free sheaf, i.e. $E = L^2 \mathcal{I}_X \mathcal{P}_{\hat{x}}$ for some $X \in \mathrm{Hilb}^n(\mathbb{T})$ and $\hat{x} \in \hat{\mathbb{T}}$, or*

2. *E is a sheaf with torsion, in which case, $E = L \mathcal{I}_{X'} \mathcal{P}_{\hat{x}} \ltimes \mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}}$ where $X' \in \mathrm{Hilb}^m(\mathbb{T})$, $0 \leq m < (n - 2)/2$ and $\alpha := 4 - n + m$, or*

3. *E is a two-step complex with $H^{-1}(E) = L^{-1} \mathcal{P}_{\hat{x}}$ and $H^0(E)$ a μ-stable locally-free sheaf with $\mathrm{ch}(H^0(E)) = (2, \ell, 0)$ only when $n = 5$.*

Proof   Throughout the proof, we suppress all twists. If E is a torsion-free sheaf then $E = L^2 \mathcal{I}_X$ for some $X \in \mathrm{Hilb}^n(\mathbb{T})$; see Lemma 2.4.1(b).

If E is atomic and has torsion then it must be supported on a curve since all torsion sheaves supported in dimension zero have infinite μ_t-slope and would destabilise E for all $t > 0$. Let $T \subset E$ be the torsion subsheaf of E and consider $0 \to T \to E \to F \to 0$ where F is torsion-free and the Chern characters read

$$(0, d\ell, \alpha - 1) \to (1, 2\ell, 4 - n) \to (1, (2 - d)\ell, 5 - n - \alpha) \quad \text{with } d > 0.$$

$F \in \mathcal{T}_0$ implies $d = 1$ and so $F \cong L \mathcal{I}_{X'}$ for some $X' \in \mathrm{Hilb}^m(\mathbb{T})$. By assumption, there is a $t > 0$ such that $\mu_t(E) < \mu_t(F) \Leftrightarrow m < (n - 2)/2$ and $t < \sqrt{n - 2m - 2}$. Notice that extensions of this kind exist since

$$\chi(L \mathcal{I}_{X'}, \mathcal{O}_D(\alpha)) = 1 - n + m < -n/2 < 0 \Rightarrow \dim \mathrm{Ext}^1(L \mathcal{I}_{X'}, \mathcal{O}_D(\alpha)) \neq 0.$$

Now suppose $H^{-1}(E) \neq 0$ with $\mathrm{ch}(H^{-1}(E)) = (r, c\ell, \chi)$ with $r \geq 1$. Since $H^0(E) \in \mathcal{T}_0$ and $H^{-1}(E) \in \mathcal{F}_0$ we are forced to have

$$\mathrm{ch}(H^0(E)) = (r + 1, (2 + c)\ell, 4 - n + \chi) \quad \text{with } -2 < c \leq 0.$$

If $c = 0$ then $H^{-1}(E)[1]$ has infinite μ_t-slope and destabilises E for all $t > 0$; contradiction. Therefore, $c = -1$ and $H^{-1}(E)$ is μ-semistable. Indeed, if D was a potential μ-destabilising object then $\deg(D) = 0$ and the composite map $D[1] \to H^{-1}(E)[1] \to E$ would destabilise E for all $t > 0$; contradiction. Thus,

by Bogomolov, we have $\chi \leq 1$ and E is μ_t-stable for some $t > 0$ only if

$$\mu_t(E) < \mu_t(H^0(E)) \quad \Leftrightarrow \quad 0 < (2r+1)t^2 < 4 - n + 2\chi \leq 6 - n \quad \Leftrightarrow \quad n < 6.$$

By Lemma 3.3.3, we know that if $H^0(E)$ has torsion then it must be supported in dimension zero (on $\mathcal{O}_Z$, say) and since $\deg(H^0(E)) = 2$ we have that $H^0(E)/\mathcal{O}_Z$ is μ-semistable by Lemma 3.2.10. By Bogomolov, we have $\chi \leq n - 4 + |Z|$ and E is μ_t-stable for some $t > 0$ if and only if

$$\mu_t(E) < \mu_t(H^0(E)/\mathcal{O}_Z) \Leftrightarrow 0 < (2r+1)t^2 < 4 - n + 2(\chi - |Z|) \leq n - 4 \Leftrightarrow n > 4.$$

Therefore, $n = 5$ and $\chi = 1$ which forces $r = 1$ and $|Z| = 0$, i.e. $H^0(E)$ is torsion-free. Now, the torsion sequence reduces to

$$0 \to H^0(E) \to H^0(E)^{**} \to \mathcal{O}_Y \to 0$$

where $H^0(E)^{**}$ is μ-semistable (Lemma 3.2.10) and $\mathrm{ch}(H^0(E)^{**}) = (2, \ell, |Y|)$. Thus, by Bogomolov, we have $|Y| \leq 1/2$ which implies $|Y| = 0$ and $H^0(E) \cong H^0(E)^{**}$, i.e. $H^0(E)$ is in fact locally-free. It remains to show that $H^0(E)$ is μ-stable. We know, a priori, that $H^0(E)$ is μ-semistable so suppose we have a semi-destabilising sequence $A \to H^0(E) \to B$ with $\mu(A) \geq 1 \geq \mu(B)$. Since A is a proper subsheaf we must have $\mathrm{rk}(A) = 1$ which implies $\deg(A) \geq 2$ and hence $\deg(B) \leq 0$; contradicting the fact that $H^0(E) \in \mathcal{T}_0$. The existence of such two-step complexes is dealt with in Lemma 3.3.8. $\qquad\square$

Lemma 3.3.5. *Suppose $E \in \mathcal{A}_0$ with $\deg(E) = 4$ is a μ_t-stable sheaf for some $t > 0$ and $0 \to K \to E \to Q \to 0$ is a destabilising sequence in $\mathcal{A}_0$. Then K must be a sheaf with $\deg(K) = 2$ and Q must be atomic, i.e. Q cannot be a two-step complex.*

Proof Taking cohomology of the destabilising sequence gives rise to a long exact sequence in $\mathrm{Coh}(\mathbb{T})$

$$0 \to H^{-1}(Q) \to K \to E \to H^0(Q) \to 0$$

and we see that K must be a sheaf. Setting $s = 0$ in the generalized rank function, we see that

$$0 < r_0(K) \leq r_0(E) \Leftrightarrow 0 < \deg(K) \leq 4.$$

However, if $\deg(K) = 4$ then $\deg(Q) = 0$ which implies $\mu_t(Q) = \infty$ and nothing can destabilise E; contradiction. Therefore, $\deg(K) = 2$. Replacing Q by one of the sheaves in the associated graded object of a factor in its Harder-Narasimhan

filtration, we can assume Q to be μ_t-stable for some $t > 0$. If either $H^{-1}(Q)$ or $H^0(Q)$ are zero then there is nothing to prove so suppose, for a contradiction, that Q is a two-step complex, i.e. $H^{-1}(Q) \neq 0$ and $H^0(Q) \neq 0$. Then splitting the long exact sequence via $K/H^{-1}(Q) \in \mathcal{T}_0$ gives rise to two short exact sequences in $\mathcal{A}_0$

$$0 \to K/H^{-1}(Q) \to E \to H^0(Q) \to 0, \quad 0 \to K \to K/H^{-1}(Q) \to H^{-1}(Q)[1] \to 0.$$

Applying the same reasoning as above to the map $K/H^{-1}(Q) \hookrightarrow E$ we see that $0 < \deg(K/H^{-1}(Q)) \leq 4$. However, if $\deg(K/H^{-1}(Q)) = 4$ then $\deg(H^0(Q)) = 0$ contradicting the fact that $E \in \mathcal{T}_0$. Thus we have $\deg(K/H^{-1}(Q)) = 2$ and $\deg(H^{-1}(Q)) = 0$. But then $0 \to H^{-1}(Q)[1] \to Q \to H^0(Q) \to 0$ destabilises Q for all $t > 0$; contradiction. Therefore, Q must be an atomic object. $\square$

The μ_t-stability of $L^2\mathcal{I}_X\mathcal{P}_{\hat{x}}$ is completely determined by the configuration of $X \in \mathrm{Hilb}^n(\mathbb{T})$:

Lemma 3.3.6. *Let $X' \in \mathrm{Hilb}^m(\mathbb{T})$. Then the objects $L^2\mathcal{I}_X \in \mathcal{A}_0$ are destabilised by $L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}}$ for $0 \leq m < (n-2)/2$ and $t \leq \sqrt{n - 2m - 2}$ if and only if X contains a collinear subscheme of colength m. If $n \neq 5$ and X does not contain a collinear subscheme of colength m for $0 \leq m < (n-2)/2$ then $L^2\mathcal{I}_X$ is μ_t-stable for all $t > 0$; if $n = 5$, then $L^2\mathcal{I}_X$ is destabilised by a μ-stable locally-free sheaf K with $\mathrm{ch}(K) = (2, \ell, 0)$ for $t \leq 1/\sqrt{3}$ if and only if every collection of four points in X contains a unique collinear length 3. For sufficiently general configurations of points, $L^2\mathcal{I}_X$ is μ_t-stable for all $t > 0$.*

Proof   Recall, from the definition of collinearity, that there is a non-zero map $L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}} \to L^2\mathcal{I}_X$ if and only if X contains a collinear subscheme of colength m. Thus, the first claim follows immediately from the destabilising condition:

$$\mu_t(L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}}) \geq \mu_t(L^2\mathcal{I}_X) \quad \Leftrightarrow \quad m \leq \frac{2 - \chi(L^2\mathcal{I}_X) - t^2}{2} < \frac{n-2}{2}$$

$$\text{and} \quad t \leq \sqrt{2\chi(L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}}) - \chi(L^2\mathcal{I}_X)} = \sqrt{n - 2m - 2}.$$

For the second claim, suppose X does not contain a collinear subscheme of colength m for $0 \leq m < (n-2)/2$ and we have a destabilising sequence $K \to L^2\mathcal{I}_X \to Q$ in $\mathcal{A}_0$ with Q μ_t-stable for some $t > 0$. By Lemma 3.3.5, we see that K must be a torsion-free sheaf of degree 2 and Q is atomic, i.e. $Q = H^0(Q)$ or $Q = H^{-1}(Q)[1]$. If Q is a sheaf then $K \cong L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}}$ for some $X' \in \mathrm{Hilb}^m(\mathbb{T})$ by Lemma 2.4.1(b) contradicting our assumption on collinear subschemes. Therefore, $Q = H^{-1}(Q)[1]$ and Q is μ-semistable. Indeed, if D was a potential μ-destabilising

object then $\deg(D) = 0$ and the composite map $D[1] \to H^{-1}(Q)[1] \xrightarrow{\sim} Q$ would destabilise Q for all $t > 0$; contradiction. Thus, by Bogomolov, we have $\chi(K) \leq 0$ and $\chi(H^{-1}(Q)) \leq 1$. By additivity of the Euler characteristic, we have

$$\chi(K) = \chi(L^2 \mathcal{I}_X) + \chi(H^{-1}(Q)) \leq 5 - n.$$

Using both bounds for $\chi(K)$ we get

$$\mu_t(K) \geq \mu_t(L^2 \mathcal{I}_X) \Leftrightarrow 0 < (2r - 1)t^2 \leq n - 4 + 2\chi(K) \leq \begin{cases} 6 - n \\ n - 4 \end{cases}$$

which implies $n = 5$ is the only possibility. Notice that this also forces $\chi(K) = 0$ and $\chi(H^{-1}(Q)) = 1$ which in turn implies that $\text{rk}(K) = 2$. In other words, when $n = 5$, the map $K \to L^2 \mathcal{I}_X$ is a surjection in $\text{Coh}(\mathbb{T})$ and an injection in $\mathcal{A}_0$ with quotient $Q[1] = L^{-1}[1]$ which (if it exists) destabilises $L^2 \mathcal{I}_X$ for $t \leq 1/\sqrt{3}$:

$$0 \to K \to L^2 \mathcal{I}_X \to L^{-1}[1] \to 0$$

$$(2, \ell, 0) \to (1, 2\ell, -1) \to (1, -\ell, 1)[1].$$

Applying the standard Fourier-Mukai transform to this sequence produces

$$0 \to \mathcal{O}_{\hat{D}}(-1) \to \Phi(L^2 \mathcal{I}_X) \to \hat{L} \to 0$$

and [Mac11, Theorem 9.2] states that $\Phi_0(L^2 \mathcal{I}_X)$ has a torsion subsheaf of this kind if and only if every collection of four points in X contains a unique collinear length three. The fact that K is locally-free and μ-stable follows in exactly the same way as Proposition 3.3.4(3). $\qquad\square$

Lemma 3.3.7. *The objects $E = L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}} \ltimes \mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}} \in \mathcal{A}_0$ can only be desta-bilised by $L\mathcal{I}_Z \mathcal{P}_{\hat{x}}$ for some $Z \in \text{Hilb}^p(\mathbb{T})$ with $m < p < (n - 2)/2$ and $t \leq \sqrt{n - 2p - 2}$. In particular, E is μ_t-stable for all*

$$\sqrt{n - 2m - 4} < t < \sqrt{n - 2m - 2}.$$

Proof First of all, let us observe that E is destabilised by $\mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{x}}$ for $t \geq \sqrt{n - 2m - 2}$ and so we restrict our attention to $t < \sqrt{n - 2m - 2}$. Suppose we have a destabilising sequence $K \to E \to Q$ in $\mathcal{A}_0$ then by taking cohomology, one sees that $K \in \mathcal{T}_0$ is a sheaf with $\text{tors}(K) \subset \text{tors}(E) = \mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}}$, i.e. $\text{tors}(K) = \mathcal{O}_{D_z}(\beta)\mathcal{P}_{\hat{x}}$ for some $\beta \leq \alpha$. (K cannot have torsion supported on points since $\mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}}$ is torsion-free on its support.) But this implies $\deg(K/\text{tors}(K)) \leq 0$ contradicting the fact that $K \in \mathcal{T}_0$; unless $K = \mathcal{O}_{D_z}(\beta)\mathcal{P}_{\hat{x}}$ but this will

never destabilise E in the specified range since $\beta \leq \alpha \Rightarrow \mu_t(\mathcal{O}_{D_z}(\beta)\mathcal{P}_{\hat{x}}) \leq \mu_t(\mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{x}})$. Hence K is torsion-free and μ-semistable (by Lemma 3.2.10). If we factor the map $K \to E$ through its image $K/H^{-1}(Q)$ then we see that $K/H^{-1}(Q)$ is torsion-free and so $\mathrm{rk}(K/H^{-1}(Q)) = 1$. Indeed, observe that $K/H^{-1}(Q) \to E$ is an injection in $\mathcal{A}_0$ and the same argument as for K goes through unchanged. Therefore, $K/H^{-1}(Q) \in \mathcal{T}_0$ and $\mathrm{Hom}(K/H^{-1}(Q), L\mathcal{I}_{X'}) \neq 0$ implies $\deg(K/H^{-1}(Q)) = 2$ and $\deg(H^{-1}(Q)) = 0$. Replacing K by one of the sheaves in the associated graded object of a factor in its Harder-Narasimhan filtration, we can assume K to be μ_t-stable with maximal μ_t-slope. But since $K \to K/H^{-1}(Q) \to H^{-1}(Q)[1]$ is a short exact sequence in $\mathcal{A}_0$ with $\mu_t(K/H^{-1}(Q)) < \mu_t(H^{-1}(Q)) = \infty$, we must have $\mu_t(K) < \mu_t(K/H^{-1}(Q))$; contradicting the maximality of K. Therefore, $Q = H^0(Q)$ and so $K \cong L\mathcal{I}_Z\mathcal{P}_{\hat{x}}$ for some $Z \in \mathrm{Hilb}^p(\mathbb{T})$ by Lemma 2.4.1(b) which only destabilises E for $m < p < (n-2)/2$ and $t^2 \leq n - 2m - 4$. □

Lemma 3.3.8. *The two-step complexes $E \in \mathcal{A}_0$ with $H^{-1}(E) = L^{-1}\mathcal{P}_{\hat{x}}$ and $H^0(E)$ a μ-stable locally-free sheaf with $\mathrm{ch}(H^0(E)) = (2, \ell, 0)$ (which only happens when $n = 5$) are μ_t-stable for all $0 < t < 1/\sqrt{3}$.*

Proof Using the fact that $\Phi_0(L^{-1}\mathcal{P}_{\hat{x}}) = \hat{L}\mathcal{P}_{-x}[-1]$ (see [Muk81] or [Mac11]) and $\Phi_0(H^0(E)) \cong \mathcal{O}_{\hat{D}_x}(-1)\mathcal{P}_y$ for some $x, y \in \mathbb{T}$, the spectral sequence $\Phi^p(H^q(E)) \Rightarrow \Phi^{p+q}(E)$ reduces to the following short exact sequence in $\mathcal{A}_0$

$$0 \to \hat{L}\mathcal{P}_{-x} \to \Phi(E) \to \mathcal{O}_{\hat{D}_x}(-1)\mathcal{P}_y \to 0.$$

Therefore, $\Phi(E) \in \mathcal{T}_0$ is a sheaf. To see that it is torsion-free, observe that any torsion must be supported on $\hat{D}_x$ of degree less than -1 (since $\mathcal{O}_{\hat{D}_x}(-1)\mathcal{P}_y$ is Gieseker-stable and $\hat{L}\mathcal{P}_{-x}$ is torsion-free); but this contradicts Bogomolov on $\Phi(E)/\mathrm{tors}(E)$. In other words, $\Phi(E) \cong \hat{L}^2\mathcal{I}_{\hat{X}}\mathcal{P}_z$ for some $\hat{X} \in \mathrm{Hilb}^5(\hat{\mathbb{T}})$ and $z \in \mathbb{T}$ by Lemma 2.4.1(b). We know these objects are μ_t-stable for $t > \sqrt{3}$ by Lemma 3.3.6 and so we can conclude that E is μ_t-stable for $0 < t < 1/\sqrt{3}$ by Lemma 3.2.7. (Observe that E is destabilised by its own cohomology when $t \geq 1/\sqrt{3}$.) □

As a summary of the previous lemmas, we have the following

Theorem 3.3.9. *The objects $E \in \mathcal{A}_0$ with numerical invariants*

$$\mathrm{ch}(E) = (1, 2\ell, 4 - n)$$

that are μ_t-stable for some $t > 0$ are either

(a) *$L^2 \mathcal{I}_X \mathcal{P}_{\hat{x}}$ for some $X \in \mathrm{Hilb}^n(\mathbb{T})$ and $\hat{x} \in \hat{\mathbb{T}}$, or*

(b) *$L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}} \ltimes \mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}}$ where $X' \in \mathrm{Hilb}^m(\mathbb{T})$, $m < (n-2)/2$ and $\alpha := 4 - n + m$, or*

(c) *$E := E^{-1} \xrightarrow{f} E^0$ where $H^{-1}(E) = \ker(f) = L^{-1}\mathcal{P}_{\hat{x}}$ and $H^0(E) = \mathrm{coker}(f)$ is a μ-stable locally-free sheaf with $\mathrm{ch}(H^0(E)) = (2, \ell, 0)$ only when $n = 5$.*

Moreover, if E is an object of type (a) and E is not μ_t-stable for some $t > 0$, then E is destabilised by $L\mathcal{I}_{X'}\mathcal{P}_{\hat{y}}$ for some $X' \in \mathrm{Hilb}^m(\mathbb{T})$ with $m < (n-2)/2$ and $t \leq \sqrt{n - 2m - 2}$ if and only if X contains a collinear subscheme of colength m; if $n = 5$ then there is a rank two destabiliser if and only if the configuration of X is very specific. Sufficiently general configurations are μ_t-stable for all $t > 0$. If E is an object of type (b) then (generically) they are μ_t-stable for all $t < \sqrt{n - 2m - 2}$ but a small (codim ≥ 2) subvariety of these extensions are destabilised by $L\mathcal{I}_Z$ for some $Z \in \mathrm{Hilb}^p(\mathbb{T})$ with $m < p < (n-2)/2$ and $0 < t \leq \sqrt{n - 2p - 2}$; of course, the extension itself is unstable for all $t \geq \sqrt{n - 2m - 2}$. If E is an object of type (c) then E is in fact μ_t-stable for all $0 < t < 1/\sqrt{3}$; for $t \geq 1/\sqrt{3}$, E is destabilised by its own cohomology.

