ANDREAS V. WEINERT

LIE ALGEBRAS, VERTEX ALGEBRAS AND AUTOMORPHIC FORMS

LIE ALGEBRAS, VERTEX ALGEBRAS AND AUTOMORPHIC FORMS

ANDREAS V. WEINERT

MSc. Master of Science by Research

August 2008

"Die Hoffnung sie nährt mich, sie nährt ja die halbe Welt, und ich habe sie mein Lebtag zur Nachbarin gehabt; was wäre sonst aus mir geworden."

— Ludwig van Beethoven

Dedicated with gratitude and admiration to Sarah Lampe.

ABSTRACT

In some special cases the denominator identities of generalised Kac-Moody algebras are automorphic forms on orthogonal groups. This project gives an overview of the fundamental concepts of Lie algebras, vertex algebras and automorphic forms and then proceeds to discuss the recent classification results on generalised Kac-Moody algebras and automorphic forms.

CONTENTS

PREFACE

The study of generalised Kac-Moody algebras, vertex algebras and automorphic forms requires a great deal of mathematical maturity in several fundamental areas. I believe that it is an exciting area of new mathematics and I intend to give an overview of some of the important results.

The first chapter begins by reviewing the classification of finite-dimensional simple Lie algebras to get a feeling for the more general infinite-dimensional case and how those Kac-Moody algebras can be obtained. Next, we introduce the notion of a vertex algebra in the second chapter and give an introduction to modular forms. These concepts will later on be combined with the idea of the moonshine for the monster in the third chapter and will allow us to discuss recent classification results of generalised Kac-Moody algebras and automorphic forms in the fourth chapter.

The first chapter is intended to be accessible for the keen final-year undergraduate mathematician. From then on, the project demands a greater mathematical maturity of the reader. Given an article of this length, it is not possible to avoid making certain assumptions, but it is hoped that the depths of the results can still be appreciated.

All in all, I hope to have composed a dissertation that not only illustrates the fascinating intersection of these diverse mathematical disciplines, but can also be appreciated by postgraduates working in this area, as well as lecturers.

Special thanks go to my supervisor Dr. Nils Scheithauer, for his insight and generosity in guiding me through this project. Thanks to the EPSRC for the financial support and to Andrew Ranicki and the School of Mathematics.

Andreas V. Weinert, Edinburgh

1

LIE ALGEBRAS

"Grau, teurer Freund, ist alle Theorie,
Und grün des Lebens' goldner Baum."

— Johann Wolfgang von Goethe, Faust I

Since generalised Kac-Moody algebras are natural generalisations of finite-dimensional simple Lie algebras, this chapter will review the well-known results for finite-dimensional Lie algebras and explain how we can then derive the infinite-dimensional case.

1.1 FINITE-DIMENSIONAL LIE ALGEBRAS

1.1.1 *Introduction to Lie algebras*

Let us begin by defining a Lie algebra.

Definition 1.1.1. A *Lie algebra* $\mathfrak{g}$ is an algebra with a bilinear map:

$$(x, y) \mapsto [x, y],$$

the *Lie bracket*, satisfying $[x, x] = 0$, and the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,$$

for all $x, y, z \in \mathfrak{g}$.

The product $[x, y]$ is often referred to as the commutator of x and y. In fact, we will see that any associative algebra can be made into a Lie algebra by defining the Lie bracket as $[x, y] = xy - yx$. Using $[x + y, x + y] = 0$, we clearly see that $[x, y] = -[y, x]$. Throughout this paper, the base field will always be $\mathbb{C}$ if not stated otherwise.

Similar to the notion of a subgroup and a normal subgroup in group theory, we have that the subalgebra of a Lie algebra $\mathfrak{g}$ is a subspace

$\mathfrak{k}$ which satisfies $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, and an ideal of $\mathfrak{g}$ is a subspace $\mathfrak{h}$ such that $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. Due to anti-commutativity, we have that every ideal is two-sided.

Example 1.1.2. Let $M_n(\mathbb{C})$ be the algebra of complex $n \times n$ matrices. Together with the Lie bracket $[A, B] = AB - BA$ with $A, B \in M_n(\mathbb{C})$, this gives us a Lie algebra. Also, $\mathfrak{sl}_{n+1}$, which are the elements of $M_{n+1}(\mathbb{C})$ with trace equal to 0 form a Lie algebra under the prescribed product.

We can also form quotients if we consider a Lie algebra and its ideals, calling $\mathfrak{g}/\mathfrak{h}$ the quotient or factor algebra of $\mathfrak{g}$. Similarly, we can talk about the kernel of Lie algebra homomorphisms and obtain corresponding isomorphism theorems. A simple example is the isomorphism $\mathfrak{gl}_n/\mathfrak{sl}_n \cong \mathbb{F}$, where an element gets sent to its trace, i.e. to the sum of its diagonal entries.

1.1.2 *Cartan subalgebra and the Killing form*

Before we define a Killing form to obtain more information about semisimple Lie algebras using matrices, we should revisit the decomposition of a $\mathfrak{g}$-module V into its weight spaces. Firstly, note that if $\mathfrak{h}$ is a subalgebra of a Lie algebra $\mathfrak{g}$, then the *idealiser* of $\mathfrak{h}$ is the set of all $x \in \mathfrak{g}$ such that $[\mathfrak{h}, x] \subseteq \mathfrak{h}$.

Definition 1.1.3. A *Cartan subalgebra* of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ which is nilpotent and equal to its own idealiser in $\mathfrak{g}$.

Now, given a Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$, we can regard $\mathfrak{g}$ as a $\mathfrak{h}$-module. Then since $\mathfrak{h}$ is nilpotent, we obtain the decomposition of $\mathfrak{g}$ into weight spaces

$$\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_{\lambda},$$

for the weights λ. Since one can show that $\mathfrak{h} = \mathfrak{g}_0$, the null component, we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{\beta} \oplus \dots$$

where $\alpha, \beta, \dots$ are non-zero weights.

Definition 1.1.4. Consider $\mathfrak{g}$ as a $\mathfrak{h}$-module. Then the non-zero weights of the $\mathfrak{h}$-module $\mathfrak{g}$ are called the *roots* of $\mathfrak{g}$ and the composition above is called the *Cartan decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

The Cartan decomposition can also be written as $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_\alpha \mathfrak{g}_\alpha)$, calling $\mathfrak{g}_\alpha$ the root space of α. Let us now introduce the notion of the Killing form to get a better understanding of the Cartan decomposition. Given two elements $x, y \in \mathfrak{g}$, the *Killing form* associates an element $\kappa(x, y) \in \mathbb{C}$ by

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y).$$

This implies that we can choose a basis and then just take the trace from our matrices. The Killing form associates a number to matrices independent from the choice of basis, because the trace of an operator is independent of choice of basis. It is a symmetric bilinear form on $\mathfrak{g}$. Moreover, the Killing form is *invariant* in the following sense

$$\kappa([x, y], z) = \kappa(x, [y, z]).$$

Define the linear map $\mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g}$ by $(\mathrm{ad}\, x)(y) = [x, y]$. Then $\mathrm{ad}\, x$ is a *derivation* of $\mathfrak{g}$ for every $x \in \mathfrak{g}$ and the *adjoint representation* is defined to be the mapping $x \mapsto \mathrm{ad}\, x$. We denote a derivation D from $\mathfrak{g}$ to itself, satisfying $[x, y]D = [xD, y] + [x, yD]$ for all x and y in $\mathfrak{g}$.

Example 1.1.5. Let us have a look at the Killing form of $\mathfrak{sl}_2(\mathbb{C})$. The standard basis consists of the three matrices e, f and h:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is an easy calculation to get $[e, f] = h$, $[h, f] = -2f$ and $[h, e] = 2e$ and the adjoint representations of $\mathrm{ad}\, e$, $\mathrm{ad}\, f$ and $\mathrm{ad}\, h$:

$$\mathrm{ad}\, e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; \mathrm{ad}\, f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \; \mathrm{ad}\, h = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Now, we can compute the Killing form

$$\kappa(e, h) = \mathrm{tr}(\mathrm{ad}\, e \circ \mathrm{ad}\, h) = \mathrm{tr} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 4,$$

and relative to our basis e, h and f, obtain the matrix for κ with $\det(\kappa) = -128$:

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

Note that the Killing form is non-degenerate since its determinant is non-zero.

Consider a subspace $\mathfrak{k}$ of a Lie algebra $\mathfrak{g}$. Then we can define the *orthogonal complement* $\mathfrak{k}^{\perp}$ as the set of elements of $\mathfrak{g}$ such that $\kappa(x, y) = 0$ for all $y \in \mathfrak{k}$. We now have a convenient way of expressing the non-degeneracy of the Killing form. We define the Killing form to be non-degenerate if and only if $\mathfrak{g}^{\perp} = 0$ and use this definition to state the following theorem.

Theorem 1.1.6 (Cartan's Theorem)**.** *A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form of $\mathfrak{g}$ is non-degenerate.*

Furthermore, we can decompose a semisimple Lie algebra into a direct sum of simple Lie algebras which means that the classification of semisimple Lie algebras will follow from the classification of the simple Lie algebras. The classification takes advantage of the so called Dynkin diagrams, but before taking a closer look at them, we will have to collect some more facts about the Cartan decomposition and its roots.

Given $h \in \mathfrak{h}$, we can define $h^* : \mathfrak{h} \to \mathbb{C}$ by $h^*(x) = \kappa(h, x)$ for all $x \in \mathfrak{h}$. Sending h to h^* can be seen as an isomorphism of vector spaces between $\mathfrak{h}$ and $\mathfrak{h}^*$. For each root $\alpha \in \mathfrak{h}^*$ there exists a unique $h_\alpha \in \mathfrak{h}$ such that $\alpha(x) = \kappa(h_\alpha, x)$. Also, if α, β and $\alpha + \beta$ are roots, then

$$h_{\alpha+\beta} = h_\alpha + h_\beta \text{ and } h_{-\alpha} = -h_\alpha.$$

The vectors h_α span the Cartan subalgebra $\mathfrak{h}$. Moreover, h_α is an element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ for any root α.

Furthermore, the dimension of the root spaces $\mathfrak{g}_\alpha$ is 1 for any root α, although the space $\mathfrak{h} = \mathfrak{g}_0$ need not be one-dimensional. Let α and β be two roots with $\beta \neq \pm\alpha$. The *α-chain* through β is the sequence

$$\beta - s\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \ldots, \beta + t\alpha,$$

where $\beta + t\alpha$ and $\beta - s\alpha$ are roots, but $\beta + (t+1)\alpha$ and $\beta - (s+1)\alpha$ are not. It can be shown that

$$2\frac{\kappa(h_\alpha, h_\beta)}{\kappa(h_\alpha, h_\alpha)} = s - t \in \mathbb{Z},$$

and furthermore that $\kappa(h_\alpha, h_\beta) \in \mathbb{Q}$ for all roots α and β [Car, pp.50-51]. In addition, if α and $\lambda\alpha$ are both roots, for $\lambda \in \mathbb{C}$, then $\lambda = \pm 1$. The next section will put these results in context with the Cartan decomposition to get one step closer to the classification result.

1.1.3 *The Cartan matrix and Dynkin diagrams*

The Cartan decomposition of a semisimple Lie algebra $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where Δ is the set of roots. If we are given a basis for $\mathfrak{h}$, say $\{h_1, h_2, \dots, h_l\}$, then we can denote by $\mathfrak{h}_0$ the set of all rational linear combinations of $h_1, h_2, \dots, h_l$, where $h_\alpha \in \mathfrak{h}_0$ for all roots α. Now, we want to put an ordering on the roots as follows. Let

$$x = \lambda_1 h_1 + \lambda_2 h_2 + \ldots + \lambda_l h_l \in \mathfrak{h}_0,$$

where we define $x > 0$ if the first non-zero λ_i is positive. Also, given any two $x, y \in \mathfrak{h}_0$, $x > y$ if $x - y > 0$, and $x < 0$ when $-x > 0$. So for a root α, we define α to be *positive* if h_α is positive. Given two roots, α and β, then $\alpha > \beta$ if $h_\alpha > h_\beta$.

Definition 1.1.7. A *simple root* is a positive root which cannot be expressed as the sum of positive roots.

From this definition, we get that every positive root is the sum of simple roots and there are exactly l fundamental roots whose corresponding vectors span $\mathfrak{h}_0$, i.e. form a basis of $\mathfrak{h}_0$ over $\mathbb{Q}$. We can also think of $\mathfrak{h}_0$ as an inner product space over $\mathbb{Q}$ where the Killing form, which is positive definite on $\mathfrak{h}_0$, can be taken as an inner product. Defining the norm of $x \in \mathfrak{h}_0$ to be equal to the square root of the Killing form, i.e. $|x| = \sqrt{\kappa(x,x)}$, then the angle $\theta_{x,y}$ between $x, y \in \mathfrak{h}_0$ is given by

$$\kappa(x, y) = |x||y|\cos(\theta_{x,y}).$$

Also, the vectors which correspond to the simple roots will be denoted by $q_1, q_2, \ldots, q_l$, where any two vectors are inclined at an obtuse angle. Thus, $\{q_1, q_2, \ldots, q_l\}$ is often referred to as an *obtuse basis* of $\mathfrak{h}_0$.

