PBW Deformations and Quiver GIT for Noncommutative Resolutions

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Joseph Harry Karmazyn*)

Publications

This thesis is based upon work of the author contained in two papers, one published in the Journal of Algebra [Kar14b] (Available online with DOI:10.1016/j.jalgebra.2014.05.007) and the other currently a preprint [Kar14a] (Available online at arXiv:1407.5005v1). The results of Chapter 3 are drawn from the first paper and the results of Chapter 4 are drawn from the second. In both cases the arguments are unchanged, but the preliminary materials from both papers have been arranged in Section 2 and there have been significant changes to presentation and ordering of material in order to provide a connected story with minimal replication.

Acknowledgements

First, and foremost, I would like to thank my supervisors, Michael Wemyss and Iain Gordon, for much guidance, patience, and direction, and for providing endless encouragement throughout my time in Edinburgh. I need to thank the rest of the Edinburgh department, which has been a brilliant place both to study and to be a part of more generally. I am grateful to the Engineering and Physical Sciences Research Council, who have funded my studies, and I want to thank my family for regularly trekking up to Edinburgh and to whom, as always, I owe more than I could say. Finally, thank you Rachel, for everything.

Lay Summary

Algebraic geometry traditionally seeks to understand geometric problems by translating them to problems in commutative algebra. More recent developments also seek to understand geometric objects via noncommutative algebra, and to find classes of noncommutative algebras that generalise the commutative algebras appearing in algebraic geometry. This thesis seeks to reach a better understanding of some aspects of these geometrically motivated noncommutative algebras.

We first work with a class of noncommutative algebras whose properties mirror those of certain nice geometric spaces. This class includes examples which are known to have interesting deformations, and we seek to understand similar deformations for the entire class of algebras and show that the deformations also have some of these nice properties.

We then consider a class of noncommutative algebras constructed from objects on geometric spaces known as 'tilting bundles'. The method of quiver GIT produces geometric spaces from noncommutative algebras, and we show that quiver GIT can reconstruct the original geometric space and tilting bundle from the corresponding noncommutative algebra in a range of examples.

Abstract

In this thesis we first investigate PBW deformations of Koszul, Calabi-Yau algebras, and we then study moduli spaces of representations of algebras defined by tilting bundles. These classes of algebras are generalisations of the skew group algebras appearing as noncommutative resolutions in the McKay correspondence, and the results we prove are motivated by corresponding results for skew group algebras.

Koszul, Calabi-Yau algebras are Morita equivalent to path algebras with relations defined by superpotentials, and we classify which PBW deformations of these algebras still have relations defined by a superpotential and show that these deformations also retain the Calabi-Yau property. As an application of these results we show that symplectic reflection algebras are Calabi-Yau and can be interpreted as path algebras with relations defined by a superpotential.

We then investigate when a variety with a tilting bundle can be produced as a moduli space of representations of an algebra defined by the tilting bundle. We find a set of conditions ensuring a variety and tilting bundle can be reconstructed in such a manner, and we show that these conditions hold in a large class of examples, which includes situations arising in the minimal model Program where the variety may be singular. As an example we show that the minimal resolution of a rational surface singularity can be produced as a moduli space for a noncommutative algebra such that the tautological bundle is a tilting bundle defining the noncommutative algebra.

Contents

Chapter 1

Introduction

The McKay correspondence is a striking series of results deeply intertwining algebra and geometry. Starting from the original combinatorial observation of McKay it has grown a many-faceted theory, building bridges between noncommutative algebra and resolutions of quotient singularities on many levels. These approaches generalise to harness noncommutative algebras towards geometric applications in a wide range of different examples.

This introduction discusses how various aspects of the McKay correspondence and its generalisations motivate the main results proved later in the thesis. As these results naturally lead in two different directions they are split into two chapters in the thesis, 3 and 4, and there are more in-depth introductions to specific results at the start of each of the relevant chapters. This introductory section describes the joint origin of these investigations.

1.1 The $\mathrm{SL}_2(\mathbb{C})$ McKay correspondence

Any finite subgroup $G < \mathrm{SL}_2(\mathbb{C})$ defines an invariant ring $\mathbb{C}[x,y]^G$ and a skew group algebra $\mathbb{C}[x,y] \rtimes G$. This section recall the definitions of these algebras and how the McKay correspondence links them to minimal resolutions of quotient singularities.

1.1.1 The skew group algebra and the preprojective algebra

For $V = \mathbb{C}^n$, a finite subgroup $G < \mathrm{GL}(V)$ defines an action of G on $\mathbb{C}[V]$ by the contragredient action $g \cdot p := p^g$, where $p^g(v) = p(g^{-1}v)$, and hence defines a commutative algebra of invariant polynomials $\mathbb{C}[V]^G = \{p \in \mathbb{C}[V] : p^g = p \text{ for all } g \in G\}$. The action of G on $\mathbb{C}[V]$ can also be used to define a noncommutative algebra.

Definition 1.1. The *skew group algebra* $\mathbb{C}[V] \rtimes G$ is defined as the vector space $\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}G$ with multiplication

$$(p_1 \otimes g_1) \cdot (p_2 \otimes g_2) = (p_1 p_2^{g_1} \otimes g_1 g_2)$$

for $p_i \in \mathbb{C}[V]$ and $g_i \in G$, extended by linearity to $\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}G$.

The algebras $\mathbb{C}[V]^G$ and $\mathbb{C}[V] \rtimes G$ are closely related. Direct calculation shows that the centre of $\mathbb{C}[V] \rtimes G$ is isomorphic to $\mathbb{C}[V]^G$. Further, $\mathbb{C}[V]$ is a finitely generated $\mathbb{C}[V]^G$-module by [Ben93, Theorem 1.3.1], and hence $\mathbb{C}[V] \rtimes G$ is a module-finite $\mathbb{C}[V]^G$-algebra. An element $g \in G < \mathrm{GL}(V)$ is a *pseudo-reflection* if it has exactly one eigenvalue that is not equal to 1,

and a subgroup $G < \mathrm{GL}(V)$ is *small* if it contains no pseudo-reflections. When $G < \mathrm{GL}(V)$ is a small subgroup the skew group algebra is isomorphic to $\mathrm{End}_{\mathbb{C}[V]^G}(\mathbb{C}[V])$, where $\mathbb{C}[V]$ is considered as a $\mathbb{C}[V]^G$-module (see [IT13, Section 3]).

Indeed, each irreducible representation ρ of G defines a $\mathbb{C}[V]^G$-module $M_\rho := (\mathbb{C}[V] \otimes_{\mathbb{C}[V]\rtimes G} \rho^\vee)^G$, and as the regular representation decomposes into $\mathbb{C}G \cong \bigoplus_\rho \rho^{\dim \rho}$ it follows that $\mathbb{C}[V] \cong (\mathbb{C}[V] \rtimes G)^G$ decomposes into $\mathbb{C}[V] \cong \bigoplus_\rho M_\rho^{\dim \rho}$ as a left-$\mathbb{C}[V]^G$-module, where the sums are taken over representatives of all isomorphism classes of irreducible representations of G. Then, for small G, it follows that $\mathbb{C}[V] \rtimes G \cong \mathrm{End}_{\mathbb{C}[V]^G}(\bigoplus_\rho M_\rho^{\oplus \dim \rho})$, and the skew group algebra is Morita equivalent to the algebra $A := \mathrm{End}_{\mathbb{C}[V]^G}(\bigoplus_\rho M_\rho)$. This algebra can be presented as the path algebra of a quiver with relations, with one vertex corresponding to each module M_ρ. An advantage of this perspective is that this quiver can be calculated combinatorially.

Let $\rho_0, \rho_1, \ldots, \rho_{r-1}$ denote representatives of the isomorphism classes of irreducible representations of a finite group $G < \mathrm{GL}(V)$ where ρ_0 is the trivial representation. Then $\rho_i \otimes_{\mathbb{C}} V$ is a finite dimensional G representation, so decomposes into irreducible representations $\rho_i \otimes_{\mathbb{C}} V \cong \bigoplus_j \rho_j^{\oplus c_{ij}}$ defining the positive integers c_{ij}.

Definition 1.2. The *McKay Quiver* associated to a finite subgroup $G < \mathrm{GL}(V)$ is defined by having as vertices 0,1, $\ldots$, $r-1$ in correspondence with the representations $\rho_0, \rho_1, \ldots, \rho_{r-1}$, and c_{ij} arrows from vertex i to vertex j.

In the particular case that $G < \mathrm{SL}_2(\mathbb{C})$ the algebra $A := \mathrm{End}_{\mathbb{C}[x,y]^G}(\bigoplus_\rho M_\rho)$ can be presented as a quiver with relations in the following combinatorial manner.

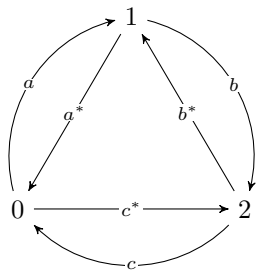
Proposition 1.3. *([CBH98]) The algebra $A = \mathrm{End}_{\mathbb{C}[x,y]^G}(\bigoplus_\rho M_\rho)$ can be presented as the* preprojective algebra *$A \cong \mathbb{C}Q/\Lambda$ where Q is the McKay quiver and the relations Λ are defined as follows: first chose an orientation between any two vertices in the quiver and label the arrows in this direction, then for each arrow labelled α from i to j choose an unlabelled arrow from j to i and label it α^* (this can be done as $c_{ij} = c_{ji}$ for $G < \mathrm{SL}_2(\mathbb{C})$). The relations Λ are generated by the element $\sum[\alpha, \alpha^*]$, with the sum taken over the arrows originally labelled.*

This gives a straightforward combinatorial recipe to present A as a path algebra with relations for any finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

Example 1.4. (Type A_2) Consider the finite group

$$G = \frac{1}{3}(1,2) := \left\langle \begin{pmatrix} \varepsilon_3 & 0 \\ 0 & \varepsilon_3^2 \end{pmatrix} \right\rangle < \mathrm{SL}_2(\mathbb{C})$$

where ε_3 is a primitive third root of unity. The group G is isomorphic to the cyclic group of order 3 hence has three irreducible representations all of dimension one: $\rho_i : g \mapsto \varepsilon_3^i$. As such the McKay quiver has three vertices $0, 1$, and 2. Then $V = \mathbb{C}^2 \cong \rho_1 \oplus \rho_2$ and $\rho_i \otimes V = \rho_{i-1} \oplus \rho_{i+1}$ (working modulo 3) hence $c_{ij} = 0$ if $i = j$ and $c_{ij} = 1$ otherwise. Choosing the orientation $0 \to 1 \to 2 \to 0$ the preprojective algebra is presented as the following McKay quiver with relations.



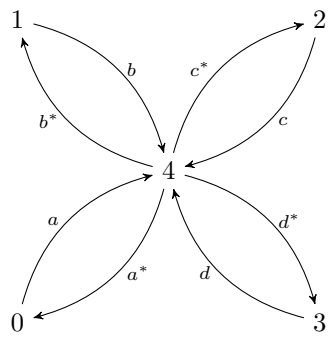
$$e_1 \sum[\alpha, \alpha^*] = aa^* - b^*b$$
$$e_2 \sum[\alpha, \alpha^*] = bb^* - c^*c$$
$$e_0 \sum[\alpha, \alpha^*] = cc^* - a^*a$$

These generating relations are recovered from $\sum[\alpha, \alpha^*]$ by multiplying by the trivial path e_i at each vertex i. This splits the single generating relation into three generating relations, one applying at each vertex.

Example 1.5. (Type D_4) Consider the binary dihedral group of order 8,

$$G := \left\langle \begin{pmatrix} \varepsilon_4 & 0 \\ 0 & \varepsilon_4^3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle < \mathrm{SL}_2(\mathbb{C}),$$

which has four one-dimensional representations $\rho_0, \rho_1, \rho_2, \rho_3$ and one two-dimensional representation ρ_4. The corresponding preprojective algebra is the following quiver with relations



$$aa^* + bb^* + cc^* + dd^* = 0$$
$$a^*a = b^*b = c^*c = d^*d = 0$$

1.1.2 Linking noncommutative algebra and geometry: the exceptional divisor and the McKay quiver

For $G < \mathrm{SL}_2(\mathbb{C})$, having introduced the invariant algebra $\mathbb{C}[x, y]^G$, the skew group algebra $\mathbb{C}[x, y] \rtimes G$, and the preprojective algebra A we now recall how these algebras link to resolutions of quotient singularities through the McKay Correspondence.

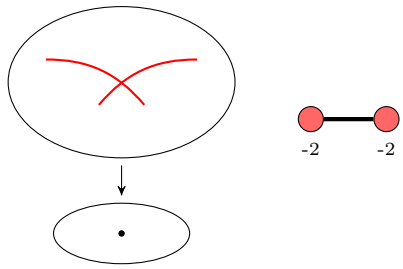
The commutative algebra $\mathbb{C}[x, y]^G$ defines the affine surface quotient singularity $\mathbb{C}^2/G := \mathrm{Spec}(\mathbb{C}[x, y]^G)$. This variety is singular with an isolated singular point at the origin (when G is non-trivial), and the minimal resolution $\pi : X \to \mathbb{C}^2/G$ of such a singularity exists since $\mathbb{C}^2/G$ has dimension 2 (see Section 2.4.1 for details).

An important combinatorial invariant of the minimal resolution is the *exceptional divisor*, which is defined to be the preimage of the singular point. For any finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ the exceptional divisor is a collection of $\mathbb{P}^1$ curves, each curve has self intersection number -2, and the dual intersection graph of this collection is a Dynkin diagram of type A, D, or E (see [Dur79]).

McKay observed the following link between the McKay quiver and the exceptional divisor.

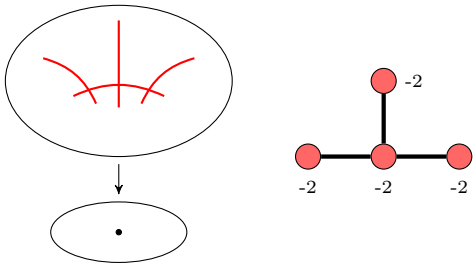
Theorem 1.6 ([McK80]). *For a finite subgroup of* $\mathrm{SL}_2(\mathbb{C})$ *the underlying graph of the McKay quiver is an affine Dynkin diagram, with extending vertex corresponding to the trivial representation, and has the same Dynkin type as the dual graph of the exceptional divisor in the minimal resolutions of the corresponding quotient surface singularity.*

Example 1.7. (Type A_2) In the type A_2 example above $\mathbb{C}[x,y]^G \cong \mathbb{C}[A,B,C]/(AC-B^3)$, and the exceptional divisor in the minimal resolution is a A_2 type tree,



which should be compared to the graph underlying the McKay quiver calculated in Example 1.4 after removing the 0 vertex. The construction of the minimal resolution of a cyclic quotient surface singularity is discussed in Section 2.4.2.

Example 1.8. (Type D_4) Similarly, continuing the type D_4 example from above $\mathbb{C}[x,y]^G \cong \mathbb{C}[A,B,C]/(A^2+B(C^2+B^2))$, and the exceptional divisor in the minimal resolution is a D_4 type tree,



which should be compared to the graph underlying the McKay quiver calculated in Example 1.5 after the 0 vertex has been removed.

1.1.3 Linking noncommutative algebra and geometry: moduli space construction and derived equivalence

This observation of McKay leads to the idea that there should be some link between the skew group algebra $\mathbb{C}[x,y] \rtimes G$ (or preprojective algebra A) and the minimal resolution of the singularity $\mathbb{C}^2/G$. Affine algebraic geometry builds the affine quotient singularity from the invariant algebra in a standard way: $\mathbb{C}^2/G := \operatorname{Spec} \mathbb{C}[x,y]^G$. In particular, the closed points of $\operatorname{Spec} \mathbb{C}[x,y]^G$ parametrise maximal ideals in $\mathbb{C}[x,y]^G$ and the construction of $\operatorname{Spec} \mathbb{C}[x,y]^G$ induces an equivalence of abelian categories between $\mathbb{C}[x,y]^G$-modules and quasicoherent sheaves on $\mathbb{C}^2/G$.

A similar construction can be extended to the noncommutative skew group algebra $\mathbb{C}[x,y] \rtimes G$. A variety can be constructed as a moduli space parametrising certain G-ideals and this moduli space construction induces not an abelian equivalence, but a derived equivalence between modules and quasicoherent sheaves.

Such a construction was achieved by Ito and Nakamura [IN96] who realised the minimal resolution $\mathbb{C}^2/G$ as the G-Hilbert scheme G-Hilb$(\mathbb{C}^2)$, a moduli space parametrising certain G-invariant ideals in $\mathbb{C}[x,y]$. Building on work of Gonzalez-Sprinberg and Verdier [GSV83], it was then shown by Kapranov and Vasserot [KV00] that the G-Hilb$(\mathbb{C}^2)$ construction induces a derived equivalence. We refer to Section 4.2.2 for the definition of G-Hilb$(\mathbb{C}^2)$ as a fine moduli space but for this introduction we will just note two important properties of this variety. Firstly, the closed points of G-Hilb$(\mathbb{C}^2)$ parametrise certain G invariant ideals in $\mathbb{C}[x,y]$:

$$\left\{\text{Closed points } p \in G\text{-Hilb}(\mathbb{C}^2)\right\} \longleftrightarrow \{I \trianglelefteq \mathbb{C}[x,y] \ \mid \ \mathbb{C}[x,y]/I \cong \mathbb{C}G \text{ as a } \mathbb{C}G\text{-module}\}$$

or equivalently parametrise the quotient $\mathbb{C}[x,y] \rtimes G$-modules

$$\longleftrightarrow \left\{ \begin{array}{c} \text{Surjections } \mathbb{C}[x,y] \to M \to 0 \\ \text{of left } \mathbb{C}[x,y] \rtimes G\text{-modules} \\ \text{such that } M \cong \mathbb{C}G \text{ as a } \mathbb{C}G\text{-module} \end{array} \right\}.$$

Secondly, the association of a $\mathbb{C}[x,y] \rtimes G$-module $M = \mathbb{C}[x,y]/I$ to each closed point of G-Hilb$(\mathbb{C}^2)$ glues into a global construction: G-Hilb$(\mathbb{C}^2)$ is equipped with a tautological vector bundle T such that the fibre at a point x is the $\mathbb{C}[x,y] \rtimes G$ module corresponding to the point x. In particular, T has a decomposition

$$T \cong \bigoplus T_\rho^{\oplus \dim \rho}$$

where the sum is taken over irreducible representations of G, $T_{triv} \cong \mathcal{O}_X$, and $\operatorname{rk} T_\rho = \dim \rho$.

This description provides a strong connection between the skew group algebra and the minimal resolution.

Theorem 1.9. *With notation as above,*

i) [IN96] The variety $X = G$-Hilb$(\mathbb{C}^2)$ is the minimal resolution of $\mathbb{C}^2/G$.

ii) [KV00] The tautological bundle T induces an equivalence between the derived category of quasicoherent sheaves on G-Hilb$(\mathbb{C}^2)$ and the derived category of left $\mathbb{C}[x,y] \rtimes G$-modules.

Hence this moduli description both constructs the minimal resolution of $\mathbb{C}^2/G$ from $\mathbb{C}[x,y] \rtimes G$ and produces a derived equivalence between G-Hilb$(\mathbb{C}^2)$ and $\mathbb{C}[x,y] \rtimes G$. Indeed, T is a tilting bundle as described in Section 2.3.1.

An alternative form of this result can be stated in terms of the preprojective algebra $A \cong \mathbb{C}Q/\Lambda$ by translating modules across the Morita equivalence. Define P_0 to be the submodule of $A \cong \mathbb{C}Q/\Lambda$ generated by all paths leaving the vertex 0 (corresponding to the $\mathbb{C}[x,y] \rtimes G$ module $\mathbb{C}[x,y]$ under the Morita equivalence). Then there exists a moduli space $\mathcal{M}_A$ parametrising certain A-modules,

$$\{\text{closed points in } \mathcal{M}_A\} \longleftrightarrow \left\{ \begin{array}{c} \text{Surjections } P_0 \to M \to 0 \text{ of left } A\text{-modules} \\ \text{such that } e_i M \text{ has dimension } \dim \rho_i. \end{array} \right\},$$

such that $\mathcal{M}_A \cong G$-Hilb$(\mathbb{C}^2)$ as varieties. This construction carries the tautological bundle $T' = \bigoplus T_\rho$ rather than the tautological bundle $T = \bigoplus T_\rho^{\oplus \dim \rho}$ carried by G-Hilb$(\mathbb{C}^2)$. This moduli space $\mathcal{M}_A$ can be constructed by quiver GIT, and the general construction of a quiver GIT quotient is discussed further in Section 2.2.2.

Theorem 1.10. *With notation as above,*

i) [CB00] The variety $\mathcal{M}_A$ is the minimal resolution of $\mathbb{C}^2/G$.

ii) The tautological bundle T' induces an equivalence between the derived category of quasi-coherent sheaves on $\mathcal{M}_A$ and the derived category of left A-modules.

We note that ii) is a restatement of Kapranov and Vasserot's result for the tautological bundle T', and Crawley-Boevey's result in [CB00] holds for more general quiver GIT quotients than the one specified here.

1.2 Path algebras with superpotentials

This section briefly outlines the results we will prove in Chapter 3. A more complete introduction is given in Section 3.1.

In the $\mathrm{SL}_2(\mathbb{C})$ McKay correspondence the skew group algebras are related to minimal resolutions of quotient singularities, and the concept of a 'noncommutative crepant resolution' (NCCR) was introduced by Van den Bergh to formalise such a situation [VdB04a] (see Section 2.4.3). In this language the skew group algebras for finite subgroups $G < \mathrm{SL}_n(\mathbb{C})$ are noncommutative crepant resolutions of the invariant algebras $\mathbb{C}[x_1, \ldots, x_n]^G$ [VdB04a, Example 1.1].

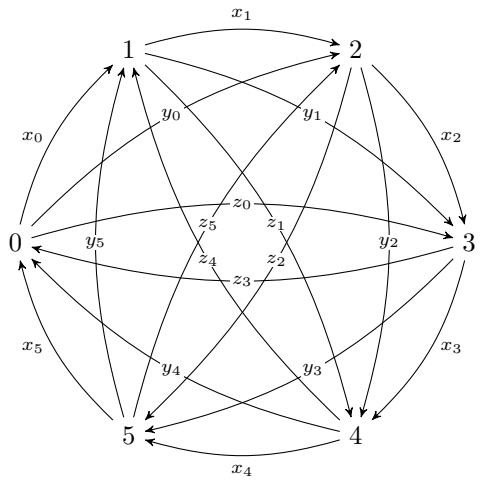
In general, any skew group algebra is Morita equivalent to the path algebra of the corresponding McKay quivers with certain relations [BSW10, Section 3]. The McKay quiver can always be constructed combinatorially from the data of the irreducible representations of G and the given representation $G < \mathrm{SL}_n(\mathbb{C})$. When $G < \mathrm{SL}_2(\mathbb{C})$ the relations can also be defined combinatorially by a single element $\sum[\alpha, \alpha^*]$ as described in Proposition 1.3. For general skew group algebras Bocklandt, Schedler, and Wemyss [BSW10] showed that the relations can always be efficiently calculated via a single *superpotential* element in the path algebra, which generalises the element $\sum[\alpha, \alpha^*]$. The notion of a superpotential is recalled in Section 2.2.1. Moreover, they also show that, up to Morita equivalence, the wider class of Koszul, Calabi-Yau algebras (see Sections 2.3.3 and 2.3.4) can be presented as path algebras of quivers with relations defined by a superpotential in a similar way, and that the superpotential element actually encodes the entire Koszul resolution. These results are recalled in Section 3.2.1. This class of Koszul, Calabi-Yau algebras includes many interesting examples of noncommutative crepant resolutions.

Example 1.11. (G=$\frac{1}{6}(1,2,3)$) Let G be the group

$$G := \left\langle \begin{pmatrix} \varepsilon_6 & 0 & 0 \\ 0 & \varepsilon_6^2 & 0 \\ 0 & 0 & \varepsilon_6^3 \end{pmatrix} \right\rangle < \mathrm{SL}_3(\mathbb{C}),$$

where ε_6 is a primitive 6th root of unity. The algebra A, defined as the path algebra of the

following McKay quiver with relations, is Koszul, 3-Calabi-Yau, and a NCCR of $\mathbb{C}[x,y,z]^G$.



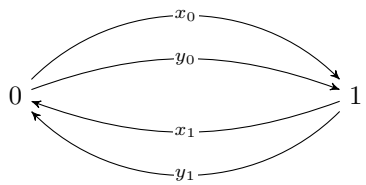
$$y_{i+1}x_i = x_{i+2}y_i$$
$$z_{i+2}y_i = y_{i+3}z_i$$
$$x_{i+3}z_i = z_{i+1}x_i$$
$$\text{for } i \in \mathbb{Z}/(6).$$

These relations can be efficiently expressed via the superpotential

$$\Phi := \sum_{i\in\mathbb{Z}/(6)} (z_{i+3}y_{i+1}x_i + x_{i+5}z_{i+2}y_i + y_{i+4}x_{i+3}z_i - y_{i+4}z_{i+1}x_i - x_{i+5}y_{i+3}z_i - z_{i+3}x_{i+2}y_i)$$

from which the relations are recovered by differentiating with respect to arrows in the quiver. We refer the reader to Section 2.2.1 and Example 2.8 for details.

Example 1.12. (Conifold quiver) The algebra presented as the following quiver with relations is Koszul, 3-Calabi-Yau, and a NCCR of the algebra $\mathbb{C}[X,Y,Z,W]/(XZ-YW)$.



$$x_0y_1y_0 = y_0y_1x_0$$
$$y_0x_1x_0 = x_0x_1y_0$$
$$y_1y_0x_1 = y_1y_0x_1$$
$$x_1x_0y_1 = y_1x_0x_1$$

These relations can be recovered from the superpotential

$$\begin{aligned}\Phi :&= \sum (x_0y_1y_0x_1 + x_1y_0y_1x_0 + y_0x_1x_0y_1 + y_1x_0x_1y_0)\\ &- \sum (x_0x_1y_0y_1 + x_1x_0y_1y_0 + y_0y_1x_1x_0 + y_1y_0x_1x_0)\,,\end{aligned}$$

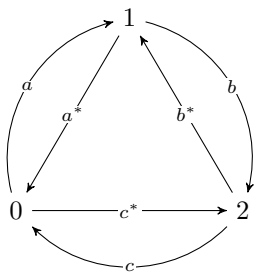
as is later calculated in Example 2.9.

We are motivated by these results to consider interesting phenomenon which occur for skew group algebras and investigate how they generalise to all Koszul, Calabi-Yau algebras, particularly properties that are controlled by cohomological data and the Koszul resolution. One phenomenon exhibited by the preprojective algebras is the existence of deformations constructed by Crawley-Boevey and Holland [CBH98].

Proposition 1.13 ([CBH98])**.** *Let $A \cong \mathbb{C}Q/\sum[\alpha,\alpha^*]$ be a preprojective algebra. Then the corresponding* deformed preprojective algebras *are defined to be $A_\lambda := \mathbb{C}Q/\Pi_\lambda$ where Π_λ is the two-sided ideal generated by the single element $\sum[\alpha,\alpha^*] - \sum \lambda_i e_i$, for some $\lambda_i \in \mathbb{C}$. These are filtered algebras with associated graded algebra isomorphic to A.*

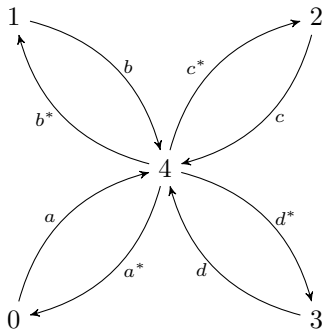
These are still path algebras of the McKay quiver with relations generated by the single element $\sum[\alpha, \alpha^*] - \sum \lambda_i e_i$, where the second sum is taken over vertices i with e_i denoting the trivial path at vertex i.

Example 1.14. (Type A_2) The deformed preprojective algebras of type A_2 can be presented as the path algebra of the following quiver with relations.



$$aa^* - b^*b = \lambda_1 e_1$$
$$bb^* - c^*c = \lambda_2 e_2$$
$$cc^* - a^*a = \lambda_0 e_0$$

Example 1.15. (Type D_4) The deformed preprojective algebras of type D_4 can be presented as the path algebra of the following quiver with relations.



$$aa^* + bb^* + cc^* + dd^* = \lambda_4 e_4.$$
$$a^*a = \lambda_0 e_0$$
$$b^*b = \lambda_1 e_1$$
$$c^*c = \lambda_2 e_2$$
$$d^*d = \lambda_3 e_3$$

Deformed preprojective algebras are PBW deformations of preprojective algebras (the definition of a PBW deformation is recalled in Section 3.2.2) and the PBW deformations of any Koszul algebra are classified by a result of Braverman and Gaitsgory [BG96], which is also recalled in Section 3.2.2. Deformed preprojective algebras are just the tip of a larger class of interesting PBW deformations. For example, symplectic reflection algebras are defined as PBW deformations of skew group algebras for finite subgroups of $\mathrm{Sp}_{2n}(\mathbb{C})$.

Motivated by these examples, in Chapter 3 we will investigate PBW deformations for general Koszul, CY algebras where we are able to use superpotentials to understand the Koszul resolution and reinterpret the results of Braverman and Gaitsgory. In Section 3.3 we give a condition for a PBW deformation of a Calabi-Yau, Koszul path algebra with relations given by a superpotential to have relations given by a superpotential and prove that these deformations are Calabi-Yau in certain cases. In Section 3.4 we apply these methods to symplectic reflection algebras to show that every symplectic reflection algebra is Morita equivalent to a path algebra with relations generated by the higher derivations of an inhomogeneous superpotential, and is Calabi-Yau regardless of the deformation parameter. A more complete introduction to the results proved in Chapter 3 is given in Section 3.1.

We would also like to be able to treat $G < \mathrm{GL}_2(\mathbb{C})$ quotient surface singularities in a similar manner, however we will see that the skew group algebras are not so closely linked to the geometry of these more general surface quotient singularities. The results of [BSW10] allow

the efficient calculation of the skew group algebras when $G < \mathrm{GL}_2(\mathbb{C})$ as path algebras with relations defined by a twisted superpotential. For G a finite subgroup of $\mathrm{GL}_2(\mathbb{C})$, we consider PBW deformations of this path algebra with relations, and in Section 3.5 we show there are no non-trivial PBW deformations when G is a small subgroup not contained in $\mathrm{SL}_2(\mathbb{C})$. This is despite there being nontrivial deformations of both the associated quotient singularities and their minimal resolutions, and so the skew group algebras do not provide an approach to study these geometric deformations for general finite subgroups of $\mathrm{GL}_2(\mathbb{C})$. This could be expected as it is well-known that the skew group algebras are not generally derived equivalent to minimal resolutions of quotient singularities unless $G < \mathrm{SL}_2(\mathbb{C})$. For a solution to this problem see [Kar].

1.3 Moduli spaces and the $\mathrm{GL}_2(\mathbb{C})$ McKay correspondence

In this section we discuss the motivation for Chapter 4 arising from the McKay correspondence. The results in Chapter 4 also apply in more general situations, and a more complete introduction to the results of Chapter 4 is given in Section 4.1.

Having discussed the McKay correspondence for a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ it seems natural to attempt to generalise from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{GL}_2(\mathbb{C})$. When G is a small, finite subgroup of $\mathrm{GL}_2(\mathbb{C})$ the algebras $\mathbb{C}[x,y]^G$ and $\mathbb{C}[x,y] \rtimes G$ and the minimal resolution $\pi : X \to \mathbb{C}^2/G = \mathrm{Spec}(\mathbb{C}[x,y]^G)$ can still be defined, the quotient $\mathbb{C}^2/G$ is still an isolated singularity, and the preimage of the singular point defines an exceptional divisor in X which is a tree of $\mathbb{P}^1$ curves. As such this situation looks very similar to the $\mathrm{SL}_2(\mathbb{C})$ McKay correspondence, and it is natural to consider whether the results there generalise to this case.

Firstly, G-$\mathrm{Hilb}(\mathbb{C}^2)$ still gives a method to construct the minimal resolution.

Theorem 1.16. *([Ish02]) For a small, finite subgroup $G < \mathrm{GL}_2(\mathbb{C})$ the G-Hilbert scheme G-$\mathrm{Hilb}(\mathbb{C}^2)$ is isomorphic to the minimal resolution of $\mathbb{C}^2/G$ as a variety.*

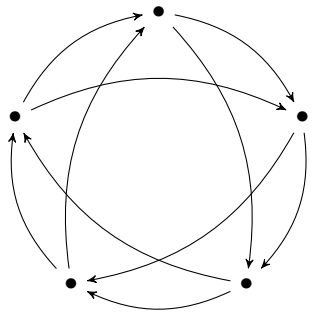
However, the tautological bundle of this moduli construction does not induce a derived equivalence between the minimal resolution and $\mathbb{C}[x,y] \rtimes G$. Indeed, there are now far more representations of G than components of the exceptional divisor. We recall two examples analogous to the A_2 and D_4 examples above.

Example 1.17. (Type $A_{5,2}$) Consider the cyclic group

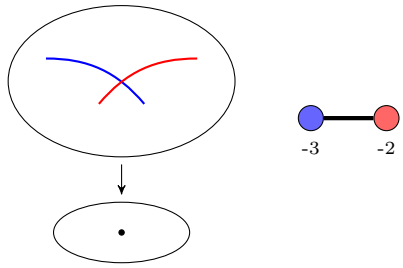
$$G = \frac{1}{5}(1,2) := \left\langle \begin{pmatrix} \varepsilon_5 & 0 \\ 0 & \varepsilon_5^2 \end{pmatrix} \right\rangle < \mathrm{GL}_2(\mathbb{C})$$

where ε_5 is a primitive 5th root of unity. The group G is abelian of order 5 so has 5 irreducible

representations, all of dimension one, and it has McKay quiver



whereas the minimal resolution has the following exceptional divisor

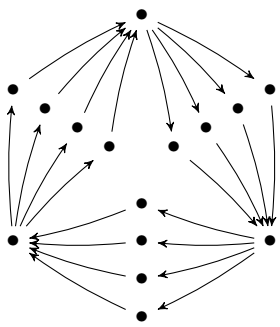


which consists of only two components, a -3 curve meeting with a -2 curve. There is a discussion on how to calculate the minimal resolution of such cyclic quotient surface singularities in Section 2.4.2.

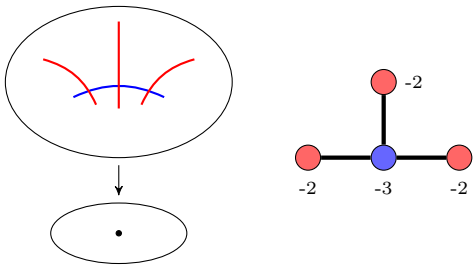
Example 1.18. (Type $D_{5,2}$) Consider the group

$$G = \left\langle \begin{pmatrix} \varepsilon_4 & 0 \\ 0 & \varepsilon_4^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_4 \\ \varepsilon_4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_6 \\ \varepsilon_6 & 0 \end{pmatrix} \right\rangle < \mathrm{GL}_2(\mathbb{C})$$

where ε_r is a primitive r^{th} root of unity. This group G is the direct product of the quaternion group of order 8 with a cyclic group of order 3, and it has 24 elements and 15 irreducible representations, 3 of dimension two and 12 of dimension one. The McKay quiver is



whereas the minimal resolution has the following exceptional divisor



which consists of only four components, with three -2 curves meeting a central -3 curve [Rie77, Section 3]. This minimal resolution is calculated in Example 4.36.

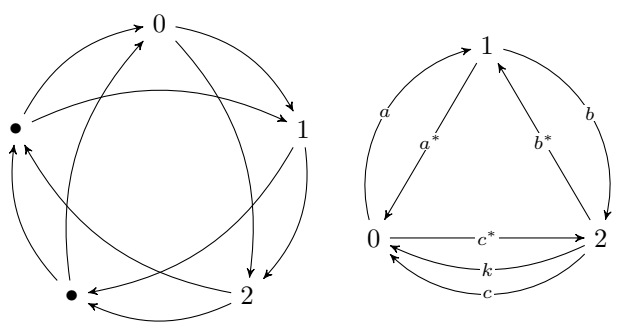
From this perspective the algebra $\mathrm{End}_{\mathbb{C}[x,y]^G}(\bigoplus_\rho M_\rho)$ is too large, and rather than considering the endomorphism algebra of the sum over the M_ρ for all irreducible representations it would be better to restrict to the sum over only certain *special representations* introduced by Wunram.

Definition 1.19. Let $R = \mathbb{C}[x,y]^G$. An irreducible representation ρ is *special* if $M_\rho \otimes_R \omega_R$ is a maximal Cohen-Macaulay R-module after torsion is factored out, where ω_R is the canonical module for R.

Wunram showed that there is a 1 to 1 correspondence between the non-trivial special representations and components of the exceptional divisor [Wun88], and this motivates the following definition.

Definition 1.20 ([Wem11b])**.** The *reconstruction algebra* associated to a finite, small subgroup $G < \mathrm{GL}_2(\mathbb{C})$ is defined to be $A := \mathrm{End}_{\mathbb{C}[x,y]^G}(\bigoplus M_\rho)$, where the sum is now taken only over special representations.

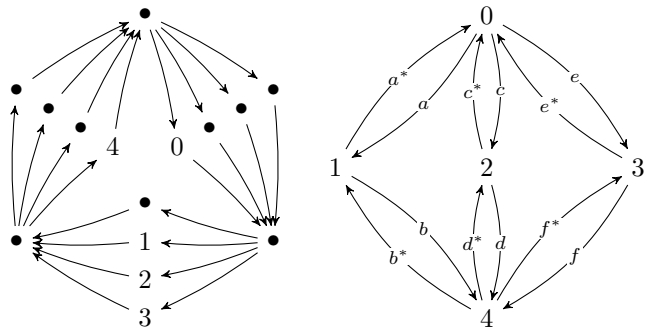
Example 1.21. (Type $A_{5,2}$) Continuing the $A_{5,2}$ example we mark the vertices corresponding to special representations in the McKay quiver following [Wun88], and present the corresponding reconstruction algebra as a quiver with relations [Wem11a].



$$aa^* = b^*b$$
$$bb^* = c^*k$$
$$kc^* = a^*a$$
$$c^*c = bak$$
$$kba = cc^*$$

Example 1.22. (Type $D_{5,2}$) Continuing the $D_{5,2}$ example we mark the vertices corresponding to special representations in the McKay quiver following [IW10], and present the corresponding

reconstruction algebra as a quiver with relations as in [Wem12].



$$a^*a = c^*c = e^*e$$
$$aa^* = b^*b,$$
$$cc^* = d^*d,$$
$$ee^* = f^*f$$
$$bb^* = dd^* = ff^*$$
$$ba - cd = fe$$

In fact, special modules were defined more generally by Wunram for (germs of) rational surface singularities, a generalisation of quotient surface singularities (see Section 2.4.1 for a definition of rational surface singularities).

Definition 1.23. Suppose that $\mathrm{Spec}(R)$ is a rational surface singularity and $\hat{R}$ is the germ at the singular point. Then a maximal Cohen-Macaulay $\hat{R}$-module M is said to be *special* if $\mathrm{Ext}^1_{\hat{R}}(M, \hat{R}) = 0$.

We note that this criterion for specialness is not the one originally given by Wunram but is equivalent by a result of Iyama and Wemyss [IW10, Theorem 2.7]. The reconstruction algebra can also be defined in this level of generality.