In terms of moduli functors and walls, we can rephrase this as

Corollary 3.3.10. *In the one-parameter family of stability conditions $(\mathcal{A}_0, \mu_t)$, the moduli functor $\mathcal{M}_t(1, 2\ell, 4 - n)$ has $\lfloor (n-1)/2 \rfloor$ walls for all $n \in \mathbb{Z}_{\geq 0}$ except for $n = 5$ when there is an extra wall. The highest wall is at $\sqrt{n - 2}$ and, except for $n = 5$, the lowest is at $\sqrt{1 + (n + 1 \mod 2)}$.*

We can extend this result to $(\mathcal{A}_s, \mu_t)$ for $0 < s < 2$ using the following observation. Suppose $E \in \mathcal{A}_s$ with $\mathrm{ch}(E) = (1, 2\ell, 4 - n)$ is $\mu_{s,t}$-stable for some $t > 0$, $0 < s < 2$ and we have a destabilising sequence $K \to E \to Q$ in $\mathcal{A}_s$ with $\mathrm{ch}(K) = (r, c\ell, \chi)$. If K is supported in codimension two then E would destabilised by K for all $t > 0$; contradiction. Therefore K is supported in codimension ≤ 1 and $r_s(K) > 0$. In particular,

$$0 < r_s(K) \leq r_s(E) \quad \Leftrightarrow \quad 0 < \deg(K) - 2s\mathrm{rk}(K) \leq \deg(E) - 2s\mathrm{rk}(E)$$
$$\Leftrightarrow \quad 2s\mathrm{rk}(K) < \deg(K) \leq \deg(E) + 2s(\mathrm{rk}(K) - \mathrm{rk}(E))$$
$$\text{i.e.} \quad sr < c \leq 2 + s(r - 1) < 2r \quad \text{since } s < 2.$$

The destabilising condition is given by

$$\mu_{s,t}(K) \geq \mu_{s,t}(E) \Leftrightarrow 0 < (2r - c)t^2 \leq (c - 2r)s^2 - (\chi + r(n - 4))s + 2\chi + c(n - 4).$$

Completing the square shows us that the centre of the semicircle is

$$s = -\frac{\chi + r(n-4)}{2(2r-c)}, \qquad t = 0.$$

Suppose that the centre lies on the positive s-axis between 0 and 2. Then we have

$$0 < -\frac{\chi + r(n-4)}{2(2r-c)} < 2 \Leftrightarrow \chi < -r(n-4) \quad \text{since } c - 2r < 0.$$

But now the destabilising condition reduces to

$$
\begin{aligned}
0 < (2r-c)t^2 \;&\leq\; (c-2r)s^2 - (\chi + r(n-4))s + 2\chi + c(n-4) \\
&=\; (c-2r)s^2 + \chi(2-s) + (n-4)(c-sr) \\
&<\; (c-2r)s^2 - r(n-4)(2-s) + (n-4)(c-sr) \\
&\qquad\qquad\qquad\qquad\qquad \text{using } \chi < -r(n-4) \\
&=\; (c-2r)s^2 + (n-4)(c-2r)
\end{aligned}
$$

which is impossible for $n \geq 4$ (since $c < 2r$). Thus, we have

Proposition 3.3.11. *For all $n \geq 4$, the only walls associated to the Chern character $(1, 2\ell, 4-n)$ in the region $0 \leq s < 2$ are those which intersect the line $s = 0$.*

Remark 3.3.12. Let $\iota : \mathbb{T} \xrightarrow{\sim} \Delta \subset \mathbb{T} \times \mathbb{T}$ be the diagonal embedding of $\mathbb{T}$. Then $\iota_* L^m$ is the Fourier-Mukai kernel corresponding to the (trivial) automorphism of twisting by L^m. In our notation, this kernel gives rise to an equivalence

$$\Phi_{\iota_* L^m} : \mathcal{A}_s \xrightarrow{\sim} \mathcal{A}_{s+m} \quad \text{for all } s \in \mathbb{Q}.$$

That is, twisting by L^m just translates the wall and chamber structure m units to the right. In particular, by Lemma 3.3.1(iv) we know that $L\mathcal{I}_X$ is $\mu_{s,t}$-stable for all $t > 0$ and $0 \leq s < 1$. Therefore, we see that $L^2\mathcal{I}_X$ is $\mu_{s,t}$-stable for all $t > 0$ and $1 \leq s < 2$, i.e. there are no walls for $L^2\mathcal{I}_X$ in the region $1 \leq s < 2$ for any $X \in \mathrm{Hilb}^n(\mathbb{T})$. Similarly, by Lemma 3.3.1(i) we see that there are no walls for L^m in the whole of the (s,t)-plane.

3.4 Flat Families

This section and the next borrow heavily from the ideas of [ABL07, Sections 4 & 5].

By Proposition 3.2.8 and Corollary 3.3.10, we see that for $t > \sqrt{n-2}$, the moduli functor $\mathcal{M}_t(1, 2\ell, 4-n)$ is represented by the moduli space of Gieseker-stable sheaves on $\mathbb{T}$ of the form $L^2 \mathcal{I}_X \mathcal{P}_{\hat{x}}$ for some $X \in \mathrm{Hilb}^n(\mathbb{T})$ and $\hat{x} \in \hat{\mathbb{T}}$. As t crosses the critical values

$$t_m = \sqrt{n - 2m - 2} \quad \text{where} \quad 0 \leq m < \frac{n-2}{2}$$

the μ_t-stability changes. More precisely, the moduli space $\mathcal{M}$ undergoes a birational surgery known as a Mukai flop; see section 3.5. The critical values t_m correspond to the cases where X contains a collinear subscheme of colength m.

Let $X' \in \mathrm{Hilb}^m(\mathbb{T})$ and $\alpha := 4 - n + m$. The goal of this section is to produce flat families of objects in $\mathcal{A}_0$ parametrising extensions of the form:

$$0 \to L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}} \to E \to \mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}} \to 0$$

and

$$0 \to \mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}} \to F \to L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}} \to 0$$

which are exchanged under wall-crossing.

Definition 3.4.1. Let Σ be a smooth quasi-projective scheme. An object $\mathcal{E}_\Sigma \in \mathcal{D}(\mathbb{T} \times \Sigma)$ is a family of objects in $\mathcal{D}(\mathbb{T})$ parametrised by Σ.

Definition 3.4.2. A family $\mathcal{E}_\Sigma$ is a flat family of objects in $\mathcal{A}_0$ if the (derived) restriction to the fibres:

$$\mathcal{E}_\sigma := i^*_{\mathbb{T} \times \{\sigma\}} \mathcal{E}_\Sigma$$

are objects of $\mathcal{A}_0$ for all closed points $\sigma \in \Sigma$ (via the isomorphism $\mathbb{T} \times \{\sigma\} \cong \mathbb{T}$).

Remark 3.4.3. Notice that $L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}}, \mathcal{O}_{D_z}\mathcal{P}_{\hat{y}} \in \mathcal{A}_0 \cap \mathrm{Coh}(\mathbb{T})$ and so we can use the classical notion of a flat family, i.e. a Σ-flat sheaf $\mathcal{E} \in \mathrm{Coh}(\mathbb{T} \times \Sigma)$.

Definition 3.4.4. Let $\mathrm{Hilb}^n(\mathbb{T})$ be the Hilbert scheme of length n subschemes of $\mathbb{T}$. If $Z \subset \mathbb{T}$ is a finite subscheme of length n, we shall abuse notation and denote the corresponding point in $\mathrm{Hilb}^n(\mathbb{T})$ by Z as well. More precisely, $\mathrm{Hilb}^n(\mathbb{T})$ represents the Hilbert functor $\mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Set}$

$$\mathcal{H}ilb^n_{\mathbb{T}} : \Sigma \mapsto \left\{ \begin{array}{c|c} Z \subset \mathbb{T} \times \Sigma & Z \text{ is flat over } \Sigma \text{ and} \\ \text{closed subscheme} & P_\sigma(Z) = n \text{ for all } \sigma \in \Sigma \end{array} \right\}$$

where $P_\sigma(Z)(m) := \chi(\mathcal{O}_{Z_\sigma}(m)) = \chi(\mathcal{O}_{Z_\sigma} \otimes_{\mathcal{O}_Z} \pi^*_{\mathbb{T}}(L^m))$; see [HL10, Theorem 2.2.4]. In particular, this means there is a universal subscheme $\mathcal{Z} \subset \mathbb{T} \times \mathrm{Hilb}^n(\mathbb{T})$, i.e. for every $Z \subset \mathbb{T} \times \Sigma$ which is Σ-flat with $\dim H^0(Z, \mathcal{O}_Z) = n$ there is a unique

morphism $f_Z : \Sigma \to \text{Hilb}^n(\mathbb{T})$ such that $Z = (\text{id}_\mathbb{T} \times f_Z)^{-1}(\mathcal{Z})$. The underlying set of $\mathcal{Z}$ is given by

$$\mathcal{Z} := \{(X, Z) : X \in Z\}.$$

Example 3.4.5. Torsion-free sheaves with Chern character $(1, \ell, 1 - m)$ move in a moduli of dimension $2 + 2m$. Let $X' \in \text{Hilb}^m(\mathbb{T})$ and consider the sheaf $\widetilde{L\mathcal{I}_{X'}} \to \mathbb{T} \times (\text{Hilb}^m(\mathbb{T}) \times \hat{\mathbb{T}})$ given by

$$\widetilde{L\mathcal{I}_{X'}} := \pi_1^* L \otimes \pi_{12}^* \mathcal{I}_\mathcal{Z} \otimes \pi_{13}^* \mathcal{P},$$

where $\mathcal{P}$ is the Poincaré line bundle over $\mathbb{T} \times \hat{\mathbb{T}}$, π_i and π_{ij} are the projections to the ith and ij^{th} factors respectively and $\mathcal{I}_\mathcal{Z}$ is the ideal sheaf of the tautological universal subscheme $\mathcal{Z} \subset \mathbb{T} \times \text{Hilb}^n(\mathbb{T})$. The existence of such a family is guaranteed by Corollary 2.2.4 and the universal properties of $\mathcal{P}$ and $\mathcal{Z}$ ensure that $\widetilde{L\mathcal{I}_{X'}}$ is also a universal family of objects in $\mathcal{A}_0 \cap \text{Coh}(\mathbb{T})$.

Example 3.4.6. Pure sheaves with Chern character $(0, \ell, \alpha - 1)$ move in a moduli of dimension 4. Recall that $\mathcal{O}_{D_z}(1)$ denotes a degree one line bundle supported on the divisor $D_z := \tau_z D_L \in |L\mathcal{P}_{-\hat{z}}|$. Without loss of generality we may assume that $e \in D_z$ and hence $x \in \tau_x D_z$ for all $x \in \mathbb{T}$. Thus, we can use the same idea as for the Hilbert scheme to define:

$$\mathcal{D} := \big\{(x, y) \in \mathbb{T} \times \mathbb{T} \mid x \in \tau_y D_z\big\}.$$

This is a universal subscheme in $\mathbb{T} \times \mathbb{T}$ and the diagonal $\Delta \subset \mathcal{D}$ makes sense because of our initial assumption. Setting $\mathcal{K} := \ker(\mathcal{O}_\mathcal{D} \to \mathcal{O}_\Delta)$ we get a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_\mathcal{D} \to \mathcal{O}_\Delta \to 0$$

which on the fibres $\mathbb{T} \times \{y\}$ reads

$$0 \to \mathcal{O}_{D_z}(-1) \to \mathcal{O}_{D_{z+y}} \to \mathcal{O}_y \to 0.$$

Consider the sheaf $\widetilde{\mathcal{O}_{D_z}(\alpha)} \to \mathbb{T} \times (\mathbb{T} \times \hat{\mathbb{T}})$ given by

$$\widetilde{\mathcal{O}_{D_z}(\alpha)} := \begin{cases} \pi_{12}^* \mathcal{K} \otimes \pi_{13}^* \mathcal{P} \otimes \pi_1^* L^m & \text{when } \alpha = 2m - 1, \ m \in \mathbb{Z} \\ \pi_{12}^* \mathcal{O}_\mathcal{D} \otimes \pi_{13}^* \mathcal{P} \otimes \pi_1^* L^m & \text{when } \alpha = 2m, \ m \in \mathbb{Z}. \end{cases}$$

Then by the universal properties of $\mathcal{D}$ and $\mathcal{P}$, we see that $\widetilde{\mathcal{O}_{D_z}(\alpha)}$ is a universal family of objects in $\mathcal{A}_0 \cap \text{Coh}(\mathbb{T})$.

The family of extensions that we are interested in are supported on projective

bundles:

Lemma 3.4.7. *Let $X' \in \mathrm{Hilb}^m(\mathbb{T})$ and set $\alpha := 4 - n + m$. Then for $n > 2$ and $m < (n-2)/2$*

$$\mathrm{Ext}^i_{\mathbb{T}} \left(\mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}}, L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}} \right) = 0 \quad \text{for } i = 0, 2.$$

Proof. We have $\mathrm{Ext}^0_{\mathbb{T}} \left(\mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}}, L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}} \right) \cong \mathrm{Hom}_{\mathcal{A}_0}(\mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}}, L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}})$ and since $\mathcal{A}_0$ is t-invariant, we may conclude that this is zero if we can find a value of $t > 0$ such that $\mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}}, L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}}$ are both μ_t-stable and $\mu_t(\mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}}) > \mu_t(L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}})$. But $\mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}}$ and $L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}}$ are μ_t-stable in $\mathcal{A}_0$ for all $t > 0$ (Lemma 3.3.1) and

$$\mu_t(\mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}}) > \mu_t(L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}}) \Leftrightarrow t > \sqrt{n - 2m - 2} > 0.$$

Similarly, we see that $\mathrm{Ext}^2_{\mathbb{T}} \left(\mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}}, L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}} \right) \cong \mathrm{Ext}^0_{\mathcal{A}_0} \left(L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}}, \mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}} \right)^*$ is equivalent to maps in $\mathcal{A}_0$ between the same objects but in the other direction. The same argument shows that, under the assumptions, we can always find a $t > 0$ such that $\mu_t(L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}}) > \mu_t(\mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}})$. $\qquad\square$

Lemma 3.4.8 (Universal Extension). *Let F_1, F_2 be coherent $\mathcal{O}_{\mathbb{T}}$-modules and let $E = \mathrm{Ext}^1(F_2, F_1)$. Since elements $\xi \in E$ correspond to extensions*

$$0 \to F_1 \to F_\xi \to F_2 \to 0,$$

the space $\Sigma = \mathbb{P}(E^\vee)$ parametrizes all non-split extensions of F_2 by F_1 up to scalars.

Moreover, there exists a universal extension

$$0 \to p^* F_1 \otimes q^* \mathcal{O}_\Sigma(1) \to \mathcal{F} \to p^* F_2 \to 0$$

on the product $\mathbb{T} \times \Sigma$ (with projections p and q to $\mathbb{T}$ and Σ, respectively), such that for each rational point $[\xi] \in E$, the fibre $\mathcal{F}_\xi$ is isomorphic to F_ξ.

