# The Fourier restriction phenomenon in thin sets

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Doctor of Philosophy
University of Edinburgh
2010

# Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Christos Papadimitropoulos*)

*To my family*

# Abstract

We study the Fourier restriction phenomenon in settings where there is no underlying proper smooth subvariety. We prove an $(L^p, L^2)$ restriction theorem in general locally compact abelian groups and apply it in groups such as $(\mathbb{Z}/p^L\mathbb{Z})^n$, $\mathbb{R}$ and locally compact ultrametric fields $K$.

The problem of existence of Salem sets in a locally compact ultrametric field $(K, |\cdot|)$ is also considered. We prove that for every $0 < \alpha < 1$ and $\epsilon > 0$ there exist a set $E \subset K$ and a measure $\mu$ supported on $E$ such that the Hausdorff dimension of $E$ equals $\alpha$ and $|\widehat{\mu}(x)| \leq C|x|^{-\frac{\alpha}{2}+\epsilon}$.

We also establish the optimal extension of the Hausdorff-Young inequality in the compact ring of integers $R$ of a locally compact ultrametric field $K$. We shall prove the following: For every $1 \leq p \leq 2$ there is a Banach function space $F^p(R)$ with $\sigma$-order continuous norm such that
(i) $L^p(R) \subsetneq F^p(R) \subsetneq L^1(R)$ for every $1 < p < 2$.
(ii) The Fourier transform $\mathcal{F}$ maps $F^p(R)$ to $\ell^{p'}$ continuously.
(iii) $L^p(R)$ is continuously included in $F^p(R)$ and $F^p(R)$ is continuously included in $L^1(R)$.
(iv) If $Z$ is a Banach function space with the same properties as $F^p(R)$ above, then $Z$ is continuously included in $F^p(R)$.
(v) $F^1(R) = L^1(R)$ and $F^2(R) = L^2(R)$.

# Acknowledgements

The present study would have hardly been completed had my supervisor Professor Jim Wright not been a scholar of such knowledge, professionalism, compassion and most of all patience. To him I duly owe my deepest gratitude.

I would like to thank my second supervisor Professor Tony Carbery for his guidance and support.

Many thanks to my friends Mike, Daniel, and Graeme with whom I shared the same room.

I would also like to thank the following persons who have been of great help in many ways during my studies in Scotland: Fr Raphael Pavouris, Fr John Maitland-Moir, Fr Avraamy Neyman, Fr John Rafan, Deacon Luke Jeffrey, Matthaios Tsolakis and his family, Themis Pavouri, Marion Jeffrey, the Orthodox Community of St Andrew in Edinburgh.

I am also grateful to my family who gave me so much love.

It is on purpose that I left until last the person to whom I owe the most throughout my studies in the University of Athens: Professor Vasili Nestoridis. Not only has he offered me knowledge but also honoured me through recommending me to the University of Edinburgh.

# Contents

# Chapter 1

# Introduction

The Fourier transform of a function $f \in L^p(\mathbb{R}^d),\ p > 1$, can be very singular on hyperplanes of $\mathbb{R}^d,\ d \geq 2$. It is easy to construct an example of $L^p$ function whose Fourier transform is infinite on a hyperplane. However it is a remarkable fact that, given a smooth submanifold $M$ of $\mathbb{R}^d$ with appropriate curvature, there is a range of $p$'s depending on $M$ such that the Fourier transform of an $L^p(\mathbb{R}^d)$ function can be restricted on $M$. This is known as the Fourier restriction phenomenon and was initially discovered by Stein [34] in the early 60's. More precisely the restriction problem can be formulated by the following inequality

$$\left(\int_{M_0} |\widehat{f}(x)|^q d\sigma(x)\right)^{1/q} \leq C_{p,q,M_0} \left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{1/p}, \qquad (1.1)$$

where $M_0$ is an open subset of $M$ with compact closure in $M$ and $d\sigma$ is the induced Lebesgue measure in $M$. The Fourier restriction phenomenon has been studied by many mathematicians who have developed various methods to prove restriction estimates. However much work has still to be done since, in most of the cases, it is difficult to improve the range of $p$'s and $q$'s where (1.1) holds and it is even more demanding to get some sharp results. In chapter 2 we shall give a little more history and discuss some positive results. The article [40] provides further information concerning what is known so far and the remaining open problems.

The Fourier restriction phenomenon is related to other problems in harmonic analysis such as the Bochner-Riesz summability [12], [4], [39], [35] and the Kakeya problem [5], [3], [43]. Moreover the restriction theorems have many applications to other areas of mathematics. Perhaps the best known application is the one concerning *a priori* estimates in partial differential equations for the wave and Schrödinger equations; this connection was first observed by Strichartz [37]. The restriction phenomenon is also related to certain problems in number theory; we mention the article of Green and Tao [14] and the references therein.

As we have already pointed out the restriction of the Fourier transform of a function on a submanifold of $\mathbb{R}^d,\ d \geq 2$, is closely related to the geometric curvature of the submanifold. For this reason, it was thought that the restriction problem is an aspect only of the $d$-dimensional Euclidean space with $d \geq 2$. However, recently Mockenhaupt [24] considered the restriction phenomenon on the real line where there is no underlying smooth subvariety. The key thing that Mockenhaupt used is that there are thin subsets of $\mathbb{R}$ carrying measures whose Fourier transform decays at infinity with a certain rate. It turns out that this is sufficient to yield Fourier restriction results on fractal sets in $\mathbb{R}$. However, the geometric curvature has a twofold role in the restriction problem on $\mathbb{R}^d$, $d \geq 2$. First it allows the Fourier transform of certain measures to have decay properties; secondly it implies necessary conditions for the range of $p$'s and $q$'s where (1.1) holds. And although the first implication of curvature is treated in the case of thin sets in $\mathbb{R}$ as described above via the decay of the Fourier transforms of measures, the counterpoint for the second one concerning necessary conditions is not well understood in the one dimensional case. It is believed that long arithmetic progressions in fractal sets might play the role of curvature and imply some necessary conditions. However there is very little known concerning this matter and that is why we do not really understand the restriction problem in $\mathbb{R}$. We refer to recent work of Laba and Pramanik [20] which provides information about arithmetic progressions in thin sets.

Another surprising fact is that the Fourier restriction phenomenon has recently been investigated in discrete settings where the geometric curvature is completely non-existent. More precisely some authors have considered the restriction problem in $(\mathbb{Z}/N\mathbb{Z})^n$. When $N$ is a prime number the problem has been investigated by Mockenhaupt and Tao [27], and for general $N$ by Wright [44]. We mention that the methods of proofs between the above two cases are different. The reason for this is that for general integers $N$, one can define a certain kind of 'scaled balls' using the divisors of $N$. We shall give more details later.

The Fourier restriction phenomenon in $\mathbb{R}$ has proved to be a very useful tool in establishing the optimal extension of the Hausdorff-Young inequality on the torus. The Hausdorff-Young inequality asserts that the Fourier transform $\mathcal{F} : L^p(\mathbb{T}) \to \ell^{p'}$, $1 \leq p \leq 2$, $1/p + 1/p' = 1$, is a bounded operator. The question under consideration is to extend $\mathcal{F}$ continuously and in an optimal way (to be specified later) keeping the range space $\ell^{p'}$ fixed. The problem was completely solved recently by Mockenhaupt and Ricker [26]. Their work blends nicely vector measure theory and harmonic analysis. Among other arguments, the authors applied Fourier restriction theorems in Salem sets of the real line. We shall define Salem sets in Section 3.1; roughly speaking they are sets with the property that carry a measure whose Fourier transform obeys an optimal decay estimate. The authors of [26] raised the question whether it is possible to extend their results in other settings than $\mathbb{R}$, replacing $\mathbb{T}$ by other compact sets. A natural setting, as one may expect, is a general locally compact field $K$. It turns out that much of the Fourier and functional analysis in [26] can be readily extended to any locally compact field (see [31]). Slightly less obvious is the establishing of the restriction phenomenon in this setting. Perhaps more non-trivial, and even more interesting is the question of the existence of Salem sets in a general locally compact field; this forms the main part of this thesis.

In Chapter 2 we start by stating the Fourier restriction phenomenon on $\mathbb{R}^d$, $d \geq 2$, and especially on the unit sphere $S^{d-1}$. We then focus on the $(L^p, L^2)$ restriction problem and state Greenleaf's result [15] and Mockenhaupt's theorem [24]. The latter includes the 1-dimensional case about fractals discussed earlier. Next we prove a general $(L^p, L^2)$ restriction theorem in the setting of locally compact groups. This allows us to get restriction results on $(\mathbb{Z}/N\mathbb{Z})^n$, $N = p^L$, $L \in \mathbb{N}$, $L > 1$. More precisely we shall prove inequalities such as

$$\left( \frac{1}{|S|} \sum_{\xi \in S} |\widehat{f}(\xi)|^2 \right)^{1/2} \leq C \left( \sum_{x \in (\mathbb{Z}/p^L\mathbb{Z})^n} |f(x)|^t \right)^{1/t} , \quad 1 \leq t < t_0 , \tag{1.2}$$

where $S = \{(x_1, .., x_k, g_{k+1}(x_1, .., x_k), .., g_n(x_1, .., x_k)) : x_i \in \mathbb{Z}/p^L\mathbb{Z}\}$ and $g_i$ are polynomials satisfing certain properties. At the end of this chapter we consider

$$S = \{(x, x^m) : x \in \mathbb{Z}/p^L\mathbb{Z}\} ,$$

and prove that

$$\left( \frac{1}{p^L} \sum_{x=0}^{p^L-1} |\widehat{f}(x, x^m)|^2 \right)^{1/2} \leq C \left( \sum_{x=0}^{p^L-1} \sum_{y=0}^{p^L-1} |f(x, y)|^t \right)^{1/t} ,$$

for every $1 \leq t < \frac{2m+2}{2m+1}$. Using some scaling arguments we shall verify that the range of $t$'s is almost sharp.

In chapter 3 we give the definition of the Fourier dimension of a set and its relation with the Hausdorff dimension. We describe briefly how Frostman's lemma and capacitary dimension imply that the Fourier dimension cannot exceed the Hausdorff dimension. We then define Salem sets which is the main theme of this chapter. Salem sets were introduced by R.Salem in 1950 [33]. He constructed a random Cantor type set in $\mathbb{R}$ which almost surely has the property its Fourier dimension equals its Hausdorff dimension. Mockenhaupt [24] in 2000 obtained $(L^p, L^2)$ Fourier restriction results on Salem sets constructed in [33]. Motivated by the fact that restriction estimates can be obtained on fractal sets, we consider the set of the well approximable numbers

$$E(\alpha) = \{x \in \mathbb{R} : |nx - m| \leq \frac{1}{n^{1+\alpha}} \text{ for infinitely many rationals } \frac{m}{n}\} ,$$

$\alpha > 0$. Kaufman [19] proved that $E(\alpha)$ contains a Salem set with the same Hausdorff dimension as $E(\alpha)$. He constructed a measure $\mu$ supported on $E(\alpha)$ satisfying the property

$$|\widehat{\mu}(x)| \leq C \frac{\log(e + |x|)}{(1 + |x|)^{\frac{1}{2+\alpha}}} .$$

The main aim of this chapter is to prove the following regularity property

$$\mu(I) \leq C_\delta |I|^{\frac{2}{2+\alpha} - \delta} , \quad \forall\, I \subseteq \mathbb{R} \text{ interval} ,$$

and establish the $(L^p, L^2)$ restriction inequality

$$\int_{E(\alpha)} |\widehat{f}|^2 d\mu \leq C\|f\|^2_{L^p(\mathbb{R})} ,$$

for a certain range of $p$'s.

In Chapter 4 we start our study on locally compact fields. This chapter serves as a background for the chapters to come. The main references are [38] and [42]. The reader may wish to refer to these references for any well-known results stated without proof. However we provide some proofs wherever we think that these are not so complicated but needed for us to understand better the theory of local fields. We also point out that the notation used in this chapter is kept throughout the following chapters 5 and 6. Most of the material in Chapter 4 is devoted to a certain class of local fields, the so called ultrametric local fields. The topological structure of an ultrametric local field is quite strange. However, due to this structure, many analysis arguments and computations are simpler than the ones in the Euclidean setting. This becomes clearer in Section 4.3 where we study the notions of Fourier and Hausdorff dimensions in an ultrametric local field.

Chapter 5 forms the main part of this thesis. We follow Salem's probabilistic approach [33] and prove the existence of Salem sets in an ultrametric local field. In Section 5.1 we provide a series of lemmas which shall be used to establish the main result. In Section 5.2 we shall prove the following theorem [31]: *Let* $(K, |\cdot|)$ *be an ultrametric local field. For every* $0 < \alpha < 1$ *and* $\epsilon > 0$ *there are a set* $E$ *and a measure* $\mu$ *supported on* $E$ *such that the Hausdorff dimension of* $E$ *equals* $\alpha$ *and* $|\widehat{\mu}(x)| \leq C_\epsilon |x|^{-\frac{\alpha}{2}+\epsilon}$ *for every* $x \in K$. The proof relies on a Cantor type construction. For a fixed $0 < \alpha < 1$ and $\epsilon > 0$, one constructs a family of Cantor type sets, endows this family with an appropriate measure and proves that almost every set $E$ of the family with respect to this measure satisfies the above theorem.

In Chapter 6 we deal with the problem of the optimal extension of the Hausdorff-Young inequality in an ultrametric local field (this part of the thesis appears in [31]). We are motivated by a recent work of Mockenhaupt and Ricker [26] who proved the following: *For every* $1 \leq p \leq 2$ *there is a Banach function space* $F^p(\mathbb{T})$ *with* $L^p(\mathbb{T}) \subsetneq F^p(\mathbb{T}) \subsetneq L^1(\mathbb{T})$ *for every* $1 < p < 2$, *and such that the Fourier transform* $\mathcal{F} : F^p(\mathbb{T}) \to \ell^{p'}$, $1/p + 1/p' = 1$, *is bounded with* $\|\widehat{f}\|_{\ell^{p'}} \leq \|f\|_{F^p(\mathbb{T})}$ *for every* $1 \leq p \leq 2$. *The norm* $\|\cdot\|_{F^p(\mathbb{T})}$ *is* $\sigma$*-order continuous and satisfies* $\|f\|_{F^p(\mathbb{T})} \leq 4\|f\|_{L^p(\mathbb{T})}$ *and* $\|f\|_{L^1(\mathbb{T})} \leq \|f\|_{F^p(\mathbb{T})}$. *Moreover if* $Z$ *is any Banach function space over* $(\mathbb{T}, \mathcal{B}(\mathbb{T}), dt)$ *with the same properties as* $F^p(\mathbb{T})$, *then* $Z$ *is contained in* $F^p(\mathbb{T})$ *and* $\|f\|_{F^p(\mathbb{T})} \leq C\|f\|_Z$. *For* $p = 1, 2$ *we have* $F^1(\mathbb{T}) = L^1(\mathbb{T})$ *and* $F^2(\mathbb{T}) = L^2(\mathbb{T})$. In [26] the authors also proved that the optimal domain $F^p(\mathbb{T})$ for the Hausdorff-Young inequality can be described by

$$\begin{aligned} F^p(\mathbb{T}) &= \{f \in L^1(\mathbb{T}) : \int_{\mathbb{T}} |fh| < \infty \ \ \forall\, h \in L^{p'}(\mathbb{T}) \ \text{ with } \ \widehat{h} \in \ell^p\} \\ &= \{f \in L^1(\mathbb{T}) : \widehat{f\mathbf{1}_A} \in \ell^{p'} \ \ \forall\, A \in \mathcal{B}(\mathbb{T})\} \\ &= \{f \in L^1(\mathbb{T}) : \widehat{fg} \in \ell^{p'} \ \ \forall\, g \in L^\infty(\mathbb{T})\} \ . \end{aligned}$$

The aim of Chapter 6 is to transfer these results in the ultrametric local field setting. We mention that most of the proofs rely on vector measure theory. However the strict containment of $L^p$ in $F^p$ $(1 < p < 2)$ is proved using Fourier restriction results on Salem sets. Therefore we start this chapter by establishing an $(L^p, L^2)$ restriction theorem on Salem sets constructed in Chapter 5.

# Chapter 2

# The Fourier restriction phenomenon

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$, $\widehat{f}(x) = \int_{\mathbb{R}^n} f(y)e^{-2\pi i x\cdot y}dy$, is continuous and bounded and as such it can be restricted as a continuous function to any set $S \subseteq \mathbb{R}^n$. However, the Fourier transform defines a unitary map from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$ and therefore $\widehat{f}$ can be completely arbitrary on sets of measure zero. Generally, the Hausdorff-Young inequality guarantees that if $f \in L^p(\mathbb{R}^n)$ and $1 \leq p \leq 2$, then $\widehat{f} \in L^{p'}(\mathbb{R}^n)$ where $1/p + 1/p' = 1$. Hence, at first sight $\widehat{f}$ is defined only almost everywhere. It is a remarkable observation of Stein from 1960's that one can restrict the Fourier transform of a general $L^p(\mathbb{R}^n)$ function for some $p > 1$ to the unit sphere $S^{n-1}$, at least as a function in $L^2(S^{n-1})$. This is known as the Fourier restriction phenomenon. Subsequently in the last four decades many mathematicians have investigated on which sets $S$ of measure zero the Fourier transform of an $L^p(\mathbb{R}^n)$ function can be restricted to and for which values of $p$. If $S$ is a hyperplane in $\mathbb{R}^n$ and $p > 1$, then it is easy to construct an $L^p(\mathbb{R}^n)$ function whose Fourier transform on $S$ is always infinity. Therefore we are interested in sets $S$ with non-zero curvature. The case of $S = S^{n-1}$, the unit sphere in $\mathbb{R}^n$, is of particular interest since it is connected to other areas of Harmonic analysis, see [12]. If we denote by $d\sigma$ the measure on $S^{n-1}$ induced by Lebesgue measure on $\mathbb{R}^n$, then the $(L^p, L^2)$ restriction problem to $S^{n-1}$ is formulated by the *a priori* inequality

$$\|\widehat{f}\|_{L^2(S^{n-1},d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{2.1}$$

for every $1 \leq p \leq \frac{2n+2}{n+3}$. The range of $p$'s is optimal. For $n = 2$ this was established by Fefferman [12]. For $n \geq 3$ Tomas [41] proved (2.1) for $1 \leq p < \frac{2n+2}{n+3}$, and Stein [34] provided the endpoint estimate. Thus the $(L^p, L^2)$ restriction problem to $S^{n-1}$ is completely solved. However the general $(L^p, L^q)$ restriction problem to $S^{n-1}$ is still open. It is conjectured that

$$\|\widehat{f}\|_{L^q(S^{n-1},d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

if and only if $1 \leq p < \frac{2n}{n+1}$ and $p' \geq \frac{n+1}{n-1}q$. When $n = 2$ this conjecture was proved by Fefferman [12].

The restriction problem can be extended to other submanifolds of $\mathbb{R}^n$. In the case of compact subsets of the paraboloid the general $(L^p, L^q)$ restriction problem is conjectured to hold with the same range of $p$'s and $q$'s as for the sphere (see [40]). The restriction problem for the cone gives a different range of $p$'s; this is because cone has one vanishing principal curvature.

## 2.1 A general $(L^p, L^2)$ restriction theorem

Wishing to include many more submanifolds and other situations, it is better to think of the Fourier restriction phenomenon with respect to a given measure. Specifically, given a compactly supported Borel measure $\mu$ on the dual group $\widehat{G}$ of a locally compact abelian group $G$ with Haar measure $dm$, one could formulate the $(L^p, L^2)$ Fourier restriction problem as establishing the *a priori* inequality

$$\|\widehat{f}\|_{L^2(\widehat{G},d\mu)} \leq C\|f\|_{L^p(G,dm)} \tag{2.2}$$

in some range $1 \leq p \leq p_0$. In this direction, Greenleaf [15] proved that if $S \subseteq \mathbb{R}^n$, $n \geq 2$, is a submanifold of dimension $\alpha < n$ and is endowed with a smooth measure $d\mu = \psi d\sigma$ satisfying $|\widehat{\mu}(x)| \leq C|x|^{-\beta/2}$, then (2.2) holds with $G = \mathbb{R}^n$ whenever $1 \leq p \leq p_0$, $p_0 = \frac{4n-4\alpha+2\beta}{4n-4\alpha+\beta}$. Note that this result extends the Tomas- Stein $(L^p, L^2)$ restriction theorem for the sphere.

It might seem that the restriction problem is closely related to the classical notion of curvature and therefore the problem seems to be an aspect of $\mathbb{R}^n$ with $n \geq 2$. However Mockenhaupt, using Tomas' argument in [41], extended the $(L^p, L^2)$ restriction estimate of Greenleaf in the following way.

**Theorem 2.1.1.** *([24])*
*If $\mu$ is a compactly supported measure in $\mathbb{R}^n$ such that $\mu(B_r(x)) \leq Cr^\alpha$ $\forall r > 0$ $(\alpha < n)$, and $|\widehat{\mu}(x)| \leq C|x|^{-\beta/2}$, then (2.2) holds in the range $1 \leq p < p_0$, $p_0 = \frac{4n-4\alpha+2\beta}{4n-4\alpha+\beta}$.*

The advantage of this theorem is that it is applicable for $n = 1$, giving $(L^p, L^2)$ Fourier restriction results for thin sets in $\mathbb{R}$. However it does not include the endpoint $p_0$. On the other hand, one can use an argument due to Carbery to say more at $p_0$ and prove that a restricted estimate holds at $p_0$; specifically, (2.2) holds for characteristic functions $f = \mathbf{1}_E$ at $p = p_0$.

The restriction problem has also been considered in situations where $\mathbb{R}$ is replaced by other fields or rings. In the cases of $(\mathbb{Z}/N\mathbb{Z})^n$ and ${Q_p}^n$, where $Q_p$ is the field of $p$-adic numbers, Fourier restriction estimates have been obtained by Wright [44]. Mockenhaupt and Tao [27] have established restriction estimates in $F^n$ where $F$ is a finite field. We now present Carbery's argument in a sufficiently abstract setting to include several interesting examples and situations which we will need later. Our aim is to give an $(L^p, L^2)$ Fourier restriction theorem on locally compact abelian (LCA) groups. Let $G$ be a LCA group and $m_G$ a Haar measure. We denote by $\widehat{G}$ the dual group of $G$ and let $m_{\widehat{G}}$ be the dual measure of $m_G$. Let $B_r^G(0)$, $B_r^{\widehat{G}}(0)$ be families of sets ('balls'), $r > 0$, on $G$ and $\widehat{G}$ respectively.