Definition 1.24. The *reconstruction algebra* associated to the germ $\hat{R}$ of rational surface singularity $\mathrm{Spec}(R)$ is defined to be $A = \mathrm{End}_{\hat{R}}(\oplus M_\rho)$ where the sum is taken over the indecomposable special maximum Cohen-Macaulay $\hat{R}$-modules.

The reconstruction algebras were defined by Wemyss, who also proved that the graph underlying the reconstruction algebra associated to $\hat{R}$ with vertex corresponding to $\hat{R}$ removed matches the dual graph of the exceptional divisor [Wem11b]. This result generalises the observation of McKay in the $\mathrm{SL}_2(\mathbb{C})$ McKay correspondence.

There is also a derived equivalence relating reconstruction algebras and minimal resolutions of rational surface singularities, generalising the derived equivalence of the $\mathrm{SL}_2(\mathbb{C})$ McKay correspondence.

Theorem 1.25 ([VdB04b,Wem11b])**.** *Suppose that G is a finite, small subgroup of $\mathrm{GL}_2(\mathbb{C})$ and $\pi : X \to \mathbb{C}^2/G$ is the minimal resolution of the quotient singularity. Then there is a vector bundle T on X that induces an equivalence between the derived category of quasicoherent sheaves on X and the derived category of right modules for the reconstruction algebra A associated to G.*

More generally, if $\pi : X \to \mathrm{Spec}(R)$ is the minimal resolution of a rational surface singularity then there is a vector bundle T on X that induces a derived equivalence between X and $\mathrm{End}_X(T)^{\mathrm{op}}$ such that the completion of $\mathrm{End}_X(T)^{\mathrm{op}}$ with respect to the isolated singular point of $\mathrm{Spec}(R)$ is Morita equivalent to the reconstruction algebra associated to $\hat{R}$.

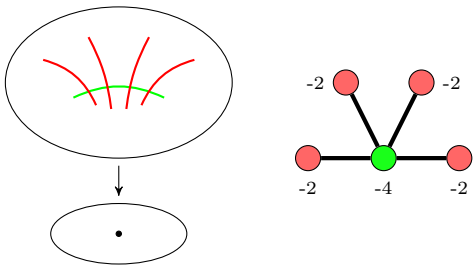
We will refer to the algebra $\mathrm{End}_X(T)^{\mathrm{op}}$ as a *reconstruction algebra* associated to R. In all of our examples the completion of $\mathrm{End}_X(T)^{\mathrm{op}}$ will be isomorphic to the corresponding reconstruction algebra, not just Morita equivalent. Such a T can be chosen whenever there is a good $\mathbb{C}^*$-action on the singularity R. In this situation the reconstruction algebra associated to

$\mathbb{C}[x,y]^G$ matches the reconstruction algebra associated to G, and the completion of a reconstruction algebra associated to R is isomorphic to the reconstruction algebra associated to $\hat{R}$. We give an example of reconstruction algebras associated to a family of non-quotient rational surface singularities.

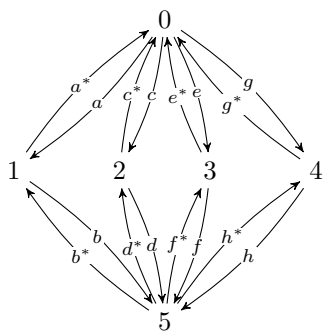
Example 1.26. (Non-quotient example) Let R be defined as the subalgebra of $\mathbb{C}[t, X_1, X_2, X_3, Y_0]$ subject to the relations given by the two by two minors of the matrix

$$\begin{pmatrix} t^2 & X_1 & X_2 & X_3 \\ Y_0 & t^2 & X_2 + t^2 & \lambda X_3 + t^2 \end{pmatrix}$$

for some $\lambda \in \mathbb{C}\backslash\{0,1\}$. This defines a rational surface singularity $\mathrm{Spec}(R)$ such that the exceptional divisor in the minimal resolution is not of Dynkin type; it is of the form



with four -2 curves intersecting a central -4 curve, see [Wah77, Corollary 3.6]. This minimal resolution is calculated in Example 4.37. The corresponding reconstruction algebra is the path algebra of the following quiver with relations [IW].



$$a^*a = c^*c = e^*e = g^*g,$$
$$aa^* = b^*b, \quad cc^* = d^*d,$$
$$ee^* = f^*f, \quad gg^* = hh^*,$$
$$bb^* = dd^* = ff^* = h^*h,$$
$$ba - cd = ef, \text{ and } ba - \lambda cd = hg$$

As this is not a quotient singularity there is no associated McKay quiver.

These results generalise the derived equivalence of the $\mathrm{SL}_2(\mathbb{C})$ McKay correspondence to all rational surface singularities, with reconstruction algebras replacing preprojective or skew group algebras. However, this derived equivalence is directly constructed from the geometry rather than realised via a moduli space construction as in the original McKay correspondence, and it is natural to attempt to extend the moduli approach of the $\mathrm{SL}_2(\mathbb{C})$ McKay correspondence to this setting.

Whilst, by Theorem 1.16, G-Hilb$(\mathbb{C}^2)$ does provide a construction of the minimal resolution of a quotient singularity as a variety, it does not induce a derived equivalence and G-Hilb$(\mathbb{C})$ can not be defined for non-quotient rational surface singularities. To remedy this it was conjectured that for any rational surface singularity the moduli space constructed by quiver GIT

for the reconstruction algebra should reproduce the minimal resolution (for a certain choice of dimension vector and stability condition) such that the tautological bundle induces the derived equivalence defined by T. However, this was previously only known for certain quotient singularity cases that could be verified by direct calculation [Cra11,Wem11a,Wem12,Wem13]. We note that for quotient singularities it is generally easier to calculate the quiver GIT quotient of the reconstruction algebra than G-Hilb($\mathbb{C}^2$): see the $D_{5,2}$ example above and compare quiver GIT for the full McKay quiver containing rank two vertices and the quiver of the reconstruction algebra which contains only rank one vertices.

Attempting to solve this, and other similar problems, motivates Chapter 4. One of the main results of Chapter 4, Corollary 4.31, shows that the quiver GIT quotient of the reconstruction algebra associated to a rational surface singularity does indeed construct the minimal resolution, and that the tautological bundle for this construction does induce a derived equivalence. This is a natural extension of the philosophy of realising derived equivalences via moduli constructions from the $\mathrm{SL}_2(\mathbb{C})$ McKay correspondence and also provides a moduli interpretation for the minimal resolutions of all rational surface singularities.

Chapter 2

Preliminary material

This chapter contains definitions and concepts which will be vital in both Chapters 3 and 4. Much of the material in this chapter is amalgamated from the preliminary sections of the author's paper [Kar14b] and preprint [Kar14a].

2.1 Notation and assumptions

We first give some general notational assumptions which will hold throughout the rest of the thesis. We will always work over the field $\mathbb{C}$. For a $\mathbb{C}$-algebra A we let A^{op} denote the opposite algebra of A, and we define $A^e := A \otimes_{\mathbb{C}} A^{\mathrm{op}}$. Then we let A-mod denote the category of finitely generated left A-modules, so A^e-mod is equivalent to the category of finitely generated A-A-bimodules. Whenever we work with schemes we will assume they are Noetherian and defined over $\mathbb{C}$. We denote the category of coherent sheaves on a scheme X by $\operatorname{Coh} X$, we denote the skyscraper sheaf of a closed point $x \in X$ by $\mathcal{O}_x$, and for a locally free sheaf $\mathcal{F} \in \operatorname{Coh} X$ we let $\mathcal{F}^\vee$ denote the dual $\mathcal{H}\mathrm{om}_X(\mathcal{F}, \mathcal{O}_X)$.

2.2 Quivers, superpotentials, and quiver GIT

2.2.1 Quivers, superpotentials, and differentiation

Here we set notation for a quiver, its path algebra, and its path algebra with relations. Certain elements of the path algebra are defined to be superpotentials, and we recall a construction which produces relations on a quiver from a superpotential by differentiation. We follow the set up of [BSW10].

Quivers

Definitions 2.1. A *quiver* is a directed multigraph, denoted by $Q = (Q_0, Q_1)$, with Q_0 the set of vertices and Q_1 the set of arrows. The set of arrows is equipped with head and tail maps $h, t : Q_1 \to Q_0$ which take an arrow to the vertices that are its head and tail respectively.

A *non-trivial path* in the quiver is defined to be a sequence of arrows

$$p = a_r \dots a_2 a_1$$

with $a_i \in Q_1$ satisfying $h(a_i) = t(a_{i+1})$ for $1 \leq i \leq r-1$.

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \cdots \xrightarrow{a_r} \bullet$$

We will reuse the notation of the head and tail maps for the head and tail of a path, defining $h(p) = h(a_r)$ and $t(p) = t(a_1)$ when $p = a_r \dots a_2 a_1$. There is also a *trivial path* e_i at each vertex $i \in Q_0$, which has both head and tail equal to i. A path p is called *closed* if $h(p) = t(p)$. The *pathlength* of a nontrivial path $p = a_r \dots a_1$, where each a_i is an arrow, is defined to be r. A trivial path is defined to have pathlength 0. We will denote the pathlength of a path p by $|p|$.

Definition 2.2. The *path algebra* of the quiver Q, denoted $\mathbb{C}Q$, is defined as follows: as a $\mathbb{C}$-vector space $\mathbb{C}Q$ has a basis given by the paths in the quiver and an associative multiplication is defined by concatenation of paths

$$p.q = \begin{cases} pq & \text{if } h(q) = t(p) \\ 0 & \text{otherwise.} \end{cases}$$

We define S to be the subalgebra of this generated by the trivial paths, and V to be the $\mathbb{C}$-vector subspace of $\mathbb{C}Q$ spanned by the arrows, $a \in Q_1$. Then S is a semisimple algebra with one simple module for each vertex.

As $a = e_{h(a)}.a.e_{t(a)}$ for any arrow a it follows that V has the structure of a left $S^e := S \otimes_{\mathbb{C}} S^{\mathrm{op}}$ module. Then $\mathbb{C}Q$ can be identified with the tensor algebra $T_S(V) = S \oplus V \oplus (V \otimes_S V) \oplus \dots$ by equating the path $a_r \dots a_1$ with the element $a_r \otimes_S \cdots \otimes_S a_1$.

The algebra $T_S(V) = \mathbb{C}Q$ is equipped with a grading and filtration by pathlength; the graded part in degree n is $T_S(V)^n := V^{\otimes_S n}$ and the filtered part in degree n is $F^n(T_S(V)) := V^{\otimes_S n} \oplus \cdots \oplus V \oplus S$.

Given $\Lambda \subset T_S(V)$ define $I(\Lambda)$ to be the two sided ideal in $T_S(V)$ generated by Λ, and define

$$\frac{\mathbb{C}Q}{\Lambda} := \frac{\mathbb{C}Q}{I(\Lambda)}$$

which we refer to as the path algebra with *relations* Λ.

We define $< \Lambda >$ to be the S^e submodule of $T_S(V)$ generated by Λ.

Superpotentials

We define superpotentials and twisted superpotentials for a path algebra $\mathbb{C}Q = T_S(V)$ in both the homogeneous and inhomogeneous cases. Superpotentials will be a central object of study in Chapter 3, but twisted superpotentials will only appear in Section 3.5. We then recall a construction that produces relations on the path algebra of the quiver from the superpotential.

Definition 2.3. Let $\Phi_n = \Sigma_p c_p p \in T_S(V)^n$ be an element of the path algebra of pathlength n expressed as a linear combination of all paths p of length n with coefficients c_p in $\mathbb{C}$.

Then Φ_n satisfies the n *superpotential condition* if $c_{aq} = (-1)^{n-1} c_{qa}$ for all $a \in Q_1$ and paths q. We use the convention that $c_0 = 0$ so that if $t(a) \neq h(b)$, then $ab = 0$ and $c_{ab} = c_0 = 0$. If Φ_n satisfies the n superpotential condition we will call it a *homogeneous superpotential of degree* n. In particular, this requires $c_p = 0$ if p is not a closed path.

Let $\sigma \in Aut_{\mathbb{C}}(\mathbb{C}Q)$ be a graded automorphism, so that $\sigma(T_S(V)^n) = T_S(V)^n$ for all $n \geq 0$, and assume that $\sigma(h(p)) = h(\sigma(p))$, $\sigma(t(p)) = t(\sigma(p))$ for any path p. The element Φ_n satisfies

the *twisted n superpotential condition* if $c_{\sigma(a)q} = (-1)^{n-1}c_{qa}$ for any $a \in Q_1$ and path q. We will say Φ_n is a *σ-twisted homogeneous superpotential of degree n* if it satisfies the *twisted n superpotential condition.*

Definition 2.4. Consider an element $\Phi = \Sigma_{m\leq n}\Sigma_{|p|=m}c_p p \in F^n(T_S(V))$ expressed as the linear combination of all paths of length $\leq n$.

Then Φ is an *inhomogeneous superpotential of degree n* if it satisfies the condition $c_{aq} = (-1)^{n-1}c_{qa}$ for any path q and arrow a, and for some p of length n the coefficient c_p is non-zero. An inhomogeneous superpotential Φ can be split into homogeneous parts $\Phi = \phi_n + \phi_{n-1} + \cdots + \phi_0$ where each of the ϕ_m satisfies the n-superpotential condition and ϕ_n is non-zero. Note that if n is even and m odd then $\phi_m = 0$, and any $\phi_0 \in S$ satisfies the n superpotential condition.

Let σ be a graded automorphism as above. The element Φ is a *σ-twisted inhomogeneous superpotential of degree n* if it satisfies the condition $c_{\sigma(a)q} = (-1)^{n-1}c_{qa}$ for any path q and arrow a. A twisted inhomogeneous superpotential Φ can be written in homogeneous parts $\Phi = \phi_n + \phi_{n-1} + \cdots + \phi_0$ where each of the ϕ_m satisfies the n-twisted superpotential condition.

Definition 2.5. Let p be a path in $\mathbb{C}Q$ and $a \in Q_1$. Then the left/right derivative of p by a is defined to be

$$\delta_a p = \begin{cases} q & \text{if } p = aq, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad p\delta'_a = \begin{cases} q & \text{if } p = qa, \\ 0 & \text{otherwise.} \end{cases}$$

This definition is extended such that if $q = a_n \dots a_1$ is a path of length n then $\delta_q = \delta_{a_1} \dots \delta_{a_n}$, and $\delta'_q = \delta'_{a_1} \dots \delta'_{a_n}$. We will call the operation δ_q left differentiation by the path q, and δ'_q right differentiation by the path q.

Note that for Φ a degree n inhomogeneous superpotential, $b, c \in Q_1$, and p any path

$$\delta_b \Phi = (-1)^{n-1}\Phi\delta'_b\,,$$

$$\delta_b \Phi \delta'_c = (-1)^{n-1}\delta_b\delta_c\Phi = (-1)^{n-1}\delta_{cb}\Phi\,,$$

and

$$\delta_p \Phi = \sum_{a\in Q_1} a\delta_a\delta_p\Phi = \sum_{a\in Q_1} \delta_p\Phi\delta'_a a\,.$$

Definition 2.6. Given an inhomogeneous superpotential Φ of degree n, S^e-modules $\mathcal{W}_{n-k} \subset \mathbb{C}Q$ are defined by

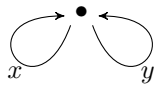
$$\mathcal{W}_{n-k} :=< \delta_p\Phi : |p| = k > .$$

The algebra $\mathcal{D}(\Phi, k)$ is then defined to be

$$\mathcal{D}(\Phi, k) := \frac{\mathbb{C}Q}{I(\mathcal{W}_{n-k})}$$

We will call this the *superpotential algebra* $\mathcal{D}(\Phi, k)$.

Example 2.7. Consider the quiver Q with one vertex $\bullet$ and two arrows x and y. Then $\mathbb{C}Q$ is the free algebra on two elements, $\mathbb{C} < x, y >$



We consider the following superpotentials on Q.

i) $\Phi_2 = xy - yx$ is a degree 2 homogeneous superpotential.

ii) $\Phi = xy - yx - e_\bullet$ is an inhomogeneous superpotential of degree 2.

iii) $\Phi_3 = xyx + xxy + yxx$ is a degree 3 homogeneous superpotential.

These define the following superpotential algebras.

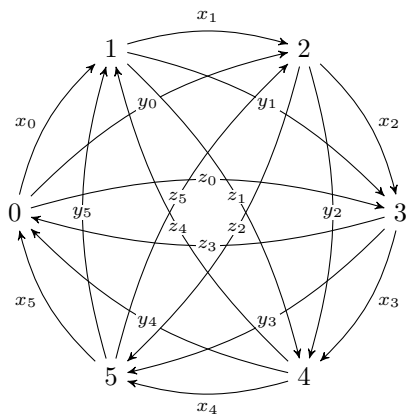
i) $\mathcal{D}(\Phi_2, 0) = \mathbb{C}[x, y]$, as $\mathcal{W}_2 =< xy - yx >$.

ii) $\mathcal{D}(\Phi, 0) = A_1(\mathbb{C})$, the first Weyl algebra, as $\mathcal{W}_2 = \{xy - yx = e_\bullet\}$.

iii) $\mathcal{D}(\Phi_3, 1)$ is $\mathbb{C} < x, y > /\mathcal{W}_2$ where $\mathcal{W}_2 = \{yx + xy = 0, xx = 0\}$.

We also realise the two examples of Section 1.2 as superpotential algebras.

Example 2.8. (G=$\frac{1}{6}(1, 2, 3)$) Consider the quiver



and superpotential

$$\Phi_3 := \sum_{i \in \mathbb{Z}/(6)} (z_{i+3}y_{i+1}x_i + x_{i+5}z_{i+2}y_i + y_{i+4}x_{i+3}z_i - y_{i+4}z_{i+1}x_i - x_{i+5}y_{i+3}z_i - z_{i+3}x_{i+2}y_i).$$

To calculate $\mathcal{D}(\Phi_3, 1)$ we differentiate Φ_3 by all paths of length one to produce the following generating relations:

$$\begin{aligned} \delta_{x_{i+5}}\Phi_3 &= z_{i+2}y_i - y_{i+3}z_i, \\ \delta_{y_{i+4}}\Phi_3 &= x_{i+3}z_i - z_{i+1}x_i, \text{ and} \\ \delta_{z_{i+3}}\Phi_3 &= y_{i+1}x_i - x_{i+2}y_i. \end{aligned}$$

Hence $\mathcal{D}(\Phi, 1)$ is isomorphic to the algebra in Example 1.11.

Example 2.9. (Conifold quiver) Consider the quiver

$$\begin{tikzcd} 0 \arrow[r, bend left=60, "x_0"] \arrow[r, bend left=20, "y_0"] & 1 \arrow[l, bend left=20, "x_1"] \arrow[l, bend left=60, "y_1"] \end{tikzcd}$$

and homogeneous superpotential

$$\begin{aligned} \Phi_4 := & \sum (x_0y_1y_0x_1 + x_1y_0y_1x_0 + y_0x_1x_0y_1 + y_1x_0x_1y_0) \\ & - \sum (x_0x_1y_0y_1 + x_1x_0y_1y_0 + y_0y_1x_1x_0 + y_1y_0x_1x_0) \,. \end{aligned}$$

Differentiating by paths of length one in the quiver we calculate:

$$\begin{aligned} \delta_{x_0}\Phi_4 &= y_1y_0x_1 - x_1y_0y_1, \\ \delta_{y_0}\Phi_4 &= x_1x_0y_1 - y_1x_0x_1, \\ \delta_{x_1}\Phi_4 &= x_0y_1y_0 - y_0y_1x_0, \text{ and} \\ \delta_{y_1}\Phi_4 &= y_0x_1x_0 - x_0x_1y_0. \end{aligned}$$

Hence $\mathcal{D}(\Phi_4, 1)$ is isomorphic to the algebra in Example 1.12.

2.2.2 Quiver GIT

We now set the notation required for quiver geometric invariant theory and recall the construction of a quiver GIT quotient, following the paper of King [Kin94].

Definitions 2.10. Let $Q = (Q_1, Q_0)$ be a quiver.

i) A *dimension vector* for Q is defined to be an element $d \in \mathbb{N}^{Q_0}$ assigning a non-negative integer to each vertex.

ii) A *dimension d representation* of Q is defined by assigning to each vertex i the vector space $V_i = \mathbb{C}^{d(i)}$, to each arrow a a linear map $\phi_a : V_{t(a)} \to V_{h(a)}$, and to each trivial path e_i the linear map id_{V_i}.

iii) A *morphism*, ψ, between two finite dimensional representations (V_i, ρ_a) and (W_i, χ_a) is given by a linear map $\psi_i : V_i \to W_i$ for each vertex i such that for every arrow a we have $\chi_a \circ \psi_{t(a)} = \psi_{h(a)} \circ \rho_a$.

iv) The *representation variety*, $\mathrm{Rep}_d(Q)$, is defined to be the set of all representations of Q of dimension d, and we note that this is an affine variety.

We then suppose that the quiver has relations Λ defining the algebra $A = \mathbb{C}Q/\Lambda$.

v) A *representation of the quiver with relations*, (Q, Λ), is a representation of Q such that the linear maps assigned to the arrows satisfy the relations among the paths in the quiver. We recall that a representation of a quiver with relations corresponds to a left $\mathbb{C}Q/\Lambda$-module.

vi) The *representation scheme* $\mathrm{Rep}_d(Q,\Lambda)$ is the closed subscheme of the affine variety $\mathrm{Rep}_d(Q)$ cut out by the ideal corresponding to the relations Λ. Closed points of $\mathrm{Rep}_d(Q,\Lambda)$ correspond to dimension d representations of (Q,Λ).

We can now define the action of a reductive group on the affine scheme $\mathrm{Rep}_d(Q,\Lambda)$. For $\{\phi_a : a \in Q_1\}$, a dimension d representation, there is an action of $\mathrm{GL}_{d(i)}(\mathbb{C})$ at vertex i by base change:

$$g.\phi_a = \begin{cases} g \circ \phi_a & \text{if } t(a) = i; \\ \phi_a \circ g^{-1} & \text{if } h(a) = i; \\ 0 & \text{otherwise.} \end{cases}$$

Then $G := \mathrm{GL}_d(\mathbb{C}) := \prod_{i \in Q_0} \mathrm{GL}_{d(i)}(\mathbb{C})$ acts on $\mathrm{Rep}_d(Q,\Lambda)$ with kernel $\mathbb{C}^* = \Delta$ and orbits of G correspond to isomorphism classes of representations.

Definition 2.11. The *affine quotient* with dimension vector d is defined to be

$$\mathrm{Rep}_d(Q,\Lambda)/\!\!/G := \mathrm{Spec}(\mathbb{C}[\mathrm{Rep}_d(Q,\Lambda)]^G).$$

In order to consider more general GIT quotients we now recall the definition of stability conditions.

Definitions 2.12.

i) A *stability condition* is defined to be an element $\theta \in \mathbb{Z}^{Q_0}$ assigning an integer to each vertex of Q. For a finite dimensional representation M let d_M be the dimension vector of M, and define $\theta(M) = \sum_{i \in Q_0} \theta(i) d_M(i)$.

ii) A finite dimensional representation M is *θ-semistable* if $\theta(M) = 0$ and any subrepresentation $N \subset M$ satisfies $\theta(N) \geq 0$.

iii) A θ-semistable representation M is *θ-stable* if the only subrepresentations $N \subset M$ with $\theta(N) = 0$ are M and 0. A stability θ is *generic* if all θ-semistable representations are stable.

iv) For a stability condition θ define $\mathrm{Rep}_d(Q,\Lambda)^s_\theta$ to be the set of θ-stable representations, and $\mathrm{Rep}_d(Q,\Lambda)^{ss}_\theta$ to be the set of θ-semistable representations.

Definition 2.13. Every finite dimensional θ-semistable representation M has a Jordan-Holder filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each M_i is θ-semistable and each quotient is θ-stable. Two θ-semistable representations are defined to be *S-equivalent* if their Jordan-Holder filtrations have matching composition factors.

We note that θ-stable objects have length one filtrations hence are S-equivalent if and only if they are isomorphic.

Any character of G is given by powers of the determinant character and is of the form

$$\chi_\theta(g) := \prod_{i \in Q_0} \det(g_i)^{\theta_i}$$

for some collection of integers θ_i. We will restrict our attention to characters which are trivial on the kernel Δ of the action on $\mathrm{Rep}_d(Q,\Lambda)$, which translates to the condition $\sum \theta(i)d(i) = 0$. Hence these characters are in correspondence with stabilities on $\mathrm{Rep}_d(Q,\Lambda)$.

Note that $\mathrm{Rep}_d(Q,\Lambda)$ is affine, and recall that $f \in \mathbb{C}[\mathrm{Rep}_d(Q,\Lambda)]$ is a *semi-invariant of weight* χ if $f(g.x) = \chi(g)f(x)$ for all $g \in G$ and all $x \in \mathrm{Rep}_d(Q,\Lambda)$. We denote the set of such f as $\mathbb{C}[\mathrm{Rep}_d(Q,\Lambda)]^{G,\chi}$.

Definition 2.14 ([Kin94])**.** The *quiver GIT quotient*, for dimension vector d and stability condition θ, is defined to be the scheme

$$\mathcal{M}^{ss}_{d,\theta} := \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[\mathrm{Rep}_d(Q,\Lambda)]^{G,\chi_\theta^n}\right).$$

It is immediate from this definition that for any stability condition θ the quiver GIT quotient $\mathcal{M}^{ss}_{d,\theta}$ is projective over the affine quotient $\mathcal{M}^{ss}_{d,0} = \mathrm{Spec}(\mathbb{C}[\mathrm{Rep}_d(Q,\Lambda)]^G)$.

2.3 Derived categories and homological algebra

This section recalls properties of the derived category, tilting bundles, N-Koszul algebras, and Calabi-Yau algebras.

2.3.1 Derived categories and tilting

Consider a triangulated $\mathbb{C}$-linear category $\mathfrak{C}$ with small direct sums. A subcategory is *localising* if it is triangulated and also closed under all small direct sums. A localising subcategory is necessarily closed under direct summands [Nee01, Proposition 1.6.8]. An object $T \in \mathfrak{C}$ *generates* if the smallest localising category containing T is $\mathfrak{C}$.

Definitions 2.15. Let $\mathfrak{C}$ be a triangulated category closed under small direct sums. An object T in $\mathfrak{C}$ is *tilting* if:

i) $\mathrm{Ext}^k_{\mathfrak{C}}(T,T) = 0$ for $k \neq 0$.

ii) T generates $\mathfrak{C}$.

iii) The functor $\mathrm{Hom}_{\mathfrak{C}}(T,-)$ commutes with small direct sums.

For X a quasi-projective scheme, let $D(X)$ denote the derived category of quasicoherent sheaves on X and $D^b(X)$ denote the bounded derived category of coherent sheaves. Then $D(X)$ is closed under small direct sums [Nee96, Example 1.3] and $D(X)$ is compactly generated with compact objects the perfect complexes [Nee96, Proposition 2.5]. We let $\mathrm{Perf}(X)$ denote the category of perfect complexes on X. When X is smooth the category of perfect complexes coincides with $D^b(X)$.

For an algebra A, we let $D(A)$ denote the derived category of left modules over A and $D^b(A)$ the bounded derived category of finitely generated left A-modules. When $D(X)$ has tilting object a sheaf, T, then $A := \mathrm{End}_X(T)^{\mathrm{op}}$. When T is a locally free coherent sheaf on X then T is a *tilting bundle*, and this gives a derived equivalence between $D(X)$ and $D(A)$.

Theorem 2.16 ([HVdB07, Theorem 7.6], [BH13, Remark 1.9])**.** *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$ with tilting bundle T on X, and define $A = \mathrm{End}_X(T)^{\mathrm{op}}$. Then:*

i) The functor $T_* := \mathbb{R}\mathrm{Hom}_X(T, -)$ is an equivalence between $D(X)$ and $D(A)$. An inverse equivalence is given by the left adjoint $T^* = T \otimes^{\mathbb{L}}_A (-)$.

ii) The functors T_*, T^* remain equivalences when restricted to the bounded derived categories of finitely generated modules and coherent sheaves.

iii) If X is smooth then A has finite global dimension.

Moreover the equivalence T_* is R-linear and A is a finite R-algebra.

2.3.2 Quivers and tilting bundles

We recall the construction of a quiver with relations from a tilting bundle.

Let $X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Given a tilting bundle T' on X and a decomposition into indecomposable summands $T' = \bigoplus_{i=0}^n E_i^{\oplus \alpha_i}$, with E_i and E_j non-isomorphic for $i \neq j$, then $T = \bigoplus_{i=0}^n E_i$ is also a tilting bundle on X and $\mathrm{End}_X(T')^{\mathrm{op}}$ is Morita equivalent to $\mathrm{End}_X(T)^{\mathrm{op}}$. Hence we will always assume, without loss of generality, that our tilting bundles have a given multiplicity free decomposition into indecomposables, $T = \bigoplus_{i=0}^n E_i$.

Recall from Theorem 2.16 that $A = \mathrm{End}_X(T)^{\mathrm{op}}$ is a finite R-algebra for R a finite type commutative $\mathbb{C}$-algebra, and we wish to present A as the path algebra of a quiver with relations such that each indecomposable E_i corresponds to the unique idempotent $e_i = id_{E_i} \in \mathrm{Hom}_X(E_i, E_i) \subset A = \mathrm{End}_X(T)^{\mathrm{op}}$ that is the trivial path at vertex i. In particular $1 = \sum e_i$ and we have a diagonal inclusion $\bigoplus_{i=0}^n e_i R \subset A$.

Indeed, we can construct a quiver by creating a vertex i corresponding to each idempotent e_i. We then choose a finite set of generators of $e_i A e_j$ as an R-module, which is possible as A is finite R-module, and create corresponding arrows from vertex j to i for all $0 \leq i, j \leq n$. We then consider a presentation of R over $\mathbb{C}$ with finitely many generators, possible as it has finite type, and add arrows corresponding to each generator of R at each vertex. If we call this quiver Q, then by this construction there is a surjection of R-algebras $\mathbb{C}Q \to A$ given by mapping each trivial path to the corresponding idempotent and each arrow to the corresponding generator. We then take the kernel of this map, I, and $\mathbb{C}Q/I \cong A$ as an R-algebra.

We note that this presentation has many unpleasant properties, for example it may be the case that the ideal of relations I is not a subset of the paths of length greater than 1. In nice situations it is possible to simplify the presentation, see for example the situation considered in [BP08, Section 1].

We also note that there is a decomposition into projective modules $A = \bigoplus_{i=0}^n \mathrm{Hom}_X(T, E_i)$, where the module $\mathrm{Hom}_X(T, E_i)$ corresponds to paths in the quiver starting at vertex i.

2.3.3 Calabi-Yau algebras

This section recalls the definition of Calabi-Yau algebras following [Gin] and [AIR], the definition of self dual used in [BSW10], and notes that the existence of a self dual finite projective A^e-module resolution of length n implies an algebra is n-Calabi-Yau.

Let $D(A^e)$ denote the unbounded derived category of left A^e modules. There are two A^e-module structures on $A \otimes_{\mathbb{C}} A$: the *inner structure* given by $(a \otimes b)(x \otimes y) = (xb \otimes ay)$ and the *outer structure* given by $(a \otimes b)(x \otimes y) = (ax \otimes yb)$. For any A^e-module M, by considering A^e with the outer structure $\mathrm{Hom}_{A^e}(M, A^e)$ can be made an A^e-module using the inner structure.

Definition 2.17. Let $n \geq 2$. Then A is *n bimodule Calabi-Yau (n-CY)* if A has a finite length resolution by finitely generated projective A^e-modules and

$$\mathbb{R}\mathrm{Hom}_{A^e}(A, A^e)[n] \cong A \quad \text{ in } D(A^e).$$

In the paper [BSW10] self-duality of a complex is used to define the Calabi-Yau condition, and we recall the definition here. Denote $(-)^\vee := \mathrm{Hom}_{A^e}(-, A^e) : A^e\text{-Mod} \to A^e\text{-Mod}$.

Definition 2.18. A complex of A^e-modules, $\mathcal{C}^\bullet$, of length n to defined to be self dual, written

$$\mathrm{Hom}_{A^e}(\mathcal{C}^\bullet, A^e) \cong \mathcal{C}^{n-\bullet},$$

if there exist A^e-module isomorphisms α_i such that the following diagram commutes:

$$\begin{tikzcd}
C_n \arrow[r, "d_n"] \arrow[d, "\alpha_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\alpha_{n-1}"] & \cdots \arrow[r, "d_2"] & C_1 \arrow[r, "d_1"] \arrow[d, "\alpha_1"] & C_0 \arrow[d, "\alpha_0"] \\
C_0^\vee \arrow[r, "-d_1^\vee"] & C_1^\vee \arrow[r, "-d_2^\vee"] & \cdots \arrow[r, "-d_{n-1}^\vee"] & C_{n-1}^\vee \arrow[r, "-d_n^\vee"] & C_n^\vee
\end{tikzcd}$$

By construction the existence of a length n self dual projective A^e-module resolution of A implies that A is n-CY, and we will use this later in Section 3.3 to show algebras are n-CY.

We recall some properties of Calabi-Yau algebras.

Lemma 2.19. *Let A be n-CY $\mathbb{C}$-algebra. Then:*

i) If there exists a non-zero finite dimensional A-module, then A has global dimension n.

ii) For $X, Y \in D(A)$ with finite dimensional total homology

$$\mathrm{Hom}_{D(A)}(X, Y) \cong \mathrm{Hom}_{D(A)}(Y, X[n])^*$$

where $*$ denotes the $\mathbb{C}$-dual.

Proof. These are some of the standard properties of CY algebras, see for example [AIR, Proposition 2.4] and [BT07, Section 2] for proofs. □

2.3.4 Koszul algebras

The concept of an N-Koszul algebra was introduced by Berger and is defined and studied in the papers [Ber01], [BDVW03], [BG06], [BM06], and [GMMVZ04]. It generalises the concept of a Koszul algebra, which is defined here to be a 2-Koszul algebra.

Definition 2.20. Let S be a semisimple ring, V a left S^e-module, and $T_S(V)$ the tensor algebra. An algebra is *N-homogeneous* if it is given in the form $A = \frac{T_S(V)}{I(\Lambda)}$ for Λ an S^e-submodule of $V^{\otimes_S N}$.

We now define the *Koszul N-complex*. For N-homogeneous $A = \frac{T_S(V)}{I(\Lambda)}$ define

$$\begin{aligned}
K_i &= \cap_{j+N+k=i} \left(V^{\otimes j} \otimes_S \Lambda \otimes_S V^{\otimes k}\right) \qquad i \geq N \\
K_i &= V^{\otimes_S i} \qquad 0 < i < N \\
K_0 &= S
\end{aligned}$$

and let

$$K^i(A) = A \otimes_S K_i \otimes_S A.$$

Now the N-complex (in an N-complex $d^N = 0$ rather than $d^2 = 0$)

$$\ldots \xrightarrow{d} K^n(A) \xrightarrow{d} \ldots \xrightarrow{d} K^0(A) \xrightarrow{\mu} A \to 0.$$

can be defined. To define the differentials let μ be multiplication and $d_l, d_r : A \otimes_S V^{\otimes_S i} \otimes_S A \to A \otimes_S V^{\otimes_S (i-1)} \otimes_S A$ be defined by

$$d_l(\alpha \otimes v_1 v_2 \ldots v_i \otimes \beta) = \alpha v_1 \otimes v_2 \ldots v_i \otimes \beta \qquad \text{and}$$
$$d_r(\alpha \otimes v_1 \ldots v_{i-1} v_i \otimes \beta) = \alpha \otimes v_1 \ldots v_{i-1} \otimes v_i \beta.$$

As $\Lambda \subset V^{\otimes_S N}$ we see $K_i \subset V^{\otimes_S i}$ and hence can consider the restriction of d_l, d_r to $K^i(A)$. Now choose $q \in \mathbb{C}$ a primitive N^{th} root of unity and define $d : K^i(A) \to K^{i-1}(A)$ as $d = d_l|_{K^i(A)} - (q)^{i-1} d_r|_{K^i(A)}$. Then d defines an N differential as d_l and d_r commute and $d_l^N = d_r^N = 0$.

A complex can be associated to such an N-complex by contracting several terms together. This is done by splitting the N-complex into sections of N consecutive differentials from the right, and each one of these is collapsed to a term with two differentials by keeping the right most differential and composing the other $N-1$. This defines a complex of the form

$$\cdots \to K^{N+1}(A) \xrightarrow{d} K^N \xrightarrow{d^{N-1}} K^0(A) \xrightarrow{\mu} A \to 0$$

with $d = (d_l - d_r)|_{K^j(A)}$ and $d^{N-1} = (d_l^{N-1} + d_l^{N-2} d_r + \cdots + d_r^{N-2})|_{K^j(A)}$.

A N-homogeneous algebra A is *N-Koszul* if this complex is exact, and hence gives an A^e-module resolution of A.

We call 2-Koszul algebras *Koszul*, and in this case the N-complex is in fact a complex.

2.4 Singularities, resolutions, and noncommutative resolutions

This section recalls several definitions relating to resolutions of singularities, both geometric and noncommutative.

2.4.1 Resolutions of singularities

Definitions 2.21. Let Y be a (possibly singular) variety. A smooth variety X with a projective birational map $\pi : X \to Y$ that is bijective over the smooth locus of Y is called a *resolution* of Y. A resolution, X, is a *minimal resolution* of Y if any other resolution factors through it. In general minimal resolutions do not exist, but they always exist for surfaces, [Lip69, Corollary 27.3]. A resolution, X, is a *crepant resolution* of Y if $\pi^* \omega_Y = \omega_X$, where ω_X and ω_Y are the canonical classes of X and Y, which we assume are normal. In general crepant resolutions do not exist. A singularity, Y, is *rational* if for any resolution $\pi : X \to Y$

$$\mathbb{R}\pi_* \mathcal{O}_X \cong \mathcal{O}_Y.$$

If this holds for one resolution it holds for all resolutions, [Vie77, Lemma 1].

In particular, we recall quotient singularities. Let $V = \mathbb{C}^n$ and $G < \mathrm{GL}(V)$ be a finite group. The group $G < \mathrm{GL}(V)$ is *small* if it contains no pseudo-reflections. There is a corresponding *quotient singularity* V/G which is affine with ring of functions

$$R := \mathbb{C}[V]^G$$

which has a natural grading given by $\mathbb{C}[V] = \mathbb{C}[x_1, \dots x_n]$ with each x_i in grade 1. Quotient singularities are known to be rational, [Vie77, Proposition 1].

2.4.2 Explicit resolutions of cyclic surface quotient singularities

This section recalls that cyclic surface quotient singularities and their minimal resolutions can be explicitly calculated by simple combinatorics. This includes the cyclic examples A_2 and $A_{5,2}$ considered Sections 1.1 and 1.3.

Definition 2.22. For r and a coprime integers such that $0 < a < r$, define the cyclic group

$$\frac{1}{r}(1,a) := \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix} \right\rangle < \mathrm{GL}_2(\mathbb{C})$$

where ε is a primitive r^{th} root of unity. Then the associated *cyclic quotient surface singularity* is the affine variety $\mathrm{Spec}(\mathbb{C}[x,y]^{\frac{1}{r}(1,a)})$. Whenever we use the notation $\frac{1}{r}(1,a)$ we assume that a and r are coprime integers such that $0 < a < r$.

Hirzebruch-Jung continued fractions provide the combinatorial data needed to describe both the cyclic surface quotient singularities and their minimal resolutions.

Definition 2.23. For a pair of coprime integers a and r such that $0 < a < r$, the *Hirzebruch-Jung continued fraction* of $\frac{r}{a}$ is defined to be the expression

$$\frac{r}{a} = \alpha_1 - \cfrac{1}{\alpha_2 - \cfrac{1}{\cdots - \cfrac{1}{\alpha_k}}} = [\alpha_1, \dots, \alpha_k]$$

where the α_i are positive integers which are ≥ 2.