Proof See [HL10, Example 2.1.12]. $\qquad\square$

The objects of the extensions that we are interested in are not rigid but move in moduli of their own. Nevertheless, we can still mimic the construction above to get

Proposition 3.4.9. *Let $X' \in \mathrm{Hilb}^m(\mathbb{T})$ and set $\alpha := 4 - n + m$. Then the projective bundle (Lemma 3.4.7)*

$$\mathbb{P}_m \to (\mathrm{Hilb}^m(\mathbb{T}) \times \hat{\mathbb{T}}) \times (\mathbb{T} \times \hat{\mathbb{T}}) \quad \text{with fibres} \quad \mathbb{P}\left(\mathrm{Ext}^1_{\mathcal{A}_0} \left(\mathcal{O}_{D_z}(\alpha) \mathcal{P}_{\hat{y}}, L\mathcal{I}_{X'} \mathcal{P}_{\hat{x}} \right) \right)$$

supports a universal family $\mathcal{E}_m$ (on $\mathbb{T} \times \mathbb{P}_m$) of extensions of objects in $\mathcal{A}_0$ of the form

$$0 \to L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}} \to E \to \mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}} \to 0.$$

The dual projective bundle $\mathbb{P}_m^\vee$ supports a universal family $\mathcal{F}_m$ of extensions of the form

$$0 \to \mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}} \to F \to L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}} \to 0.$$

Proof. Let the following maps denote the relevant projections:

$$
\begin{array}{ccc}
& \mathbb{T} \times \mathbb{P}_m \xrightarrow{\ \ p\ \ } \mathbb{P}_m & \\
& \rho \downarrow & \\
& \mathbb{T} \times ((\mathrm{Hilb}^m(\mathbb{T}) \times \hat{\mathbb{T}}) \times (\mathbb{T} \times \hat{\mathbb{T}})) & \\
{}^{\pi_{123}}\swarrow & & \searrow^{\pi_{145}} \\
\mathbb{T} \times (\mathrm{Hilb}^m(\mathbb{T}) \times \hat{\mathbb{T}}) & & \mathbb{T} \times (\mathbb{T} \times \hat{\mathbb{T}})
\end{array}
$$

Then, by Lemma 3.4.8, there is a universal extension on $\mathbb{T} \times \mathbb{P}_m$ given by

$$0 \to \rho^* \widetilde{L\mathcal{I}_{X'}}_{123} \otimes p^* \mathcal{O}_{\mathbb{P}_m}(1) \to \mathcal{E}_m \to \rho^* \widetilde{\mathcal{O}_{D_z}(\alpha)}_{145} \to 0 \qquad (\dagger)$$

where $\widetilde{L\mathcal{I}_{X'}}_{123}, \widetilde{\mathcal{O}_{D_z}(\alpha)}_{145}$ are the pull-backs of $\widetilde{L\mathcal{I}_{X'}}, \widetilde{\mathcal{O}_{D_z}(\alpha)}$ via the projections π_{123}, π_{145} respectively. This universal extension has the property that each

$$i_{\mathbb{T} \times \epsilon}^*(\dagger): \quad 0 \to L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}} \to E \to \mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}} \to 0$$

is the short exact sequence (in $\mathcal{A}_0$) corresponding to the extension (modulo scalars):

$$\epsilon \in \mathbb{P}\left(\mathrm{Ext}^1_{\mathcal{A}_0}\left(\mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}}, L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}}\right)\right).$$

Similarly, there is a universal extension on $\mathbb{T} \times \mathbb{P}_m^\vee$ given by

$$0 \to (\rho^\vee)^* \widetilde{\mathcal{O}_{D_z}(\alpha)}_{145} \otimes (p^\vee)^* \mathcal{O}_{\mathbb{P}_m^\vee}(1) \to \mathcal{F}_m \to (\rho^\vee)^* \widetilde{L\mathcal{I}_{X'}}_{123} \to 0$$

where $\rho^\vee: \mathbb{T} \times \mathbb{P}_m^\vee \to \mathbb{T} \times ((\mathrm{Hilb}^m(\mathbb{T}) \times \hat{\mathbb{T}}) \times (\mathbb{T} \times \hat{\mathbb{T}}))$ and $p^\vee: \mathbb{T} \times \mathbb{P}_m^\vee \to \mathbb{P}_m^\vee$ are the projections. $\qquad \square$

Example 3.4.10. The moduli space of Gieseker-stable sheaves with Chern character $(1, 2\ell, 4 - n)$ is represented by $\mathrm{Hilb}^n(\mathbb{T}) \times \hat{\mathbb{T}}$ ([Muk84, Theorem 0.3]) and the universal family of coherent sheaves $\mathcal{U} \to \mathbb{T} \times (\mathrm{Hilb}^n(\mathbb{T}) \times \hat{\mathbb{T}})$ which realises this is given by

$$\mathcal{U} := \pi_1^* L^2 \otimes \pi_{12}^* \mathcal{I}_\mathcal{Z} \otimes \pi_{13}^* \mathcal{P}.$$

In general, we would like to conclude that, away from the set of critical values $\{t_m\}$, there is a fine moduli space M_t representing the functor $\mathcal{M}_t$. As we have just established, there is a universal sheaf $\mathcal{U}_t$ on $\mathbb{T} \times M_t$ which realises this for $t > \sqrt{n-2}$. In the next section, we try to keep track of this universal sheaf as t crosses the critical values and the relevant surgeries are performed.

3.5   Wall Crossing

In the remarkable [Muk84], Mukai proves that moduli spaces $\mathcal{M} = \mathcal{M}_S(r, c_1, \mathrm{ch}_2)$ of μ-stable coherent sheaves on an abelian (or K3) surface S are symplectic. More precisely, there is a skew-symmetric, non-degenerate bilinear form on the tangent bundle coming from Serre duality

$$\omega \; : \; \mathrm{Ext}^1_{\mathcal{O}_S}(E, E) \times \mathrm{Ext}^1_{\mathcal{O}_S}(E, E) \to H^2(S, \mathcal{O}_S) \cong \mathbb{C}.$$

In the same paper, Mukai goes on to prove the following

Theorem 3.5.1. *Let $\mathcal{M}$ be a symplectic variety, and let P be a $\mathbb{P}^n$-bundle contained in $\mathcal{M}$ in codimension $n \geq 2$. Then there is a birational map, denoted $elm_P : \mathcal{M} \dashrightarrow \mathcal{M}'$, called the elementary transformation along P, with the following properties:*

1) $\mathcal{M}'$ contains the dual $\mathbb{P}^n$ bundle P' of P and has a symplectic structure ω' which coincides with ω outside of P', and

2) elm_P is the composite of the blowing up $\alpha^{-1} : \mathcal{M} \dashrightarrow \widetilde{\mathcal{M}}$ along P and the blowing down of the exceptional divisor $D := \alpha^{-1}(P)$ onto P'.

Proof   See [Muk84, Theorem 0.7]. □

In Proposition 3.4.9, we showed that the extensions which became μ_t-unstable, after crossing a critical value, were supported on projective bundles. If we can show that these bundles live in codimension ≥ 2 then Theorem 3.5.1 provides a way of excising the bundle supporting the unstable extensions and gluing in the dual bundle, which we know supports stable extensions for that range of t.

Lemma 3.5.2. *The projective bundles $\mathbb{P}_m$ (of Proposition 3.4.9), which support the universal families of extensions $0 \to L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}} \to E \to \mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}} \to 0$, satisfy the necessary condition for a Mukai flop when $n \geq 4$; namely*

$$\text{fibre dimension} = \mathrm{codim}(\mathbb{P}_m) \geq 2.$$

Proof. As usual, let $X' \in \mathrm{Hilb}^m(\mathbb{T})$ with $m = 0, 1, 2, \ldots, < (n-2)/2$ and set $\alpha := 4 - n + m$. By Theorem 2.2.5 and Lemma 2.4.6 we calculate the following:

Sheaf	Moduli	Dimension
E	$\mathcal{M}_t(1, 2\ell, 4-n)$	$2+2n$
$L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}}$	$\mathcal{M}_t(1, \ell, 1-m)$	$2+2m$
$\mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}}$	$\mathcal{M}_t(0, \ell, \alpha-1)$	4
fibre	$\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{x}}, L\mathcal{I}_{X'}\mathcal{P}_{\hat{y}}))$	$n-m-2$
$\mathbb{P}_m$	"$L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}} + \mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}} +$ fibre"	$4+n+m$

Therefore, we have

$$\text{fibre dimension} = n - m - 2 = \mathrm{codim}(\mathbb{P}_m)$$

and the necessary condition is satisfied when $n - m \geq 4 \Leftrightarrow n \geq 4$. Indeed, when $n = 2k$ we have

$$m < \frac{n-2}{2} = k - 1 \Rightarrow m \leq k - 2 \Rightarrow n - m \geq 2k + (2-k) = k + 2 \geq 4 \Leftrightarrow k \geq 2.$$

We know that $k = 0, 1$ corresponds to $L, L^2\mathcal{I}_Q$ respectively (where $Q \in \mathrm{Hilb}^2(\mathbb{T})$) which are both μ_t-stable for all $t > 0$. Similarly, when $n = 2k + 1$ we have

$$m < \frac{n-2}{2} = k - \frac{1}{2} \Rightarrow m \leq k - 1 \Rightarrow n - m \geq 2k + 1 + (1-k) = k + 2 \geq 4 \Leftrightarrow k \geq 2.$$

The cases $k = 0, 1$ correspond to $L^2\mathcal{I}_P, L^2\mathcal{I}_Y$ respectively (where $P \in \mathrm{Hilb}^1(\mathbb{T})$ and $Y \in \mathrm{Hilb}^3(\mathbb{T})$); $L^2\mathcal{I}_P$ is μ_t-stable for all $t > 0$ but $L^2\mathcal{I}_Y$ is destabilised by L when $t \leq 1$ if and only if Y is collinear. See section 4.1.4 for more details. $\qquad\square$

Remark 3.5.3. Notice that for collinear Y, the appropriate surgery on $\mathcal{M}_t$ as t passes over the critical value ($t = 1$) is a codimension 1 operation; this will be dealt with as a special case in section 4.1.4.

Tying in with the theme of preservation of stability, we have the following

Corollary 3.5.4. *Let $n \geq 4$ and $X \in \mathrm{Hilb}^n(\mathbb{T})$ be generic. Then $L^2\mathcal{I}_X$ is a μ-stable $\Phi_{\mathcal{P}}$-WIT$_1$ sheaf with μ-stable transform.*

Proof By Lemma 3.5.2, we see that every wall on the line $s = 0$ has codimension at least one. In particular, for sufficiently generic configurations of $X \in \mathrm{Hilb}^n(\mathbb{T})$, the sheaves $L^2\mathcal{I}_X$ will not be affected by any walls. Therefore, the stability cannot change. The fact that $L^2\mathcal{I}_X$ is WIT$_1$ follows from Proposition 3.3.2. $\qquad\square$

In order to show that there is a fine moduli space M_t representing $\mathcal{M}_t$ for all values of t away from the critical set $\{t_m\}$ we proceed in exactly the same way as the proof of [ABL07, Theorem 5.1], modifying the details to our setting as we go. Rather than regurgitate their proof here, we will provide the following

Summary 3.5.5. As already observed, the moduli space $\mathcal{M}_t(1, 2\ell, 4 - n)$ for $t > t_0 = \sqrt{n-2}$ is the "classical" moduli space of Gieseker-stable sheaves of the form $L^2 \mathcal{I}_X \mathcal{P}_{\hat{x}}$ for some $X \in \mathrm{Hilb}^n(\mathbb{T})$ and $\hat{x} \in \hat{\mathbb{T}}$. By Example 3.4.10, we have an explicit universal sheaf $\mathcal{U}_{t_0 + \epsilon}$ which realises $M_0 := M_{t_0 + \epsilon} = \mathrm{Hilb}^n(\mathbb{T}) \times \hat{\mathbb{T}}$ as a fine moduli space. The proof now tracks this sheaf through the elementary modification along $\mathbb{P}_0$ to produce a new object in $\mathcal{U}' \in \mathcal{D}(\mathbb{T} \times \mathcal{M}')$. Remarkably, $\mathcal{U}'$ is in fact a universal sheaf which together with $\mathcal{M}'$ agrees with $(\mathcal{M}_{t_0 - \epsilon}, \mathcal{U}_{t_0 - \epsilon})$, i.e. we have constructed a fine moduli space $M_1 := M_{t_0 - \epsilon}$. Using induction, we can show this is indeed the case around each critical value:
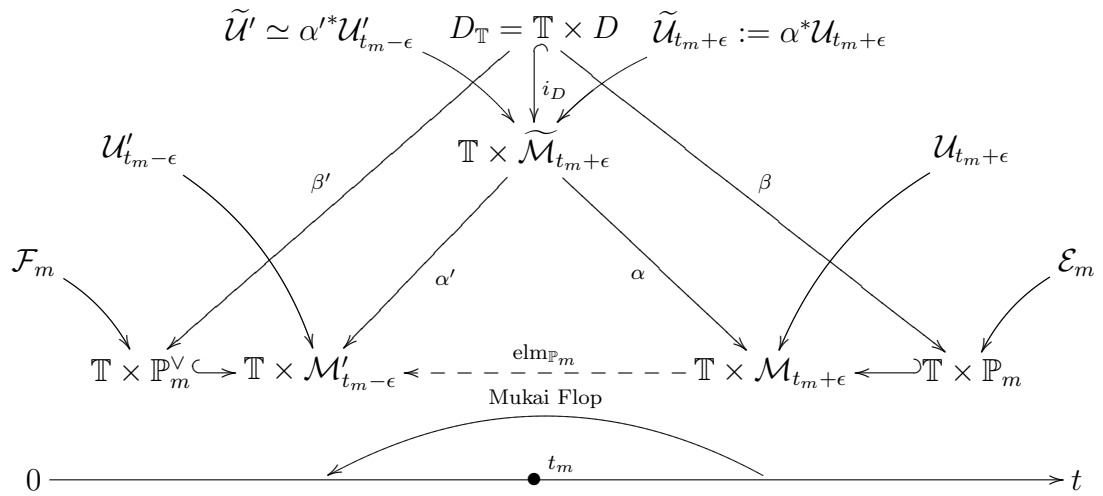
Remark 3.5.6. The main difference in our version of the proof is that we do not require [ABL07, Lemma 5.4]. This is somewhat reassuring given that we expect the Lemma is generally false in our situation. In fact, following discussions with Aaron Bertram, we realised that they do not need it for their proof of [ABL07, Theorem 5.1] either(!). The issue is that we cannot hope to find an isomorphism between $\mathcal{U}|_{\mathbb{T} \times \mathbb{P}_m}$ and $\mathcal{E}_m$ since $\mathcal{U}_{t_m + \epsilon}$ and $\mathcal{U}_{t_m + \epsilon} \otimes \mathcal{L}_{\mathcal{M}_{t_m + \epsilon}}$ give equivalent universal families for any line bundle $\mathcal{L}_{\mathcal{M}_{t_m + \epsilon}}$ on $\mathcal{M}_{t_m + \epsilon}$. But we can find an explicit Poincaré-type object $\mathcal{L}$ on $(\mathrm{Hilb}^m(\mathbb{T}) \times \hat{\mathbb{T}}) \times (\mathbb{T} \times \hat{\mathbb{T}})$ such that $\mathcal{U}|_{\mathbb{T} \times \mathbb{P}_m} \cong \mathcal{E}_m \otimes \rho^* \mathcal{L}$. Namely,

$$\mathcal{L} := \left(\left(\mathrm{id} \times \phi_L^{-1} \right) \circ \pi_{12} \right)^* \mathcal{I}_{\mathcal{Z}} \otimes \pi_{34}^* \mathcal{P}$$

where $\phi_L : \mathbb{T} \xrightarrow{\sim} \hat{\mathbb{T}}$ is the identification given by the polarization L.

Curiously, we can prove an analogue of [ABL07, Lemma 5.4] when $n = 4$:

Lemma 3.5.7. *For $n = 4$ and $m = 0$, the following map is surjective for all*

$t > 0$

$$\mathrm{Pic}\left(\mathcal{M}_t(1, 2\ell, 4 - n)\right) \twoheadrightarrow \mathrm{Pic}(\mathbb{P}_0)/\mathrm{Pic}((\mathrm{Hilb}^m(\mathbb{T}) \times \hat{\mathbb{T}}) \times (\mathbb{T} \times \hat{\mathbb{T}})) \cong \mathbb{Z}.$$

Proof Recall for $n = 4$ there is only one wall at $t_0 = \sqrt{2}$ corresponding to $Z \subset D_v$ being collinear. For all $t > t_0$, we know that $E \in \mathcal{M}_t$ is torsion-free and μ_t-stable (Theorem 3.3.9). That is, $E = L^2 \mathcal{I}_Z \mathcal{P}_{\hat{x}}$ for some $Z \in \mathrm{Hilb}^4(\mathbb{T})$ and $\hat{x} \in \hat{\mathbb{T}}$. The Fourier-Mukai transform of such objects is completely understood; see [Mac11, Section 8]. In particular, for $Z \subset D_v$ and $\sigma := \sum Z$ the Mukai spectral sequence reduces to

$$0 \to L\mathcal{P}_{-\hat{v}} \to L^2 \mathcal{I}_{-Z} \to \mathcal{O}_{D_{-v}} \mathcal{P}_{2\hat{v} - \hat{\sigma}} \to 0$$

which lies in the fibre of $\mathbb{P}_0$ over $(-\hat{v}, -v, 2\hat{v} - \hat{\sigma}) \in \hat{\mathbb{T}} \times \mathbb{T} \times \hat{\mathbb{T}}$.

Recall that sheaves of the form $L^2 \mathcal{I}_Z$ naturally sit inside the twisted structure sequence $0 \to L^2 \mathcal{I}_Z \to L^2 \to \mathcal{O}_Z \to 0$ which under the standard Fourier-Mukai transform becomes

$$0 \to \Phi^0(L^2 \mathcal{I}_Z) \to \widehat{L^2} \xrightarrow{f} \mathcal{H}_Z \to \Phi^1(L^2 \mathcal{I}_Z) \to 0.$$

Taking determinants, we see that

$$[\det(f)] \in \mathbb{P}\mathrm{Hom}(\det(\widehat{L^2}), \det(\mathcal{H}_Z)) \cong \mathbb{P}H^0(\hat{L}^2 \mathcal{P}_\sigma) \cong \mathbb{P}\Phi^0(\hat{L}^2)_\sigma =: \mathbb{P}(\widehat{\hat{L}^2})_\sigma$$

by semicontinuity. Therefore, we have a natural map:

$$\pi : \mathcal{M}_{t \geq \sqrt{2}}(1, 2\ell, 0) \to |\hat{L}^2 \mathcal{P}_\sigma| \; ; \; L^2 \mathcal{I}_Z \mapsto [\det(f)]$$

where the image is the fibre of $\mathbb{P}(\widehat{\hat{L}^2})$ over $\sigma \in \mathbb{T}$:

$$
\begin{array}{ccc}
\mathbb{P}H^0(\hat{L}^2 \mathcal{P}_\sigma) \cong \mathbb{P}^3 & \hookrightarrow & \mathbb{P}(\widehat{\hat{L}^2}) \\
\downarrow{\scriptstyle \mathrm{fibre}} & & \downarrow \\
\sigma = \Sigma Z & \hookrightarrow & \mathbb{T}
\end{array}
$$

This map is defined regardless of whether Z is collinear or not. However, if $Z \subset D_v$ is collinear, then the map $L^2 \mathcal{I}_{-Z} \mapsto \mathbb{P}(\widehat{\hat{L}^2})_{-\sigma}$ factors through $(\mathbb{P}_0)_{(-\hat{v}, -v, 2\hat{v} - \hat{\sigma})}$ for some $v \in \mathbb{T}$. That is, we have an inclusion on the fibres

$$\mathbb{P}^2 \cong \mathbb{P}\mathrm{Ext}^1(\mathcal{O}_{D_{-v}} \mathcal{P}_{2\hat{v} - \hat{\sigma}}, L\mathcal{P}_{-\hat{v}}) \hookrightarrow \mathbb{P}(\widehat{\hat{L}^2})_{-\sigma} \cong \mathbb{P}^3.$$

$$\text{Pic}\left(\mathcal{M}_t(1, 2\ell, 0)\right) \longrightarrow \text{Pic}(\mathbb{P}(\widehat{\hat{L}^2}))/\text{Pic}(\mathbb{T})$$

$$\text{Pic}(\mathbb{P}_0)/\text{Pic}(\hat{\mathbb{T}} \times \mathbb{T} \times \hat{\mathbb{T}}) \cong \mathbb{Z}.$$

The isomorphism with $\mathbb{Z}$ follows from the fact that $\mathbb{P}_0$ is a projective bundle over $\hat{\mathbb{T}} \times \mathbb{T} \times \hat{\mathbb{T}}$. Surjectivity follows since the line bundle $\pi^* \mathcal{O}_{\mathbb{P}(\widehat{\hat{L}^2})_{-\sigma}}(1) = \pi^* \mathcal{O}_{\mathbb{P}^3}(1)$ carries over to a line bundle on $\mathcal{M}_t$ (across the Mukai flop at $t = \sqrt{2}$), which agrees with $\pi^* \mathcal{O}_{(\mathbb{P}_0)_{(-\hat{v}, -v, 2\hat{v}-\hat{\sigma})}}(1) = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$ and so this line bundle on $\mathcal{M}_t$ generates the relative Picard group of $\mathbb{P}_0$ over $\hat{\mathbb{T}} \times \mathbb{T} \times \hat{\mathbb{T}}$. $\square$

We have the following analogue of [ABL07, Theorem 5.1]

Theorem 3.5.8. *Let $(\mathbb{T}, L)$ be an irreducible principally polarized abelian surface with $\text{Pic}(\mathbb{T}) = \mathbb{Z}[\ell]$ and consider objects $E \in \mathcal{A}_0$ with $\text{ch}(E) = (1, 2\ell, 4 - n)$ where $n \in \mathbb{Z}_{\geq 0}$ and E is μ_t-stable for some $t > 0$. Then we have a set of critical values:*

$$\left\{ t_m = \sqrt{n - 2m - 2} \ : \ 0 \leq m < \frac{n-2}{2} \right\},$$

away from which, there is a smooth proper moduli space

$$\mathcal{M}_t := \mathcal{M}_t(1, 2\ell, 4 - n)$$

which together with a suitable coherent sheaf $\mathcal{U}_t$ on $\mathbb{T} \times \mathcal{M}_t$ represents the functor: isomorphism classes of flat families of μ_t-stable objects in $\mathcal{A}_0$.