Let us define the reflection in a root α by $r_\alpha : \mathfrak{h}_0 \to \mathfrak{h}_0$ where

$$r_\alpha(x) = x - 2\frac{\kappa(h_\alpha, x)}{\kappa(h_\alpha, h_\alpha)}h_\alpha.$$

Definition 1.1.8. The *Weyl group* is the group generated by all reflections r_α.

We can now introduce the Cartan matrix and classify all such matrices by classifying the associated Dynkin diagrams. In doing so we classify the simple Lie algebras and hence the semisimple Lie algebras. Firstly, the angle between any two vectors h_α, h_β with $\alpha \neq \pm\beta$ is one of the following: $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}$ or $\frac{5\pi}{6}$. This can be seen from

$$2\frac{\kappa(h_\alpha, h_\beta)}{\kappa(h_\alpha, h_\alpha)}2\frac{\kappa(h_\alpha, h_\beta)}{\kappa(h_\beta, h_\beta)} = 4\cos^2(\theta_{\alpha,\beta}),$$

recalling that

$$2\frac{\kappa(h_\alpha, h_\beta)}{\kappa(h_\alpha, h_\alpha)} \in \mathbb{Z}$$

and that $0 \leqslant \cos^2(\theta_{\alpha,\beta}) \leqslant 1$. They have the same length if h_α and h_β are inclined at $\frac{\pi}{3}$ or $\frac{2\pi}{3}$. If the ratio of their lengths is $\sqrt{2}$, then they are inclined at $\frac{\pi}{4}$ or $\frac{3\pi}{4}$, and if the ratio of their lengths is $\sqrt{3}$, then h_α and h_β are inclined at $\frac{\pi}{6}$ or $\frac{5\pi}{6}$. Note that we obtain no information about their relative length if $\theta_{\alpha,\beta} = \frac{\pi}{2}$.

Definition 1.1.9. For any two fundamental vectors q_i, q_j, define

$$a_{ij} = 2\frac{\kappa(q_i, q_j)}{\kappa(q_i, q_i)}.$$

The $l \times l$ matrix $C = (c_{ij})$ is called the *Cartan matrix* of $\mathfrak{g}$ and the integers c_{ij} are called *Cartan integers*. The Cartan matrix has the following properties:

(i) $c_{ii} = 2$ for all i.

(ii) $c_{ij} \in \{0, -1, -2, -3\}$ if $i \neq j$.

(iii) If $c_{ij} = -2$ or -3, then $c_{ji} = -1$

(iv) $c_{ij} = 0$ if and only if $c_{ji} = 0$.

A *Dynkin diagram* of a semisimple Lie algebra $\mathfrak{g}$ is a graph with l vertices where the ith vertex is joined to the jth vertex by a bond of strength $N_{ij} = c_{ij}c_{ji}$ for $i \neq j$. In case the bond strength is greater than one, there is an arrow pointing to the shorter of the two roots so that the graph completely determines the Weyl group. The Dynkin diagram is not necessarily connected, however, an arbitrary Dynkin diagram is clearly the union of connected Dynkin diagrams. The Cartan matrix splits up accordingly into blocks and the simple roots split up into disjoint subsets which are mutually orthogonal. The split then corresponds to the decomposition of the semisimple Lie algebra $\mathfrak{g}$ as a direct sum of simple Lie algebras. Thus, a semisimple Lie algebra $\mathfrak{g}$ has a connected Dynkin diagram if and only if $\mathfrak{g}$ is simple. To determine the distinct Dynkin diagrams it is important to introduce a quadratic form which is defined in terms of the Dynkin diagrams. This real quadratic form is called the *angle form* of $\mathfrak{g}$

$$\sum_{i,j=1}^{l} \omega_{ij}x_i x_j \text{ where } \omega_{ij} = \begin{cases} 2 & \text{if } i = j \\ -\sqrt{N_{ij}} & \text{if } i \neq j. \end{cases}$$

Note that $\omega_{ij}\omega_{ji} = c_{ij}c_{ji}$. So the matrix $\omega = (w_{ij})$ has 2s on the diagonal and $0, -1, -\sqrt{2}, -\sqrt{3}$ on the off-diagonal. Also, the matrix is symmetric and entirely determined by the Dynkin diagram. Since a quadratic form is said to be *positive definite* if all the leading minors of its symmetric matrix (c_{ij}) have positive determinant, we have that the angle form is positive definite.

The connected Dynkin diagram of any simple Lie algebra satisfies the following conditions:

(A) The graph is connected.

(B) Any two distinct vertices are joined by a bond strength $0, 1, 2$ or 3.

(C) The corresponding quadratic form is positive definite.

We wish to determine which graphs satisfy the conditions above and which occur as Dynkin diagrams. By showing that the resultant list has no redundant entries (i.e. all possible Dynkin diagrams obtained give rise to simple Lie algebras), one obtains the classification of finite-dimensional simple Lie algebras over $\mathbb{C}$. The following picture shows the only graphs which satisfy our conditions, the classical Dynkin diagrams.

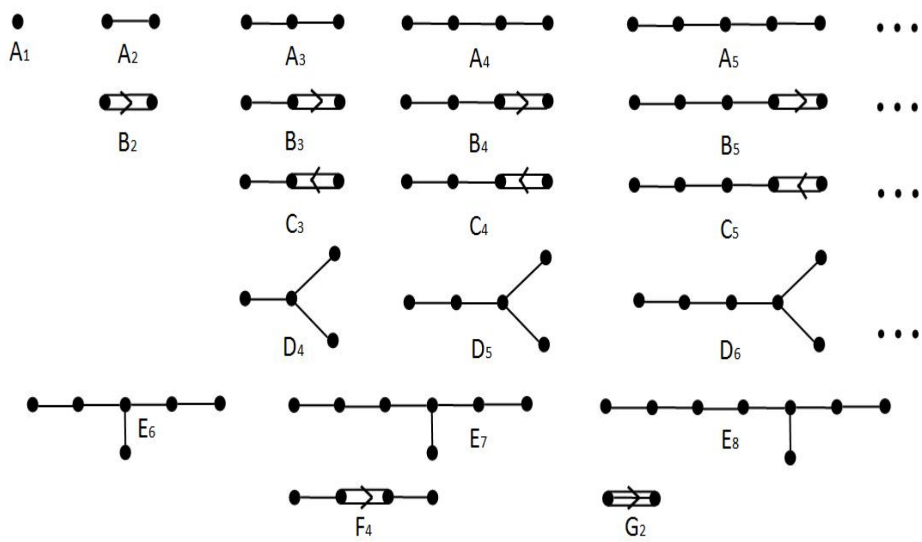

Figure 1. Dynkin diagrams

We can now recover any simple Lie algebra from its Dynkin diagram by reconstructing its Cartan matrix. The infinite families A_n, B_n, C_n and D_n correspond to $\mathfrak{sl}_{n+1}(\mathbb{C})$ the special linear group, $\mathfrak{so}_{2n+1}(\mathbb{C})$ the odd orthogonal Lie algebra, $\mathfrak{sp}_{2n}(\mathbb{C})$ the symplectic Lie algebra, and $\mathfrak{so}_{2n}(\mathbb{C})$ the even orthogonal Lie algebra respectively. The other simple Lie algebras are called *exceptional Lie algebras*.

Theorem 1.1.10 (Cartan, Killing). *With five exceptions, every finite-dimensional simple Lie algebra* $\mathfrak{g}$ *over* $\mathbb{C}$ *is isomorphic to one of the classical Lie algebras:*

$$\mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n(\mathbb{C}) \text{ and } \mathfrak{sp}_{2n}(\mathbb{C}).$$

The five exceptional Lie algebras are known as E_6, E_7, E_8, F_4 *and* G_2*.*

We will give an example of a simple root system and its corresponding Cartan matrix.

Example 1.1.11. Let us look at the Dynkin diagram A_2. There are two fundamental vectors, say q_1 and q_2. Since the Dynkin diagram contains only single bonds, the Cartan matrix is uniquely determined and looks as follows:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We also know that $\frac{|q_1|^2}{|q_2|^2} = \frac{c_{21}}{c_{12}} = 1$ which implies that $|q_1| = |q_2|$. Also, the angle satisfies

$$4\cos^2(\theta_{1,2}) = c_{12}c_{21} = 1,$$

which gives us $\theta_{1,2} = \frac{2\pi}{3}$, because the fundamental vectors form an obtuse basis. If we continue geometrically, then we draw the fundamental vectors to scale and reflect them. Algebraically, we repeatedly apply the fundamental reflections to the fundamental vectors until we obtain no new vectors. Since we must also have the negatives of all vectors, this gives us $q_1, -q_1, q_2, -q_2, q_1 + q_2, -q_1 - q_2$. There are six roots and hence the dimension of A_2 is 8. Also, it can be shown that $(r_1r_2)^3 = 1$ and $r_1^2 = r_2^2 = 1$, which is a presentation of the symmetric group S_3. Thus, the Weyl group of A_2 is isomorphic to S_3.

A result by Serre reconstructs Lie algebras directly from its Cartan matrix. In group theory, it is common to describe a group by generators and relations that they satisfy. Serre's construction can be seen as the analogous case for Lie algebras. The standard generators e_i, h_i and f_i for $i = 1, \dots, l$ satisfy the following properties:

(i) $[e_i, f_j] = \delta_{ij}h_i$

(ii) $[h_i, e_j] = c_{ij}e_j, \quad [h_i, f_j] = -c_{ij}f_j$

(iii) $(\mathrm{ad}\, e_i)^{1-2c_{ij}/c_{ii}}(e_j) = 0$ and $(\mathrm{ad}\, f_i)^{1-2c_{ij}/c_{ii}}(f_j) = 0$ for $i \neq j$.

Serre's theorem shows that the relations above entirely determine the Lie algebra and, up to isomorphism, there is just one Lie algebra for each root system.

Theorem 1.1.12 (Serre's Theorem). *Let C be a Cartan matrix of a root system and $\mathfrak{g}$ be a complex Lie algebra which is generated by the elements e_i, h_i and f_i for $1 \leqslant i \leqslant l$ with the relations above. Then $\mathfrak{g}$ is finite-dimensional and semisimple with Cartan subalgebra $\mathfrak{h}$ spanned by $\{h_1, \dots, h_l\}$ and its root system has Cartan matrix C.*

Let us again have a quick glance at the representations of these algebras. We have seen that given a finite-dimensional simple Lie algebra, a finite-dimensional representation decomposes into weight spaces, where the representation is characterised by the highest weight in the decomposition. We can calculate the irreducible characters of a simple Lie algebra $\mathfrak{g}$ by using Weyl's character formula

$$\text{char } V = \frac{\sum_{w \in W} \det(w) w(e^{\rho+\lambda})}{e^{\rho} \prod_{\alpha>0} (1 - e^{\alpha})},$$

where W is the Weyl group, λ denotes the highest weight of the irreducible representation and α runs over all positive simple roots. Moreover, if we apply Weyl's character formula to the trivial representation, then we obtain the so called *denominator identity*, i.e.

$$e^{\rho} \prod_{\alpha>0} (1 - e^{\alpha}) = \sum_{w \in W} \det(w) w(e^{\rho}).$$

1.2 INFINITE-DIMENSIONAL LIE ALGEBRAS

If we apply the construction of complex semisimple Lie algebras given by Serre's theorem to a more general matrix instead of the Cartan matrix, then we can construct new families of Lie algebras. In the finite-dimensional case, the resultant Lie algebras are exactly those we have already; the genuinely new Lie algebras are infinite-dimensional. We define Kac-Moody algebras and generalised Kac-Moody algebras using Serre's relations and these generalised Cartan matrices.

1.2.1 *Kac-Moody algebras*

Recall that given a finite-dimensional simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, $\mathfrak{g}$ decomposes into a direct sum as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha},$$

where Δ denotes the set of roots and the dimension of $\mathfrak{h}$ is equal to l.

The *symmetrised Cartan matrix* $A = (a_{ij})$ satisfies the following properties.

(1.) $a_{ii} > 0$

(2.) $a_{ij} = a_{ji}$

(3.) $a_{ij} \leqslant 0$ for $i \neq j$

(4.) $2a_{ij}/a_{ii} \in \mathbb{Z}$.

(5.) A is positive definite.