Remark 2.24. The Hirzebruch-Jung continued fraction of $\frac{r}{a}$ can be calculated by the recurrence relation

$$\alpha_{i+1} = \lceil x_i \rceil, \qquad x_{i+1} = \frac{1}{\alpha_{i+1} - x_i},$$

where we set $x_0 = r/a$ and halt when $\alpha_{i+1} = x_i$.

Examples 2.25. The fraction $\frac{3}{2}$ has Hirzebruch-Jung continued fraction $[2,2]$ as $\frac{3}{2} = 2 - \frac{1}{2}$. Similarly: $\frac{3}{1} = [3]$, $\frac{5}{2} = [3,2]$, and $\frac{5}{3} = [2,3]$.

The cyclic quotient surface singularity $\mathrm{Spec}(\mathbb{C}[x,y]^{\frac{1}{r}(1,a)})$ and its minimal resolution can be

explicitly defined via the combinatorics of the Hirzebruch-Jung continued fractions

$$\frac{r}{a} = [\alpha_1, \ldots, \alpha_k] \qquad \text{and} \qquad \frac{r}{r-a} = [\beta_1, \ldots, \beta_l]\,.$$

The fraction $[\beta_1, \ldots, \beta_l]$ is used to define the singularity and the fraction $[\alpha_1, \ldots, \alpha_k]$ is used to define its minimal resolution.

In particular, the affine variety $\mathrm{Spec}(\mathbb{C}[x,y]^{\frac{1}{r}(1,a)})$ can be explicitly described by giving generators and relations for the ring $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$.

Theorem 2.26 ([Rie74, Section 1])**.** *The ring $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$ has a generating set of invariant polynomials $Z_i = x^{\gamma_i} y^{\delta_i}$ for $0 \leq i \leq l+1$ where γ_i and δ_i are defined by the recurrence relations*

$$\begin{array}{lll} \gamma_0 = r & \gamma_1 = r - a & \gamma_i = \beta_{i-1}\gamma_{i-1} - \gamma_{i-2} \quad \textit{for } 2 \leq i \leq l+1 \\ \delta_0 = 0 & \delta_1 = a & \delta_i = \beta_{i-1}\delta_{i-1} - \delta_{i-2} \quad \textit{for } 2 \leq i \leq l+1 \end{array}$$

where the β_i are defined by the Hirzebruch-Jung continued fraction $r/(r-a) = [\beta_1, \ldots, \beta_l]$.

Examples 2.27.

i) The ring $\mathbb{C}[x,y]^{\frac{1}{3}(1,2)}$ has generators

$$Z_0 = x^3, Z_1 = xy, \text{ and } Z_2 = y^3.$$

ii) The ring $\mathbb{C}[x,y]^{\frac{1}{5}(1,2)}$ has generators

$$Z_0 = x^5, Z_1 = x^3 y, Z_2 = xy^2, \text{ and } Z_3 = y^5.$$

Theorem 2.28 ([Rie74, Section 2])**.** *The ring $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$ is isomorphic to the quotient of $\mathbb{C}[Z_0, \ldots, Z_{l+1}]$ by the ideal generated by the relations*

$$Z_i Z_j = Z_{i+1} \left(\prod_{k=i+1}^{j-1} Z_k^{\beta_k - 2} \right) Z_{j-1}$$

for $1 \leq i+1 < j \leq l+1$, where the β_k are defined by the Hirzebruch-Jung continued fraction $r/(r-a) = [\beta_1, \ldots, \beta_l]$.

Examples 2.29.

i) The ring $\mathbb{C}[x,y]^{\frac{1}{3}(1,1)}$ is isomorphic to $\mathbb{C}[Z_0, \ldots, Z_2]/J$ where J is generated by the element

$$Z_0 Z_2 - Z_1^3.$$

ii) The ring $\mathbb{C}[x,y]^{\frac{1}{5}(1,2)}$ is isomorphic to $\mathbb{C}[Z_0, \ldots, Z_3]/J$ where J is generated by the elements

$$Z_0 Z_2 - Z_1^2, Z_1 Z_3 - Z_2^3, \text{ and } Z_0 Z_3 - Z_1 Z_2^2.$$

In the following theorem the minimal resolution of a cyclic quotient surface singularity is described by explicitly giving affine charts and transition maps.

Theorem 2.30 ([Rie74, Section 3]). *The minimal resolution of the cyclic quotient singularity* $\mathrm{Spec}(\mathbb{C}[x,y]^{\frac{1}{r}(1,a)})$ *is given by* $k+1$ *affine charts,* $U_j = \mathrm{Spec}(\mathbb{C}[X_j, Y_j])$*, for* $0 \leq j \leq k$ *and transition maps*

$$X_j = Y_{j-1}^{-1} \qquad X_{j-1} = X_i^{\alpha_j} Y_j$$
$$Y_j = Y_{j-1}^{\alpha_j} X_{j-1} \qquad Y_{j-1} = X_j^{-1}$$

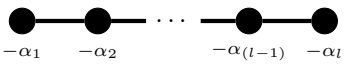
where the α_i *are defined by the Hirzebruch-Jung continued fraction* $r/a = [\alpha_1, \ldots, \alpha_l]$*. In birational coordinates* $X_0 = x^n, Y_0 = yx^{-a}$*, and all other* X_i, Y_i *can be calculated by the transition maps. The map* $\pi : X \to \mathrm{Spec}(\mathbb{C}[x,y]^{\frac{1}{r}(1,a)})$ *is defined by the morphisms for each affine chart*

$$\mathbb{C}[x,y]^{\frac{1}{r}(1,a)} \cong \mathbb{C}[Z_1, \ldots, Z_l]/\Lambda \to \mathbb{C}[U_j]$$
$$Z_i \mapsto X_j^{b_i^j} Y_j^{c_i^j}$$

where b_i^j *and* c_i^j *are defined by the recurrence relations*

$$\begin{array}{llll} b_0^0 = 1 & b_1^0 = 1 & b_i^j = \beta_{i-1} b_{i-1}^j - b_{i-2}^j & \text{for } 2 \leq i \leq l+1, \\ & & b_i^j = \alpha_j b_i^{j-1} - c_i^{j-1} & \text{for } 1 \leq j \leq k, \\ c_0^0 = 0 & c_1^0 = 1 & c_i^j = \beta_{i-1} c_{i-1}^j - c_{i-2}^j & \text{for } 2 \leq i \leq l+1, \\ & & c_i^j = b_i^{j-1} & \text{for } 1 \leq j \leq k. \end{array}$$

and the β_i *are defined by the Hirzebruch continued fraction* $r/(r-a) = [\beta_1, \ldots, \beta_l]$*. In addition, the continued fraction* $r/a = [\alpha_1, \ldots, \alpha_l]$ *describes the self intersection numbers of the components of the exceptional divisor, which has the following dual graph.*



Examples 2.31. This example illustrates Theorem 2.30 and calculates the minimal resolutions of the cyclic quotient surface singularities appearing in Section 1. We give birational coordinates in $\mathbb{C}(x,y)^{\frac{1}{r}(1,a)}$ for the generators of each $\mathbb{C}[U_j]$, from which the transition maps between the charts and the map $X \to \mathrm{Spec}(\mathbb{C}[x,y]^{\frac{1}{r}(1,a)})$ can be calculated.

i) The minimal resolution X of $\mathrm{Spec}(\mathbb{C}[x,y]^{\frac{1}{3}(1,2)})$ is defined by the following charts and birational coordinates.

$$\begin{array}{lll} U_0 = \mathrm{Spec}(\mathbb{C}[X_0, Y_0]) & X_0 = x^3 & Y_0 = yx^{-2} \\ U_1 = \mathrm{Spec}(\mathbb{C}[X_1, Y_1]) & X_1 = x^2 y^{-1} & Y_1 = y^2 x^{-1} \\ U_2 = \mathrm{Spec}(\mathbb{C}[X_2, Y_2]) & X_2 = xy^{-2} & Y_2 = y^3 \end{array}$$

In the following table we exhibit the map $\pi : X \to \mathrm{Spec}(\mathbb{C}[x,y]^{\frac{1}{3}(1,2)})$ by giving the image of the generators of $\mathbb{C}[x,y]^{\frac{1}{3}(1,2)}$ in the coordinate ring of each affine chart, $\mathbb{C}[U_j]$.

$\mathbb{C}[x,y]^{\frac{1}{3}(1,2)}$	$\mathbb{C}[X_0,Y_0]$	$\mathbb{C}[X_1,Y_1]$	$\mathbb{C}[X_2,Y_2]$
$Z_0 = x^3$	X_0	$X_1^2Y_1$	$X_2^3Y_2^2$
$Z_1 = xy$	X_0Y_0	X_1Y_1	X_2Y_2
$Z_2 = y^3$	$X_0^2Y_0^3$	$X_1Y_1^2$	Y_2

ii) The minimal resolution, X, of $\mathrm{Spec}(\mathbb{C}[x,y]^{\frac{1}{5}(1,2)})$ is defined by the following charts and birational coordinates.

$$\begin{aligned} U_0 &= \mathrm{Spec}(\mathbb{C}[X_0,Y_0]) & X_0 &= x^5 & Y_0 &= yx^{-2} \\ U_1 &= \mathrm{Spec}(\mathbb{C}[X_1,Y_1]) & X_1 &= x^2y^{-1} & Y_1 &= y^3x^{-1} \\ U_2 &= \mathrm{Spec}(\mathbb{C}[X_2,Y_2]) & X_2 &= xy^{-3} & Y_2 &= y^5 \end{aligned}$$

In the following table we exhibit the map $\pi : X \to \mathrm{Spec}(\mathbb{C}[x,y]^{\frac{1}{5}(1,2)})$ by giving the image of the generators of $\mathbb{C}[x,y]^{\frac{1}{5}(1,2)}$ in the coordinate ring of each affine chart, $\mathbb{C}[U_j]$.

$\mathbb{C}[x,y]^{\frac{1}{5}(1,2)}$	$\mathbb{C}[X_0,Y_0]$	$\mathbb{C}[X_1,Y_1]$	$\mathbb{C}[X_2,Y_2]$
$Z_0 = x^5$	X_0	$X_1^3Y_1$	$X_2^5Y_2^3$
$Z_1 = x^3y$	X_0Y_0	$X_1^2Y_1$	$X_2^3Y_2^2$
$Z_2 = xy^2$	$X_0Y_0^2$	X_1Y_1	X_2Y_2
$Z_3 = y^5$	$X_0^2Y_0^5$	$X_1Y_1^2$	Y_2

2.4.3 Noncommutative crepant resolutions

This section briefly recalls the definition, due to Van den Bergh [VdB04a], of a noncommutative crepant resolution (NCCR) of an algebra.

Suppose that R is a commutative, Noetherian $\mathbb{C}$-algebra, and A a module-finite R-algebra. Then A is *homologically homogeneous* if $\mathrm{gldim} A_P = \dim R_P$ and A_P is maximal Cohen-Macaulay for all prime ideals $P \in \mathrm{Spec}(R)$.

Definition 2.32. A *noncommutative crepant resolution* of a normal Gorenstein domain R is an homologically homogeneous R-algebra A of the form $A = \mathrm{End}_R(M)$ for M a reflexive R-module.

A major example are skew group algebras $\mathbb{C}[x_1,\dots,x_n] \rtimes G$ with $G < \mathrm{SL}_n(\mathbb{C})$ and $R = \mathbb{C}[x_{,1},\dots,x_n]^G$, [VdB04a, Example 1.1.].

Skew group algebras can be constructed as superpotential algebras for the corresponding McKay quivers, and are examples of Koszul, Calabi-Yau algebras.

Theorem 2.33 ([BSW10, Theorem 3.2, Lemma 6.1])**.** *Let V be a n-dimensional $\mathbb{C}$-vector space and G be a finite subgroup of $\mathrm{GL}(V)$. Then $\mathbb{C}[V] \rtimes G$ is Morita equivalent to $\mathcal{D}(\Phi_n, n-2)$ for some homogeneous superpotential Φ_n of degree n attached to the McKay quiver of $G < \mathrm{GL}(V)$. Moreover, $\mathcal{D}(\Phi_n, n-2)$ is n-CY and Koszul for $G \leq \mathrm{SL}(V)$.*

There is a recipe to compute a superpotential Φ_n such that $\mathbb{C}[V] \rtimes G$ is Morita equivalent to the superpotential algebra $\mathcal{D}(\Phi_n, n-2)$ attached to the McKay quiver of $G < \mathrm{GL}_n(\mathbb{C})$. The recipe is given in [BSW10, Theorem 3.2]. When G is abelian the superpotential algebra $\mathcal{D}(\Phi_n, n-2)$ is in fact isomorphic to $\mathbb{C}[V] \rtimes G$.

Chapter 3

PBW deformations, Superpotentials, and CY algebras

This chapter contains a largely unchanged exposition of the main results of the author's paper [Kar14b].

3.1 Introduction to Chapter 3

In this chapter we consider path algebras of quivers with relations produced from higher order derivations of a superpotential, continuing from the discussion of Section 1.2. We first recall several results from the paper [BSW10] of Bocklandt, Schedler, and Wemyss. In particular, in [BSW10] a complex $\mathcal{W}^\bullet$ is defined only depending on the superpotential, and a path algebra with relations is shown to be N-Koszul and CY if and only if it is of the form $\mathcal{D}(\Phi_n, k)$ for a superpotential Φ_n with $\mathcal{W}^\bullet$ defining a projective resolution.

Skew group algebras, $\mathbb{C}[V] \rtimes G$, for G a finite subgroup of $\mathrm{SL}(V)$, are Morita equivalent to path algebras of this form, so are a motivating case. These are 2-Koszul and CY, and hence their relations can be given by a superpotential, as noted in Theorem 2.33 above, and they appear in the $\mathrm{SL}_2(\mathbb{C})$ McKay correspondence as noncommutative crepant resolutions.

In this chapter we prove two results concerning the PBW deformations of $(n-k)$-Koszul, $(k+2)$-CY algebras of the form $\mathcal{D}(\Phi_n, k)$. First we prove Theorem 3.11, which classifies which PBW deformations of an algebra $\mathcal{D}(\Phi_n, k)$ defined by a homogeneous superpotential are of the form $\mathcal{D}(\Phi, k)$ for an inhomogeneous superpotential $\Phi = \Phi_n + \phi_{n-1} + \phi_{n-2}$. This was previously known for the case of PBW deformations of $\mathcal{D}(\Phi_n, 1)$, [BT07, Theorem 3.1, 3.2], but we extend this to include higher differentials, $k > 1$.

Second, in Theorem 3.13, we prove that certain classes of these PBW deformations of a 2-Koszul, n-CY $\mathcal{D}(\Phi_n, n-2)$ are n-CY. This is already known in the 3-CY case due to [BT07] which proves that any PBW deformation of an N-Koszul, 3-CY $\mathcal{D}(\Phi_n, 1)$ given by inhomogeneous superpotential is 3-CY,[BT07, Theorem 3.6]. In the case of a one vertex quiver there is a also result, [WZ, Theorem 3.1], which finds a necessary and sufficient condition for a PBW deformation of a Noetherian, 2-Koszul, n-CY algebra to be n-CY. Our results extend and unify both.

We give an application of these results to symplectic reflection algebras. Symplectic re-

flection algebras are defined in [EG02] as PBW deformations of certain skew group algebras $\mathbb{C}[V] \rtimes G$, and hence we consider the Morita equivalent path algebra with relations. Applying the previous results, and a result of [EG02], we deduce that these path algebras with relations are of the form $\mathcal{D}(\Phi_{2n} + \phi_{2n-2}, 2n-2)$ and are $2n$-CY.

We go on to consider PBW deformations of the path algebras with relations Morita equivalent to $\mathbb{C}[x,y] \rtimes G$ when G is a finite subgroup of $\mathrm{GL}_2(\mathbb{C})$ not contained is $\mathrm{SL}_2(\mathbb{C})$. We show that if G is small there are no PBW deformations.

3.1.1 Summary of main results in Chapter 3

1. A classification of the PBW deformations of a $(k+2)$-CY, $(n-k)$-Koszul, superpotential algebra $\mathcal{D}(\Phi_n, k)$ whose relations are given by inhomogeneous superpotentials. These are proved to be the PBW deformations satisfying one additional property, which we call the zeroPBW condition, Definition 3.9, and the corresponding superpotentials are shown to be k-coherent superpotentials, Definition 3.8.

   Theorem. *(Theorem 3.11) Let $A = \mathcal{D}(\Phi_n, k)$, for Φ_n a homogeneous superpotential of degree n, be $(n-k)$-Koszul and $(k+2)$-CY. Then the zeroPBW deformations of A correspond exactly to the algebras $\mathcal{D}(\Phi', k)$ defined by k-coherent inhomogeneous superpotentials of the form $\Phi' = \Phi_n + \phi_{n-1} + \cdots + \phi_k$.*

2. A proof that these PBW deformations are CY in certain cases. We consider $A = \mathcal{D}(\Phi_n, n-2)$ which is n-CY and 2-Koszul, and a zeroPBW deformation, $\mathcal{A}$, which by the previous results is of the form $\mathcal{A} := \mathcal{D}(\Phi', n-2)$ for $\Phi' = \Phi_n + \phi_{n-1} + \phi_{n-2}$.

   Theorem. *(Theorem 3.13) Suppose $\phi_{n-1} = 0$, then $\mathcal{A}$ is n-CY.*

3. An application to symplectic reflection algebras. Let H be a path algebra with relations Morita equivalent to an undeformed symplectic reflection algebra. So H is 2-Koszul and $2n$-CY and $H = \mathcal{D}(\Phi_{2n}, 2n-2)$ for some homogeneous degree $2n$ superpotential Φ_{2n}.

   Theorem. *(Theorem 3.17) Any PBW deformation of H is a zeroPBW deformation and of the form $\mathcal{D}(\Phi', n-2)$ for $\Phi' = \Phi_{2n} + \phi_{2n-2}$ an inhomogeneous superpotential. Hence any PBW deformation of H is $2n$-CY, and all symplectic reflection algebras are Morita equivalent to $2n$-CY algebras of the form $\mathcal{D}(\Phi', 2n-2)$.*

3.2 Preliminary material on Superpotentials and PBW deformations

This section recalls results from the paper [BSW10] that will be required in later sections, and also recalls the definition of PBW deformations.

3.2.1 Superpotentials and Higher Order Derivations

Let Q be a quiver with path algebra $\mathbb{C}Q$, let Φ_n be a homogeneous superpotential of degree n, and define $A = \mathcal{D}(\Phi_n, n-2)$. For $i = 0, \ldots n$ we have S^e-modules $\mathcal{W}_i$ as in Definition 2.6 above, and we can define a complex $\mathcal{W}^\bullet$ by

$$0 \to A \otimes_S \mathcal{W}_n \otimes_S A \xrightarrow{d_n} A \otimes_S \mathcal{W}_{n-1} \otimes_S A \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_1} A \otimes_S \mathcal{W}_0 \otimes_S A \to 0. \tag{3.1}$$

To define the differential first define $d_i^l, d_i^r : A \otimes_S \mathcal{W}_i \otimes_S A \to A \otimes_S \mathcal{W}_{i-1} \otimes_S A$ by

$$\begin{aligned} d_i^l(\alpha \otimes \delta_p \Phi_n \otimes \beta) &= \sum_{a \in Q_1} \alpha a \otimes \delta_a \delta_p \Phi_n \otimes \beta \\ d_i^r(\alpha \otimes \delta_p \Phi_n \otimes \beta) &= \sum_{a \in Q_1} \alpha \otimes \delta_p \Phi_n \delta'_a \otimes a\beta \end{aligned}$$

for p any path of length $n-i$. The differential d_i is defined as

$$d_i = \epsilon_i(d_i^l + (-1)^i d_i^r) \text{ where } \epsilon_i := \begin{cases} (-1)^{i(n-i)} & \text{if } i < (n+1)/2, \\ 1 & \text{otherwise.} \end{cases}$$

This defines a complex as d^l, d^r commute and have square 0. We show this for d^l, the other case being similar. Writing $\Phi_n = \sum_{|t|=n} c_t t$, we have

$$\begin{aligned} d_{j-1}^l \circ d_j^l(1 \otimes \delta_p \Phi_n \otimes 1) &= \sum_{a,b} ab \otimes \delta_{ab} \delta_p \Phi_n \otimes 1 \\ &= \sum_{a,b,q} ab \otimes c_{pabq} q \otimes 1 \\ &= \sum_{a,b,q} (-1)^{(n-1)(j-2)} c_{qpab} ab \otimes q \otimes 1 \\ &= \sum_{q} (-1)^{(n-1)j} \delta_{qp} \Phi_n \otimes q \otimes 1 = 0 \end{aligned}$$

where $a, b \in Q_1$, p is a path of length $(n-j)$, q a path of length $j-2$, and the sums are taken over all such a, b and q.

Theorem 3.1 ([BSW10, Section 6])**.** *Let $A = \mathcal{D}(\Phi_n, n-2)$, then $\mathcal{W}^\bullet$ is a self dual complex of projective A^e-modules.*

Theorem 3.2 ([BSW10, Theorem 6.2])**.** *Let $A = \frac{\mathbb{C}Q}{\Lambda}$ be a path algebra with relations. Then A is 2-Koszul and n-CY if and only if A is of the form $\mathcal{D}(\Phi_n, n-2)$ for some homogeneous superpotential Φ_n of degree n and the attached complex $\mathcal{W}^\bullet$ is a resolution of A. In this case the resolution equals the Koszul complex of Definition 2.20 with each $\mathcal{W}_i = K_i$.*

The complex (3.1) is the relevant complex for $\mathcal{D}(\Phi_n, n-2)$, where the relations are obtained by differentiation by paths of length $n-2$. More generally, differentiating by paths of length k, another complex is needed. Let $N = n-k$, and in this case we define an N-complex $\widetilde{\mathcal{W}}^\bullet$, again making use of the S^e-modules $\mathcal{W}_j$.

$$0 \to A \otimes_S \mathcal{W}_n \otimes_S A \xrightarrow{d_n} A \otimes_S \mathcal{W}_{n-1} \otimes_S A \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} A \otimes_S \mathcal{W}_0 \otimes_S A \to 0 \tag{3.2}$$

with differential $d_i : A \otimes_S \mathcal{W}_i \otimes_S A \to A \otimes_S \mathcal{W}_{i-1} \otimes_S A$ defined by $d_i = d_i^l + (q)^i d_i^r$, for q a primitive N^{th} root of unity. This can be contracted into a 2-complex, $\widehat{\mathcal{W}}^\bullet$

$$\begin{aligned} 0 \to & A \otimes_S \mathcal{W}_{mN+1} \otimes_S A \xrightarrow{d} A \otimes_S \mathcal{W}_{mN} \otimes_S A \xrightarrow{d^{N-1}} \\ & A \otimes_S \mathcal{W}_{(m-1)N+1} \otimes_S A \xrightarrow{d} \dots \xrightarrow{d} A \otimes_S \mathcal{W}_0 \otimes_S A \to 0 \end{aligned}$$

by composing the differentials in $\widetilde{\mathcal{W}}^\bullet$, as in Definition 2.20, where m is the largest integer such that $m \leq n/N$.

Theorem 3.3 ([BSW10, Theorem 6.8])**.** *Let $A = \frac{\mathbb{C}Q}{\Lambda}$ be a path algebra with relations. Then A is $(n-k)$-Koszul and $(k+2)$-Calabi Yau if and only if it is of the form $\mathcal{D}(\Phi_n, k)$ where Φ_n is an homogeneous superpotential of degree n and $\widehat{\mathcal{W}}^\bullet$ is a resolution of A. In this case $\widehat{\mathcal{W}}^\bullet$ equals the Koszul $(n-k)$-complex as in Definition 2.20.*

3.2.2 PBW deformations

This section recalls the definition of the PBW deformations of a graded algebra A. We will be considering PBW deformations of A an N-Koszul algebra relative to S a semisimple algebra, and we will make use of the setup and results of [BG06].

Let S be a semisimple $\mathbb{C}$-algebra, V a left S^e-module, and $T_S(V)$ the tensor algebra of V over S. Consider the $(\mathbb{Z}_{\geq 0})$ grading on $T_S(V)$ with degree n part $T_S(V)_n = V^{\otimes_S n}$, and filtered parts $F^n = V^{\otimes_S n} \oplus \cdots \oplus V \oplus S$. For Λ an S^e-submodule of $V^{\otimes_S N}$ define $I(\Lambda) = \sum_{i,j\geq 0} V^{\otimes i}\Lambda V^{\otimes j}$ to be the two sided ideal in $T_S(V)$ generated by Λ, which is graded by $I(\Lambda)_n = I(\Lambda) \cap V^{\otimes_S n}$. Define $A := \frac{T_S(V)}{\Lambda} := \frac{T_S(V)}{I(\Lambda)}$, and as Λ is homogeneous this is a graded algebra with degree n part $A_n := \frac{V^{\otimes_S n}}{I(\Lambda)_n}$.

Now define the projection map $\pi_N : F^N \to V^{\otimes_S N}$, and let Π be an S^e-submodule of F^N such that $\pi(\Pi) = \Lambda$. Let $I(\Pi)$ be the 2 sided ideal in $T_S(V)$ generated by Π. This is not graded but is filtered by $I(\Pi)^n = I(\Pi) \cap F^n$, and hence $\mathcal{A} = \frac{T_S(V)}{I(\Pi)}$ is also a filtered algebra with $\mathcal{A}^n = \frac{F^n}{I(\Pi)^n}$.

We can construct the associated graded algebra $gr(\mathcal{A})$, which is graded with $gr(\mathcal{A})_n = \frac{\mathcal{A}^n}{\mathcal{A}^{n-1}}$. Identify $gr(\mathcal{A})_n$ with the S^e-module $\frac{F^n}{I(\Pi)^n + F^{n-1}}$ by noting $I(\Pi)^n \cap F^{n-1} = I(\Pi)^{n-1}$, and consider the maps $\phi_n : V^{\otimes n} \to gr(\mathcal{A})_n$ defined as the composition $V^{\otimes n} \hookrightarrow F^n \to \frac{F^n}{I(\Pi)^n + F^{n-1}}$. This allows us to define a surjective algebra morphism $\phi = \bigoplus_{n\geq 0} \phi_n : T_S(V) \to gr(\mathcal{A})$. Now $\phi_N(\Lambda) = 0$ as $\Pi + F^{N-1} = \Lambda \oplus F^{N-1}$, and hence this defines a surjective morphism of $\mathbb{Z}_{\geq 0}$-graded S^e-modules

$$p : A \to gr(\mathcal{A}).$$

Definition 3.4. The filtered algebra $\mathcal{A} = \frac{T_S(V)}{\Pi}$ is a *PBW deformation* of $A = \frac{T_S(V)}{\Lambda}$ if p is an isomorphism. The S^e-module Π is of *PBW type* if $\mathcal{A}$ is a PBW deformation of A.

The papers [BT07] and [BG06] prove a collection of conditions on Π equivalent to it being PBW type which we state here.

First note that for $\Pi \subset F^N$ to be of PBW type it must be the case that $\Pi \cap F^{N-1} = \{0\}$. Hence any PBW type Π can be given as $\Pi = \{r - \theta(r) : r \in \Lambda\}$ for an S^e-module map $\theta : \Lambda \to F^{N-1}$. Such a map can be written in homogeneous components as $\theta = \theta_{N-1} + \cdots + \theta_0$, with $\theta_j : \Lambda \to V^{\otimes_S j}$.

Consider the map $\psi(\theta_j)$ defined for each θ_j as

$$\psi(\theta_j) := id \otimes \theta_j - \theta_j \otimes id : (V \otimes_S \Lambda) \cap (\Lambda \otimes_S V) \to V^{\otimes_S (j+1)}$$

where $(V \otimes_S \Lambda) \cap (\Lambda \otimes_S V) \subset V^{\otimes_S (N+1)}$

Theorem 3.5 ([BG06, Section 3])**.** *The S^e-module Π is of PBW type if and only if the following conditions are satisfied:*

PBW1) $\Pi \cap F^{N-1} = \{0\}$,

PBW2) $\operatorname{Im} \psi(\theta_{N-1}) \subset \Lambda$,

PBW3) $\operatorname{Im}(\theta_j(\psi(\theta_{N-1})) + \psi(\theta_{j-1})) = \{0\}$ *for all* $1 \leq j \leq N-1$,

PBW4) $\operatorname{Im}\, \theta_0(\psi(\theta_{N-1})) = \{0\}$.

Example 3.6. Let $A = \mathcal{D}(\Phi_n, k)$ be a N-Koszul $(k+2)$-CY superpotential algebra, where $N = n - k$. Then Theorem 3.3 tells us that the Koszul N-complex equals the complex $\widehat{\mathcal{W}^\bullet}$ of Definition 2.20 and $\mathcal{W}_i = K_i$. Hence $(V \otimes_S \Lambda) \cap (\Lambda \otimes_S V) = \mathcal{W}_{N+1}$, and $\Lambda = \mathcal{W}_N$. We can also use the expression $\delta_p \Phi_n = \sum_{a \in Q_1} a \otimes \delta_a \delta_p \Phi_n = \sum_{a \in Q_1} \delta_p \Phi_n \delta'_a \otimes a$ to give an explicit form for $\psi(\theta_j)$ with $j = 0, \ldots, N-1$:

$$\begin{aligned} \psi(\theta_j) : \mathcal{W}_{N+1} &\to V^{\otimes_S (j+1)} \\ \delta_p \Phi &\mapsto \sum_{a \in Q_1} a \otimes \theta_j(\delta_a \delta_p \Phi) - \theta_j(\delta_p \Phi \delta'_a) \otimes a \quad \text{where } |p| = k-1 \end{aligned}$$

Theorem 3.5 was proved by Braverman and Gaitsgory [BG96] in the case of S being a field and A being 2-Koszul. They also show that in this situation any PBW deformation produces a graded deformation. With the notation as above a graded deformation A_t of A is a graded $\mathbb{C}[t]$ algebra with t in degree 1 such that A_t is free as a module over $\mathbb{C}[t]$ and there is an isomorphism $A_t/tA_t \to A$. It is shown that for a PBW deformation $\mathcal{A}$ there is a graded deformation of A with fibre at $t = 1$ canonically isomorphic to $\mathcal{A}$, [BG96, Theorem 4.1].

In the case when $A = \mathcal{D}(\Phi_n, n-2)$ is 2-Koszul and S a field there is also the following theorem of Wu and Zhu, [WZ], which classifies when PBW deformations of a Noetherian n-CY algebra are also n-CY. In our language this applies when there is only a single vertex in the quiver.

Theorem 3.7. *Let Φ_n be a homogeneous superpotential on a single vertex quiver. Let $A = \mathcal{D}(\Phi_n, n-2)$. If A is n-CY Noetherian and Koszul and $\mathcal{A}$ is a PBW deformation given by θ_0, θ_1 then $\mathcal{A}$ is n-CY if and only if*

$$\sum_{i=0}^{n-2} (-1)^i id^{\otimes i} \otimes \theta_1 \otimes id^{\otimes (n-2-i)}(\Phi_n) = 0$$

Proof. See [WZ, Theorem 3.1]. This proves such a PBW deformation is n-CY if and only if

$$\sum_{i=0}^{n-2} (-1)^i id^{\otimes i} \otimes \theta_1 \otimes id^{\otimes (n-2-i)} : \cap_i V^{\otimes i} \otimes R \otimes V^{\otimes (n-2-i)} \to V^{\otimes n-1}$$

is 0. Note that $\cap_i V^{\otimes i} \otimes R \otimes V^{\otimes (n-2-i)} = \mathcal{W}_n =< \Phi_n >$ in this case, giving the result. □

3.3 Results for PBW deformations of Koszul, CY algebras

We state and prove our main results concerning the PBW deformations of a $(k+2)$-CY, $(n-k)$-Koszul, superpotential algebra $A := \mathcal{D}(\Phi_n, k)$, where Φ_n is a homogeneous superpotential. We will give a condition for a PBW deformation to be of the form $\mathcal{D}(\Phi', k)$ for an inhomogeneous superpotential $\Phi' = \Phi_n + \phi_{n-1} + \cdots + \phi_k$ and prove that inhomogeneous superpotentials with

degree n part Φ_n define PBW deformations of A. In the Koszul case we will also show that certain PBW deformations of such an n-CY algebra are also n-CY.

Throughout this section consider $\Phi_n \in \mathbb{C}Q$ to be a homogeneous superpotential of degree n on some quiver Q. This will have inhomogeneous superpotentials associated to it, denoted $\Phi' = \Phi_n + \phi_{n-1} + \cdots + \phi_k$.

We introduce two new definitions we make use of in the proof.

Definition 3.8. We will call Φ', an inhomogeneous superpotential, *k-coherent* if for any collection of $\lambda_p \in \mathbb{C}$

$$\sum_{|p|=k} \lambda_p \delta_p \Phi_n = 0 \in \mathbb{C}Q \Rightarrow \sum_{|p|=k} \lambda_p \delta_p \Phi' = 0 \in \mathbb{C}Q$$

Definition 3.9. Let $\mathcal{A} = \frac{T_S(V)}{\Pi}$ and $A = \frac{T_S(V)}{\Lambda}$ be as in Definition 3.4. We will say that $\mathcal{A}$ is a *zeroPBW deformation* of A if it is a PBW deformation which also satisfies the *zeroPBW condition*

$$\mathrm{Im}(\psi(\theta_{N-1})) = \{0\}$$

where ψ and θ are defined as in Section 3.2.2.

We will say Π is of *zeroPBW type* if $\mathcal{A}$ is a zeroPBW deformation of A.

Lemma 3.10. *An S^e-module P is of zeroPBW type if and only if*

PBW1) $P \cap F^{N-1} = \{0\}$, *and*

ZPBW) $\mathrm{Im}(\psi(\theta_j)) = \{0\}$ *for* $j = 0 \ldots N-1$.

Proof. By definition P is of zeroPBW type if and only if it is of PBW type and also satisfies the zeroPBW condition. It is of PBW type if and only if it satisfies conditions PBW1,2,3,4) of Theorem 3.5. Satisfying the zeroPBW condition is equivalent to reducing the conditions PBW2,3,4) to the condition ZPBW). □

3.3.1 Deformations of superpotential algebras

We state and prove our results relating superpotentials and PBW deformations.

Theorem 3.11. *Let $A = \frac{\mathbb{C}Q}{\Lambda}$ be a path algebra with relations which is $(n-k)$-Koszul and $(k+2)$-CY. Then $A = \mathcal{D}(\Phi_n, k)$, for Φ_n a homogeneous superpotential of degree n, and the zeroPBW deformations of A correspond exactly to the algebras $\mathcal{D}(\Phi', k)$ defined by k-coherent inhomogeneous superpotentials of the form $\Phi' = \Phi_n + \phi_{n-1} + \cdots + \phi_k$.*

Proof. As A is $(k+2)$-CY and $(n-k)$-Koszul by Theorem 3.3 $A = \mathcal{D}(\Phi_n, k)$ for Φ_n a degree n homogeneous superpotential.

We define inverse maps between Π of zeroPBW type and coherent superpotentials Φ' which will give us the correspondence. We note that zeroPBW deformations exactly correspond to Π of zeroPBW type.

We first define a map, $\mathcal{F}$, taking Π of zeroPBW type to k-coherent superpotentials. We define $\mathcal{F}(\Pi)$ as an element of $\mathbb{C}Q$, then show it is k-coherent and a superpotential.

Any zeroPBW deformation defined by Π is given by a map $\theta = \theta_{n-k-1} + \cdots + \theta_0$ with $\theta_j : \Lambda \to V^{\otimes_S j}$ such that Π and θ satisfy the conditions PBW1) and ZPBW) of Lemma 3.10. We construct $\mathcal{F}(\Pi) := \Phi_n + \phi_{n-1}(\theta_{n-k-1}) + \cdots + \phi_k(\theta_0)$ by defining

$$\phi_{n-j}(\theta_{n-k-j}) := -\sum_{|p|=k} p\theta_{n-k-j}(\delta_p\Phi_n) =: \sum_{|s|=n-j} c_s s$$

and show that this is a k-coherent superpotential. For any p with $|p| = k$ it follows from this definition that $\delta_p\phi_{n-j}(\theta_{n-k-j}) = -\theta_{n-k-j}(\delta_p\Phi_n)$.

Firstly $\mathcal{F}(\Pi)$ is not k-coherent precisely when there exist coefficients $\lambda_p \in \mathbb{C}$ such that $\sum_{|p|=k} \lambda_p\delta_p\Phi_n = 0$ with $\omega = \sum_{|p|=k} \lambda_p\delta_p\Phi'(\theta) \neq 0$. But this only occurs when there exists $\omega \in \Pi \cap F^{n-1}$ which is non zero. But as Π is of zeroPBW type PBW1) holds, and $\Pi \cap F^{n-1} = \{0\}$. Hence $\mathcal{F}(\Pi)$ is k-coherent.

Now we show that $\mathcal{F}(\Pi)$ is a superpotential. As Π is of zeroPBW type by Lemma 3.10 $\operatorname{Im}\psi(\theta_j) = \{0\}$ for $j = 0, \ldots, n-k-1$. We evaluate $\psi(\theta_j)$ on elements of the form $\delta_q\Phi_n$ where $|q| = k-1$, as in Example 3.6, and use $\theta_j(\delta_p\Phi_n) = -\delta_p\phi_{k+j}(\theta_j)$ to deduce that $\mathcal{F}(\Pi)$ is a superpotential.

$$\begin{aligned}
0 = \psi(\theta_j)(\delta_q\Phi_n) &= \sum_a \big(a\theta_j(\delta_{qa}\Phi_n) + (-1)^n\theta_j(\delta_{aq}\Phi_n)a\big) \\
&= -\sum_a \big(a\delta_{qa}\phi_{k+j}(\theta_j) + (-1)^n\delta_{aq}\phi_{k+j}(\theta_j)a\big) \\
&= -\sum_{a,p} \big(c_{qap}ap + (-1)^n c_{aqp}pa\big) \\
&= -\sum_{a,p} \big((c_{qap_1\ldots p_j} + (-1)^n c_{p_jqap_1\ldots p_{j-1}})ap\big)
\end{aligned}$$

with the sums over all $a \in Q_1$ and $p = p_1 \ldots p_j$ of length j. Considering the coefficient of a path ap we see $\phi_{k+j} = \sum_s c_s s$ satisfies the n superpotential condition

$$c_{qap_1\ldots p_j} = (-1)^{n-1} c_{p_jqap_1\ldots p_{j-1}}.$$

Hence each ϕ_{j+k} satisfies the n superpotential condition, and $\mathcal{F}(\Pi)$ is a inhomogeneous superpotential of degree n. The zeroPBW deformation defined by Π is, by construction, $\mathcal{D}(\mathcal{F}(\Pi), k)$ as $\Pi =< \{r - \theta(r) : r \in \Lambda\} >=< \{\delta_p\mathcal{F}(\Pi) : |p| = k\} >$.

Now we define a map, $\mathcal{G}$, sending k-coherent superpotentials $\Phi' = \Phi_n + \phi_{n-1} + \cdots + \phi_{n-k}$ to Π of zeroPBW type. We define $\mathcal{G}(\Phi')$ as an S^e-module, show that $\mathcal{G}(\Phi')$ satisfies the PBW1) condition, and then define a map $\theta^{\Phi'}$ as in Lemma 3.10. We then show that $\mathcal{G}(\Phi')$ is of zeroPBW type.