Moreover, if $X' \in \text{Hilb}^m(\mathbb{T})$ and $\alpha := 4 - n + m$ then there are flat families of objects in $\mathcal{A}_0$ parameterising extensions of the form

$$0 \to L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}} \to E \to \mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}} \to 0 \quad (*)$$

and

$$0 \to \mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}} \to F \to L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}} \to 0 \quad (**)$$

which are exchanged under the wall-crossing. In particular, these families are supported by projective bundles $\mathbb{P}_m$ and $\mathbb{P}_m^\vee$ respectively, where

$$\mathbb{P}_m \to (\text{Hilb}^m(\mathbb{T}) \times \hat{\mathbb{T}}) \times (\mathbb{T} \times \hat{\mathbb{T}}) \quad \text{with fibres} \quad \mathbb{P}\left(\text{Ext}^1_{\mathcal{A}_0}\left(\mathcal{O}_{D_z}(\alpha)\mathcal{P}_{\hat{y}}, L\mathcal{I}_{X'}\mathcal{P}_{\hat{x}}\right)\right).$$

For $t > t_m$, the μ_t-stable objects are given by extensions $(*)$ and for $t < t_m$ by extensions $(**)$; Serre duality exchanges these under a Mukai flop

$$\mathbb{P}_m^\vee \subset \mathcal{M}_{t_m - \epsilon} \dashleftarrow \mathcal{M}_{t_m + \epsilon} \supset \mathbb{P}_m.$$

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We have an extra critical value at $t = 1/\sqrt{3}$ if and only if $n = 5$. In this case, extensions of the form $0 \to \Phi_0\left(\mathcal{O}_{\hat{D}_z}(-1)\mathcal{P}_y\right) \to E \to L^{-1}\mathcal{P}_{\hat{x}}[1] \to 0$ are replaced with two-step complexes $0 \to L^{-1}\mathcal{P}_{\hat{x}}[1] \to F \to \Phi_0\left(\mathcal{O}_{\hat{D}_z}(-1)\mathcal{P}_y\right) \to 0$ and the relevant projective bundle $\mathbb{P}$ is given by

$$\mathbb{P} \to \mathbb{T} \times (\mathbb{T} \times \hat{\mathbb{T}}) \quad \text{with fibres} \quad \mathbb{P}(\mathrm{Ext}^1_{\mathcal{A}_0}(L^{-1}\mathcal{P}_{\hat{x}}[1], \Phi_0(\mathcal{O}_{\hat{D}_z}(-1)\mathcal{P}_y)).$$

3.6   Projectivity

If we number the walls $i = 0, \ldots, d = \lfloor (n-3)/2 \rfloor$ from the greatest t downwards then we have $\lfloor (n+1)/2 \rfloor$ potential moduli spaces M_i, with $M_0 = \mathrm{Hilb}^n(\mathbb{T}) \times \hat{\mathbb{T}}$ (and analogously for $n = 5$):

Theorem 3.6.1. *For any $t > 0$, the moduli space of μ_t-stable objects in $\mathcal{A}_0$ with Chern character $(1, 2\ell, 4 - n)$ is a smooth complex projective variety for each non-negative integer n.*

Proof. The fact that the M_i are fine moduli spaces given by smooth varieties follows from Theorem 3.5.8. Notice when $n = 0, 1$ or 2 there are no walls and hence one moduli space which is evidently projective. The case $n = 3$ will be dealt with as a special case in Theorem 4.1.4 below but for this section, we will assume that $n \geq 4$. In which case, objects in $\Phi_0(M_{d+1}) = M_{t \gg t_0}(n-4, 2\ell, -1)$ are represented by sheaves and so M_{d+1} is projective; see Corollary 3.2.9, Proposition 3.3.2 and the comments which follow. To show that the other spaces M_i are projective we observe that (for $n \geq 4$) each chamber intersects the the real line and close to the real line we can find a suitable Fourier-Mukai transform which sends stable objects to ideal sheaves, therefore identifying the moduli space with the Hilbert scheme.

More precisely, Proposition 3.3.11 tells us that the only walls in the region $0 \leq s < 2$ are those which intersect the line $s = 0$. In particular, every destabiliser (apart from $n = 5$) is of the form $L\mathcal{I}_{X'}$ for some $X' \in \mathrm{Hilb}^m(\mathbb{T})$ with $m < n/2 - 1$ and the condition for the corresponding wall is given by

$$t^2 + \left(s + \frac{n-m-3}{2}\right)^2 - \left(\frac{n-m-3}{2}\right)^2 - (n - 2m - 2) = 0.$$

One can observe that these semicircles all satisfy $0 < (\text{centre} + \text{radius}) \leq 1$ with equality precisely when $m = 0$. In other words, there are no walls in the region

$1 < s < 2$ and every semicircle intersects the line $t = 0$ for some $s \in (0, 1]$. For the case $n = 10$, the resulting semicircles are illustrated in the following picture:
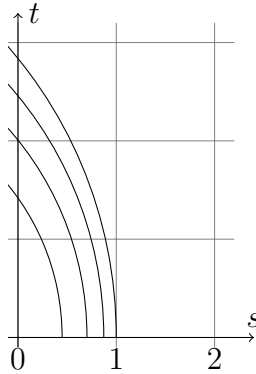
Wall and chamber structure for $n = 10$.

The semicircles are nested and intersect the s-axis in distinct points. Thus, for each moduli space M_i we can always find a rational number $s \in \mathbb{Q}$ which lies between the ith and $i + 1^{\text{st}}$ wall on the line $t = 0$ by "sliding down the wall":
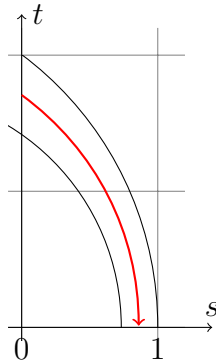
Sliding down the wall.

Now, by Corollary 3.2.9, we can use the Fourier-Mukai transform Φ_{-s} to identify $\mathcal{M}_{s,t \ll 1}(1, 2\ell, 4 - n)$ with $\mathcal{M}_{s',t \gg 1}(\Phi^H_{-s}(1, 2\ell, 4 - n))$ which is projective by Proposition 3.2.8. The Chern character $(1, 2\ell, 4 - n)$ is primitive and so by Corollary 2.2.4 we know that $M_{s',t \gg 1}(\Phi^H_{-s}(1, 2\ell, 4 - n))$ is a fine moduli space of torsion-free sheaves provided it is not empty. This is taken care of in Proposition 3.6.2.

$\square$

Proposition 3.6.2. *Let $0 < s < 1$ be a rational number. If $n \geq 4$ there is some $X \in \mathrm{Hilb}^n(\mathbb{T})$ such that $\Phi_{-s}(L^2 \mathcal{I}_X)$ is a torsion-free sheaf in $\mathcal{A}_{s'} = \Phi_{-s}(\mathcal{A}_s)$.*

Proof   Let $X \in \mathrm{Hilb}^n(\mathbb{T})$ be such that it does not contain a collinear subscheme of colength m for $0 \leq m < n/2 - 1$. Suppose for a contradiction that $\Phi_{-s}^{-1}(L^2\mathcal{I}_X) \neq 0$ and consider the natural short exact sequence in $\mathcal{A}_{s'}$

$$0 \to \Phi_{-s}^{-1}(L^2\mathcal{I}_X)[1] \to \Phi_{-s}(L^2\mathcal{I}_X) \to \Phi_{-s}^0(L^2\mathcal{I}_X) \to 0.$$

Applying the inverse transform yields a short exact sequence in $\mathcal{A}_s$

$$0 \to \hat{\Phi}_{-s}(\Phi_{-s}^{-1}(L^2\mathcal{I}_X)[1]) \to L^2\mathcal{I}_X \to \hat{\Phi}_{-s}(\Phi_{-s}^0(L^2\mathcal{I}_X)) \to 0$$

from which we can see that $K := \hat{\Phi}_{-s}(\Phi_{-s}^{-1}(L^2\mathcal{I}_X)[1]) \in \mathcal{T}_s$ is a (shifted) WIT$_{-1}$ torsion-free sheaf. Thus, if $\mathrm{ch}(K) = (r, c\ell, \chi)$ and $\mathrm{ch}(\mathcal{E}_x) = (a^2, -ab\ell, b^2)$ with $s = b/a$ then

$$
\begin{aligned}
0 \quad &< \quad \chi(K\mathcal{E}_x) = a^2\chi + b^2 r - 2cab \\
&< \quad a^2\chi + b^2 r - 2srab \quad \text{since } K \in \mathcal{T}_s \Rightarrow c > sr \\
&= \quad a^2\chi - b^2 r \quad \Rightarrow \quad \chi > s^2 r.
\end{aligned}
$$

Also, by Corollary 3.2.5, we have $\mu(K) < \mu(L^2\mathcal{I}_X) \Leftrightarrow c < 2r$. Now, the destabilising condition for $K \hookrightarrow L^2\mathcal{I}_X$ in $\mathcal{A}_s$ is given by

$$\mu_t(K) \geq \mu_t(L^2\mathcal{I}_X) \Leftrightarrow 0 < (2r - c)t^2 \leq (c - 2r)s^2 + \chi(2 - s) + (n - 4)(c - sr).$$

But the above inequalities tell us that

$$(c - 2r)s^2 + \chi(2 - s) + (n - 4)(c - sr) > (sr - 2r)s^2 + s^2 r(2 - s) = 0,$$

i.e. K destabilises $L^2\mathcal{I}_X$ for some $t > 0$ and $0 < s < 1$. By Theorem 3.3.9, we see that K must be of the form $L\mathcal{I}_{X'}$ for some $X' \in \mathrm{Hilb}^m(\mathbb{T})$ with $m < (n - 2)/2$; unless $n = 5$, in which case we just choose $s > 1/3$. In particular, K is a destabiliser if and only if X contains a collinear subscheme of colength m; contradiction. Therefore, $\Phi_{-s}(L^2\mathcal{I}_X)$ is a $\mu_{s',t}$-stable sheaf for all $t \gg 1$ and hence must be torsion-free by Proposition 3.2.8. $\square$

Chapter 4

Wall and Chamber Structure Computations

4.1 Examples

Let us now consider the low values of n in more detail. For convenience, let us recall the destabilising condition for $L^2 \mathcal{I}_X$. That is, if $K \hookrightarrow L^2 \mathcal{I}_X$ in $\mathcal{A}_s$ for some $X \in \text{Hilb}^n(\mathbb{T})$ and $s \in \mathbb{Q}$ then

$$\mu_{s,t}(K) \geq \mu_{s,t}(L^2 \mathcal{I}_X) \Leftrightarrow 0 < (2r-c)t^2 \leq (c-2r)s^2 + ((4-n)r-\chi)s + 2\chi - (4-n)c$$

where $c < 2r$ by Corollary 3.2.5 and the corresponding semicircle has

$$\text{centre} = \left(\frac{(4-n)r - \chi}{2(2r-c)}, 0 \right)$$

and

$$\text{radius} = \sqrt{(\text{centre} - 2)^2 - n}.$$

Notice that the radius is positive whenever centre $< 2 - \sqrt{n}$ or centre $> 2 + \sqrt{n}$ but this last inequality can never be satisfied since $L^2 \mathcal{I}_X \in \mathcal{T}_s \Rightarrow s < 2$ and $\text{Hom}(K, L^2 \mathcal{I}_X) \neq 0$ by assumption. By Lemma 3.3.1(iv) and Remark 3.3.12 we see that there are no walls for $L^2 \mathcal{I}_X$ in the region $1 \leq s < 2$ for any $X \in \text{Hilb}^n(\mathbb{T})$. Therefore, no semicircle can intersect the line $s = 1$ and we must have

$$
\begin{aligned}
\text{centre} \pm \text{radius} \leq 1 \quad &\Leftrightarrow \quad \text{centre} \pm \sqrt{(\text{centre} - 2)^2 - n} \leq 1 \\
&\Leftrightarrow \quad (\text{centre} - 2)^2 - n \leq (1 - \text{centre})^2 \\
&\Leftrightarrow \quad (3 - n)/2 \leq \text{centre}.
\end{aligned}
$$

In other words, the centre of any potential wall is confined to lie between

$$\frac{3-n}{2} \leq \text{centre} < 2 - \sqrt{n}.$$

4.1.1 $n = 0$

By Lemma 3.3.1(i) and Remark 3.3.12 we see that there are no walls in the whole of the (s,t)-plane.

4.1.2 $n = 1$

Let $(P, \hat{x}) \in \text{Hilb}^1(\mathbb{T}) \times \hat{\mathbb{T}}$. Plugging $n = 1$ into the formulae above, we see that any wall must satisfy $1 \leq \text{centre} < 1$; contradiction. Therefore, there are no walls for $L^2 \mathcal{I}_P$ in the whole (s,t)-plane for any $P \in \text{Hilb}^1(\mathbb{T})$.

4.1.3 $n = 2$

Let $(Q, \hat{x}) \in \text{Hilb}^2(\mathbb{T}) \times \hat{\mathbb{T}}$. Following [Mac11, Proposition 6.3], we see that the Mukai spectral sequence for $L^2 \mathcal{I}_Q$ gives rise to the following short exact sequence in $\mathcal{A}_s$ for $0 \leq s < 1$

$$0 \to \hat{\Phi}_0(\mathcal{H}_Q L^{-1}) \to L^2 \mathcal{I}_{-Q} \to \mathcal{P}_{\hat{x}}[1] \to 0$$

where $x = \sum Q$ and $\hat{\Phi}_0(\mathcal{H}_Q L^{-1})$ is a μ-semistable bundle with $\text{ch}(\hat{\Phi}_0(\mathcal{H}_Q L^{-1})) = (2, 2\ell, 2)$. Since $\hat{\Phi}_0(\mathcal{H}_Q L^{-1})$ and $\mathcal{P}_{\hat{x}}$ are WIT_{-1} and WIT_1 respectively, we can use Theorem 2.5.7 to see that

$$\text{Hom}_{\mathcal{A}_s}(\hat{\Phi}_0(\mathcal{H}_Q L^{-1}), \mathcal{P}_{\hat{x}}[1]) \cong \text{Ext}^1_{\mathbb{T}}(\hat{\Phi}_0(\mathcal{H}_Q L^{-1}), \mathcal{P}_{\hat{x}}) \cong \text{Ext}^{-1}_{\mathbb{T}}(\mathcal{H}_Q L^{-1}, \mathcal{O}_{-\hat{x}}) = 0$$

and

$$\text{Hom}_{\mathcal{A}_s}(\mathcal{P}_{\hat{x}}[1], \hat{\Phi}_0(\mathcal{H}_Q L^{-1}) \cong \text{Ext}^{-1}_{\mathbb{T}}(\mathcal{P}_{\hat{x}}, \hat{\Phi}_0(\mathcal{H}_Q L^{-1})) = 0.$$

Also, we can observe that $\dim \mathcal{M}(1, 2\ell, 2) = 2 + v(E)^2 = 6$ and $\hat{\Phi}_0(\mathcal{H}_Q L^{-1})$ moves in a 4-dimensional family whilst $\mathcal{P}_{\hat{x}}[1]$ moves in a 2-dimensional family, i.e. this is a codimension zero wall with $(\text{centre}, \text{radius}) = (1/2, 1/2)$.

Now, consider the composite Fourier-Mukai transform

$$\Psi := \hat{\Phi}_0 \circ L^{-2} \circ \Phi_0 \circ L^{-4} : \mathcal{D}(\mathbb{T}) \xrightarrow{\sim} \mathcal{D}(\mathbb{T})$$

where we implicitly understand that L^m represents the Fourier-Mukai transform

corresponding to twisting by L^m. Using the fact that

$$\Phi_0^H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Phi_{\iota_* L^m}^H = \begin{pmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ m^2 & 2m & 1 \end{pmatrix}$$

we see that

$$\Psi^H = \begin{pmatrix} 49 & -28 & 4 \\ 28 & -15 & 2 \\ 16 & -8 & 1 \end{pmatrix} \quad \Rightarrow \quad \text{ch}(\Psi(L^2 \mathcal{I}_Q)) = \text{ch}(L^2 \mathcal{I}_Q).$$

Lemma 4.1.1. *Let* $K_i := \Psi(K_{i-1})$ *with* $K_0 := \hat{\Phi}_0(\mathcal{H}_Q L^{-1})$ *and* $Q_i := \Psi(Q_{i-1})$ *with* $Q_0 := \mathcal{P}_{\hat{x}}$. *Then* K_i *and* Q_i *are* Ψ-IT$_0$ *for all* $i \geq 1$.