Now, if we Apply Serre's construction to a matrix which satisfies conditions $1. - 4.$, but which is not necessarily positive definite, then we obtain a *Kac-Moody algebra*. They are in general infinite-dimensional, but their theory can be seen as similar to the finite-dimensional case. Kac-Moody algebras can be classified under certain assumptions on their Cartan matrices. If we assume that the determinant of the matrix vanishes and the proper principal minors are positive or, equivalently, that we have an indecomposable matrix A of rank $l - 1$, then the construction gives us the affine Kac-Moody algebras. Similarly to the finite-dimensional case, we have a character formula for Kac-Moody algebras, which takes into account that the multiplicity of a root α, i.e. the dimension of the root space $\mathfrak{g}_\alpha$, is here not necessarily 1. The *Weyl-Kac character formula* is

$$\text{char } V = \frac{\sum_{w \in W} \det(w) w(e^{\rho+\lambda})}{e^\rho \prod_{\alpha>0}(1-e^\alpha)^{\mathrm{mult}(\alpha)}}, \tag{1.1}$$

where $\mathrm{mult}(\alpha)$ denotes the multiplicity of the root α. If we apply the character formula to the trivial representation, then the character of $\lambda = 0$ gives us $e^0 = 1$ and hence the denominator identity looks as follows

$$e^\rho \prod_{\alpha>0}(1-e^\alpha)^{\mathrm{mult}(\alpha)} = \sum_{w \in W} \det(w) w(e^\rho).$$

In general, Kac-Moody algebras can be decomposed into 3 classes, namely finite-dimensional, affine and indefinite Kac-Moody algebras. Without going into details, we can note that an affine Kac-Moody algebra can be written as

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

a tensor product of a finite-dimensional Lie algebras with the algebra of Laurent polynomials in one variable and realised as a so called *loop algebra*, which we shall meet in the next chapter. The denominator identities of the affine Kac-Moody algebras give sum expansions of infinite products.

Proposition 1.2.1. *For complex* z *and* q *with* $z \neq 0$ *and* $|q| < 1$*, the denominator identity of the affinisation of* $\mathfrak{sl}_2(\mathbb{C})$ *is*

$$\prod_{n>0}(1-q^{2n})(1-q^{2n-1}z)(1-q^{2n-1}z^{-1}) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} z^n,$$

the Jacobi triple product identity.

Sketch of the proof:

Take the denominator identity (1), which is valid for any symmetrisable Cartan matrix. Then given the affinisation of $\mathfrak{sl}_2(\mathbb{C})$, its system of all positive roots can be described in terms of the two fundamental roots α_1 and α_2 as follows:

$$\Delta^+ = \{n(\alpha_1+\alpha_2), (n-1)\alpha_1 + n\alpha_2, n\alpha_1 + (n-1)\alpha_2\} \text{ for } n \in \mathbb{Z}.$$

The multiplicity of each root is equal to 1 and we can define two variables z and q by $z^{-1}q = e^{-\alpha_2}$, $zq = e^{-\alpha_1}$. Substituting this into (1) and considering the action of the Weyl group on the roots, we get

$$1 + \sum_{n=1}^{\infty}(-1)^n(z^n q^{n^2} + z^{-n}q^{n^2}) = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n^2},$$

which is equal to

$$\prod_{n>0}(1-q^{2n})(1-q^{2n-1}z)(1-q^{2n-1}z^{-1}).$$

1.2.2 *Generalised Kac-Moody algebras*

Borcherds weakened the conditions on the symmetrised Cartan matrix even further to obtain the *generalised Kac-Moody algebras*. If we have matrices which satisfy condition 2., 3., 4. if $a_{ii} > 0$, then using Serre's construction yields the generalised Kac-Moody algebras.

More formally, the real matrix $A = (a_{ij})$ satisfies the following properties:

(1.) $a_{ij} = a_{ji}$

(2.) $a_{ij} \leqslant 0$ for $i \neq j$

(3.) $2a_{ij}/a_{ii} \in \mathbb{Z}$.

The generalised Kac-Moody algebra, associated to A is then defined as the Lie algebra with generators $\{e_i, h_i, f_i\}$ satisfying the following relations:

(i) $[e_i, f_j] = \delta_{ij} h_i$

(ii) $[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j$

(iii) $(\text{ad}\, e_i)^{1-2a_{ij}/a_{ii}}(e_j) = (\text{ad}\, f_i)^{1-2a_{ij}/a_{ii}}(f_j) = 0$
if $a_{ij} > 0$ for $i \neq j$.

(iv) $[e_i, e_j] = [f_i, f_j] = 0$ if $a_{ij} = 0$.

Note that the generalised Kac-Moody algebra is a Kac-Moody algebra if all the diagonal entries of the matrix A are positive. The set of roots Δ (as in the finite-dimensional case) can be written as the union of the positive roots and the negative roots, i.e. $\Delta = \Delta^+ \cup \Delta^-$. We say that $\alpha \in \Delta$ is a *real root* if $\kappa(\alpha, \alpha)$ is positive and is an *imaginary root* if $\kappa(\alpha, \alpha)$ is negative. Note that the standard definition of the Killing form cannot be used, since the trace of an infinite-dimensional matrix is usually not defined. Nevertheless, the problem can be fixed by imposing the properties of invariance, symmetry and bilinearity. Let $\kappa(\ ,\ )$ be a symmetric, bilinear form on any Lie algebra $\mathfrak{g}$, invariant in the sense that

$$\kappa([x, y], z) = \kappa(x, [y, z]), \text{ for } x, y, z \in \mathfrak{g}.$$

If $\mathfrak{g}$ is finite-dimensional, then $\kappa(\ ,\ )$ is a multiple of the Killing form as defined before.

Borcherds discovered a character formula for generalised Kac-Moody algebras. The corresponding denominator identity is given by

$$e^{\rho} \prod_{\alpha > 0} (1 - e^{\alpha})^{\text{mult}(\alpha)} = \sum_{w \in W} \det(w) w(e^{\rho} \sum_{\alpha} \epsilon(\alpha) e^{\alpha}),$$

where $\epsilon(\alpha)$, for α in the root lattice, is $(-1)^n$ if α is the sum of n pairwise orthogonal imaginary simple roots and 0 otherwise. We will give further explanation of this in the next chapters, but want to emphasize that generalised Kac-Moody algebras can be seen as a natural generalisation of the finite-dimensional case where imaginary roots are allowed.

In Chapter 3 it will be shown that their denominator identities are sometimes automorphic forms.

2

VERTEX ALGEBRAS

"There is no point in having an idea that is so complicated that nobody can understand it. I remember I used to give talks on vertex algebras, and usually nobody turned up. Then there was this one time when I got a really big audience. But there had been a misprint, and the title read "vortex algebras", not "vertex algebras". The audience was made up of fluid physicists, and when they realised it was a misprint, they weren't interested either in what I had to say.".

— Richard E. Borcherds

2.1 VERTEX ALGEBRAS

Vertex algebras find their application in the representation theory of affine Lie algebras and play an important role in string theory, sporadic group theory and the geometric Langlands correspondence. This chapter will give a brief introduction to vertex algebras and some of their applications. References are for example [Kac2] and [Fre1].

2.1.1 *Background and definitions*

To get a first intuition, we can think of a vertex algebra as a vector space where each element has an associated formal operator - a *vertex operator* - that acts on the space, satisfying certain conditions.
A vertex algebra and its representation theory, i.e. its modules and intertwiners, is a mathematically rigorous description of a two-dimensional conformal field theory. By $\mathrm{End}(V)[\![z]\!]$, we denote an algebra with respect to multiplication of formal series.

We let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Then a formal series

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \in \operatorname{End}(V)[\![z, z^{-1}]\!]$$

is called a *field* if for each $v \in V$, we have that $a_n v = 0$ for n large enough.

Since the product of two nonzero elements might not be well defined, we cannot multiply two formal series in $\operatorname{End}(V)[\![z, z^{-1}]\!]$, in general.

Example 2.1.1. Let $V = \mathbb{C}$. Define $a(z) = z^{21} - 17z^{100}$ and $b(z) = \sum_{n=-\infty}^{\infty} z^n$. Then

$$\begin{aligned} a(z)b(z) &= \sum_{n\in\mathbb{Z}} z^{n+21} - 17 \sum_{n\in\mathbb{Z}} z^{n+100} \\ &= \sum_{n\in\mathbb{Z}} z^n - 17 \sum_{n\in\mathbb{Z}} z^n \\ &= -16b(z). \end{aligned}$$

Hence, $(a(z) + 16)\, b(z) = 0$.

This example highlights an interesting fact. We cannot simply cancel terms and do not even have an integral domain. Therefore we have to be careful between the differences of ordinary multiplication and the *normally ordered product* of two fields, say $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ and $b(w) = \sum_{n\in\mathbb{Z}} b_n w^{-n-1}$ which is defined as follows:

$$\begin{aligned} :a(z)b(w): &= \sum_{n\in\mathbb{Z}} \left(\sum_{m<0} a_m b_n z^{-m-1} + \sum_{m\geqslant 0} b_n a_m z^{-m-1} \right) w^{-n-1} \\ &= a(z)_+ b(w) + b(w) a(z)_-, \end{aligned}$$

where

$$\begin{aligned} a(z)_+ &= \sum_{m<0} b_m z^{-m-1} \quad \text{and} \\ a(z)_- &= \sum_{m\geqslant 0} a_m z^{-m-1}. \end{aligned}$$

Since $:a(z)b(z):$ is a well-defined field, the space of fields forms an algebra under the operation of multiplication with respect to the normal ordering as defined above. This operation is not necessarily commutative or associative and therefore we adopt the convention to take the product under normal ordering from right to left, i.e. $: a(z)b(z)c(z) :=: a(z)\,(: b(z)c(z) :) :$.

As a next step, it is important to define locality. Two fields $a(z)$ and $b(z)$, as formal series in $\mathrm{End}(V)[\![z^{\pm 1}, w^{\pm 1}]\!]$, are said to be *local* if

$$(z-w)^N [a(z), b(w)] = 0,$$

for N large enough.

Let V be a vector space. We can now obtain a so-called vertex algebra by defining a state-field correspondence which associates to each state $a \in V$ a field $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$ satisfying certain properties. More formally:

Definition 2.1.2. A *vertex algebra* is a vector space V with a vacuum vector $|0\rangle$, a translation operator T and a state-field correspondence

$$\begin{aligned} V \to\ & \mathrm{End}(V)[\![z, z^{-1}]\!] \\ a \mapsto Y(a,z) = & \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \end{aligned}$$

which satisfies the following axioms:

(i) The vacuum axiom: $Y(|0\rangle, z) = 1$ and $Y(a,z)|0\rangle\,|_{z=0} = a$.

(ii) The translation axiom: T on V is defined as $[T, Y(a,z)] = \delta_z Y(a,z)$.

(iii) The locality axiom: All fields are mutually local.

Let us now have a look at a first example of a commutative vertex algebra.

Example 2.1.3 (Fre1). Let V be a $\mathbb{Z}$-graded commutative algebra $Z = \oplus_{n=0}^{\infty} V_n$ with an identity element and a derivation T. We want to show that V can canonically be given the structure of a vertex algebra. We take the identity of V as the vacuum vector $|0\rangle$ and define the operation Y as follows

$$Y(a,z) = \sum_{n \geqslant 0} \frac{1}{n!} m(T^n a) z^n,$$

where $m(b)$ is the multiplication operator by b on V, for all $b \in V$. All of the axioms of a vertex algebra are satisfied. For example, the locality axiom is satisfied in a strong sense: If we take any $a, b \in V$, then $[Y(a,z), Y(b,w)] = 0$. Thus, the locality condition holds for $N = 0$. It can actually be shown that the notion of a $\mathbb{Z}$-graded algebra with an identity, a derivation and $\dim(V_n) < \infty$ is equivalent to the notion of a commutative vertex algebra .

Note that throughout this chapter we will be using two different notations for the same thing, namely $a(z)$ and $Y(a,z)$. The reason for this is that both are commonly used in the recommended references and in some cases one of them turns out to be more convenient.

2.1.2 *Basic properties*

When Borcherds first defined vertex algebras, he took the following identity as a main axiom.

Theorem 2.1.4 (Borcherds' identity)**.** *Let* $k, m, n \in \mathbb{Z}$. *Then*

$$\sum_{j\geqslant 0}\binom{m}{j}(a_{n+j}b)_{m+k-j}c = \sum_{j\geqslant 0}\binom{n}{j}(a_{m+n-j}b_{k+j}c - (-1)^n b_{n+k-j}a_{m+j}c).$$

Borcherds' identity follows from locality and can be proven by contour integration. It is the most important identity in a vertex algebra. There are several special cases of this identity and in what follows, we will mention two of them. If we let $n = 0$, then we obtain the following commutator formula

$$[a_m, b_k] = \sum_{j\geqslant 0}\binom{m}{j}(a_j b)_{m+k-j},$$

and by setting $m = 0$, we obtain the associativity formula

$$(a_n b)_k = \sum_{j\geqslant 0}(-1)^j\binom{n}{j}(a_{n-j}b_{k+j} - (-1)^n b_{n+k-j}a_j).$$

The *Antisymmetry* condition implies that $Y(a,z)b = e^{zT}Y(b,-z)a$ for $a, b \in V$ and can also be derived from Borcherds' identity. Let us now have a look at some of the standard definitions. A *subalgebra* of a vertex algebra V is a subspace U which contains $|0\rangle$ such that

$$a_n U \subset U$$

for all $a \in U, n \in \mathbb{Z}$. Also, a *homomorphism* of a vertex algebra V to a vertex algebra V' is a linear map $f : V \to V'$ such that

$$f(a_n b) = f(a)_n f(b).$$

Then an *ideal* J of a vertex algebra V is a T-invariant subspace which does not contain $|0\rangle$ such that $a_n J \subset J$, for all $a \in V$ and $n \in \mathbb{Z}$. The ideal J is a two-sided ideal by antisymmetry. Thus, we can form the quotient V/J which is a vertex algebra as well. In addition to that, the projection $V \to V/J$ is a vertex algebra homomorphism.