The map $\mathcal{G}$ is defined by $\mathcal{G}(\Phi') :=< \{\delta_p\Phi' : |p| = k\} >$. We first check that $\mathcal{G}(\Phi')$ satisfies PBW1). Indeed as Φ' is k-coherent there are no $\omega = \sum_{|p|=k} \lambda_p\delta_p\Phi' \neq 0$, with $\lambda_p \in \mathbb{C}$, satisfying $\sum_{|p|=k} \lambda_p\delta_p\Phi_n = 0$. This implies $\mathcal{G}(\Phi'_n) \cap F^{n-1} = \{0\}$, hence $\mathcal{G}(\Phi'_n)$ satisfies PBW1).

Then, as $\mathcal{G}(\Phi'_n)$ satisfies PBW1), $\mathcal{G}(\Phi') =< \{r - \theta^{\Phi'}(r) : r \in \Lambda\} >$ where $\theta^{\Phi'}$ are the maps defined by $\theta^{\Phi'} := \sum_{j=0}^{n-k-1} \theta_j^{\phi_{j+k}}$ with each $\theta_j^{\phi_{j+k}}$ defined on elements $\delta_p\Phi_n$ by

$$\begin{aligned}
\theta_j^{\phi_{j+k}} : \Lambda &\to V^{\otimes_S j} \\
\delta_p\Phi_n &\mapsto -\delta_p\phi_{j+k}
\end{aligned}$$

We next show that $\mathcal{G}(\Phi')$ is of zeroPBW type. By Lemma 3.10, to show that $\mathcal{G}(\Phi')$ is of zeroPBW type it is enough to show that $\mathcal{G}(\Phi')$ and $\theta^{\Phi'}$ satisfy the conditions PBW1) and

ZPBW). We have already shown PBW1) is satisfied, so need only check ZPBW); $\operatorname{Im}\psi(\theta_j^{\phi_{j+k}}) = \{0\}$ for $j = 0, \dots, n-k-1$. As in Example 3.6 $(V \otimes_S \Lambda) \cap (\Lambda \otimes_S V) = \mathcal{W}_{n-k+1} = \langle \delta_p \Phi_n : |p| = k-1 \rangle$, hence we calculate $\psi(\theta_j^{\phi_{j+k}})(\delta_q \Phi_n)$, for $j = 0, \dots, n-k-1$, where $|q| = k-1$.

$$\begin{aligned}
\psi(\theta_j^{\phi_{j+k}})(\delta_q \Phi_n) &= \sum_a \left(a\theta_j^{\phi_{j+k}}(\delta_{qa}\Phi_n) + (-1)^n \theta_j^{\phi_{j+k}}(\delta_{aq}\Phi_n) a \right) \\
&= -\sum_a \left(a\delta_{qa}\phi_{j+k} + (-1)^n \delta_{aq}\phi_{j+k} a \right) \\
&= -\sum_{a,p} \left(c_{qap} ap + (-1)^n c_{aqp} pa \right) \\
&= -\sum_{a,p} \left((c_{qap_1\dots p_j} + (-1)^n c_{p_j q a p_1 \dots p_{j-1}}) ap \right)
\end{aligned}$$

where we write $\phi_j = \sum_{|t|=j} c_t t$, and the sums are over all $a \in Q_1$ and $p = p_1 \dots p_j$ of length j. Hence as ϕ_j satisfies the n superpotential condition

$$c_{qap_1\dots p_r} = (-1)^{n-1} c_{p_r q a p_1 \dots p_{r-1}}$$

then $\psi(\theta_j^{\phi_{j+k}})(\delta_q \Phi_n) = 0$ for all q and j. Therefore $\operatorname{Im}\psi(\theta_j^{\phi_{k+j}}) = \{0\}$ for all j, and $\mathcal{G}(\Phi')$ is of zeroPBW type.

We note that as $\mathcal{G}(\Phi') =< \{\delta_p \Phi' : |p| = k\} >$ by construction the zeroPBW deformation defined by $\mathcal{G}(\Phi')$ is indeed $\mathcal{D}(\Phi', k)$.

Now the maps $\mathcal{F}$ and $\mathcal{G}$ are inverses by construction, and provide a bijection between zeroPBW deformations and k-coherent superpotentials of the form $\Phi_n + \phi_{n-1} + \dots + \phi_k$. Moreover the zeroPBW deformation defined by P equals $\mathcal{D}(\mathcal{F}(\Pi), k)$, and the zeroPBW deformation defined by $\mathcal{G}(\Phi')$ equals $\mathcal{D}(\Phi', k)$. □

Remark 3.12. *Let $A = \mathcal{D}(\Phi_n, n-2)$ be a 2-Koszul n-CY algebra, and $\mathcal{A} = \mathcal{D}(\Phi', n-2)$, as in Theorem 3.11, be a zeroPBW deformation of defined by θ_1, θ_0. Then if n is even $\theta_1 = 0$.*

Proof. Note that if n is even then any ϕ_{n-1} satisfying the n superpotential condition is 0, as observed after Definition 2.4. In particular a zeroPBW deformation defined by θ_0, θ_1 corresponds to a superpotential $\Phi_n + \phi_{n-1} + \phi_{n-2}$ by the bijection of Theorem 3.11, and superpotentials with $\phi_{n-1} = 0$ correspond to zeroPBW deformations with $\theta_1 = 0$. Hence for even n we have $\phi_{n-1} = 0$ and so $\theta_1 = 0$. □

3.3.2 CY property of deformations

We now consider when zeroPBW deformations are CY algebras. We consider the case where $A = \mathcal{D}(\Phi_n, n-2)$ is n-CY and 2-Koszul. We prove a result only in the case of a zeroPBW deformation with $\theta_1 = 0$, which covers all even dimensional cases by Remark 3.12.

We then briefly mention two results concerning PBW deformations of CY algebras. Our results are weaker, but work in a more general setting. One is the result of [WZ, Theorem 3.1], which is quoted as Theorem 3.7 above, giving a complete characterisation of CY PBW deformations of a Noetherian CY algebra over a field. The other is the paper [BT07] which proves that the zeroPBW deformations of a 3-CY superpotential algebra are 3-CY PBW deformations.

Theorem 3.13. *Let $\mathcal{A}$ be a zeroPBW deformation of A with $\theta_1 = 0$. Then $\mathcal{A}$ is n-CY.*

Proof. By Theorem 3.11 such a deformation is given by a superpotential $\Phi' = \Phi_n + \phi_{n-2} = \sum_t c_t t$ and $\mathcal{A} = \mathcal{D}(\Phi', n-2)$. We construct a resolution for $\mathcal{A}$ and show it is self-dual. We recall the complex $\mathcal{W}^\bullet$, defined as (3.1) in Section 3.2.1, and, by Theorem 3.1, that $\mathcal{W}^\bullet$ is an A^e-module resolution of A. We then define a complex $\mathcal{W}^\bullet_{\mathcal{A}}$

$$0 \to \mathcal{A} \otimes_S \mathcal{W}_n \otimes_S \mathcal{A} \xrightarrow{d_n} \mathcal{A} \otimes_S \mathcal{W}_{n-1} \otimes_S \mathcal{A} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} \mathcal{A} \otimes_S \mathcal{W}_0 \otimes_S \mathcal{A} \xrightarrow{\mu} \mathcal{A} \to 0.$$

The S^e-modules $\mathcal{W}_k$, and differential d, are defined as in Section 3.2.1, i.e. $\mathcal{W}_j =< \{\delta_p \Phi_n : |p| = n-j\} >$, and $d_i = \epsilon_i(d_i^l + (-1)^i d_i^r)$ where

$$\epsilon_i := \begin{cases} (-1)^{i(n-i)} & \text{if } i < (n+1)/2 \\ 1 & \text{otherwise} \end{cases}$$

and $d_i^l, d_i^r : \mathcal{A} \otimes_S \mathcal{W}_i \otimes_S \mathcal{A} \to \mathcal{A} \otimes_S \mathcal{W}_{i-1} \otimes_S \mathcal{A}$ are defined by

$$\begin{aligned} d_i^l(\alpha \otimes \delta_p \Phi_n \otimes \beta) &= \sum_{a \in Q_1} \alpha a \otimes \delta_a \delta_p \Phi_n \otimes \beta \\ d_i^r(\alpha \otimes \delta_p \Phi_n \otimes \beta) &= \sum_{a \in Q_1} \alpha \otimes \delta_p \Phi_n \delta'_a \otimes a\beta \end{aligned}$$

for p any path of length $n-i$. We will show this is a self-dual resolution of $\mathcal{A}$.

We first check this is a complex, checking $d_{j-1} \circ d_j = 0$ for $j = 2, \dots, n$ and that $\mu \circ d_1 = 0$. In the calculations $|p| = n-j$, and the sums are taken over all $a, b \in Q_1$ and paths q of length $j-2$.

$$\begin{aligned} d_{j-1} \circ d_j(1 \otimes \delta_p \Phi_n \otimes 1) &= \epsilon_j \epsilon_{j-1} \sum_{a,b} (ab \otimes \delta_{pab} \Phi_n \otimes 1 - 1 \otimes \delta_p \Phi_n \delta'_{ab} \otimes ab) \\ &= \epsilon_j \epsilon_{j-1} \sum_{a,b,q} (c_{pabq} ab \otimes q \otimes 1 - 1 \otimes q \otimes c_{pqab} ab) \\ &= \epsilon_j \epsilon_{j-1} \sum_{a,b,q} ((-1)^{(n-1)j} c_{qpab} ab \otimes q \otimes 1 - 1 \otimes q \otimes c_{pqab} ab) \\ &= \epsilon_j \epsilon_{j-1} \sum_{q} ((-1)^{(n-1)j} \delta_{qp} \Phi_n \otimes q \otimes 1 - 1 \otimes q \otimes \delta_{pq} \Phi_n) \\ &= -\epsilon_j \epsilon_{j-1} \sum_{q} ((-1)^{(n-1)j} \delta_{qp} \phi_{n-2} \otimes q \otimes 1 - 1 \otimes q \otimes \delta_{pq} \phi_{n-2}) \\ &= -\epsilon_j \epsilon_{j-1} \sum_{q} ((-1)^{(n-1)j} c_{qp} 1 \otimes q \otimes 1 - 1 \otimes q \otimes 1 c_{pq}) \\ &= -\epsilon_j \epsilon_{j-1} \sum_{q} (c_{pq} 1 \otimes q \otimes 1 - 1 \otimes q \otimes 1 c_{pq}) = 0. \end{aligned}$$

$$\begin{aligned} \mu \circ d_1(1 \otimes \delta_p \Phi_n \otimes 1) &= \epsilon_1 \mu(\sum_a a \otimes \delta_{pa} \Phi_n \otimes 1 - 1 \otimes \delta_p \Phi_n \delta'_a \otimes a) \\ &= \epsilon_1 \sum_a (c_{pa} a - (-1)^{n-1} c_{ap} a) = \epsilon_1 \sum_a (c_{pa} a - c_{pa} a) = 0. \end{aligned}$$

Since $\mathcal{A}$ is a PBW deformation of A we have $gr\mathcal{A} \cong A$ and so $gr\mathcal{W}^\bullet_{\mathcal{A}} \cong \mathcal{W}^\bullet$, where we use

the product filtration, $F^n(\mathcal{A}\otimes_S \mathcal{W}_j\otimes_S \mathcal{A}) := \sum_{i+j+k\leq n} F^i(\mathcal{A})\otimes_S \mathcal{W}_j \otimes_S F^k(\mathcal{A})$. Since $\mathcal{W}^\bullet$ is exact, so is $\mathcal{W}^\bullet_{\mathcal{A}}$,[Sjö73, Lemma 1], and since $\mathcal{W}^\bullet$ consists of finitely generated projectives so does $\mathcal{W}^\bullet_{\mathcal{A}}$,[MRS01, Lemmas 6.11, 6.16]. Hence $\mathcal{W}^\bullet_{\mathcal{A}}$ gives a resolution of $\mathcal{A}$ by finitely generated projectives. Now we need only check that this is still self dual, implying $\mathcal{A}$ is n-CY.

We now construct isomorphisms α_k between the complex $\mathcal{W}^\bullet_{\mathcal{A}}$ and its dual

$$\begin{tikzcd}
\mathcal{A}\otimes_S \mathcal{W}_n\otimes_S \mathcal{A} \arrow[r, "d_n"] \arrow[d, "\alpha_n"] & \mathcal{A}\otimes_S \mathcal{W}_{n-1}\otimes_S \mathcal{A} \arrow[r, "d_{n-1}"] \arrow[d, "\alpha_{n-1}"] & \cdots \arrow[r, "d_2"] & \mathcal{A}\otimes_S \mathcal{W}_1\otimes_S \mathcal{A} \arrow[r, "d_1"] \arrow[d, "\alpha_1"] & \mathcal{A}\otimes_S \mathcal{W}_0\otimes_S \mathcal{A} \arrow[d, "\alpha_0"] \\
(\mathcal{A}\otimes_S \mathcal{W}_0\otimes_S \mathcal{A})^\vee \arrow[r, "-d_1^\vee"] & (\mathcal{A}\otimes_S \mathcal{W}_1\otimes_S \mathcal{A})^\vee \arrow[r, "-d_2^\vee"] & \cdots \arrow[r, "-d_{n-1}^\vee"] & (\mathcal{A}\otimes_S \mathcal{W}_{n-1}\otimes_S \mathcal{A})^\vee \arrow[r, "-d_n^\vee"] & (\mathcal{A}\otimes_S \mathcal{W}_n\otimes_S \mathcal{A})^\vee
\end{tikzcd}$$

such that all the squares commute.

We will use the notation and result of [Boc08, Section 4], working over $\mathbb{C}$. For T a finite dimensional S^e-module let F_T be the $\mathcal{A}^e$-module $\mathcal{A}\otimes_S T\otimes_S \mathcal{A}$, and let T^* denote the $\mathbb{C}$ dual of T. Then

$$\begin{aligned} F_{T^*} &\to F_T^\vee \\ (1\otimes\phi\otimes 1) &\mapsto \left((1\otimes p\otimes 1)\mapsto \sum_{e,f\in Q_0}\phi(epf)e\otimes_{\mathbb{C}} f\right)\end{aligned}$$

gives an isomorphism of $\mathcal{A}^e$-modules. Moreover, as S is semisimple, tensoring $\mathcal{A}\otimes_S(-)\otimes_S\mathcal{A}$ is flat, hence constructing an isomorphism of S^e-modules $\mathcal{W}_j\to\mathcal{W}^*_{n-j}$ will give us an isomorphism of $\mathcal{A}^e$-modules $F_{\mathcal{W}_j}\to F_{\mathcal{W}^*_{n-j}}$, and composing with the above isomorphism an will give us an isomorphism $F_{\mathcal{W}_j}\to F^\vee_{\mathcal{W}_{n-j}}$.

For $|p| = n-j$ define $\partial_p\in\mathcal{W}^*_{n-j}$ by $\partial_p(\delta_q\Phi_n) = c_{qp}$, where $\Phi_n = \sum_{|t|=n} c_t t$ and $|q| = j$. Then there are S^e-module homomorphisms

$$\begin{aligned} \eta_j : \mathcal{W}_j &\to \mathcal{W}^*_{n-j} \\ \delta_p\Phi_n &\mapsto \gamma_j\partial_p\end{aligned}$$

for γ_j arbitrary nonzero constants.

To see η_j is injective suppose $\eta_j(\sum\lambda_p\delta_p\Phi_n) = 0$. Then $\sum_p\lambda_p\partial_p(\delta_q\Phi_n) = \sum_p\lambda_p c_{qp} = 0$ for all q. Hence $\sum_p\lambda_p\delta_p\Phi_n = \sum_{p,q}\lambda_p c_{pq} q = \sum_q(\sum_p\lambda_p c_{pq})q = 0$, so the map is injective.

There are $\mathbb{C}$-vector space isomorphisms between $e\mathcal{W}_j f$ and $(f\mathcal{W}_{n-j}e)^*$ for $e,f\in Q_0$ arising from the pairings,

$$\begin{aligned} e\mathcal{W}_j f\otimes_{\mathbb{C}} f\mathcal{W}_{n-j}e &\to \mathbb{C} \\ e\delta_p\Phi_n f\otimes f\delta_q\Phi_n e &\mapsto \gamma_j c_{qp} = \eta_j(e\delta_p\Phi_n f)(f\delta_q\Phi_n e) \\ &\quad = (-1)^{(n-1)|q|}\frac{\gamma_j}{\gamma_{n-j}}\eta_{n-j}(f\delta_q\Phi_n e)(e\delta_p\Phi_n f)\end{aligned}$$

where we note that $\partial_p(\delta_q\Phi_n) = c_{qp} = (-1)^{(n-1)|q|}c_{pq} = (-1)^{(n-1)|q|}\partial_q(\delta_p\Phi_n)$, and to be non-zero we require $h(p) = t(q)$ and $t(p) = h(q)$.

As $e\mathcal{W}_j f$ and $f\mathcal{W}_{n-j}e$ are finite dimensional $\mathbb{C}$-vector spaces the injectivity of η_j and η_{n-j} implies this pairing is perfect, thus η_j is an isomorphism.

Now we use this to define isomorphisms α_j

$$\begin{aligned}
\alpha_j : \mathcal{A} \otimes_S \mathcal{W}_j \otimes_S \mathcal{A} &\to (\mathcal{A} \otimes_S \mathcal{W}_{n-j} \otimes_S \mathcal{A})^\vee \\
(a_1 \otimes \delta_p\Phi_n \otimes a_2) &\mapsto \big((b_1 \otimes \delta_q\Phi_n \otimes b_2) \mapsto (b_1a_2 \otimes \eta_j(\delta_p\Phi_n)(\delta_q\Phi_n) \otimes a_1b_2)\big) \\
&= \big((b_1 \otimes \delta_q\Phi_n \otimes b_2) \mapsto (\gamma_j c_{qp} b_1 h(p) a_2 \otimes_{\mathbb{C}} a_1 t(p) b_2)\big)
\end{aligned}$$

where we note that for this to be non-zero $h(p) = t(q) = h(a_2) = t(b_1)$ and $h(q) = t(p) = h(b_2) = t(a_1)$.

It remains to check $\alpha_{j-1} \circ d_j = -d^\vee_{n-j+1} \circ \alpha_j$, i.e. the following diagram commutes;

$$\begin{tikzcd}
\mathcal{A} \otimes_S \mathcal{W}_j \otimes_S \mathcal{A} \arrow[r, "d_j"] \arrow[d, "\alpha_j"] & \mathcal{A} \otimes_S \mathcal{W}_{j-1} \otimes_S \mathcal{A} \arrow[d, "\alpha_{j-1}"] \\
(\mathcal{A} \otimes_S \mathcal{W}_{n-j} \otimes_S \mathcal{A})^\vee \arrow[r, "-d^\vee_{n-j+1}"] & (\mathcal{A} \otimes_S \mathcal{W}_{n-j+1} \otimes_S \mathcal{A})^\vee
\end{tikzcd}$$

If $|p| = n - j$ and $|q| = j - 1$ then

$$\begin{aligned}
& (\alpha_{j-1} \circ d_j(1 \otimes \delta_p\Phi \otimes 1))(1 \otimes \delta_q\Phi \otimes 1) \\
= \; & \alpha_{j-1}(\epsilon_j \sum_a a \otimes \delta_{pa}\Phi \otimes 1 + (-1)^{j+(n-1)} 1 \otimes \delta_{ap}\Phi \otimes a)(1 \otimes \delta_q\Phi \otimes 1) \\
= \; & \gamma_{j-1}\epsilon_j \sum_a c_{qpa} t(q) \otimes a + \gamma_{j-1}\epsilon_j (-1)^{j+n-1} \sum_a c_{qap} a \otimes h(q)
\end{aligned}$$

whilst

$$\begin{aligned}
& (-d^\vee_{n-j+1} \circ \alpha_j(1 \otimes \delta_p\Phi \otimes 1))(1 \otimes \delta_q\Phi \otimes 1) \\
= \; & \alpha_j(1 \otimes \delta_p\Phi \otimes 1)(-\epsilon_{n-j+1} \sum_a a \otimes \delta_{qa}\Phi \otimes 1 + (-1)^j 1 \otimes \delta_{aq}\Phi \otimes a) \\
= \; & -\gamma_j\epsilon_{n-j+1} \sum_a c_{qap} a \otimes t(p) - \gamma_j\epsilon_{n-j+1}(-1)^j \sum_a c_{aqp} h(p) \otimes a
\end{aligned}$$

where the sums are taken over all $a \in Q_1$. Thus for these to be equal we require

$$-\gamma_j\epsilon_{n-j+1} c_{qap} = \gamma_{j-1}\epsilon_j (-1)^{j+n-1} c_{qap}$$

and

$$-\gamma_j\epsilon_{n-j+1}(-1)^j c_{aqp} = \gamma_{j-1}\epsilon_j c_{qpa} = (-1)^{n-1}\gamma_{j-1}\epsilon_j c_{aqp}$$

for all $a \in Q_1$. These follow if $(-1)^n\gamma_{j-1}\epsilon_j = \gamma_j\epsilon_{n-j+1}(-1)^j$ for $j = 1, \ldots, n$. As the γ_j were arbitrary non-zero scalars, we can choose the γ_j so that this is satisfied, and the proof is completed. □

Let Q be a single vertex quiver and $A = \mathcal{D}(\Phi_n, n-2)$ be a Noetherian, Koszul, n-CY algebra. We consider a PBW deformation, $\mathcal{A}$, given by θ_1, θ_0, and set $\phi_{n-1} = \sum_p p\theta_1(\delta_p \Phi_n) = \sum c_q q$. If the deformation is a zeroPBW deformation ϕ_{n-1} has the n-superpotential property.

Theorem 3.14. *Keeping the above notation and assumptions,*

i) Any zeroPBW deformation of A is n-CY.

ii) Let $n = 2$ or 3. Then the zeroPBW deformations of A are exactly the n-CY PBW deformations of A, and moreover any superpotential $\Phi' = \Phi_n + \phi_{n-1} + \phi_{n-2}$ is $(n-2)$-coherent.

Proof. i) Let $\mathcal{A}$ be a PBW deformation of A, defined by a map θ. Then define $\Phi'(\theta) = \Phi_n + \phi_{n-1} + \phi_{n-2}$ by

$$\phi_{n-2+j} := -\sum_{|p|=n-2} p\theta_j(\delta_p \Phi_n).$$

Write $\phi_{n-1} = \sum_t c_t t$. We will show that $\mathcal{A}$ is n-CY if and only if

$$0 = \sum_{i=0}^{n-2} (-1)^{in} c_{q_{i+2}\dots q_{n-1}q_1\dots q_{i+1}}$$

for any $q = q_1 \dots q_{n-1}$, with $q_j \in Q_1$. In particular this shows any zeroPBW deformation is n-CY, as then $\Phi'(\theta)$ is a superpotential and when n is even $\phi_{n-1} = \sum c_t t = 0$ and when n is odd the terms cancel in pairs by the superpotential property.

Referring to Theorem 3.7 $\mathcal{A}$ is n-CY if and only if

$$\begin{aligned}
0 &= \sum_{i=0}^{n-2} (-1)^i id^{\otimes i} \otimes \theta_1 \otimes id^{\otimes(n-2-i)}(\Phi_n) \\
&= \sum_i (-1)^i \sum_{p=p_1\dots p_{n-2}} p_1 \dots p_i \theta_1(\delta_{p_1\dots p_i} \Phi_n \delta'_{p_{i+1}\dots p_{n-2}}) p_{i+1} \dots p_{n-2} \\
&= \sum_i (-1)^{i+(n-1)(n-2-i)} \sum_p p_1 \dots p_i \theta_1(\delta_{p_{i+1}\dots p_{n-2}p_1\dots p_i} \Phi_n) p_{i+1} \dots p_{n-2} \\
&\quad \Big(\text{using } \theta_1(\delta_q \Phi_n) = \delta_q \phi_{n-1} \text{ and writing } \phi_{n-1} = \sum_{|r|=n-1} c_r r\Big) \\
&= \sum_i (-1)^{in} \sum_{p,a} p_1 \dots p_i c_{p_{i+1}\dots p_{n-2}p_1\dots p_i a} a p_{i+1} \dots p_{n-2} \\
&= \sum_i (-1)^{in} \sum_q c_{q_{i+2}\dots q_{n-1}q_1\dots q_i q_{i+1}} q_1 \dots q_i q_{i+1} q_{i+2} \dots q_{n-1}.
\end{aligned}$$

Considering the coefficients of the paths, which are linearly independent, and calculating the coefficient of $q_1 \dots q_{n-1}$ we find the condition on ϕ_{n-1} to be

$$0 = \sum_{i=0}^{n-2} (-1)^{in} c_{q_{i+2}\dots q_{n-1}q_1\dots q_{i+1}}$$

for all $q_1 \dots q_{n-1}$.

ii) By Theorem 3.7 we have a condition for $\mathcal{A}$ to be n-CY. In the $n = 3$ case the condition gives that $\operatorname{Im} id \otimes \theta_1 - \theta_1 \otimes id = \psi(\theta_1) = \{0\}$ which is the zeroPBW condition. When $n = 2$ it

gives the condition $\theta_1 = 0$ which is the zeroPBW condition for even n.

Moreover when $n = 3$ any superpotential, Φ', is 1-coherent. In particular the fact that A is 3-CY gives a duality in $\mathcal{W}^\bullet$ between the 1st and 2nd terms - the arrows $a \in Q_1$ and the relations $\delta_a \Phi_3$. Hence as the arrows are linearly independent so are the relations, and $\sum_{a \in Q_1} \lambda_a \delta_a \Phi_3 = 0 \Rightarrow \lambda_a = 0 \Rightarrow \sum \lambda_a \delta_a \Phi' = 0$, so Φ' is 1-coherent. When $n = 2$ the superpotential is only differentiated by paths of length 0, so it is clearly 0-coherent. □

A similar result in the 3-CY case is already known in the context of a general quiver due to the following result of Berger and Taillefer. We note that their result is for potentials rather than superpotentials, where a potential is defined to be an element $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$, and there is a map into $\mathbb{C}Q$ defined by $\mathfrak{c} : a_n \dots a_1 \mapsto \sum_i a_i \dots a_1 a_n \dots a_{i+1}$, see [BT07, Section 2].

Theorem 3.15 ([BT07, Section 3])**.** *Let $A = \mathcal{D}(\mathfrak{c}(W_{N+1}), 1)$ be N Koszul and 3-CY, with W_{N+1} a potential on some quiver Q. Then a PBW deformation $\mathcal{A}$ of A is 3-CY if it is a zeroPBW deformation. Moreover the zeroPBW deformations correspond to $\mathcal{D}(\mathfrak{c}(W'), 1)$ for $W' = W_{N+1} + W_N + \dots + W_1$ an inhomogeneous potential with each W_i in grade i.*

We also note that Theorem 3.15 makes use of the results of [Boc08], in particular using the bimodule resolution of a graded 3-CY algebra given in [Boc08, Section 4.2], whereas our results use the bimodule resolution of a Koszul n-CY algebra given in [BSW10, Theorem 6.8].

3.4 Application: symplectic reflection algebras

In this section we recall the definition of symplectic reflection algebras and deduce they are Morita equivalent to CY superpotential algebras by applying Theorems 2.33, 3.11, and 3.13. We go on to calculate some examples, and consider the interpretation of the parameters of a symplectic reflection algebra in the superpotential setting.

Let V be a $2n$ dimensional $\mathbb{C}$-vector space, equipped with symplectic form ω and $G < \mathrm{Sp}(V)$ a symplectic reflection group which acts faithfully on V preserving the symplectic form. Then $G < \mathrm{Sp}(V)$ is indecomposable if there is no G-stable splitting $V = V_1 \oplus V_2$ with $\omega(V_1, V_2) = 0$. The symplectic reflection algebras are the PBW deformations of $\mathbb{C}[V] \rtimes G$ relative to $\mathbb{C}G$ and were defined and classified by Etingof and Ginzburg [EG02].

Theorem 3.16. *([EG02, Theorem 1.3]) Any PBW deformation of such an indecomposable $\mathbb{C}[V] \rtimes G$ is of the form*

$$H_{t,\mathbf{c}} = \frac{T_{\mathbb{C}G}(V^*)}{< [x,y] - \kappa_{t,\mathbf{c}}(x,y) : x, y \in V^* >}$$

where $\kappa_{t,\mathbf{c}}(x,y) = t\omega_{V^*}(x,y) - \Sigma_s \omega_s(x,y)\mathbf{c}(s)s$ with the sum taken over the symplectic reflections s, and $\mathbf{c}$ a complex valued class function on symplectic reflections. The symplectic form ω_{V^*} on V^* is induced from ω on V, and ω_s is defined as ω_{V^*} restricted to $(id - s)(V^*)$.

In particular, they fall into the even dimensional case considered in Section 3.3 above, with the PBW parameter θ_1 equal to zero.

Two infinite classes of symplectic reflection algebras are the rational Cherednik and wreath product algebras.

- **Rational Cherednik Algebras.** Let $\mathfrak{h}$ be a finite dimensional vector space and G a finite subgroup of $\mathrm{GL}(\mathfrak{h})$. Then $V = \mathfrak{h} \oplus \mathfrak{h}^*$ has the natural symplectic form $\omega((x,f),(y,g)) = f(y) - g(x)$, and an action of G as a symplectic reflection group. Then the symplectic reflection algebras given by PBW deformations of $\mathbb{C}[V] \rtimes G$ are the Rational Cherednik Algebras.

- **Wreath Product Algebras.** Let K be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, and S_n the symmetric group of order n. Then the wreath product group $G = S_n \wr K$ is a symplectic reflection group acting on $V = (\mathbb{C}^2)^n$.

3.4.1 Symplectic reflection algebras as superpotential algebras

Here we show that symplectic reflection algebras are Morita equivalent to superpotential algebras.

Theorem 3.17. *Let $H_{t,\mathbf{c}}$ be as in Theorem 3.16.*

i) $H_{0,0}$ is Morita equivalent to $A = \mathcal{D}(\Phi_{2n}, 2n-2)$ for some homogeneous superpotential Φ_{2n}, and is 2n-CY and Koszul.

ii) Any $H_{t,\mathbf{c}}$ is Morita equivalent to $\mathcal{A} = \mathcal{D}(\Phi', 2n-2)$ for $\Phi' = \Phi_{2n} + \phi_{2n-2}$ an inhomogeneous superpotential which is $(2n-2)$-coherent.

iii) Any $(2n-2)$-coherent superpotential of the form $\Phi' = \Phi_{2n} + \phi_{2n-2}$, gives an algebra $\mathcal{A} = D(\Phi', 2n-2)$ that is Morita equivalent to a symplectic reflection algebra $H_{t,\mathbf{c}}$.

iv) All $H_{t,\mathbf{c}}$ are 2n-CY algebras.

Proof. i): $H_{0,0}$ is just $\mathbb{C}[V] \rtimes G$. Hence Theorem 2.33 applies, and $H_{0,0}$ is Morita equivalent to a $2n$-CY, Koszul algebra $A := \mathcal{D}(\Phi_{2n}, 2n-2)$ for the McKay quiver for (G, V).

ii) and iii): The Morita equivalence between $\mathbb{C}[V] \rtimes G$ and A switches $\mathbb{C}G$ with S, respects the gradings, and respects the Koszul resolutions. Hence any zeroPBW deformation of $\mathbb{C}[V] \rtimes G$ corresponds to a zeroPBW deformation of A, and ii) and iii) follow from Theorem 3.11 once we note by Theorem 3.16 all PBW deformations of $\mathbb{C}[V] \rtimes G$ are zeroPBW.

iv): From ii) and iii) we know any symplectic reflection algebra is Morita equivalent to some $\mathcal{A} := \mathcal{D}(\Phi_{2n} + \phi_{2n-2}, 2n-2)$. By Theorem 3.13 $\mathcal{A}$ is $2n$-CY: $\mathcal{A}$ has a finite length resolution by finitely generated projective modules and $\mathbb{R}\mathrm{Hom}(\mathcal{A}, \mathcal{A}^e)[n] \cong \mathcal{A}$. Then the Morita equivalent symplectic reflection algebra, $H_{t,\mathbf{c}}$, also has a finite length resolution by finitely generated projective modules, as these properties are preserved under Morita equivalence. The isomorphism in the derived category $\mathbb{R}\mathrm{Hom}(\mathcal{A}, \mathcal{A}^e)[n] \cong \mathcal{A}$ transfers to the isomorphism $\mathbb{R}\mathrm{Hom}(H_{t,\mathbf{c}}, H_{t,\mathbf{c}}^e)[n] \cong H_{t,\mathbf{c}}$ under the Morita equivalence, hence $H_{t,\mathbf{c}}$ is $2n$-CY. □

We illustrate the above theorem in three examples.

Example 3.18. Consider the symplectic reflection algebra corresponding to the group S_3 acting on $\mathfrak{h} \oplus \mathfrak{h}^*$ in the manner of a rational Cherednik algebra, where the representation $\mathfrak{h}$ is given by

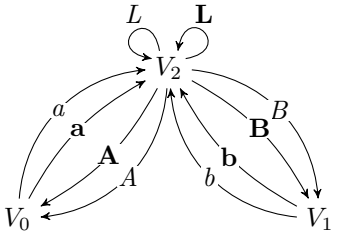
$$S_3 = \left\langle g = \begin{pmatrix} \varepsilon_3 & 0 \\ 0 & \varepsilon_3^2 \end{pmatrix}, h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

with ε_3 is a primitive third root of unity.

This has irreducible representations with character table

S_3	ccl(e)	ccl(h)	ccl(g)
ch V_0	1	1	1
ch V_1	1	-1	1
ch V_2	2	0	-1
ch V	4	0	-2

with $\mathfrak{h} = V_2$ and $V = \mathfrak{h} \oplus \mathfrak{h}^*$. As such the McKay quiver is;



Now following the calculation in [BSW10, Theorem 3.2] we choose a G-equivariant basis to calculate the superpotential to accompany this quiver

$$\begin{aligned}\Phi_4 = \quad & -A\mathbf{a}A\mathbf{a} + 2\mathbf{A}aA\mathbf{a} + 4A\mathbf{L}L\mathbf{a} - 4\mathbf{A}LL\mathbf{a} - 4Ab\mathbf{B}\mathbf{a} + 2A\mathbf{b}B\mathbf{a} + 2\mathbf{A}bB\mathbf{a} \\ & -\mathbf{A}a\mathbf{A}a - 4A\mathbf{L}\mathbf{L}a + 4\mathbf{A}L\mathbf{L}a + 2A\mathbf{b}\mathbf{B}a + 2\mathbf{A}b\mathbf{B}a - 4\mathbf{A}\mathbf{b}Ba - 8LL\mathbf{L}\mathbf{L} \\ & +4bB\mathbf{L}\mathbf{L} - 4b\mathbf{B}L\mathbf{L} + 4L\mathbf{b}B\mathbf{L} + 4\mathbf{b}\mathbf{B}LL - 2bB\mathbf{b}\mathbf{B} + b\mathbf{B}b\mathbf{B} + \mathbf{b}B\mathbf{b}B \\ & \qquad +\text{cyclic permutations.}\end{aligned}$$

We differentiate this

$$\begin{aligned}
\delta_{\mathbf{A}}\Phi &= 2aA\mathbf{a} - 4LL\mathbf{a} + 2bB\mathbf{a} - 2a\mathbf{A}a + 4L\mathbf{L}a + 2b\mathbf{B}a - 4\mathbf{b}Ba \\
\delta_{A}\Phi &= 2\mathbf{a}\mathbf{A}a - 4\mathbf{L}\mathbf{L}a + 2\mathbf{b}\mathbf{B}a - 2\mathbf{a}A\mathbf{a} + 4\mathbf{L}L\mathbf{a} + 2\mathbf{b}B\mathbf{a} - 4b\mathbf{B}\mathbf{a} \\
\delta_{\mathbf{B}}\Phi &= 2bB\mathbf{b} + 4LL\mathbf{b} + 2aA\mathbf{b} - 2b\mathbf{B}b - 4L\mathbf{L}b + 2\mathbf{a}A\mathbf{b} - 4\mathbf{a}Ab \\
\delta_{B}\Phi &= 2\mathbf{b}\mathbf{B}b + 4\mathbf{L}\mathbf{L}b + 2\mathbf{a}\mathbf{A}b - 2\mathbf{b}B\mathbf{b} - 4\mathbf{L}Lb + 2a\mathbf{A}b - 4a\mathbf{A}\mathbf{b} \\
\delta_{\mathbf{a}}\Phi &= 2A\mathbf{a}A - 2\mathbf{A}aA - 4A\mathbf{L}L + 4\mathbf{A}LL + 4Ab\mathbf{B} - 2A\mathbf{b}B - 2\mathbf{A}bB \\
\delta_{a}\Phi &= 2\mathbf{A}a\mathbf{A} - 2A\mathbf{a}\mathbf{A} - 4\mathbf{A}L\mathbf{L} + 4A\mathbf{L}\mathbf{L} + 4\mathbf{A}\mathbf{b}B - 2\mathbf{A}b\mathbf{B} - 2A\mathbf{b}\mathbf{B} \\
\delta_{\mathbf{b}}\Phi &= 2B\mathbf{b}B - 2\mathbf{B}bB - 4B\mathbf{L}L + 4\mathbf{B}LL + 4Ba\mathbf{A} - 2B\mathbf{a}A - 2\mathbf{B}aA \\
\delta_{b}\Phi &= 2\mathbf{B}b\mathbf{B} - 2B\mathbf{b}\mathbf{B} - 4\mathbf{B}L\mathbf{L} + 4B\mathbf{L}\mathbf{L} + 4\mathbf{B}\mathbf{a}A - 2\mathbf{B}a\mathbf{A} - 2B\mathbf{a}\mathbf{A} \\
\delta_{\mathbf{L}}\Phi &= -4L\mathbf{a}A + 4\mathbf{L}aA - 4aA\mathbf{L} + 4a\mathbf{A}L - 8\mathbf{L}LL + 8LL\mathbf{L} + 4\mathbf{L}bB - 4bB\mathbf{L} + 4b\mathbf{B}L - 4L\mathbf{b}B \\
\delta_{L}\Phi &= -4\mathbf{L}a\mathbf{A} + 4L\mathbf{a}\mathbf{A} - 4\mathbf{a}\mathbf{A}L + 4\mathbf{a}A\mathbf{L} - 8L\mathbf{L}\mathbf{L} + 8\mathbf{L}\mathbf{L}L + 4L\mathbf{b}\mathbf{B} - 4\mathbf{b}\mathbf{B}L + 4\mathbf{b}B\mathbf{L} - 4\mathbf{L}b\mathbf{B}
\end{aligned}$$

and go on to calculate the second differentials which tell us generators for the relations. Note there is redundancy amongst these generating relations.

$$\begin{array}{cccc}
\delta_{\mathbf{A}\mathbf{a}}\Phi = 0 & \delta_{\mathbf{A}a}\Phi = 2A\mathbf{a} - 2\mathbf{A}a & \delta_{A\mathbf{a}}\Phi = -2A\mathbf{a} + 2\mathbf{A}a & \delta_{Aa}\Phi = 0 \\
\delta_{\mathbf{A}\mathbf{b}}\Phi = -4Ba & \delta_{\mathbf{A}b}\Phi = 2B\mathbf{a} + 2\mathbf{B}a & \delta_{A\mathbf{b}}\Phi = 2B\mathbf{a} + 2\mathbf{B}a & \delta_{Ab}\Phi = -4Ba \\
\delta_{\mathbf{A}\mathbf{L}}\Phi = 0 & \delta_{\mathbf{A}L}\Phi = -4L\mathbf{a} + 4\mathbf{L}a & \delta_{A\mathbf{L}}\Phi = 4L\mathbf{a} - 4\mathbf{L}a & \delta_{AL}\Phi = 0
\end{array}$$

$$\begin{array}{cccc}
\delta_{\mathbf{B}\mathbf{a}}\Phi = -4Ab & \delta_{\mathbf{B}a}\Phi = 2A\mathbf{b} + 2\mathbf{A}b & \delta_{B\mathbf{a}}\Phi = 2A\mathbf{b} + 2\mathbf{A}b & \delta_{Ba}\Phi = -4\mathbf{A}\mathbf{b} \\
\delta_{\mathbf{B}\mathbf{b}}\Phi = 0 & \delta_{\mathbf{B}b}\Phi = 2B\mathbf{b} - 2\mathbf{B}b & \delta_{B\mathbf{b}}\Phi = -2B\mathbf{b} + 2\mathbf{B}b & \delta_{Bb}\Phi = 0 \\
\delta_{\mathbf{B}\mathbf{L}}\Phi = 0 & \delta_{\mathbf{B}L}\Phi = 4L\mathbf{b} - 4\mathbf{L}b & \delta_{B\mathbf{L}}\Phi = -4L\mathbf{b} + 4\mathbf{L}b & \delta_{BL}\Phi = 0
\end{array}$$

$$\begin{aligned}
\delta_{\mathbf{La}}\Phi &= 0 & \delta_{\mathbf{L}a}\Phi &= -4A\mathbf{L} + 4\mathbf{A}L \\
\delta_{\mathbf{Lb}}\Phi &= 0 & \delta_{\mathbf{L}b}\Phi &= -4B\mathbf{L} + 4\mathbf{B}L \\
\delta_{\mathbf{LL}}\Phi &= 4aA{+}4bB{-}8LL & \delta_{\mathbf{L}L}\Phi &= -4\mathbf{a}A{+}8L\mathbf{L}{-}4\mathbf{b}B
\end{aligned}$$

$$\begin{aligned}
\delta_{L\mathbf{a}}\Phi &= 4A\mathbf{L} - 4\mathbf{A}L & \delta_{La}\Phi &= 0 \\
\delta_{L\mathbf{b}}\Phi &= 4B\mathbf{L} - 4\mathbf{B}L & \delta_{Lb}\Phi &= 0 \\
\delta_{L\mathbf{L}}\Phi &= -4a\mathbf{A}{+}8\mathbf{L}L{-}4b\mathbf{B} & \delta_{LL}\Phi &= 4\mathbf{aA}{+}4\mathbf{bB}{-}8\mathbf{LL}
\end{aligned}$$

$$\begin{aligned}
\delta_{\mathbf{aA}}\Phi &= -2aA{+}4LL{-}2bB & \delta_{a\mathbf{A}}\Phi &= 2a\mathbf{A}{-}4L\mathbf{L}{-}2b\mathbf{B}{+}4\mathbf{b}B \\
\delta_{\mathbf{a}A}\Phi &= 2\mathbf{a}A{-}4\mathbf{L}L{+}4b\mathbf{B}{-}2\mathbf{b}B & \delta_{aA}\Phi &= -2\mathbf{aA}{+}4\mathbf{LL}{-}2\mathbf{bB} \\
\delta_{\mathbf{bB}}\Phi &= -2aA{+}4LL{-}2bB & \delta_{b\mathbf{B}}\Phi &= 2b\mathbf{B}{-}4L\mathbf{L}{-}2a\mathbf{A}{+}4\mathbf{a}A \\
\delta_{\mathbf{b}B}\Phi &= 2\mathbf{b}B{-}4\mathbf{L}L{+}4a\mathbf{A}{-}2\mathbf{a}A & \delta_{bB}\Phi &= -2\mathbf{aA}{+}4\mathbf{LL}{-}2\mathbf{bB}
\end{aligned}$$

Any other unmentioned derivatives are 0. We have our quiver Q, superpotential Φ_4 and relations $\Lambda = \mathcal{W}_2$ such that $A := D(\Phi_4, 2) = \frac{\mathbb{C}Q}{\Lambda}$. Note that for x, y arrows the $\delta_{xy}\Phi$ are not linearly independent; in fact many are 0.