We then define

$$B_r^G(x) = x + B_r^G(0) \text{ and } B_r^{\widehat{G}}(\xi) = \xi + B_r^{\widehat{G}}(0). \tag{2.3}$$

We assume that if $(r_k)$ is any sequence of real numbers with $r_k \to \infty$, then

$$\bigcup_k B_{r_k}^{\widehat{G}}(\xi) = \widehat{G} \quad \forall\, \xi \in \widehat{G}. \tag{2.4}$$

Let $\mu$ be a finite measure on $\widehat{G}$ such that

$$\mu(B_r^{\widehat{G}}(\xi)) \leq Cr^\alpha \quad \forall\, r > 0 \quad \forall\, \xi \in \widehat{G} \quad \text{and} \tag{2.5}$$

$$\text{if} \quad x \notin B_r^G(0)\ , \qquad \text{then} \quad |\widehat{\mu}(-x)| \leq Cr^{-\beta/2}. \tag{2.6}$$

We also assume that there is a family of functions $\phi_r : G \to \mathbb{C}$ with the following properties:

$$\sup_r \|\phi_r\|_{L^\infty} < \infty\ , \tag{2.7}$$

$$\phi_r \equiv 1 \quad on \quad B_r^G(0)\ , \tag{2.8}$$

$$|\widehat{\phi_r}(\xi)| \leq Cr^n \quad \forall\, \xi \in \widehat{G},\ (\alpha < n)\ , \tag{2.9}$$

$$|\widehat{\phi_r}(-\xi)| \leq C\left(\frac{r}{2^j}\right)^n \quad \forall\, \xi \in B_{\frac{2^j}{r}}^{\widehat{G}}(0) \backslash B_{\frac{2^{j-1}}{r}}^{\widehat{G}}(0),\ j \in \mathbb{N}. \tag{2.10}$$

Remarks:

1. The translation-invariant families of 'balls' $\{B_r^G(x)\}_{r>0}, \{B_r^{\widehat{G}}(\xi)\}_{r>0}$ do not necessarily arise from a metric. This is important for certain applications discussed below.

2. The family $\{\phi_r\}_{r>0}$ of functions should be thought of as 'smooth' approximations to $\mathbf{1}_{B_r^G(0)}$; properties (2.7)-(2.10) encoding the regularity property (2.5) and Fourier decay property (2.6) for the Haar measure $m_G$ on $G$. Note that (2.7) and (2.8) clearly hold for the family $\phi_r = \mathbf{1}_{B_r^G(0)}$ and for this family property (2.9) would follow from a regularity property $m_G(B_r^G(0)) \leq Cr^n$ implying $m_G(B_r^G(x)) \leq Cr^n$ by (2.3). And although (2.10) can hold for $\mathbf{1}_{B_r^G(0)}$ in many situations as we shall see below, it does not hold for $G = \mathbb{R}^n$ and $n \geq 2$. However in this case, if $\phi \in \mathbb{S}(\mathbb{R}^n)$ (Schwartz class) with $\phi \equiv 1$ on $B_1(0)$ then $\phi_r(x) := \phi(\frac{x}{r})$ satisfies properties (2.7) through (2.10).

**Theorem 2.1.2.** *Under the above assumptions the $(L^{p_0}, L^2)$ Fourier restriction estimate (2.2) holds for characteristic functions $f = \mathbf{1}_E$ on $G$, where $p_0 = \frac{4n-4\alpha+2\beta}{4n-4\alpha+\beta}$, and the strong type $(L^p, L^2)$ estimate (2.2) holds for $1 \leq p < p_0$.*

*Proof.* Let $S \subseteq \widehat{G}$ be the support of $\mu$. We introduce the operator $R$ where $R(f)$ is the restriction of $\widehat{f}$ on $S$, that is,

$$R(f)(\xi) = \int_G f(x)\overline{\langle x, \xi \rangle} dm_G(x), \quad \xi \in S.$$

The question is whether the following two estimates hold:

$$\|R(\mathbf{1}_E)\|_{L^2(d\mu)} \leq C m_G(E)^{1/p_0} \ , \tag{2.11}$$

$$\|R(f)\|_{L^2(d\mu)} \leq C\|f\|_{L^p(G,dm_G)} \ \ \forall\ 1 \leq p < p_0 \ . \tag{2.12}$$

The formal adjoint of $R$, $R^*$, is given by

$$R^*g(x) = \int_S g(\xi)\langle \xi, x\rangle d\mu(\xi) = \widehat{gd\mu}(-x) \ .$$

Therefore $R^*R(f) = f * k$ with $k(x) = \widehat{\mu}(-x)$. We will prove that

$$\int |R^*R(\mathbf{1}_E)(x)\mathbf{1}_F(x)|dm_G(x) \leq C m_G(E)^{1/p_0} m_G(F)^{1/p_0} \ \ \forall\ E, F \subseteq G. \tag{2.13}$$

This is sufficient to yield (2.11) because

$$\|R(\mathbf{1}_E)\|^2_{L^2(d\mu)} = \langle R(\mathbf{1}_E), R(\mathbf{1}_E)\rangle_{L^2(d\mu)} = \langle \mathbf{1}_E, R^*R(\mathbf{1}_E)\rangle_{L^2(G)}$$

and so applying (2.13) for $F = E$ we get (2.11).
To prove (2.13) we write $R^*R(f) = T_1(f) + T_2(f)$ where

$$T_1(f) = f * (k(1-\phi_r)) \ \text{ and } \ T_2(f) = f * (k\phi_r) \ .$$

The value of $r$ will be determined later.

$$\int |T_1(\mathbf{1}_E)(x)\mathbf{1}_F(x)|dm_G(x) \leq \sup_{x\in G} |\mathbf{1}_E * (k(1-\phi_r))(x)| \cdot m_G(F).$$

Applying (2.6), (2.7),(2.8) we get

$$\int |T_1(\mathbf{1}_E)(x)\mathbf{1}_F(x)|dm_G(x) \leq C r^{-\frac{\beta}{2}} m_G(E) m_G(F). \tag{2.14}$$

Using the Cauchy-Schwartz inequality and Plancherel theorem we have

$$\int |T_2\mathbf{1}_E(x)\mathbf{1}_F(x)|dm_G(x) \leq \left(\int |\widehat{\mathbf{1}_E}(\xi)|^2|\widehat{k\phi_r}(\xi)|^2 dm_{\widehat{G}}(\xi)\right)^{1/2} m_G(F)^{1/2} \ ,$$

$$\text{with } \ |\widehat{k\phi_r}(\xi)| = |\widehat{\phi_r} * \widehat{k}(\xi)| \leq \int_{\widehat{G}} |\widehat{\phi_r}(\xi - x)|d\mu(x) \ . \tag{2.15}$$

Using (2.4) the last integral equals

$$\int_{B^{\widehat{G}}_{\frac{1}{r}}(\xi)} |\widehat{\phi_r}(\xi - x)|d\mu(x) + \sum_{j=1}^{\infty} \int_{B^{\widehat{G}}_{\frac{2j}{r}}(\xi)\setminus B^{\widehat{G}}_{\frac{2j-1}{r}}(\xi)} |\widehat{\phi_r}(\xi - x)|d\mu(x) \ .$$

By (2.3), (2.9),(2.10) the last quantity is bounded by

$$Cr^n \mu(B^{\widehat{G}}_{\frac{1}{r}}(\xi)) + \sum_{j=1}^{\infty} C \left(\frac{r}{2^j}\right)^n \mu(B^{\widehat{G}}_{\frac{2^j}{r}}(\xi)) \ ,$$

which by (2.5) is less than

$$Cr^{n-\alpha} + Cr^{n-\alpha} \sum_{j=1}^{\infty} \left(\frac{1}{2^{n-\alpha}}\right)^j \ .$$

Hence, since $\alpha < n$, we have $|\widehat{k\phi_r}(\xi)| \leq Cr^{n-\alpha}$. Applying this to (2.15) we get

$$\int |T_2(\mathbf{1}_E)(x)\mathbf{1}_F(x)| dm_G(x) \leq Cr^{n-\alpha} m_G(E)^{\frac{1}{2}} m_G(F)^{\frac{1}{2}} \ . \tag{2.16}$$

From (2.14),(2.16) we choose $r > 0$ such that

$$r^{-\frac{\beta}{2}} m_G(E) m_G(F) = r^{n-\alpha} m_G(E)^{\frac{1}{2}} m_G(F)^{\frac{1}{2}} \ .$$

This gives us $r = m_G(E)^{\frac{1}{2n-2\alpha+\beta}} m_G(F)^{\frac{1}{2n-2\alpha+\beta}}$. Therefore

$$\int |R^* R \mathbf{1}_E(x) \mathbf{1}_F(x)| dm_G(x) \leq C m_G(E)^{\frac{4n-4\alpha+\beta}{4n-4\alpha+2\beta}} m_G(F)^{\frac{4n-4\alpha+\beta}{4n-4\alpha+2\beta}} \ ,$$

i.e. $R^*R$ is of restricted weak type $(p_0, p_0')$ with $p_0 = \frac{4n-4\alpha+2\beta}{4n-4\alpha+\beta}$.
Since the measure $\mu$ is finite we have that $R^*R$ is of strong type $(1, \infty)$. Hence, interpolation gives us that $R^*R$ is of strong type $(p, p')$ for every $1 \leq p < p_0$. Applying Holder inequality we get

$$\begin{aligned} \|R(f)\|^2_{L^2(d\mu)} &= \langle f, R^* R f\rangle_{L^2(G)} \\ &\leq \|f\|_{L^p(G)} \|R^* R f\|_{L^{p'}(G)} \\ &\leq C\|f\|^2_{L^p(G)}. \end{aligned}$$

Therefore (2.12) is true and the theorem is proved.

□

As discussed above, this theorem applies to the Euclidean case $\mathbb{R}^n$ simply taking the balls to be the usual balls in $\mathbb{R}^n$ and $\phi_r(x) := \phi(\frac{x}{r})$, a scaled family of a fixed Schwartz function $\phi$, and so includes Theorem 2.1.1 as a special case, even giving an endpoint estimate. More generally, using Theorem 2.1.2, one can get restriction estimates in $K^n$ where $K$ is a local field. As we shall see later in this thesis, every local field is endowed with a natural norm which yields the family of balls needed in Theorem 2.1.2. As far as the family of functions $\phi_r$ is concerned, it depends on the norm of $K$: if this norm has the archimedean property then $K$ is either $\mathbb{R}$ or $\mathbb{C}$ (see Theorem 4.2.13 in Chapter 4) and this case is treated above. However, if the norm of $K$ is non-archimedean and $G = K$, then one can take $\phi_r$ to be the characteristic function on the ball of radius $r$ centered at $0 \in K$ (For more details we refer to Section 6.1 in Chapter 6).

## 2.2 The restriction phenomenon in $(\mathbb{Z}/p^L\mathbb{Z})^n$

As another application of Theorem 2.1.2, we show how one can establish $(L^p, L^2)$ restriction results in the group $(\mathbb{Z}/N\mathbb{Z})^n$. When $N$ is restricted to be prime, such results appear in [27] but here we follow [44] where results and examples are given for general $N$. Following [44] we will define a natural 'norm' on $\mathbb{Z}/N\mathbb{Z}$ below which allows us to define isotropic and anisotropic balls in $(\mathbb{Z}/N\mathbb{Z})^n$, making various euclidean scaling arguments carry over to this discrete setting. As formulated by Mockenhaupt and Tao in [27], the Fourier restriction problem in the setting of the compact abelian group $G = (\mathbb{Z}/N\mathbb{Z})^n$, endowed with counting measure, is the following: equip $\widehat{G} \simeq G$ with normalised counting measure and let $S = \{g = 0\} \subset \widehat{G}$ be the zero set of $g \in \mathbb{Z}[X_1, \ldots, X_n]$. The $(L^t, L^q)$ Fourier restriction problem for $S$ is formulated by the inequality

$$\left(\frac{1}{|S|}\sum_{\xi\in S}|\widehat{f}(\xi)|^q\right)^{1/q} \leq C_{t,q}\left(\sum_{x\in G}|f(x)|^t\right)^{1/t} , \tag{2.17}$$

where $C_{t,q}$ is essentially independent of $N$ (from now on we use the letter $t$ instead of $p$ for exponents in Fourier restriction formulae because we intend to use the letter $p$ for prime numbers).

Let $r \in \mathbb{Z}/N\mathbb{Z}$ and $\vec{x} = (x_1, .., x_n) \in G$. We define

$$|r| := \frac{N}{\gcd(r, N)} \text{ and } \|\vec{x}\| := \frac{N}{\gcd(x_1, .., x_n, N)} .$$

We also define $x \leq_R y$ if and only if $x|y$. For $r = d,\ d|N$ we consider

$$B_r^G(0) := \{\vec{x} = (x_1, .., x_n) \in G : \|\vec{x}\| \leq_R r\} = \{(x_1, .., x_n) \in G : \frac{N}{d}|x_i \ \forall i\}.$$

In the dual space $\widehat{G}$, the 'balls' $B_r^{\widehat{G}}(0)$ are defined in a different way and the radius is not a divisor of $N$ but a reciprocal of the divisors of $N$. So, for $r = \frac{1}{d},\ d|N$ we define

$$B_r^{\widehat{G}}(0) = \{\vec{x} = (x_1, .., x_n) \in \widehat{G} : \|\vec{x}\| \leq_R |\frac{1}{r}|\} = \{(x_1, .., x_n) \in \widehat{G} : d|x_i \ \forall i\}.$$

For the following theorem we shall consider $N = p^L,\ L \geq 2,\ p$ prime.

**Theorem 2.2.1.** *Let $G = (\mathbb{Z}/p^L\mathbb{Z})^n$ and let $S \subset \widehat{G}$ be the algebraic variety*

$$S = \{\ (x_1, .., x_k, g_{k+1}(x_1, .., x_k), ..., g_n(x_1, .., x_k))\ :\ x_i \in \mathbb{Z}/p^L\mathbb{Z}\ \} ,$$

*where $g_j$ are polynomials with integer coefficients. We assume that*

$$|\frac{1}{p^{Lk}}\sum_{x_1=0}^{p^L-1}\cdots\sum_{x_k=0}^{p^L-1} e^{2\pi i(y_1x_1+..+y_kx_k+y_{k+1}g_{k+1}+..+y_ng_n)/p^L}| \leq C\|\vec{y}\|^{-\frac{\beta}{2}} , \tag{2.18}$$

*$\vec{y} = (y_1,..,y_n) \in G$. Then the $(L^{t_0}, L^2)$ restriction estimate (2.17) holds for characteristic functions $f = \mathbf{1}_E$ on $G$, where $t_0 = \frac{4n-4k+2\beta}{4n-4k+\beta}$, and the strong type $(L^t, L^2)$ estimate (2.17) holds for $1 \leq t < t_0$.*

*Proof.* We shall apply Theorem 2.1.2 of which the assumptions we should verify. The families of balls $B_r^G(0)$, $B_r^{\widehat{G}}(0)$ have already been defined for certain radii. We now define them for every $r > 0$. Let $\ell$ be an integer such that $1 \leq \ell \leq L$. For $r \geq p^L$, $p^{\ell-1} \leq r < p^\ell$, $0 < r < 1$ we define $B_r^G(0)$ to be $G$, $B_{p^{\ell-1}}^G(0)$, $\emptyset$ respectively. In the dual space $\widehat{G}$, for $r \geq 1$ we have $B_r^{\widehat{G}}(0) := \widehat{G}$, for $p^{-\ell} \leq r < p^{-\ell+1}$ we define $B_r^{\widehat{G}}(0) := B_{p^{-\ell}}^{\widehat{G}}(0)$, and for $r < p^{-L}$ we have $B_r^{\widehat{G}}(0) := \emptyset$. Hence the assumption (2.4) is satisfied.
The set $S$ is endowed with the normalized counting measure which plays the role of $\mu$ in Theorem 2.1.2. Clearly the cardinality of $S$ equals $p^{Lk}$. Hence

$$\mu(f) = \frac{1}{p^{Lk}}\sum_{x_1=0}^{p^L-1}\cdots\sum_{x_k=0}^{p^L-1} f(x_1,..,x_k,g_{k+1},..,g_n).$$

We now verify the regularity of $\mu$. Let $B_r^{\widehat{G}}(0)$ with $r = p^{-\ell}$. Then

$$\mu(B_{p^{-\ell}}^{\widehat{G}}(0)) = \frac{1}{p^{Lk}}\sum_{x_1=0}^{p^L-1}\cdots\sum_{x_k=0}^{p^L-1} \mathbf{1}_{B_{p^{-\ell}}^{\widehat{G}}(0)}(x_1,..,x_k,g_{k+1},..,g_n).$$

From the definition of $B_{p^{-\ell}}^{\widehat{G}}(0)$ we get that the only surviving $x_j$'s in the above sum are at most those which are multiples of $p^\ell$, i.e. $x_j = n_jp^\ell$, $0 \leq n_j \leq p^{L-\ell}-1$. Therefore

$$\mu(B_{p^{-\ell}}^{\widehat{G}}(0)) \leq \frac{1}{p^{Lk}}\sum_{n_1=0}^{p^{L-\ell}-1}\cdots\sum_{n_k=0}^{p^{L-\ell}-1} 1 = (p^{-\ell})^k.$$

Similarly, for $r \neq p^{-\ell}$ the regularity property of $\mu$ is proved as above. One can also prove the regularity of $\mu$ on balls centered at any $\xi \in \widehat{G}$.
Next we verify the Fourier decay assumption (2.6). Let

$$\vec{y} = (y_1,..,y_n) \notin B_r^G(0), \ \ p^\ell \leq r < p^{\ell+1}, \ \ 0 \leq \ell \leq L-1.$$

From the definition of $B_r^G(0)$ we have that there is $i_0$ such that $p^{L-\ell}$ does not divide $y_{i_0}$. This implies that

$$\|\vec{y}\| > r ,$$

because otherwise $\gcd(y_1,..,y_n,p^L) \geq p^Lr^{-1} > p^{L-\ell-1}$ or equivalently $p^{L-\ell}|y_i$ for every $i$ which cannot be true. Therefore using (2.18) we have that

$$|\widehat{\mu}(-\vec{y})| \leq C r^{-\beta/2} \ .$$

The cases $r \geq p^L$ and $r < 1$ are trivial.
We now introduce the family of functions $(\phi_r)$ to be $\phi_r = \mathbf{1}_{B_r^G(0)}$. Clearly the assumptions (2.7),(2.8) of Theorem 2.1.2 are satisfied. We now check the assumption (2.9). We have that

$$\widehat{\phi_r}(\vec{\xi}) = \sum_{x_1=0}^{p^L-1} \cdots \sum_{x_n=0}^{p^L-1} \mathbf{1}_{B_r^G(0)}(\vec{x}) e^{-2\pi i(\xi_1 x_1 + .. + \xi_n x_n)/p^L} .$$

Let $p^\ell \leq r < p^{\ell+1},\ 0 \leq \ell \leq L-1$. The surviving $x_j$'s in the above sum are those with the property $p^{L-\ell}|x_j$ , i.e. $x_j = \ell_j p^{L-\ell},\ 0 \leq \ell_j \leq p^\ell - 1$. Therefore

$$\widehat{\phi_r}(\vec{\xi}) = \sum_{\ell_1=0}^{p^\ell-1} \cdots \sum_{\ell_n=0}^{p^\ell-1} e^{-2\pi i(\xi_1 \ell_1 + .. + \xi_n \ell_n)/p^\ell} = p^{\ell n} \mathbf{1}_{\{\vec{\xi}: p^\ell | \xi_j \ \forall j\}}.$$ Hence

$$\widehat{\phi_r}(\vec{\xi}) = p^{\ell n} \mathbf{1}_{B^{\widehat{G}}_{p^{-\ell}}(0)} \ . \tag{2.19}$$

Therefore $|\widehat{\phi_r}(\vec{\xi})| \leq r^n$.
Let now $r \geq p^L$. Then $B_r^G(0) = G$ and the same argument as above gives us

$$\widehat{\phi_r}(\vec{\xi}) = p^{Ln} \mathbf{1}_{B^{\widehat{G}}_{p^{-L}}(0)} = p^{Ln} \mathbf{1}_{\{(0,..,0)\}} \ .$$

Hence $|\widehat{\phi_r}(\vec{\xi})| \leq p^{Ln} < r^n$.
Last if $r < 1$ then $B_r^G(0) = \emptyset$ and therefore $\widehat{\phi_r}(\vec{\xi}) = 0$.

Next we verify the assumption (2.10). Let

$$\vec{\xi} \in B^{\widehat{G}}_{\frac{2^j}{r}}(0) \backslash B^{\widehat{G}}_{\frac{2^{j-1}}{r}}(0) \ .$$

To avoid trivialities we can assume that $p^\ell \leq r < p^{\ell+1},\ 0 \leq \ell \leq L$.

A) Let $p^\ell \geq \frac{r}{2^{j-1}}$. Then from (2.19) we get $\widehat{\phi_r}(\vec{\xi}) = 0$.

B) Let $\frac{r}{2^{j-1}} > p^\ell$. If $p^\ell \geq \frac{r}{2^j}$ then (2.19) implies that

$$\begin{aligned} \widehat{\phi_r}(\vec{\xi}) &\leq p^{\ell n} \\ &< 2^n (\frac{r}{2^j})^n \ . \end{aligned}$$

If $\frac{r}{2^j} > p^\ell$ then $\widehat{\phi_r}(\vec{\xi}) = 0$ because $B^{\widehat{G}}_{\frac{2^j}{r}}(0) = B^{\widehat{G}}_{\frac{2^{j-1}}{r}}(0) = B^{\widehat{G}}_{p^{-\ell-1}}(0)$.
Hence in any case the assumption (2.10) is satisfied and the theorem is proved by applying Theorem 2.1.2 .