We can define a *tensor product* of two vertex algebras U and V in the following way: Let the space of states be defined as $U \otimes V$. The vacuum is $|0\rangle \otimes |0\rangle$. and the translation operator is defined to be $T \otimes 1 + 1 \otimes T$. Then the fields are

$$\begin{aligned} Y(a \otimes b, z) &= Y(a, z) \otimes Y(b, z) \\ &= \sum_{m,n \in \mathbb{Z}} a_m \otimes b_n z^{-(m+n)-2}. \end{aligned}$$

That is equivalent to

$$(a \otimes b)_k = \sum_{m \in \mathbb{Z}} a_m \otimes b_{-m+k-1},$$

and given this structure, $U \otimes V$ is a vertex algebra.

Finally, we want to define a conformal vertex algebra.
A *conformal vector* of a vertex algebra V is a vector w such that the field

$$Y(w, z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-1}$$

gives a representation of the *Virasoro algebra* of central charge c, i.e.

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n} \frac{m^3 - m}{12} c$$

and

(i) $L_{-1} = T$

(ii) L_0 is diagonalisable on V.

We have that c is called the *central charge* of w and a *conformal vertex algebra* is defined to be a vertex algebra with a conformal vector w.

Now, let V be a conformal vertex algebra such that the eigenvalues of L_0 are integral and L_1 acts locally nilpotent.
Define $V_h = \{v \in V | L_0 v = hv\}$. Then for $a \in V_h$, we can define the *adjoint vertex operator* by

$$Y^*(a,z) = Y(e^{zL_1}(-z^{-2})^{L_0} a, z^{-1}).$$

It follows that

$$Y^*(a,z) = \sum_{n \in \mathbb{Z}} a_n^* z^{-n-1}$$

with

$$a_n^* = (-1)^h \sum_{m \geqslant 0} \left(\frac{L_1^m}{m!} a \right)_{2h-n-m-z}.$$

As an example, we have $L_n^* = w_{n+1}^* = w_{1-n} = L_{-n}$. Borcherds identity then implies

Theorem 2.1.5. *Let* $k, m, n \in \mathbb{Z}$. *Then*

$$\sum_{j \geqslant 0} \binom{m}{j} (b_{n+j} a)_{m+k-j}^* c = \sum_{j \geqslant 0} \binom{n}{j} (-1)^j \left(a_{k+j}^* b_{m+n-j}^* - (-1)^n b_{m+j}^* a_{n+k-j}^* \right) c.$$

A bilinear form $(\ ,\ )$ on V is called *invariant* if

$$(Y(a,z)b, c) = (b, Y^*(a,z)c),$$

or equivalently

$$(a_n b, c) = (b, a_n^* c).$$

It can be shown that the space of invariant bilinear forms on V is naturally isomorphic to the dual of V_0/L_1V_1. This implies that we can construct invariant bilinear forms by defining them on V_0/L_1V_1.

We will discuss the two vertex algebras which are regarded as the simplest and best understood: lattice vertex algebras and Wess-Zumino Witten models and then explain how this relates to infinite-dimensional Lie algebras.

2.1.3 *Lattice vertex algebras*

Let us briefly review some of the basic concepts surrounding lattices before proceeding to construct the vertex algebras associated to an integral lattice.

Definition 2.1.6. A *lattice* L is a free $\mathbb{Z}$-module together with a real valued bilinear form.

A lattice $L \subset \mathbb{R}^n$ is called *integral* if $x \cdot y \in \mathbb{Z}$ for all $x, y \in L$.
Also, the *dual lattice*, denoted by L^*, is defined as

$$\begin{aligned} L^* &= \text{ Hom } (L, \mathbb{Z}) \\ &= \{x \in \mathbb{R}^n | x \cdot y \in \mathbb{Z} \text{ for all } y \in L\}. \end{aligned}$$

A lattice is said to be *unimodular* if $L^* = L$.

Definition 2.1.7. An integral lattice L is *even* if $x^2 = x \cdot x \equiv 0(\mod 2)$.

Let L be an integral lattice and define the underlying complex vector space as $H = L \otimes \mathbb{C}$. Then the Heisenberg algebra with central element c is defined as

$$\hat{H} = H \otimes \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}c$$

with the products

$$[h_1(m), h_2(n)] = m\delta_{m+n}(h_1, h_2)c.$$

Note that $h_1(m) = h_1 \otimes t^m$. Also, $\hat{H}^- = H \otimes \mathbb{C}[t^{-1}]t^{-1}$ is an abelian subalgebra of $\hat{H}$ and $S(\hat{H}^-)$ denotes the symmetric algebra of polynomials in $\hat{H}^-$.

We let $\epsilon : L \times L \to \{\pm 1\}$ be a 2-cocyle, i.e.

$$\begin{aligned} \epsilon(0, \alpha) = \epsilon(\alpha, 0) = 1 \\ \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma) = \epsilon(\alpha + \beta, \gamma)\epsilon(\alpha, \beta) \end{aligned}$$

and denote the twisted group algebra by $\mathbb{C}[L]_\epsilon$. The twisted group algebra has a basis $\{e^\alpha | \alpha \in L\}$ and with the products

$$e^\alpha e^\beta = \epsilon(\alpha, \beta) e^{\alpha+\beta},$$

and the unit $1 = e^0$ it forms an associative algebra. Now, the vector space

$$V = S(\hat{H}^-) \otimes \mathbb{C}[L]_\epsilon$$

decomposes as

$$V = V_0 \oplus V_1$$

where $V_j = S(\hat{H}^-) \otimes \mathbb{C}[L_j]_\epsilon$.

Therefore, the vector space $\hat{H}$ has a natural action on the vector space V which can be used to define a vertex algebra structure. Let c act as the identity and $k(m)$ acts by left multiplication on $S(\hat{H}^-)$ if $m < 0$. If $m > 0$, then $k(m)$ acts as derivation on $S(\hat{H}^-)$ and trivial on $\mathbb{C}[L]_\epsilon$. Finally, $k(0)$ acts on $\mathbb{C}[L]_\epsilon$ by $k(0)e^\alpha = (k, \alpha)e^\alpha$.

Define the vacuum vector $|0\rangle$ as $1 \otimes e^0$ and
$T : V \to V$ by $T|0\rangle = 0$, $[T, k(m)] = -mk(m-1)$ and $T(e^\alpha) = \alpha(-1)e^\alpha$.

Then we define the vertex operator of e^α as

$$e^\alpha(z) = e^\alpha(z)^+ + e^\alpha(z)^-$$

where

$$e^\alpha(z)^+ = e^\alpha \exp\left(\sum_{m>0} \alpha(-m)\frac{z^m}{m}\right) = e^\alpha c_\alpha \sum_{m \geqslant 0} S_m(\alpha) z^m$$

and

$$e^\alpha(z)^- = z^{\alpha(0)} \exp\left(-\sum_{m>0} \alpha(m)\frac{z^{-m}}{m}\right).$$

S_n are called the Schur polynomials and for $k(-n-1), n \geqslant 0$, we put

$$k(-n-1)(z) = \delta_z^{(n)} k(z)$$

with $k(z) = \sum_{n \in \mathbb{Z}} k(n) z^{-n-1}, k \in H$.

The normal ordered product of the Heisenberg generators $: k_1(n_1) \dots k_h(n_h) :$ is defined, as before, by placing all $k(n)$ with $n < 0$ to the left of those with $n \geqslant 0$. Therefore, $(k_1(-n_1-1) \dots k_h(-n_h-1)e^\alpha)(z)$

$$= e^\alpha(z)^+ : (k_1(-n_1-1)(z) \dots k_h(-n_h-1)(z) : e^\alpha(z)^-.$$

This definition can be linearly extended to V so that we obtain our state-field correspondence $V \to \operatorname{End}(V)[\![z, z^{-1}]\!]$, as desired.

Theorem 2.1.8. *With the structure defined above, V is a vertex algebra and different 2-cocyles satisfying the condition $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{\alpha\beta + \alpha^2\beta^2}$ give isomorphic vertex algebras.*

Example 2.1.9. V is graded by L and

$$(V^\alpha)_m (V^\beta) \subset V^{\alpha+\beta}.$$

Example 2.1.10. Also, the first 3 Schur polynomials are

$$S_0(\alpha) = 1$$
$$S_1(\alpha) = \alpha(-1)$$
$$S_3(\alpha) = \frac{1}{2}(\alpha(-2) + \alpha(-1)\alpha(-1)).$$

Since a *super Lie algebra* is simply a Lie algebra with a $\mathbb{Z}_2$-grading where the bracket is commutative on the odd part, the above definitions and results can be easily generalised to the supercase.

2.1.4 *Wess-Zumino Witten models*

Here is another another example of a vertex algebra, namely the Wess-Zumino Witten models. Essentially, we will show that the *vacuum representation* of a loop algebra carries a vertex algebra structure.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. Then up to a constant factor, $\mathfrak{g}$ has a unique non-degenerate, invariant, symmetric bilinear form $(\, , \,)$. We can normalise $(\, , \,)$ in such a way that the highest root θ of our Lie algebra $\mathfrak{g}$ has norm 2, i.e. $\theta^2 = (\theta, \theta) = 2$. Then this bilinear form $(\, , \,)$ is related to the Killing form by the dual Coxeter number, i.e. $(\, , \,) = \frac{1}{2h^\vee}(\, , \,)_\kappa$, where $h^\vee > 0$ denotes the dual Coxeter number.

We define the *loop algebra* of $\mathfrak{g}$ as the Lie algebra

$$L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$$

with products

$$[x_m, y_n] = [x, y]_{m+n}$$

where, as before, we write $x_m = x \otimes t^m$.

$L\mathfrak{g}$ has unique 1-dimensional central extension

$$\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$$

with products

$$[x_m, y_n] = [x, y]_{m+n} + m\delta_{m+n}(x, y)K.$$

Note that $\hat{\mathfrak{g}}$ is not the full affine Kac-Moody algebra corresponding to $\mathfrak{g}$, because the derivation term is missing.

To obtain the vacuum representation, we define a 1-dimensional representation $\mathbb{C}v_k$ of

$$\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K = \mathfrak{g} \otimes \mathbb{C}[t]t \oplus \mathfrak{g} \oplus \mathbb{C}K$$

by $\mathfrak{g} \otimes \mathbb{C}[t]v_k = 0$ and $Kv_k = kv_k$, where $k \in \mathbb{C}$.

The *vacuum representation of level* k of $\hat{\mathfrak{g}}$ is the induced representation

$$\begin{aligned} V_k(\mathfrak{g}) &= \operatorname{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}K}\mathbb{C}v_k \\ &= U(\hat{\mathfrak{g}})_{U(\mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}K)} \otimes \mathbb{C}v_k \\ &= U(\hat{\mathfrak{g}}) \otimes_{\mathbb{C}} \mathbb{C}v_k/J, \end{aligned}$$

where J is the vector space generated by $ba \otimes v_k - b \otimes av_k$ for $b \in U(\mathfrak{g})$, $a \in \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ and $\hat{\mathfrak{g}}$ acts by left multiplication.

We have that $\hat{\mathfrak{g}}$ decomposes into subalgebras as follows

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t^{-1}]t^{-1} \oplus \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K.$$

Then by the Poincaré-Birkhoff-Witt Theorem

$$U(\hat{\mathfrak{g}}) = U(\mathfrak{g} \otimes \mathbb{C}[t^{-1}]t^{-1}) \otimes U(\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K).$$

Hence,

$$V_k(\mathfrak{g}) \cong U(\mathfrak{g} \otimes \mathbb{C}[t^{-1}]t^{-1}).$$

Let $\{J^a\}_{a=1,\ldots,d}$ be an ordered basis of $\mathfrak{g}$. Then the Poincaré-Birkhoff-Witt Theorem implies that $V_k(\mathfrak{g})$ has a basis of monomials of the form

$$J^{a_1}_{n_1} \ldots J^{a_m}_{n_m} v_k,$$

with $n_1 \leqslant \ldots \leqslant n_m < 0$ and if $n_i = n_{i+1}$, then $a_i \leqslant a_{i+1}$.

This enables us to define our vertex operators and to set up our vertex algebras as follows. Define

(i) $|0\rangle = V_k$

(ii) $T : V_k(\mathfrak{g}) \to V_k(\mathfrak{g})$. by $T|0\rangle = 0$ and $[T, J^a_n] = -nJ^a_{n-1}$.