We now calculate the zeroPBW deformations, which by Theorem 3.17 correspond to 2-coherent superpotentials $\Phi' = \Phi_4 + \phi_2$. Writing $\phi_2 = \sum c_{xy}xy$ the PBW deformations are parametrised by the c_{xy} such that Φ' is a 2-coherent superpotential. We see that Φ' is a superpotential if $c_{xy} = -c_{yx}$ for all arrows x, y. A superpotential Φ' defines a map

$$\begin{aligned}
\Lambda &\to \Pi \\
\delta_{xy}\Phi_4 &\mapsto \delta_{xy}\Phi_4 + c_{xy}e_{h(x)}e_{t(y)}
\end{aligned}$$

and Φ' is a 2-coherent superpotential if the $\delta_{xy}\Phi' = \delta_{xy} + c_{xy}e_{h(x)}$ satisfy the linear relations that the $\delta_{xy}\Phi_4$ did. For instance we require that $c_{Aa} = 0$ and that $c_{A\mathbf{a}} = -c_{\mathbf{A}a}$.

Making these calculations in this example we see the only non-zero c and dependency relations, are

$$\begin{aligned}
c_{\mathbf{a}A} &= -c_{A\mathbf{a}} = c_{\mathbf{A}a} = -c_{a\mathbf{A}} \\
c_{\mathbf{b}B} &= -c_{B\mathbf{b}} = c_{\mathbf{B}b} = -c_{b\mathbf{B}} \\
c_{\mathbf{a}A} + c_{\mathbf{b}B} &= c_{\mathbf{L}L} = -c_{L\mathbf{L}}.
\end{aligned}$$

So there are 2 degrees of freedom in our parameters, exactly as for the $t, \mathbf{c}$ in $H_{t,\mathbf{c}}$ for S_3 acting on $\mathfrak{h} \oplus \mathfrak{h}^*$.

Example 3.19. Consider the symplectic reflection algebra corresponding to the dihedral group of order 8, G, acting on $\mathfrak{h} \oplus \mathfrak{h}$ in the manner of a rational Cherednik algebra. This acts on $\mathfrak{h}$ by

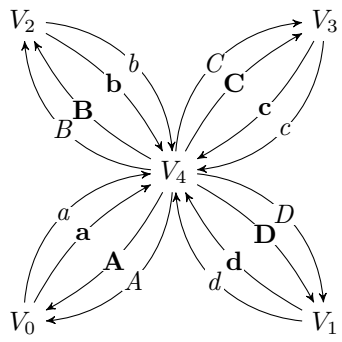
$$G = \left\langle \sigma = \begin{pmatrix} \varepsilon_4 & 0 \\ 0 & \varepsilon_4^3 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

where ε_4 is a primitive fourth root of unity.

This has the character table

G	ccl(e)	ccl(σ)	ccl(σ^2)	ccl(τ)	ccl$(\tau\sigma)$
ch V_0	1	1	1	1	1
ch V_1	1	1	1	-1	-1
ch V_2	1	-1	1	1	-1
ch V_3	1	-1	1	-1	1
ch V_4	2	0	-2	0	0

with $V_4 = \mathfrak{h}$ and $V = \mathfrak{h} \oplus \mathfrak{h}^*$. As such the McKay quiver is



and by choosing a G-equivariant basis we calculate the superpotential

$$\begin{aligned}\Phi = \quad & -A\mathbf{a}A\mathbf{a} + 2\mathbf{A}aA\mathbf{a} - 4Ad\mathbf{Da} + 2A\mathbf{d}D\mathbf{a} + 2\mathbf{A}dD\mathbf{a} + 2A\mathbf{b}B\mathbf{a} - 2\mathbf{A}bB\mathbf{a} + 2A\mathbf{c}C\mathbf{a} \\ & -2\mathbf{A}cC\mathbf{a} - \mathbf{A}a\mathbf{A}a - 4\mathbf{Ad}Da + 2\mathbf{A}d\mathbf{D}a + 2A\mathbf{d}\mathbf{D}a + 2\mathbf{A}b\mathbf{B}a - 2A\mathbf{b}\mathbf{B}a + 2\mathbf{A}c\mathbf{C}a \\ & -2A\mathbf{c}\mathbf{C}a - D\mathbf{d}D\mathbf{d} + 2\mathbf{D}dD\mathbf{d} + 2D\mathbf{b}B\mathbf{d} - 2\mathbf{D}bB\mathbf{d} + 2D\mathbf{c}C\mathbf{d} - 2\mathbf{D}cC\mathbf{d} - \mathbf{D}d\mathbf{D}d \\ & +2\mathbf{D}b\mathbf{B}d - 2D\mathbf{b}\mathbf{B}d + 2\mathbf{D}c\mathbf{C}d - 2D\mathbf{c}\mathbf{C}d - B\mathbf{b}B\mathbf{b} + 2\mathbf{B}bB\mathbf{b} - 4Bc\mathbf{C}\mathbf{b} + 2B\mathbf{c}C\mathbf{b} \\ & +2\mathbf{B}cC\mathbf{b} - 4\mathbf{Bc}Cb - \mathbf{B}b\mathbf{B}b + 2\mathbf{B}c\mathbf{C}b + 2B\mathbf{c}\mathbf{C}b - C\mathbf{c}C\mathbf{c} + 2\mathbf{C}cC\mathbf{c} - \mathbf{C}c\mathbf{C}c \\ & +\text{cyclic permutations.}\end{aligned}$$

Since $n = 4$, we differentiate twice in order to calculate the relations and the possible PBW deformations.

$$\begin{array}{cccc}
\delta_{\mathbf{Aa}}\Phi = 0 & \delta_{\mathbf{A}a}\Phi = 2A\mathbf{a} - 2\mathbf{A}a & \delta_{A\mathbf{a}}\Phi = -2A\mathbf{a} + 2\mathbf{A}a & \delta_{Aa}\Phi = 0 \\
\delta_{\mathbf{Ab}}\Phi = 0 & \delta_{\mathbf{A}b}\Phi = -2B\mathbf{a} + 2\mathbf{B}a & \delta_{A\mathbf{b}}\Phi = 2B\mathbf{a} - 2\mathbf{B}a & \delta_{Ab}\Phi = 0 \\
\delta_{\mathbf{Ac}}\Phi = 0 & \delta_{\mathbf{A}c}\Phi = -2C\mathbf{a} + 2\mathbf{C}a & \delta_{A\mathbf{c}}\Phi = 2C\mathbf{a} - 2\mathbf{C}a & \delta_{Ac}\Phi = 0 \\
\delta_{\mathbf{Ad}}\Phi = -4Da & \delta_{\mathbf{A}d}\Phi = 2D\mathbf{a} + 2\mathbf{D}a & \delta_{A\mathbf{d}}\Phi = 2D\mathbf{a} + 2\mathbf{D}a & \delta_{Ad}\Phi = -4\mathbf{Da} \\
\\
\delta_{\mathbf{Ba}}\Phi = 0 & \delta_{\mathbf{B}a}\Phi = -2A\mathbf{b} + 2\mathbf{A}b & \delta_{B\mathbf{a}}\Phi = 2A\mathbf{b} - 2\mathbf{A}b & \delta_{Ba}\Phi = 0 \\
\delta_{\mathbf{Bb}}\Phi = 0 & \delta_{\mathbf{B}b}\Phi = 2B\mathbf{b} - 2\mathbf{B}b & \delta_{B\mathbf{b}}\Phi = -2B\mathbf{b} + 2\mathbf{B}b & \delta_{Bb}\Phi = 0 \\
\delta_{\mathbf{Bc}}\Phi = -4Cb & \delta_{\mathbf{B}c}\Phi = 2C\mathbf{b} + 2\mathbf{C}b & \delta_{B\mathbf{c}}\Phi = 2C\mathbf{b} + 2\mathbf{C}b & \delta_{Bc}\Phi = -4\mathbf{Cb} \\
\delta_{\mathbf{Bd}}\Phi = 0 & \delta_{\mathbf{B}d}\Phi = -2D\mathbf{b} + 2\mathbf{D}b & \delta_{B\mathbf{d}}\Phi = 2D\mathbf{b} - 2\mathbf{D}b & \delta_{Bd}\Phi = 0 \\
\\
\delta_{\mathbf{Ca}}\Phi = 0 & \delta_{\mathbf{C}a}\Phi = -2A\mathbf{c} + 2\mathbf{A}c & \delta_{C\mathbf{a}}\Phi = 2A\mathbf{c} - 2\mathbf{A}c & \delta_{Ca}\Phi = 0 \\
\delta_{\mathbf{Cb}}\Phi = -4Bc & \delta_{\mathbf{C}b}\Phi = 2B\mathbf{c} + 2\mathbf{B}c & \delta_{C\mathbf{b}}\Phi = 2B\mathbf{c} + 2\mathbf{B}c & \delta_{Cb}\Phi = -4\mathbf{Bc} \\
\delta_{\mathbf{Cc}}\Phi = 0 & \delta_{\mathbf{C}c}\Phi = 2C\mathbf{c} - 2\mathbf{C}c & \delta_{C\mathbf{c}}\Phi = -2C\mathbf{c} + 2\mathbf{C}c & \delta_{Cc}\Phi = 0 \\
\delta_{\mathbf{Cd}}\Phi = 0 & \delta_{\mathbf{C}d}\Phi = -2D\mathbf{c} + 2\mathbf{D}c & \delta_{C\mathbf{d}}\Phi = 2D\mathbf{c} - 2\mathbf{D}c & \delta_{Cd}\Phi = 0 \\
\\
\delta_{\mathbf{Da}}\Phi = -4Ad & \delta_{\mathbf{D}a}\Phi = 2A\mathbf{d} + 2\mathbf{A}d & \delta_{D\mathbf{a}}\Phi = 2A\mathbf{d} + 2\mathbf{A}d & \delta_{Da}\Phi = -4\mathbf{Ad} \\
\delta_{\mathbf{Db}}\Phi = 0 & \delta_{\mathbf{D}b}\Phi = -2B\mathbf{d} + 2\mathbf{B}d & \delta_{D\mathbf{b}}\Phi = 2B\mathbf{d} - 2\mathbf{B}d & \delta_{Db}\Phi = 0 \\
\delta_{\mathbf{Dc}}\Phi = 0 & \delta_{\mathbf{D}c}\Phi = -2C\mathbf{d} + 2\mathbf{C}d & \delta_{D\mathbf{c}}\Phi = 2C\mathbf{d} - 2\mathbf{C}d & \delta_{Dc}\Phi = 0 \\
\delta_{\mathbf{Dd}}\Phi = 0 & \delta_{\mathbf{D}d}\Phi = 2D\mathbf{d} - 2\mathbf{D}d & \delta_{D\mathbf{d}}\Phi = -2D\mathbf{d} + 2\mathbf{D}d & \delta_{Dd}\Phi = 0
\end{array}$$

$$\delta_{\mathbf{aA}}\Phi = -2aA - 2dD + 2bB + 2cC \qquad \delta_{a\mathbf{A}}\Phi = 2a\mathbf{A} - 2d\mathbf{D} - 2b\mathbf{B} - 2c\mathbf{C} + 4\mathbf{d}D$$
$$\delta_{\mathbf{a}A}\Phi = 2\mathbf{a}A - 2\mathbf{d}D - 2\mathbf{b}B - 2\mathbf{c}C + 4d\mathbf{D} \qquad \delta_{aA}\Phi = -2\mathbf{aA} - 2\mathbf{dD} + 2\mathbf{bB} + 2\mathbf{cC}$$

$$\delta_{\mathbf{bB}}\Phi = 2aA + 2dD - 2bB - 2cC \qquad \delta_{b\mathbf{B}}\Phi = -2a\mathbf{A} - 2d\mathbf{D} + 2b\mathbf{B} - 2c\mathbf{C} + 4\mathbf{c}C$$
$$\delta_{\mathbf{b}B}\Phi = -2\mathbf{a}A - 2\mathbf{d}D + 2\mathbf{b}B - 2\mathbf{c}C + 4c\mathbf{C} \qquad \delta_{bB}\Phi = 2\mathbf{aA} + 2\mathbf{dD} - 2\mathbf{bB} - 2\mathbf{cC}$$

$$\delta_{\mathbf{cC}}\Phi = 2aA + 2dD - 2bB - 2cC \qquad \delta_{c\mathbf{C}}\Phi = -2a\mathbf{A} - 2d\mathbf{D} - 2b\mathbf{B} + 2c\mathbf{C} + 4\mathbf{b}B$$
$$\delta_{\mathbf{c}C}\Phi = -2\mathbf{a}A - 2\mathbf{d}D - 2\mathbf{b}B + 2\mathbf{c}C + 4b\mathbf{B} \qquad \delta_{cC}\Phi = 2\mathbf{aA} + 2\mathbf{dD} - 2\mathbf{bB} - 2\mathbf{cC}$$

$$\delta_{\mathbf{dD}}\Phi = -2aA - 2dD + 2bB + 2cC \qquad \delta_{d\mathbf{D}}\Phi = -2a\mathbf{A} + 2d\mathbf{D} - 2b\mathbf{B} - 2c\mathbf{C} + 4\mathbf{a}A$$
$$\delta_{\mathbf{d}D}\Phi = -2\mathbf{a}A + 2\mathbf{d}D - 2\mathbf{b}B - 2\mathbf{c}C + 4a\mathbf{A} \qquad \delta_{dD}\Phi = -2\mathbf{aA} - 2\mathbf{dD} + 2\mathbf{bB} + 2\mathbf{cC}$$

Any other unmentioned derivatives are 0. So we have our quiver Q, superpotential Φ_4 and relations $\Lambda = \mathcal{W}_2$ such that $A := D(\Phi_4, 2) = \frac{\mathbb{C}Q}{\Lambda}$. We now look for deformations parametrised by c_{xy} with map

$$\begin{array}{ccc} \Lambda & \to & \Pi \\ \delta_{xy}\Phi_4 & \mapsto & \delta_{xy}\Phi_4 + c_{xy}e_{h(a)} \end{array}$$

such that the c_{xy} define the deformation $\frac{\mathbb{C}Q}{\Pi}$ given by the 2-coherent inhomogeneous superpotential $\Phi_4 + \sum_{xy} c_{xy}xy$, as in Theorem 3.11.

Making these calculations in this example we see that the only non-zero c_{xy}, and dependency relations, are

$$\begin{aligned} c_{\mathbf{a}A} &= -c_{A\mathbf{a}} = c_{\mathbf{A}a} = -c_{a\mathbf{A}} \\ c_{\mathbf{b}B} &= -c_{B\mathbf{b}} = c_{\mathbf{B}b} = -c_{b\mathbf{B}} \\ c_{\mathbf{c}C} &= -c_{C\mathbf{c}} = c_{\mathbf{C}c} = -c_{c\mathbf{C}} \\ c_{\mathbf{d}D} &= -c_{D\mathbf{d}} = c_{\mathbf{D}d} = -c_{d\mathbf{D}} \\ c_{\mathbf{a}A} + c_{\mathbf{d}D} &= -c_{b\mathbf{B}} - c_{c\mathbf{C}} = c_{\mathbf{b}B} + c_{\mathbf{c}C}. \end{aligned}$$

Hence there are 3 degrees of freedom in our parameters, exactly as for the parameters t and $\mathbf{c}$ in the symplectic reflection algebra for the dihedral group of order 8 acting on $\mathfrak{h} \oplus \mathfrak{h}^*$.

Example 3.20. A special case of path algebras Morita equivalent to symplectic reflection algebras are the deformed preprojective algebras of [CBH98]. These can be given as superpotential algebras in the $n = 2$, differentiation by paths of length 0, case.

We consider a skew group algebras $\mathbb{C}[x, y] \rtimes G$ for G is a finite subgroup of $\mathrm{Sp}_2(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C})$. Construct the McKay quiver and label the arrows in a particular way; between any two vertices we choose a direction, label the arrows in this direction $a_1, ..., a_k$, and the arrows in the opposite direction $a_1^*, \ldots a_k^*$. Then $\mathbb{C}[x, y] \rtimes G$ is Morita equivalent to $A = \mathcal{D}(\Phi_2, 0)$ for the homogeneous superpotential, $\Phi_2 = \sum[a, a^*]$. This is the preprojective algebra

$$A = \mathcal{D}(\Phi_2, 0) = \frac{\mathbb{C}Q}{\sum[a, a^*]}$$

as appeared previously in Proposition 1.3.

Now we consider the PBW deformations of A. We apply Theorem 3.17, and deduce PBW deformations correspond to 0-coherent inhomogeneous superpotentials, $\Phi_2 + \phi_0$. We note that $\phi_0 := -\sum_{i \in Q_0} \lambda_i e_i \in S$ can in fact be arbitrary as any element of S satisfies the superpotential property, and the 0-coherent property is always satisfied. Hence we recover that PBW deformations of the preprojective algebra are precisely the deformed preprojective algebras that appeared in Definition 1.13

$$\mathcal{A} = \mathcal{D}(\Phi_2 + \phi_0, 0) = \frac{\mathbb{C}Q}{\sum [a, a^*] - \sum \lambda_i e_i},$$

which are parametrised by a scalar, λ_i, for each vertex, i. By Theorem 3.17, these are 2-CY.

It is possible to interpret the parameter t of the symplectic reflection algebras in these superpotential examples, where it corresponds to a one-dimensional subspace in the parameter space of coherent superpotentials. This can be identified by considering finite dimensional representations. First we note that representations isomorphic to $\mathbb{C}G$ only exist when $t = 0$.

Lemma 3.21. *Suppose $H_{t,\mathbf{c}}$ has a representation isomorphic to $\mathbb{C}G$ as a $\mathbb{C}G$ module. Then $t = 0$.*

Proof. If such a representation appears the matrices involved must satisfy the relations for $H_{t,\mathbf{c}}$. In particular, by taking the trace of these relations and noting that the character of the regular representation evaluates to 0 everywhere but at the identity where it is $|G|$ we find $0 = \mathrm{tr}([x, y]) = \mathrm{tr}(t\omega(x, y) - \Sigma_s \omega_s(x, y)\mathbf{c}(s)s) = t\omega(x, y)|G|$. For non-trivial ω, $\omega(x, y) \neq 0$ for some x and y hence $t = 0$. □

Such a representation always exists. Let $t = 0$, and define the centre of $H_{0,\mathbf{c}}$ to be $Z_{0,\mathbf{c}}$.

Lemma 3.22. *There always exists a representation of $H_{0,\mathbf{c}}$ isomorphic to $\mathbb{C}G$.*

Proof. $H_{0,\mathbf{c}}$ is a prime, finitely generated, $\mathbb{C}$-algebra, which is finitely generated as a module over $Z_{0,\mathbf{c}}$ [Bro03, 4.4,4.5]. Hence the Azumaya locus is non-empty [BG02, III.1.7], and so there exists a representation of dimension $|G|$ [MRS01, 13.7.14]. By [EG02, Theorem 1.7] any representation of dimension $|G|$ is isomorphic to $\mathbb{C}G$. □

Hence if we identify in our parameter space a subspace of dimension one for which there are no representations isomorphic to $\mathbb{C}G$ then this space corresponds to the parameter t. We do this in the three examples considered above.

Example 3.23. For S_3 with deformed relations, calculated in Example 3.18, we find the following relation in the path algebra

$$\begin{aligned} 0 &= \delta_{\mathbf{A}a}\Phi' - \delta_{a\mathbf{A}}\Phi' + \delta_{\mathbf{a}A}\Phi' - \delta_{A\mathbf{a}}\Phi' + \delta_{\mathbf{B}b}\Phi' - \delta_{b\mathbf{B}}\Phi' + \delta_{\mathbf{b}B}\Phi' - \delta_{B\mathbf{b}}\Phi' \\ &= 4[A, \mathbf{a}] + 4[a, \mathbf{A}] + 4[b, \mathbf{B}] + 4[B, \mathbf{b}] + 8[L, \mathbf{L}] \\ &\quad -c_{\mathbf{A}a}e_{V_0} + c_{a\mathbf{A}}e_{V_2} - c_{\mathbf{a}A}e_{V_2} + c_{A\mathbf{a}}e_{V_0} - c_{\mathbf{B}b}e_{V_1} + c_{b\mathbf{B}}e_{V_2} - c_{\mathbf{b}B}e_{V_2} + c_{B\mathbf{b}}e_{V_1}. \end{aligned}$$

Hence, taking the trace of a regular representation and considering the dependency relations in Example 3.18, we find the condition

$$\begin{aligned} 0 &= c_{\mathbf{A}a} - 2c_{a\mathbf{A}} + 2c_{\mathbf{a}A} - c_{A\mathbf{a}} + c_{\mathbf{B}b} - 2c_{b\mathbf{B}} + 2c_{\mathbf{b}B} - c_{B\mathbf{b}} \\ &= 6c_{\mathbf{A}a} + 6c_{\mathbf{B}b} \end{aligned}$$

is necessary for the existence of a representation isomorphic to the regular representation, so the parameter t corresponds to the parameter $c_{\mathbf{A}a} + c_{\mathbf{B}b}$.

Example 3.24. For the dihedral group of order 8 with undeformed relations, calculated in Example 3.19, we find the following relation in the path algebra

$$\begin{aligned} 0 &= \delta_{\mathbf{A}a}\Phi - \delta_{a\mathbf{A}}\Phi + \delta_{\mathbf{a}A}\Phi - \delta_{A\mathbf{a}}\Phi + \delta_{\mathbf{B}b}\Phi - \delta_{b\mathbf{B}}\Phi + \delta_{\mathbf{b}B}\Phi - \delta_{B\mathbf{b}}\Phi \\ &\quad + \delta_{\mathbf{C}c}\Phi - \delta_{c\mathbf{C}}\Phi + \delta_{\mathbf{c}C}\Phi - \delta_{C\mathbf{c}}\Phi + \delta_{\mathbf{D}d}\Phi - \delta_{d\mathbf{D}}\Phi + \delta_{\mathbf{d}D}\Phi - \delta_{D\mathbf{d}}\Phi \\ &= 8[A,\mathbf{a}] + 8[a,\mathbf{A}] + 8[b,\mathbf{B}] + 8[B,\mathbf{b}] + 8[C,\mathbf{c}] + 8[c,\mathbf{C}] + 8[d,\mathbf{D}] + 8[D,\mathbf{d}]. \end{aligned}$$

Hence by taking traces for the deformation relations we find the condition

$$\begin{aligned} 0 &= c_{\mathbf{A}a} - 2c_{a\mathbf{A}} + 2c_{\mathbf{a}A} - c_{A\mathbf{a}} + c_{\mathbf{B}b} - 2c_{b\mathbf{B}} + 2c_{\mathbf{b}B} - c_{B\mathbf{b}} \\ &+ c_{\mathbf{C}c} - 2c_{c\mathbf{C}} + 2c_{\mathbf{c}C} - c_{C\mathbf{c}} + c_{\mathbf{D}d} - 2c_{d\mathbf{D}} + 2c_{\mathbf{d}D} - c_{D\mathbf{d}} \\ &= 6c_{\mathbf{A}a} + 6c_{\mathbf{B}b} + 6c_{\mathbf{C}c} + 6c_{\mathbf{D}d} \end{aligned}$$

is necessary for the existence representation isomorphic to the regular representation, and we see that t corresponds to the parameter $c_{\mathbf{A}a} + c_{\mathbf{B}b} + c_{\mathbf{C}c} + c_{\mathbf{D}d}$.

Example 3.25. Finally, for the deformed preprojective algebra we have the relation $\sum[\alpha, \alpha^*] = \sum_i \lambda_i e_i$ and so, taking traces, we see that a representation isomorphic to $\mathbb{C}G$ can only occur when $0 = \sum \lambda_i \dim V_i$. This is a result of Crawley-Boevey and Holland in [CBH98, Lemma 4.1].

3.5 Application: PBW deformations of skew group rings for $\mathrm{GL}_2(\mathbb{C})$

So far we have been considering subgroups of $\mathrm{SL}_n(\mathbb{C})$, requiring non-twisted superpotentials to calculate the corresponding skew group algebras. This section considers algebras Morita equivalent to $\mathbb{C}[x_1, \ldots, x_n] \rtimes G$ for G a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ and investigates when PBW deformations exist for such algebras.

We recall the following theorem.

Theorem 3.26 ([BSW10, 3.2, 6.1 and 6.8])**.** *Let $W = \mathbb{C}^n$, and G be a finite subgroup of $\mathrm{GL}(W)$. Then $\mathbb{C}[W] \rtimes G$ is Morita equivalent to $\mathcal{D}(\Phi_n, n-2)$ for the McKay quiver associated to $G < \mathrm{GL}(W)$, Φ_n a twisted homogeneous superpotential, and twist automorphism given by $(-) \otimes_{\mathbb{C}} \det W$. There is a recipe to construct the twisted superpotential.*

Working with homogeneous superpotentials, $\Phi_n = \sum c_p p$, we have $c_p = 0$ for any p that is not a closed path. This is no longer the case for twisted homogeneous superpotentials; here we find $c_p = 0$ unless $h(p) = \sigma(t(p))$. Since the twist for $\mathbb{C}[W] \rtimes G$ is given by tensoring by $\det W$ the coefficient c_p is nonzero only for paths from W_i to $\det W \otimes_{\mathbb{C}} W_i$, where the W_i are the irreducible representations corresponding to vertices in the McKay quiver.

There are two different cases of finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ we consider, those that contain pseudo-reflections, and those that do not. Those that do not are known as *small* subgroups.

As a particular case we will consider $\mathrm{GL}_2(\mathbb{C})$, where differentiation is by paths of length 0, and so our relations are a sum of paths with tail W_i and head $\det W \otimes_{\mathbb{C}} W_i$.

Theorem 3.27. *Let G be a small finite subgroup of $\mathrm{GL}_2(\mathbb{C})$, which is not contained in $\mathrm{SL}_2(\mathbb{C})$. Then $\mathbb{C}[x,y] \rtimes G$ has no nontrivial (relative to $\mathbb{C}G$) PBW deformations.*

Proof. The algebra $\mathbb{C}[x,y] \rtimes G$ can be written as $\frac{T_{\mathbb{C}G}(\mathbb{C}^{2*} \otimes_{\mathbb{C}} \mathbb{C}G)}{<[x,y]\otimes 1>}$ and is Morita equivalent to a path algebra with relations $\mathbb{C}Q/\Lambda = \frac{T_S(V)}{\Lambda} = \mathcal{D}(\Phi_2, 0)$ for some twisted homogeneous superpotential Φ_2, where we use notation as in as in Section 2.2.1. The Morita equivalence switches $\mathbb{C}G$ with S and respects the gradings and Koszul resolutions. Hence considering PBW deformations as in Section 3.2.2 we see that under the Morita equivalence any PBW deformation of $\mathbb{C}[x,y] \rtimes G$ would give a PBW deformation of $\mathbb{C}Q/\Lambda$, noting that in one case we are considering PBW deformations relative to $\mathbb{C}G$, and in the other relative to S.

Hence it is enough to show that the Morita equivalent twisted superpotential algebra $A := \mathcal{D}(\Phi_2, 0)$ has no nontrivial PBW deformations.

There can only possibly exist PBW deformations if there exists some non zero θ_0, θ_1 as in Section 3.2.2. But $\theta_1 \in \mathrm{Hom}_{S^e}(\Lambda, V)$ and $\theta_0 \in \mathrm{Hom}_{S^e}(\Lambda, S)$, so if both these sets are $\{0\}$ there are no nontrivial PBW deformations.

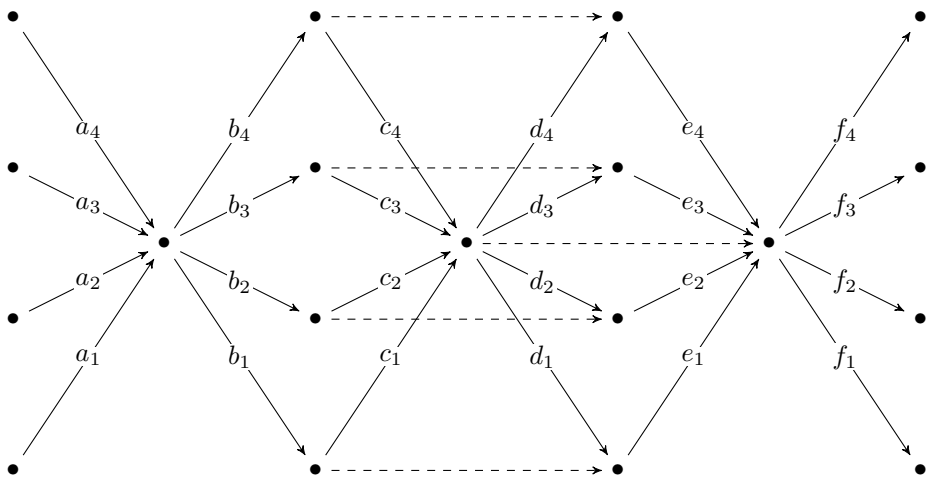
Define the *distance* between two vertices in the quiver to be the minimal length of a path from one to the other. It is shown in Lemma 3.30 below that the tail and head of any relation are vertices which are distance greater than one apart. Hence, as S^e module maps preserve heads and tails, the sets $\mathrm{Hom}_{S^e}(\Lambda, V)$ and $\mathrm{Hom}_{S^e}(\Lambda, S)$ are both $\{0\}$ and there are no nontrivial PBW deformations. □

We look at examples of a small and non small subgroup, using the calculations from [BSW10]. We let ε_m denote a primitive m^{th} root of unity.

Example 3.28. Consider the small subgroup $D_{5,2}$, which also appears in Section 1.3, with representation

$$\left\langle \begin{pmatrix} \varepsilon_4 & 0 \\ 0 & \varepsilon_4^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_4 \\ \varepsilon_4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_6 \\ \varepsilon_6 & 0 \end{pmatrix} \right\rangle < \mathrm{GL}_2(\mathbb{C}).$$

This is given as an example in [BSW10, Example 5.4]. It has McKay quiver



where the first column of vertices equals the final column of vertices. Tensoring by the determinant representation maps from a vertex to the next vertex in line to the right, as indicated by the dashed arrows, wrapping around from the right side of the diagram to the left.

The relations are generated by

$$b_i a_i = 0, \quad d_i c_i = 0, \quad f_i e_i = 0 \quad \text{for } i = 1, 2, 3, 4$$

and

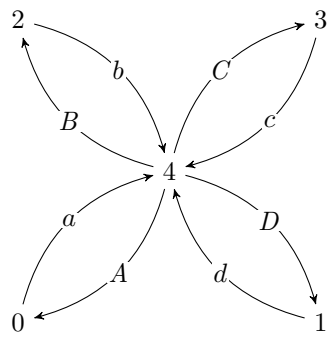
$$\sum_{j=1}^{4} c_j b_j = 0, \quad \sum_{j=1}^{4} e_j d_j = 0, \quad \sum_{j=1}^{4} a_j f_j = 0.$$

Consider the head and tail of any element of Λ. These vertices are related by the twist, and we see the shortest path between the tail and head is always length 2. Then $\mathrm{Hom}_{S^e}(\Lambda, V)$ and $\mathrm{Hom}_{S^e}(\Lambda, S)$ are both $\{0\}$, hence there can be no nontrivial PBW deformations.

Example 3.29. We now consider the dihedral group of order 8 as the non-small subgroup of $\mathrm{GL}_2(\mathbb{C})$ with representation

$$G = \left\langle g = \begin{pmatrix} \varepsilon_4 & 0 \\ 0 & \varepsilon_4^3 \end{pmatrix}, h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

and McKay quiver and relations, Q and Λ, displayed below.



$$Da = 0 \quad Cb = 0$$
$$Ad = 0 \quad Bc = 0$$
$$aA + dD - bB - cC = 0$$

Then PBW deformations of $\mathbb{C}Q/\Lambda$ are classified by $\theta_1, \theta_0 : \Lambda \to A$ which are S^e-module maps; $\theta_1 \in \mathrm{Hom}_{S^e}(\Lambda, V)$ and $\theta_0 \in \mathrm{Hom}_{S^e}(\Lambda, S)$.

As S^e-module maps preserve heads and tails we see that θ_1 must be zero, and θ_0 must be zero on all relations but the central one $aA+dD-bB-cC$. At this relation $\theta_0(aA+dD-bB-cC) = \lambda e_4$ for some $\lambda \in \mathbb{C}$. In this case $(\Lambda \otimes_S V) \cap (V \otimes_S \Lambda) = 0$, and so any such θ_0 gives us a PBW deformation. Hence there is a one parameter collection of PBW deformations.

We use the classification of McKay quivers for small finite subgroups of $\mathrm{GL}_2(\mathbb{C})$ [AR86] to prove the following lemma, which is used in the proof of Theorem 3.27.

Lemma 3.30. *Let $G < \mathrm{GL}_2(\mathbb{C})$ be a small finite subgroup, given by a representation $W \cong \mathbb{C}^2$. Let W_i be an irreducible representation of G. Then the shortest path from W_i to $\det W \otimes W_i$ has length ≥ 2.*

Proof. We outline this case by case by examining the McKay quivers, showing there are no length 0 or 1 paths between a vertex in the quiver and the vertex related by tensoring by the determinant.

We list the small finite subgroups of $\mathrm{GL}_2(\mathbb{C})$ up to conjugacy, as in [AR86, Section 2].

Let $Z_n = \left\langle g = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \right\rangle$, and ε be a primitive n^{th} root of unity. Any finite small subgroup of $\mathrm{GL}_2(\mathbb{C})$ is, up to conjugacy, one of the following:

1. Z a cyclic subgroup, $Z = \left\langle g = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^q \end{pmatrix} \right\rangle$ for $1 \leq q < n$.

2. $Z_n D = \{zd \mid z \in Z_n, d \in D\}$ for D a finite, non cyclic, subgroup of $\mathrm{SL}_2(\mathbb{C})$.

3. $\bar{H}$. To define $\bar{H}$ let $D < SL_2(\mathbb{C})$ be a binary dihedral group, with A a cyclic subgroup of index 2, and define $H < Z_{2n} \times D$ to be

$$H = \{(z, d) \in Z \times D \,|\, d + A = z + Z_m \text{ in } Z_2 = D/A = Z_{2n}/Z_n\}.$$

Then $\bar{H}$ is the image of H under the map $H \to GL_2(\mathbb{C})$.

4. $\bar{K}$. To define $\bar{K}$ let $D < SL_2(\mathbb{C})$ be the binary tetrahedral group, with A a normal binary dihedral subgroup of index 3, let $n \geq 3$, and define $K < Z_{3n} \times D$ to be

$$K = \{(z, d) \in Z \times D \,|\, d + A = z + Z_n \text{ in } Z_3 = D/A = Z_{3n}/Z_n\}.$$

Then $\bar{K}$ is the image of K under the map $K \to GL_2(\mathbb{C})$.

We note that if $n = 1, 2$ then $\bar{K}$ is the binary tetrahedral group with defining representation containing pseudo reflections, so is not small.

The McKay quivers for these groups are described in [AR86, Proposition 7], and we look at the determinant representation in each case and show tensoring by it relates vertices at least distance two apart.

We first consider cyclic subgroups. Let Z be as above. Such a representation is in $\mathrm{SL}_2(\mathbb{C})$ only when $q + 1 = n$. We suppose $q + 1 \neq n$.

Such a group has n irreducible one dimensional representations, which we label $W_0, \ldots W_{n-1}$, where W_i is given by $g \mapsto \varepsilon^i$. The defining representation is reducible as $\mathbb{C}^2 = W_1 \oplus W_q$ with $0 < q < n - 1$ as G is small and $q + 1 \neq n$, and its determinant is the representation W_{1+q}. Hence the McKay quiver has n vertices corresponding to the W_i and at vertex i has two arrows to vertices $i + 1$ and $i + q$ modulo n. The relations on the McKay quiver have head and tail related by tensoring by the determinant, hence any relations with tail W_i have head W_{i+q+1} module n. We see that the two vertices W_i and W_{i+q+1} are distance 2 apart; they are not distance 0 as $i \neq i + 1 + q$ module n, and they are not distance 1 as the only arrows from i are to $i + 1$ or $i + q$, and neither of these equals $i + 1 + q$ modulo n.