□

There are many examples of algebraic varieties $S$ for which the condition

(2.18) is true. For $k=1$, there is a classical estimate due to Hua [16]:

$$|\frac{1}{p^L}\sum_{x=0}^{p^L-1} e^{2\pi i \frac{h(x)}{p^L}}| \leq C_m \frac{1}{p^{L/m}} \ , \tag{2.20}$$

where $h(x) = b_1x + \cdots + b_m x^m,\ b_m \neq 0,\ \gcd(b_1,..,b_m,p^L) = 1$. In the case where

$$\gcd(b_1,..,b_m,p^L) = p^s,\ s \neq 0,$$

by a simple argument, Hua's estimate implies

$$|\frac{1}{p^L}\sum_{x=0}^{p^L-1} e^{2\pi i \frac{h(x)}{p^L}}| \ \leq \ C_m \frac{1}{p^{(L-s)/m}} \ = \ C_m \|\vec{b}\|^{-1/m} \ , \tag{2.21}$$

where $\vec{b} = (b_1,..,b_m)$. The estimate (2.21) allows us to verify (2.18) for the algebraic variety

$$S = \{(g_1(x),..,g_n(x)) : x \in \mathbb{Z}/p^L\mathbb{Z}\} \ ,$$

where $g_1(x) = x$ and $g_j$ are polynomials with integer coefficients which satisfy the following assumptions. First of all we assume that $g_i \neq cg_j$ for every $i \neq j$, for if we have $g_{i_0} = cg_{j_0}$ then the exponential sum on the line $\{cy_{i_0} = -y_{j_0}, y_i = 0\ \forall i \neq i_0, j_0\}$ equals 1 and we do not have any decay. However, this necessary condition is not sufficient. So we further make the following two assumptions:

(A) no monomial of $g_j$ is a monomial of $g_i,\ i \neq j$,

(B) if $g_j = \sum_{\nu=1}^{m_j} c_\nu^j x^\nu,\ c_{m_j}^j \neq 0$, then $\gcd(c_1^j,..,c_{m_j}^j,p^L) = 1$ for every $j = 1,..,n$.

Assumption (B) implies that

$$\gcd(y_1c_1^1,..,y_1c_{m_1}^1,..,y_nc_1^n,..,y_nc_{m_n}^n,p^L) = \gcd(y_1,..,y_n,p^L). \tag{2.22}$$

If we set $h(x) = y_1g_1(x) + \cdots + y_ng_n(x) = \sum_{i=1}^m b_i x^i$ then by (2.22) and assumption (A) we have that $\|\vec{b}\| = \|\vec{y}\|$. Therefore (2.21) implies that

$$|\frac{1}{p^L}\sum_{x=0}^{p^L-1} e^{2\pi i (y_1g_1(x)+\cdots+y_ng_n(x))/p^L}| \leq C_m \|\vec{y}\|^{-1/m} \ ,$$

where $m = \max\{m_1,..,m_n\}$, i.e. the maximum of the degrees of $g_1,..,g_n$.

To verify (2.18) for $k>1$ and certain algebraic varieties $S$, the arguments are the same as in the case $k=1$. The key point is a generalization of Hua estimate for polynomials of $k$ variables $\vec{x} = (x_1,..,x_k)$. It is known [1] that if

$$h(\vec{x}) = \sum_{n_1=0}^{m_1} \cdots \sum_{n_k=0}^{m_k} b_{n_1,..,n_k} x_1^{n_1} \cdots x_k^{n_k} \ \text{ with } \ \gcd(\{b_{n_1,..,n_k} : n_1,..,n_k\}, p^L) = 1 \ ,$$

then

$$|\frac{1}{p^{Lk}}\sum_{x_1=0}^{p^L-1}\cdots\sum_{x_k=0}^{p^L-1}e^{2\pi i h(\vec{x})/p^L}|\leq C_m\frac{L^{k-1}}{p^{L/m}}\ ,$$

where $m=\max\{m_1,..,m_k\}$. As in the case $k=1$, if

$$\gcd(\{b_{n_1,..,n_k}:n_1,..,n_k\},p^L)=p^s,\ \ s\neq 0,$$

then we can express the above estimate in terms of $\|\vec{b}\|$, where $\vec{b}$ is the vector with coordinates $b_{n_1,..,n_k}$ as $n_1,..,n_k$ vary. In this case we have

$$\begin{aligned}|\frac{1}{p^{Lk}}\sum_{x_1=0}^{p^L-1}\cdots\sum_{x_k=0}^{p^L-1}e^{2\pi i h(\vec{x})/p^L}| &\leq C_m\frac{(L-s)^{k-1}}{p^{(L-s)/m}}\\ &\leq C_m\|\vec{b}\|^{-\beta/2}\end{aligned}$$

for every $\beta<\frac{2}{m}$. This estimate implies the condition (2.18) for

$$S=\{(g_1(\vec{x}),..,g_k(\vec{x}),..,g_n(\vec{x})):\vec{x}\in(\mathbb{Z}/p^L\mathbb{Z})^k\}\ ,$$

where $g_j(\vec{x})=x_j\ \forall j=1,..,k$ and $g_j$ are polynomials which satisfy the assumptions (A) and (B) (of case $k=1$) transferred in the present setting in an obvious way.

So far we have discussed positive results of the restriction problem (2.17). We now consider the algebraic variety

$$\Sigma=\{(x,x^m):x\in\mathbb{Z}/p^L\mathbb{Z}\}\ ,\quad m\geq 2\ ,$$

and we aim to limit the range of $(t,q)$ for which the $(L^t,L^q)$ Fourier restriction phenomenon on $\Sigma$ can hold. Then taking $q=2$ we shall observe that, for the set $\Sigma$, the result of Theorem 2.2.1 is almost sharp. The $(L^t,L^q)$ restriction problem to $\Sigma$ is formulated by the inequality

$$\left(\frac{1}{p^L}\sum_{x=0}^{p^L-1}|\widehat{f}(x,x^m)|^q\right)^{1/q}\leq C_{t,q}\left(\sum_{x=0}^{p^L-1}\sum_{y=0}^{p^L-1}|f(x,y)|^t\right)^{1/t}\ ,\tag{2.23}$$

where $C_{t,q}$ is independent of $p$ and $L$.
Let $\widehat{f}(x,y)=\mathbf{1}_B(x,y)$ where

$$B=\{(x,y)\in(\mathbb{Z}/p^L\mathbb{Z})^2:p^s|x,p^{sm}|y\},\ \ 1\leq s\leq L-1\ .$$

The only $x$'s which contribute in the sum on the left hand side of (2.23) are those which are multiples of $p^s$, i.e. $x=p^sx',0\leq x'\leq p^{L-s}-1$. Therefore

$$\left(\frac{1}{p^L}\sum_{x=0}^{p^L-1}|\widehat{f}(x,x^m)|^q\right)^{1/q}=\left(\frac{1}{p^L}\sum_{x'=0}^{p^{L-s}-1}1\right)^{1/q}=\frac{1}{p^{s/q}}\ .\tag{2.24}$$

To compute the right hand side of (2.23) we should first compute $f(x,y)$.

$$\begin{aligned} f(x,y) &= \frac{1}{p^{2L}} \sum_{u=0}^{p^L-1} \sum_{v=0}^{p^L-1} \mathbf{1}_B(u,v) e^{2\pi i (xu+yv)/p^L} \\ &= \frac{1}{p^{2L}} \sum_{u'=0}^{p^{L-s}-1} \sum_{v'=0}^{p^{L-ms}-1} e^{2\pi i (xp^s u' + yp^{sm} v')/p^L} \\ &= \frac{1}{p^{2L}} \sum_{u'=0}^{p^{L-s}-1} e^{2\pi i \frac{xu'}{p^{L-s}}} \sum_{v'=0}^{p^{L-ms}-1} e^{2\pi i \frac{yv'}{p^{L-sm}}} \\ &= \frac{1}{p^{(1+m)s}} \mathbf{1}_D(x,y) \ , \end{aligned}$$

where $D = \{(x,y) : p^{L-s}|x, \ p^{L-sm}|y\}$. Therefore the right hand side of (2.23) becomes

$$\frac{1}{p^{(1+m)s}} \left( \sum_{x'=0}^{p^s-1} \sum_{y'=0}^{p^{sm}-1} 1 \right)^{1/t} = \frac{1}{p^{(1+m)s}} \cdot p^{(1+m)s/t} = \frac{1}{p^{(1+m)s/t'}} \ , \tag{2.25}$$

where $t'$ is the dual exponent of $t$, i.e. $1/t + 1/t' = 1$. Substituting (2.24) and (2.25) to (2.23) we get the following necessary condition for the $(L^t, L^q)$ restriction problem to $\Sigma$

$$\frac{1}{q} \geq \frac{1+m}{t'} \ .$$

For $q = 2$ this becomes

$$1 \leq t \leq \frac{2m+2}{2m+1} \ . \tag{2.26}$$

On the other hand, from (2.21) we have that the exponent $\beta$ of the Fourier decay assumption (2.18) for $\Sigma$ is $2/m$. Moreover $k = 1$ and $n = 2$. Hence the endpoint $t_0$ of Theorem 2.2.1 is

$$t_0 = \frac{4 \cdot 2 - 4 + 2 \cdot \frac{2}{m}}{4 \cdot 2 - 4 + \frac{2}{m}} = \frac{2m+2}{2m+1} \ .$$

Therefore (2.26) implies that the $(L^t, L^2)$ restriction result to $\Sigma$ is almost sharp.

Remark: The exponents for the restriction problem associated to $(x, x^m)$ are the same as those as in the euclidean setting (see [35]).

# Chapter 3

# The $(L^p, L^2)$ restriction phenomenon on Salem sets in the real line

In this chapter we introduce the notion of *Salem* sets and consider the Fourier restriction problem to such sets. In the first section we shall describe Salem's construction [33] and Mockenhaupt's work [24] on the $(L^p, L^2)$ restriction problem to thin sets of the real line, especially to Salem sets. The latter was the first to consider the $(L^p, L^2)$ restriction problem on the real line and especially to Salem sets. In the second section, motivated by the surprising fact that the restriction phenomenon can happen on the real line where the classical notion of curvature does not make much sense, we consider the set of the well-approximable numbers $E(\alpha)$, a deterministic Salem set, and apply either Theorem 2.1.1 or 2.1.2 in order to get $(L^p, L^2)$ Fourier restriction results on this set. Actually, after Kaufman's work [19] who proved the existence of a measure $\mu$ supported on $E(\alpha)$ whose Fourier transform $\widehat{\mu}$ obeys a proper decay, the only remaining thing for us is to prove that $\mu$ satisfies a regularity property (see condition 2.5 of Theorem 2.1.2). We note that the sharpness of the above Fourier restriction results is still an open problem. The lack of curvature and therefore of scaling arguments makes the decision of the sharpness of such restriction estimates a difficult task. It is likely that one needs to understand whether long arithmetic progressions exist in such thin sets.

## 3.1 Salem's construction

We begin with a short review of the Hausdorff and Fourier dimension of a compact set $E \subseteq \mathbb{R}^d$ (see [18],[23]).

1) Frostman's lemma states that the Hausdorff measure of order $\alpha$ of a compact set $E \subseteq \mathbb{R}^d$ is positive if and only if $E$ carries a probability measure $\mu$ such that

$$\mu(B_r(x)) \leq C r^{\alpha} \ , \tag{3.1}$$

where $B_r(x)$ denotes a ball of radius $r$ centered at $x$.

2) Given a positive measure $\mu \neq 0$, the $\alpha$-energy of $\mu$ is defined as

$$\mathbb{E}_\alpha(\mu) = \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} \ .$$

3) The capacitarian dimension of a compact set $E \subseteq \mathbb{R}^d$ is defined as the supremum of numbers $\alpha$ such that there is a positive, finite measure $\mu$ supported on $E$ satisfying $\mathbb{E}_\alpha(\mu) < \infty$.

4) Let now $\mu$ be a positive, finite measure such that $\mu(B_r(x)) \leq Cr^\alpha$ and $\beta < \alpha$. Then

$$\int \frac{d\mu(x)}{|x-y|^\beta}$$

is uniformly bounded with respect to $y$, hence $\mathbb{E}_\beta(\mu) < \infty$. On the other hand if $\mathbb{E}_\alpha(\mu) < \infty$, then the regularity property (3.1) holds for a suitable restriction of $\mu$. Therefore, by Frostman's lemma, the capacitarian dimension of a compact set $E \subseteq \mathbb{R}^d$ is nothing but the Hausdorff dimension $\dim_H E$.

5) Since the Fourier transform of $|x|^{-\alpha}$ , $(0 < \alpha < d)$, is $C|\xi|^{\alpha-d}$, one can show that

$$\mathbb{E}_\alpha(\mu) = C\int_{\mathbb{R}^d} |\xi|^{\alpha-d}|\widehat{\mu}(\xi)|^2 d\xi \ .$$

Therefore the Hausdorff dimension $\dim_H E$ is characterized by the property that for every $\beta < \dim_H E$ there is a positive, finite measure $\mu \neq 0$ supported on $E$ such that $\int |x|^{\beta-d}|\widehat{\mu}(x)|^2 dx < \infty$ and no such measure exists if $\beta > \dim_H E$.

6) The Fourier dimension $\dim_F E$ of a compact set $E \subseteq \mathbb{R}^d$ is defined as the supremum of exponents $\alpha \in [0, d]$ so that there is a positive, finite measure $\mu \neq 0$, supported in $E$, satisfying

$$|\widehat{\mu}(x)| \ \leq \ C\,|x|^{-\alpha/2} \ .$$

From the above discussion one can see that the Fourier dimension never exceeds the Hausdorff dimension

$$\dim_F A \leq \dim_H A \ .$$

We would like to note that the dimension $d$ (of the Euclidean space $\mathbb{R}^d$) is crucial in the definition of Fourier dimension. To see this, let us consider the interval $I = [0, 1]$. Regarding $I$ as a subset of $\mathbb{R}$, we have that its Fourier dimension equals 1. However if we assume that $I$ is imbedded in $\mathbb{R}^2$ then $\dim_F I = 0$ because for every measure $\mu$ supported on $I$, $\widehat{\mu}(x_1, x_2)$ is independent of $x_2$ and therefore we do not have any decay in the direction of $y$-axis.

**Definition 3.1.1.** A set $A \subseteq \mathbb{R}^d$ is called a *Salem* set if $\dim_F A = \dim_H A$.

Except some trivial cases (e.g. spheres), Salem sets are very rare as deterministic sets. For $d = 1$ the only deterministic Salem set with $0 < \dim_F A < 1$ is the set of well-approximable numbers [19] which we will study in the next section.

For the rest of this chapter we consider $d = 1$.

One may have the impression that Salem sets are quite exceptional. However, in a way the opposite is true. Salem [33] succeeded in defining a random set on the real line which enjoys this property almost surely. We now describe his construction.

Let $[A, B]$ be an interval of length $L$. Let also

$$0 \leq a_1 < \cdots < a_N < 1$$

and $\xi$ be a number satisfying

$$\xi > 0,\ a_1 + \xi < a_2\ , \cdots,\ a_{N-1} + \xi < a_N,\ \text{and } a_N + \xi < 1\ . \tag{3.2}$$

We consider the $N$ disjoint intervals $[A + La_j, A + La_j + L\xi],\ j = 1, .., N$ and we call them *white* intervals. The $N+1$ complementary intervals with respect to $[A, B]$ will be called *black* intervals. This dissection of $[A, B]$ will be said to be of type $(N, a_1, .., a_N, \xi)$.

We now start from the interval $[0, 2\pi]$ and we fix $a_1, .., a_N$ as above. Let also $(\xi_i)$ be a sequence where each $\xi_i$ satisfies condition (3.2). We carry out the following Cantor type construction. At the first step we perform a dissection of type $(N, a_1, .., a_N, \xi_1)$ on the interval $[0, 2\pi]$, we keep the white intervals

$$[2\pi a_j, 2\pi a_j + 2\pi\xi_1]\ ,$$

and remove the black intervals. At the second step we operate a dissection of type $(N, a_1, .., a_N, \xi_2)$ on each of the white intervals of the previous step and removing the black intervals we get the $N^2$ white intervals

$$[2\pi a_j + 2\pi a_i \xi_1\ ,\ 2\pi a_j + 2\pi a_i \xi_1 + 2\pi \xi_1 \xi_2]\ .$$

We continue inductively and at the $n$-th step we have $N^n$ white intervals each of length $2\pi\xi_1\xi_2\cdots\xi_n$. When $n \to \infty$ the procedure gives us a perfect, nowhere dense set $E$ whose elements are given by the formula

$$x = 2\pi a_{\epsilon_0} + 2\pi a_{\epsilon_1}\xi_1 + .. + 2\pi a_{\epsilon_n}\xi_1 \cdots \xi_n + \ldots \quad ,$$

where each $\epsilon_j$ takes all values $1, 2, .., N$.

We also consider the sequence of functions $(F_n)$ where $F_n$ is continuous which satisfies the following properties:

- $F_n(x) = 0$ for $x \leq 0$ and $F_n(x) = 1$ for $x \geq 2\pi$.
- $F_n$ is linearly increasing by $1/N^n$ on each of the white intervals of the $n$-th step.
- $F_n$ is constant on the black intervals of the $n$-th step.

The limit $F(x) = \lim_n F_n(x)$, the so called *Lebesgue function*, is a continuous non-decreasing function and clearly the corresponding measure $dF$ is supported on $E$. The Fourier-Stieltjes transform

$$\int_{\mathbb{R}} e^{ixy} dF(y)$$

is the limit as $n \to \infty$ of

$$\sum \frac{1}{N^n} e^{2\pi i x(a_{\epsilon_0}+a_{\epsilon_1}\xi_1+\cdots+a_{\epsilon_{n-1}}\xi_1\cdots\xi_{n-1})} \ ,$$

where the sum is taken over all the $N^n$ possible combinations of $\epsilon_0, \ldots, \epsilon_{n-1}$. Thus, writing

$$P(x) = \frac{1}{N}(e^{2\pi i a_1 x} + \cdots + e^{2\pi i a_N x}) \ ,$$

we have

$$\widehat{dF}(x) = P(x) \prod_{n=1}^{\infty} P(\xi_1 \cdots \xi_n x) \ .$$

Obviously $E$ and $dF$ depend on $N, a_1, .., a_N$ and $(\xi_i)$ . Salem succeeded in proving the following.

**Proposition 3.1.2.** *([33]) For every $\alpha > 0$ and $\epsilon > 0$ there exist $N, a_1, .., a_N, (\xi_i)$ such that the corresponding set $E$ has Hausdorff dimension $\alpha$ and the measure $dF$ satisfies*

$$|\widehat{dF}(x)| \leq C|x|^{-\frac{\alpha}{2}+\epsilon} \ . \tag{3.3}$$

Salem's approach was probabilistic and he actually proved that the estimate (3.3) holds for almost every $(\xi_i)$ with respect to a suitable measure.

Mockenhaupt [24] verified that the measure $dF$ satisfies also a regularity property. In fact, he proved that

$$|F(x) - F(y)| \leq C|x-y|^{\alpha} \ . \tag{3.4}$$

He then applied (3.3),(3.4) to his Theorem 2.1.1 and proved the following.

**Proposition 3.1.3.** *([24]) Let $p < \frac{4-2\alpha}{4-3\alpha}$. Then for $\epsilon > 0$ sufficiently small there exist $N, a_1, .., a_N, (\xi_i)$ such that the corresponding set $E$ has Hausdorff dimension $\alpha$ and the measure $dF$ satisfies the a priori inequality*

$$\int |\widehat{f}|^2 dF \leq C\|f\|^2_{L^p(\mathbb{R})} \ .$$

## 3.2 The set of well-approximable numbers

The set of well-approximable numbers is defined as

$$E(\alpha) = \{x \in \mathbb{R} : |x - \frac{m}{n}| < n^{-2-\alpha} \text{ for infinitely many rationals } \frac{m}{n}\} \ ,$$

where $\alpha > 0$. Jarník [17] established that $E(\alpha)$ has Hausdorff dimension $2/2+\alpha$. Kaufman [19] proved that $E(\alpha)$ contains a Salem set with the same Hausdorff dimension as $E(\alpha)$. It is the only deterministic Salem set in $\mathbb{R}$ of dimension strictly between 0 and 1. We shall describe Kaufman's work in a slightly modified way, see [2].

Let $F$ be a Schwartz function such that $F \geq 0$, $\int F = 1$ and supp$F = [-1/2, 1/2]$. We fix $M > 0$ and we denote by $P_M$ the set of prime numbers in the interval $[M, 2M]$. By the prime number theorem we have that there are constants $C_1, C_2$ independent on $M$ such that

$$C_1 \frac{M}{\log M} \leq |P_M| \leq C_2 \frac{M}{\log M} \ . \tag{3.5}$$

We shall also use the fact that, for a non-zero integer $k$, the number of its prime divisors belonging in $P_M$ is less than $\frac{\log |k|}{\log M}$, i.e.

$$|\{(p, m) \in P_M \times \mathbb{Z} : k = pm\}| \leq \frac{\log |k|}{\log M} \ . \tag{3.6}$$

This is an immediate consequence of the factorization of $|k|$ by prime numbers.

Next we set $F_M(x) = (2M)^{1+\alpha} F((2M)^{1+\alpha} x)$ and let $\widetilde{F_M}$ be the periodic extension of $F_M$ in $\mathbb{R}$ of period 1. Let also

$$g_M(x) = \frac{1}{|P_M|} \sum_{p \in P_M} \widetilde{F_M}(px) \ .$$

**Remark 3.2.1.** If $x$ satisfies $g_M(x) > 0$, then there exist $p \in P_M$ and $n \in \mathbb{Z}$ such that $|px - n| \leq \frac{1}{2}(2M)^{-1-\alpha}$ and therefore $|x - \frac{n}{p}| \leq p^{-2-\alpha}$.