(iii) $Y(|0\rangle, z) = 1$ and $Y(J^a_{-1}|0\rangle, z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$.

In addition to that, one can show that the fields $Y(J^a_{-1}|0\rangle, z)$ are mutually local.

Theorem 2.1.11. $V_k(\mathfrak{g})$ *is a vertex algebra.*

In general, $V_k(\mathfrak{g})$ is not irreducible as a $\hat{\mathfrak{g}}$-module. In fact, $V_k(\mathfrak{g})$ is not irreducible, if k is a non-negative integer. But the quotient $L_k(\mathfrak{g}) = V_k(\mathfrak{g})/J_k$ by a maximal proper submodule is irreducible. If for example $k = 0$, then $L_k(\mathfrak{g})$ is the trivial 1-dimensional representation of $\hat{\mathfrak{g}}$. If $k \neq 0$, then J_k is an ideal of the vertex algebra $V_k(\mathfrak{g})$ and $L_k(\mathfrak{g})$ is a *simple* vertex algebra.

Now, for *positive* integer k, the vertex algebra $L_k(\mathfrak{g})$ is called the WZW-model (Wess-Zumino-Witten-model) of $\mathfrak{g}$ of level k.

Also, the *moonshine module* $V^\natural$ has the structure of a vertex operator algebra. Constructed by Frenkel, Lepowsky and Meurmann in 1984 [Fre2], the moonshine module $V^\natural$ can be written as

$$V^\natural = \bigoplus_{n \in \mathbb{Z}} V_n.$$

Let Λ be the Leech lattice and start with the so-called chiral vertex algebra $V(\Lambda)$ for the torus $\mathbb{R}^{24}/\Lambda$. Then by using the ± 1-symmetry of the Leech lattice, the moonshine module $V^\natural$ was constructed as the orbifold of the vertex algebra $V(\Lambda)$ and is the direct sum of the invariant part $V^\natural_+ = V(\Lambda)^1_+$ and the twisted part $V^\natural_- = V(\Lambda)^{-1}_+$. It is of central charge $c = 24$, with $V^\natural_0 = \mathbb{C}1$ and $V^\natural_1 = \{0\}$ and more importantly the *Monster* is the automorphism group of $V^\natural$. In the next chapter, we will explain its relation to modular forms and Borcherds' famous proof of the moonshine conjecture.

2.2 CONSTRUCTION OF THE FAKE MONSTER ALGEBRA

In this section we will study a generalised Kac-Moody algebra called the *fake monster algebra*, which describes the physical state of a bosonic string moving on a 26-dimensional torus. We will construct this algebra by using cohomology groups of the BRST-operator.

Let

$$L = II_{25,1} \oplus \mathbb{Z}_\sigma,$$

where $II_{25,1}$ is the unique even unimodular Lorentzian lattice of dimension 26 and $\sigma^2 = 1$. The vertex algebra V_L associated to L represents the Fock space of a bosonic string moving on the torus $\mathbb{R}^{25,1}/II_{25,1}$:

$$V_L = V_{II_{25,1} \oplus \mathbb{Z}_\sigma}.$$

We choose a 2-cocycle $\epsilon : L \times L \to \{1\}$ such that $\epsilon(\sigma, \sigma) = 1$. Then we define *ghosts* to be $b = e^{-\sigma}$ and $a = e^{\sigma}$ and the *ghost current* as

$$j^N = c_{-1} b = \sigma(-1).$$

Also the ghost number operator is

$$j_0^N = \sigma(0).$$

We have $b_n b = c_n c = 0$ for $n \geqslant 0$ and $c_n b = S_{-n}(\sigma) e^0$ so that $c_0 b = 1$ and $c_n b = 0$ when for $n \geqslant 1$. Therefore, the commutator formula implies that

$$\{b_n, b_m\} = \{c_n, c_m\} = 0$$

and

$$\{b_m, c_n\} = \delta_{m+n+1}.$$

Let $\{x^\mu\}$ be a basis of $II_{25,1} \otimes \mathbb{R}$, $\{x_\mu\}$ be the corresponding dual basis and $g^{\mu\nu} = (x^\mu, x^\nu)$. The vector

$$\omega^M = \frac{1}{2} x^\mu(-1) x_\mu(-1)$$

generates a representation of the Virasoro algebra of central charge 26. Now we define the vector

$$\omega^G = \frac{1}{2}\sigma(-1)\sigma(-1) + \frac{3}{2}\sigma(-2),$$

which generates a representation of the Virasoro algebra of central charge $1 - 12\frac{9}{4} = -26$. Then the following vector is a conformal vector of central charge 0:

$$\omega = \omega^M + \omega^G.$$

Note that L_1 acts locally nilpotent and L_0 has integral eigenvalues on V_L, because the vector $e^{n\sigma}$ has always integral weight $n(n-3)/2$.

Let us now define the BRST- current and its operator. The BRST-current is defined as

$$j^{BRST} = c_{-1}(\omega^M + \frac{1}{2}\omega^G),$$

and the BRST-operator is

$$Q = j_0^{BRST}.$$

Theorem 2.2.1. *The operators satisfy the following relations*

$$Q^2 = 0 \qquad [j_0^N, Q] = Q$$
$$\{Q, b_{n+1}\} = L_n \qquad [Q, L_n] = 0.$$

The vector $e^{3\sigma}$ has conformal weight 0 and is not contained in $L_1 V_1$. The ghost b has conformal weight 2 and is annihilated by L_1 so that

$$b_n^* = b_{2-n}.$$

Therefore, we can define an invariant bilinear form on V_L by setting

$$(1, e^{3\sigma}) = 1.$$

Similarly, we take

$$c_n^* = -c_{-4-n}$$
$$k(n) = k(-1)_n^* = -k(-n)$$
$$Q^* = -Q.$$

The bilinear form $(\ ,\ )$ is non-degenerate on V_L. The space

$$C = V_L \cap \ker b_1 \cap \ker L_0$$

is invariant under Q and graded by the lattice $II_{25,1}$ and the ghost number

$$C = \bigoplus_{\alpha \in II_{25,1}} C_\alpha^n, \text{ where } n \in \mathbb{Z}.$$

Note that C_α^n is of finite dimension.

We have the following finite sequence

$$\ldots \to C_\alpha^{n-1} \xrightarrow{Q} C_\alpha^n \xrightarrow{Q} C_\alpha^{n+1} \to \ldots$$

with cohomology groups H_α^n. Let

$$H = \bigoplus_{\alpha \in II_{25,1}} H_\alpha^n, \text{ where } n \in \mathbb{Z}.$$

We have that $H^1 = \oplus_{\alpha \in II_{25,1}} H_\alpha^1$ is the space of physical states of the compactified bosonic string.

Theorem 2.2.2. *Let $\alpha \neq 0$. Then the cohomology groups H^n_α are all trivial, except for $n = 1$.*

Since $\{c_{-1}b_1\} = 1$ and $b_1^* = b_1$, we have that $(\ , \ )$ is trivial on $\ker b_1$. We can now define a non-trivial bilinear form on C by

$$(u, v)_C = (c_{-2}u, v).$$

This bilinear form induces a form $\langle \, , \rangle$ on H which pairs non-degenerate H^{1+n}_α with $H^{1-n}_{-\alpha}$ and is symmetric on H^1. The pairing is given by

$$\langle k(-1)|c\rangle, k(-1)|c\rangle\rangle = k^2$$

so that the map

$$\begin{aligned} H^1_\alpha &\to II_{25,1} \otimes \mathbb{R} \\ k(-1)|c\rangle &\mapsto k \end{aligned}$$

is an isometry of vector spaces. Recall that $\Delta(z) = q\prod_{n>0}(1-q^n)^{24}$ where $q = e^{2\pi i n z}$. By the Euler-Poincaré principle we obtain

$$\dim H^1_\alpha = [1/\Delta](-\alpha^2/2) \text{ for } \alpha \neq 0.$$

If we now define a product on C in the following way

$$[u, v] = (b_0 u)_0 v,$$

then this product projects down to H and to $G = H^1$. Finally, if we define

$$G = \bigoplus_{\alpha \in II_{25,1}} H^1_\alpha,$$

then we obtain the following results.

Theorem 2.2.3. *(i) G is a Lie algebra under this product.*

(ii) $\langle \, , \, \rangle$ is a non-degenerate symmetric invariant bilinear form on G.

(iii) G decomposes as

$$G = \bigoplus_{\alpha \in II_{25,1}} G_\alpha$$

with

$$\textit{dim } G_\alpha = [1/\Delta](-\alpha^2/2)$$

for $\alpha \neq 0$. In particular, the norms of the roots are bounded above by 2.

(iv) $[G_\alpha, G_\beta] \subset G_{\alpha+\beta}$

(v) G *is a generalised Kac-Moody algebra.*

(vi) *The denominator identity of* G *is*

$$e^\rho \prod_{\alpha \in II_{25,1}^+} (1-e^\alpha)^{[1/\Delta](-\alpha^2/2)} = \sum_{w \in W} det\,(w) w \left(e^\rho \prod_{n>0} (1-e^{n\rho})^2 \right),$$

where ρ *is a primitive norm* 0 *vector in* $II_{25,1}$ *corresponding to the Leech lattice and* W *is the reflection group of* $II_{25,1}$.

The construction we explained in this section can be generalised as follows. The lattice $II_{25,1}$ decomposes as $II_{25,1} = \Lambda \oplus II_{1,1}$ so that

$$V_L = V_\Lambda \otimes V_{II_{1,1}} \otimes V_{\mathbb{Z}\sigma}.$$

We can replace V_Λ by any other vertex algebra of central charge 24 and the construction we have shown is still possible. We could have for example taken the monster vertex algebra to get the monster algebra. Because of the factor $V_{II,1}$, different vertex algebras can give isomorphic generalised Kac-Moody algebras.

3

MODULAR FORMS

"The real voyage of discovery consists,
not in seeking new landscapes,
but in having new eyes."

— Marcel Proust

This chapter will give a gentle introduction into the theory of modular forms and automorphic forms. We will start with a review of the basic definitions and properties, following [Ser1] and [Dia] very closely, and then explain how this relates to the framework from the material discussed in the previous chapters.

3.1 BASIC DEFINITIONS

The general linear group $GL_n(R)$ is defined as the set of invertible $n \times n$ matrices over any commutative ring R. We will be interested in the special linear group $SL_2(\mathbb{Z})$ over the ring of integers and its subgroups. Let $\Gamma = SL_2(\mathbb{Z})$ be the *modular group*, which can be represented as 2×2 matrices of the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{for } a, b, c, d \in \mathbb{Z} \quad \text{and} \quad \det M = 1.$$

The *upper-half plane* $\mathcal{H}$ is a simply connected Riemann surface defined as the set of points with imaginary part strictly greater than 0:

$$\mathcal{H} = \{z \in \mathbb{C} | \text{ Im } (z) > 0\}.$$

Then Γ acts on $\mathcal{H}$ as follows:

$$Mz = \frac{az + b}{cz + d}.$$

This mapping is referred to as *fractional linear transformation*. Since

$$\text{Im}\,(Mz) = \frac{\text{Im}\,(z)}{|cz+d|^2},$$

we have that $\mathcal{H}$ is stable under the action of Γ, namely it maps the upper-half plane to itself. The identity matrix $\mathbf{1}$ maps any point of $\mathcal{H}$ back to itself. Also, $MM'(z) = M(M'(z))$ for all $M, M' \in \Gamma$ and $z \in \mathcal{H}$. Thus, it is indeed a group action. Furthermore, since the element $-\mathbf{1}$ acts trivially on $\mathcal{H}$ as well, we actually have the projective linear group $PSL_2(\mathbb{Z}) = SL_2(Z)/\{\pm 1\}$ acting *faithfully* on $\mathcal{H}$, which means that every element other than the identity acts in a non-trivial way. To prove this, we can suppose that $z = \frac{az+b}{cz+d}$ for all $z \in \mathcal{H}$. Then for all z, we have that

$$cz^2 + (d-a)z - b = 0,$$

which implies that $a = d$ and $b = c = 0$. Since the only scalar matrices with determinant 1 are $\pm\mathbf{1}$, we have that only the identity acts trivially, as desired.

The following two matrices S and T generate Γ and can therefore be seen as the building blocks of our modular group:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If we consider the action of the two matrices under our fractional linear transformation, we have

$$Sz = \frac{-1}{z} \quad \text{and} \quad Tz = z+1.$$

In addition to that, $S^2 = \mathbf{1}$ and $(ST)^3 = \mathbf{1}$. With the help of these identities, one can show that there exists a *fundamental domain* D for the action of Γ on $\mathcal{H}$. Let G be a subgroup of Γ and recall that two points z_1, z_2 are said to be G-equivalent if there exists a $M \in G$ such that $z_2 = Mz_1$. Then the formal definition of a fundamental domain D is as follows.