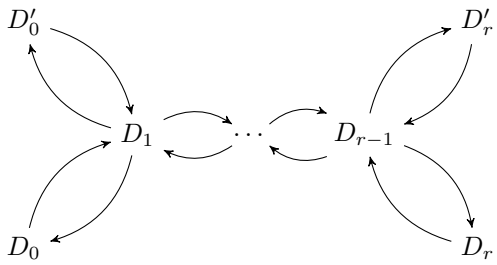
All the remaining groups are constructed by taking a subgroup of $Z_n \times D$ and then taking the image of this under the map to $\mathrm{GL}_2(\mathbb{C})$. If we calculate the McKay quiver for the subgroup then the image in $\mathrm{GL}_2(\mathbb{C})$ has McKay quiver which is a subquiver. Hence for our purposes it is enough to calculate the McKay quivers for the various subgroups of $Z_n \times D$.

We first do this for the case $Z_n D$. We note that this is contained in $\mathrm{SL}_2(\mathbb{C})$ for $n = 1, 2$, hence we assume $n > 2$. In this case we consider the McKay quiver of $Z_n \times D$. Let the irreducible representations of D be labeled $D_0, \ldots, D_{r-1}$ where D_0 is trivial, and D_1 is the given 2 dimensional representation. Let R_i, for $i = 0, \ldots n - 1$, be the n one dimensional irreducible

representations of Z_n with R_i given by $g \mapsto \varepsilon^i$. Then $Z_n \times D$ has nr irreducible representations given by $R_i \otimes D_j$ for $0 \leq i < n$ and $0 \leq j < r$. Then we consider the McKay quiver for the defining representation $R_1 \otimes D_1$, as this corresponds to the defining representation in $\mathrm{GL}_2(\mathbb{C})$. In particular, the McKay quiver has n groups of r representations labeled by representation of Z_n, with group i corresponding to the set of representations $\{R_i \otimes D_j \,|\, j = 0, \dots r-1\}$. By definition any arrows in the quiver go from group i to group $i+1$ modulo n. As the defining representation is $R_1 \otimes D_1$ the determinant representation is $R_2 \otimes D_0$ and the determinant maps from group i to group $i+2$ modulo n. In particular, as $n > 2$, any two vertices related by this are not distance zero or one apart.

For the cases $\bar{H}$ and $\bar{K}$ we take the McKay quiver for $Z_n \times D$, make some identifications to account for certain irreducible representations being identified for the subgroups H, K.

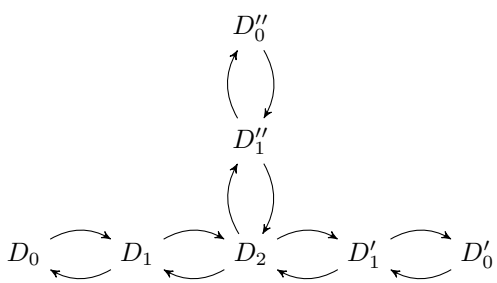
We first consider H. In this case we label the representations of the binary dihedral group as



where D_1 is the representation in $\mathrm{SL}_2(\mathbb{C})$ and D_0 is the trivial representation.

Now we label the representations of Z_{2n} as R_i for $i = 0 \dots 2n-1$ as above, and $Z_{2n} \times D$ has $2n(r+2)$ irreducible representations given by their tensor products. The defining representation in $\mathrm{GL}_2(\mathbb{C})$ is given as $R_1 \otimes D_1$, and hence the determinant representation is $R_2 \otimes D_0$. Now all representations of $Z_{2n} \times D$ are still irreducible for H but some are identified, [AR86, Proposition 7 (d)]. The representations which are identified are $R_i \otimes D_j$ with $R_{n+i} \otimes D_j$ for $j = 1 \dots r-1$, modulo $2n$, and $R_i \otimes D_j$ with $R_{n+i} \otimes D_j'$ modulo $2n$ for $j = 0, r$. The McKay quiver for H is the McKay quiver for $Z_{2n} \times D$ with these identifications made. In this case we group the irreducible representations into n groups labeled by the representation of Z_{2n} modulo n, so arrows go from group i to $i+1$ and determinant from group i to $i+2$ module n. Hence, for $n > 2$, the head and tail of two vertices related by the determinant are not distance zero or one apart. When $n = 1, 2$ then $H = D$ and then the representation is contained in $\mathrm{SL}_2(\mathbb{C})$.

The final case is to consider $K < Z_{3n} \times D$. Again we label the representations of Z_{3n} by R_i for $i = 0 \dots 3n-1$, and we label the representations of D by



where D_0 is the trivial representation, and D_1 the defining representation.

The representation defining the group in $\mathrm{GL}_2(\mathbb{C})$ is $R_1 \otimes D_1$, and the determinant representation is $R_2 \otimes D_0$.

Now all the irreducible representations for $Z_{3n} \times D$ remain so for K, however some are identified [AR86, Proposition 7 (f)]. This time triples are identified: $R_i \otimes D_j \cong R_{i+n} \otimes D'_j \cong R_{i+2n} \otimes D''_j$ modulo $3n$, for $j = 0, 1$ and $R_i \otimes D_2 \cong R_{i+n} \otimes D_2 \cong R_{i+2n} \otimes D_2$ modulo $3n$, as representations of K.

Once again we note that this splits the quiver into n groups, labelled by the representation of Z_{3n} module n, with arrows from group i to $i+1$ and determinant from i to $i+2$ modulo n. Hence, as $n > 2$, the determinant maps between vertices which are not distance 0 or 1 apart.

Hence for any small finite subgroup of $\mathrm{GL}_2(\mathbb{C})$ not contained in $\mathrm{SL}_2(\mathbb{C})$ the determinant in the McKay quiver maps between vertices which are distance greater than one apart.

□

We remark that, while we do not know a complete proof without the use of the classification assuming $W_i \neq \det W \otimes W_i$ there is an elementary character-theoretic proof of Lemma 3.30.

Lemma 3.31. *Let $G < \mathrm{GL}_2(\mathbb{C})$ be a small finite subgroup, given by a representation $W \cong \mathbb{C}^2$. Let W_i be an irreducible representation of G. Then, assuming $W_i \neq \det W \otimes W_i$, there are no arrows from W_i to $\det W \otimes W_i$ hence the shortest path from W_i to $\det W \otimes W_i$ has length ≥ 2.*

Proof. We suppose that G is a small finite subgroup of $\mathrm{GL}_2(\mathbb{C})$, and that $W_i \neq \det W \otimes W_i$. Hence there are no length 0 paths from W_i to $\det W \otimes W_i$. We show that there are no length one paths W_i to $\det W \otimes W_i$

We let $< -, - >$ denote the standard inner product for characters of G, let $\mathrm{ch}\, W$ denote the character of a representation, and suppose that the determinant representation of W has order d. We note that for any two characters A, B

$$< \mathrm{ch}(W)A, B >=< A, \mathrm{ch}(W)\,\mathrm{ch}(\det W)^{d-1}B >$$

as if $g \in G$ has representation $W(g)$ with eigenvalues ϵ_1, ϵ_2 then

$$\mathrm{tr}\, W(g^{-1}) = (\epsilon_1^{-1} + \epsilon_2^{-1}) = (\epsilon_1\epsilon_2)^{-1}(\epsilon_1 + \epsilon_2) = \mathrm{tr}(\det W(g))^{-1}\,\mathrm{tr}(W(g))$$

Suppose there were an arrow from W_i to $\det W \otimes W_i$, then W_i is a summand of $W_i \otimes W$ and

$$1 \leq a =< \mathrm{ch}(W_i)\,\mathrm{ch}(W), \mathrm{ch}(\det W)\,\mathrm{ch}(W_i) >=< \mathrm{ch}(W_i), \mathrm{ch}(W)\,\mathrm{ch}(W_i) > .$$

By considering dimensions we see $a = 1$, and so $W \otimes W_i = W_i \oplus \det(W) \otimes W_i$. Now suppose that $g \in G$, $W(g)$ has eigenvalues ϵ_1, ϵ_2, and $W_i(g)$ has eigenvalues $\mu_1, \dots \mu_r$. Then

$$(\epsilon_1 + \epsilon_2)(\mu_1 + \dots \mu_r) = \mathrm{ch}(W \otimes W_i)(g) = \mathrm{ch}(W_i \oplus \det(W) \otimes W_i)(g) = (1 + \epsilon_1\epsilon_2)(\mu_1 + \dots \mu_r).$$

As W_i is an irreducible representation, and G is nontrivial, we can choose $g \in G$ nontrivial such that $\mathrm{ch}\, W_i(g) = \mu_1 + \cdots + \mu_r \neq 0$. The eigenvalues of such a g satisfy $1 + \epsilon_1\epsilon_2 = \epsilon_1 + \epsilon_2$. Hence 1 is an eigenvalue of $W(g)$, so g is a pseudo reflection, which is a contradiction as G is small. □

Chapter 4

Quiver GIT for algebras defined by tilting bundles

This chapter contains the main results of the author's preprint [Kar14a]. Throughout this chapter all schemes will be over $\mathbb{C}$ and a variety will be a scheme which is separated, reduced, irreducible and of finite type over $\mathbb{C}$. We will often work in the generality of schemes, X, arising from projective morphisms $\pi : X \to \mathrm{Spec}(R)$ of finite type schemes over $\mathbb{C}$. Such schemes are quasi-projective over $\mathbb{C}$, and hence separated, so are a slight generalisation of varieties projective over an affine base in that they may not be reduced or irreducible. We recall that a finite type scheme is Noetherian, and that any morphism of Noetherian schemes is quasi-compact.

4.1 Introduction to Chapter 4

Any variety X equipped with a tilting bundle T induces a derived equivalence between the bounded derived category of coherent sheaves on X and the bounded derived category of finitely generated left modules for the algebra $A := \mathrm{End}_X(T)^{\mathrm{op}}$. This situation is similar to the case of an affine variety $\mathrm{Spec}(R)$ where we can construct the commutative algebra $R = \mathrm{End}_X(\mathcal{O}_X)^{\mathrm{op}}$ and there is an abelian equivalence between coherent sheaves on $\mathrm{Spec}(R)$ and finitely generated left R-modules. However, whereas in the affine case we can recover the variety $\mathrm{Spec}(R)$ from the algebra R, it is not so clear how to recover the variety X from the algebra A. One possibility is to present A as the path algebra of a quiver with relations, construct the quiver representation moduli space of A by quiver GIT for some dimension vector and stability condition, and attempt to relate this moduli space back to X.

While this approach may not work in general there are many examples where this is known to be successful, such as del Pezzo surfaces [Kin,CW13], minimal resolutions of Kleinian singularities [Kro89,CS98,CB00] as appear in the $\mathrm{SL}_2(\mathbb{C})$ McKay correspondence, and crepant resolutions of Gorenstein quotient singularities in dimension 3 [BKR01,CI04], which lead us to hope it may work in some other interesting settings.

In this chapter we will determine conditions for X to be a fine moduli space for the quiver representation moduli functor $\mathcal{F}_A$, (Section 4.2.3), and this will allow us to prove that X is a quiver GIT quotient for a specific stability condition and dimension vector in a large class of examples. These examples include applications to the minimal model Program and to

resolutions of rational surface singularities.

This problem was also considered by Bergman and Proudfoot, [BP08], who study embeddings of closed points and tangent spaces to show that a smooth variety is a connected component of the quiver GIT quotient for 'great' stability condition and dimension vector. However, their approach cannot be extended to singular varieties and it can be difficult to identify which conditions are 'great'. The methods developed in this chapter have the advantages of applying to singular varieties, such as those occurring in the minimal model Program, and allowing us to identify a specific stability condition and dimension vector in applications.

4.1.1 Comparing moduli functors

In developing methods to understand quiver GIT moduli functors we are inspired by the following result of Sekiya and Yamaura [SY13].

Theorem ([SY13, Theorem 4.20])**.** *Let B be an algebra with tilting module T. Define $A = \mathrm{End}_B(T)^{\mathrm{op}}$, suppose that both A and B are presented as path algebras of quivers with relations, and let $\mathcal{F}_A$ and $\mathcal{F}_B$ denote quiver representation moduli functors on A and B for some choice of stability conditions and dimension vectors. Then if the tilting equivalences*

$$\begin{tikzcd} D^b(B\text{-mod}) \arrow[r, bend left, "\mathbb{R}\mathrm{Hom}_B(T,-)"] & D^b(A\text{-mod}) \arrow[l, bend left, "T \otimes^{\mathbb{L}}_A (-)"] \end{tikzcd}$$

restrict to a bijection between $\mathcal{F}_B(\mathbb{C})$ and $\mathcal{F}_A(\mathbb{C})$ then $\mathcal{F}_B$ is naturally isomorphic to $\mathcal{F}_A$.

This leads us to the idea of working with a moduli functor for which X is a fine moduli space instead of working with X itself, and we then prove the following variant of Sekiya and Yamaura's result.

Theorem (Theorem 4.15)**.** *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of varieties. Suppose X is equipped with a tilting bundle T, define $A = \mathrm{End}_X(T)^{\mathrm{op}}$, and suppose that A is presented as a quiver with relations. Let $\mathcal{F}_A$ be a quiver representation moduli functor on A for some stability condition and dimension vector. Then if the tilting equivalences*

$$\begin{tikzcd} D^b(\mathrm{Coh}\, X) \arrow[r, bend left, "\mathbb{R}\mathrm{Hom}_X(T,-)"] & D^b(A\text{-mod}) \arrow[l, bend left, "T \otimes^{\mathbb{L}}_A (-)"] \end{tikzcd}$$

restrict to a bijection between $\mathcal{F}_X(\mathbb{C})$ and $\mathcal{F}_A(\mathbb{C})$ then $\mathcal{F}_X$ is naturally isomorphic to $\mathcal{F}_A$.

We recall the definitions of the moduli functors $\mathcal{F}_A$ and $\mathcal{F}_X$ in Sections 4.2.3 and 4.2.4. The moduli functor $\mathcal{F}_X$ is similar to the Hilbert functor of one point on a variety, which is well-known to be represented by X, but for lack of a reference in this setting we provide a proof.

Theorem (Theorem 4.17)**.** *Let $\pi : X \to \operatorname{Spec}(R)$ be a projective morphism of varieties. Then there is an natural isomorphism between the functor of points $\operatorname{Hom}_{\mathfrak{Sch}}(-, X)$ and the moduli functor $\mathcal{F}_X$. In particular X is a fine moduli space for $\mathcal{F}_X$.*

Combining these two results we have a method to show when a variety X with tilting bundle T can be recovered via quiver GIT as the fine moduli space for the quiver representation moduli functor for the algebra $A = \operatorname{End}_X(T)^{\mathrm{op}}$.

Corollary 4.1. *Let $\pi : X \to \operatorname{Spec}(R)$ be a projective map of varieties and suppose X has a tilting bundle T. Define $A = \operatorname{End}_X(T)^{\mathrm{op}}$, suppose that A is presented as a quiver with relations, and let $\mathcal{F}_A$ be a quiver representation moduli functor on A for some stability condition θ and dimension vector d. Then if the tilting equivalences*

$$\begin{tikzcd} D^b(\operatorname{Coh} X) \arrow[r, bend left, "\mathbb{R}\operatorname{Hom}_X(T,-)"] & D^b(A\text{-mod}) \arrow[l, bend left, "T \otimes^{\mathbb{L}}_A (-)"] \end{tikzcd}$$

restrict to a bijection between the skyscraper sheaves on X and the θ-semistable A-modules with dimension vector d then X is isomorphic to the quiver GIT quotient of A for the stability condition θ and dimension vector d.

4.1.2 Applications

To give an application of this theorem we need a class of varieties with tilting bundles and well-understood tilting equivalences. For this, we consider the situation arising in following theorem of Van den Bergh.

Theorem 4.2 ([VdB04b, Theorem A])**.** *Let $\pi : X \to \operatorname{Spec}(R)$ be a projective morphism of Noetherian schemes such that $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$ and π has fibres of dimension ≤ 1. Then there are tilting bundles T_0 and $T_1 = T_0^\vee$ on X such that the derived equivalences $\mathbb{R}\operatorname{Hom}_X(T_i, -) : D^b(\operatorname{Coh} X) \to D^b(A_i\text{-mod})$ restrict to equivalences of abelian categories between ${}^{-i}\operatorname{Per}(X/R)$ and A_i-mod, where $A_i = \operatorname{End}_X(T_i)^{\mathrm{op}}$.*

This gives us a large class of varieties with well-understood tilting equivalences. We recall the definition of ${}^{-i}\operatorname{Per}(X/R)$ for $i = 0, 1$ in Definition 4.25. We then show that in this situation there is a particular choice of dimension vector d_{T_0} and stability condition θ_{T_0} such that X occurs as the quiver GIT quotient of A_0.

Corollary (Corollary 4.28)**.** *Suppose we are in the situation of Theorem 4.2 and that X and $\operatorname{Spec}(R)$ are both varieties. Then X is isomorphic to the quiver GIT quotient of $A_0 = \operatorname{End}_X(T_0)^{\mathrm{op}}$ for dimension vector d_{T_0} and stability condition θ_{T_0}, and T_0 is the dual of the tautological bundle on X.*

See Section 4.5.1 for the definitions of θ_{T_0} and d_{T_0}. We note they are easy to define and depend only on a decomposition of T into indecomposable summands.

4.1.3 Applications to the minimal model program

The class of varieties in the above corollary includes flips and flops of dimension 3 in the minimal model program. In the setting of smooth, projective 3-folds flops were constructed as components of moduli spaces and shown to be derived equivalent in the work of Bridgeland [Bri02], and this work was extended to include projective 3-folds with Gorenstein terminal singularities by Chen [Che02]. These results were reinterpreted more generally via tilting bundles by Van den Bergh [VdB04b]. We can now reinterpret these results once again by combining Corollary 4.28 with Van den Bergh's results.

It is immediate from Corollary 4.28 that if $\pi : X \to \mathrm{Spec}(R)$ is either a flipping or flopping contraction with fibres of dimension ≤ 1 then both X and its flip/flop can be reconstructed as quiver GIT quotients. Further, in the case of flops, the following corollary shows that both X and its flop can be constructed as quiver GIT quotients of algebras arising from tilting bundles on X.

Corollary (Corollary 4.30)**.** *Suppose we are in the situation of Corollary 4.28 and that $\pi : X \to \mathrm{Spec}(R)$ is a flopping contraction with flop $\pi' : X' \to \mathrm{Spec}(R)$. Then X is the quiver GIT quotient of the algebra $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$ for dimension vector d_{T_0} and stability condition θ_{T_0}, and the flop X' is the quiver GIT quotient of the algebra $A_1 = \mathrm{End}_X(T_1)^{\mathrm{op}}$ for dimension vector d_{T_1} and stability condition θ_{T_1}.*

This fits into a general philosophy of having a preferred stability condition defined by a tilting bundle and realising all minimal models via quiver GIT by changing the tilting bundle rather than changing the stability condition.

4.1.4 Applications to resolutions of rational surface singularities

Minimal resolutions of affine rational surface singularities automatically satisfy the conditions of Corollary 4.28 hence provide another class of examples. These appear in the generalisation of the $\mathrm{SL}_2(\mathbb{C})$ McKay correspondence discussed in Section 1.3.

Corollary (Example 4.31)**.** *Suppose that $\pi : X \to \mathrm{Spec}(R)$ is the minimal resolution of a rational surface singularity. Then there is a tilting bundle T_0 on X such that X is the quiver GIT quotient of $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$ for dimension vector d_{T_0} and stability condition θ_{T_0}. Moreover, $T_0^\vee$ is the tautological bundle of this construction.*

For quotient surface singularities this result was already known when either $G < \mathrm{SL}_2(\mathbb{C})$ [CB00], or when G was a cyclic or dihedral subgroup of $\mathrm{GL}_2(\mathbb{C})$ [Cra11,Wem11a,Wem12, Wem13], but is new in all other cases. This corollary provides a moduli space interpretation with tilting tautological bundle for minimal resolutions of all rational surface singularities, expanding the moduli space interpretation as $G\text{-Hilb}^G(\mathbb{C}^2)$ for minimal resolutions of quotient surface singularities, see [IN96,Ish02], which only has a tilting tautological bundle when $G < \mathrm{SL}_2(\mathbb{C})$.

4.2 Moduli functors

In this section we recall the functor of points, moduli spaces, $G\text{-Hilb}(\mathbb{C}^2)$, and the moduli functor associated to the quiver GIT construction.

4.2.1 Functor of points and moduli spaces

We recall the definition of the functor of points and the definitions of fine and coarse moduli spaces. Let $\mathfrak{Sch}$ denote the category of finite type schemes over $\mathbb{C}$, let $\mathfrak{Sets}$ denote the category of sets, and let $\mathfrak{R}$ denote the category of finite type commutative $\mathbb{C}$-algebras. Suppose $X \in \mathfrak{Sch}$, then the functor of points for X is defined to be the functor

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{Sch}}(-, X) : \mathfrak{R} &\to \mathfrak{Sets} \\ S &\mapsto \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(S), X) \end{aligned}$$

and by Yoneda's lemma this gives an embedding of $\mathfrak{Sch}$ into the category of functors from $\mathfrak{R}$ to $\mathfrak{Sets}$.

A functor $\mathcal{F} : \mathfrak{R} \to \mathfrak{Sets}$ is *representable* if there is some $Y \in \mathfrak{Sch}$ and a natural isomorphism $\nu : \mathcal{F} \to \mathrm{Hom}_{\mathfrak{Sch}}(-, Y)$. Then Y is said to be a *fine moduli space* for $\mathcal{F}$. A scheme Y is said to be a *coarse moduli space* for $\mathcal{F}$ if there is a natural transformation $\nu : \mathcal{F} \to \mathrm{Hom}_{\mathfrak{Sch}}(-, Y)$ such that $\nu_{\mathbb{C}} : \mathcal{F}(\mathbb{C}) \to \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(\mathbb{C}), Y)$ is a bijection and for any scheme Y' with a natural transformation $\nu' : \mathcal{F} \to \mathrm{Hom}_{\mathfrak{Sch}}(-, Y')$ there is a unique morphism $Y \to Y'$ factoring ν' through ν.

Moduli functors and the functor of points could be defined in terms of functors $\mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Sets}$ rather than functors $\mathfrak{R} \to \mathfrak{Sets}$, however as schemes are defined by local affine structure there is a one to one correspondence between contravariant functors from $\mathfrak{Sch}$ to $\mathfrak{Sets}$ and covariant functors from $\mathfrak{R}$ to $\mathfrak{Sets}$ so either viewpoint is equivalent. We choose the one above to automatically simplify later arguments and definitions to considering affine cases. One advantage of the alternative description is that it is clear to see that if X is a fine moduli space for a moduli functor $\mathcal{F}$ then there is a *tautological element* in $\mathcal{F}(X)$ corresponding to $id \in \mathrm{Hom}_{\mathfrak{Sch}}(X, X)$ under the natural isomorphism.

4.2.2 G-Hilbert schemes

We recall the definitions of the Hilbert scheme of points on $\mathbb{C}^n$ and the G-Hilbert scheme for $G < \mathrm{GL}_n(\mathbb{C})$ as examples of fine moduli spaces.

The Hilbert scheme of r points on $\mathbb{C}^n$ parametrises zero dimensional subschemes $Z \subset \mathbb{C}^n$ such that $\mathrm{H}^0(\mathcal{O}_Z)$ has dimension r. This is equivalent to parametrising ideals $I < \mathbb{C}[x_1, \ldots, x_n]$ such that $\mathbb{C}[x_1, \ldots, x_n]/I$ has dimension r.

More precisely, $\mathrm{Hilb}^r(\mathbb{C}^n)$ is the fine moduli space for the moduli functor:

$$\begin{aligned} \mathcal{F} : \mathfrak{R} &\to \mathfrak{Sets} \\ R &\mapsto \left\{ \begin{array}{c} \text{Ideals } I \trianglelefteq \mathbb{C}[x_1, \ldots, x_n] \otimes_{\mathbb{C}} R \text{ such that} \\ M := \mathbb{C}[x_1, \ldots, x_n]/I \text{ is flat as an } R\text{-module} \\ \text{and } M \text{ has fibres over } R \text{ of dimension } r. \end{array} \right\}. \end{aligned}$$

A given finite group $G < \mathrm{GL}_n(\mathbb{C})$ defines an action of G on $\mathbb{C}^n$. The G-Hilbert scheme $G\text{-Hilb}(\mathbb{C}^n)$ is the subscheme of $\mathrm{Hilb}^{|G|}(\mathbb{C}^n)$ consisting of G *clusters*, which are defined to be G invariant subschemes $Z \subset \mathbb{C}^n$ such that $\mathrm{H}^0(\mathcal{O}_Z)$ is the regular representation of G. Equivalently, this parametrises ideals I of $\mathbb{C}[x_1, \ldots, x_n]$ such that $\mathbb{C}[x_1, \ldots, x_n]/I$ is isomorphic to the regular representation $\mathbb{C}G$ as a $\mathbb{C}G$-module.

More precisely G-Hilb$(\mathbb{C}^n)$ is the fine moduli space for the following moduli functor:

$$\mathcal{F} : \mathfrak{R} \to \mathfrak{Sets}$$
$$R \mapsto \left\{ \begin{array}{c} \text{Ideals } I \trianglelefteq \mathbb{C}[x_1, \ldots, x_n] \otimes_{\mathbb{C}} R \text{ such that} \\ M := \mathbb{C}[x_1, \ldots, x_n]/I \text{ is flat as an } R\text{-module} \\ \text{and } M \text{ has fibres over } R \text{ isomorphic to } \mathbb{C}G. \end{array} \right\}.$$

4.2.3 Quiver GIT and moduli functors

We recall the definition of a moduli functor of quiver representations for which the quiver GIT quotient is a moduli space. Let A be a $\mathbb{C}$-algebra of finite type. Suppose that A is presented as a quiver with relations and for $B \in \mathfrak{R}$ define $A^B := A \otimes_{\mathbb{C}} B$. We recall that left A-modules correspond to quiver representations. For a stability condition θ and dimension vector d the *quiver representation moduli functor* is defined as in [SY13, Definition 4.1],

$$\mathcal{F}^{(s)s}_{A,d,\theta} : \mathfrak{R} \to \mathfrak{Sets}$$
$$B \mapsto \left\{ M \in A^B\text{-mod} \;\middle|\; \begin{array}{l} \bullet\ M \text{ is a finitely generated and flat } B\text{-module.} \\ \bullet\ \text{The } A\text{-module } B/\mathfrak{m} \otimes_B M \text{ has} \\ \text{dimension vector } d \text{ and is } \theta\text{-(semi)stable} \\ \text{for all maximal ideals } \mathfrak{m} \text{ of } B. \end{array} \right\} \Big/ \sim$$

where the equivalence $\sim$ is defined by S-equivalence at fibres; M and N are equivalent if $B/\mathfrak{m} \otimes_B M$ and $B/\mathfrak{m} \otimes_B N$ are S-equivalent A-modules for all maximal ideal $\mathfrak{m}$ of B. By [SY13, Remark 4.4] this functor coincides with the definition of King in [Kin94], and hence this functor has a coarse moduli space.

Theorem 4.3 ([Kin94, Proposition 5.2])**.** *The quiver GIT quotient scheme $\mathcal{M}^{ss}_{d,\theta}$ is a coarse moduli space for $\mathcal{F}^{ss}_{A,d,\theta}$.*

If we restrict to stable representations then the functor has a fine moduli space.

Theorem 4.4 ([Kin94, Proposition 5.3])**.** *Suppose d is indivisible and let $\mathcal{M}^{s}_{d,\theta}$ be the open subscheme of $\mathcal{M}^{ss}_{d,\theta}$ corresponding to the stable points. Then $\mathcal{M}^{s}_{d,\theta}$ is a fine moduli space for $\mathcal{F}^{s}_{A,d,\theta}$.*

We note that when d is indivisible and all semistable points are stable the two functors coincide and $\mathcal{M}^{ss}_{d,\theta} = \mathcal{M}^{s}_{d,\theta}$ is a fine moduli space. We will often just refer to either functor as $\mathcal{F}_A$, recalling the choices of θ, d and semistability/stability only when necessary. We also note that the tautological element for $\mathcal{F}^{s}_{A,d,\theta}$ is a vector bundle on $\mathcal{M}^{s}_{d,\theta}$ with each fibre corresponding to a θ-stable representation of A with dimension vector d which we refer to as the *tautological bundle.*

4.2.4 Geometric moduli functors

We define a similar functor for a scheme, X, arising in a projective morphism, $\pi : X \to \mathrm{Spec}(R)$, of finite type schemes over $\mathbb{C}$.

We first introduce several pieces of notation which we will frequently use. Let $\rho : X \to \mathrm{Spec}(\mathbb{C})$ denote the structure morphism. For $B \in \mathfrak{R}$ we define $X^B := X \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(B)$ and

consider the following pullback diagram

$$\begin{tikzcd} X^B \arrow[r, "\rho^X"] \arrow[d, "\rho^B"'] & X \arrow[d, "\rho"] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

which defines the morphisms ρ^B and ρ^X from the structure morphism $\rho : X \to \mathrm{Spec}(\mathbb{C})$. We note that X^B is also of finite type over $\mathbb{C}$, and has a projective morphism $\pi^B : X^B \to \mathrm{Spec}(R \otimes_{\mathbb{C}} B)$, see [BH13, Remark 1.7]. Also, if X has a tilting bundle T the following result, which is a particular case of the result [BH13, Proposition 2.9] of Buchweitz and Hille, defines a tilting bundle T^B on X^B.

Proposition 4.5 ([BH13, Proposition 2.9])**.** *If T is a tilting bundle on X and $A = \mathrm{End}_X(T)^{\mathrm{op}}$ then $T^B := \mathbb{L}\rho^{X*}T$ is a tilting bundle on X^B, and $A^B := \mathrm{End}_{X^B}(T^B)^{\mathrm{op}} = A \otimes_{\mathbb{C}} B$.*

We introduce a further piece of notation. For any $B \in \mathfrak{R}$ we let $\mathrm{MaxSpec}(B)$ denote the closed points of $\mathrm{Spec}(B)$, and for any $p \in \mathrm{MaxSpec}(B)$ there is a closed immersion $i_p : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(B)$ and a pullback diagram

$$\begin{tikzcd} X \arrow[r, hook, "j_p"] \arrow[d, "\rho"'] & X^B \arrow[d, "\rho^B"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, hook, "i_p"] & \mathrm{Spec}(B) \end{tikzcd} \qquad (i_p/j_p)$$

which we later refer to as the diagram (i_p/j_p).

We can now define the geometric moduli functor. We define $\mathcal{F}_X(\mathbb{C})$ to be the set of skyscraper sheaves of X considered up to isomorphism, and define the moduli functor

$$\begin{aligned} &\mathcal{F}_X : \mathfrak{R} \to \mathfrak{Sets} \\ &B \mapsto \left\{ \mathcal{E} \in D^b(X^B) \;\middle|\; \begin{array}{l} \bullet\ \mathbb{L}j_p^*\mathcal{E} \in \mathcal{F}_X(\mathbb{C}) \text{ for all } p \in \mathrm{MaxSpec}(B). \\ \bullet\ \mathbb{R}\rho_*^B \mathcal{RH}\mathrm{om}_{X^B}(\mathbb{L}\rho^{X*}\mathcal{F}, \mathcal{E}) \in \mathrm{Perf}(B) \text{ for all } \mathcal{F} \in \mathrm{Perf}(X). \end{array} \right\} \Big/ \sim \end{aligned}$$

where the equivalence $\sim$ is defined by equivalence at fibres; $\mathcal{E}_1$ is equivalent to $\mathcal{E}_2$ if $\mathbb{L}j_p^*\mathcal{E}_1$ is equivalent to $\mathbb{L}j_p^*\mathcal{E}_2$ in $\mathcal{F}_X(\mathbb{C})$ for all $p \in \mathrm{MaxSpec}(B)$. We later prove in Theorem 4.17 that X is a fine moduli space for this functor, and in Lemma 4.16 iii) we show that the definition of fibrewise equivalence is the same as defining $\mathcal{E}_1$ and $\mathcal{E}_2$ to be equivalent if there exists a line bundle L on $\mathrm{Spec}(B)$ such that $\rho^{B*}L \otimes_{X^B} \mathcal{E}_1 \cong \mathcal{E}_2$.

Remark 4.6. It follows immediately from Lemmas 4.7 and 4.8, which we state below, that if X has a tilting bundle T the set $\mathcal{F}_X(B)$ is equivalent to the set

$$\left\{ \mathcal{E} \in \mathrm{Coh}(X^B) \;\middle|\; \begin{array}{l} \bullet\ \mathcal{E} \text{ is flat as a } B\text{-module.} \\ \bullet\ j_p^*\mathcal{E} \in \mathcal{F}_X(\mathbb{C}) \text{ for all } p \in \mathrm{MaxSpec}(B). \\ \bullet\ \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \in \mathrm{Perf}(B). \end{array} \right\} \Big/ \sim .$$

Lemma 4.7. *Suppose X has a tilting bundle T. Then for $\mathcal{E} \in D^b(X^B)$ the condition*

- $\mathbb{R}\rho^B_* \mathcal{RH}\mathrm{om}_{X^B}(\mathbb{L}\rho^{X*}\mathcal{F}, \mathcal{E}) \in \mathrm{Perf}(B)$ *for any* $\mathcal{F} \in \mathrm{Perf}(X)$

is equivalent to the condition

- $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \in \mathrm{Perf}(B)$.

Proof. Define $\mathfrak{T}$ to be the subset of $\mathrm{Perf}(X)$ consisting of objects $\mathcal{G} \in \mathrm{Perf}(X)$ such that $\mathbb{R}\mathrm{Hom}_{X^B}(\mathbb{L}\rho^{B*}\mathcal{G}, \mathcal{E}) \in \mathrm{Perf}(B)$. Then $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \in \mathrm{Perf}(B)$ if and only if $T \in \mathfrak{T}$. By [Nee92, Lemma 2.2] as T is a tilting bundle and $\mathfrak{T}$ is closed under shifts, triangles, and direct summands $\mathfrak{T}$ contains T if and only if $\mathfrak{T} = \mathrm{Perf}(X)$. □

We recall the following lemma.

Lemma 4.8 ([Bri99, Lemma 4.3])**.** *Let $f : X \to Y$ be a morphism of finite type schemes over $\mathbb{C}$, and for each closed point $y \in Y$ let j_y denote the inclusion of the fibre $f^{-1}(y)$. Suppose $\mathcal{E} \in D^b(X)$ is such that $\mathbb{L}j_y^*\mathcal{E}$ is a sheaf for all y. Then $\mathcal{E}$ is a coherent sheaf on X which is flat over Y.*

Remark 4.9. In the definition of the moduli functor $\mathcal{F}_X$ we could change the set $\mathcal{F}_X(\mathbb{C})$ of skyscraper sheaves up to isomorphism to, for example, the set of perverse point sheaves as defined by Bridgeland, [Bri02, Section 3], to obtain a functor mirroring Bridgeland's perverse point sheaf moduli functor. Indeed, the results of Section 4.3 and Theorem 4.15 do not rely on the fact that $\mathcal{F}_X(\mathbb{C})$ consists of skyscraper sheaves up to isomorphism, but Theorem 4.17 and our applications in Section 4.5 do.

4.3 Base change lemmas for moduli functors

In this section we give a series of lemmas required to prove the main results in the next section.

4.3.1 Derived base change

We first recall the following property, which we will make use of several times.

Lemma 4.10. *Let $f : X \to Y$ be a quasi-compact, separated morphism of Noetherian schemes over $\mathbb{C}$. Then if $T \in \mathrm{Perf}(Y)$*

$$\mathbb{L}f^*\mathcal{RH}\mathrm{om}_Y(T, \mathcal{E}) \cong \mathcal{RH}\mathrm{om}_X(\mathbb{L}f^*T, \mathbb{L}f^*\mathcal{E}).$$

for any $\mathcal{E} \in D^b(Y)$.

Proof. We consider the two functors

$$\mathrm{Hom}_{D^b(X)}(\mathbb{L}f^*\mathcal{RH}\mathrm{om}_Y(T, \mathcal{E}), -) : D^b(X) \to \mathfrak{Sets}, \text{ and}$$
$$\mathrm{Hom}_{D^b(X)}(\mathcal{RH}\mathrm{om}_X(\mathbb{L}f^*T, \mathbb{L}f^*\mathcal{E}), -) : D^b(X) \to \mathfrak{Sets}.$$

We will show these are naturally isomorphic, hence that $\mathbb{L}f^*\mathcal{RH}\mathrm{om}_Y(T, \mathcal{E}) \cong \mathcal{RH}\mathrm{om}_X(\mathbb{L}f^*T, \mathbb{L}f^*\mathcal{E})$ as they represent the same functor under the Yoneda embedding. This follows from the chain

of natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{D(X)}(\mathbb{L}f^*\mathcal{RH}\mathrm{om}_Y(T,\mathcal{E}),-) &\cong \mathrm{Hom}_{D(Y)}(\mathcal{RH}\mathrm{om}_Y(T,\mathcal{E}),\mathbb{R}f_*(-)) && \text{(adjunction)}\\
&\cong \mathrm{Hom}_{D(Y)}(\mathcal{E},T\otimes^{\mathbb{L}}_Y \mathbb{R}f_*(-)) && (T \text{ perfect})\\
&\cong \mathrm{Hom}_{D(Y)}(\mathcal{E},\mathbb{R}f_*(\mathbb{L}f^*T\otimes^{\mathbb{L}}_X(-))) && \text{(projection)}\\
&\cong \mathrm{Hom}_{D(X)}(\mathbb{L}f^*\mathcal{E},\mathbb{L}f^*T\otimes^{\mathbb{L}}_Y(-)) && \text{(adjunction)}\\
&\cong \mathrm{Hom}_{D(X)}(\mathcal{RH}\mathrm{om}_X(\mathbb{L}f^*T,\mathbb{L}f^*\mathcal{E}),-). && (\mathbb{L}f^*T \text{ perfect})
\end{aligned}$$

□

We then recall the following derived base change results.

Lemma 4.11. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$, and let $B, C \in \mathfrak{R}$. Consider the following pullback diagram for a morphism $u : \mathrm{Spec}(B) \to \mathrm{Spec}(C)$, where we use the notation of Section 4.2.4.*

$$\begin{tikzcd}
X^B \arrow[r, "v"] \arrow[d, "\rho^B"'] & X^C \arrow[d, "\rho^C"] \\
\mathrm{Spec}(B) \arrow[r, "u"] & \mathrm{Spec}(C)
\end{tikzcd}$$

Suppose $\mathcal{E} \in D^b(X^C)$. Then

$$\mathbb{L}u^*\mathbb{R}\rho^C_*\mathcal{E} \cong \mathbb{R}\rho^B_*\mathbb{L}v^*\mathcal{E}.$$

Suppose further that X has a tilting bundle T and define $A = \mathrm{End}_X(T)^{\mathrm{op}}$. If $\mathbb{R}\mathrm{Hom}_{X^C}(T^C,\mathcal{E})$ is an A^C-module which is flat as a C-module then

$$B \otimes_C \mathbb{R}\mathrm{Hom}_{X^C}(T^C,\mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B,\mathbb{L}v^*\mathcal{E})$$

as A^B-modules.