Proof Observe that K_0 is a semi-rigid object, i.e. $\chi(K_0, K_0) = 0$. By Theorem 2.4.7, we see that K_0 gives rise to a Fourier-Mukai transform with kernel $\mathcal{K}$ (say) such that $K_0 = \mathcal{K}_x := \Phi_{\mathcal{K}}(\mathcal{O}_x)$. Let $\mathcal{E}$ be the kernel corresponding to the Fourier-Mukai transform Ψ defined above, i.e. $\Psi := \Phi_{\mathcal{E}}$. Then

$$K_1 := \Psi(K_0) = \Psi_{\mathcal{E}}(\Phi_{\mathcal{K}}(\mathcal{O}_x)) = (\Psi_{\mathcal{E}} \circ \Phi_{\mathcal{K}})(\mathcal{O}_x) = \mathcal{F}_x$$

where $\mathcal{F}$ is the composite Fourier-Mukai kernel given by Proposition 2.3.3. By Proposition 2.5.11, $K_1 = \mathcal{F}_x$ is a μ-stable vector bundle and $\chi(K_1) > 0$ implies K_1 is either IT$_0$ or IT$_2$. But $\deg(K_1) > 0$ and so K_1 is forced to be IT$_0$ by Lemma 2.6.2. A similar argument shows that Q_1 is IT$_0$.

Suppose K_i is IT$_0$ with $\chi(K_i, K_i) = 0$, i.e. $\text{ch}(K_i) = (a^2, ab\ell, b^2)$ for two coprime integers $a > 0$ and b; notice that $\text{ch}(K_1) = 2(5^2, 15\ell, 3^2)$ and so $a > b$ for K_1. Then

$$\text{ch}(K_{i+1}) = \begin{pmatrix} 49 & -28 & 4 \\ 28 & -15 & 2 \\ 16 & -8 & 1 \end{pmatrix} \begin{pmatrix} a^2 \\ ab \\ b^2 \end{pmatrix} = \begin{pmatrix} (7a - 2b)^2 \\ (7a - 2b)(4a - b) \\ (4a - b)^2 \end{pmatrix}.$$

In particular, K_{i+1} is semi-rigid as well and the statement follows by induction. $\square$

Corollary 4.1.2. Ψ *gives rise to an infinite series of codimension zero walls converging to* $2 - \sqrt{2}$.

Proof Let $E_i := \Psi(E_{i-1})$ with $E_0 := L^2 \mathcal{I}_Q$. By Lemma 4.1.1, we see that $0 \to K_i \to E_i \to Q_i[1] \to 0$ is a short exact sequence in $\mathcal{A}_s$ for $\mu(Q_i)/2 \leq s <$

$\mu(K_i)/2$. Each short exact sequence gives rise to a wall and by definition, E_i must be stable for all points outside the corresponding semicircle. Therefore, the problem reduces to understanding the how the Chern character of K_i and Q_i change. If $\mathrm{ch}(K_n) = 2(a_n^2, a_n b_n \ell, b_n^2)$ then from the matrix calculation above, we have $a_{n+1} = 7a_n - 2b_n$ and $b_{n+1} = 4a_n - b_n$ and similarly for Q_n. To show that the K_i actually destabilises E_i for all $i \geq 0$ we need to show that the corresponding semicircles have positive radii. We can check using our formula at the beginning of this section that the first wall has radius $1/2$. Now, suppose the semicircle corresponding to K_n has positive radius, i.e. $b_n^2 + (6 - 4\sqrt{2})a_n^2 + (2\sqrt{2} - 4)a_n b_n > 0$. Then, from the recursion relations, we have

$$
\begin{aligned}
& b_{n+1}^2 + (6 - 4\sqrt{2})a_{n+1}^2 + (2\sqrt{2} - 4)a_{n+1}b_{n+1} \\
=\ & (4a_n - b_n)^2 + (6 - 4\sqrt{2})(7a_n - 2b_n)^2 + (2\sqrt{2} - 4)(7a_n - 2b_n)(4a_n - b_n) \\
=\ & (17 - 12\sqrt{2})\left(b_n^2 + (6 - 4\sqrt{2})a_n^2 + (2\sqrt{2} - 4)a_n b_n \right) > 0.
\end{aligned}
$$

Therefore, by induction, we see that every K_i does indeed give rise to a genuine codimension zero wall. Suppose $\mu(K_n)/2 = b_n/a_n$ converges to a limit x, say. Then

$$
x = \lim_{n \to \infty}\left(\frac{b_{n+1}}{a_{n+1}} \right) = \lim_{n \to \infty}\left(\frac{4a_n - b_n}{7a_n - 2b_n} \right) = \lim_{n \to \infty}\left(\frac{4 - \frac{b_n}{a_n}}{7 - 2\frac{b_n}{a_n}} \right) = \frac{4 - x}{7 - 2x}
$$

$$
\Rightarrow x^2 - 4x + 2 = 0, \quad \text{i.e. } x = 2 \pm \sqrt{2}
$$

where we are forced to choose $2 - \sqrt{2}$ since $\mu(K_i) < \mu(E_i) = 2$ by Corollary 3.2.5. If $\mathrm{ch}(Q_n) = (c_n^2, c_n d_n \ell, d_n^2)$ then a similar calculation shows that $\mu(Q_n)/2 = d_n/c_n$ converges to the same limit. $\qquad \square$

The infinite series of codimension zero walls converge so quickly that we can only illustrate the first two (see Remarks 4.1.3 for an explanation of the dashed semicircle):
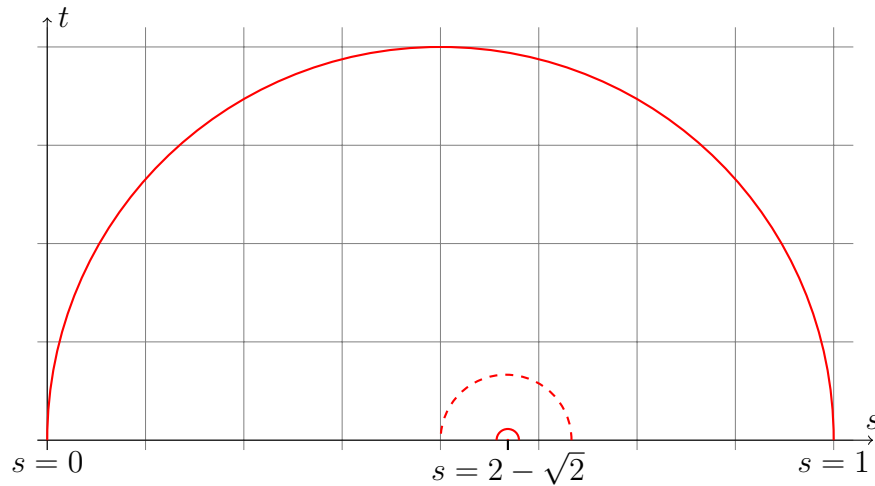
Remarks 4.1.3. It seems we have directly detected an autoequivalence $\Psi \in$ Aut$(\mathcal{D}(\mathbb{T}))$ of infinite order; of course, a generic element of Aut$(\mathcal{D}(\mathbb{T}))$ will have infinite order. As yet, we are unable to provide a link to the geometry of $\mathbb{T}$.

The example above is very special in the sense that the Chern characters of K_i and Q_i obey very specific equations: $c_n^2 - 2a_n^2 = -1$, $d_n^2 - 2b_n^2 = -2$ and $c_n d_n - 2a_n b_n = -2$. The theory of Pell's equations essentially says that if there is one solution to these equations then there is an infinite number of solutions; see [Bar03] for more details. The Mukai spectral sequence provides us with the first solution and allows the induction to start.

We will actually see a similar phenomena for all non-square values of n. In particular, it is always possible to write down a Fourier-Mukai transform which, when iterated, provides an infinite series of codimension zero walls converging to $2 - \sqrt{n}$; the powers of L in our expression for Ψ come from the repeating block of the continued fraction for $2 - \sqrt{n}$. The codimension zero wall which allows the process to start is the vertical wall at $s = 2$ corresponding to the short exact sequence $\mathcal{O}_X \to L^2 \mathcal{I}_X[1] \to L^2[1]$ in $\mathcal{A}_s$ for $s \geq 2$. The image of this wall under the Ψ considered above (when $X \in \mathrm{Hilb}^2(\mathbb{T})$) actually gives us a wall with centre and radius $(7/12, 1/12)$ which we have illustrated as a dashed line. Since no wall can intersect the line $s = 1$ (Remark 3.3.12) we know that there can be no other codimension zero walls. Thus, given the first (vertical) wall and the wall corresponding to the Mukai spectral sequence, we can use Ψ to generate them all. Furthermore, one can actually write down a transform to take you from the first wall at $s = 2$ to the wall with centre and radius $(1/2, 1/2)$; namely $\Psi' := L \circ \Phi_0 \circ L^{-1}$ when $n = 2$. Therefore, one only needs the vertical wall together with Ψ and Ψ' to generate all the walls. For $n \geq 4$, Proposition 3.3.2 states that for generic $X \in \mathrm{Hilb}^n(\mathbb{T})$, the object $L^2 \mathcal{I}_X$ is WIT$_1$ and so the Mukai spectral sequence does not provide the first semicircular wall. However, for all non-square values of n it is possible to find alternative semi-homogeneous

presentations of the (twisted) ideal sheaf; by this we mean exhibiting $L^2 \mathcal{I}_X$ as a kernel or a cokerenel of a map between semi-homogeneous sheaves.

4.1.4 $n = 3$

Let $(Y, \hat{x}) \in \text{Hilb}^3(\mathbb{T}) \times \hat{\mathbb{T}}$ and set $s = 0$. By Theorem 3.5.8, there is one critical value at $t = 1$ and thus two moduli spaces $M_0 := M_{t>1}(1, 2\ell, 1)$ and $M_1 := M_{t<1}(1, 2\ell, 1)$. In particular, by Theorem 3.3.9, we know that $L^2 \mathcal{I}_Y \mathcal{P}_{\hat{x}}$ is destabilised by $L\mathcal{P}_{\hat{y}}$ for $t \leq 1$ if and only if Y is collinear; for generic Y, $L^2 \mathcal{I}_Y \mathcal{P}_{\hat{x}}$ is μ_t-stable for all $t > 0$. By Proposition 3.4.9 and Lemma 3.5.2, there is a $\mathbb{P}^1$-bundle $M_0 \supset \mathbb{P}_0 \to \hat{\mathbb{T}} \times \mathbb{T} \times \hat{\mathbb{T}}$ supporting sheaves $E = \mathcal{O}_{D_z}(1)\mathcal{P}_{\hat{y}} \ltimes L\mathcal{P}_{\hat{x}}$ which must be replaced by the dual bundle when we cross the critical value. Notice that $\mathbb{P}_0^\vee \subset M_1$ is a codimension one subvariety and so M_0 and M_1 are birational; the map is given by identifying the points corresponding to non-collinear Y's.

Theorem 4.1.4. *The composite functor of duality followed by the standard Fourier-Mukai transofrm identifies the two moduli spaces in question. More precisely, we have the following isomorphism*

$$\Phi_0 \circ \mathbb{D} : M_{t>1}(1, 2\ell, 1) \xrightarrow{\sim} M_{t<1}(1, 2\ell, 1).$$

Proof   Applying $\mathcal{H}om(-, \mathcal{O}_{\mathbb{T}})$ to the twisted structure sequence $0 \to L^2 \mathcal{I}_Y \to L^2 \mathcal{P}_{\hat{\rho}} \to \mathcal{O}_Y \to 0$ (where $\rho = \sum Y$) yields the following long exact sequence

$$
\begin{aligned}
0 \quad &\to \quad \mathcal{E}xt^0(\mathcal{O}_Y, \mathcal{O}_{\mathbb{T}}) \to \mathcal{E}xt^0(L^2 \mathcal{P}_{\hat{\rho}}, \mathcal{O}_{\mathbb{T}}) \to \mathcal{E}xt^0(L^2 \mathcal{I}_Y, \mathcal{O}_{\mathbb{T}}) \to \\
&\to \quad \mathcal{E}xt^1(\mathcal{O}_Y, \mathcal{O}_{\mathbb{T}}) \to \mathcal{E}xt^1(L^2 \mathcal{P}_{\hat{\rho}}, \mathcal{O}_{\mathbb{T}}) \to \mathcal{E}xt^1(L^2 \mathcal{I}_Y, \mathcal{O}_{\mathbb{T}}) \to \\
&\to \quad \mathcal{E}xt^2(\mathcal{O}_Y, \mathcal{O}_{\mathbb{T}}) \to \mathcal{E}xt^2(L^2 \mathcal{P}_{\hat{\rho}}, \mathcal{O}_{\mathbb{T}}) \to \mathcal{E}xt^2(L^2 \mathcal{I}_Y, \mathcal{O}_{\mathbb{T}}) \to 0.
\end{aligned}
$$

By [Huy06, Corollary 3.40] and [Har77, Proposition 6.3] we see that $\mathcal{E}xt^0(\mathcal{O}_Y, \mathcal{O}_{\mathbb{T}}) = 0 = \mathcal{E}xt^1(\mathcal{O}_Y, \mathcal{O}_{\mathbb{T}})$ and $\mathcal{E}xt^2(\mathcal{O}_Y, \mathcal{O}_{\mathbb{T}}) \cong \mathcal{O}_Y$. Also, since $L^2 \mathcal{P}_{\hat{\rho}}$ is locally-free we have $\mathcal{E}xt^i(L^2 \mathcal{P}_{\hat{\rho}}, \mathcal{O}_{\mathbb{T}}) \cong \mathcal{E}xt^i(\mathcal{O}_{\mathbb{T}}, L^{-2}\mathcal{P}_{-\hat{\rho}}) \cong L^{-2}\mathcal{P}_{-\hat{\rho}}$ when $i = 0$ and zero otherwise; see [Har77, Proposition 6.7 & Proposition 6.3]. Now we can read off the following identities

$$\mathcal{E}xt^0(L^2 \mathcal{I}_Y, \mathcal{O}_{\mathbb{T}}) \cong \mathcal{E}xt^0(L^2 \mathcal{P}_{\hat{\rho}}, \mathcal{O}_{\mathbb{T}}) \cong L^{-2}\mathcal{P}_{-\hat{\rho}},$$

$$\mathcal{E}xt^1(L^2 \mathcal{I}_Y, \mathcal{O}_{\mathbb{T}}) \cong \mathcal{E}xt^2(\mathcal{O}_Y, \mathcal{O}_{\mathbb{T}}) \cong \mathcal{O}_Y$$

and $\mathcal{E}xt^i(L^2 \mathcal{I}_Y, \mathcal{O}_{\mathbb{T}}) = 0$ for all $i \neq 0, 1$. In particular, if we let $\mathbb{D} := R\mathcal{H}om(-, \mathcal{O}_{\mathbb{T}})[1] : \mathcal{D}(\mathbb{T}) \xrightarrow{\sim} \mathcal{D}(\mathbb{T})$ be the (shifted) derived dual functor then $\mathbb{D}(L^2 \mathcal{I}_Y) \in \mathcal{A}_0$ is a two-step complex with $H^{-1}(\mathbb{D}(L^2 \mathcal{I}_Y)) = L^{-2}\mathcal{P}_{-\hat{\rho}}$ and $H^0(\mathbb{D}(L^2 \mathcal{I}_Y)) = \mathcal{O}_Y$. The

Mukai spectral sequence $\Phi_0^p(H^q(E)) \Rightarrow \Phi_0^{p+q}(E)$ applied to an object $E \in \mathcal{A}_s$ gives rise to a long exact sequence

$$0 \to \Phi_0^0(H^{-1}(E)) \to \Phi_0^{-1}(E) \to \Phi_0^{-1}(H^0(E)) \to \Phi_0^1(H^{-1}(E)) \to \Phi_0^0(E).$$

Setting $E := \mathbb{D}(L^2\mathcal{I}_Y)$ we get

$$0 \to \Phi_0^{-1}(E) \to \mathcal{H}_Y \to \Phi_0(L^{-2}\mathcal{P}_{-\hat{\rho}}) \to \Phi_0^0(E)$$

since $L^{-2}\mathcal{P}_{-\hat{\rho}}$ is WIT_1 and $\mathcal{O}_Y$ is WIT_{-1}. Notice that the map $\mathcal{H}_Y \to \Phi_0(L^{-2}\mathcal{P}_{-\hat{\rho}})$ must be an injection since it is the transform of the (non-zero) injection $\mathcal{O}_Y \to L^{-2}\mathcal{P}_{-\hat{\rho}}[2]$; this last map is an injection because its dual is the surjection $L^2\mathcal{P}_{\hat{\rho}} \to \mathcal{O}_Y$. Therefore, $\Phi_0^{-1}(E) = 0$ and $\Phi_0(\mathbb{D}(L^2\mathcal{I}_Y)) \in \mathcal{A}_0$ is a WIT_0 sheaf with $\mathrm{ch}(\Phi_0(\mathbb{D}(L^2\mathcal{I}_Y))) = (1, 2\ell, 1)$. If $Y \subset D_x$ is collinear then we have a short exact sequence $0 \to L\mathcal{P}_{\hat{x}} \to L^2\mathcal{I}_Y \to \mathcal{O}_{D_x}(1) \to 0$ which we can track through the same process to get $0 \to \mathcal{O}_{D_x}(1) \to \Phi_0(\mathbb{D}(L^2\mathcal{I}_Y)) \to L\mathcal{P}_{\hat{x}} \to 0$; just use the fact that $\mathbb{D}(\mathcal{O}_{D_x}(1)) \cong \mathcal{O}_{D_x}(1)$ ([Huy06, Corollary 3.40]) and $\mathcal{O}_{D_x}(1)$ is WIT_0. $\qquad\square$

For completeness, observe that we have a fourth moduli space $\widehat{M_0} := \Phi_0(M_0) \cong M_{t<1}(-1, 2\ell, -1)$. A generic point of $\widehat{M_0}$ is represented by the two-step complex $\mathbb{D}(L^2\mathcal{I}_Y)$ described in Theorem 4.1.4 but there is a codimension one subvariety consisting of two-step complexes $E \in \mathcal{A}_0$ with $H^{-1}(E) \cong L^{-1}\mathcal{P}_{\hat{x}}$ and $H^0(E) \cong \mathcal{O}_{D_z}(1)\mathcal{P}_{\hat{y}}$ for some $(\hat{x}, \hat{y}, z) \in \hat{\mathbb{T}} \times \hat{\mathbb{T}} \times \mathbb{T}$. Let $Y \subset D_x$ be a collinear subscheme of length three and $\rho = \sum Y$ then we can illustrate our observations with the following picture:
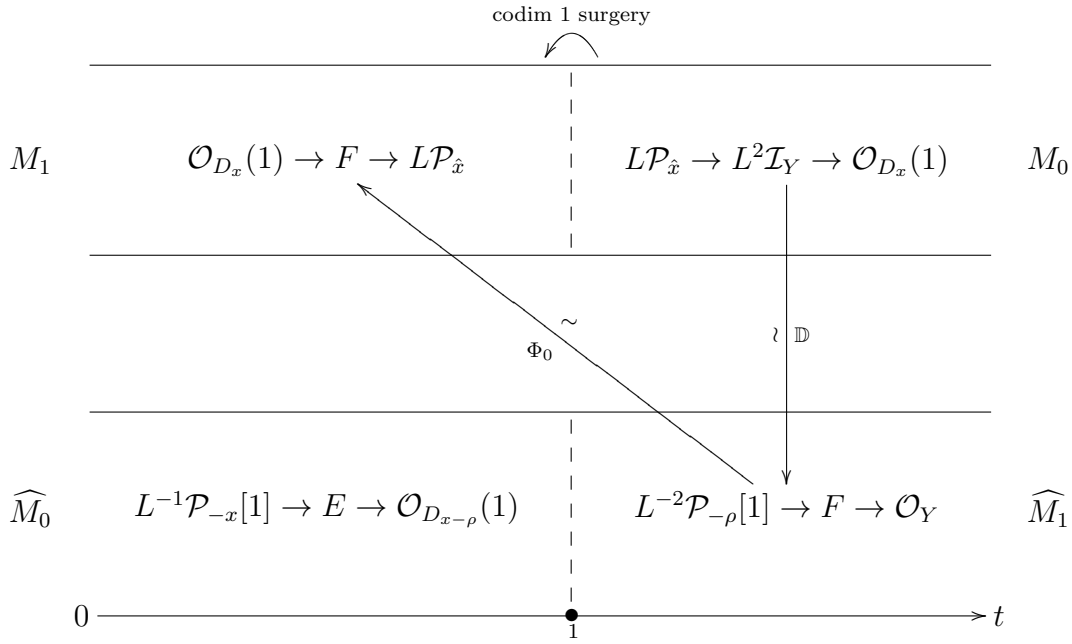
Remark 4.1.5. Using the calculation in [Mac11, Theorem 7.3] we can write down the map $M_0 \to M_1$ explicitly at a reduced scheme $\{p, q, y\} =: Y \in \mathrm{Hilb}^3(\mathbb{T})$ as

$$\begin{pmatrix} -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

thought of as acting on the "vector" $(p, q, y, \hat{x})$. In particular, it is not(!) the extension of the birational map $M_0 \dashrightarrow M_1$.