Definition 3.1.1 (Kob)**.** Let D be a closed region in $\mathcal{H}$. Let every $z \in \mathcal{H}$ be G-equivalent to a point in D, but no two distinct points z_1, z_2 in the interior of D being G-equivalent. Then D is a fundamental domain for the subgroup G of Γ, where two boundary points are permitted to be G-equivalent.

Theorem 3.1.2. *The region* D*, defined as*

$$D = \{z \in \mathcal{H} | \frac{-1}{2} \leqslant \ Re\ (z) \leqslant \frac{1}{2} \text{ and } |z| \geqslant 1\},$$

is the fundamental domain for Γ*.*

The following picture shows the fundamental domain D for Γ.

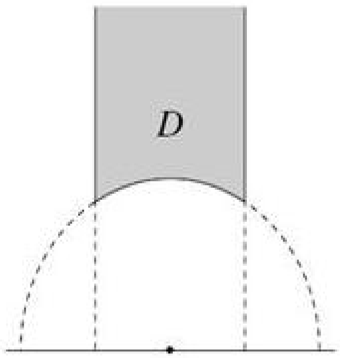

Figure 2. Fundamental Domain for $SL_2(\mathbb{Z})$

The proof of the theorem can be found in any of the recommended texts and is based on the idea that one can shift and translate any point into the fundamental domain by repeatedly applying S and T. Before we define modular forms, it is important to describe a few subgroups of Γ.

Let N be a positive integer. Then we define

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | a \equiv d \equiv 1(\mod N), b \equiv c \equiv 0(\text{mod } N) \right\}.$$

This subgroup is called the *principal congruence subgroup of* Γ *of level* N and in fact a normal subgroup of Γ. It is the kernel of the group homomorphism from Γ to $SL_2(\mathbb{Z}/N\mathbb{Z})$, reducing each entry modulo N. The following lemma from [Shi] summarises this fact as follows.

Lemma 3.1.3. *Let* $f : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ *be defined by* $f(M) = \alpha(\textit{mod } N)$*, then the sequence*

$$1 \to \Gamma(N) \to SL_2(\mathbb{Z}) \xrightarrow{f} SL_2(\mathbb{Z}/N\mathbb{Z}) \to 1$$

is exact.

Any subgroup of $SL_2(\mathbb{Z})$, which contains $\Gamma(N)$ for some N, is called a *congruence subgroup* of Γ. The most important congruence subgroups are

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma | c \equiv 0(\text{mod } N) \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma | a, d \equiv 1(\text{mod } N), c \equiv 0(\text{mod } N) \right\}.$$

If one wants to abbreviate the defined groups, then one can write these two congruence subgroups as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\text{mod } N) \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (\text{mod } N) \right\},$$

where $*$ means that there are no congruence conditions modulo N on this entry.

Given an element $z \in \mathcal{H}$, an *isotropy subgroup* of z is denoted by Γ_z and defined as

$$\Gamma_z = \{M \in \Gamma | Mz = z\}.$$

Thus, for a point in the upper-half plane and the modular group, we have the following lemma.

Lemma 3.1.4. *Let* $z \in D$*. Then* $\Gamma_z = \pm I$*, except in the following cases:*

(i) $\Gamma = \pm\{I, S\}$ *if* $z = i$.

(ii) $\Gamma = \pm\{I, ST, (ST)^2\}$ *if* $z = \omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.

(iii) $\Gamma = \pm\{I, TS, (TS)^2\}$ *if* $z = -\bar{\omega} = \frac{1}{2} + \frac{\sqrt{-3}}{2}$.

The points in $\mathcal{H}$ with non-trivial isotropy subgroups are called *elliptic points*.

3.2 MODULAR FORMS

In this section, we will review some of the basic properties of modular forms. The advanced reader can skip the first sections, returning to them for definitions and notation as needed.

Definition 3.2.1. Let k be an integer. A meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is called *weakly modular* of weight k with respect to Γ if it satisfies the following relation:

$$f(Mz) = (cz + d)^k f(z),$$

where $M \in \Gamma$ and $z \in \mathcal{H}$. A weakly modular function f is *modular* if it is meromorphic at infinity.

So considering the behaviour of weakly modular functions under the action of Γ, we see that to understand these functions on $\mathcal{H}$, it suffices to understand them on a fundamental domain D.

More precisely, since S and T generate Γ, one can easily show that for a meromorphic function to be weakly modular it suffices to satisfy the two relations

$$f(z + 1) = f(z) \text{ and } f(-1/z) = z^k f(z).$$

This implies that f can be expressed as

$$f = \sum_{-\infty}^{+\infty} a_n e^{2\pi i n z} = \sum_{-\infty}^{+\infty} a_n q^n.$$

We say that f is holomorphic at $i\infty$ if f has a Laurent expansion in a neighborhood of the origin with $a_n = 0$ for $n < 0$.

Definition 3.2.2. Let k be an integer. A weakly modular function f which is holomorphic on $\mathcal{H}$ and holomorphic at $i\infty$ is said to be a *modular form* of weight k.

Given this definition, we can write f as a series in the form

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} = \sum_{n=0}^{\infty} a_n q^n.$$

The series converges if $|q| < 1$ which is equivalent to the condition that $\text{Im}\,(z) > 0$. Now, if we are given two modular forms, say of weight k and l, then their product will be a modular form of weight $k + l$. In fact, the set of modular forms is a graded ring. If we denote the space of modular forms of weight k by $\mathcal{M}_k(SL_2(\mathbb{Z}))$, then

$$\mathcal{M}(SL_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k(SL_2(\mathbb{Z})).$$

Definition 3.2.3. Let k be an integer. A modular form f is called a *cusp form* of weight k if the leading coefficient of its Fourier Series is equal to 0. Therefore, f can be written in the form

$$f(z) = \sum_{n=1}^{\infty} a_n q^n,$$

where $q^n = e^{2\pi n z}$ and similarly to above, $\mathcal{S}_k(SL_2(\mathbb{Z}))$ denotes the vector space of cusp forms of weight k.

Given the Fourier expansion, the constant term a_0 is called the value of f at $i\infty$, denoted by $f(i\infty)$. The function f is a cusp form, if $a_0 = 0$ and the smallest n such that $a_n \neq 0$ is called the order of the zero of f at $i\infty$. Also, the cusp forms $\mathcal{S}_k(SL_2(\mathbb{Z}))$ are a vector subspace of the modular forms $\mathcal{M}_k(SL_2(\mathbb{Z}))$. Since a product of a modular form and a cusp form (*Spitzenform* in German) will still have a leading coefficient equal to zero, it should be clear that the graded ring

$$\mathcal{S}(SL_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} \mathcal{S}_k(SL_2(\mathbb{Z}))$$

is an ideal in $\mathcal{M}(SL_2(\mathbb{Z}))$.

Let us give an example of a modular form of weight k.

Example 3.2.4. Let us consider the Eisenstein series. Let k be an even integer and let m, n not both be zero. We have that

$$E_k(z) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \frac{1}{(mz+n)^k}$$

is a modular form of weight k for $k \geqslant 4$.

Proof. By reordering the sum, one can notice that the Eisenstein series transform nicely under the action of the modular group:

$$E_k(Tz) = E_k(z+1) = E_k(z)$$
$$E_k(Sz) = E_k\left(\frac{-1}{z}\right) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \frac{1}{(-m+nz)^k} = z^{-k} E_k(z).$$

Using the partial fraction expansion of the cotangent, we can show that the Fourier expansion of the Eisenstein series is given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

with $q = e^{2\pi i z}$ and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. ☐

Example 3.2.5. The function

$$j(z) = 1728 \frac{E_4(z)^3}{E_4(z)^3 - E_6(z)^2} = q^{-1} + 744 + 196884q + 21493760q^2 + \dots$$

is a modular function of weight 0.

Example 3.2.6. If we put $\Delta(z) = E_4(z)^3 - 27E_6(z)^2$, then $\Delta(z)$ is a cusp form of weight 12.

Definition 3.2.7. The *Dedekind eta function* $\eta(z)$ is defined as

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

The Dedekind eta function satisfies the transformation law, i.e.

$$\eta(-1/z) = \sqrt{-iz}\, \eta(z).$$

An interesting fact follows from $\eta^{24}(-1/z) = z^{12} \eta^{24}(z)$.

By comparing the coefficients of the Fourier expansion of our delta function and the Dedekind eta function, we obtain the following identity [Dia, page 20]:

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \eta^{24}(z).$$

We are now able to have a look at spaces of modular forms. To determine the spaces $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$, we will state a few well-known theorems which do not only help to identify when two modular forms turn out to be the same, but also give some insight into the structure of spaces of modular forms in general.

Theorem 3.2.8 (Kob). *Let $f(z)$ be a non-zero modular function of weight k for Γ. For a point $z \in \mathcal{H}$, let $\nu_z(f)$ denote the order of the zero of f at the point z. Let $\nu_\infty(f)$ denote the index of the first non-vanishing term in the q-expansion of $f(z)$. Then*

$$\nu_\infty(f) = \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\omega(f) + \sum_{z \in \Gamma \backslash \mathcal{H}, z \neq i, \omega} \nu_z(f) = \frac{k}{12}.$$

The proof to this theorem can be found in any of the recommended texts and can also be generalised to congruence subgroups. The idea of the proof is to count the zeros and the poles in $\Gamma \backslash \mathcal{H}$ by using contour integration along the boundary of the fundamental domain D. A fairly detailed version of the proof is given in [Kob] and [Ser].

As a consequence of this theorem, we obtain the following result.

Corollary 3.2.9 (Kob). *Let k be an even integer. Then*

(i) The only modular forms of weight 0 for Γ are constants, i.e. $\mathcal{M}_0(\Gamma) = \mathbb{C}$.

(ii) $\mathcal{M}_k(\Gamma) = 0$ if k is negative or $k = 2$.

(iii) $\mathcal{M}_k(\Gamma)$ is one-dimensional, generated by E_k, if $k = 4, 6, 8, 10$ or 14. In other words, $\mathcal{M}_k(\Gamma) = \mathbb{C}E_k$ for those values of k.

(iv) $\mathcal{S}_k(\Gamma) = 0$ if $k < 0$ or $k = 14$. $\mathcal{S}_{12}(\Gamma) = \mathbb{C}\Delta$ and for $k > 14$, we have $\mathcal{S}_k(\Gamma) = \Delta\mathcal{M}_{k-12}(\Gamma)$. This implies that the cusp forms of weight k are obtained by multiplying modular forms of weight $k - 12$ by the function $\Delta(z)$.

(v) $\mathcal{M}_k(\Gamma) = \mathcal{S}_k(\Gamma) \oplus \mathbb{C}E_k$ *for* $k > 2$.

From the previous theorem, one can see that the terms on the left-hand side will all be non-negative for any given modular form. Thus, by taking different values of k, there are only so many possible ways of choosing $\nu_z(f)$ so that the theorem still holds.

As a final result for this section, the algebra of modular forms $\mathcal{M}(\Gamma)$ can be described as follows.

Theorem 3.2.10 (Ebe)**.** *The algebra* $\mathcal{M}(\Gamma)$ *of modular forms is isomorphic to the polynomial algebra* $\mathbb{C}[E_4, E_6]$ *of complex polynomials in the Eisenstein series* E_4 *and* E_6*. Thus, any* $f \in \mathcal{M}(\Gamma)$ *can be written in the form*

$$f(z) = \sum_{4i+6j=k} c_{i,j} E_4(z)^i E_6(z)^j.$$

A nice proof of this theorem can be found in [Lan], using induction on k. Similarly, one can consider the algebra of modular forms for different congruence subgroups, for example $\Gamma_0(2)$, and determine whether they form a polynomial algebra or not.

Corollary 3.2.11. *Let* k *be a positive integer. The dimension of* $\mathcal{M}(\Gamma)$ *is*

$$\dim \mathcal{M}(\Gamma) = \begin{cases} [\frac{k}{6}] & \text{if } k \equiv 1 (\text{mod } 6) \\ [\frac{k}{6}] + 1 & \text{if } k \not\equiv 1 (\text{mod } 6). \end{cases}$$

We define the *factor of automorphy* $j(M, z) \in \mathbb{C}$ for a matrix $M \in \Gamma$ and $z \in \mathcal{H}$ as

$$j(M, z) = cz + d.$$

Note that it is possible to generalise the factor of automorphy to other matrix groups.

Definition 3.2.12 (Dia)**.** Let k be an integer and let $M \in \Gamma$. Then the *weight-*k *operator* $[M]_k$ on functions $f : \mathcal{H} \to \mathbb{C}$ is defined as

$$(f[M]_k)(z) = j(M, z)^k f(M(z)), \text{ for } z \in \mathcal{H}.$$

If we have second look at our Figure 1, the fundamental domain $\mathcal{D}$, then we can identify the two vertical sides by T and the two arcs of the circle $|z| = 1$ by S. This identification yields a topological space $\Gamma \setminus \mathcal{H}$ which looks like a punctured 2-sphere. Therefore, we can extend our definition of the upper-half plane in the following way.