Proof. As X^C is flat over $\mathrm{Spec}(C)$, for any $x \in X^C$ and any $b \in \mathrm{Spec}(B)$ such that $\rho^C(x) = u(b) = c$ we have that $\mathrm{Tor}_i^{\mathcal{O}_{C,c}}(\mathcal{O}_{B,b},\mathcal{O}_{X^C,x}) = 0$ for all $i \neq 0$. Hence X^C and $\mathrm{Spec}(B)$ are Tor independent over $\mathrm{Spec}(C)$, and so the first result follows from [Sta, Lemma 35.16.3(Tag 08IB)].

The second result follows by applying the first result and the previous lemma:

$$\begin{aligned}
B \otimes_C \mathbb{R}\mathrm{Hom}_{X^C}(T^C,\mathcal{E}) &\cong \mathbb{L}u^*\mathbb{R}\rho^C_*\mathcal{RH}\mathrm{om}_{X^C}(T^C,\mathcal{E})\\
&\cong \mathbb{R}\rho^B_*\mathbb{L}v^*\mathcal{RH}\mathrm{om}_{X^C}(T^C,\mathcal{E}) && (\mathbb{L}u^*\mathbb{R}\rho^C_* \cong \mathbb{R}\rho^B_*\mathbb{L}v^*)\\
&\cong \mathbb{R}\rho^B_*\mathcal{RH}\mathrm{om}_{X^B}(\mathbb{L}v^*T^C,\mathbb{L}v^*\mathcal{E}) && \text{(Lemma 4.10)}\\
&\cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B,\mathbb{L}v^*\mathcal{E}).
\end{aligned}$$

□

The following corollary is also useful.

Corollary 4.12. *Let X be a scheme of finite type over $\mathbb{C}$, and let $B \in \mathfrak{R}$. Suppose that $\mathcal{E} \in D^b(X^B)$ is such that $\mathbb{R}\rho^B_*\mathcal{E} \in D^b(B)$, and that for any $p \in \mathrm{MaxSpec}(B)$ with diagram*

(i_p/j_p) *we have that* $\mathbb{R}\rho_*\mathbb{L}j_p^*\mathcal{E}$ *is a coherent sheaf on* $\mathrm{Spec}(\mathbb{C})$. *Then* $\mathbb{R}\rho_*^B\mathcal{E}$ *is a flat coherent sheaf on* $\mathrm{Spec}(B)$.

Proof. By assumption $\mathbb{R}\rho_*^B\mathcal{E} \in D^b(\mathrm{Spec}(B))$, hence by Lemma 4.8 if $\mathbb{L}i_p^*\mathbb{R}\rho_*^B\mathcal{E}$ is a sheaf for all embeddings of closed points, $i_p : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(B)$, then $\mathbb{R}\rho_*^B\mathcal{E}$ is a flat coherent sheaf on $\mathrm{Spec}(B)$. For any such $p \in \mathrm{MaxSpec}(B)$ by Theorem 4.11 $\mathbb{L}i_p^*\mathbb{R}\rho_*^B\mathcal{E} \cong \mathbb{R}\rho_*\mathbb{L}j_p^*\mathcal{E}$ which is a sheaf by the hypothesis. □

4.3.2 Natural transformations

In this section let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Suppose that X has a tilting bundle T and that $A = \mathrm{End}_X(T)^{\mathrm{op}}$ is presented as a quiver with relations. Choose some stability condition θ and dimension vector d in order to define $\mathcal{F}_A$. We aim to define a natural transformation, η, between the moduli functors $\mathcal{F}_X$ and $\mathcal{F}_A$ defined in sections 4.2.4 and 4.2.3. We define $\eta : \mathcal{F}_X \to \mathcal{F}_A$ by

$$\begin{aligned} \eta_B : \mathcal{F}_X(B) &\to \mathcal{F}_A(B) \\ \mathcal{E} &\mapsto \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \end{aligned}$$

for any $B \in \mathfrak{R}$, and we must check when this is well defined.

Lemma 4.13. *Suppose* $\eta_{\mathbb{C}}$ *is well defined. Then* η *is well defined and is a natural transformation.*

Proof. To prove that η is well defined we must check the following for any $B \in \mathfrak{R}$ and any $\mathcal{E} \in \mathcal{F}_X(B)$:

i) $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E})$ is a B-module which is flat and finitely generated.

ii) For all maximal ideals $\mathfrak{m}$ of B the A-module $B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E})$ is in $\mathcal{F}_A(\mathbb{C})$.

iii) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent in $\mathcal{F}_X(B)$ then $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{E}_1)$ and $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{E}_2)$ are equivalent in $\mathcal{F}_A(B)$.

Firstly we check i). It follows from the definition of $\mathcal{F}_X(B)$ that $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \in \mathrm{Perf}(B) \subset D^b(B)$. Then by Lemma 4.12 if $\mathbb{R}\rho_*\mathbb{L}j_p^*\mathcal{R}\mathcal{H}\mathrm{om}_{X^B}(T^B, \mathcal{E})$ is a sheaf on $\mathrm{Spec}(\mathbb{C})$ for all $p \in \mathrm{MaxSpec}(B)$ with diagrams (i_p/j_p) then

$$\mathbb{R}\rho_*^B\mathcal{R}\mathcal{H}\mathrm{om}_{X^B}(T^B, \mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E})$$

is a flat and finitely generated B-module. For all $p \in \mathrm{MaxSpec}(B)$ with diagrams (i_p/j_p)

$$\mathbb{R}\rho_*\mathbb{L}j_p^*\mathcal{R}\mathcal{H}\mathrm{om}_{X^B}(T^B, \mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E}),$$

by Lemma 4.10 and $\mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E}) \in \mathcal{F}_A(\mathbb{C})$ as $\mathbb{L}j_p^*\mathcal{E} \in \mathcal{F}_X(\mathbb{C})$ by the definition of $\mathcal{F}_X(B)$ and $\eta_{\mathbb{C}}$ is well defined. Hence $\mathbb{R}\rho_*\mathbb{L}j_p^*\mathcal{R}\mathcal{H}\mathrm{om}_{X^B}(T^B, \mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E})$ is a coherent sheaf on $\mathrm{Spec}(\mathbb{C})$, so we have proved i).

Secondly, to prove ii), we note for any maximal ideal $\mathfrak{m}$ of B we have a corresponding closed point $p \in \mathrm{MaxSpec}(B)$ and diagram (i_p/j_p). Then we assume that $\mathcal{E} \in \mathcal{F}_X(B)$, and for each maximal ideal we have $B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E})$ by Lemma 4.11 as

$\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E})$ is a flat B module. Hence $B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \in \mathcal{F}_A(\mathbb{C})$ as $\eta_{\mathbb{C}}$ is well defined and $\mathbb{L}j_p^*\mathcal{E} \in \mathcal{F}_X(\mathbb{C})$ by the definition of $\mathcal{F}_X(B)$.

Similarly, any maximal ideal $\mathfrak{m}$ of B defines a closed point $p \in \mathrm{MaxSpec}(B)$ and a diagram (i_p/j_p). Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be equivalent elements of $\mathcal{F}_X(B)$, then as $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_i)$ are flat B-modules

$$B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_i) \cong \mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E}_i)$$

by Lemma 4.11. As $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent in $\mathcal{F}_X(B)$ we know that $\mathbb{L}j_p^*\mathcal{E}_1 \cong \mathbb{L}j_p^*\mathcal{E}_2$, and hence the two A-modules $B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_1)$ and $B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_2)$ are S-equivalent as $\eta_{\mathbb{C}}$ is well defined. This shows that $\eta_B(\mathcal{E}_1)$ and $\eta_B(\mathcal{E}_2)$ are equivalent in $\mathcal{F}_X(B)$ and proves part iii).

We now show that η is a natural transformation. Suppose that $B, C \in \mathfrak{R}$ and $u : \mathrm{Spec}(B) \to \mathrm{Spec}(C)$, then we have the base change diagram

$$\begin{tikzcd} X^B \arrow[r, "v"] \arrow[d, "\rho^B"'] & X^C \arrow[d, "\rho^C"] \\ \mathrm{Spec}(B) \arrow[r, "u"] & \mathrm{Spec}(C) \end{tikzcd}$$

and we consider the diagram

$$\begin{tikzcd} \mathcal{F}_X(C) \arrow[r, "\mathbb{R}\mathrm{Hom}_{X^C}(T^C, -)"] \arrow[d, "\mathbb{L}v^*"'] & \mathcal{F}_A(C) \arrow[d, "B \otimes_C (-)"] \\ \mathcal{F}_X(B) \arrow[r, "\mathbb{R}\mathrm{Hom}_{X^B}(T^B, -)"] & \mathcal{F}_A(B) \end{tikzcd}$$

and to show that η is natural we must check that this commutes. For $\mathcal{E} \in \mathcal{F}_X(C)$

$$B \otimes_C \mathbb{R}\mathrm{Hom}_{X^C}(T^C, \mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathbb{L}v^*\mathcal{E})$$

as A^B-modules by Lemma 4.11 as $\mathbb{R}\mathrm{Hom}_{X^C}(T^C, \mathcal{E})$ is a flat C-module. Hence η is natural. □

Lemma 4.14. *With the assumptions as in Lemma 4.13, if $\eta_{\mathbb{C}}$ is also injective then η_B is injective for all $B \in \mathfrak{R}$. If $\eta_{\mathbb{C}}$ is also bijective with inverse $T \otimes_A^{\mathbb{L}} (-)$, then η_B is bijective for all $B \in \mathfrak{R}$.*

Proof. Let $B \in \mathfrak{R}$. We first assume that $\eta_{\mathbb{C}}$ is injective and show this implies that η_B is injective. We suppose that $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{F}_X(B)$ and $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_1)$ is equivalent to $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_2)$, hence for all maximal ideals $\mathfrak{m}$ of B the A-modules $B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_1)$ and $B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_2)$ are S-equivalent. We then note that each $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_i)$ is a flat B-module, and that any maximal ideal $\mathfrak{m}$ of B defines a closed point $p \in \mathrm{MaxSpec}(B)$ and diagram (i_p/j_p), so by Lemma 4.11

$$\mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E}_i) \cong B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_i)$$

as A-modules. Hence $\mathbb{L}j_p^*\mathcal{E}_1 \cong \mathbb{L}j_p^*\mathcal{E}_2$ for all $p \in \mathrm{MaxSpec}(B)$ by the injectivity of $\eta_{\mathbb{C}}$, so $\mathcal{E}_1$ is

equivalent to $\mathcal{E}_2$ in $\mathcal{F}_X(B)$.

We now suppose that $\eta_{\mathbb{C}}$ is bijective with inverse $T \otimes^{\mathbb{L}}_A (-)$, and we show that η_B is surjective. We consider $M \in \mathcal{F}_A(B)$ and note that as T^B is a tilting bundle there exists some $\mathcal{E} \in D^b(X^B)$ such that $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \cong M$. If we can show that $\mathcal{E} \in \mathcal{F}_X(B)$ then we have proved that η_B is surjective. We check first that $\mathbb{L}j_p^*\mathcal{E} \in \mathcal{F}_X(\mathbb{C})$ for any $p \in \mathrm{MaxSpec}(B)$ and diagram (i_p/j_p), and then we check that $\mathbb{R}\mathrm{Hom}_{X^B}(\mathbb{L}\rho^{B*}\mathcal{G}, \mathcal{E}) \in \mathrm{Perf}(B)$ for any $\mathcal{G} \in \mathrm{Perf}(X)$.

For any maximal ideal $\mathfrak{m}$ of B there is a corresponding closed point $p \in \mathrm{MaxSpec}(B)$ and diagram (i_p/j_p), and by Lemma 4.11

$$B/\mathfrak{m} \otimes_B M \cong \mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E})$$

as M is flat over B. As $B/\mathfrak{m} \otimes_B M \cong \mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E}) \in \mathcal{F}_A(\mathbb{C})$ and $\eta_{\mathbb{C}}$ is bijective with inverse $T \otimes^{\mathbb{L}}_A (-)$ it follows that $\mathbb{L}j_p^*\mathcal{E} \cong T \otimes^{\mathbb{L}}_A \mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E}) \in \mathcal{F}_X(\mathbb{C})$.

The second condition holds by Lemma 4.7 as M is a flat and finitely generated B-module so $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \cong M \in \mathrm{Perf}(B)$. Hence $\mathcal{E} \in \mathcal{F}_X(B)$ and η_B is surjective. □

4.4 Results comparing geometric and quiver representation moduli functors

In this section we state our main result, which follows from the previous lemmas, and we also show that the moduli functor $\mathcal{F}_X$ is represented by X. We will find several applications of these results in the next section.

Theorem 4.15. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Suppose X is equipped with a tilting bundle T, define $A = \mathrm{End}_X(T)^{\mathrm{op}}$, and suppose that A is presented as a quiver with relations. If there exists a stability condition θ and dimension vector d defining the moduli functor $\mathcal{F}_A := \mathcal{F}^{ss}_{A,\theta,d}$ such that the tilting equivalence*

$$\begin{tikzcd} D^b(X) \arrow[r, bend left, "\mathbb{R}\mathrm{Hom}_X(T,-)"] & D^b(A) \arrow[l, bend left, "T \otimes^{\mathbb{L}}_A (-)"] \end{tikzcd}$$

restricts to a bijection between $\mathcal{F}_X(\mathbb{C})$ and $\mathcal{F}_A(\mathbb{C})$, then the map $\eta : \mathcal{F}_X \to \mathcal{F}_A$ defined by $\eta_B : \mathcal{E} \mapsto \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E})$ is a natural isomorphism.

Proof. This follows from Lemmas 4.13 and 4.14. □

We now prove that the moduli functor $\mathcal{F}_X$ has X as a fine moduli space. This closely follows the proof of the more general result [CG13, Theorem 2.10] of Calabrese and Groechenig, which we split into the following lemma and theorem in our setting.

Lemma 4.16. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Suppose that $B \in \mathfrak{R}$ and that $\mathcal{E} \in \mathcal{F}_X(B)$. Then:*

i) $\mathcal{E}$ is a coherent sheaf on X^B that is flat over $\mathrm{Spec}(B)$, and $\mathbb{R}\rho^B_*\mathcal{E}$ is a line bundle on $\mathrm{Spec}(B)$.

ii) Let $\iota : Z \to X^B$ be the schematic support of $\mathcal{E}$. Then $\rho^B \circ \iota : Z \to \mathrm{Spec}(B)$ is an isomorphism.

iii) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent objects in $\mathcal{F}_X(B)$ then there exists a line bundle L on $\mathrm{Spec}(B)$ such that $\mathcal{E}_1 \otimes \rho^{B*}L \cong \mathcal{E}_2$.

Proof. Firstly, as $\mathcal{E} \in \mathcal{F}_X(B)$ it is a coherent sheaf on X^B which is flat over $\mathrm{Spec}(B)$ by Remark 4.6. Since $\mathcal{O}_X \in \mathrm{Perf}(X)$, by the definition of $\mathcal{F}_X(B)$ we know that $\mathbb{R}\rho^B_*\mathcal{E} = \mathbb{R}\mathrm{Hom}_{X^B}(\mathcal{O}_{X^B}, \mathcal{E}) \in \mathrm{Perf}(B) \subset D^b(B)$. It follows that $\mathbb{R}\rho^B_*\mathcal{E}$ is a flat coherent sheaf on $\mathrm{Spec}(B)$ by Corollary 4.12 as for all $p \in \mathrm{MaxSpec}(B)$ with diagrams (i_p/j_p) $\mathbb{R}\rho_*\mathbb{L}j_p^*\mathcal{E} = \mathbb{C}$ as $\mathbb{L}j_p^*\mathcal{E}$ is a skyscraper sheaf. As $\mathbb{L}i_p^*\mathbb{R}\rho^B_*\mathcal{E} = \mathbb{C}$ the flat coherent sheaf $\mathbb{R}\rho^B_*\mathcal{E}$ has rank 1 and is a line bundle on $\mathrm{Spec}(B)$.

To prove ii) let Z denote the schematic support of $\mathcal{E}$ with closed immersion $\iota : Z \to X^B$, and let $\mathcal{G} := \iota^*\mathcal{E}$ denote the sheaf on Z such that $\iota_*\mathcal{G} \cong \mathcal{E}$. We then have the diagram

$$\begin{tikzcd} Z \arrow[r, hook, "\iota"] \arrow[dr, "\psi"'] & X^B \arrow[r, "\rho^X"] \arrow[d, "\rho^B"] & X \arrow[d, "\rho"] \\ & \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

where we define $\psi = \rho^B \circ \iota$. We recall that X^B is projective over affine and ρ^B can be factored into

$$X^B \xrightarrow{\pi^B} \mathrm{Spec}(R \otimes_{\mathbb{C}} B) \xrightarrow{\alpha^B} \mathrm{Spec}(B)$$

where π^B is projective, and α^B is affine. We then see that as ι is a closed immersion, hence proper, $\pi^B \circ \iota$ is a proper map and it has affine fibres, as the fibres are all empty or points, so is an affine morphism by [Ryd, Theorem 8.5]. We then conclude that $\psi = \alpha^B \circ (\pi^B \circ \iota)$ is an affine morphism as it is the composition of two affine morphisms, in particular ψ_* is exact.

We recall that $\psi_*\mathcal{G}$ is defined as an $\mathcal{O}_B$-module via its definition as an $\psi_*\mathcal{O}_Z$-module by the map of rings

$$\mathcal{O}_B \to \psi_*\mathcal{O}_Z \to \mathrm{End}_{\psi_*\mathcal{O}_Z}(\psi_*\mathcal{G}) \to \mathrm{End}_{\mathcal{O}_B}(\psi_*\mathcal{G}).$$

Then as $\psi_*\mathcal{G} \cong \mathbb{R}\rho^B_*\mathcal{E}$ is a line bundle this series of maps composes to an isomorphism, hence the first map is injective and the last surjective. We also note that the last map is the forgetful map so is also injective, thus is an isomorphism. Hence the middle map is surjective. Then as the support of $\mathcal{G}$ is Z, the middle map is also injective, hence is an isomorphism, so in fact the first map must also be an isomorphism. In particular this implies $\mathcal{O}_B \cong \psi_*\mathcal{O}_Z$ and as ψ is affine it follows that $Z \cong \mathrm{Spec}(B)$ and ψ is an isomorphism.

To prove iii) we begin by noting that as $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent in $\mathcal{F}_X(B)$ they share the same support $\iota : Z \cong \mathrm{Spec}(B) \to X^B$ and there exists $\mathcal{G}_i := \iota^*\mathcal{E}_i$ such that $i_*\mathcal{G}_i \cong \mathcal{E}_i$. Hence, using the isomorphism of part ii), we see that the $\mathcal{G}_i$ are line bundles on $Z \cong \mathrm{Spec}(B)$, and we

define a line bundle $L = \psi_*(\mathcal{G}_1^\vee \otimes \mathcal{G}_2)$ on $\mathrm{Spec}(B)$. Then

$$\begin{aligned}
\mathcal{E}_2 &\cong \iota_* \mathcal{G}_2 \\
&\cong \iota_* \left(\mathcal{G}_1 \otimes (\mathcal{G}_1^\vee \otimes \mathcal{G}_2)\right) \\
&\cong \iota_* \left(\mathcal{G}_1 \otimes \iota^* \rho^{B*} L\right) && (\psi = \rho^B \circ \iota \text{ an isomorphism}) \\
&\cong \mathcal{E}_1 \otimes \rho^{B*} L, && (\text{projection formula})
\end{aligned}$$

□

Theorem 4.17. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Then there is a natural isomorphism between the functor of points $\mathrm{Hom}_{\mathfrak{Sch}}(-, X)$ and the moduli functor $\mathcal{F}_X$. In particular X is a fine moduli space for $\mathcal{F}_X$ with tautological bundle $\Delta_* \mathcal{O}_X$, where $\Delta : X \to X \times_{\mathrm{Spec}(\mathbb{C})} X$ is the diagonal map.*

Proof. Consider

$$\mu : \mathrm{Hom}_{\mathfrak{Sch}}(-, X) \to \mathcal{F}_X$$

defined by

$$\mu_C : (g : \mathrm{Spec}(C) \to X) \mapsto ((\Gamma_g)_* \mathcal{O}_C)$$

for $C \in \mathfrak{R}$, where $\Gamma_g : \mathrm{Spec}(C) \to X^C$ is the graph of g. The graph is a closed immersion as X is separated, and hence Γ_g is affine and $(\Gamma_g)_*$ is exact.

We now show this is a well defined natural transformation. To show that it is well defined we consider a morphism $g : \mathrm{Spec}(C) \to X$ and check that $(\Gamma_g)_* \mathcal{O}_C \in \mathcal{F}_X(C)$. Firstly, as Γ_g is a closed immersion it is proper, hence $(\Gamma_g)_* \mathcal{O}_C$ is a coherent sheaf [Sta, Lemma 29.17.2 (Tag 0205)]. Further, as Γ_g is a closed immersion and $\mathcal{O}_C$ is flat over $\mathrm{Spec}(C)$ it follows by considering stalks that $(\Gamma_g)_* \mathcal{O}_C$ is also flat over $\mathrm{Spec}(C)$. Then as Γ_g is affine $j_p^*(\Gamma_g)_* \mathcal{O}_C \cong (\Gamma_{g \circ i_p})_* i_p^* \mathcal{O}_C$ for all $p \in \mathrm{MaxSpec}(C)$ with diagrams (i_p / j_p) by [Sta, Lemma 29.5.1 (Tag 02KE)], hence

$$\mathbb{L} j_p^* (\Gamma_g)_* \mathcal{O}_C \cong j_p^* (\Gamma_g)_* \mathcal{O}_C \cong (\Gamma_{g \circ i_p})_* i_p^* \mathcal{O}_C \cong \mathcal{O}_{g(p)}.$$

Secondly, for any $\mathcal{F} \in \mathrm{Perf}(X)$ both $\mathbb{L} g^* \mathcal{F}$ and its derived dual $\mathbb{R}\mathrm{Hom}_C(\mathbb{L} g^* \mathcal{F}, \mathcal{O}_C)$ are in $\mathrm{Perf}(C)$ so $\mathbb{R}\mathrm{Hom}_{X^C}(\mathbb{L} \rho^{X*} \mathcal{F}, (\Gamma_g)_* \mathcal{O}_C) \cong \mathbb{R}\mathrm{Hom}_C(\mathbb{L} g^* \mathcal{F}, \mathcal{O}_C) \in \mathrm{Perf}(C)$. Hence μ_C is well defined as $(\Gamma_g)_* \mathcal{O}_{\mathrm{Spec}(C)} \in \mathcal{F}_A(C)$ for any $g \in \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(C), X)$. It is natural as if $B, C \in \mathfrak{R}$ with a morphism $u : \mathrm{Spec}(B) \to \mathrm{Spec}(C)$ and $g : \mathrm{Spec}(C) \to X \in \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(C), X)$ we have the diagram

$$\begin{tikzcd}
X^B \arrow[r, "v"] \arrow[d, shift left, "\rho^B"] & X^C \arrow[r, "\rho^X"] \arrow[d, shift left, "\rho^C"] & X \arrow[d, "\rho"] \\
\mathrm{Spec}(B) \arrow[r, "u"] \arrow[u, bend left, "\Gamma_{g \circ u}"] & \mathrm{Spec}(C) \arrow[r] \arrow[u, bend left, "\Gamma_g"] \arrow[ur, "g"] & \mathbb{C}
\end{tikzcd}$$

where $g = \rho^X \circ \Gamma_g$, $g \circ u = \rho^X \circ v \circ \Gamma_{g \circ u}$ and the squares can be seen to be pullback squares using the universal property of pullback squares and the fact that $\rho^B \circ \Gamma_{g \circ u}$ is the identity. As

above, since Γ_g and $\Gamma_{g\circ u}$ are closed immersions

$$(\Gamma_{g\circ u})_* u^*\mathcal{E} \cong v^*(\Gamma_g)_*\mathcal{E}$$

for any $\mathcal{E} \in \mathrm{Coh}(\mathrm{Spec}(C))$ by [Sta, Lemma 29.5.1 (Tag 02KE)]. Hence

$$\mu_B(g \circ u) \cong \Gamma_{(g\circ u)*}\mathcal{O}_B \cong \Gamma_{(g\circ u)*}u^*\mathcal{O}_C \cong v^*(\Gamma_g)_*\mathcal{O}_C \cong v^*\mu_B(g).$$

To show it is a natural isomorphism we need to check that μ_B is bijective for all $B \in \mathfrak{R}$. We do this now by constructing an inverse ν_B. For $B \in \mathfrak{R}$, given $\mathcal{E} \in \mathcal{F}_X(B)$ we consider its support Z, and we then have the diagram

$$\begin{tikzcd}
Z \arrow[r, hook, "\iota"] \arrow[dr, "\psi"'] & X^B \arrow[r, "\rho^X"] \arrow[d, "\rho^B"] & X \arrow[d, "\rho"] \\
& \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(\mathbb{C})
\end{tikzcd}$$

where we define $\psi = \rho^B \circ \iota$. We recall that ψ is an isomorphism from Lemma 4.16 ii), and we then consider the map $\rho^X \circ \iota \circ \psi^{-1} : \mathrm{Spec}(B) \to X \in \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(B), X)$, and our inverse is defined by sending $\mathcal{E} \in \mathcal{F}_X(B)$ to this element of $\mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(B), X)$:

$$\begin{aligned}
\nu_B : \mathcal{F}_X(B) &\to \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(B), X) \\
\mathcal{E} &\mapsto \left(\rho_X \circ \iota \circ \psi^{-1} : \mathrm{Spec}(B) \to X\right).
\end{aligned}$$

This is an inverse as

$$\nu_B \circ \mu_B(g) = \nu_B(\Gamma_{g_*}\mathcal{O}_B) = g$$

and

$$\mu_B \circ \nu_B(\mathcal{E}) = \mu_B\left((\rho_X \circ \iota \circ \psi^{-1}) : \mathrm{Spec}(B) \to X^B\right) = \Gamma_{(\rho_X \circ \iota \circ \psi^{-1})_*}(\mathcal{O}_B)$$

where we note that $\Gamma_{(\rho_X \circ \iota \circ \psi^{-1})_*}(\mathcal{O}_B)$ is equivalent to $\mathcal{E}$ in $\mathcal{F}_X(B)$ as they agree at all fibres. Hence $\mathrm{Hom}_{\mathfrak{Sch}}(-, X)$ is naturally isomorphic $\mathcal{F}_X$.

Finally, under this identification the identity morphism $id \in \mathrm{Hom}_{\mathfrak{Sch}}(X, X)$ is mapped to the bundle $\Gamma_{id*}\mathcal{O}_X = \Delta_*\mathcal{O}_X$, so this is the tautological element. □

Remark 4.18. Combining Theorems 4.15 and 4.17 we can deduce that if there exists a stability condition θ and dimension vector d such that $\mathbb{R}\mathrm{Hom}_X(T, -)$ and $T \otimes^{\mathbb{L}}_A (-)$ restrict to bijections between $\mathcal{F}_X(\mathbb{C})$ and $\mathcal{F}_A(\mathbb{C})$, then X is a fine moduli space for the functor $\mathcal{F}_A$. Further, in this situation the tautological bundle on X is in fact $T^\vee$, as can be seen by translating the tautological element $\Delta_*\mathcal{O}_X$ across the natural isomorphism between $\mathcal{F}_X$ and $\mathcal{F}_A$, so $\mathrm{End}_X(T^\vee) \cong \mathrm{End}_X(T)^{\mathrm{op}} = A$ and the dual of the tautological bundle is the tilting bundle T.

4.5 Applications

Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$, suppose X has a tilting bundle T, and suppose that $A = \mathrm{End}_X(T)^{\mathrm{op}}$ is presented as a quiver with relations.

In this section we will introduce a dimension vector d_T and stability condition θ_T defined by a decomposition of the tilting bundle and give general conditions for the map $\eta : \mathcal{F}_X \to \mathcal{F}_A$ introduced in the previous sections to be a natural isomorphism for this stability condition and dimension vector. We will then use these general conditions to produce the applications outlined in the introduction.

4.5.1 Dimension vectors and stability

We introduce a certain dimension vector and stability condition defined from a decomposition of a tilting bundle and then, using Theorem 4.15, give a criterion for η to be a natural isomorphism with respect to this stability condition and dimension vector. In order to do this we make the following assumption on T, a tilting bundle on a scheme X.

Assumption 4.19. The tilting bundle T has a decomposition into non-isomorphic indecomposables $T = \bigoplus_{i=0}^{n} E_i$ such that there is a unique indecomposable, E_0, isomorphic to $\mathcal{O}_X$.

We then consider a presentation of $A = \mathrm{End}_X(T)^{\mathrm{op}}$ as the path algebra of a quiver with relations such that each indecomposable E_i corresponds to a vertex i of the quiver, as in Section 2.3.2. In particular the 0 vertex in the quiver corresponds to the summand $\mathcal{O}_X$.

Definitions 4.20. Suppose T is a tilting bundle T with decomposition $T = \bigoplus_{i=0}^{n} E_i$.

i) The dimension vector d_T is defined by

$$d_T(i) = \mathrm{rk}\, E_i.$$

In particular $d_T(0) = 1$ as E_0 is assumed to be isomorphic to $\mathcal{O}_X$.

ii) The stability condition θ_T is defined by

$$\theta_T(i) = \begin{cases} -\sum_{i \neq 0} \mathrm{rk}\, E_i & \text{if } i = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 4.21. *The stability condition θ_T has the following properties:*

i) Let $P_0 := \mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_X)$ and M be an A-module with dimension vector d_T. Then $\mathrm{Hom}_A(P_0, M)$ is one dimensional, and M is θ_T-stable if and only if there is a surjection $P_0 \to M \to 0$.

ii) The stability θ_T is generic for A-modules of dimension d_T.

Proof. The A-module P_0 is the projective module consisting of paths in the quiver starting at the vertex 0. For any representation M with dimension vector d_T a homomorphism from P_0 to M is determined by the image of the trivial path $e_0 \in P_0$ in the vector space $\mathbb{C} \subset M$ at vertex 0, which we denote by 1_0. This is as any path p starting at 0 must be sent to the evaluation in M of the linear map corresponding to p on the element 1_0. Hence $\mathrm{Hom}_A(P_0, M) = \mathbb{C}$, and any nonzero element of $\mathrm{Hom}_A(P_0, M)$ is surjective precisely when the linear maps in M corresponding to paths starting at 0 form a surjection from the vector space at the zero vertex onto M. By the definition of θ_T the module M is θ_T-semistable if and only if any proper submodule N has $d_N(0) = 0$, and this property is equivalent to the linear maps in M corresponding to paths starting at 0 forming a surjection. This proves part i).

We now prove ii). It is clear by the definitions of θ_T and d_T that any dimension d_T module M can have no proper submodules $N \subset M$ such that $\theta_T(N) = 0$ as if N is a nontrivial submodule, either $d_N(0) = 0$ and $\theta_T(N) > 0$, or $d_N(0) = 1$ and $N = M$. □

We now give conditions for $\eta : \mathcal{F}_X \to \mathcal{F}_A$ to be a natural isomorphism for this stability condition and dimension vector. We note that there is an abelian category $\mathcal{A}$ corresponding to the abelian category A-mod under the tilting equivalence between $D^b(X)$ and $D^b(A)$ such that T is a projective generator of $\mathcal{A}$. Then $\mathbb{R}\mathrm{Hom}_X(T, -)$ and $T \otimes^{\mathbb{L}}_A (-)$ define an equivalence of abelian categories between $\mathcal{A}$ and A-mod. Our conditions are defined on this category $\mathcal{A}$.

Lemma 4.22. *Take the dimension vector d_T and stability condition θ_T as above. Suppose the following conditions hold:*

i) The structure sheaf $\mathcal{O}_X$ is in $\mathcal{A}$, and for any closed point $x \in X$ the skyscraper sheaf $\mathcal{O}_x$ is in $\mathcal{A}$.

ii) For all closed points $x \in X$ there is a surjection $\mathcal{O}_X \to \mathcal{O}_x$ in $\mathcal{A}$.

Then η is a well defined natural transformation and η_B is injective for all $B \in \mathfrak{R}$. Suppose further that the following condition also holds:

iii) The set

$$S := \left\{ \mathcal{E} \in \mathcal{A} \;\middle|\; \begin{array}{l} \bullet\ \mathbb{R}\mathrm{Hom}_X(T, \mathcal{E}) \text{ has dimension vector } d_T. \\ \bullet\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x) = 0 \text{ for all closed points } x \in X. \end{array} \right\}$$

is empty.

Then η is a natural isomorphism.

Proof. We first assume that conditions i) and ii) hold and prove that $\eta_{\mathbb{C}}$ is well defined and injective. Then it follows from Lemmas 4.13 and 4.14 that η is a natural transformation and η_B is injective for all $B \in \mathfrak{R}$.

Any element of $\mathcal{F}_X(\mathbb{C})$ is a skyscraper sheaf on X up to isomorphism. For any closed point $x \in X$ the A-module $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x)$ has dimension vector d_T, hence the map $\eta_{\mathbb{C}}$ is well defined if and only if all $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x)$ are θ_T-semistable A-modules. By condition i) they are A-modules. By considering the surjections of condition ii), $\mathcal{O}_X \to \mathcal{O}_x \to 0$ in $\mathcal{A}$, and applying the abelian equivalence $\mathbb{R}\mathrm{Hom}_X(T, -)$ we see that all $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x)$ are θ_T-stable by Lemma 4.21 i). Hence $\eta_{\mathbb{C}}$ is well defined.

By Lemma 4.21 ii) θ_T is generic so $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x)$ and $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_y)$ are S-equivalent if and only if they are isomorphic, then as $\mathbb{R}\mathrm{Hom}_X(T, -)$ is an equivalence $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x) \cong \mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_y)$ implies $\mathcal{O}_x$ and $\mathcal{O}_y$ are isomorphic, so $\eta_{\mathbb{C}}$ is injective.

We now also assume that condition iii) holds and prove that $\eta_{\mathbb{C}}$ is also surjective with inverse $T \otimes^{\mathbb{L}}_A (-)$. It then follows from Theorem 4.15 that η is a natural isomorphism. Take an A-module, M, with dimension vector d_T and which is θ_T-stable. As M is θ_T-stable by Lemma 4.21 ii) there is a surjection

$$P_0 \to M \to 0$$

which under the abelian equivalence gives an exact sequence in $\mathcal{A}$

$$\mathcal{O}_X \to \mathcal{E} \to 0$$

where $\mathcal{E} \cong M \otimes^{\mathbb{L}}_A T \in D^b(X)$. Then by condition iii) there must be some closed point $x \in X$ such that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x) \neq 0$. We then apply $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{O}_x)$ to the surjection $\mathcal{O}_X \to \mathcal{E} \to 0$ to obtain an injection

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_X, \mathcal{O}_x) = \mathbb{C}$$

and hence the surjection $\mathcal{O}_X \to \mathcal{O}_x \to 0$ factors through $\mathcal{E}$, and there is a surjection $\mathcal{E} \to \mathcal{O}_x \to 0$. We then apply the abelian equivalence functor $\mathbb{R}\mathrm{Hom}_X(T, -)$ to obtain a surjection of finite dimensional A-modules

$$M \to \mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x) \to 0$$

and by comparing dimension vectors we see that the map is an isomorphism, hence that $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x) \cong M$ and $T \otimes^{\mathbb{L}}_A M \cong \mathcal{O}_x$. □

Corollary 4.23. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Let T be a tilting bundle on X which defines an equivalence of an abelian category $\mathcal{A}$ with A-mod, where $A = \mathrm{End}_X(T)^{\mathrm{op}}$. Choose the stability condition θ_T and dimension vector d_T as above, define $\mathcal{F}_A = \mathcal{F}^{ss}_{A,d_T,\theta_T}$, and assume that conditions i) and ii) of Lemma 4.22 hold for $\mathcal{A}$. Then:*

i) The map $\eta : \mathcal{F}_X \to \mathcal{F}_A$ defined in Section 4.3.2 is a natural transformation and induces a morphism $f : X \to \mathcal{M}^{ss}_{d_T,\theta_T}$ between X and the quiver GIT quotient of A for stability condition θ_T and dimension vector d_T. This morphism is a monomorphism in the sense of [Sta, Definition 25.23.1 (Tag 01L2)].

ii) If condition iii) of Lemma 4.22 also holds for $\mathcal{A}$ then the morphism f is an isomorphism.

Proof. We note that $\mathcal{M}^{ss}_{d_T,\theta_T} = \mathcal{M}^{s}_{d_T,\theta_T}$ as θ_T is generic by Lemma 4.21 ii) and that $\mathcal{M}^{ss}_{d_T,\theta_T}$ is a fine moduli space for $\mathcal{F}_A$ by Theorem 4.4 as the dimension vector d_T is indivisible. The map $\eta : \mathcal{F}_X \to \mathcal{F}_A$ is a natural transformation as conditions i) and ii) of Lemma 4.22 hold for $\mathcal{A}$. It then follows that there is a corresponding morphism $f : X \to \mathcal{M}^{ss}_{d_T,\theta_T}$ as $\mathcal{F}_A$ is represented by $\mathcal{M}^{ss}_{d_T,\theta_T}$ and $\mathcal{F}_X$ is represented by X by Theorem 4.17. For all $B \in \mathfrak{R}$ the map η_B is injective by Lemma 4.22, hence the corresponding morphism, f, is a monomorphism.

If condition iii) of Lemma 4.22 also holds for $\mathcal{A}$ then η is actually a natural isomorphism by Lemma 4.22. Hence f is an isomorphism, proving ii). □

Remark 4.24. While we make no further use of the monomorphism property we note that it can be a useful notion as proper monomorphisms are exactly closed immersions, [Sta, Lemma 40.7.2 (Tag 04XV)], and étale monomorphisms are exactly open immersions, [Sta, Theorem 40.14.1 (Tag 025G)].

4.5.2 One dimensional fibres

To apply Lemma 4.22 and Corollary 4.23 we need a class of varieties with tilting bundles such that we understand the abelian categories $\mathcal{A}$. Such a class was introduced in Theorem 4.2; if $\pi : X \to \mathrm{Spec}(R)$ is a projective morphism of Noetherian schemes such that $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$ and the fibres of π have dimension ≤ 1 then there exist tilting bundles T_i on X such that the abelian category $\mathcal{A}$ is ${}^{-i}\mathrm{Per}(X/R)$, defined as follows.

Definition 4.25 ([VdB04b, Section 3])**.** Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of Noetherian schemes such that $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$ and π has fibres of dimension ≤ 1. Define $\mathfrak{C}$ to be

the abelian subcategory of $\operatorname{Coh} X$ consisting of $\mathcal{F} \in \operatorname{Coh} X$ such that $\mathbb{R}\pi_*\mathcal{F} \cong 0$. For $i = 0, 1$ the abelian category ${}^{-i}\mathrm{Per}(X/R)$ is defined to contain $\mathcal{E} \in D^b(X)$ which satisfy the following conditions:

i) The only non-vanishing cohomology of $\mathcal{E}$ lies in degrees -1 and 0.

ii) $\pi_*\mathcal{H}^{-1}(\mathcal{E}) = 0$ and $\mathbb{R}^1\pi_*\mathcal{H}^0(\mathcal{E}) = 0$, where $\mathcal{H}^j$ denotes taking the jth cohomology sheaf.

iii) For $i = 0$, $\operatorname{Hom}_X(C, \mathcal{H}^{-1}(\mathcal{E})) = 0$ for all $C \in \mathfrak{C}$.

iv) For $i = 1$, $\operatorname{Hom}_X(\mathcal{H}^0(\mathcal{E}), C) = 0$ for all $C \in \mathfrak{C}$.

We note that the abelian categories ${}^{-i}\mathrm{Per}(X/R)$ are hearts of t-structures on $D^b(X)$ and short exact sequences in ${}^{-i}\mathrm{Per}(X/R)$ correspond to triangles in $D^b(X)$ whose vertices are in ${}^{-i}\mathrm{Per}(X/R)$.