Away from $s = 0$, we find other walls. Indeed, the Mukai spectral sequence reads

$$0 \to L\mathcal{P}_{-\hat{x}} \to L^2\mathcal{I}_{-Y} \to \mathcal{O}_{D_{-x}}(1) \to 0 \quad \text{when } Y \subset D_x \text{ is collinear and}$$

$$0 \to \mathcal{H}_Y \to \hat{\Phi}_0(L^{-2})\mathcal{P}_{-\hat{\rho}} \to L^2\mathcal{I}_{-Y} \to 0 \quad \text{when } Y \text{ is generic.}$$

The first sequence gives rise to the codimension one wall we already know about and one can calculate that the corresponding semicircle has $(\text{centre}, \text{radius}) = (0, 1)$. The second sequence needs to be turned once to give a short exact sequence $0 \to \hat{\Phi}_0(L^{-2})\mathcal{P}_{-\hat{\rho}} \to L^2\mathcal{I}_{-Y} \to \mathcal{H}_Y[1] \to 0$ in $\mathcal{A}_s$ for $0 \leq s < 1/2$. In exactly the same way as we did for $n = 2$, we can show that this is also a codimension zero wall with $(\text{centre}, \text{radius}) = (1/4, 1/4)$.

This time, consider the composite Fourier-Mukai transform

$$\Psi := \Phi_0 \circ L^{-4} : \mathcal{D}(\mathbb{T}) \overset{\sim}{\to} \mathcal{D}(\mathbb{T})$$

and observe that

$$\Psi^H = \begin{pmatrix} 16 & -8 & 1 \\ 4 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \text{ch}(\Psi(L^2\mathcal{I}_Y)) = \text{ch}(L^2\mathcal{I}_Y).$$

Lemma 4.1.6. *Let $K_i := \Psi(K_{i-1})$ with $K_0 := \hat{\Phi}_0(L^{-2})$, $Q_i := \Psi(Q_{i-1})$ with $Q_0 := \mathcal{H}_Y$, $F_i := \Psi(F_{i-1})$ with $F_0 := L$ and $G_i := \Psi(G_{i-1})$ with $G_0 := \mathcal{O}_{D_x}(1)$. Then K_i, Q_i, F_i and G_i are Ψ-IT$_0$ for all $i \geq 1$.*

Proof Proceed in exactly the same way as Lemma 4.1.1 for K_i, Q_i and F_i. Notice this time, however, that if K_i is a semi-rigid IT$_0$ sheaf with $\text{ch}(K_i) = (a^2, ab\ell, b^2)$ for two coprime integers $a > 0$ and b then

$$\text{ch}(K_{i+1}) = \begin{pmatrix} 16 & -8 & 1 \\ 4 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a^2 \\ ab \\ b^2 \end{pmatrix} = \begin{pmatrix} (4a-b)^2 \\ a(4a-b) \\ a^2 \end{pmatrix}.$$

The method in Lemma 4.1.1 does not work for G_i because they move in a 4-dimensional family rather than a 2-dimensional one. Observe that G_0 is Φ_0-WIT$_0$ by Corollary 2.6.13 and since $\mathcal{O}_{D_x}(1)L^{-4} = \mathcal{O}_{D_x}(-7)$ we see that G_0 is Ψ-WIT$_{-1}$ by Corollary 2.6.12. Now, applying Ψ to $0 \to F_0 \to L^2\mathcal{I}_Y \to G_0 \to 0$ yields a short exact sequence $0 \to F_1 \to E_1 \to G_1[1] \to 0$ in $\mathcal{A}_s$ for $1/8 \leq s < 1/3$ where E_1 is a two-step complex with $H^{-1}(E_1) = \mathcal{H}_Y$ and $H^0(E_1) = \Phi_0(L^{-2})$. But $G_1 \in \mathcal{A}_s$ for $s < 1/8$ and so $\Psi^i(G_1) = 0$ for all $i \neq -1, 0$. Thus, applying Ψ again produces $0 \to F_2 \to E_2 \to G_2[1] \to 0$ in $\mathcal{A}_s$ for $\mu(G_2)/2 \leq s < \mu(F_2)/2$ since $\Psi^0(G_1[1]) \cong \Psi^1(G_1) = 0$. That is, G_1 is WIT$_0 \Rightarrow$ IT$_0$. By induction, we see that G_i is IT$_0$ for all $i \geq 1$. $\qquad\square$

Corollary 4.1.7. *Ψ gives rise to an infinite series of codimension zero and one walls which both converge to $2 - \sqrt{3}$.*

Proof Let $E_i := \Psi(E_{i-1})$ with $E_0 := L^2\mathcal{I}_Y$. By Lemma 4.1.6, we see that $0 \to K_i \to E_i \to Q_i[1] \to 0$ is a short exact sequence in $\mathcal{A}_s$ for $\mu(Q_i)/2 \leq s < \mu(K_i)/2$ and $0 \to F_i \to E_i \to G_i[1] \to 0$ is a short exact sequence in $\mathcal{A}_s$ for $\mu(G_i)/2 \leq s < \mu(F_i)/2$. As before, these short exact sequences give rise to walls and by definition, E_i must be stable for all points outside the corresponding semicircles.

Therefore, the problem reduces to understanding the how the Chern character of K_i and F_i change. If $\operatorname{ch}(K_n) = (a_n^2, a_n b_n \ell, b_n^2)$ then from the matrix calculation above, we have $a_{n+1} = 4a_n - b_n$ and $b_{n+1} = a_n$. To show that the K_i (and F_i) actually destabilise E_i for all $i \geq 0$ we need to show that the corresponding semicircles have positive radii. A similar calculation to the one in Corollary 4.1.2 shows that the radius (of either the codimension zero or one wall) gets multiplied by $7 - 4\sqrt{3}$ each time. Therefore, by induction, we see that every K_i and F_i does indeed give rise to a genuine codimension zero and one wall respectively. Suppose $\mu(K_n)/2 = b_n/a_n$ converges to a limit x, say. Then

$$ x = \lim_{n \to \infty} \left(\frac{b_{n+1}}{a_{n+1}} \right) = \lim_{n \to \infty} \left(\frac{a_n}{4a_n - b_n} \right) = \lim_{n \to \infty} \left(\frac{1}{4 - \frac{b_n}{a_n}} \right) = \frac{1}{4 - x} $$

$$ \Rightarrow x^2 - 4x + 1 = 0, \quad \text{i.e. } x = 2 \pm \sqrt{3} $$

where we are forced to choose $2 - \sqrt{3}$ since $\mu(K_i) < \mu(E_i) = 2$ by Corollary 3.2.5. The Chern character of Q_n is given by $\operatorname{ch}(Q_n) = (1 - a_n^2, (2 - a_n b_n)\ell, 1 - b_n^2)$ and a similar calculation shows that $\mu(Q_n)/2$ converges to the same limit. $\square$

Thus we can give a complete description of the wall and chamber structure for $n = 3$:
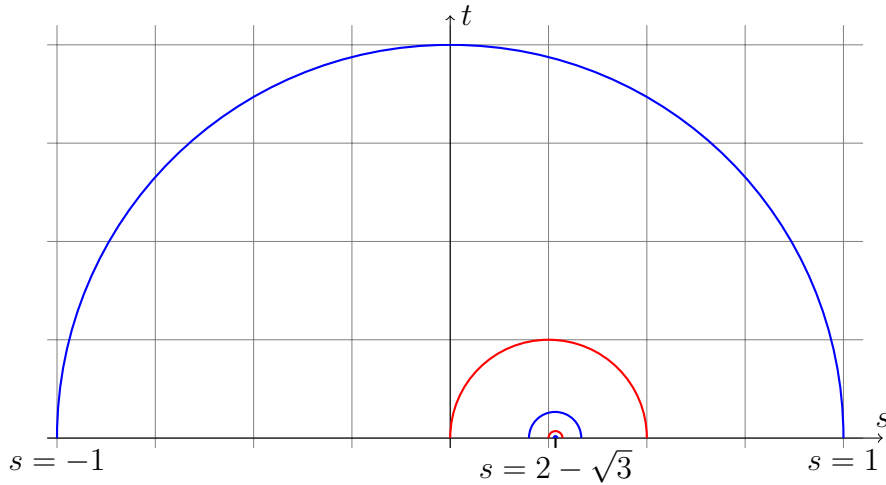
Remark 4.1.8. Observe that any accumulation point for an infinite series of walls like the ones above must be irrational. Indeed, if the accumulation point s were rational then the Fourier-Mukai transform Φ_{-s} would force us to have an infinite series of walls going off to infinity; contradicting the fact that we have already found the maximal wall.

4.1.5 $n = 4$

Let $(Z, \hat{x}) \in \mathrm{Hilb}^4(\mathbb{T}) \times \hat{\mathbb{T}}$. As before, there is one critical value at $t = \sqrt{2}$ and two moduli spaces $M_0 := M_{t>1}(1, 2\ell, 0)$ and $M_1 := M_{t<1}(1, 2\ell, 0)$. Objects of the form $L^2 \mathcal{I}_Z \mathcal{P}_{\hat{x}}$ are destabilised by $L\mathcal{P}_{\hat{y}}$ for $t \leq \sqrt{2}$ if and only if Z is collinear; for generic Z, $L^2 \mathcal{I}_Z \mathcal{P}_{\hat{x}}$ is μ_t-stable for all $t > 0$. The collinear Z's live in a codimension 2 subvariety and so, by [ABL07], we can construct M_1 as a Mukai flop of M_0.

This time, the Fourier-Mukai transform gives us an isomorphism $\widehat{M_1} := \Phi_0(M_1) \cong M_{t>1/\sqrt{2}}(0, 2\ell, -1)$ which consists of pure torsion sheaves of rank 1 and degree 3 supported on a translate of a divisor in the linear system $|\hat{L}^2|$; see [Mac11, Theorem 8.3]. In particular, the moduli space M_1 is projective by Theorem 2.2.2. (The points of $\widehat{M_1}$ are harder to describe because the linear system $|\hat{L}^2|$ has singular and reducible elements.) For the Chern character $(0, 2\ell, -1)$ there is exactly one wall at $t = 1/\sqrt{2}$ where we need to glue in the transforms of the collinear Z's, that is, objects $E \in \mathcal{A}_0$ with $H^{-1}(E) \cong L^{-1}\mathcal{P}_x$ and $H^0(E) \cong L\mathcal{I}_P \mathcal{P}_y$ for some $(x, y, P) \in \mathbb{T} \times \mathbb{T} \times \mathrm{Hilb}^1(\mathbb{T})$. Let $Z \subset D_x$ be a collinear subscheme of length four and $\sigma = \sum Z$ then we can illustrate our observations with the following picture:
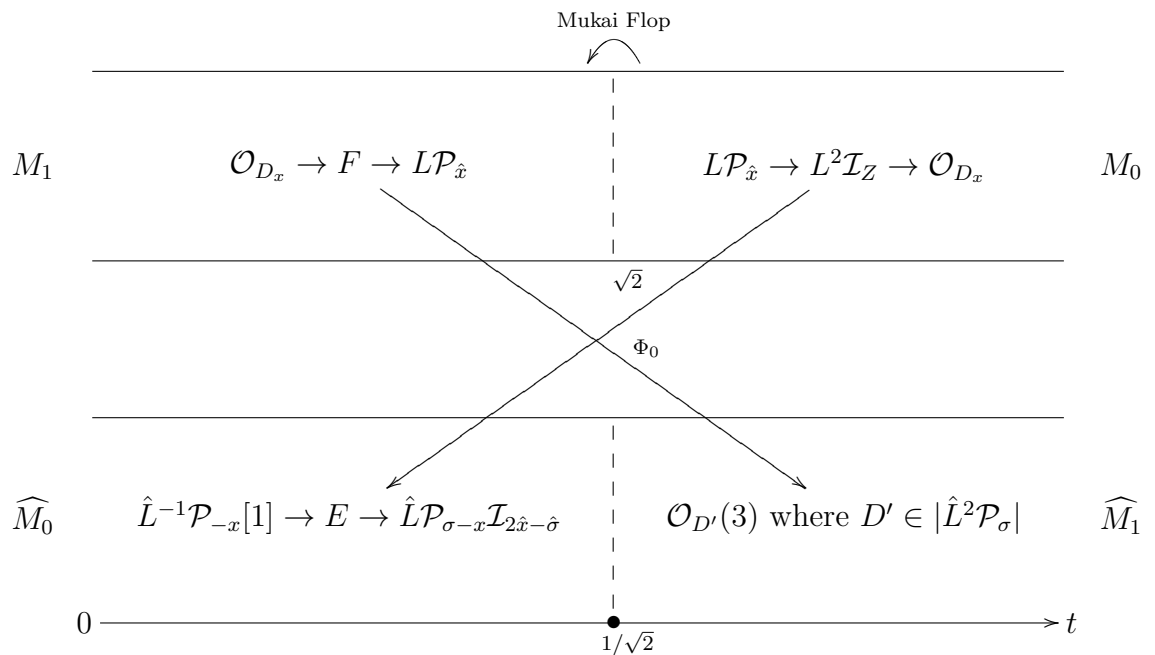
By Theorem 3.3.9, there is only one wall for $L^2 \mathcal{I}_Z$ on the line $s = 0$. Since any wall associated to $L^2 \mathcal{I}_Z$ in the region $0 \leq s < 2$ must intersect the line $s = 0$ (Proposition 3.3.11), the wall corresponding to the collinear Z's must be the only wall in this region. Thus, any other wall in the (s, t)-plane must actually satisfy

centre $\pm$ radius < 0 and so we can improve our bounds to

$$\frac{4 - n}{4} \leq \text{centre} < 2 - \sqrt{n}.$$

Setting $n = 4$, we arrive at the contradiction $0 \leq \text{centre} < 0$. Therefore, there is only one wall for $L^2 \mathcal{I}_Z$ in the whole (s,t)-plane. In particular, the wall and chamber structure for $n = 4$ looks like:
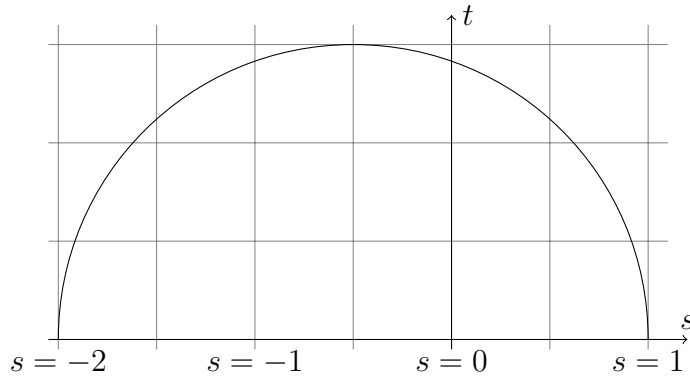

Remark 4.1.9. In this picture, like the others, we have not drawn the vertical wall at $s = 2$. However, in this case it turns out to be the only codimension zero wall.

4.1.6 $n = 5$

Let $(W, \hat{x}) \in \text{Hilb}^5(\mathbb{T}) \times \hat{\mathbb{T}}$. This case is special since it is the only case where the standard Fourier-Mukai transform acts on the space

$$\Phi_0 : \mathcal{M}_t(1, 2\ell, -1) \to \mathcal{M}_{1/t}(1, 2\ell, -1).$$

Because of this, the space has three walls and four moduli spaces which are identified in the following way:
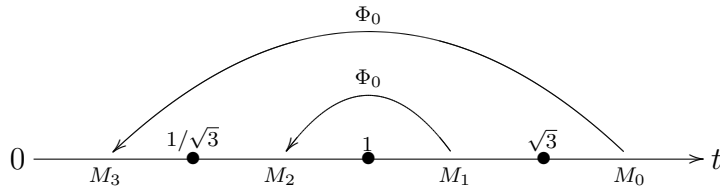
We try to represent the surgeries for $n = 5$ in the diagram below. In particular, the vertical lines indicate walls and the horizontal lines indicate strata in each moduli space. The letters A, B, C, D and E indicate sheaves of a particular type and their corresponding hatted letters are the transformed spaces. To the right of a wall in regions A, B, C and E we have torsion-free sheaves characterized by

the geometric property indicated. The codimensions of the spaces are as follows: $\operatorname{codim}_{M_0}(A) = 3$, $\operatorname{codim}_{M_0}(B) = 2$, $\operatorname{codim}_{M_0}(C) = 3$ and $\operatorname{codim}_{\operatorname{Flop}(M_0, \mathbb{P}_0)}(D) = 2$ which implies $\operatorname{codim}_{M_1}(D) = 5$.