Since $\int F_M = 1$ and $F_M \geq 0$ we get

$$|\widehat{F_M}(m)| \leq 1 \ , \tag{3.7}$$

and by partial integration we have that

$$|\widehat{F_M}(m)| \leq C_\alpha \frac{M^{2+2\alpha}}{m^2} \ , \ m \in \mathbb{Z} \setminus \{0\} \ . \tag{3.8}$$

Using the Fourier series expansion of $\widetilde{F_M}$ we get

$$\begin{aligned} \widehat{g_M}(k) &= \frac{1}{|P_M|} \sum_{p \in P_M} \int_{-1/2}^{1/2} \widetilde{F_M}(px) e^{-2\pi i k x} dx \\ &= \frac{1}{|P_M|} \sum_{p \in P_M} \sum_{m \in \mathbb{Z}} \widehat{F_M}(m) \int_{-1/2}^{1/2} e^{2\pi i (pm - k) x} dx \ . \end{aligned}$$

Therefore we have

$$\widehat{g_M}(k) = \begin{cases} 0 & , \quad k \neq pm, p \in P_M, m \in \mathbb{Z} \\ \frac{1}{|P_M|} \sum_{p \in P_M} \sum\limits_{\substack{m \in \mathbb{Z} \\ k = mp}} \widehat{F_M}(m) & , \quad \text{otherwise} \end{cases} \tag{3.9}$$

**Lemma 3.2.2.** *For $M$ sufficiently large we have*

*(i) $|\widehat{g_M}(k)| \leq C_\alpha \frac{\log M}{M}$, for every $k \in \mathbb{Z} \setminus \{0\}$.*

*(ii) $|\widehat{g_M}(k)| \leq C_\alpha \frac{\log |k|}{|k|^{\frac{1}{2+\alpha}}}$, $|k| \geq M^{2+\alpha}$.*

*Proof.* When $0 < |k| \leq M^{2+\alpha}$, then from (3.5),(3.6),(3.7) we get

$$|\widehat{g_M}(k)| \leq C\frac{\log|k|}{M} \leq C_\alpha \frac{\log M}{M}.$$

Let us now assume that $|k| \geq M^{2+\alpha}$. The formula (3.9), using (3.5),(3.8), gives us

$$\begin{aligned} |\widehat{g_M}(k)| &\leq C_\alpha \frac{\log M}{M} \sum_{p\in P_M} \sum_{\substack{m\in\mathbb{Z} \\ k=mp}} \frac{M^{2+2\alpha}}{m^2} \\ &= C_\alpha (\log M) M^{1+2\alpha} \sum_{p\in P_M} \sum_{\substack{m\in\mathbb{Z} \\ k=mp}} \frac{p^2}{k^2} \ . \end{aligned}$$

Since $p < 2M$, using (3.6) we further have that the last quantity is bounded by

$$C_\alpha \frac{\log|k|}{k^2} M^{3+2\alpha} \ .$$

Since now $|k| \geq M^{2+\alpha}$ we eventually get

$$|\widehat{g_M}(k)| \leq C_\alpha \frac{\log|k|}{|k|^{\frac{1}{2+\alpha}}} \ . \tag{3.10}$$

For $|k| \geq M^{2+\alpha}$ and $M$ sufficiently large the function $\frac{\log|k|}{|k|^{\frac{1}{2+\alpha}}}$ is decreasing. Therefore from (3.10) we also have

$$|\widehat{g_M}(k)| \leq C_\alpha \frac{\log M}{M} \ .$$

□

We set the function

$$\theta(x) = \frac{\log(3+|x|)}{(1+|x|)^{\frac{1}{2+\alpha}}} \log\log(3+|x|) \ .$$

If $|y|$ is big enough we have

$$\theta(y) < \theta(x) \ , \qquad \text{for every } x \text{ such that } |x| < |y| \ . \tag{3.11}$$

**Lemma 3.2.3.** *For every $\psi \in C_c^2$ and $\delta > 0$ there exists $M_0 > 0$ such that*

$$|\widehat{\psi g_M}(x) - \widehat{\psi}(x)| \leq \delta\theta(x) \ , \qquad \forall\ x \in \mathbb{R},\ \forall\ M \geq M_0 \ .$$

**Note:** The $\widehat{\ }$ notation here denotes the Fourier transform although we have used this already to denote the Fourier coefficients of a function. Hopefully the context will make it clear which one we are using in any particular circumstance.

*Proof.*

$$\begin{aligned}
\widehat{\psi g_M}(x) - \widehat{\psi}(x) &= \int \psi(y) g_M(y) e^{-2\pi i x y} dy - \widehat{\psi}(x) \\
&= \sum_{k\in\mathbb{Z}} \widehat{g_M}(k) \int \psi(y) e^{-2\pi i (x-k) y} dy - \widehat{\psi}(x) \\
&= \sum_{k\in\mathbb{Z}} \widehat{g_M}(k) \widehat{\psi}(x-k) - \widehat{\psi}(x) \\
&= \sum_{k\neq 0} \widehat{g_M}(k) \widehat{\psi}(x-k)
\end{aligned}$$

because $\widehat{g_M}(0) = 1$ by (3.9). Since $\psi \in C_c^2$ we get

$$|\widehat{\psi g_M}(x) - \widehat{\psi}(x)| \leq C \sum_{k\neq 0} \frac{|\widehat{g_M}(k)|}{(1+|k-x|)^2} \ . \tag{3.12}$$

When $|x| \leq M^{2+\alpha}$, then using Lemma 3.2.2(i) and (3.12) we get

$$|\widehat{\psi g_M}(x) - \widehat{\psi}(x)| \leq C \frac{\log M}{M} \ .$$

For $M$ sufficiently large we have

$$C \frac{\log M}{M} \leq \delta\theta(M^{2+\alpha}) \ .$$

Now the lemma follows because from (3.11) we have $\theta(M^{2+\alpha}) \leq \theta(x)$ provided that $M$ is big enough.

Let us now assume that $|x| > M^{2+\alpha}$. We shall split the sum in (3.12) into two parts.

a) Let $k : |k - x| \geq \frac{|x|}{2}$.

We further split and consider $|k| \geq \frac{3|x|}{2}$. Then Lemma 3.2.2(ii) gives us

$$\begin{aligned}
\sum_{|k|\geq\frac{3|x|}{2}} \frac{|\widehat{g_M}(k)|}{(1+|k-x|)^2} &\leq C_\alpha \sum_{|k|\geq\frac{3|x|}{2}} \frac{\log|k|}{|k|^{\frac{1}{2+\alpha}}} \frac{1}{(1+|k-x|)^2} \\
&\leq C_\alpha \frac{1}{1+|x|} \sum_{|k|\geq\frac{3|x|}{2}} \frac{\log|k|}{|k|^{\frac{1}{2+\alpha}}} \frac{1}{|k|} \\
&= C_\alpha \frac{1}{1+|x|} \ .
\end{aligned}$$

Clearly, for $M$ sufficiently large (and hence $|x|$ large) we have $C_\alpha \frac{1}{1+|x|} \leq \delta\theta(x)$ .

Let now $|k| < \frac{3|x|}{2}$ (but still $|k-x| \geq \frac{|x|}{2}$). Then

$$\begin{aligned} \sum_{|k|<\frac{3|x|}{2}} \frac{|\widehat{g_M}(k)|}{(1+|k-x|)^2} &\leq C \sum_{|k|<\frac{3|x|}{2}} \frac{1}{(1+|x|)^2} \\ &\leq C \frac{1}{1+|x|} , \end{aligned}$$

because from (3.7),(3.9) we get $|\widehat{g_M}(k)| \leq 1$ for every $k$. As above, choosing $M$ big enough we have

$$C \frac{1}{1+|x|} \leq \delta\theta(x) .$$

b) For $k : |k-x| \leq \frac{|x|}{2}$ .

Then $|k| \geq \frac{|x|}{2}$ and therefore $|k| > \frac{M^{2+\alpha}}{2}$ . Lemma 3.2.2(ii) holds (adjusting the constant $C_\alpha$) and gives us

$$\sum_{|k|\geq\frac{|x|}{2}} \frac{|\widehat{g_M}(k)|}{(1+|k-x|)^2} \leq C_\alpha \sum_{|k|\geq\frac{|x|}{2}} \frac{\log|k|}{|k|^{\frac{1}{2+\alpha}}} \frac{1}{(1+|k-x|)^2} . \tag{3.13}$$

If $M$ is sufficiently large then the function

$$\frac{\log|x|}{|x|^{\frac{1}{2+\alpha}}}$$

is decreasing for $|x| \geq M^{2+\alpha}$ . Hence the right hand side of (3.13) is further bounded by

$$C_\alpha \frac{\log|x|}{(1+|x|)^{\frac{1}{2+\alpha}}} ,$$

which is less than $\delta\theta(x)$ for $M$ big enough.

□

**Lemma 3.2.4.** *Let $(\mu_n)$ be a sequence of Borel measures on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If $c \leq \|\mu_n\| \leq C,\ c > 0,\ C < \infty$, and the sequence of Fourier transforms $(\widehat{\mu_n})$ converges pointwise, then there exists a Borel measure $\mu \neq 0$ such that $\mu_n \xrightarrow{w^*} \mu$.*

*Proof.* By Alaoglu's theorem there exists a measure $\mu \neq 0$ and a subsequence $(\mu_{n_k})_k$ of $(\mu_n)_n$ such that $\mu_{n_k} \xrightarrow{w^*} \mu$. Hence

$$\widehat{\mu_{n_k}} \to \widehat{\mu} .$$

Let $(\mu_{n_\ell})_\ell$ be an arbitrary subsequence of $(\mu_n)_n$. By the same argument as above we get $\mu_{n_{\ell_m}} \xrightarrow{w^*} \nu$ and $\widehat{\mu_{n_{\ell_m}}} \to \widehat{\nu}$ for some measure $\nu$ and subsequence $(\mu_{n_{\ell_m}})_m$ of $(\mu_{n_\ell})_\ell$. Since $(\widehat{\mu_n})_n$ is convergent we have that $\widehat{\mu} = \widehat{\nu}$ and therefore $\mu = \nu$. The Lemma now follows because the closed unit ball of $M(\mathbb{T})$ is metrizable in the $w^*$-topology. □

**Proposition 3.2.5.** *There exists a non-zero measure $\mu$ supported on $E(\alpha)$ such that*

$$|\widehat{\mu}(x)| \leq C\theta(x),$$

*where* $\theta(x) = \frac{\log(3+|x|)}{(1+|x|)^{\frac{1}{2+\alpha}}} \log\log(3+|x|)$.

**Note:** In fact, Kaufman [19] established the stronger result that

$$|\widehat{\mu}(x)| \leq C(1+|x|)^{-1/(2+\alpha)} \log(e+|x|) \ .$$

The above proposition is a modified version of Kaufman's theorem and is due to Bluhm [2].

*Proof.* We aim to use inductively Lemma 3.2.3. Let $\psi_0$ be an arbitrary function in $C_c^2(\mathbb{R})$ such that $\psi_0 \geq 0$. We apply Lemma 3.2.3 with $\psi = \psi_0$ and $\delta = 1/2$, and we take $M_1$ such that

$$|\widehat{\psi_0 g_{M_1}}(x) - \widehat{\psi_0}(x)| < \frac{1}{2}\theta(x) \ .$$

At the $k$th step we apply Lemma 3.2.3 with $\psi = \psi_0 g_{M_1} \cdots g_{M_{k-1}}$ and $\delta = \frac{1}{2^k}$. Setting

$$d\mu_k = \psi_0 g_{M_1} \cdots g_{M_k} dx,$$

we have

$$|\widehat{\mu_k}(x) - \widehat{\mu_{k-1}}(x)| \leq \frac{1}{2^k}\theta(x) \ . \tag{3.14}$$

We shall prove that the sequence $(\mu_k)$ satisfies the assumptions of Lemma 3.2.4 . From (3.14) the sequence $(\widehat{\mu_k})$ is convergent. We now choose $\psi_0 \in C_c^2, \psi_0 \geq 0$, such that

$$\widehat{\psi_0}(0) = \int \psi_0 > \theta(0) + 1.$$

Then (3.14) gives us

$$|\widehat{\mu_k}(0) - \widehat{\psi_0}(0)| \leq \theta(0) \sum_{j=1}^{k} \frac{1}{2^j} \ .$$

Hence $1 \leq \|\mu_k\| \leq \theta(0) + \widehat{\psi_0}(0)$. By Lemma 3.2.4 there exists a measure $\mu \neq 0$ such that $\mu_k \xrightarrow{w^*} \mu$. Choosing the sequence $(M_k)$ such that $M_k > 2M_{k-1}$[1],we have by Remark 3.2.1 that $\mu$ is supported on $E(\alpha)$. Taking into account that $|\widehat{\psi_0}(x)| \leq C\theta(x)$ and $\widehat{\mu_k} \to \widehat{\mu}$, (3.14) implies that

$$|\widehat{\mu}(x)| \leq C\theta(x) \ ,$$

and the proposition is proved. □

Next we aim to prove that the measure $\mu$ satisfies a regularity property. This, together with the Fourier decay estimate of Proposition 3.2.5, will enable us to

[1] Lemma 3.2.3 allows us to do so.

get restriction results for $\mu$ on $E(\alpha)$. We will see that the regularity property of $\mu$ depends on the choice of cut-off function $\psi_0$ in the construction of $\mu$ above. The construction allows the mass of the measure $\mu$ to accumulate substantially at the origin, so much that if $\psi_0$ is identically one near the origin, the regularity property of $\mu$ would not be lined with the dimension of $E(\alpha)$. We will want to avoid this and so we will eventually choose $\psi_0$ to have support away from 0 and in this case we will show that the corresponding measure $\mu$ satisfies the desired regularity property: for all $\delta > 0$, there is a constant $C_\delta$ so that

$$\mu(I) \leq C_\delta |I|^{\frac{2}{2+\alpha}-\delta} \tag{3.15}$$

holds for all intervals $I$.

Let $(M_k)$ be a sequence, produced inductively as above, yielding the measures $\mu_k$ and $\mu$. Whenever is needed we modify $M_k$ considering a larger one. We are allowed to do so by Lemma 3.2.3 .The function $\psi_0$, appeared in the proof of Proposition 3.2.5, is considered to be supported in $[-1/2, 1/2]$. Let

$$|P_{M_k}| = n_k \text{ and } P_{M_k} = \{p_{1,k}, \ldots, p_{n_k,k}\} \ .$$

From Remark 3.2.1 the support of $g_{M_k}$, restricted in $[-1/2, 1/2]$, consists of intervals of length

$$\frac{1}{p_{j,k}(2M_k)^{1+\alpha}} \ ,$$

centered at

$$m_j/p_{j,k} \ ,$$

$m_j = 0, \pm 1, \ldots, \pm[\frac{p_{j,k}}{2}]$ , $j = 1, \ldots, n_k$ , where $[\cdot]$ denotes the integer part. We call each such an interval a *k-interval*. For a fixed $j$ there are $p_{j,k}$ *k-intervals* of length

$$\frac{1}{p_{j,k}(2M_k)^{1+\alpha}} \ ,$$

which are uniformly distributed in $[-1/2, 1/2]$.

**Remark 3.2.6.** If $I_1, I_2$ are *k-intervals* with $0 \notin I_1 \bigcap I_2$, then $I_1 \bigcap I_2 = \emptyset$.

*Proof.* For $i = 1, 2$ let $I_i$ be centered at $m_{j_i}/p_{j_i,k}$ and $|I_i| = \frac{1}{p_{j_i,k}(2M_k)^{1+\alpha}}$. Then

$$\begin{aligned} \left|\frac{m_{j_1}}{p_{j_1,k}} - \frac{m_{j_2}}{p_{j_2,k}}\right| &= \frac{|m_{j_1}p_{j_2,k} - m_{j_2}p_{j_1,k}|}{p_{j_1,k}p_{j_2,k}} \\ &\geq \frac{1}{p_{j_1,k}p_{j_2,k}} \\ &> \frac{|I_1| + |I_2|}{2} \ . \end{aligned}$$

□

We denote by $I_0$ the biggest 1-*interval* centered at zero.

Let $I$ be an 1-*interval* such that $I \bigcap I_0 = \emptyset$. We fix

$$p_{j_2,2} \in P_{M_2}, \dots, p_{j_k,k} \in P_{M_k} \ .$$

There are $p_{j_2,2}|I|$ 2-*intervals* of length

$$\frac{1}{p_{j_2,2}(2M_2)^{1+\alpha}}$$

contained in $I$ (because, as we have observed, these intervals are uniformly distributed in [-1/2,1/2]). We denote by $J_2$ the union of these intervals. Hence

$$|J_2| = \frac{1}{(2M_2)^{1+\alpha}}|I| \ .$$

Similarly there are $p_{j_3,3}|J_2|$ 3-*intervals* of length

$$\frac{1}{p_{j_3,3}(2M_3)^{1+\alpha}}$$

contained in $J_2$. If we denote their union by $J_3$, then

$$\begin{aligned} |J_3| &= \frac{1}{(2M_3)^{1+\alpha}}|J_2| \\ &= \frac{1}{(2M_3)^{1+\alpha}}\frac{1}{(2M_2)^{1+\alpha}}|I| \ . \end{aligned}$$

We continue inductively and we get that

$$|J_k| = \frac{1}{(2M_k)^{1+\alpha}} \cdots \frac{1}{(2M_2)^{1+\alpha}}|I| \ ,$$

where $J_k$ is the union of $k$-*intervals* of length

$$\frac{1}{p_{j_k,k}(2M_k)^{1+\alpha}}$$

contained in $J_{k-1}$.
The above procedure was made for fixed $p_{j_2,2} \in P_{M_2}, \dots, p_{j_k,k} \in P_{M_k}$. Taking into account all the possible combinations, we have that the support of

$$\psi_0 g_{M_1} \cdots g_{M_k}$$

in $I$ has Lebesgue measure equal to

$$|P_{M_2}| \cdots |P_{M_k}| \frac{1}{(2M_k)^{1+\alpha}} \cdots \frac{1}{(2M_2)^{1+\alpha}}|I| \ .$$

We also have that $\psi_0 g_{M_1}\cdots g_{M_k}$, restricted on $[-1/2,1/2]\setminus I_0$, is bounded by

$$C\frac{1}{|P_{M_1}|}(2M_1)^{1+\alpha}\cdots\frac{1}{|P_{M_k}|}(2M_k)^{1+\alpha}\ . \tag{3.16}$$

Therefore

$$\begin{aligned}\mu_k(I) &= \int_I \psi_0 g_{M_1}\cdots g_{M_k}dx\\ &\leq C\frac{1}{|P_{M_1}|}(2M_1)^{1+\alpha}|I|\ .\end{aligned}$$

From (3.5) we get

$$\mu_k(I)\leq C(\log M_1)M_1^{\alpha}|I|\ . \tag{3.17}$$

Since $I$ is an 1-*interval* we have $|I|\leq\frac{1}{M_1^{2+\alpha}}$ and hence $M_1\leq|I|^{-\frac{1}{2+\alpha}}$. Therefore from (3.17) we get

$$\begin{aligned}\mu_k(I) &\leq C(\log|I|^{-1})|I|^{\frac{2}{2+\alpha}}\\ &\leq C_\delta|I|^{\frac{2}{2+\alpha}-\delta}\ .\end{aligned}$$

Next we check the regularity in the case that $I$ is an $i$-*interval*, $i\in\mathbb{N}$, such that $I\bigcap I_0=\emptyset$. The same arguments as before give us that the support of

$$\psi_0 g_{M_1}\cdots g_{M_k}$$

in $I$ $(k>i)$ has Lebesgue measure equal to

$$|P_{M_{i+1}}|\cdots|P_{M_k}|\frac{1}{(2M_k)^{1+\alpha}}\cdots\frac{1}{(2M_{i+1})^{1+\alpha}}|I|\ .$$

Hence using (3.16) we have

$$\mu_k(I)\leq C\frac{1}{|P_{M_1}|}(2M_1)^{1+\alpha}\cdots\frac{1}{|P_{M_i}|}(2M_i)^{1+\alpha}|I|\ . \tag{3.18}$$

Replacing $1/|P_{M_i}|$ by $(\log M_i)/M_i$ and choosing $M_i$ so large that

$$\frac{1}{|P_{M_1}|}(2M_1)^{1+\alpha}\cdots\frac{1}{|P_{M_{i-1}}|}(2M_{i-1})^{1+\alpha}<\log M_i\ ,$$

we get

$$\mu_k(I)\leq C(\log M_i)^2M_i^{\alpha}|I|\ . \tag{3.19}$$

Since $I$ is an $i$-*interval* we have $M_i\leq|I|^{-\frac{1}{2+\alpha}}$. Hence (3.19) gives us

$$\mu_k(I)\leq C_\delta|I|^{\frac{2}{2+\alpha}-\delta}\ .$$

Let now $I$ be an arbitrary interval (i.e. not an $i$-*interval*) such that $I\bigcap I_0=\emptyset$.

We consider $i \in \mathbb{N}$ such that $I$ contains *i-intervals* and

$$|I| < \frac{1}{(2M_{i-1})^{2+\alpha}} \ . \tag{3.20}$$

Let us assume that $I$ contains $N$ *i-intervals*, say $I_1, \ldots, I_N$. Since the total number of *i-intervals* is bounded by

$$p_{1,i} + \cdots + p_{n_i,i}$$

and they are uniformly distributed, we have that

$$N \le (p_{1,i} + \cdots + p_{n_i,i})|I| \ .$$

Hence

$$N \le 2M_i|P_{M_i}||I| \ . \tag{3.21}$$

We also have that $|I_j| \le \frac{1}{M_i^{2+\alpha}}$ for every $j = 1, \ldots, N$. Therefore using (3.18) we get

$$\begin{aligned} \mu_k(I) &\le C\frac{1}{|P_{M_1}|}(2M_1)^{1+\alpha} \cdots \frac{1}{|P_{M_i}|}(2M_i)^{1+\alpha}\sum_{j=1}^{N}|I_j| \\ &\le C\frac{1}{|P_{M_1}|}(2M_1)^{1+\alpha} \cdots \frac{1}{|P_{M_i}|}(2M_i)^{1+\alpha}N\frac{1}{M_i^{2+\alpha}} \ . \end{aligned}$$

Using (3.21) and after cancellation we get

$$\mu_k(I) \le C\frac{1}{|P_{M_1}|}(2M_1)^{1+\alpha} \cdots \frac{1}{|P_{M_{i-1}}|}(2M_{i-1})^{1+\alpha}|I| \ .$$

Replacing $1/|P_{M_{i-1}}|$ by $(\log M_{i-1})/M_{i-1}$ and choosing $M_{i-1}$ sufficiently large we have

$$\mu_k(I) \le C(\log M_{i-1})^2 M_{i-1}^{\alpha}|I| \ .$$

This is the same type of estimate as in (3.17) and (3.19). Therefore by (3.20) and using the standard arguments we get

$$\mu_k(I) \le C_\delta |I|^{\frac{2}{2+\alpha}-\delta} \ .$$

Since $\mu_k \xrightarrow{w^*} \mu$ we also have the same estimate for $\mu$.

It remains to check the regularity of $\mu$ on intervals which contain zero. In this case $g_{M_i}$ is bounded by $(2M_i)^{1+\alpha}$ and not by $(2M_i)^{1+\alpha}/|P_{M_i}|$ as before, because *i-intervals* containing zero overlap. Consequently, the regularity exponent of $\mu$ is reduced from $2/(2+\alpha)$ to $1/(1+\alpha)$ . More precisely, the same approach as above gives us that

$$\mu(I) \le C_\delta |I|^{\frac{1}{2+\alpha}-\delta} \ ,$$

for every interval $I$ containing zero. We aim to deal with this reduction of the exponent of regularity of $\mu$ because it causes the range of $p$'s where the Fourier restriction phenomenon in $E(\alpha)$ holds to become smaller. To deal with this, we consider $\psi_0$ supported in $[-1/2, 1/2]$ and in the complement of a small neighborhood of zero. Then we pick $M_1$ sufficiently large such that $I_0 \subseteq [-1/2, 1/2] \setminus \text{supp}\psi_0$. This guarantees that the measure $\mu$ is supported away from $I_0$ and hence (3.15) holds.

From Proposition 3.2.5 we also get

$$|\widehat{\mu}(x)| \leq C_\delta(1 + |x|)^{-\frac{1}{2+\alpha}+\delta} \ . \tag{3.22}$$

Combining (3.15),(3.22) with either Theorem 2.1.1 or 2.1.2 we get the following $(L^p, L^2)$ Fourier restriction estimate on $E(\alpha)$.