Definition 3.2.13 (Sil)**.** The extended upper half-plane $\bar{\mathcal{H}}$ is the union of the upper half-plane $\mathcal{H}$ and the rational points including a point at infinity,

$$\bar{\mathcal{H}} = \mathcal{H} \cup \mathbb{Q} \cup \{i\infty\}.$$

The point at infinity can be visualised as far up the imaginary axis and is sometimes also denoted simply as "$\{\infty\}$". The action of Γ is transitive on the points $\mathbb{Q} \cup \{i\infty\}$, called *cusps*, since any fraction $\frac{a}{c}$, with $(a, c) = 1$, can be completed to a matrix which lies in Γ. Thus, all the rational points lie in the same equivalence class as $\{i\infty\}$. If we consider a subgroup of Γ, say Γ', then the action on $\mathbb{Q} \cup \{i\infty\}$ is generally not transitive. We have usually more than just one equivalence class when dealing with congruence subgroups of Γ. It is then common to use the most convenient representative of the equivalence class and say that for example "Γ has a single cusp at $i\infty$". It is obvious that ∞ in the case for $SL_2(\mathbb{Z})$ could be replaced with any other fraction $\frac{a}{c}$.

It can be proved that $SL_2(\mathbb{Z}) \setminus \bar{\mathcal{H}}$ has only one cusp and for any congruence subgroup Γ' of Γ, we have that $\Gamma' \setminus \bar{\mathcal{H}}$ has finitely many cusps. Note that there exist more modular forms when dealing with congruence subgroups. Let $\frac{a}{c} \in \bar{\mathcal{H}}$ be a representative of a cusp and let M_0 be the corresponding element in Γ such that $M_0(i\infty) = \frac{a}{c}$. Then we have the following definition of modular forms for congruence subgroups.

Definition 3.2.14 (Kob)**.** Let k be an integer and let $\Gamma' \leqslant \Gamma$ be a congruence subgroup of level N. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is called a *modular form of weight* k *for* Γ', if

$$(f[M]_k)(z) = f(z) \text{ for all } M \in \Gamma'$$

and if for any $M_0 \in \Gamma$,

$$(f[M_0]_k)(z) \text{ has the form } \sum_{n=0}^{\infty} a_n q^n \text{ with } a_n = 0 \text{ for } n < 0.$$

The last condition describes what it means for a modular form to be holomorphic at a cusp. Hence, in addition to being holomorphic on $\mathcal{H}$, a modular form for Γ' must be holomorphic at all cusps. A cusp form for Γ' must vanish at all cusps, i.e. $a_0 = 0$ for every $M_0 \in \Gamma$.

3.2.1 *Moonshine for the Monster*

The proof of the classification of finite simple groups was published around 1980 and comprises the work of more than a hundred mathematicians, consisting of several thousand journal pages. After the proof had been announced, several mathematicians tried to simplify parts of the proofs and discovered several gaps. It was only a few years ago that M. Aschbacher and S.D. Smith succeeded in closing the last gaps in their 1,300-page pre-print.

Theorem 3.2.15. *Classification Theorem [Asc] Each finite simple group is isomorphic to one of the following groups:*

(1) A group of prime order.

(2) An alternating group A_n *for* $n \geqslant 5$*.*

(3) A group of Lie type.

(4) One of the 26 *sporadic groups.*

An example of a group of Lie type is $PSL_2(\mathbb{F}_q)$, that is the quotient of $SL_2(\mathbb{F}_q)$ by its centre. The sporadic groups do not fit into any of the categories and range from the Mathieu group M_{11} to the largest sporadic simple group, the so-called *Monster* $\mathbb{M}$. The monster group contains all but 6 of the other sporadic groups as subgroups or subquotients. This group was predicted by Fischer and Griess in 1973 and constructed by Griess in 1982. Whereas the M_{11} is of order 7920, the Monster $\mathbb{M}$ has order

$$|\mathbb{M}| = 2^{45} \times 3^{20} \times 5^9 \times 7^6 \times 11^2 \times 13^3 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71 \approx 8 \times 10^{53}.$$

The character table of the Monster can be found in the Atlas and the number of conjugacy classes is equal to 194. Now, the dimension of the first few irreducible representations of the monster group are

$$1,\ 196883,\ 21296876,\ 842609326,\ \ldots.$$

Almost 30 years ago, J. McKay noticed that

$$\begin{aligned} 1 &= 1 \\ 196884 &= 196883 + 1 \\ 21493760 &= 21296876 + 196883 + 1. \end{aligned}$$

Recall the j-function from the previous section.

$$j(z) = \frac{\theta_{E_8}(z)^3}{\Delta(z)} = 1728 \frac{E_4(z)^3}{E_4(z)^3 - E_6(z)^2} = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots.$$

It appears that the coefficients of the elliptic modular function j are related to the dimensions of the irreducible representations of $\mathbb{M}$. This led Conway and Norton in 1979 [Con1] to the *moonshine conjecture*. Moonshine here refers to what was considered to be the completely unlikely, *outlandish* relation between sporadic groups and modular functions. The conjecture states that there exists an infinite-dimensional $\mathbb{Z}$-graded $\mathbb{M}$-module

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

for each element g in the Monster such that the McKay-Thompson series

$$T_g(z) = \sum_{n \in \mathbb{Z}} \mathrm{tr}\,(g|V_n) q^n$$

is a modular function of weight 0 for genus 0 subgroups of $SL_2(\mathbb{Z})$. The first essential step in the proof of this conjecture was made in 1988, when Frenkel-Lepowsky-Meurman [Fre] constructed a graded infinite-dimensional representation of $\mathbb{M}$, namely $V^\natural$. We have already encountered the rich algebraic structure of $V^\natural$ and, indeed, Aut $(V^\natural) \cong \mathbb{M}$. They constructed a suitable candidate for V, with $T_1(z) = j(z) - 744$, but could not prove in general that the McKay-Thompson series satisfied the required properties. This was eventually proved by Borcherds applying the theory of generalised Kac-Moody algebras [Bor1]. He constructed, using the module V, a generalised Kac-Moody algebra, namely the monster Lie algebra of chapter 2. The denominator identity of this Lie algebra

$$\frac{1}{q_1} \prod_{\substack{n_1 > 0 \\ n_2 \in \mathbb{Z}}} (1 - q_1^{n_1} q_2^{n_2})^{[j](n_1 n_2)} = j(z_1) - j(z_2)$$

gives a product expansion of the function j. The exponent $[j](n)$ denotes the coefficients at q^n in the Fourier expansion of j. Borcherds showed that the monster acts naturally on the monster Lie algebra and calculated the corresponding twisted denominator identities. These identities imply that the McKay-Thompson series are modular functions of weight 0 for genus 0 groups. Borcherds received the Fields Medal for his astonishing work in 1998 and described his feeling in an interview with the following words (Guardian, 28 August, 1998): "I was over the moon when I proved the moonshine conjecture. If I get a good result I spend several days feeling really happy about it. I sometimes wonder if this is the feeling you get when you take certain drugs. I don't actually know, as I have not tested this theory of mine."

3.3 AUTOMORPHIC FORMS ON ORTHOGONAL GROUPS

The autormophic forms on orthogonal groups are meromorphic functions on Grassmannians transforming nicely under discrete subgroups of the orthogonal groups $O_{n,2}(\mathbb{R})$.

Let L be an even lattice of signature $(n, 2)$ with $n \geqslant 2$ and $V = L \otimes \mathbb{R}$. Note that a negative definite form is the negative of a positive form with signature $(0, n)$. The orthogonal group $O^+(V)$ acts on the Grassmannian

$$D = \{Z \subset V | \dim (Z) = 2, Z \text{ negative definite}\},$$

D has a realisation as a tube domain $H \subset \mathbb{C}^n$ and the above definition of an automorphic form carries over to $\Gamma \subset O^+(L)$ as follows:

Definition 3.3.1. A meromorphic function f on $\mathcal{H}$ is called an automorphic form of weight k for a subgroup Γ of finite index in $O^+(L)$ if

$$f(Mz) = j(M, z)^k f(z)$$

for all $M \in \Gamma$. Here $j(M, z)$ is an automorphy factor for $O^+(V)$. If f has weight $n/2 - 1$, then we say that f has singular weight.

Borcherds' singular theta correspondence is a map from modular forms for the Weil representation to automorphic functions on orthogonal groups [Bor5]. In the following, we want to give a little more detail on how this lift works.

Let L be an even lattice of even rank and L' the dual lattice of L. The discriminant form of L is the finite abelian group $D = L'/L$ with quadratic form $\gamma^2/2 \bmod 1$. The level of D is the smallest positive integer N such that $N\gamma^2/2 = 0 \bmod 1$ for all $\gamma \in D$. Now, the Weil representation of $SL_2(\mathbb{Z})$ on $\mathbb{C}[D]$ is defined as

$$\rho_D(T)e^\gamma = e(-\gamma^2/2)e^\gamma$$

and

$$\rho_D(S)e^\gamma = \frac{e(\text{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} e((\gamma, \beta))e^\beta$$

where S and T are the two generators of $SL_2(\mathbb{Z})$.

Definition 3.3.2. A holomorphic function

$$F(z) = \sum_{\gamma \in D} F_\gamma(z)e^\gamma$$

on $\mathcal{H}$ with values in $\mathbb{C}[D]$ is called an *almost* modular form for ρ_D of weight k if

$$F(Mz) = j(M, z)^k \rho_D(M)F(z)$$

for all $M \in SL_2(\mathbb{Z})$ and F is meromorphic at $i\infty$.

Example 3.3.3. Let L be a positive definite even lattice of even rank $2k$. For $\gamma \in D$, we can define the theta function as

$$\theta_{\gamma+L}(z) = \sum_{\alpha \in \gamma + L} q^{\alpha^2/2}.$$

Then $\theta(z) = \sum_{\gamma \in D} \theta_{\gamma+L}(z)e^\gamma$ is a modular form for the dual Weil representation of weight k, which is holomorphic at $i\infty$.

Suppose D has level N. Then we can construct modular forms which transform under the Weil representation by lifting scalar valued modular forms on $\Gamma_0(N)$. The identity e^0 in $\mathbb{C}[D]$ is up to a character value invariant under $\Gamma_0(N)$. Thus, if f is a modular form on $\Gamma_0(N)$ of weight k and character χ_D, then

$$F(z) = \sum_{M \in \Gamma_0(N) \backslash SL_2(\mathbb{Z})} f|_M(z)\rho_D(M^{-1})e^0$$

is a modular form for ρ_D of weight k. This map can be used to construct Eisenstein series for the Weil representation by lifting scalar valued Eisenstein series on $\Gamma_0(N)$.

Borcherds' singular theta corrpesondence maps modular forms with poles at cusps transforming under the Weil representation of $SL_2(\mathbb{Z})$ to automorphic forms on orthogonal groups. Since these automorphic functions can be written as infinite products, they are called *automorphic products*. If we take a modular form F for ρ_D and integrate it with an Eisenstein series E, then by exponentiation, we obtain the automorphic product Ψ corresponding to F.

When we discuss the classification of generalised Kac-Moody algebras, we will see the reflective automorphic products play a very important role. The zeros and poles of automorphic products lie on divisors $D_\lambda = \{Z \in D | Z \perp \lambda\}$ where λ is a primitive vector of positive norm in L. Also, a *root* of L is a primitive vector α of positive norm such that the reflection $\sigma_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$ is an automorphism of L. Then an automorphic product Ψ is called *reflective* if its divisors are zeros of order 1 and are orthogonal to roots of L. Also, Ψ is called *completely reflective* if Ψ is reflective and all roots of L give zeros of order 1. We will see examples of automorphic products in the beginning of the next chapter.

4

RECENT RESULTS

"Inanimate objects can be classified scientifically into three major categories; those that don't work, those that break down and those that get lost."

— Russell Baker

4.1 RECENT RESULTS ON GENERALISED KAC-MOODY ALGEBRAS AND AUTOMORPHIC FORMS

In this chapter we will discuss some recent results concerning the classification of automorphic products and generalised Kac-Moody algebras. We will review and state some of Scheithauer's results [Sch3] and then give some open problems with a more detailed strategy on how to proceed.

4.1.1 *Denominator identities as automorphic forms*

The denominator identities of generalised Kac-Moody algebras are sometimes automorphic forms.

The Matthieu group M_{23} acts on the Leech lattice Λ. Given an element g of order N, we can write its characteristic polynomial as

$$\prod_{k|N}(x^k - 1)^{b_k}.$$

If N is square-free, then $b_k = 24/\sigma_1(N)$ for all $k|N$. Recall that

$$\eta(z) = q^{1/24} = \prod_{n>0}(1 - q^n).$$

Then we define a modular form for each element g as

$$\eta_g(z) = \prod_{k|N}\eta(kz)^{b_k}.$$

We lift the modular form $f_g = 1/\eta_g$ to a vector valued modular form F_g on the lattice

$$\Lambda^g \oplus II_{1,1} \oplus II_{1,1}(N).$$

Next, we apply the singular theta correspondence to obtain an automorphic form Ψ_g. This can be represented by the following diagram

$$g \mapsto 1/\eta_g \mapsto F_g \mapsto \Psi_g.$$

An explicit calculation yields that Ψ_g has singular weight. Also, the theta correspondence gives the product expansion of Ψ_g at the different cusps.