- For $W \subset D_x$, we have the collinears

$$A = L\mathcal{P}_{\hat{x}} \to L^2 \mathcal{I}_W \to \mathcal{O}_{D_x}(-1)$$
$$\hat{A} = \hat{L}^{-1} \mathcal{P}_{-x}[1] \to \Phi_0(L^2 \mathcal{I}_W) \to \Phi_0(\mathcal{O}_{D_x}(-1)).$$

- The collinear length fours

$$B = L\mathcal{I}_P \mathcal{P}_{\hat{x}} \to L^2 \mathcal{I}_W \to \mathcal{O}_{D_x}$$
$$\hat{B} = \mathcal{O}_{\hat{D}_{-x}} \mathcal{P}_{p-x} \to \Phi_0(L^2 \mathcal{I}_W) \to \hat{L}\mathcal{I}_{-\hat{x}} \mathcal{P}_x.$$

- The special collinear length threes

$$C = \Phi_0(\mathcal{O}_{\hat{D}_x}(-1)) \to L^2 \mathcal{I}_W \to L^{-1} \mathcal{P}_{\hat{x}}[1]$$
$$\hat{C} = \mathcal{O}_{\hat{D}_x}(-1) \to \Phi_0(L^2 \mathcal{I}_W) \to \hat{L}\mathcal{P}_{-x}.$$

- Torsion extensions $F = L\mathcal{P}_{\hat{x}} \ltimes \mathcal{O}_{D_x}(-1)$ with a lift $\operatorname{Hom}(L\mathcal{I}_P \mathcal{P}_{\hat{x}}, F) \neq 0$

$$D = L\mathcal{I}_P \mathcal{P}_{\hat{x}} \to F \to \mathcal{O}_{D_x}$$
$$\hat{D} = \mathcal{O}_{\hat{D}_{-x}} \mathcal{P}_{p-x} \to \hat{F} \to \hat{L}\mathcal{I}_{-\hat{x}} \mathcal{P}_x.$$

- For generic configurations, we have

$$
\begin{aligned}
E &= L^2 \mathcal{I}_W \mathcal{P}_{\hat{x}} \text{ is WIT}_0 \text{ with} \\
\hat{E} &= \Phi_0(L^2 \mathcal{I}_W \mathcal{P}_{\hat{x}}) \cong \hat{L}^2 \mathcal{I}_{W'} \mathcal{P}_y \\
&\quad \text{for some } (W', y) \in \text{Hilb}^5(\hat{\mathbb{T}}) \times \mathbb{T}.
\end{aligned}
$$

The wall and chamber structure for $n = 5$ looks like:
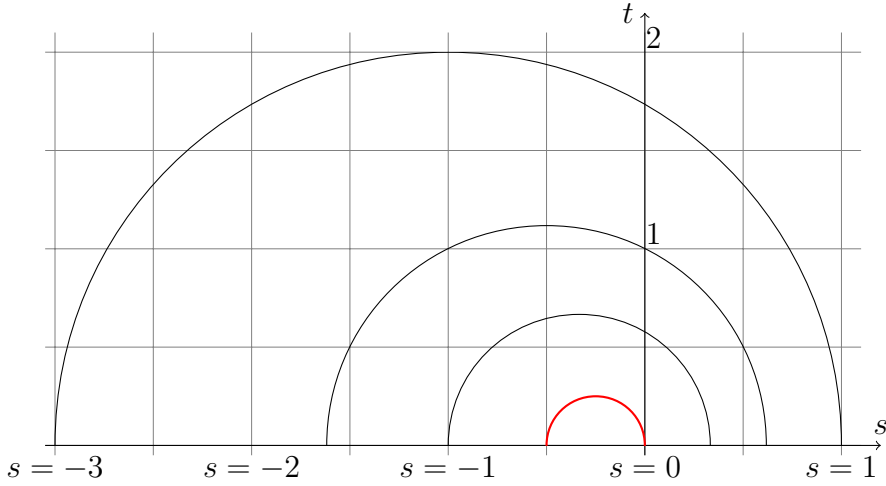
Remarks 4.1.10. Observe that it is the middle wall at $t = 1$ which realises the family of examples in Corollary 2.6.11, whose μ-stability was not preserved, as explicit wall-crossing.

The red semicircle is the codimension zero wall corresponding to taking the transform of the twisted structure sequence or alternatively, it is the image of the vertical wall at $s = 2$ under the standard Fourier-Mukai transform. In a similar way to before, we can cook up a Fourier-Mukai transform that produces an infinite series of walls converging to $2 - \sqrt{5}$.

The codimensions of A, B and C in M_0 follow from Lemma 3.5.2. To see that $\text{codim}_{\text{Flop}(M_0, \mathbb{P}_0)}(D) = 2$, consider the following completed diagram:

$$
\begin{array}{ccc}
L\mathcal{I}_P & =\!=\!= & L\mathcal{I}_P \\
\downarrow & & \downarrow \\
\mathcal{O}_{D_x}(-1) \longrightarrow F \longrightarrow L \\
\| \qquad\qquad \downarrow \qquad \downarrow \\
\mathcal{O}_{D_x}(-1) \longrightarrow \mathcal{O}_{D_x} \longrightarrow \mathcal{O}_P
\end{array}
$$

The existence of a lift $L\mathcal{I}_P \to F$ is equivalent to the pullback extension being split, i.e. we are interested in those classes in $\text{Ext}^1(L, \mathcal{O}_{D_x}(-1))$ which map to

zero in $\mathrm{Ext}^1(L\mathcal{I}_P, \mathcal{O}_{D_x}(-1))$. Applying $\mathrm{Hom}(-, \mathcal{O}_{D_x}(-1))$ to the last column we get

$$0 \to \mathrm{Ext}^1(\mathcal{O}_P, \mathcal{O}_{D_x}(-1)) \to \mathrm{Ext}^1(L, \mathcal{O}_{D_x}(-1)) \to \mathrm{Ext}^1(L\mathcal{I}_P, \mathcal{O}_{D_x}(-1)).$$

Since $\chi(\mathcal{O}_P, \mathcal{O}_{D_x}(-1)) = 0$ we see by Riemann-Roch that

$$\begin{aligned} \mathrm{ext}^1(\mathcal{O}_P, \mathcal{O}_{D_x}(-1)) &= \mathrm{ext}^0(\mathcal{O}_P, \mathcal{O}_{D_x}(-1)) + \mathrm{ext}^2(\mathcal{O}_P, \mathcal{O}_{D_x}(-1)) \\ &= \mathrm{ext}^0(\mathcal{O}_{D_x}(-1), \mathcal{O}_P)^* = 1 \text{ if } P \subset D_x \end{aligned}$$

Therefore, $\mathrm{Ext}^1(\mathcal{O}_P, \mathcal{O}_{D_x}(-1))$ is a codimension two subspace of $\mathrm{Ext}^1(L, \mathcal{O}_{D_x}(-1))$ which is codimension three in the whole space. Thus, D has codimension five in the whole space.

4.1.7 $n \geq 6$

Let $(X, \hat{x}) \in \mathrm{Hilb}^n(\mathbb{T}) \times \hat{\mathbb{T}}$ for $n \geq 6$. By Corollary 3.3.10, there are $\lfloor (n-1)/2 \rfloor$ walls corresponding to destabilisers of the form $L\mathcal{I}_{X'}$ for some $X' \in \mathrm{Hilb}^m(\mathbb{T})$ with $0 \leq m < (n-2)/2$ (Theorem 3.3.9). By Theorem 3.5.8, the resulting $\lfloor (n+1)/2 \rfloor$ moduli spaces are all smooth projective varieties related via a series of Mukai flops. Using [Mac11, Section 10], we can give a complete description of the transform spaces as well.

4.2 Realising the Non-Preservation of Stability as Explicit Wall-Crossing

Given the technology developed in Chapter 3, we are now in a position to answer the question posed at the beginning of Section 2.6. Recall that we manufactured examples of stable sheaves which, under the standard Fourier-Mukai transform, became unstable; see Corollary 2.6.7, Corollary 2.6.9 and Corollary 2.6.11. The aim of this Chapter is to realise these examples as explicit wall-crossing in $\mathrm{Stab}(\mathcal{D}(\mathbb{T}))$. In particular, we will analyse the Bridgeland-stability of objects

$$E \in \mathcal{A}_0 \text{ with } \mathrm{ch}(E) = (2r+1, 2\ell, -(2r+1)) \text{ where } r \in \mathbb{Z}_{\geq 0}.$$

We have already realised the example with $r = 0$ in the last chapter and so it remains to consider the examples with $r = 1$ and $r = 2$ respectively.

Proposition 4.2.1. *Suppose $E \in \mathcal{A}_0$ with $\mathrm{ch}(E) = (3, 2\ell, -3)$ is μ_t-stable for some $t > 0$ and $0 \to K \to E \to Q \to 0$ is a destabilising sequence in $\mathcal{A}_0$. Then either*

1. *E is a torsion-free sheaf, i.e.*

   - $E = L\mathcal{I}_Y \mathcal{P}_{\hat{x}} \ltimes \Phi_0(L\mathcal{I}_Y \mathcal{P}_{\hat{y}})$ *for some* $Y \in \text{Hilb}^3(\mathbb{T})$,

   - $E = L\mathcal{I}_Z \mathcal{P}_{\hat{x}} \ltimes \Phi_0(\mathcal{O}_{D_x}(-1)\mathcal{P}_{\hat{y}})$ *for some* $Z \in \text{Hilb}^4(\mathbb{T})$,

   - $E = \mathcal{O}_{D_x}(-1)\mathcal{P}_{\hat{y}} \ltimes \Phi_0(L\mathcal{I}_Z \mathcal{P}_{\hat{x}})$,

   - $E = \mathcal{O}_{D_x}(-2)\mathcal{P}_{\hat{y}} \ltimes \Phi_0(\mathcal{O}_{D_x}(-2)\mathcal{P}_{\hat{x}})$,

   - $E = \Phi_0(L\mathcal{I}_Y \mathcal{P}_{\hat{y}}) \ltimes L\mathcal{I}_Y \mathcal{P}_{\hat{x}}$ *or*

   - $E = \Phi_0(\mathcal{O}_{D_x}(-1)\mathcal{P}_{\hat{y}}) \ltimes L\mathcal{I}_Z \mathcal{P}_{\hat{x}}$.

2. *E is a sheaf with torsion, i.e.*

   - $E = \Phi_0(L\mathcal{I}_Z \mathcal{P}_{\hat{x}}) \ltimes \mathcal{O}_{D_x}(-1)\mathcal{P}_{\hat{y}}$ *for some* $Z \in \text{Hilb}^4(\mathbb{T})$ *or*

   - $E = \Phi_0(\mathcal{O}_{D_x}(-2)\mathcal{P}_{\hat{x}}) \ltimes \mathcal{O}_{D_x}(-2)\mathcal{P}_{\hat{y}}$.

Proof Suppose $E \in \mathcal{A}_s$ is a torsion-free sheaf and let $\text{ch}(K) = (r, c\ell, \chi)$ as usual. Then taking cohomology shows that K must be torsion-free and setting $s = 0$ we see that $c = 1$ and Q is atomic by Lemma 3.3.5. For $r \geq 3$, we must have $Q = H^{-1}(Q)[1]$ giving rise to a wall. If we cross this wall then we have to glue in two-step complexes of the form $0 \to Q[1] \to F \to K \to 0$ but no such objects can exist. Indeed, suppose E is a two-step complex and consider the short exact sequence $0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0$ in $\mathcal{A}_0$. Since $H^0(E) \in \mathcal{T}_0$ and $H^{-1}(E) \in \mathcal{F}_0$ we are forced to have $\deg(H^{-1}(E)) = -2$ or 0 but if $\deg(H^{-1}(E)) = 0$ then $H^{-1}(E)[1]$ has infinite μ_t-slope and destabilises E for all $t > 0$; contradiction. Therefore, $\deg(H^{-1}(E)) = -2$ and $H^{-1}(E)$ is μ-semistable. To see this, observe that if D was a potential μ-destabilising object then $\deg(D) = 0$ and the composite map $D[1] \to H^{-1}(E)[1] \to E$ would destabilise E for all $t > 0$; contradiction. Thus, by Bogomolov, we have $\chi(H^{-1}(E)) \leq 1$ and E is μ_t-stable for some $t > 0$ if and only if $\mu_t(E) < \mu_t(H^0(E))$, i.e.

$$0 < (2\text{rk}(H^{-1}(E)) + 3)t^2 < 2\chi(H^{-1}(E)) - 3 \leq -1 \; ; \text{ contradiction.}$$

In other words, E is never represented as a two-step complex and $r \leq 3$ with $Q = H^0(Q)$. Notice that Q cannot have torsion supported on points because nothing could then destabilise E and if it had torsion supported on a curve then $\deg(Q/\text{tors}(Q)) \leq 0$ contradicting the fact that $Q \in \mathcal{T}_0$. Therefore, Q is either itself a torsion sheaf $\mathcal{O}_{D_x}(\alpha)$ or it is torsion-free and μ-semistable by Lemma 3.2.10. Finally, observe that $c = 1$ implies K is μ-semistable by Lemma 3.2.10. Thus if $r \geq 2$ then, by Bogomolov, we have $\chi \leq 0$ and together with the fact

that E is μ_t-stable for some $t > 0$ we see that $\mu_t(K) < \mu_t(E) \Leftrightarrow \chi \geq -1$, i.e. $\chi = -1$ or 0. Therefore, when $r = 2$ we have

$$0 \to \Phi_0(L\mathcal{I}_Y \mathcal{P}_{\hat{y}}) \to E \to L\mathcal{I}_Y \mathcal{P}_{\hat{x}} \to 0 \quad \text{stable for some } t > 1 \text{ or}$$

$$0 \to \Phi_0(\mathcal{O}_{D_x}(-1)\mathcal{P}_{\hat{y}}) \to E \to L\mathcal{I}_Z \mathcal{P}_{\hat{x}} \to 0 \quad \text{stable for some } t > \sqrt{3}$$

and when $r = 3$ we have

$$0 \to \Phi_0(L\mathcal{I}_Z \mathcal{P}_{\hat{x}}) \to E \to \mathcal{O}_{D_x}(-1)\mathcal{P}_{\hat{y}} \to 0 \quad \text{stable for some } t > 1/\sqrt{3} \text{ or}$$

$$0 \to \Phi_0(\mathcal{O}_{D_x}(-2)\mathcal{P}_{\hat{x}}) \to E \to \mathcal{O}_{D_x}(-2)\mathcal{P}_{\hat{y}} \to 0 \quad \text{stable for some } t > 1.$$

When $r = 1$, we see that $\mathrm{ch}(Q) = (2, \ell, -3 - \chi)$. Thus, by Bogomolov, we have $-3 - \chi \leq 0 \Leftrightarrow \chi \geq -3$. Together with the destabilising condition $\mu_t(K) < \mu_t(E) \Leftrightarrow \chi \leq -2$ we get $\chi = -3$ or -2. That is,

$$0 \to L\mathcal{I}_Y \mathcal{P}_{\hat{x}} \to E \to \Phi_0(L\mathcal{I}_Y \mathcal{P}_{\hat{y}}) \to 0 \quad \text{stable for some } t < 1 \text{ or}$$

$$0 \to L\mathcal{I}_Z \mathcal{P}_{\hat{x}} \to E \to \Phi_0(\mathcal{O}_{D_x}(-1)\mathcal{P}_{\hat{y}}) \to 0 \quad \text{stable for some } t < \sqrt{3}$$

If E has torsion then it must be supported on a curve since all torsion sheaves supported in dimension zero have infinite μ_t-slope and would destabilise E for all $t > 0$. Let $T \subset E$ be the torsion subsheaf of E and consider $0 \to T \to E \to F \to 0$ where F is torsion-free and the Chern characters read

$$(0, d\ell, \alpha) \to (3, 2\ell, -3) \to (3, (2-d)\ell, -3 - \alpha) \quad \text{with } d > 0.$$

$F \in \mathcal{T}_0$ implies $d = 1$ and so by Lemma 3.2.10, F is μ-semistable. Thus, by Bogomolov, we have $-3 - \alpha \leq 0 \Leftrightarrow \alpha \geq -3$. Since E is μ_t-stable for some $t > 0$, we have $\mu_t(E) < \mu_t(F) \Leftrightarrow \alpha \leq -2$, i.e. $\alpha = -3$ or -2. That is,

$$0 \to \mathcal{O}_{D_x}(-2)\mathcal{P}_{\hat{y}} \to E \to \Phi_0(\mathcal{O}_{D_x}(-2)\mathcal{P}_{\hat{x}}) \to 0 \quad \text{stable for some } t < 1 \text{ or}$$

$$0 \to \mathcal{O}_{D_x}(-1)\mathcal{P}_{\hat{y}} \to E \to \Phi_0(L\mathcal{I}_Z \mathcal{P}_{\hat{x}}) \to 0 \quad \text{stable for some } t < 1/\sqrt{3}.$$

$\square$

Therefore, on the $s = 0$ ray, we have calculated all the potential destabilisers of E. The destabilisers and corresponding walls are summarised in the following table:

	Euler Characteristic χ	
Rank r	0	-1
2	$\sqrt{3}$	1
3	1	$\frac{1}{\sqrt{3}}$

where we only have rows for $r = 2$ and 3 because the $r = 0$ and 1 rows are obtained by taking the transform. As before, we try to represent the surgeries in a diagram where the vertical lines indicate walls and the horizontal lines indicate strata:
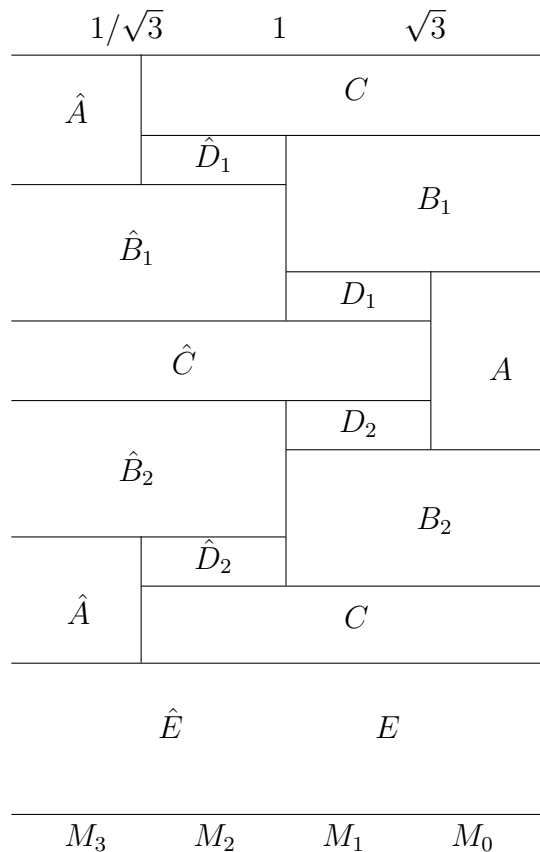
Letting hatted letters represent the transform under Φ_0 and suppressing translates and twists, we have