**Theorem 3.2.7.** *Let* $1 \leq p < \frac{2+2\alpha}{1+2\alpha}$*. Then for* $\delta > 0$ *sufficiently small there exists a constant* $C$ *depending only on* $\delta$ *such that*

$$\int_{E(\alpha)} |\widehat{f}|^2 d\mu \leq C\|f\|^2_{L^p(\mathbb{R})} \ .$$

We would like to note that it is not known whether the result in Theorem 3.2.7 is sharp or not. From the dual form of restriction problem

$$\int_{\mathbb{R}} |\widehat{fd\mu}|^{p'} dx \leq C\|f\|^{p'}_{L^2(d\mu)} \ ,$$

and using (3.22) we cannot exclude the possibility that the $(L^p, L^2)$ restriction phenomenon holds for $p' > 2 + \alpha$ or equivalently

$$p < \frac{2+\alpha}{1+\alpha} \ .$$

# Chapter 4

# Local fields

This chapter serves as a background for the chapters to come. We describe the theory of locally compact fields emphasizing those tools which we will use in the following chapters. The basic references are [38], [42] and [32] and the reader may wish to refer to these references for any well known results stated below without proof. We start with an important example, the field $Q_p$ of $p$-adic numbers. This will help us to understand the general theory of local fields which is the theme of Section 4.2. In Section 4.3 we deal with the Hausdorff and Fourier dimensions in a local ultrametric field. We shall see that all the Real analysis about these two notions and the connections between them, discussed in Section 3.1 of Chapter 3, can be easily transferred into the local field setting.

## 4.1 The field $Q_p$ of $p$-adic numbers

We give a short description of $p$-adic numbers following Folland's book [13]. Let $p$ be a prime and $r \in \mathbb{Q},\ r \neq 0$. From the prime factorization theorem we have that $r$ can uniquely be written as

$$r = p^m \frac{r_1}{r_2} \ ,$$

where $m \in \mathbb{Z},\ p$ divides neither $r_1$ nor $r_2$, $\gcd(r_1, r_2) = 1,\ r_1 \in \mathbb{Z},\ r_2 \in \mathbb{N}$. Then we define

$$|r|_p = p^{-m} \ .$$

We also define $|0|_p = 0$. One can verify that $|\cdot|_p$ satisfies

$$|x+y|_p \leq \max\{|x|_p, |y|_p\} \text{ and } |xy|_p = |x|_p|y|_p \ . \tag{4.1}$$

Therefore $d_p(x,y) = |x-y|_p$ is a metric in $\mathbb{Q}$ and the arithmetic operations are $d_p$-continuous. It follows that these operations extend to the completion $Q_p$ of $\mathbb{Q}$ with respect to $d_p$. $Q_p$ is called the field of $p$-adic numbers. The $p$-adic norm $|\cdot|_p$ can also be extended to $Q_p$ and still satisfies equation (4.1). $Q_p$ is described in a more concrete way as the following proposition shows.

**Proposition 4.1.1.** *If $m \in \mathbb{Z}$ and $c_j \in \{0, 1, .., p-1\}$, then $\sum_m^\infty c_j p^j$ converges* *in $Q_p$. Moreover every $p$-adic number can be written as such series,*

$$Q_p = \{\sum_{j=m}^{\infty} c_j p^j : m \in \mathbb{Z}, c_j = 0, 1, .., p-1\}.$$

The algorithms for adding, multiplying, dividing and subtracting $p$-adic expansions $\sum_m^\infty c_j p^j$ are similar to the case of real numbers in decimal form with obvious modifications using mod $p$ arithmetic. For example

$$-\sum_{j=m}^{\infty} c_j p^j = (p - c_m)p^m + \sum_{j=m+1}^{\infty} (p - 1 - c_j)p^j.$$

If $x = \sum_m^\infty c_j p^j$ with $c_m \neq 0$, then

$$|x|_p = p^{-m}.$$

Hence the $p$-adic norm $|\cdot|_p$ takes on only the values $p^k,\ k \in \mathbb{Z}$, and 0.

For $x \in Q_p,\ k \in \mathbb{Z}$, we define the balls

$$\bar{B}_{p^k}(x) = \{y \in Q_p : |y - x|_p \le p^k\},$$

and we note that

$$\bar{B}_{p^k}(x) = x + \bar{B}_{p^k}(0),$$

where

$$\bar{B}_{p^k}(0) = \{\sum_{j=-k}^{\infty} c_j p^j : c_j = 0, 1, .., p-1\} .$$

For every $k \in \mathbb{Z}$ the ball $\bar{B}_{p^k}(0)$ is an additive subgroup of $Q_p$. Therefore if $m < -k$ then

$$\sum_{j=m}^{\infty} c_j p^j + \bar{B}_{p^k}(0) = \sum_{j=m}^{-k-1} c_j p^j + \bar{B}_{p^k}(0),$$

i.e. the part of $p$-adic expansion after the $(-k)$th coordinate contributes nothing. It is equivalent to say that

$$\bar{B}_{p^k}(x) = \bar{B}_{p^k}(y) \text{ if and only if } y \in \bar{B}_{p^k}(x).$$

If $m < n$, then $\bar{B}_{p^n}(0)$ is a union of $p^{n-m}$ balls of radius $p^m$,

$$\bar{B}_{p^n}(0) = \bigcup \left(c_{-n}p^{-n} + \cdots + c_{-m-1}p^{-m-1} + \bar{B}_{p^m}(0)\right),$$

where every $c_j$ takes all the values $0, 1, .., p-1$. Therefore $\bar{B}_{p^n}(0)$ is totally bounded, and since it is closed in $Q_p$ we conclude that $\bar{B}_{p^n}(0)$ is compact. Hence $Q_p$ is locally compact.

By (4.1), if $k \le 0$, then $\bar{B}_{p^k}(0)$ is a subring of $Q_p$. The ball $\bar{B}_1(0)$ is called the ring of $p$-adic integers and is denoted by $Z_p$,

$$Z_p = \{\sum_{j=0}^{\infty} c_j p^j : c_j = 0, 1, .., p-1\}.$$

We now describe the dual groups $\widehat{Q_p}, \widehat{Z_p}$ of $(Q_p, +)$ and $(Z_p, +)$ respectively. Let $\xi_1$ be the character defined by

$$\langle \sum_{j=-\infty}^{\infty} c_j p^j, \xi_1 \rangle = e^{2\pi i \sum_{-\infty}^{-1} c_j p^j} ,$$

where $c_j = 0$ for sufficiently large $j \leq 0$. The kernel of $\xi_1$ is $Z_p$ and hence it is constant on cosets of $Z_p$. For $y \in Q_p$ we define $\xi_y$ by

$$\langle x, \xi_y \rangle = \langle xy, \xi_1 \rangle.$$

Therefore $\xi_y$ equals 1 on the ball $\bar{B}_{|y|_p^{-1}}(0)$ and it is constant on cosets of this ball. It can be shown that every character on $Q_p$ is of the form $\xi_y$ and that $\widehat{Q_p}$ is isomorphic to $Q_p$ as topological groups.
The above analysis suggests that the dual group of $(Z_p, +)$ can be identified with the $p$-adic numbers of form

$$\sum_{j=-m}^{-1} c_j p^j, \ m \in \mathbb{N}, \ c_j = 0, 1, .., p-1.$$

In fact, it can be shown that $\widehat{Z_p} \cong Q_p / Z_p$.
All the above elements concerning $Q_p$ are generalized for any ultrametric local field to be defined in the next section. As we shall see in Theorem 4.2.13, $Q_p$ is the main representative among the ultrametric local fields of characteristic 0.

## 4.2 Local fields

In this section we recall some well known results about local fields. A local field $K$ is defined as a locally compact field, i.e. a field endowed with a topology where the additive group $K^+$ and multiplicative $K^* = K \setminus \{0\}$ group are locally compact abelian groups (we exclude the discrete fields from consideration). The local compactness property allows $K$ to be endowed with an absolute value $|\cdot|$. This is because if $a \in K^*$ and $dx$ is a Haar measure in $K^+$, then $d(ax)$ is also a Haar measure in $K^+$ and hence there is a constant which we denote by $|a| \in \mathbb{R}_+^*$ such that

$$d(ax) = |a| dx .$$

We define $|0| = 0$. It turns out that the so defined modular mapping $a \mapsto |a|$ is continuous on $K$ and

$$|ab| = |a||b| . \tag{4.2}$$

For $r > 0$ and $x \in K$ we define the balls

$$\begin{aligned} \bar{B}_r(x) &= \{y \in K : |y - x| \leq r\}, \\ \mathring{B}_r(x) &= \{y \in K : |y - x| < r\}. \end{aligned}$$

One can prove that $\bar{B}_r(x)$ is compact and the family $\{\bar{B}_r(x) : r > 0\}$ makes up a fundamental system of neighborhoods of $x$ in $K$.

**Proposition 4.2.1.** *The function $a \mapsto |a|$ induces on $K^*$ an open homomorphism of $K^*$ onto a closed subgroup $\Gamma$ of $\mathbb{R}_+^*$.*

**Theorem 4.2.2.** *There is a constant $A \geq 1$ such that*

$$|x + y| \leq A \max\{|x|, |y|\} \ ,$$

*for all $x, y \in K$.*

**Definition 4.2.3.** A local field $K$ is said to be *ultrametric* when

$$|x + y| \leq \max\{|x|, |y|\}. \tag{4.3}$$

This property is called *non-archimedean* or *ultrametric.*

Using the ultrametric property one can easily prove the following.

**Lemma 4.2.4.** *In an ultrametric local field the following properties hold:*
*a) Any point of a ball is a possible center of the ball.*
*b) If two balls have a common point, one is contained in the other.*
*c) Any two balls of same radius either coincide or are disjoint.*
*d) The diameter of a ball is less than or equal to its radius.*

Let us denote by $S_r(a) = \{x \in K : |x - a| = r\}$. Then, if $K$ is ultrametric, the following is true:

$$\text{if } \ x \in S_r(a), \quad \text{ then } \ \mathring{B}_r(x) \subset S_r(a). \tag{4.4}$$

This is an immediate consequense of the fact that

$$\text{if } \ |x| > |y|, \quad \text{ then } \ |x + y| = |x| \ , \tag{4.5}$$

for every $x, y \in K$. To prove (4.5) we first observe that $|x + y| \leq \max\{|x|, |y|\} = |x|$. On the other hand

$$|x| \leq \max\{|x + y|, |-y|\} = \max\{|x + y|, |y|\}$$

and since $|x| > |y|$ we get $|x| \leq |x + y|$.

From (4.4) we have that $S_r(a) = \bigcup_{x \in S_r(a)} \mathring{B}_r(x)$. Hence $S_r(a)$ is open. This implies that $\bar{B}_r(x) = \mathring{B}_r(x) \bigcup S_r(x)$ is open and $\mathring{B}_r(x) = \bar{B}_r(x) \setminus S_r(x)$ is closed. Hence we showed the following.

**Lemma 4.2.5.** *If $K$ is an ultrametric local field, then the sets $\bar{B}_r(x), \mathring{B}_r(x), S_r(x)$ are both closed and open.*

For ultrametric local fields we use the notation

$$R = \bar{B}_1(0) \text{ and } P = \mathring{B}_1(0).$$

Properties (4.2),(4.3) imply the following.

**Lemma 4.2.6.** *All balls containing* 0 *in an ultrametric local field* $K$ *are additive subgroups.* $R$ *is the unique maximal subring of* $K$ *and every ball containing* 0 *with radius less than* 1 *is an ideal of* $R$. $P$ *is the unique maximal ideal of* $R$.

**Definition 4.2.7.** $R$ is called the *ring of integers* in $K$.

**Proposition 4.2.8.** *Let* $K$ *be an ultrametric local field. Then the image* $\Gamma$ *of* $K^*$ *under* $|\cdot|$ *is of the form*

$$\theta^{\mathbb{Z}} = \{\theta^m : m \in \mathbb{Z}\},\ 0 < \theta < 1.$$

*Proof.* From Lemma 4.2.5, $P$ is closed in the compact $R$. Hence the image of $P$ under the continuous mapping $|\cdot|$ attains a maximum, say $\theta,\ 0 < \theta < 1$. So the interval $(\theta, 1) \subset \mathbb{R}$ does not intersect $\Gamma$. On the other hand, by Proposition 4.2.1, $\Gamma$ is a multiplicative subgroup of $\mathbb{R}_+^*$, and it is known that a multiplicative subgroup of $\mathbb{R}_+^*$ is either discrete or dense. Since it cannot be dense the proposition follows.

□

Let $S \subset R$ be a full set of representatives of the classes modulo $P$ in $R$. Then

$$R = \bigcup_{a \in S} (a + P). \tag{4.6}$$

Since $a + P$ is open and $R$ is compact, the last disjoint union must be finite. Therefore we get the following.

**Proposition 4.2.9.** *In an ultrametric local field the field* $R/P$ *is finite.*

Notation: We shall always denote by $q$ the number of elements of $S$.

Since $q = \#(R/P)$, we have that $q = p^c$, where $p$ is the characteristic of $R/P$. We note that $p \cdot 1_K$ belongs to $P$ and $p$ is the smallest prime with this property.

From Proposition 4.2.8 we fix $\pi \in P$ with $|\pi|$ the biggest possible value less than 1. This implies that $\pi^{-1}x \in R$ for all $x \in P$. Since we also have $\pi R \subseteq P$, we conclude that

$$P = \pi R.$$

Moreover we know that $R = \bigcup_{a \in S}(a+P)$ and hence considering any Haar measure we get

$$|R| = q|P| = q|\pi R| = q|\pi||R|.$$

Therefore, since $R$ is compact, we have $|\pi| = q^{-1}$. Let us summarize.

**Proposition 4.2.10.** *In an ultrametric local field there exists an element* $\pi \in P$ *such that*

$$P = \pi R \quad \text{and} \quad |\pi| = q^{-1}. \tag{4.7}$$

*Moreover the number* $\theta$ *of Proposition 4.2.8 equals* $q^{-1}$.

**Lemma 4.2.11.** *A local field (either ultrametric or not) is complete.*

*Proof.* Let $(x_n)$ be a Cauchy sequence in $K$. Hence $(x_n)$ is bounded and it is contained in a ball $\bar{B}_r(0)$. Since $\bar{B}_r(0)$ is compact the lemma follows.

□

**Theorem 4.2.12.** *If $K$ is an ultrametric local field, then*

$$K = \{\sum_{j=m}^{\infty} d_j \pi^j : d_j \in S,\ m \in \mathbb{Z}\}.$$

*Proof.* We shall first show that the series $\sum_{j=0}^{\infty} d_j \pi^j$ converges in $K$. Let

$$s_n = \sum_{j=0}^{n} d_j \pi^j,\ n \in \mathbb{N}.$$

For $n > k$ we have $s_n - s_k = \sum_{j=k+1}^{n} d_j \pi^j$. Since $|d_j|$ equals either 0 or 1 we get by (4.5) and (4.7) that $|s_n - s_k| \le q^{-k-1}$. Therefore $(s_n)$ is Cauchy and the claim follows by Lemma 4.2.11.
Let now $x \in K,\ x \neq 0$. We assume that $|x| = q^{-m},\ m \in \mathbb{Z}$. Then by (4.7) we have that $|\pi^{-m} x| = 1$. From (4.6) there are $d_0 \in S$ and $y \in P$ such that $d_0 \neq 0$ and

$$\pi^{-m} x = d_0 + y.$$

From Proposition 4.2.10 there exists $x' \in R$ such that $y = x'\pi$. We now appeal the same argument for $x'$. By induction, using alternatively (4.6) and Proposition 4.2.10, we get

$$\pi^{-m} x = \sum_{j=0}^{\infty} d_j \pi^j\ .$$

Multiplying by $\pi^m$ we are done.

□

The proof of Theorem 4.2.12 implies that in an ultrametric local field

$$\text{if}\ \ |x| = q^{-m}, \quad \text{then}\ \ x = \sum_{j=m}^{\infty} d_j \pi^j\ \ \text{with}\ \ d_m \neq 0.$$

Conversely, let us assume that $x = \sum_{j=0}^{\infty} d_j \pi^j$ with $d_0 \neq 0$. Then using (4.5) we have that $|s_n| = 1\ \forall\ n \in \mathbb{N}$, where $s_n = \sum_{j=0}^{n} d_j \pi^j$. Therefore, by the continuity of $|\cdot|$, $|x| = 1$. By scaling we conclude that

$$|x| = q^{-m} \text{ if and only if } x = \sum_{j=m}^{\infty} d_j \pi^j \text{ with } d_m \neq 0.$$

This gives us that

$$\bar{B}_{q^m}(0) = \{\sum_{j=-m}^{\infty} d_j \pi^j : d_j \in S\}.$$

As we saw in the previous section, the field $Q_p$ of $p$-adic numbers is an ultrametric local field of characteristic 0. It turns out that any other ultrametric local field of characteristic 0 is a finite field extension of $Q_p$, for some $p$ (see Theorem 4.2.13 below).

It can be shown that if a local field is of characteristic $p \neq 0$, then it is ultrametric. In fact, Theorem 4.2.13 below states that the only ultrametric local fields of characteristic $p$ are the so called *Formal Laurent series*. We give a brief description of these fields.

*Formal Laurent series* $F_{p^c}((X))$: Let $p$ be a prime real number, $c \in \mathbb{N}$, and $F_{p^c}$ the unique (up to isomorphism) finite field of $p^c$ elements. We define the set of formal series over $F_{p^c}$ in an indeterminate $X$ as

$$F_{p^c}((X)) = \{\sum_{j \geq m} c_j X^j : m \in \mathbb{Z}, c_j \in F_{p^c}\}.$$

One can also view the elements of this set as double sequences $(c_j)_{j=-\infty}^{\infty}$, $c_j \in F_{p^c}$, with finitely many negative $j$'s with $c_j$ non-zero. The arithmetic operations are defined by

$$(b_j) + (c_j) = (b_j + c_j) \text{ and } (b_j) \cdot (c_j) = (\sum_{n \in \mathbb{Z}} b_n c_{j-n}) \ .$$

If $x = (c_j) \in F_{p^c}((X))$ and $m$ is the smallest integer such that $c_m \neq 0$, we have

$$|x| = p^{-cm}.$$

The ring of integers $R$, denoted by $F_{p^c}[[X]]$, is the ring of formal power series over $F_{p^c}$, that is,

$$F_{p^c}[[X]] = \{\sum_{j \geq 0} c_j X^j : c_j \in F_{p^c}\} \ .$$

The next theorem classifies the local fields.

**Theorem 4.2.13.** *(classification theorem)*

***(i)*** *Let $K$ be an ultrametric local field. If its characteristic is non-zero, say char$(K) = p$, then $K$ is the field of formal Laurent series over a finite field of characteristic $p$, that is, $K = F_{p^c}((X))$ for some $c \in \mathbb{N}$. If char$(K) = 0$, then $K$ is either $Q_p$ for some prime $p$ or a finite field extension of $Q_p$.*

***(ii)*** *If $K$ is a non-ultrametric local field, then $K$ is either $\mathbb{R}$ or $\mathbb{C}$.*

Next we pass to Fourier analysis on an ultrametric local field $K$.

**Proposition 4.2.14.** *Let $\chi$ be a non-trivial character on $K^+$. There is $m \in \mathbb{Z}$ such that*

$$\chi(x) = 1 \quad on \quad \bar{B}_{q^m}(0) \ ,$$

*and $\chi$ is non-trivial on $\bar{B}_{q^{m+1}}(0)$. Moreover $\chi$ is constant on cosets of $\bar{B}_{q^m}(0)$ in $K^+$.*

*Proof.* Since $\chi$ is continuous there is a ball $\bar{B}_{q^m}(0)$ such that

$$|\chi(x) - 1| < \sqrt{2} \ ,$$

for every $x \in \bar{B}_{q^m}(0)$. We note that if $x \in \bar{B}_{q^m}(0)$, then

$$nx \in \bar{B}_{q^m}(0)$$

for every $n \in \mathbb{Z}$. Hence $|\chi(nx) - 1| < \sqrt{2}$ or equivalently

$$|\chi(x)^n - 1| < \sqrt{2} \ ,$$

for every $n \in \mathbb{Z}$ and $x \in \bar{B}_{q^m}(0)$. This cannot happen unless $\chi(x) = 1$ on $\bar{B}_{q^m}(0)$. We consider the biggest $m$ with this property. If now

$$x \in y + \bar{B}_{q^m}(0) \ ,$$

so $x = y + z, \ z \in \bar{B}_{q^m}(0)$, then $\chi(x) = \chi(y)\chi(z) = \chi(y)$.

□

If $\chi$ is a character on $K^+$, then every other character of $K^+$ is given by

$$\chi_y(x) = \chi(yx),$$

for some $y \in K$. The group of characters of $K^+$ is isomorphic to $K^+$ via the isomorphism $y \longrightarrow \chi_y$.

*Throughout this thesis $\chi$ is a fixed character on $K^+$ such that $\chi(x) = 1$ on $R$ and is non-trivial on $\bar{B}_q(0)$.*

We can find such a character $\chi$ by starting with any non-trivial character and rescaling. We also note that

$$\chi_y(x) = 1 \text{ on } \bar{B}_{|y|^{-1}}(0).$$

*In the sequel we always assume that $dx$ is the Haar measure of $K^+$ such that the measure of $R$ equals one.* Therefore, since

$$\bar{B}_{q^m}(0) = \pi^{-m}R,$$

we have that the Haar measure $dx$ of $\bar{B}_{q^m}(0)$ equals $q^m$.

If $f : K \to \mathbb{C}$ is an integrable function, then we define the Fourier transform of $f$ as

$$\widehat{f}(y) \ = \ \int_K f(x)\chi(yx)\, dx.$$

The Fourier transform is extended to measures $\mu$ on $K^+$ in the usual way,

$$\widehat{\mu}(y) = \int_K \chi(yx) d\mu(x).$$

The ball $\bar{B}_{q^m}(0)$ is a compact (additive) group and clearly every character of $K^+$ restricted on $\bar{B}_{q^m}(0)$ is a character of $\bar{B}_{q^m}(0)$. Using the fact that the dual group of a compact group is an orthogonal set we get

$$\int_{\bar{B}_{q^m}(0)} \chi(x) dx = \begin{cases} q^m & , \quad m \leq 0 \\ 0 & , \quad m > 0. \end{cases}$$

Therefore the integral $\int_{\bar{B}_{q^m}(a)} \chi(yx)dx$ after the change of variables $x \to x+a$ and then $x \to yx$ equals

$$\int_{\bar{B}_{q^m}(a)} \chi(yx)dx = \begin{cases} q^m \chi(ya) & , \quad |y| \leq q^{-m} \\ 0 & , \quad |y| > q^{-m}. \end{cases} \tag{4.8}$$

Passing to the compact ring of integers $R$, regarded as additive group, we expect its dual group to be discrete. Indeed, its dual group is isomorphically identified with the additive group $K/R$. Hence the characters of $R^+$ are of the form $\chi_y(x) = \chi(yx)$ where $y$ lies in the countable set

$$\{\sum_{j=-m}^{-1} d_j \pi^j : m \in \mathbb{N}, d_j \in S\}.$$

## 4.3 Hausdorff and Fourier dimensions in local ultrametric fields

Throughout this section $K$ is an ultrametric local field.