Theorem 4.1.1 (Sch3). *Let g be an element of M_{23} of order N. Then g corresponds naturally to a reflective automorphic product Ψ of singular weight on the lattice $\Lambda^g \oplus II_{1,1} \oplus II_{1,1}(N)$. If N is square-free, then Ψ is completely reflective and the expansion of Ψ is given by*

$$e((\rho, z)) \prod_{k|N} \prod_{\alpha \in (L \cap kL')^+} (1 - e((\alpha, z)))^{[1/\eta_g](-\alpha^2/2k)}$$
$$= \sum_{w \in W} \det(w) \eta_g((w\rho, z))$$

where W is the reflection group of the lattice $L = \Lambda^g \oplus II_{1,1}$. This identity is the denominator identity of a generalised Kac-Moody algebra.

The theorem gives a denominator identity of a generalised Kac-Moody algebra and an automorphic form of singular weight for an orthogonal group. In this way we obtain 10 generalised Kac-Moody algebras, corresponding to elements of square-free level in M_{23}, which are very similar to the fake monster Lie algebra described in chapter 2.

4.1.2 *Classification results*

The classification of Kac-Moody algebras requires certain assumptions on the Cartan matrices. In particular, we have that the Cartan matrix must be finite, which is not a reasonable assumption for generalised Kac-Moody algebras. The most interesting examples, i.e. the monster Lie algebra and the fake monster Lie algebra, have infinitely many simple roots and thus infinite Cartan matrices. Given that the denominator identities of some generalised Kac-Moody algebras are automorphic forms of singular weight for orthogonal groups, it is natural to consider whether such algebras can be classified. We will show that the 10 Lie algebras constructed in the previous section are the only generalised Kac-Moody algebras whose denominator identities are completely reflective automorphic products of singular weight on lattices of square-free level and positive signature [Sch3].

We know that reflective automorphic products come from lifting reflective modular forms, which transform under the Weil representation. Given the theorem above, we can give a necessary condition for the existence of a reflective form by pairing it with an Eisenstein series. This tool has been used to obtain the following classification result.

Theorem 4.1.2 (Sch3)**.** *Let* L *be an even lattice of signature* $(n, 2)$ *with* $n > 2$ *and square-free level* N*. Suppose* L *splits 2 hyperbolic planes. Let* G *be a real generalised Kac-Moody algebra whose denominator identity is a completely reflective automorphic product of singular weight on* L*. Then* G *can be constructed as seen before, by an element of order* N *in* M_{23}*.*

Sketch of the proof:

By lifting a vector valued modular function $F = \sum F_\gamma e^\gamma$ for the Weil representation of weight $2-k$ where $k = 1 + n/2$, we obtain the denominator identity of G. Let $E = \sum E_\gamma e^\gamma$ be the Eisenstein series for the dual Weil representation. Then $\sum F_\gamma E_\gamma$ is a scalar modular function on $SL_2(\mathbb{Z})$ of weight 2 with a pole at the cusp $i\infty$. By the residue theorem, the constant coefficient in the Fourier expansion of $\sum F_\gamma E_\gamma$ vanishes.

The equation has only very few solutions, namely only those which come from elements with square-free order in M_{23}. If L has even p-ranks n_p, the equation parametrising completely reflective automorphic products is given by

$$\frac{k}{k-2}\,\frac{1}{B_k}\prod_{p|N}\frac{1}{p^k-1}\left(\epsilon_p\left(\frac{-1}{p}\right)^{n_p/2}(p^{k-n_p/2}+p^{n_p/2})-2\right) = 1.$$

This is an equation in the indeterminates N, k, ϵ_p and n_p and has exactly 8 solutions.

4.2 OPEN PROBLEMS

Here are some interesting open problems in the theory of generlised Kac-Moody algebras and automorphic products.

(i) Scheithauer classified those generalised Kac-Moody algebras whose denominator identities are reflective automorphic products of singular weight on lattices of squarefree level. There are 10 such Lie algebras and 2 exceptional cases where it is still unknown if the Lie algebra exists. The exceptional cases can be obtained by using theorem 4.2, not assuming that L has to split 2 hyperbolic planes.

(ii) In chapter 2, we have already seen that for $N = 1$, the generalised Kac-Moody algebra can be realised as the cohomology group of the BRST-operator Q acting on the vertex superalgebra $V_{II_{25,1}} \otimes \mathbb{Z}_{\mathbb{Z}\sigma}$. Show that the 10 generalised Kac-Moody algebras whose denominator identities are reflective automorphic products of singular weight on lattices of squarefree level describe bosonic strings moving on a torus moving on suitable space times.

(iii) The denominator identity of some generalised Kac-Moody algebras are automorphic forms on orthogonal groups. Is there a relation between automorphic forms and the representation theory of these Lie algebras?

Let us describe a possible approach to problem (i) and (ii) in more detail.

By [Sch3], the exceptional cases can only occur in level 30 and 182. Thus, to attack the first problem, we consider the congruence subgroups $\Gamma_0(30)$ and $\Gamma_0(182)$. In particular, we analyse the corresponding cusp forms of weight 2 and trivial character and of weight 1 and character $\chi(j) = \left(\frac{j}{7}\right)$, respectively. Once we have constructed a basis of the vector spaces, we try to find suitable functions which can be lifted to vector valued modular forms, using [Sch3]. Applying Borcherds' singular theta correspondence [Bor5], it could then be shown that the exceptional cases in the classification correspond to existing automorphic forms and generalised Kac-Moody algebras. This result would be very important, because it would give the first example of reflective automorphic products of singular weight and generalised Kac-Moody algebras which do not correspond to automorphisms of the Leech lattice.

For the second problem, one possible approach is as follows. Since each of the 10 Lie algebras correspond to a unique element g in the automorphism group of the Leech lattice Λ, one could tensor the vertex algebra of the fixpoint lattice Λ^g with a suitable orbifold vertex algebra of $\Lambda^{g\perp}$. This yields a vertex algebra V of conformal weight 24. Then the physical state of the bosonic string with vertex algebra $V \otimes V_{II_{1,1}}$ gives a natural realisation of the generalised Kac-Moody algebras corresponding to g.

5

BIBLIOGRAPHY

The following references can be highly recommended and discuss most of the material presented in this thesis.

[Apo] Tom M. Apostol. *Modular Functions and Dirichlet Series in Number Theory*. Second Edition, Graduate Texts in Mathematics, Springer-Verlag, 1990.

[Asc] Michael Aschbacher. *The Status of the Classification of the Finite Simple Groups*. Notices of the American Mathematical Society, Volume 51, Number 7, pages 736-740, 2004.

[Bor1] Richard E. Borcherds. *Generalized Kac-Moody algebras*. J. Algebra 115, 501-512, 1988.

[Bor2] Richard E. Borcherds. *What is the Monster?*. Notices of the American Mathematical Society, Volume 49, Number 9, pages 1076-10777, 2002.

[Bor3] Richard E. Borcherds. *Automorphic forms and Lie algebras*. Current Developments in mathematics 1996, International Press 1998.

[Bor4] Richard E. Borcherds. *Monstrous Moonshine and Monstrous Lie Superalgebras*. Inventiones Mathematicae. 109, 405-444, 1992.

[Bor5] Richard E. Borcherds. *Automorphic forms with singularities on Grassmannians*. Inventiones Mathematicae. 132, 491-562, 1998.

[Bor6] Richard E. Borcherds. *The monster Lie algebra*. Adv. Math. 83 , 30-47, 1990.

[Bor7] Richard E. Borcherds. *Central extensions of generalized Kac-Moody algebras*. J. Algebra 140, 330-335, 1991.

[Bor8] Richard E. Borcherds. *A characterization of generalized Kac-Moody algebras*. J. Algebra 174, 1073-1079, 1995.

[Car] Roger Carter. *Lie Algebras of Finite and Affine Type*. Cambridge Studies in advanced mathematics (96), Cambridge University Press, 2005.

[Con1] John Horton Conway and S.P.Norton. *Monstrous moonshine*. Bulletin of the London Mathematical Society 11 (1979), 308-339.

[Con2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson *Atlas of Finite Groups*. Clarendon Press, Oxford, 1985.

[Dia] Fred Diamond and Jerry Shurman. *A First Course in Modular Forms*. Graduate Texts in Mathematics, Springer-Verlag, 2005.

[Ebe] Wolfgang Ebeling. *Lattices and Codes*. 2nd revised edition, Vieweg Wiesbaden Braunschweig, 2002.

[Erd] Karin Erdmann and Mark J. Wildon. *Introduction to Lie Algebras*. Springer Undergraduate Mathematics Series, Springer-Verlag, 2006.

[Fre1]Edward Frenkel and D. Ben-Zvi. *Vertex Algebras and Algebraic Curves*. Mathematical Surveys and Monographs 88, American Mathematical Society, 2004.

[Fre2] I.Frenkel, J. Lepowsky, and A.Meurmann. *Vertex Operator Algebras and the Monster*. Pure and Applied Mathematics, Volume 134, Academic Press, San Diego, 1988.

[Gan1] Terry Gannon. *Moonshine Beyond the Monster*. Cambridge Monographs On Mathematical Physics, Cambridge University Press, 2006.

[Gan2] Terry Gannon. *Monstrous Moonshine: The first twenty-five years*. Bulletin of the London Mathematical Society 2006, 38(1), pages 1-33.

[Hum] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Graduate Texts in Mathematics, Springer-Verlag, 1997.

[Kac1] Victor G. Kac. *Infinite Dimensional Lie Algebras*. Progress in Mathematics, Birkhäuser, 1984.

[Kac2] Victor G. Kac. *Vertex Algebras for Beginners*. University Lecture Series, Volume 10, American Mathematical Society, 1996.

[Kna] Anthony W. Knapp. *Elliptic Curves*. Mathematical Notes, Princeton University Press, 1992.

[Kob] Neal Koblitz. *Introduction to Elliptic Curves and Modular Forms*. Second Edition, Graduate Texts in Mathematics, Springer-Verlag, 1993.

[Lan] Serge Lang. *Introduction to Modular Forms*. Grundlehren der mathematischen Wissenschaften, Volume 222, Springer-Verlag, 1976.

[Miy] Toshitsune Miyake. *Modular Forms*. Springer Monographs in Mathematics, Springer-Verlag, 1989.

[Moo] Robert V. Moody and Arturo Pianzola. *Lie Algebras With Triangular Decomposition* Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley-Interscience Publication, 1995.

[Ray] Urmie Ray. *Automorphic Forms and Lie Superalgebras*. Algebra and Applications, Springer-Verlag, 2006.

[Sch1] Nils R. Scheithauer. *Generalized Kac-Moody algebras, automorphic forms and Conway's group II*. To appear in J. reine angew. Math.

[Sch2] Nils R. Scheithauer. *The Weil representation of* $SL_2(\mathbb{Z})$ *and some applications*. Preprint, submitted.

[Sch3] Nils R. Scheithauer. *On the classification of automorphic products and generalised Kac-Moody algebras*. Inventiones Mathematicae, 164, 641-678, 2006.

[Ser1] Jean-Pierre Serre. *A Course in Arithmetic*. Graduate Texts in Mathematics, Springer-Verlag, 1973.

[Ser2] Jean-Pierre Serre. *Complex Semisimple Lie Algebras*. Springer Monographs in Mathematics, Springer-Verlag, 2001.

[Shi] Goro Shimura. *Introduction to the Arithmetic Theory of Automorphic Functions*. Publications of the Mathematical Society of Japan 11, Princeton University Press, 1971.

[Sil] Joseph H. Silverman. *Advanced Topics in the Arithmetic of Elliptic Curves*. Graduate Texts in Mathematics, Springer-Verlag, 1994.

COLOPHON

This thesis was typeset with $\LaTeX\,2_\varepsilon$ using Hermann Zapf's *Palatino* and *Euler* type faces (Type 1 PostScript fonts *URW Palladio L* and *FPL* were used). The listings are typeset in *Bera Mono*, originally developed by Bitstream, Inc. as "Bitstream Vera". (Type 1 PostScript fonts were made available by Malte Rosenau and Ulrich Dirr.)

The typographic style was inspired by Robert Bringhurst's genius as presented in *The Elements of Typographic Style* (Version 2.5, Hartley & Marks, 2002).

DECLARATION

I certify that this project report has been written by me, is a record of work carried out by me, and is essentially different from work undertaken for any other purpose or assessment. In submitting this project report to the University of Edinburgh, I give permission for it to be made available for use in accordance with the regulations of the University Library. I also give permission for the title and abstract to be published and for copies of the report to be made and supplied at cost to any bona fide library or research worker. I retain the copyright of the work.

Edinburgh, United Kingdom , August 2008

Andreas V. Weinert