- Extensions which are stable for $t \geq \sqrt{3}$ and $t \leq 1/\sqrt{3}$ resp.

$$A = \widehat{\mathcal{O}_{\hat{D}}(-1)} \to E \to L\mathcal{I}_Z$$
$$\hat{A} = \mathcal{O}_{\hat{D}}(-1) \to \hat{E} \to \widehat{L\mathcal{I}_Z}.$$

- Extensions which are stable for $t \geq 1$ and $t \leq 1$ resp.

$$B_1 = \widehat{L\mathcal{I}_Y} \to E \to L\mathcal{I}_Y$$
$$\hat{B}_1 = L\mathcal{I}_Y \to \hat{E} \to \widehat{L\mathcal{I}_Y}.$$

- Extensions which are stable for $t \geq 1$ and $t \leq 1$ resp.

$$B_2 \quad = \quad \widehat{\mathcal{O}_{\hat{D}}(-2)} \to E \to \mathcal{O}_D(-2)$$

$$\hat{B}_2 \quad = \quad \mathcal{O}_{\hat{D}}(-2) \to \hat{E} \to \widehat{\mathcal{O}_D(-2)}.$$

- Extensions which are stable for $t \geq 1/\sqrt{3}$ and $t \leq \sqrt{3}$ resp.

$$C \quad = \quad \widehat{L\mathcal{I}_Z} \to E \to \mathcal{O}_D(-1)$$

$$\hat{C} \quad = \quad L\mathcal{I}_Z \to \hat{E} \to \widehat{\mathcal{O}_D(-1)}.$$

- Extensions $E = \widehat{\mathcal{O}_{\hat{D}}(-1)} \ltimes L\mathcal{I}_Z$ with a lift $\mathrm{Hom}(\widehat{L\mathcal{I}_Y}, E) \neq 0$, i.e.

$$D_1 =$$

$$\hat{D}_1 =$$

- Extensions $E = \widehat{\mathcal{O}_{\hat{D}}(-1)} \ltimes L\mathcal{I}_Z$ with a lift $\mathrm{Hom}(\widehat{\mathcal{O}_{\hat{D}}(-2)}, E) \neq 0$, i.e.

$$D_2 =$$

$$\mathcal{O}_{\hat{D}}(-2)$$

$$\hat{D}_2 = \quad \widehat{L\mathcal{I}_Z} \longrightarrow \hat{E} \longrightarrow \mathcal{O}_D(-1)$$

$$\widehat{\mathcal{O}_D(-2)}$$

Similar codimension calculations to the one below yield:

$$\mathrm{codim}_{M_0}(A) = 7 \quad \mathrm{codim}_{M_0}(B_1) = 6 \quad \mathrm{codim}_{M_0}(B_2) = 10 \quad \mathrm{codim}_{M_0}(C) = 7$$

$$\mathrm{codim}_{\mathrm{Flop}(M_0,\mathbb{P}_0)}(D_1) = 7 \quad \text{which implies} \quad \mathrm{codim}_{M_1}(D_1) = 13$$

$$\mathrm{codim}_{\mathrm{Flop}(M_0,\mathbb{P}_0)}(D_2) = 9 \quad \text{which implies} \quad \mathrm{codim}_{M_1}(D_2) = 16$$
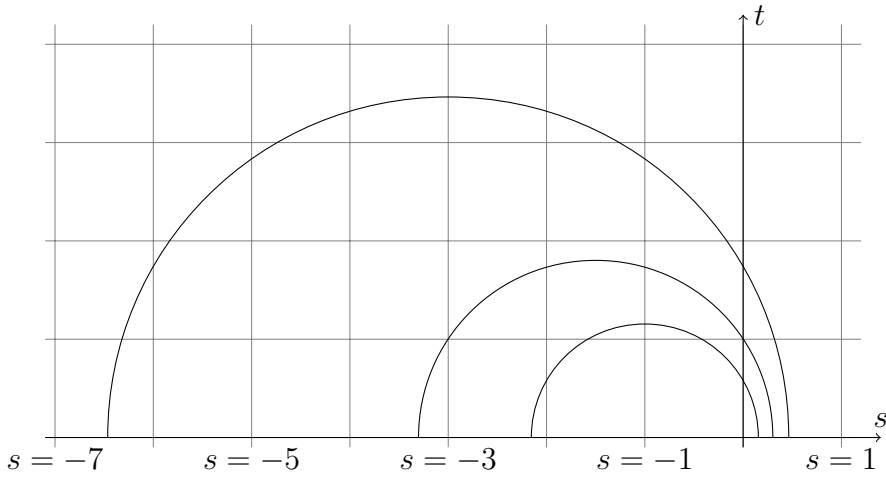
Remarks 4.2.2. Observe that it is the wall at $t = 1$ corresponding to $B_1 \to \hat{B}_1$ which realises the family of examples in Corollary 2.6.7 as explicit wall-crossing. Notice that there are two walls lying on top of each other in this picture. In the strata picture above, we drew them as separate walls because they do indeed correspond to disjoint subvarieties.

Since all the codimension calculations are similar, we will only illustrate one. To see that $\mathrm{codim}_{\mathrm{Flop}(M_0,\mathbb{P}_0)}(D_2) = 9$, consider the completed D_2 diagram:

$$\widehat{\mathcal{O}_{\hat{D}}(-2)} = \widehat{\mathcal{O}_{\hat{D}}(-2)}$$

$$L\mathcal{I}_Z \longrightarrow E \longrightarrow \mathcal{O}_{\hat{D}}(-1)$$

$$L\mathcal{I}_Z \longrightarrow \mathcal{O}_D(-2) \longrightarrow \mathcal{P}_{\hat{x}}[1]$$

Applying $\mathrm{Hom}(-, L\mathcal{I}_Z)$ to the last column we get

$$0 \to \mathrm{Ext}^1(\mathcal{P}_{\hat{x}}[1], L\mathcal{I}_Z) \to \mathrm{Ext}^1(\widehat{\mathcal{O}_{\hat{D}}(-1)}, L\mathcal{I}_Z) \to \mathrm{Ext}^1(\widehat{\mathcal{O}_{\hat{D}}(-2)}, L\mathcal{I}_Z).$$

Since $\mathrm{Ext}^1(\mathcal{P}_{\hat{x}}[1], L\mathcal{I}_Z) \cong \mathrm{Hom}(\mathcal{P}_{\hat{x}}, L\mathcal{I}_Z) \cong \mathbb{C}$ if and only if $Z \subset D_x$ is collinear and $\dim_{\mathbb{C}} \mathrm{Ext}^1(\widehat{\mathcal{O}_{\hat{D}}(-1)}, L\mathcal{I}_Z) = -\chi(\widehat{\mathcal{O}_{\hat{D}}(-1)}, L\mathcal{I}_Z) = 8$, we see that $\mathrm{Hom}(\mathcal{P}_{\hat{x}}, L\mathcal{I}_Z)$ is a codim 7 subspace of $\mathrm{Ext}^1(\widehat{\mathcal{O}_{\hat{D}}(-1)}, L\mathcal{I}_Z)$. We get an extra 2 dimensions from the geometric condition on the Z's taking us to codim 9. Indeed, any two points lie on a divisor and so it is one linear constraint to ask for three points to lie on a D_x and two constraints to ask for four. A computation similar to that of Lemma 3.5.2 shows that $\mathrm{Ext}^1(\widehat{\mathcal{O}_{\hat{D}}(-1)}, L\mathcal{I}_Z)$ has codimension $-\chi(\widehat{\mathcal{O}_{\hat{D}}(-1)}, L\mathcal{I}_Z) - 1 = 7$. Therefore, the collinear Z's form a codim 2 subspace in $\mathrm{Hilb}^4(\mathbb{T})$ and the sublocus corresponding to the destabilisers $\widehat{\mathcal{O}_{\hat{D}}(-2)}$ has codim 16 in the whole space.

Proposition 4.2.3. *Suppose $E \in \mathcal{A}_0$ with $\mathrm{ch}(E) = (5, 2\ell, -5)$ is μ_t-stable for some $t > 0$ and choose $F \in \mathcal{M}(2, \ell, -3)$ as in Lemma 2.6.8. Let $F_1 := \ker(F \to \mathcal{O}_P)$ for $P \in \mathrm{Hilb}^1(\mathbb{T})$ and $F_2 := \ker(F \to \mathcal{O}_Q)$ for $Q \in \mathrm{Hilb}^2(\mathbb{T})$. Then either*

1. *E is a torsion-free sheaf, i.e.*

   - $E = F \ltimes \hat{F}$,
   - $E = F_1 \ltimes \widehat{L\mathcal{I}_Z}$ *for some $Z \in \mathrm{Hilb}^4(\mathbb{T})$,*
   - $E = F_2 \ltimes \widehat{\mathcal{O}_D(-2)}$,
   - $E = L\mathcal{I}_Z \ltimes \hat{F}_1$ *for some $Z \in \mathrm{Hilb}^4(\mathbb{T})$,*
   - $E = L\mathcal{I}_W \ltimes \widehat{L\mathcal{I}_W}$ *for some $W \in \mathrm{Hilb}^5(\mathbb{T})$,*
   - $E = L\mathcal{I}_X \ltimes \widehat{\mathcal{O}_{\hat{D}}(-3)}$ *for some $X \in \mathrm{Hilb}^6(\mathbb{T})$,*
   - $E = \mathcal{O}_D(-2) \ltimes \hat{F}_2$,
   - $E = \mathcal{O}_D(-3) \ltimes \widehat{L\mathcal{I}_X}$ *for some $X \in \mathrm{Hilb}^6(\mathbb{T})$,*
   - $E = \mathcal{O}_D(-4) \ltimes \widehat{\mathcal{O}_D(-4)}$,
   - $E = \hat{F}_1 \ltimes L\mathcal{I}_Z$,
   - $E = \widehat{L\mathcal{I}_W} \ltimes L\mathcal{I}_W$,
   - $E = \widehat{\mathcal{O}_D(-3)} \ltimes L\mathcal{I}_X$,
   - $E = \hat{F} \ltimes F$,

- $E = \widehat{L\mathcal{I}_Z} \ltimes F_1$,

- $E = \widehat{\mathcal{O}_D(-2)} \ltimes F_2$.

2. *E is a sheaf with torsion, i.e.*

- $E = \hat{F}_2 \ltimes \mathcal{O}_D(-2)$,

- $E = \widehat{L\mathcal{I}_X} \ltimes \mathcal{O}_D(-3)$ *for some* $X \in \mathrm{Hilb}^6(\mathbb{T})$,

- $E = \widehat{\mathcal{O}_D(-4)} \ltimes \mathcal{O}_D(-4)$

Proof   Proceed in exactly the same way as Proposition 4.2.1.   □

Therefore, on the $s = 0$ ray, we have calculated all the potential destabilisers of E. The destabilisers and corresponding walls are summarised in the following table:

	Euler Characteristic χ		
Rank r	0	-1	-2
3	$\sqrt{5}$	$\sqrt{3}$	1
4	$\frac{\sqrt{5}}{\sqrt{3}}$	1	$\frac{1}{\sqrt{3}}$
5	1	$\frac{\sqrt{3}}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$

where we only have rows for $r = 3, 4$ and 5 because the $r = 0, 1$ and 2 rows are obtained by taking the transform. A similar strata picture picture can be created but given that it is very cluttered, we just give the wall and chamber structure picture instead:
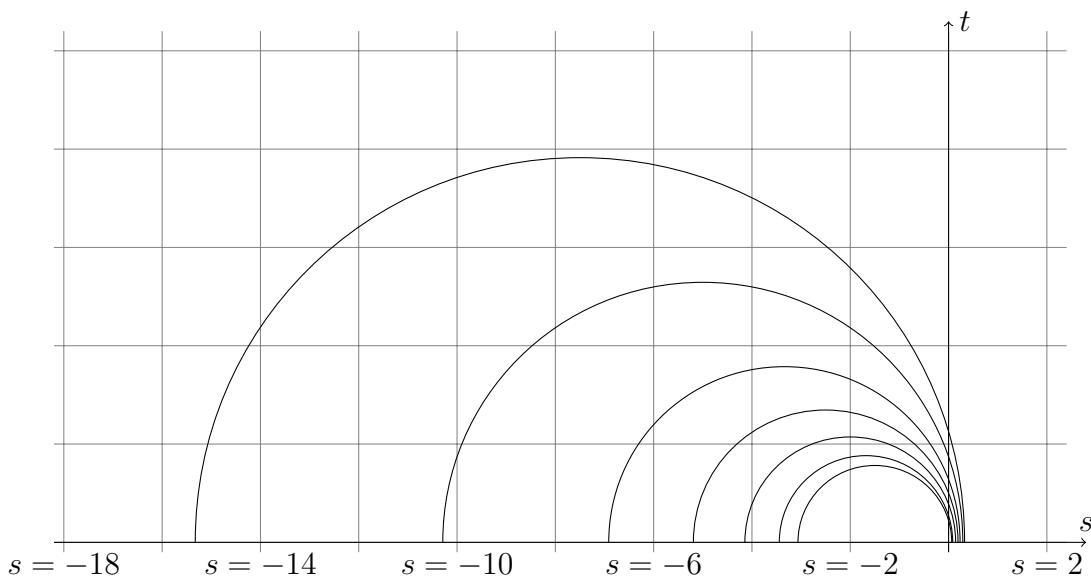
Remark 4.2.4. Observe that it is the wall at $t = 1$ corresponding to $F \ltimes \hat{F} \to \hat{F} \ltimes F$ which realises the family of examples in Corollary 2.6.7 as explicit wall-crossing.

Chapter 5

Conclusion

In this thesis, we have succeeded in realising the examples of non-preservation of stability as explicit wall-crossing. Initially, our goal was to classify all the such subvarieties where stability is not preserved but now we can see that this is essentially asking for a global understanding of the wall and chamber structure. In the long run, we hope to reveal the bigger picture but as this thesis demonstrates, it is highly non-trivial. In our case, we have provided detailed descriptions of the moduli spaces of σ-stable objects and explained how they change when we cross walls. In particular, we have been able to relate (most of) these wall-crossings to precise geometric scenarios on $\mathbb{T}$. In the future, we hope to extend these ideas to K3 surfaces where the geometry is a lot richer.

In the introduction and during the proof of projectivity we commented on how crucial the nesting behaviour of the walls was to our argument. It turns out that similar behaviour has been observed when considering configurations of n points on $\mathbb{P}^2$; see [ABCH12]. We expect that when the Picard rank is one, like in both of these examples, the walls always nest; if true, we expect similar methods would allow us to conclude that these moduli spaces (with any numerical invariants) are projective as well. Is it obvious from a minimal model point of view that the walls should nest? In other words, when running the minimal model program should you ever be presented with a choice of which birational model to go to next? Minimal models are unique in dimension two but if this result were true, it would also suggest they are unique for moduli spaces of objects on surfaces. This is not such a wild suggestion given the evidence that the derived category seems to have encoded all the birational information of the underlying variety; see [Kaw02]. For higher Picard rank, we expect a generic slice will have crossing walls but maybe it is possible to always choose a particular slice so that the walls nest? Again, if true, we could hopefully use Fourier-Mukai theory to pin down projectivity here too. It seems that in the final stages of writing this thesis, some progress has been made on the nesting conjecture; see [Mac12] for more details.

Understanding how the moduli space of slope stable sheaves on X changes as

one varies the polarization is a classical problem; see [MFK94] or [MW97]. In particular, if we choose a class of sheaves with some fixed numerical invariants then the cone of ample divisors $\mathrm{Amp}(X)$ breaks up into a series of walls and chambers:
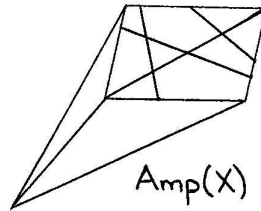

That is, the set of slope stable sheaves with the given invariants is constant in each chamber. As mentioned in the introduction, we have a similar behaviour when we consider the set of σ-stable objects on X as a function of $\sigma \in \mathrm{Stab}(\mathcal{D}(X))$. Schematically, the wall and chamber structure of $\mathrm{Stab}(\mathcal{D}(X))$ might look something like:
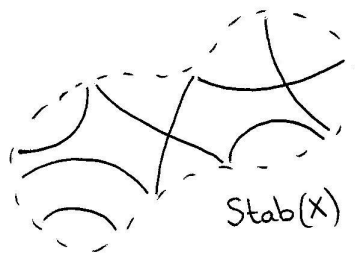
Two natural questions to explore in future work would be:

1. Are the birational maps $\mathcal{M}_X^v(\sigma) \dashrightarrow \mathcal{M}_X^v(\sigma')$ in some sense related to Mori's minimal model program?

2. Is there a map $\mathrm{Stab}(\mathcal{D}(X)) \to \mathrm{Amp}(\mathcal{M}_X^v(\sigma))$ which explains all the wall-crossings? In particular, should we think of a Fourier-Mukai transform on $\mathrm{Stab}(\mathcal{D}(X))$ as changing the polarization downstairs in $\mathrm{Amp}(X)$?
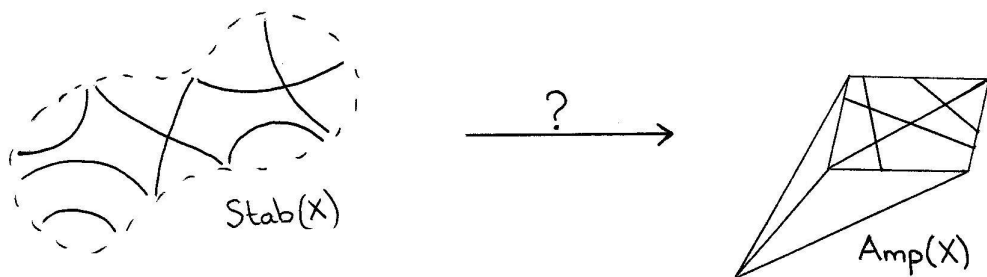
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