The existence of norm in $K$ allows us to speak of Hausdorff dimension. Moreover, Frostman's lemma is valid in any compact metric space (see [23]). Hence, if $E \subset K$ is compact, then its Hausdorff dimension $\dim_H E$ equals the supremum of numbers $\alpha \in [0,1]$ such that $E$ carries a probability measure $\mu$ satisfying

$$\mu(\bar{B}_{q^m}(x)) \leq C q^{m\alpha} \ , \tag{4.9}$$

for every $x \in K$ and $m \in \mathbb{Z}$.

The $\alpha$-energy of a measure $\mu$ in $K$ is defined as

$$\mathbb{E}_\alpha(\mu) = \int_K \int_K \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} \ .$$

**Lemma 4.3.1.** *Let $\mu$ be a positive, finite measure in $K$ such that the regularity property* (4.9) *holds. If $\beta < \alpha$ then*

$$\int_K \frac{d\mu(y)}{|x-y|^\beta}$$

*is uniformly bounded with respect to $x$ and hence $\mathbb{E}_\beta(\mu) < \infty$.*

*Proof.* We split the integral into

$$I_1 = \int_{\bar{B}_1(x)} \frac{d\mu(y)}{|x-y|^\beta} \text{ and } I_2 = \int_{|x-y|>1} \frac{d\mu(y)}{|x-y|^\beta} \ .$$

Since $\mu$ is finite we have that $I_2 < \infty$ uniformly on $x$. For the integral $I_1$ we write $\bar{B}_1(x)$ as the disjoint union of the spheres

$$S_{q^{-j}}(x) = \{y \in K : |x-y| = q^{-j}\}, \ j = 0, 1, .. \ .$$

Then

$$\begin{aligned} I_1 &= \sum_{j=0}^{\infty} q^{j\beta} \int_{S_{q^{-j}}(x)} d\mu(y) \\ &= \sum_{j=0}^{\infty} q^{j\beta} \mu(S_{q^{-j}}(x)) \\ &\leq C \sum_{j=0}^{\infty} q^{j(\beta-\alpha)} \\ &< \infty \ . \end{aligned}$$

□

**Lemma 4.3.2.** *Let $\mu$ be a positive, finite measure in $K$. If $\mathbb{E}_\alpha(\mu) < \infty$ then the regularity property (4.9) holds for a suitable restriction of $\mu$.*

*Proof.* Since $\mathbb{E}_\alpha(\mu) < \infty$, the set

$$A = \{x \in K : \int \frac{d\mu(y)}{|x-y|^\alpha} \leq M\}$$

has positive $\mu$ measure for some $M$. Let $\nu$ be the restriction of $\mu$ on $A$. Then for $x \in A$

$$\int_{\bar{B}_{q^m}(x)} \frac{d\nu(y)}{|x-y|^\alpha} \leq \int \frac{d\nu(y)}{|x-y|^\alpha} \leq M. \tag{4.10}$$

Since

$$q^{-m\alpha} \nu(\bar{B}_{q^m}(x)) \leq \int_{\bar{B}_{q^m}(x)} \frac{d\nu(y)}{|x-y|^\alpha} \ ,$$

(4.10) gives us that

$$\nu(\bar{B}_{q^m}(x)) \leq M q^{m\alpha} \quad \forall x \in A \ . \tag{4.11}$$

It remains to verify (4.11) for all $x$ belonging to $K$. Let such an $x \in K$. If $\bar{B}_{q^m}(x) \bigcap A = \emptyset$ then $\nu(\bar{B}_{q^m}(x)) = 0$. If there is $z \in \bar{B}_{q^m}(x) \bigcap A$, then by Lemma 4.2.4 we have that

$$\bar{B}_{q^m}(z) = \bar{B}_{q^m}(x) \ .$$

Since (4.11) holds for $z$ we are done.

□

Frostman's lemma together with the last two Lemmas give the following.

**Proposition 4.3.3.** *The Hausdorff dimension of a compact set $E \subset K$ equals the supremum of numbers $\alpha \in [0,1]$ such that there is a positive, finite measure $\mu$ supported on $E$ with the property $\mathbb{E}_\alpha(\mu) < \infty$.*

**Lemma 4.3.4.** ***(i)*** *Suppose $r_t(x) = \frac{1}{|x|^t} \mathbf{1}_{R \setminus \{0\}}(x)$ , $t < 1$, where $\mathbf{1}_A$ denotes the characteristic function on $A$. Then*

$$\widehat{r_t}(x) = \frac{1-q^{-1}}{1-q^{t-1}} \mathbf{1}_R(x) + \frac{1-q^{-t}}{1-q^{t-1}} \frac{1}{|x|^{1-t}} \mathbf{1}_{|\cdot|>1}(x) \ .$$

***(ii)*** *If $\mu$ is a probability compactly supported measure on $K$, then*

$$\mathbb{E}_t(\mu) = \frac{1-q^{-t}}{1-q^{t-1}} \int_K \frac{|\widehat{\mu}(x)|^2}{|x|^{1-t}} dx \ , \ 0<t<1 \ .$$

*Proof.* **(i)**

$$\begin{aligned}
\widehat{r_t}(x) &= \int_{R\setminus\{0\}} \frac{1}{|y|^t} \chi(xy)dy \\
&= \sum_{j=0}^{\infty} \int_{|y|=q^{-j}} \frac{1}{|y|^t} \chi(xy)dy \\
&= \sum_{j=0}^{\infty} q^{jt} \int_{|y|=q^{-j}} \chi(xy)dy \\
&= \sum_{j=0}^{\infty} q^{jt} \left( \int_{\bar{B}_{q^{-j}}(0)} \chi(xy)dy - \int_{\bar{B}_{q^{-j-1}}(0)} \chi(xy)dy \right) .
\end{aligned}$$

By (4.8) the last quantity equals

$$\sum_{j=0}^{\infty} q^{jt} \left( q^{-j} \mathbf{1}_{\bar{B}_{q^j}(0)}(x) - q^{-j-1} \mathbf{1}_{\bar{B}_{q^{j+1}}(0)}(x) \right) .$$

Therefore

$$\widehat{r_t}(x) = \sum_{j=0}^{\infty} q^{j(t-1)} \left( \mathbf{1}_{\bar{B}_{q^j}(0)}(x) - q^{-1} \mathbf{1}_{\bar{B}_{q^{j+1}}(0)}(x) \right) . \tag{4.12}$$

- If $x \in R$, (4.12) gives

$$\begin{aligned}
\widehat{r_t}(x) &= (1-q^{-1}) \sum_{j=0}^{\infty} q^{j(t-1)} \\
&= \frac{1-q^{-1}}{1-q^{t-1}} \ .
\end{aligned}$$

- Let now $|x| = q^m, \ m \geq 1$.

Then the sum in (4.12) should start from $j = m-1$. Hence

$$\begin{aligned}
\widehat{r_t}(x) &= q^{(m-1)(t-1)}(-q^{-1}) \ + \ (1-q^{-1}) \sum_{j=m}^{\infty} q^{j(t-1)} \\
&= q^{m(t-1)}(-q^{-t}) \ + \ \frac{1-q^{-1}}{1-q^{t-1}} q^{m(t-1)} \ .
\end{aligned}$$

Replacing $q^m$ by $|x|$ we finally get

$$\widehat{r_t}(x) = \left( \frac{1-q^{-t}}{1-q^{t-1}} \right) |x|^{t-1} \ .$$

**(ii)** Let $\Delta = \{(x,x) : x \in \text{supp}\mu\}$.

$$\begin{aligned}
\int_K \frac{|\widehat{\mu}(x)|^2}{|x|^{1-t}}dx &= \lim_{n\to\infty} \int_{|x|\leq q^n} \frac{\widehat{\mu}(x)\overline{\widehat{\mu}(x)}}{|x|^{1-t}}dx \\
&= \lim_{n\to\infty} \int\int\int_{|x|\leq q^n} \frac{\chi((y-z)x)}{|x|^{1-t}}dx d\mu(y) d\mu(z) \\
&= \lim_{n\to\infty} q^{nt} \int\int\int_{|x|\leq 1} \frac{\chi((y-z)\pi^{-n}x)}{|x|^{1-t}}dx d\mu(y) d\mu(z) \\
&= \lim_{n\to\infty} q^{nt} \, [C(\mu\times\mu)(\Delta) +
\end{aligned}$$

$$+ \iint_{\Delta^c} \int_{|x|\leq 1} \frac{\chi((y-z)\pi^{-n}x)}{|x|^{1-t}}dx d\mu(y) d\mu(z) \,] \;, \tag{4.13}$$

where in the third equality we made the change of variables

$$x' = \pi^n x \ (dx' = q^{-n}dx).$$

For $y, z \in \text{supp}\mu$, let

$$F_n(y,z) = \begin{cases} q^{nt} \int_{|x|\leq 1} \frac{\chi((y-z)\pi^{-n}x)}{|x|^{1-t}}dx & , \ y \neq z \\ 0 & , \ y = z \ . \end{cases}$$

Then, using (i), we have that

$$\lim_n F_n(y,z) = \begin{cases} B_{q,t}\frac{1}{|y-z|^t} & , \ y \neq z \\ 0 & , \ y = z \end{cases}$$

where $B_{q,t} = \frac{1-q^{t-1}}{1-q^{-t}}$.

For $|y-z| > q^{-n}$ we get from (i) that

$$F_n(y,z) = B_{q,t}\frac{1}{|y-z|^t} \ ,$$

and for $0 < |y-z| \leq q^{-n}$ we get that

$$F_n(y,z) \ = \ \frac{1-q^{-1}}{1-q^{-t}}q^{nt} \ \leq \ \frac{1-q^{-1}}{1-q^{-t}}\frac{1}{|y-z|^t} \ .$$

Therefore we have that

$$|F_n(y,z)| \leq G(y,z) \quad \text{uniformly in } n \ ,$$

where

$$G(y,z) = C_{q,t}\frac{1}{|y-z|^t}\mathbf{1}_{\text{supp}\mu\times\text{supp}\mu}(y,z) \ .$$

We note that $G$ is integrable since $t < 1$ and $\mu$ has compact support. Therefore by the Lebesgue's dominated convergence theorem we get that

$$\lim_{n\to\infty} q^{nt} \iint\limits_{\Delta^c} \int_{|x|\le 1} \frac{\chi((y-z)\pi^{-n}x)}{|x|^{1-t}} dx d\mu(y) d\mu(z) = \frac{1-q^{t-1}}{1-q^{-t}} \iint\limits_{\Delta^c} \frac{d\mu(y)d\mu(z)}{|y-z|^t} \ . \tag{4.14}$$

(a) case: $(\mu \times \mu)(\Delta) = 0$.
From (4.13) and (4.14) we get

$$\int_K \frac{|\widehat{\mu}(x)|^2}{|x|^{1-t}} dx = \frac{1-q^{t-1}}{1-q^{-t}} \int \int \frac{d\mu(y)d\mu(z)}{|y-z|^t} \ .$$

(b) case: $(\mu \times \mu)(\Delta) \neq 0$.
From (4.13) and (4.14) we have that

$$\int_K \frac{|\widehat{\mu}(x)|^2}{|x|^{1-t}} dx = \infty \ .$$

Moreover, clearly $\int \int \frac{1}{|x-y|^t} d\mu(x) d\mu(y) = \infty$.

□

We close this chapter giving the definition of Fourier dimension $\dim_F E$ of a set $E \subset K$. This is defined as the supremum of $\alpha \in [0,1]$ such that $E$ carries a measure $\mu$ with the property

$$|\widehat{\mu}(x)| \le C|x|^{-\frac{\alpha}{2}} \ .$$

By Proposition 4.3.3 and Lemma 4.3.4 we conclude that the Fourier dimension cannot exceed the Hausdorff dimension,

$$\dim_F E \le \dim_H E \ .$$

**Definition 4.3.5.** A set $E \subset K$ is called *Salem* if $\dim_F E = \dim_H E$.

In the next chapter we shall prove the existence of Salem sets in the ring of integers $R$ of an ultrametric local field.

# Chapter 5

# Salem sets in ultrametric local fields

We follow Salem's probabilistic approach [33] to proving the existence of generalized Cantor type Salem sets in an ultrametric local field $K$. We shall see that the characteristic of $K$ plays an important role and does not allow us to give a unified proof independent of the characteristic.

## 5.1 Preliminaries lemmas

We prove a series of lemmas which will play a decisive role in the proof of the main result of this chapter. In what follows, the last chapter and notation within it (e.g. $\pi,\ q,\ S$ etc) will be used without any particular reference.

**Lemma 5.1.1.** *Let $K$ be an ultrametric local field of characteristic $0$. Also let $N \in \mathbb{N},\ k \in \mathbb{N}$ with $N < q^k$. Then there are $a_1, \ldots, a_N \in R$ linearly independent over $\mathbb{Q}$ such that $|a_i - a_j| > q^{-k}$ for every $i \neq j$.*

*Proof.* Setting $S = \{c_0, \ldots, c_{q-1}\}$ as it was defined in Section 4.2, we consider the field extension

$$\mathbb{Q}(c_0, \ldots, c_{q-1}, \pi)$$

of $\mathbb{Q}$, where $\mathbb{Q}$ is regarded as a subfield of $K$. Clearly $\mathbb{Q}(c_0, \ldots, c_{q-1}, \pi)$ is a countable subfield of $K$ whereas $K$ itself is uncountable. Therefore one can find $b_1, \ldots, b_N \in K$ such that $1_K, b_1, \ldots, b_N$ are linearly independent over $\mathbb{Q}(c_0, \ldots, c_{q-1}, \pi)$. For every $b_i, i = 1, \ldots, N$, we consider its expansion

$$b_i = \sum_{j \geq m(i)} d_j(i)\pi^j \ ,$$

where $d_j(i) \in S$ and $m(i) \in \mathbb{Z}$. If $m(i)$ is a negative integer we consider

$$\overline{b_i} = b_i - \sum_{j=m(i)}^{-1} d_j(i)\pi^j \ .$$

If $m(i)$ is non-negative, then we set $\overline{b_i} = b_i$. Hence $\overline{b_i} \in R$ for every $i = 1, \ldots, N$. Since

$$\sum_{j=m(i)}^{-1} d_j(i)\pi^j \in \mathbb{Q}(c_0, \ldots, c_{q-1}, \pi) ,$$

one can easily verify that $1_K, \overline{b_1}, \ldots, \overline{b_N}$ are linearly independent over $\mathbb{Q}(c_0, \ldots, c_{q-1}, \pi)$. Setting

$$\overline{b_i}' = \pi^k \overline{b_i} ,$$

we have that the expansion of each $\overline{b_i}'$ starts from the $k$-th coordinate, that is,

$$\overline{b_i}' = \sum_{j \geq k} d'_j(i)\pi^j, \ d'_j(i) \in S.$$

Since $\#S = q$, we have that the number of the different elements $\sum_{j=0}^{k-1} d_j\pi^j$, as $d_j$ run over $S$, equals $q^k$. We choose $N$ of them, say

$$r_1, \ldots, r_N .$$

Last we set

$$a_i = r_i + \overline{b_i}' ,$$

for $i = 1, \ldots, N$. Clearly $a_1, \ldots, a_N$ fulfill the desired properties: if $i \neq j$, the expansion of $a_i - a_j$ in $K$ has a non-zero coordinate lying between the 0-th coordinate and the $(k-1)$-th coordinate, hence $|a_i - a_j| > q^{-k}$. The linear independence is clearly satisfied over $\mathbb{Q}(c_0, \ldots, c_{q-1}, \pi)$ and thus $\mathbb{Q}$. □

Lemma 5.1.1 covers only the cases where $K$ is either the field of $p$-adic numbers $Q_p$ or some finite field extension of it. We would like to have the same result when $\mathrm{char}(K) = p \neq 0$; that is, $K$ is the field of formal Laurent series over a finite field. Of course, in this case, $\mathbb{Q}$ cannot be embedded in $K$ and regarded as a subfield of it. However, as one naturally expects, we may replace $\mathbb{Q}$ by $F_p$, the unique (up to isomorphism) field of $p$ elements.

**Lemma 5.1.2.** *Let $K$ be an ultrametric local field of characteristic $p \neq 0$. Under the same conditions as in Lemma 5.1.1, there are $a_1, \ldots, a_N \in R$ linearly independent over $F_p$ such that $|a_i - a_j| > q^{-k}$ for every $i \neq j$.*

*Proof.* One could imitate the proof of Lemma 5.1.1 replacing $\mathbb{Q}$ by $F_p$. However things here are much simpler. $K$ is a Formal series field $F_{p^c}((X))$ for some $c \in \mathbb{N}$ $(q = p^c)$ and $R = F_{p^c}[[X]] = \{(c_j)_{j \geq 0} : c_j \in F_{p^c}\}$. We choose $N$ different finite sequences

$$(c_j(i))_{j=0}^{k-1}, \ i = 1, \ldots, N ,$$

where $c_j(i) \in F_{p^c}$ $(N < q^k)$. For $i = 1, \ldots, N$ we define

$$a_i = (b_j(i))_{j \geq 0} ,$$

with

$$b_j(i) = c_j(i) \text{ for every } j = 0, \ldots, k-1 ,$$

for $j = k-1+i$

$$b_{k-1+i}(i) = 1 \ ,$$

and

$$b_j(i) = 0 \text{ for every } j \neq 0, \ldots, k-1, k-1+i \ .$$

The definition of the arithmetic operations in $F_{p^c}((X))$ guarantees the linear independence of $a_1, \ldots, a_N$ over $F_p$. The property $|a_i - a_j| > q^{-k}$ is justified in the same way as in Lemma 5.1.1. □

Next we need two combinatorial lemmas.

**Lemma 5.1.3.** *Let $n_1, \ldots, n_\nu \in \mathbb{N}$ with $\nu$ an even integer, say $\nu = 2\mu$, $\mu \in \mathbb{N}$. Then*

$$\prod_{i=1}^{\nu}(n_i + 1) \leq \sum n_{i_1}(n_{i_1}+1)\cdots n_{i_\mu}(n_{i_\mu}+1) \ , \tag{5.1}$$

*where the sum is taken over all the possible choices $\{i_1, \ldots, i_\mu\}$ out of $\{1, \ldots, \nu\}$.*

*Proof.* Without loss of generality we assume that $n_1 \geq n_2 \geq \cdots \geq n_\nu (\geq 1)$. We also assume that we have carried out all the arithmetic operations in both the left and right hand sides of (5.1) and hence what we are left with are sums with summands of the form $n_{i_1} \cdots n_{i_k}$ from the left hand side and summands of the form $n_{i_1} \cdots n_{i_\lambda} n^2_{i_{\lambda+1}} \cdots n^2_{i_\mu}$ from the right hand side.

We are going to give an algorithm showing how to bound every summand of the left hand side of (5.1) by a summand of the right hand side. We will ensure that no summand of the right hand side is used more than once to bound terms of the left hand side. Let $n_{i_1} n_{i_2} \cdots n_{i_k}$ be an arbitrary summand of the left hand side of (5.1). The algorithm depends on the 'length' $k$.

**(i)** $k > \mu$.
Then there are pairs of indices $(i_j, i_{j'})$ appearing at $n_{i_1} \cdots n_{i_k}$ such that $i_{j'} = i_j + \mu$. For every such square we have $n_{i_j} n_{i_{j'}} \leq n^2_{i_j}$. So we replace the product $n_{i_j} n_{i_{j'}}$ by $n^2_{i_j}$ when $i_{j'} = i_j + \mu$ and leave unchanged the other factors $n_{i_j}$ appearing in $n_{i_1} \cdots n_{i_k}$. Counting every arising square $n^2_{i_j}$ as one factor, the outcome is a product of $\mu$ factors. Therefore it is one of the summands in the right hand side of (5.1) (after the perfomance of all the arithmetic operations there).

**(ii)** $k = \mu$.
We do not change anything since the product $n_{i_1} \cdots n_{i_k}$ appears in the right hand side of (5.1).

**(iii)** $k < \mu$.
Now we should complete the length of the product $n_{i_1} \cdots n_{i_k}$ so that we get a product of $\mu$ factors. What we do here is to bound $n_{i_1} \cdots n_{i_k}$ by $n_{i_1} \cdots n_{i_k} \prod_j n_j^2$ where the product is taken over the $j$'s with $j > \mu$, $j \neq i_1, \ldots, i_k$ and such that $\#j$'s $= \mu - k$. One might have many sets of such $j$'s. We simply pick one.

The uniqueness of the above algorithm is trivially verified: The squares $n_j^2$ in the case **(i)** appear for $1 \leq j \leq \mu$, while in the case **(iii)** for $\mu + 1 \leq j \leq \nu$ (in the case **(ii)** there are not squares at all). The uniqueness within the same case is also obvious. □

**Lemma 5.1.4.** *For* $k, n, N \in \mathbb{N}$ *set* $\Gamma_k = \{((n_1,..,n_N),(l_1,..,l_N)) : n_j, l_j \in \mathbb{N}\bigcup\{0\},\ n_1+..+n_N = n,\ l_1+..+l_N = n,\ n_j - l_j \equiv 0(mod\ k)\ \forall j = 1,..,N\}$. *Then*

$$\sum_{\Gamma_m} \frac{1}{n_1!\cdots n_N!}\frac{1}{l_1!\cdots l_N!} \le \sum_{\Gamma_2} \frac{1}{n'_1!\cdots n'_N!}\frac{1}{l'_1!\cdots l'_N!}\ \forall\ m \ge 2,\ m \in \mathbb{N}.$$

*Proof.* We may assume $m \ge 3$ is an odd integer; otherwise $\Gamma_m \subseteq \Gamma_2$ and the lemma is obvious. The idea behind the proof is that we should map every element

$$\gamma = ((n_1,..,n_N),(l_1,..,l_N)) \quad \text{of } \Gamma_m\ ,$$

to a subset

$$\Delta(\gamma) \text{ of } \Gamma_2\ ,$$

such that

$$\frac{1}{n_1!\cdots n_N!}\frac{1}{l_1!\cdots l_N!} \le \sum_{\Delta(\gamma)} \frac{1}{n'_1!\cdots n'_N!}\frac{1}{l'_1!\cdots l'_N!} \quad \text{and} \tag{5.2}$$

$$\Delta(\gamma_1)\bigcap\Delta(\gamma_2) = \emptyset \text{ whenever } \gamma_1 \ne \gamma_2. \tag{5.3}$$

The latter condition will ensure that we do not use a summand of the right hand side of lemma's inequality more than once in order to bound summands of the left hand side.