Any projective generator of the abelian category ${}^{-i}\mathrm{Per}(X/R)$ gives a tilting bundle T_i with the properties defined in Theorem 4.2, and we can assume that such a tilting bundle contains $\mathcal{O}_X$ as a summand by the following proposition.

Proposition 4.26 ([VdB04b, Proposition 3.2.7])**.** *Define $\mathfrak{V}_X$ to be the category of vector bundles $\mathcal{M}$ on X which are generated by global sections and such that $\mathrm{H}^1(X, \mathcal{M}^\vee) = 0$, and define $\mathfrak{V}_X^\vee := \{\mathcal{M}^\vee : \mathcal{M} \in \mathfrak{V}_X\}$. The projective generators of ${}^{-1}\mathrm{Per}(X/R)$ are the $\mathcal{M} \in \mathfrak{V}_X$ such that $\wedge^{\mathrm{rk}\,\mathcal{M}}\mathcal{M}$ is ample and $\mathcal{O}_X$ is a summand of $\mathcal{M}^{\oplus a}$ for some $a \in \mathbb{N}$. The projective generators of ${}^{0}\mathrm{Per}(X/R)$ are the elements of $\mathfrak{V}_X^\vee$ which are dual to projective generators of ${}^{-1}\mathrm{Per}(X/R)$.*

Hence we let T_i be a projective generator of ${}^{i}\mathrm{Per}(X/R)$ with a decomposition as required in Assumption 4.19. Then the algebra $A_i = \operatorname{End}_X(T_i)^{\mathrm{op}}$ can be presented as a quiver with relations with vertex 0 corresponding to $\mathcal{O}_X$ and the stability condition θ_{T_i} and dimension vector d_{T_i} are well defined.

We now check that the conditions of Lemma 4.22 hold for ${}^{0}\mathrm{Per}(X/R)$.

Theorem 4.27. *Let $\pi : X \to \operatorname{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$ such that π has fibres of dimension ≤ 1 and $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$. Then the abelian category $\mathcal{A} = {}^{0}\mathrm{Per}(X/R)$ satisfies conditions i), ii) and iii) of Lemma 4.22.*

Proof. We begin by checking $\mathcal{A}$ satisfies conditions i) and ii) of Lemma 4.22. All skyscraper sheaves $\mathcal{O}_x$ and the structure sheaf $\mathcal{O}_X$ are in $\mathcal{A}$ as they satisfy the conditions of Definition 4.25. Then, for any $x \in X$, the short exact sequence of sheaves $0 \to I \to \mathcal{O}_X \to \mathcal{O}_x \to 0$ corresponds to a triangle in $D^b(X)$, and the ideal sheaf I is also in $\mathcal{A}$ as $\mathbb{R}^1\pi_*I = 0$ due to the exact sequence $0 \to \pi_*I \to \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_x \to \mathbb{R}^1\pi_*I \to 0$ where $\pi_*\mathcal{O}_X \cong \mathcal{O}_R$ and the third arrow is a surjection. Hence the map $\mathcal{O}_X \to \mathcal{O}_x \to 0$ is in fact a surjection in $\mathcal{A}$. We then note, for all $x \in X$, that $\operatorname{Hom}_{\mathcal{A}}(\mathcal{O}_X, \mathcal{O}_x) \cong \operatorname{Hom}_{D^b(X)}(\mathcal{O}_X, \mathcal{O}_x) \cong \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_x) \cong \mathbb{C}$, hence $\operatorname{Hom}_{\mathcal{A}}(\mathcal{O}_X, \mathcal{O}_x) \cong \mathbb{C}$ corresponding to the map of sheaves $\mathcal{O}_X \to \mathcal{O}_x \to 0$ which is surjective in $\mathcal{A}$.

To check condition iii) suppose S is not empty and so there exists some $\mathcal{E} \in S$. In particular, $M \cong \operatorname{Hom}_{D^b(X)}(T_0, \mathcal{E})$ has dimension vector d_{T_0} so $\mathbb{R}\pi_*\mathcal{E} \cong \mathcal{O}_y$ for some $y \in \operatorname{Spec}(R)$. As $\mathcal{E} \in \mathcal{A}$ there is a short exact sequence in $\mathcal{A}$

$$0 \to \mathcal{H}^{-1}(E)[1] \to \mathcal{E} \to \mathcal{H}^0(\mathcal{E}) \to 0$$

where [1] is the shift in $D^b(X)$. Hence, for all closed points $x \in X$, there is an injection

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}^0(\mathcal{E}), \mathcal{O}_x) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x).$$

Then it follows that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}^0(\mathcal{E}), \mathcal{O}_x) = \mathrm{Hom}_{D^b(X)}(\mathcal{H}^0(\mathcal{E}), \mathcal{O}_x) = 0$ for all $x \in X$ as by assumption $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x) = 0$, and hence $\mathcal{H}^0(\mathcal{E}) = 0$ as a nonzero coherent sheaf must be supported somewhere. So $\mathcal{E} = \mathcal{H}^{-1}(\mathcal{E})[1]$, and we now seek to reach a contradiction to the existence of such an $\mathcal{E}$. The argument below should be thought of as an explicit translation to our setting of the proof of Nakamura's conjecture for the G-Hilbert scheme in [BKR01, Section 8] which derives a contradiction between the facts that the Euler pairing of a coherent sheaf shifted by [1] with a very ample line bundle must be negative whereas the Euler pairing of a G-cluster with any locally free sheaf must be positive.

We begin by noting that $\pi_*\mathcal{H}^{-1}(\mathcal{E}) = 0$ and $\mathbb{R}^1\pi_*\mathcal{H}^{-1}(\mathcal{E}) = \mathcal{O}_y$. By [VdB04b, Lemma 3.1.3] there is an injection of sheaves

$$0 \to \mathcal{H}^{-1}(\mathcal{E}) \to \mathcal{H}^{-1}(\pi^!\mathcal{O}_y)$$

and hence $\mathcal{H}^{-1}(\mathcal{E})$ is set-theoretically supported on $\pi^{-1}(y)$. In particular y corresponds to a maximal ideal $\mathfrak{m}_y$ of R and we consider the completion $R \to \hat{R} = \varprojlim(R/\mathfrak{m}_y^n)$. This produces the following pullback diagram

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "\hat{\pi}"'] & X \arrow[d, "\pi"] \\ \mathrm{Spec}(\hat{R}) \arrow[r, "i"] & \mathrm{Spec}(R) \end{tikzcd}$$

where Y is the formal fibre $Y := \varprojlim(\mathrm{Spec}(R/\mathfrak{m}_y^n) \times_{\mathrm{Spec}(R)} X)$, the morphisms i and j are both flat and affine, and the morphism $\hat{\pi}$ is projective. Then we have the following isomorphism, where we recall that the morphisms i and j are both flat and affine so we need not derive them,

$$\begin{aligned} \mathbb{R}\mathrm{Hom}_X(T_0, j_*j^*\mathcal{E}) &\cong i_*\mathbb{R}\mathrm{Hom}_Y(j^*T_0, j^*\mathcal{E}) && (j_*, j^* \text{ adjoint pair}) \\ &\cong i_*\mathbb{R}\hat{\pi}_*j^*\mathcal{R}\mathcal{H}\mathrm{om}_X(T_0, \mathcal{E}) && (\text{Lemma 4.10}) \\ &\cong i_*i^*\mathbb{R}\mathrm{Hom}_X(T_0, \mathcal{E}). && (\text{Flat base change}) \end{aligned}$$

Then as $M \cong \mathbb{R}\mathrm{Hom}_X(T_0, \mathcal{E})$ is finite dimensional and supported on $\mathfrak{m}_y$ it follows that completion in $\mathfrak{m}_y$ followed by restriction of scalars acts as the identity, see [Eis95, Theorem 2.13] and [LW12, Lemma 2.5], hence $i_*i^*M := \hat{R} \otimes_R M \cong M$. We deduce that $\mathbb{R}\mathrm{Hom}_X(T_0, j_*j^*\mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_X(T_0, \mathcal{E})$, and so $\mathcal{E} \cong j_*j^*\mathcal{E}$ as T_0 is a tilting bundle. Finally we can define $\mathcal{G} := j^*\mathcal{H}^{-1}(\mathcal{E})$ with the property that $j_*\mathcal{G}[1] \cong \mathcal{E}$.

We now note that by Lemma 4.26 there exists $P \in \mathfrak{V}_X$ such that $T_0 = P^\vee$. We then note that as P is a vector bundle generated by global sections so is j^*P, hence as $\hat{R}$ is a complete local ring there exists a short exact sequence

$$0 \to \mathcal{O}_Y^{\oplus d-1} \to j^*P \to \wedge^d j^*P \to 0$$

by [VdB04b, Lemma 3.5.1], where $d = \mathrm{rk}\, P = \mathrm{rk}\, j^*P$. Also, as $P \in \mathfrak{V}_X$, the line bun-

dle $\wedge^d P$ is ample and so the line bundle $\mathcal{L} := \wedge^d j^* P \cong j^* \wedge^d P$ is also ample as j is affine. Then by Serre vanishing, [Har77, III Theorem 5.2], there exists some $N > 0$ such that $\mathrm{Hom}_{D^b(Y)}(\mathcal{L}^{\otimes -N}, \mathcal{G}[1]) \cong \mathrm{Ext}^1_Y(\mathcal{O}_Y, \mathcal{L}^{\otimes N} \otimes \mathcal{G}) = 0$. As j^*P is generated by global sections the vector bundle $j^*P^{\oplus N}$ is also generated by global sections so again there exists a short exact sequence

$$0 \to \mathcal{O}_Y^{\oplus Nd-1} \to (j^*P)^{\oplus N} \to \mathcal{L}^{\otimes N} \to 0$$

by [VdB04b, Lemma 3.5.1]. Dualising this we obtain the short exact sequence

$$0 \to \mathcal{L}^{\otimes -N} \to (j^*T_0)^{\oplus N} \to \mathcal{O}_Y^{\oplus Nd-1} \to 0,$$

where $(j^*P)^\vee \cong j^*(P^\vee)$ by Lemma 4.10. As $\mathrm{Hom}_{D^b(Y)}(\mathcal{L}^{\otimes -N}, \mathcal{G}[1]) = 0$ applying the functor $\mathrm{Hom}_{D^b(Y)}(-, \mathcal{G}[1])$ to this sequence produces an exact sequence

$$\mathrm{Hom}_{D^b(Y)}(\mathcal{O}_Y, \mathcal{G}[1])^{\oplus Nd-1} \to \mathrm{Hom}_{D^b(Y)}(j^*T_0, \mathcal{G}[1])^{\oplus N} \to 0. \qquad (\dagger)$$

Then

$$\begin{aligned}
\dim_{\mathbb{C}} \mathrm{Hom}_{D^b(Y)}(\mathcal{O}_Y, \mathcal{G}[1]) &= \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(Y)}(\mathbb{L}j^*\mathcal{O}_X, \mathcal{G}[1]) && (\mathcal{O}_Y \cong \mathbb{L}j^*\mathcal{O}_X) \\
&= \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(X)}(\mathcal{O}_X, \mathbb{R}j_*\mathcal{G}[1]) && (\mathbb{L}j^*, \mathbb{R}j_* \text{ adjoint pair}) \\
&= \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(X)}(\mathcal{O}_X, j_*\mathcal{G}[1]) && (\text{As } j \text{ affine } \mathbb{R}j_* = j_*) \\
&= \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(X)}(\mathcal{O}_X, \mathcal{E}) = 1
\end{aligned}$$

and

$$\begin{aligned}
\dim_{\mathbb{C}} \mathrm{Hom}_{D^b(Y)}(j^*T_0, \mathcal{G}[1]) &= \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(Y)}(\mathbb{L}j^*T_0, \mathcal{G}[1]) && (T_0 \text{ locally free}) \\
&= \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(X)}(T_0, \mathbb{R}j_*\mathcal{G}[1]) && (\mathbb{L}j^*, \mathbb{R}j_* \text{ adjoint pair}) \\
&= \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(X)}(T_0, j_*\mathcal{G}[1]) && (\text{As } j \text{ affine } \mathbb{R}j_* = j_*) \\
&= \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(X)}(T_0, \mathcal{E}) \\
&= \dim_{\mathbb{C}} M = d
\end{aligned}$$

as $M \cong \mathrm{Hom}_{D^b(X)}(T_0, \mathcal{E})$ has dimension vector d_{T_0} and $d = \mathrm{rk}\, T_0$. Comparing the dimensions in the sequence $(\dagger)$ we find a contradiction since a $Nd-1$ dimensional space cannot surject onto an Nd dimensional space. Hence such an $\mathcal{E}$ cannot exist and so S is empty. □

Combining this theorem with Corollary 4.23 gives us the following result, showing that in this situation schemes can be reconstructed by quiver GIT.

Corollary 4.28. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$ such that π has fibres of dimension ≤ 1 and $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$. Let T_0 be a tilting bundle which is a projective generator of ${}^0\mathrm{Per}(X/R)$ as defined by Theorem 4.2, define $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$, suppose A_0 is presented as a quiver with relations, and choose the stability condition θ_{T_0} and dimension vector d_{T_0} as above. Then X is isomorphic to the quiver GIT quotient of $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$ for dimension vector d_{T_0} and stability condition θ_{T_0}, so X is the fine moduli space of the corresponding quiver representation moduli functor and has tautological bundle $T_0^\vee$.*

4.5.3 Example: flops in the minimal model program

The class of varieties considered in Section 4.5.2 were originally motivated by flops in the minimal model program. In the paper [Bri02] Bridgeland proves that smooth varieties in dimension three which are related by a flop are derived equivalent, and in the process constructs the flop of such a variety as a moduli space of perverse point sheaves. In this section we show that this moduli space construction can in fact be done using quiver GIT. Recall the following theorem.

Theorem 4.29 ([VdB04b, Theorems 4.4.1, 4.4.2])**.** *Suppose $\pi : X \to \operatorname{Spec}(R)$ is a projective birational map of quasi-projective Gorenstein varieties of dimension ≥ 3, with π having fibres of dimension ≤ 1, the exceptional locus of π having codimension ≥ 2, and Y having canonical hypersurface singularities of multiplicity ≤ 2. Then the flop $\pi' : X' \to \operatorname{Spec}(R)$ exists and is unique. Further X and X' are derived equivalent such that ${}^{-1}\operatorname{Per}(X/R)$ corresponds to ${}^{0}\operatorname{Per}(X'/R)$. In particular, for a tilting bundle T_1 on X which is a projective generator of ${}^{-1}\operatorname{Per}(X/R)$ there is a tilting bundle T_0' on X' which is a projective generator of ${}^{0}\operatorname{Per}(X'/R)$ such that $A_1 = \operatorname{End}_X(T_1)^{\mathrm{op}} \cong \operatorname{End}_{X'}(T_0')^{\mathrm{op}} = A_0'$ and $\pi_* T_1 \cong \pi'_* T_0'$.*

We refer the reader to [VdB04b, Theorem 4.4.1] for the definition of a flop in this setting. The results from the previous sections now imply the following corollary, showing that the variety X and its flop X' can both be constructed as quiver GIT quotients from tilting bundles on X.

Corollary 4.30. *Suppose we are in the situation of Theorem 4.29. Then X is the quiver GIT quotient of $A_0 = \operatorname{End}_X(T_0)^{\mathrm{op}}$ for stability condition θ_{T_0} and dimension vector d_{T_0}, and X' is the quiver GIT quotient of $A_1 = \operatorname{End}_X(T_1)^{\mathrm{op}}$ for stability condition θ_{T_1} and dimension vector d_{T_1}.*

Proof. Corollary 4.28 tells us both that X is the quiver GIT quotient of A_0 for stability condition θ_{T_0} and dimension vector d_{T_0}, and that X' is the quiver GIT quotient of $A_0' = \operatorname{End}_{X'}(T_0')^{\mathrm{op}}$ for stability condition $\theta_{T_0'}$ and dimension vector $d_{T_0'}$. We now relate A_0', $\theta_{T_0'}$ and $d_{T_0'}$ to A_1, θ_{T_1} and d_{T_1}.

We note that by Theorem 4.29 $A_0' \cong A_1$, and we choose a presentation of A_1 as a quiver with relations matching that of A_0' in order to identify the stability condition and dimension vector matching $\theta_{T_0'}$ and $d_{T_0'}$. In particular, there is a decomposition of $T_1 = \bigoplus_{i=0}^n E_i$ and $T_0' = \bigoplus_{i=0}^n E_i'$ such that $\pi_* E_i \cong \pi'_* E_i'$. We note that under this correspondence the vertices corresponding to $\mathcal{O}_X$ and $\mathcal{O}_{X'}$ correspond by [VdB04b, Lemma 4.2.1] as $\pi_* \mathcal{O}_X \cong \pi'_* \mathcal{O}_{X'} \cong \mathcal{O}_R$, and since π and π' are birational $\operatorname{rk}_X E_i = \operatorname{rk}_R \pi_* E_i = \operatorname{rk}_R \pi'_* E_i' = \operatorname{rk}_{X'} E_i'$. Hence $A_0' \cong A_1$, $d_{T_0'} = d_{T_1}$ and $\theta_{T_0'} = \theta_{T_1}$ so X' is the quiver GIT quotient of $A_1 = \operatorname{End}_X(T_1)^{\mathrm{op}}$ for stability condition θ_{T_1} and dimension vector d_{T_1}. □

4.5.4 Example: resolutions of rational singularities

We give a further application of Theorem 4.28 to the case of rational singularities, extending and recapturing several well-known examples as discussed in Chapter 1.

Minimal resolutions of rational affine singularities $\pi : X \to \operatorname{Spec}(R)$ satisfy the condition $\mathbb{R}\pi_* \mathcal{O}_X \cong \mathcal{O}_R$ by definition, and in the case of surface singularities it is immediate that the dimensions of the fibres of π are ≤ 1. Hence the following corollary is immediate from Corollary 4.28 ii).

Corollary 4.31. *Suppose that $\pi : X \to \mathrm{Spec}(R)$ is the minimal resolution of a rational surface singularity. Then there is a tilting bundle T_0 on X as in Theorem 4.2, and by Corollary 4.28 ii) X is the quiver GIT quotient of $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$ for dimension vector d_{T_0} and stability condition θ_{T_0}, and the tautological bundle produced on X by this moduli construction is $T_0^\vee$.*

This gives a moduli interpretation of minimal resolutions for all rational surface singularities. In certain examples the tilting bundles and algebras are well-understood and this corollary recovers previously known examples.

Example 4.32 (Kleinian Singularities)**.** Kleinian singularities are quotient singularities $\mathbb{C}^2/G$ for G a non-trivial finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, and are discussed in Section 1.1. These have crepant resolutions, and in particular $G\text{-Hilb}(\mathbb{C}^2) = X \to \mathbb{C}^2/G$ is a crepant resolution [IN96]. There is a tilting bundle T on X constructed by Kapranov and Vasserot [KV00], which, if we take the multiplicity free version, matches the T_0 of Theorem 4.2. Then $A = \mathrm{End}_X(T)^{\mathrm{op}}$ is presentable as the McKay quiver with relations, the preprojective algebra, and $G\text{-Hilb}(\mathbb{C}^2)$ is the quiver GIT quotient of the preprojective algebra for stability condition θ_T and dimension vector d_T. The crepant resolutions were previously constructed as hyper-Kähler quotients by Kronheimer [Kro89], this approach was interpreted as a GIT quotient construction by Cassens and Slodowy [CS98], and as a quiver GIT quotient by Crawley-Boevey [CB00].

Example 4.33 (Surface quotient singularities)**.** As an expansion of the previous example we consider G a non-trivial, small, finite subgroup of $\mathrm{GL}_2(\mathbb{C})$ as in Section 1.3. Then $\mathbb{C}^2/G$ is a rational singularity with a minimal resolution $\pi : G\text{-Hilb}(\mathbb{C}) = X \to \mathbb{C}^2/G$ by [Ish02]. The variety X has the tilting bundle T_0, and the algebras $A = \mathrm{End}_X(T_0)^{\mathrm{op}}$ can be presented as the path algebras of quivers with relations, the reconstruction algebras recalled in Section 1.3, which are defined and explicitly calculated in [Wem11b,Wem11a,Wem12,Wem13]. If $G < \mathrm{SL}_2(\mathbb{C})$ then this example falls into the case of Kleinian singularities above, otherwise these fall into a classification in types A (cyclic quotient surface singularities as defined in Section 2.4.2), D, T, I, and O, [Wem11b, Section 5]. It was shown by explicit calculation in [Cra11,Wem11a,Wem12, Wem13] that in types A and D the minimal resolutions X are quiver GIT quotients of A with stability condition θ_{T_0} and dimension vector d_{T_0}. Corollary 4.31 recovers these cases without needing to perform explicit calculations, and also includes the same result for the remaining cases T, I, and O.

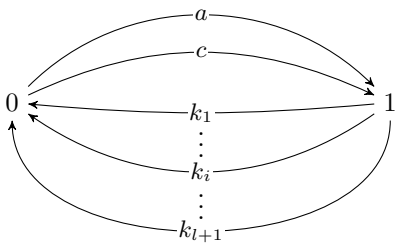
Corollary 4.34. *Suppose $G < \mathrm{GL}_2(\mathbb{C})$ is a finite, non-trivial, pseudo-reflection-free group. Then the minimal resolution of the quotient singularity $\mathbb{C}^2/G$ can be constructed as the quiver GIT quotient of the corresponding reconstruction algebra for stability condition θ_{T_0} and dimension vector d_{T_0}.*

Proof. We note that in Theorem 4.2 $T_1 = T_0^\vee$ and that $\mathrm{End}_X(T_0^\vee) \cong \mathrm{End}_X(T_0)^{\mathrm{op}}$. Hence our definition of $A = \mathrm{End}_X(T_0)^{\mathrm{op}}$ as the reconstruction algebra matches that given in [Wem11b, Wem11a,Wem12,Wem13] as $A = \mathrm{End}_X(T_1)$. Then the result is an immediate corollary of Corollary 4.31. □

Example 4.35 (Determinantal singularity)**.** We give one higher dimensional example. Let R be the $\mathbb{C}$-algebra $\mathbb{C}[X_0, \dots X_l, Y_1, \dots Y_{l+1}]$ subject to the relations generated by all two by two minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & \dots & X_i & \dots & X_l \\ Y_1 & Y_2 & \dots & Y_{i+1} & \dots & Y_{l+1} \end{pmatrix}.$$

Then $\mathrm{Spec}(R)$ is a $l+2$ dimensional rational singularity and has an isolated singularity at the origin. This has a resolution given by $\pi : X = \mathrm{Tot}\left(\bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^1}(-1)\right) \to \mathrm{Spec}(R)$, the total space of the locally free sheaf $\bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^1}(-1)$ mapping onto its affinisation. The variety X has a tilting bundle T_0 by Theorem 4.2, which, considering the bundle map $f : X \to \mathbb{P}^1$, we can identify as $T_0 = \mathcal{O}_X \oplus f^*\mathcal{O}_{\mathbb{P}^1}(-1)$. We can then present $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$ as the following quiver with relations, (Q, Λ).



$$k_i a k_j = k_j a k_i$$
$$k_i c k_j = k_j c k_i$$
$$a k_j c = c k_j a$$
$$\text{for } 1 \leq i, j \leq l+1$$

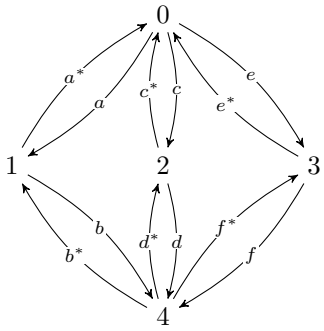
By Theorem 4.28 we know that X can be reconstructed as the quiver GIT quotient of A_0 with dimension vector $d_{T_0} = (1,1)$ and stability condition $\theta_{T_0} = (-1,1)$. In this example we will explicitly verify this. A dimension d_{T_0} representation is defined by assigning a value $\lambda_i \in \mathbb{C}$ to each k_i and $(\alpha, \gamma) \in \mathbb{C}^2$ to (a, c). The relations are all automatically satisfied so $\mathrm{Rep}_{d_{T_0}}(Q, \Lambda) = \mathbb{C}^{l+1} \times \mathbb{C}^2$. Then a representation is θ_{T_0} stable if it has no dimension $(1,0)$ submodules, so these correspond to the subvariety with $(\alpha, \gamma) \in \mathbb{C}^2/(0,0)$, hence $\mathrm{Rep}_d(Q, \Lambda)^{ss} = \mathbb{C}^{l+1} \times \mathbb{C}^2/(0,0)$. We then find that the corresponding quiver GIT quotient is given by the action of $\mathbb{C}^*$ on $\mathbb{C}^{l+1} \times \mathbb{C}^2/(0,0)$ with weights -1 on $\mathbb{C}^2/(0,0)$ and 1 on $\mathbb{C}^{l+1}$. This produces the total bundle X.

When $l = 2$ this is the motivating example of the Atiyah flop given as the opening example of [VdB04b] and A_0 is the conifold quiver discussed as an example in Section 1.2. In this case, by Theorem 4.30, we can calculate the flop as the quiver GIT quotient of $A_1 \cong A_0^{\mathrm{op}}$.

In the following two examples we explicitly calculate open affine covers of quiver GIT quotients for the reconstruction algebras appearing as Examples 1.21 and 1.26. By Corollary 4.31 this calculates a minimal resolution of the associated rational surface singularity, and we note that this calculation is straightforward and similar to the case of cyclic quotient surface singularities discussed in Section 2.4.2. In both these examples the dimension vector d_{T_0} is 1 at every vertex, and the stability condition θ_{T_0} requires a nonzero path from 0 to every other vertex for a representation to be stable.

Example 4.36. (Type $D_{5,2}$) This example is a particular case of the explicit affine covers for the quiver GIT quotients of a type D reconstruction algebra calculated in [Wem13]. The recon-

struction algebra can be presented as the path algebra of the following quiver with relations.



$$a^*a = c^*c = e^*e$$
$$aa^* = b^*b,$$
$$cc^* = d^*d,$$
$$ee^* = f^*f$$
$$bb^* = dd^* = ff^*$$
$$ba - cd = fe$$

A representation of the quiver is described by assigning to each arrow a complex number such that the relations are satisfied. In particular $\mathrm{Rep}_{d_{T_0}}(Q, \Lambda)$ is the affine variety cut from $\operatorname{Spec} \mathbb{C}[a, a^*, b, b^*, c, c^*, d, d^*, e, e^*, f, f^*]$ by the relations above. The stable representations are those with a nonzero path from vertex 0 to every other vertex, and are spanned by the five open affine charts, U_0, U_1, U_2, U_3 and U_4 which we describe below.

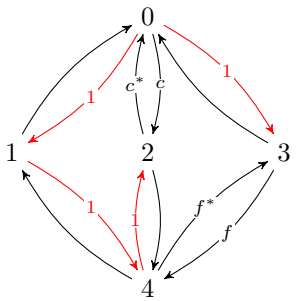


Chart U_0: This chart is defined by a, b, d^*, and e being nonzero, and rescaling by the $\mathbb{C}^*$-action at each vertex we may assume $a = b = d^* = e = 1$. This is represented in the adjacent quiver, where all unlabelled arrows in the quiver have value cc^*. Then the relations imply that this chart is defined in $\operatorname{Spec} \mathbb{C}[c_0, c_0^*, f_0, f_0^*]$ by the equations $1 - c_0^2 c_0^* = f_0$, and $c_0 c_0^* = f_0^* f_0$. Hence $U_0 = \operatorname{Spec} \mathbb{C}[c_0, c_0^*, f_0^*]/(c_0 c_0^* - f_0^*(1 - c_0^2 c_0^*))$.

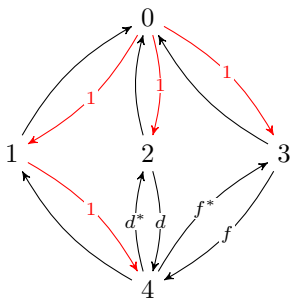


Chart U_1: This chart is defined by a, b, c, and e being nonzero, and rescaling by the $\mathbb{C}^*$-action at each vertex we may assume $a = b = c = e = 1$. This is represented in the adjacent quiver, where all unlabelled arrows in the quiver have value dd^*. The relations imply that this chart is defined in $\operatorname{Spec} \mathbb{C}[d_1, d_1^*, f_1, f_1^*]$ by the equations $1 - d_1 = f_1$, and $d_1 d_1^* = f_1^* f_1$. Hence $U_1 = \operatorname{Spec} \mathbb{C}[d_1, d_1^*, f_1^*]/(d_1 d_1^* - f_1^*(1 - d_1))$ where, considering the scaling action, we have transition maps $d_1 = c_0^2 c_0^*, d_1^* = c_0^{-1}, f_1^* = f_0^*$.

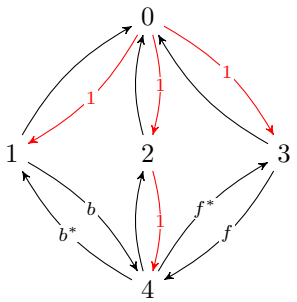


Chart U_2: This chart is defined by a, c, d, and e being nonzero, and rescaling by the $\mathbb{C}^*$-action at each vertex we may assume $a = c = d = e = 1$. This is represented in the adjacent quiver, where all unlabelled arrows in the quiver have value bb^*. Then the relations imply this chart is defined in $\operatorname{Spec} \mathbb{C}[b_2, b_2^*, f_2, f_2^*]$ by the equations $b_2 - 1 = f_2$, and $b_2 b_2^* = f_2^* f_2$. Hence $U_2 = \operatorname{Spec} \mathbb{C}[b_2, b_2^*, f_2^*]/(b_2 b_2^* - f_2^*(b_2 - 1))$ where, considering the scaling action, we have transition maps $b_2 = d_1^{-1}, b_2^* = d_1^2 d_1^*, f_2^* = f_1^* d_1^{-1}$.

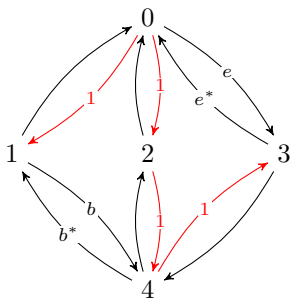


Chart U_3: This chart is defined by a, c, d, and f^* being nonzero, and rescaling by the $\mathbb{C}^*$-action at each vertex we may assume $a = c = d = f^* = 1$. This is represented in the adjacent quiver, where all unlabelled arrows in the quiver have value ee^*. Then the relations imply that this chart defined in $\mathbb{C}[b_3, b_3^*, e_3, e_3^*]$ by the equations $b_3 - 1 = e_3^2 e_3^*$, and $b_3 b_3^* = e_3^* e_3$. Hence $U_3 = \operatorname{Spec} \mathbb{C}[b_3^*, e_3, e_3^*]/((1 + e_3^2 e_3^*)b_3^* - e_3^* e_3)$ where, considering the scaling action, we have transition maps $b_3^* = b_2^*, e_3 = f_2^{*-1}$, and $e_3^* = f_2^{*-1}$.

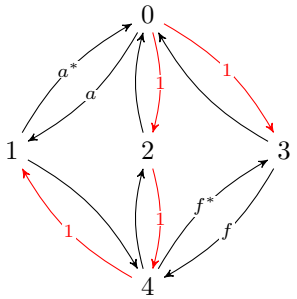
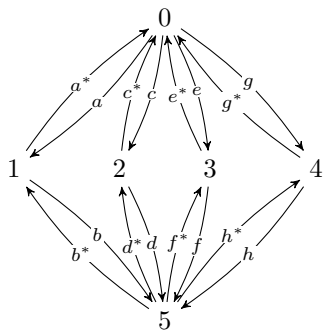


Chart U_4: This chart is defined by b^*, c, d, and e being nonzero, and rescaling by the $\mathbb{C}^*$-action at each vertex we may assume $b^* = c = d = e = 1$. This is represented in the adjacent quiver, where all unlabelled arrows in the quiver have value aa^*. Then the chart is defined in $\operatorname{Spec} \mathbb{C}[a_4, a_4^*, f_4, f_4^*]$ by the equations $a_4 - 1 = f_4$, and $a_4 a_4^* = f_4^* f_4$. Hence $U_4 = \operatorname{Spec} \mathbb{C}[a_4, a_4^*, f_4^*]/(a_4 a_4^* - f_4^*(a_4 - 1))$ where, considering the scaling action, we have transition maps $a_4 = b_2^{*-1}, a_4^* - = b_2^{*2} b_2$, and $f_4^* = f_2$.

Example 4.37. (Non-quotient example) Consider the following reconstruction algebra presented as the path algebra of a quiver with relations.



$$a^*a = c^*c = e^*e = g^*g,$$
$$aa^* = b^*b, \quad cc^* = d^*d,$$
$$ee^* = f^*f, \quad gg^* = hh^*,$$
$$bb^* = dd^* = ff^* = h^*h,$$
$$ba - cd = ef, \text{ and } ba - \lambda cd = hg$$
$$\lambda \in \mathbb{C}\backslash\{0, 1\}.$$

A representation of the quiver is described by assigning to each arrow a complex number such that the relations are satisfied. In particular $\operatorname{Rep}_{d_{T_0}}(Q, \Lambda)$ is the affine variety cut from $\operatorname{Spec} \mathbb{C}[a, a^*, b, b^*, c, c^*, d, d^*, e, e^*, f, f^*, g, g^*, h, h^*]$ by the relations above. The stable representations are those with a nonzero path from vertex 0 to every other vertex, and are spanned by the six open affine charts, U_0, U_1, U_2, U_3, U_4 and U_5 which we describe below.

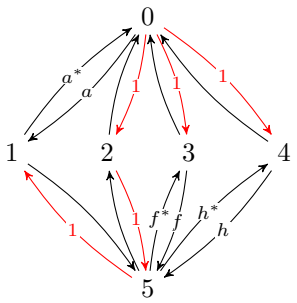


Chart U_0: This chart is defined by b^*, c, d, e, and g being nonzero, and rescaling by the $\mathbb{C}^*$-action at each vertex we may assume $b^* = c = d^* = e = g = 1$. Then the relations imply that this chart is defined in $\operatorname{Spec} \mathbb{C}[a_0, a_0^*, f_0, f_0^*, h_0, h_0^*]$ by the equations $a_0^2 a_0^* - 1 = f_0$, $a_0^2 a_0^* - \lambda = h_0$, and $c_0 c_0^* = f_0^* f_0 = h_0 h_0^*$.

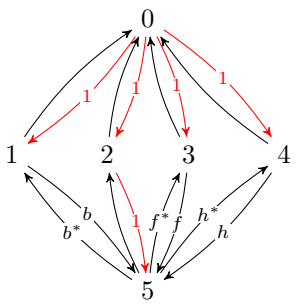


Chart U_1: This chart is defined by a, c, d, e, and g being nonzero, and rescaling by the $\mathbb{C}^*$-action at each vertex we may assume $a = c = d = e = g = 1$. The relations imply that this chart is defined in $\operatorname{Spec} \mathbb{C}[b_1, b_1^*, f_1, f_1^*, h_1, h_1^*]$ by the equations $b_1 - 1 = f_1$, $b_1 - \lambda = h_1$ and $d_1 d_1^* = f_1^* f_1 = h_1 h_1^*$. Considering the scaling action gives the transition maps $b_1 = a_0^{-1}, b_1^* = a_0^2 a_0^*, f_1 = f_0, f_1^* = f_0^*, h_1 = h_0$, and $h_1^* = h_0^*$.

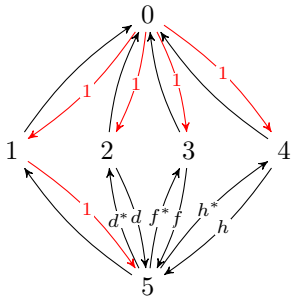


Chart U_2: This chart is defined by a, b, c, e, and g being nonzero, and rescaling by the $\mathbb{C}^*$-action at each vertex we may assume $a = b = c = e = g = 1$. Then the relations imply this chart is defined in $\operatorname{Spec} \mathbb{C}[d_2, d_2^*, f_2, f_2^*, h_2, h_2^*]$ by the equations $1 - d_2 = f_2$, $1 - \lambda d_2 = h_2$, and $d_2 d_2^* = f_2^* f_2 = h_2 h_2^*$. Considering the scaling action gives the transition maps $d_2 = b_1^{-1}, d_2^* = b_1^2 b_1^*, f_2 = f_1, f_2^* = f_1^*, h_2 = h_1$, and $h_2^* = h_1^*$.

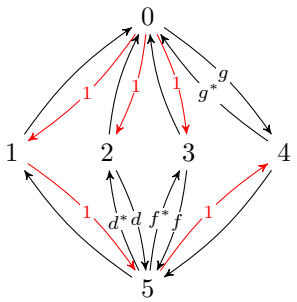


Chart U_3: This chart is defined by a, b, c, e, and h^* being nonzero, and rescaling by the $\mathbb{C}^*$-action at each vertex we may assume $a = b = c = e = h^* = 1$. Then the relations imply that this chart is defined in $\operatorname{Spec} \mathbb{C}[d_3, d_3^*, f_3, f_3^*, g_3, g_3^*]$ by the equations $1 - d_3 = f_3$, $1 - \lambda d_3 = g_3^2 g_3^*$ and $d_3 d_3^* = f_3 f_3^* = g_3 g_3^*$. Considering the scaling action gives the transition maps $d_3 = d_2, d_3^* = d_2^*, f_3 = f_2, f_3^* = f_2^*, g_3 = h_2^{*-1}$, and $g_3^* = h_2^{2*} h_2$.

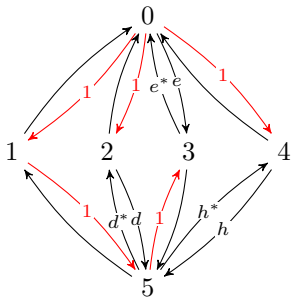


Chart U_4: This chart is defined by a, b, c, f^*, and g being nonzero, and rescaling by the $\mathbb{C}^*$-action at each vertex we may assume $a = b = c = f^* = g = 1$. Then the chart is defined in $\operatorname{Spec} \mathbb{C}[d_4, d_4^*, e_4 e_4^*, h_4, h_4^*]$ by the equations $1 - d_4 = e_4^2 e_4^*$, $1 - \lambda d_4 = g_4^2 g_4^*$, and $d_4 d_4^* = e_4 e_4^* = h_4 h_4^*$. Considering the scaling action gives the transition maps $d_4 = d_2, d_4^* = d_2^*, e_4 = f_2^{*-1}, e_4^* = f_2^{2*} f_2, g_4 = g_2$, and $g_4^* = g_2^*$.

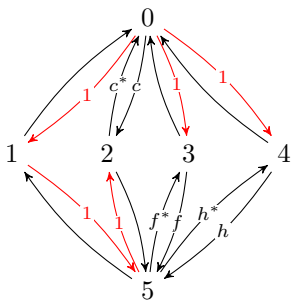


Chart U_5: This chart is defined by a, b, d^*, e, and g being nonzero, and rescaling by the $\mathbb{C}^*$-action at each vertex we may assume $a = b = d^* = e = g = 1$. Then the chart is defined in $\mathbb{C}[c_5, c_5^*, f_5, f_5^*, h_5, h_5^*]$ by the equations $1 - c_5^2 c_5^* = f_5$, $1 - \lambda c_5^2 c_5^* = h$, and $c_5 c_5^* = f_5 f_5^* = h_5 h_5^*$. Considering the scaling action gives the transition maps $c_5 = d_2^{*-1}, c_4^* = d_2^{*2} d_2, f_5 = f_2, f_5^* = f_2^*, g_5 = g_2$, and $g_5^* = g_2^*$.

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