Let $\gamma = ((n_1,..,n_N),(l_1,..,l_N))$ be a fixed element of $\Gamma_m$. Then there are

$$\lambda_1,..,\lambda_N \in \mathbb{Z}\ ,$$

such that

$$\sum_{j=1}^{N} \lambda_j = 0\ ,$$

and

$$n_j - l_j = \lambda_j m \text{ for every } j = 1,..,N\ .$$

Let

$$\nu = \#\{j : \lambda_j \text{ odd}\}\ .$$

The property $\sum_{j=1}^{N}\lambda_j = 0$ implies that $\nu$ is a nonnegative even number, say

$$\nu = 2\mu,\ \mu \in \mathbb{N}\bigcup\{0\}\ .$$

**(i)** case: $\nu = 0$.

In this case $\gamma \in \Gamma_2$. We simply consider $\Delta(\gamma) = \{\gamma\}$ and therefore (5.2), (5.3) are fulfilled.

**(ii)** case: $\nu \ne 0,\ \#\{j : \lambda_j \text{ odd},\ l_j = 0\} = 0$.

We set

$$\Delta(\gamma) = \{((n_1,..,n_N),(l_1+\epsilon_1,..,l_N+\epsilon_N)) :$$

$$\epsilon_j = 0 \text{ if } \lambda_j \text{ is even, } \epsilon_j = \pm 1 \text{ if } \lambda_j \text{ is odd, } \sum_{j=1}^{N} \epsilon_j = 0\}.$$

The set $\Delta(\gamma)$ is well-defined since the condition $\sum_{j=1}^{N} \epsilon_j = 0$ is justified by the fact that $\nu$ is an even integer. It is also clear that it is a subset of $\Gamma_2$ because

$$\sum_{j=1}^{N} l_j + \epsilon_j = n \ ,$$

and $n_j - l_j - \epsilon_j$ is an even number for every $j = 1, \ldots, N$. Without loss of generality we assume that the odd $\lambda_j$'s are

$$\lambda_1, .., \lambda_\nu \ .$$

To satisfy (5.2) it is sufficient to prove

$$\frac{1}{l_1! \cdots l_\nu!} \le \sum \frac{1}{(l_1 + \epsilon_1)! \cdots (l_\nu + \epsilon_\nu)!} \ ,$$

where the sum is taken over the set

$$\{(\epsilon_1, .., \epsilon_\nu) : \epsilon_j = \pm 1, \ \sum_{j=1}^{\nu} \epsilon_j = 0\} \ .$$

Therefore we need to prove that

$$1 \le \sum \frac{l_{i_1} \cdots l_{i_\mu}}{(l_{i_{\mu+1}} + 1) \cdots (l_{i_\nu} + 1)} \ ,$$

or equivalently

$$\prod_{i=1}^{\nu} (l_i + 1) \le \sum l_{i_1}(l_{i_1} + 1) \cdots l_{i_\mu}(l_{i_\mu} + 1) \ ,$$

where the last two sums are taken over all the possible choices $\{i_1, .., i_\mu\}$ out of $\{1, .., \nu\}$. This follows from Lemma 5.1.3, and hence (5.2) is satisfied.

To prove (5.3), let $\gamma_1, \gamma_2 \in \Gamma_m$ (both belonging to case **(ii)**).
Let

$$\gamma_1 = ((\overline{n}_1, .., \overline{n}_N), (\overline{l}_1, .., \overline{l}_N)) \ ,$$

and

$$\gamma_2 = ((n'_1, .., n'_N), (l'_1, .., l'_N)) \ .$$

We assume that $\Delta(\gamma_1) \bigcap \Delta(\gamma_2) \neq \emptyset$. Then for every $j = 1, .., N$ we have

$$\overline{n}_j = n'_j \ ,$$

and

$$\overline{l}_j + \overline{\epsilon}_j = l'_j + \epsilon'_j \ ,$$

for some choices of $\overline{\epsilon}_j, \epsilon'_j$. Using the fact that

$$\overline{n}_j - \overline{l}_j = \overline{\lambda}_j m \ ,$$

and

$$n'_j - l'_j = \lambda'_j m \ ,$$

we get

$$(\overline{\lambda}_j - \lambda'_j)m = \overline{\epsilon}_j - \epsilon'_j \ .$$

Since $m \geq 3$ and $\overline{\epsilon}_j, \epsilon'_j$ can take only the values $0, \pm 1$, we conclude that

$$\overline{\epsilon}_j = \epsilon'_j \text{ and } \overline{\lambda}_j = \lambda'_j \ ,$$

for every $j = 1, .., N$. Therefore $\gamma_1 = \gamma_2$.

**(iii)** case: $0 < \#\{j : \lambda_j \text{ odd}, \ l_j = 0\} \leq \#\{j : \lambda_j \text{ odd}, \ l_j \neq 0\}$.

Let

$$A = \{j : \lambda_j \text{ odd}, \ l_j = 0\}, \quad B = \{j : \lambda_j \text{ odd}, \ l_j \neq 0\} \ ,$$

and $C$ be some subset of $B$ with

$$\#A = \#C \ .$$

We set

$$\Delta(\gamma) = \{((n_1, .., n_N), (l_1 + \epsilon_1, .., l_N + \epsilon_N)) : \ \epsilon_j = 0 \text{ if } \lambda_j \text{ is even},$$

$$\epsilon_j = 1 \text{ if } j \in A, \epsilon_j = -1 \text{ if } j \in C, \epsilon_j = \pm 1 \text{ if } j \in B \backslash C, \ \sum_{j=1}^{N} \epsilon_j = 0\}.$$

To prove that $\Delta(\gamma)$ is a well-defined set, we should check that $\#B \backslash C$ is an even number (so that the condition $\sum_{j=1}^{N} \epsilon_j = 0$ in $\Delta(\gamma)$ is consistent with the condition $\epsilon_j = \pm 1$ for $j \in B \backslash C$; this will be sufficient because we also have $\#A = \#C$). To this end, we observe that

$$\#A + \#C + \#B \backslash C = \nu \ ,$$

and hence

$$\#B \backslash C = \nu - 2 \cdot \#A$$

is an even number since $\nu$ is even. Clearly $\Delta(\gamma)$ is a subset of $\Gamma_2$. Following the same argument as in case **(ii)**, one can prove (5.2). We note here that $\epsilon_j$ vary only for $j \in B \backslash C$ (taking values $\pm 1$) and it is for those $j$'s that we repeat the above argument in case **(ii)**. For $j \in A$ or $C$ we have the fixed values $\epsilon_j = 1$ and $\epsilon_j = -1$ respectively and we use the fact that $0! = 1!$ (when $j \in A$) and $1/l_j! \leq 1/(l_j - 1)!$ (when $j \in C$). Finally, property (5.3) is proved in the same way as in case **(ii)**.

**(iv)** case: $\#\{j : \lambda_j \text{ odd}, \ l_j = 0\} > \#\{j : \lambda_j \text{ odd}, \ l_j \neq 0\}$.

Let $A$ and $B$ be the sets defined in case **(iii)**. Let also

$$D = \{j : \lambda_j \text{ odd}, \; n_j = 0\} \ ,$$

and

$$E = \{j : \lambda_j \text{ odd}, \; n_j \neq 0\} \ .$$

We observe that $A \subseteq E$ and $D \subseteq B$. Since $\#A > \#B$ we see that

$$\#E > \#D \ .$$

We consider a subset $F$ of $E$ such that

$$\#F = \#D \ .$$

We set

$$\Delta(\gamma) = \{((n_1 + \delta_1, .., n_N + \delta_N), (l_1, .., l_N)) : \; \delta_j = 0 \text{ if } \lambda_j \text{ is even},$$

$$\delta_j = 1 \text{ if } j \in D, \delta_j = -1 \text{ if } j \in F, \delta_j = \pm 1 \text{ if } j \in E \backslash F, \; \sum_{j=1}^{N} \delta_j = 0\}.$$

As in case **(iii)** we see that $\#E\backslash F$ is an even number and hence $\Delta(\gamma)$ is a well-defined subset of $\Gamma_2$. The conditions (5.2) and (5.3) are satisfied as in cases **(ii)** and **(iii)**. We only mention that the $l_j$'s are now unchanged and it is the $\delta_j$'s, with $j \in E\backslash F$, which vary instead of $\epsilon_j$'s.

It remains to prove (5.3) when $\gamma_1, \gamma_2$ do not lie in the same case. Whenever neither $\gamma_1$ nor $\gamma_2$ belong to case **(iv)**, we do exactly what we did in case **(ii)** to see that (5.3) is satisfied. Now let

$$\gamma_1 = ((\overline{n}_1, .., \overline{n}_N), (\overline{l}_1, .., \overline{l}_N)) \in \Gamma_m \text{ be as in case } \textbf{(iv)} \ ,$$

and

$$\gamma_2 = ((n'_1, .., n'_N), (l'_1, .., l'_N)) \in \Gamma_m \text{ be as in either case } \textbf{(i)}, \textbf{(ii)} \text{ or } \textbf{(iii)} \ ,$$

such that $\Delta(\gamma_1) \bigcap \Delta(\gamma_2) \neq \emptyset$. We observe that

$$\#A \bigcap (E\backslash F) \geq \#A - \#F > \#B - \#D = \#(B \bigcap E) \geq \#B \bigcap (E\backslash F).$$

Therefore there is $j_0$ such that $\overline{n}_{j_0} - 1 = n'_{j_0}$ and $0 = \overline{l}_{j_0} = l'_{j_0} + \epsilon_{j_0}$. Hence either

$$l'_{j_0} = 1 \text{ and } \epsilon_{j_0} = -1 \ ,$$

or

$$l'_{j_0} = \epsilon_{j_0} = 0 \ .$$

Therefore, using the formulae $\overline{n}_{j_0} - \overline{l}_{j_0} = \overline{\lambda}_{j_0} m$ and $n'_{j_0} - l'_{j_0} = \lambda'_{j_0} m$, we get either

$$(\overline{\lambda}_{j_0} - \lambda'_{j_0}) m = 2 \ ,$$

or

$$(\overline{\lambda}_{j_0} - \lambda'_{j_0})m = 1 \ ,$$

but neither can be true.

□

## 5.2 The main theorem

We now turn to our main estimate which is the local field analogue of Salem's basic estimate. To keep things as unified as possible, we denote by $\mathbb{F}$ either $\mathbb{Q}$ or $F_p$ depending on which case we are in; that is, $\mathbb{F} = \mathbb{Q}$ when $\mathrm{char}(K) = 0$ and $\mathbb{F} = F_p$ when $\mathrm{char}(K) = p \neq 0$.

**Proposition 5.2.1.** *Let $K$ be an ultrametric local field, $N \in \mathbb{N}$, $m \in 2\mathbb{N}$ with $N > m$. We set $P(x) = 1/N \sum_{i=1}^{N} \chi(a_i x)$ with $a_1, \ldots, a_N \in R$ linearly independent over $\mathbb{F}$. Then there is a positive constant $T_0 = T_0(a_i, N, m)$ such that*

$$\left(\frac{1}{T}\int_{\bar{B}_T(x_0)} |P(x)|^m dx\right)^{1/m} \leq \frac{\sqrt{m}}{\sqrt{N}} \quad \forall\ T \geq T_0,\ \forall\ x_0 \in K.$$

**Note:** The square root on the right hand side cannot be improved and this is crucial.

*Proof.* Let $m = 2n,\ n \in \mathbb{N}$. Then

$$|P(x)|^m = P(x)^n \overline{P(x)}^n \ ,$$

with

$$\overline{P(x)} = \frac{1}{N}\sum_{i=1}^{N} \chi(-a_i x) \ .$$

Hence we have

$$|P(x)|^m = \frac{1}{N^m}\sum \frac{n!}{n_1! \cdots n_N!}\frac{n!}{l_1! \cdots l_N!}\chi(((n_1 - l_1)a_1 + .. + (n_N - l_N)a_N)x) \ , \quad (5.4)$$

where the sum is taken over the set

$$\Theta = \{((n_1, .., n_N), (l_1, .., l_N)) : \sum_{j=1}^{N} n_j = n,\ \sum_{j=1}^{N} l_j = n,\ n_j, l_j \in \mathbb{N} \bigcup \{0\}\} \ .$$

**(i)** case: $\mathrm{char}(K) = 2$.

Then $\mathbb{F} = F_2$. We observe that $\overline{\chi}(x) = \chi(-x) = \chi(x)$ and hence $\chi(x) \in \mathbb{R}\ \forall x \in K$. Therefore

$$|P(x)|^m = P(x)^m = \frac{1}{N^m}\sum_{m_1 + .. + m_N = m} \frac{m!}{m_1! \cdots m_N!}\chi((m_1 a_1 + .. + m_N a_N)x).$$

We split the last sum into two parts $I_1, I_2$ where $I_1$ is taken over the set

$$G = \{(m_1, .., m_N) : m_1 + .. + m_N = m,\ m_j \geq 0 \text{ even integer } \forall j\}\ ,$$

and $I_2$ is taken over those $N$-tuples which at least one $m_j$ is odd. Then we have

$$I_1 = \frac{1}{N^m} \sum_G \frac{m!}{m_1! \cdots m_N!} \quad \text{and} \quad I_2 = \frac{1}{N^m} \sum_j \lambda_j \chi(b_j x)\ ,$$

where $\lambda_j \in \mathbb{N}$ and $b_j$ are non-zero elements of $K$ because $a_1, .., a_N$ are linearly independent over $F_2$. Choosing sufficiently large $T_0$, (4.8) yields

$$\int_{\bar{B}_T(x_0)} I_2\ dx = 0 \ \ \forall\ T \geq T_0,\ \forall x_0 \in K.$$

Hence we have

$$\frac{1}{T} \int_{\bar{B}_T(x_0)} |P(x)|^m\ dx = \frac{1}{N^m} \sum_G \frac{m!}{m_1! \cdots m_N!} \ \ \forall\ T \geq T_0,\ \forall x_0 \in K. \tag{5.5}$$

Therefore it is sufficient to prove that

$$\frac{1}{N^m} \sum_G \frac{m!}{m_1! \cdots m_N!} \leq \left( \frac{\sqrt{m}}{\sqrt{N}} \right)^m . \tag{5.6}$$

Since $m = 2n$ we see that

$$\begin{aligned} \#G &= \#\{(m'_1, .., m'_N) : m'_1 + .. + m'_N = n,\ m'_j \geq 0\ \forall\ j\} \\ &= \binom{n + N - 1}{N - 1}. \end{aligned}$$

So we need to prove

$$\frac{m!}{N^m} \binom{n + N - 1}{N - 1} \leq \left( \frac{\sqrt{m}}{\sqrt{N}} \right)^m .$$

By simple arithmetic operations we see that the quotient of the left hand side with the right hand side is

$$\frac{(1 + \frac{1}{N})(1 + \frac{2}{N}) \cdots (1 + \frac{n-1}{N})}{(1 + \frac{1}{2n-1})(1 + \frac{2}{2n-2}) \cdots (1 + \frac{n-1}{n+1})}\ ,$$

which is less than 1 since $N > m = 2n$.

This completes the case when $\text{char}(K) = 2$, but before we pass to the next case, we derive a useful formula which will be used in the case when $\text{char}(K) = p > 2$. Turning back to equation (5.4), we repeat the same argument of splitting the sum into $J_1, J_2$ where $J_1$ is taken over $\Gamma_2$ (see Lemma 5.1.4 for the definition of $\Gamma_2$) and $J_2$ is taken over $\Theta \backslash \Gamma_2$ (i.e. at least one $n_j - l_j$ is odd). Therefore, as above, exploiting the linear independence of $a_1, .., a_N$ over $F_2$ and using equation

(4.8), we can find $T_1 > T_0$ ($T_0$ being the constant which appeared in equation (5.5)) such that

$$\frac{1}{T}\int_{\bar{B}_T(x_0)} |P(x)|^m dx = \frac{1}{N^m}\sum_{\Gamma_2} \frac{n!}{n_1!\cdots n_N!}\frac{n!}{l_1!\cdots l_N!} \quad \forall\, T \geq T_1,\ \forall x_0 \in K.$$

Hence, by equation (5.5), we get

$$\sum_G \frac{m!}{m_1!\cdots m_N!} = \sum_{\Gamma_2} \frac{n!}{n_1!\cdots n_N!}\frac{n!}{l_1!\cdots l_N!}. \tag{5.7}$$

**(ii)** case: $\mathrm{char}(K) = p > 2$.

We now have $\mathbb{F} = F_p$. The same arguments as in case **(i)** yield

$$\frac{1}{T}\int_{\bar{B}_T(x_0)} |P(x)|^m dx = \frac{1}{N^m}\sum_{\Gamma_p} \frac{n!}{n_1!\cdots n_N!}\frac{n!}{l_1!\cdots l_N!} \quad \forall\, T \geq T_0,\ \forall x_0 \in K\ ,$$

for some constant $T_0$. Hence the desired estimate follows by applying Lemma 5.1.4, then equation (5.7) and finally the estimate (5.6).

**(iii)** case: $\mathrm{char}(K) = 0$.

Then $\mathbb{F} = \mathbb{Q}$. In this case equation (5.4) becomes

$$|P(x)|^m = \frac{1}{N^m}\sum_{n_1+..+n_N=n}\left(\frac{n!}{n_1!\cdots n_N!}\right)^2 + \frac{1}{N^m}\sum_j \mu_j \chi(d_j x)\ ,$$

where $\mu_j \in \mathbb{N}$ and $d_j$ is a linear combination of $a_1, .., a_N$ with rational coefficients not all zero. Hence $d_j \neq 0\ \forall j$. Using once again equation (4.8) we get, for sufficiently large $T_0 > 0$, that

$$\frac{1}{T}\int_{\bar{B}_T(x_0)} |P(x)|^m dx = \frac{1}{N^m}\sum_{n_1+..+n_N=n}\left(\frac{n!}{n_1!\cdots n_N!}\right)^2 \quad \forall\, T \geq T_0,\ \forall x_0 \in K.$$

However

$$\begin{aligned}\frac{1}{N^m}\sum_{n_1+..+n_N=n}\left(\frac{n!}{n_1!\cdots n_N!}\right)^2 &\leq \frac{n!}{N^m}\sum_{n_1+..+n_N=n}\frac{n!}{n_1!\cdots n_N!}\\ &= \frac{n!}{N^n}\\ &\leq \left(\frac{\sqrt{m}}{\sqrt{N}}\right)^m\ ,\end{aligned}$$

and this completes the proof of the proposition.

□

Before passing to our result on the existence of Salem sets in the ring of integers $R$ of a local ultrametric field $K$, we need the following lemma.

**Lemma 5.2.2.** *Let $\theta \in \mathbb{R}$, and $k = [\theta]$ be the integer part of $\theta$. Then there is a sequence $(k_n), n \in \mathbb{N}$, such that $k_n \in \{k, k+1\}$ and*

$$\lim_n \frac{k_1 + \cdots + k_n}{n} = \theta \ .$$

*Proof.* We set $k_1 = k$. We consider $n_1 \in \mathbb{N}$ the smallest integer such that

$$\theta < \frac{n_1 k + n_1 - 1}{n_1} \ ,$$

and we set $k_2 = \cdots = k_{n_1} = k + 1$. We pick $n_2 \in \mathbb{N}$ the smallest integer such that

$$\frac{(n_1 + n_2)k + n_1 - 1}{n_1 + n_2} < \theta \ ,$$

and we set $k_{n_1+1} = \cdots = k_{n_1+n_2} = k$. We continue inductively and we construct a sequence $(n_\nu)_{\nu \in \mathbb{N}}$, such that $n_\nu \in \mathbb{N}$ and

$$\frac{k \sum_{j=1}^{2\nu} n_j + \sum_{j=1, j \ odd}^{2\nu-1} n_j \ - 1}{\sum_{j=1}^{2\nu} n_j} < \theta < \frac{k \sum_{j=1}^{2\nu+1} n_j + \sum_{j=1, j \ odd}^{2\nu+1} n_j \ - 1}{\sum_{j=1}^{2\nu+1} n_j} \ , \tag{5.8}$$

where $n_{2\nu}, n_{2\nu+1}$ are the smallest natural numbers for which the above inequalities hold. We claim that both sequences in (5.8) converge to $\theta$ as $\nu \to \infty$. Let us suppose that this is not true for the left hand side sequence which we denote by $\theta_\nu$. Then there is $\epsilon > 0$ such that for infinitely many $\nu$'s we have $\theta_\nu < \theta - \epsilon$, or equivalently

$$k \sum_{j=1}^{2\nu} n_j + \sum_{j=1, j \ odd}^{2\nu-1} n_j \ - 1 + \epsilon \sum_{j=1}^{2\nu} n_j < \theta \sum_{j=1}^{2\nu} n_j. \tag{5.9}$$

From the definition of $(n_\nu)_\nu$ we have that the first inequality in (5.8) (i.e. $\theta_\nu < \theta$) does not hold if we replace $n_{2\nu}$ by $n_{2\nu} - 1$. Therefore

$$\theta \sum_{j=1}^{2\nu} n_j \leq k \sum_{j=1}^{2\nu} n_j + \sum_{j=1, j \ odd}^{2\nu-1} n_j \ - 1 + \theta - k. \tag{5.10}$$

From (5.9) and (5.10) we get that

$$\epsilon \sum_{j=1}^{2\nu} n_j < \theta - k \ ,$$

for infinitely many $\nu$'s, which cannot be true. In the same way one can prove the convergence of the sequence of the right hand side of (5.8).

□

Following Salem's approach (see [33],[25]), we now construct a family of generalised Cantor type sets, using a proper probability measure, to conclude that almost every set of this family is a Salem set. The probabilistic approach relies on the freedom of choices of centres of the various balls in the construction, and not on the range of their radii since the absolute value of a local ultrametric field takes on only discrete values.

**Theorem 5.2.3.** *Let $K$ be an ultrametric local field and $R$ its ring of integers. Then for every $0 < \alpha < 1$ and $\epsilon > 0$ there is a set $E \subset R$ of Hausdorff dimension $\alpha$, which carries a measure $\mu_\epsilon$ such that $|\widehat{\mu_\epsilon}(x)| < C_\epsilon |x|^{-\frac{\alpha}{2}+\epsilon}$.*

*Proof.* Consider a large even integer $M \in \mathbb{N}$ and set $N = M^M$. As usual $q$ denotes the number of elements of the finite residue field $R/P$. Setting

$$k = [\theta] \ \text{ with } \ \theta = \frac{\log N}{\log q^\alpha},$$

we get

$$\frac{\log N}{\log q^k} < \alpha(1 + \frac{1}{k}).$$

Therefore, for sufficiently large $M$ we have $N < q^k$ (since $\alpha < 1$). By Lemmas 5.1.1 and 5.1.2 we consider $a_1, \dots, a_N \in R$ linearly independent over $\mathbb{F}$ (see comments about $\mathbb{F}$ before Proposition 5.2.1) satisfying

$$|a_i - a_j| > q^{-k} \ \forall i \neq j \ .$$

Moreover by Lemma 5.2.2 we get a sequence $(k_n)$ such that $k_n \in \{k, k+1\}$, $k_1 = k$ and

$$\lim_n \frac{k_1 + \cdots + k_n}{n} = \theta \ . \tag{5.11}$$

Also let $(\xi_n)$ be a sequence in $K$ such that

$$|\xi_n| = q^{-k_n} \ ,$$

for every $n \in \mathbb{N}$. Since $|a_i - a_j| > q^{-k_1}$, we have from Lemma 4.2.4 that the balls

$$\bar{B}_{q^{-k_1}}(a_i)$$

are mutually disjoint for $i = 1, \dots, N$. In each ball $\bar{B}_{q^{-k_1}}(a_i)$ we consider $N$ subballs

$$\bar{B}_{q^{-k_1-k_2}}(a_i + a_j\xi_1), \ j = 1, \dots, N \ .$$

To see that

$$\bar{B}_{q^{-k_1-k_2}}(a_i + a_j\xi_1) \subseteq \bar{B}_{q^{-k_1}}(a_i) \ ,$$

it is sufficient from Lemma 4.2.4 to observe that

$$|a_i + a_j\xi_1 - a_i| = |a_j||\xi_1| \leq q^{-k_1} \ .$$

Moreover these balls are also pairwise disjoint since

$$|a_j - a_{j'}||\xi_1| > q^{-k_1}q^{-k_1} \geq q^{-k_1-k_2} \ .$$

We continue inductively and after $n$ steps we get $N^n$ balls

$$\bar{B}_{q^{-k_1-\cdots-k_n}}(a_{\epsilon_0}+a_{\epsilon_1}\xi_1+\cdots+a_{\epsilon_{n-1}}\xi_1\xi_2\cdots\xi_{n-1}),\ \epsilon_j=1,\ldots,N\ ,$$

being contained in

$$\bar{B}_{q^{-k_1-\cdots-k_{n-1}}}(a_{\epsilon_0}+a_{\epsilon_1}\xi_1+\cdots+a_{\epsilon_{n-2}}\xi_1\xi_2\cdots\xi_{n-2})\ .$$

This containment is true because

$$|a_{\epsilon_{n-1}}\xi_1\cdots\xi_{n-1}|\le q^{-k_1-\cdots-k_{n-1}}\ .$$

These balls are also mutually disjoint since

$$|a_{\epsilon_{n-1}}-a_{\epsilon'_{n-1}}||\xi_1|\cdots|\xi_{n-1}|>q^{-k_1}q^{-k_1-\cdots-k_{n-1}}\ge q^{-k_1-\cdots-k_{n-1}-k_n}.$$

The limit of this Cantor type construction yields a set $E$ whose elements are of the form

$$x=a_{\epsilon_0}+a_{\epsilon_1}\xi_1+\ldots+a_{\epsilon_n}\xi_1\cdots\xi_n+\ldots\ ,\ \epsilon_j=1,\ldots,N.$$

One can see that for every $M$ the Hausdorff dimension of $E$ is bounded above by $\alpha$. Clearly the set $E$, apart from $M$, depends on the sequence $(\xi_n)$ as well. Our aim is, under a probabilistic approach, to ensure the existence of a sequence $(\xi_n)$ so that the corresponding set $E$ carries a measure $\mu$ satisfying $|\widehat{\mu}(x)|\le C_M|x|^{-\frac{\alpha}{2}(1-\frac{3}{\sqrt{M}})}$. We consider the measures

$$\mu_0=\frac{1}{N}\sum_{j=1}^{N}\delta_{a_j}\quad\text{and}\quad\mu_n=\frac{1}{N}\sum_{j=1}^{N}\delta_{a_j\xi_1\cdots\xi_n}\ ,$$

where $\delta_x$ is the Dirac measure concentrating on $x$. Then the infinite convolution $\mu=\mu_0*\mu_1*\ldots$ is a probability measure whose support is the set $E$. The Fourier-Stieltjes transform of $\mu$ is given by

$$\widehat{\mu}(x)=\prod_{n=0}^{\infty}\widehat{\mu_n}(x),\ x\in K\ ,$$

where

$$\widehat{\mu_0}(x)=\frac{1}{N}\sum_{j=1}^{N}\chi(a_jx)\quad\text{and}\quad\widehat{\mu_n}(x)=\frac{1}{N}\sum_{j=1}^{N}\chi(a_j\xi_1\cdots\xi_nx).$$

Setting

$$P(x)=\frac{1}{N}\sum_{j=1}^{N}\chi(a_jx)\ ,$$

we get the formula

$$\widehat{\mu}(x) = P(x)\prod_{n=1}^{\infty} P(\xi_1\cdots\xi_n x).$$

Writing

$$\xi_n = \pi^{k_n} + \pi^{k_n+1}\zeta_n \ ,$$

where $\zeta_n \in R$, we have that both $E$ and $\widehat{\mu}$ depend on $(\zeta_n)$. We denote by $\aleph_0$ the cardinality of $\mathbb{N}$ and consider the *Hilbert cube* $R^{\aleph_0}$ endowed with the product measure

$$d\zeta = d\zeta_1 d\zeta_2 \cdots$$

of Haar measures $d\zeta_j$ on $R$ with $|R| = 1$. Since $|P(x)| \leq 1$ we have for every $n \geq 1$,

$$\int_{R^{\aleph_0}} |\widehat{\mu}(x)|^M d\zeta \leq \int_{R^n} \prod_{j=1}^{n} |P(\xi_1\cdots\xi_j x)|^M d\zeta_n \cdots d\zeta_1 =$$

$$\int_{R^{n-1}} \prod_{j=1}^{n-1} |P(\xi_1\cdots\xi_j x)|^M \int_R |P(\xi_1\cdots\xi_n x)|^M d\zeta_n \cdots d\zeta_1.$$

We deal first with the inner integral. Substituting $\xi_n = \pi^{k_n} + \pi^{k_n+1}\zeta_n$, it is equal to

$$\int_{|\zeta_n|\leq 1} |P(\xi_1\cdots\xi_{n-1}\pi^{k_n}x + \xi_1\cdots\xi_{n-1}\pi^{k_n+1}\zeta_n x)|^M d\zeta_n = \frac{1}{T}\int_{\bar{B}_T(x_0)} |P(\zeta_n)|^M d\zeta_n \ ,$$

where

$$x_0 = \xi_1\cdots\xi_{n-1}\pi^{k_n}x \ ,$$

and

$$T = |\xi_1\cdots\xi_{n-1}\pi^{k_n+1}x| = q^{-k_1-\cdots-k_n}q^{-1}|x| \ .$$

Hence, by Proposition 5.2.1, there is $T_0 > 0$ such that

$$\int_R |P(\xi_1\cdots\xi_n x)|^M d\zeta_n \leq \left(\frac{\sqrt{M}}{\sqrt{N}}\right)^M \ ,$$

whenever $q^{-k_1-\cdots-k_n}q^{-1}|x| \geq T_0$. Equivalently this is true whenever

$$\log|x| - (k_1 + \cdots + k_n)\log q - \log q \geq \log T_0 \ .$$

We consider $c > 1$ such that

$$c(1 - \frac{1}{\sqrt{M}}) < 1 \ .$$

We set

$$n = \left[(1 - \frac{1}{\sqrt{M}})\frac{\log|x|}{\theta\log q}\right] + 1 \ ,$$

where $\theta$ equals $\frac{\log N}{\log q^\alpha}$ and the square brackets denote the integer part of the enclosed number. For the above $c, n$, if $|x|$ is sufficiently large, say

$$|x| \geq q^{L_0} ,$$

we have from (5.11) that

$$k_1 + \cdots + k_n \leq c \cdot n \cdot \theta .$$

Hence

$$\log|x| - (k_1 + \cdots + k_n)\log q - \log q \;\geq\; \log|x| - c \cdot n \cdot \theta \cdot \log q - \log q \;\geq$$

$$\left(1 - c(1 - \frac{1}{\sqrt{M}})\right)\log|x| - c \cdot \theta \cdot \log q - \log q$$

which is bigger than $\log T_0$ for sufficiently large $|x|$, say

$$|x| \geq q^{L_1},\ L_1 \in \mathbb{N},\ L_1 > L_0 .$$

Clearly, for such an $|x|$, the condition $\log|x| - (k_1 + \cdots + k_j)\log q - \log q \geq \log T_0$ is also true for every $j = 1, \ldots, n$. Therefore applying Proposition 5.2.1 $n$ times, successive integrations give

$$\begin{aligned}
\int_{R^{\aleph_0}} |\widehat{\mu}(x)|^M d\zeta \;&\leq\; \left(\frac{\sqrt{M}}{\sqrt{N}}\right)^{Mn} \\
&\leq\; M^{\frac{1}{2}(1-M)M(1-\frac{1}{\sqrt{M}})\frac{\log|x|}{\theta\log q}} \\
&=\; |x|^{-\frac{\alpha}{2}(M-1)(1-\frac{1}{\sqrt{M}})} \\
&\leq\; |x|^{-\frac{\alpha}{2}M(1-\frac{2}{\sqrt{M}})} .
\end{aligned}$$

This implies

$$\int_{R^{\aleph_0}} \left(|x|^{\frac{\alpha}{2}(1-\frac{3}{\sqrt{M}})}|\widehat{\mu}(x)|\right)^M d\zeta \leq |x|^{-\frac{\alpha}{2}\sqrt{M}} ,$$

and hence integrating with respect to $x$ on the set $\{x \in K : |x| \geq q^{L_1}\}$ we get

$$\int_{R^{\aleph_0}} \int_{|x|\geq q^{L_1}} \left(|x|^{\frac{\alpha}{2}(1-\frac{3}{\sqrt{M}})}|\widehat{\mu}(x)|\right)^M dx d\zeta < \infty ,$$

provided that $M$ is sufficiently large. This means that

$$\int_{|x|\geq q^{L_1}} \left(|x|^{\frac{\alpha}{2}(1-\frac{3}{\sqrt{M}})}|\widehat{\mu}(x)|\right)^M dx < \infty , \tag{5.12}$$

for almost every $\zeta = (\zeta_n) \in R^{\aleph_0}$. Clearly the set

$$\{\sum_{j=-m}^{-1} d_j \pi^j : d_j \in S, m \geq L_1, m \in \mathbb{N}, d_m \neq 0\}$$

is countable, which we enumerate by $\{\beta_n : n \in \mathbb{N}\}$. Then

$$\{x \in K : |x| \geq q^{L_1}\} = \bigcup_n (\beta_n + R), \tag{5.13}$$

the union being disjoint. For every

$$x \in \beta_n + R ,$$

we have from (4.5)

$$|x| = |\beta_n| ,$$

and moreover

$$\widehat{\mu}(x) = \widehat{\mu}(\beta_n) ,$$

because $\mu$ is supported on $E \subset R$ and $\chi$ equals 1 on $R$. Therefore, splitting the integral in (5.12) according to (5.13) we get

$$\sum_n \left( |\beta_n|^{\frac{\alpha}{2}(1-\frac{3}{\sqrt{M}})} |\widehat{\mu}(\beta_n)| \right)^M < \infty ,$$

and hence

$$|\widehat{\mu}(\beta_n)| \leq C_M |\beta_n|^{-\frac{\alpha}{2}(1-\frac{3}{\sqrt{M}})} ,$$

for almost every $(\zeta_n) \in R^{\aleph_0}$. Using once again (5.13) we get that

$$|\widehat{\mu}(x)| \leq C_M |x|^{-\frac{\alpha}{2}(1-\frac{3}{\sqrt{M}})} ,$$

for almost every $(\zeta_n) \in R^{\aleph_0}$ and for every $x \in K$ with $|x| \geq q^{L_1}$. Since $\widehat{\mu}$ is bounded we conclude that the above estimate of $\widehat{\mu}$ holds for every $x \in K$. Now, starting with an $\epsilon > 0$, we choose $M$ so large that

$$\frac{\alpha}{2} \frac{3}{\sqrt{M}} < \epsilon ,$$

and such that all the arguments above hold. The measure $\mu_\epsilon = \mu$ has the desired properties, completing the proof of the theorem.

□

# Chapter 6

# Optimal extension of the Hausdorff-Young inequality in ultrametric local fields

Recently Mockenhaupt and Ricker in [26] showed that the Hausdorff-Young inequality on the torus $\mathbb{T}$ can be extended in an optimal way. They proved the existence of a Banach function space $F^p(\mathbb{T})$, where $1 \leq p \leq 2$, genuinely containing the space $L^p(\mathbb{T})$ for $1 < p < 2$ and such that $\|\widehat{f}\|_{\ell^{p'}(\mathbb{Z})} \leq \|f\|_{F^p(\mathbb{T})}$ for every $f \in F^p(\mathbb{T}), 1 \leq p \leq 2,\ 1/p + 1/p' = 1$. The space $F^p(\mathbb{T})$ is also strictly contained in $L^1(\mathbb{T})$ for $1 < p \leq 2$. For $p = 1, 2$ $\ F^p(\mathbb{T}) = L^p(\mathbb{T})$. The existence and the maximality property of $F^p(\mathbb{T})$ relies on the theory of vector measures and are easily transferred to the local fields setting. However the fact that $F^p(\mathbb{T})$ is strictly larger than $L^p(\mathbb{T})$ for $1 < p < 2$ is proved using a more concrete description of $F^p(\mathbb{T})$ together with an $(L^p, L^2)$ Fourier restriction estimate on Salem sets in $\mathbb{T}$. The aim of this chapter is to transfer these results into the local ultrametric field setting replacing $\mathbb{T}$ by the ring of integers $R$. Throughout this chapter we fix $K$ an ultrametric local field.

## 6.1 $(L^p, L^2)$ restriction estimates in ultrametric local fields

We consider $E \subset R$ and $\mu_\epsilon$ as in Theorem 5.2.3. Our aim is to apply the abstract Theorem 2.1.2 in the local field setting. We shall verify the assumptions of that theorem and get an $(L^p, L^2)$ Fourier restriction result on $E$.

Using the notation of Theorem 2.1.2 we have $G = K^+ = \widehat{G}$. We also define

$$B_r^G(0) = B_r^{\widehat{G}}(0) = \bar{B}_{q^m}(0) \ ,$$

for $q^m \leq r < q^{m+1}$. Then properties (2.3) and (2.4) are clearly satisfied. Theorem 5.2.3 yields the Fourier decay property (2.6). Next we set

$$\phi_r = \mathbf{1}_{\bar{B}_{q^m}(0)} \quad \text{for} \ \ q^m \leq r < q^{m+1} \ .$$

Then (2.7) and (2.8) are fulfilled. Since

$$\widehat{\phi}_r(\xi) = \int_{\bar{B}_r(0)} \chi(\xi x) dx \ ,$$

we get from 4.8 that

$$\widehat{\phi}_r(\xi) = \begin{cases} r & , \quad |\xi| \leq r^{-1} \\ 0 & , \quad |\xi| > r^{-1}. \end{cases}$$

Hence (2.9) is satisfied with $n = 1$. If now

$$\xi \in \bar{B}_{\frac{2j}{r}}(0) \setminus \bar{B}_{\frac{2j-1}{r}}(0) \ ,$$

then $\widehat{\phi}_r(\xi) = 0$ and this implies assumption (2.10). It remains to prove the regularity property (2.5).

**Lemma 6.1.1.** *The measure* $\mu_\epsilon$ *satisfies the following regularity property: For every* $\delta > 0$ *there exists a constant* $C_\delta$ *(which also depends on* $\epsilon$ *and* $\alpha$*) such that*

$$\mu_\epsilon(\bar{B}_{q^\nu}(x_0)) \leq C_\delta q^{\nu\alpha(1-\delta)} \ ,$$

$\forall x_0 \in K, \ \forall \nu \in \mathbb{Z}$*, where* $\alpha$ *is the Hausdorff dimension of* $E$*.*

*Proof.* We focus on $x_0 \in R$ and $\nu \leq 0$ because otherwise $\mu_\epsilon(\bar{B}_{q^\nu}(x_0))$ equals either 1 or 0 and the lemma is trivially satisfied. Let $\delta > 0$. If $(k_n)$ is the sequence defined in the proof of Theorem 5.2.3, then there is $n_0 \in \mathbb{N}$ such that

$$N^n > q^{\alpha(1-\delta)(k_1+\cdots+k_n)} \qquad \forall \ n \geq n_0 \equiv n_0(\delta). \tag{6.1}$$

Let $\bar{B}_r(x_0)$ be one of the balls appearing in the construction of $E$ with

$$\begin{aligned} x_0 &= a_{\epsilon_0} + \cdots + a_{\epsilon_{n-1}} \xi_1 \cdots \xi_{n-1} \ , \\ r &= q^{-k_1-\cdots-k_n} \ , \end{aligned}$$

and such that $n \geq n_0$. Clearly $\mu_\epsilon$ is the weak limit of $\mu_0 * \cdots * \mu_m$ , $m \in \mathbb{N}$, and

$$(\mu_0 * \cdots * \mu_m)(\bar{B}_r(x_0)) = \frac{1}{N^n} \ ,$$

for every $m \geq n - 1$. Since $\bar{B}_r(x_0)$ is compact we have

$$\mu_\epsilon(\bar{B}_r(x_0)) \geq \limsup_m \ (\mu_0 * \cdots * \mu_m)(\bar{B}_r(x_0)) = \frac{1}{N^n}.$$

Moreover, $\bar{B}_r(x_0)$ being open, we have

$$\mu_\epsilon(\bar{B}_r(x_0)) \leq \liminf_m \ (\mu_0 * \cdots * \mu_m)(\bar{B}_r(x_0)) = \frac{1}{N^n}.$$

Therefore $\mu_\epsilon(\bar{B}_r(x_0)) = 1/N^n$ and by (6.1) we get $\mu_\epsilon(\bar{B}_r(x_0)) \leq r^{\alpha(1-\delta)}$. Clearly there are only finitely many balls appearing in the construction of $E$ with radius bigger than $q^{-k_1-\cdots-k_{n_0}}$. Hence, for a suitable constant $C_\delta$, we get that $\mu_\epsilon(\bar{B}_r(x_0)) \leq C_\delta r^{\alpha(1-\delta)}$ for every ball in the construction of $E$.

To get the regularity property for any ball, we need to impose a further condition on the choice of $a_i$'s in Lemmas 5.1.1 and 5.1.2. So, if

$$a_i = \sum_{j\geq 0} d_j(i)\pi^j ,$$

we assume that we choose the $a_i$'s in a uniform way so that $N/q$ of them have the same 1-st coordinate $d_0(i)$, $N/q^2$ have the same 1-st and 2-nd coordinates (i.e. $d_0(i) = d_0(i'), d_1(i) = d_1(i')$) and generally $N/q^\lambda$ of them satisfy $d_0(i) = d_0(i'), \ldots, d_{\lambda-1}(i) = d_{\lambda-1}(i'),\ 1 \leq \lambda \leq k_1$. Then an arbitrary ball $\bar{B}_r(y_0)$ with

$$r = q^{-k_1-\cdots-k_{n-1}-\lambda} \ ,\ 1 \leq \lambda \leq k_n - 1,$$

can contain at most $Nq^{-\lambda}$ balls of radius $q^{-k_1-\cdots-k_n}$ of the construction of $E$. Hence

$$\mu_\epsilon(\bar{B}_r(y_0)) \leq C_\delta N q^{-\lambda} q^{-(k_1+\cdots+k_n)\alpha(1-\delta)}.$$

We shall show that

$$Nq^{-\lambda}q^{-(k_1+\cdots+k_n)\alpha(1-\delta)} \leq q \cdot q^{-(k_1+\cdots+k_{n-1}+\lambda)\alpha(1-\delta)}.$$

Substituting $N = q^{\theta\alpha}$, with $\theta$ as in the proof of Theorem 5.2.3, it is sufficient to show $\theta\alpha - (k_n - \lambda)\alpha(1-\delta) \leq 1 + \lambda$. This is true for $\delta$ small enough because $k = [\theta],\ k_n \in \{k, k+1\}$ and $1 \leq \lambda \leq k_n - 1$.

□

**Theorem 6.1.2.** *For every $p \in [1, \frac{4-2\alpha}{4-3\alpha})$ there is $\epsilon > 0$ such that for the corresponding $E$ and $\mu_\epsilon$ in Theorem 5.2.3, we have*

$$\int_E |\widehat{f}(x)|^2 d\mu_\epsilon(x) \leq C_\epsilon \left( \int_K |f(x)|^p dx \right)^{2/p} \quad \forall\ f \in L^p(K, dx).$$

*Proof.* We apply Theorem 2.1.2. The regularity exponent in (2.5) is $\alpha(1-\delta)$ and the Fourier decay exponent of (2.6) equals $\frac{\alpha}{2} - \epsilon$. We also have $n = 1$. Therefore the endpoint $p_0$ is

$$p_0 = \frac{4 - 2\alpha - 4\epsilon + 4\alpha\delta}{4 - 3\alpha - 2\epsilon + 4\alpha\delta}.$$

Taking $\delta = \epsilon$ sufficiently small we get the desired estimate.

□

## 6.2 Optimal extension of the Hausdorff-Young inequality

The Hausdorff-Young inequality for the compact ring of integers $R$ of an ultrametric local field $K$ is given by

$$\|\widehat{f}\|_{\ell^{p'}} \leq \|f\|_{L^p(R)} \ , \qquad \frac{1}{p}+\frac{1}{p'}=1 \ ,$$

for every $1 \leq p \leq 2$. Here $\widehat{f}$ means $(\widehat{f}(\gamma_n))_n$ where $\gamma_n$ is as usual an enumeration of the set

$$\{\sum_{j=-m}^{-1} d_j \pi^j : m \in \mathbb{N},\ d_j \in S\} \ .$$

The Hausdorff-Young inequality implies that the Fourier transform map

$$\mathcal{F} : L^p(R) \to \ell^{p'} \ ,$$

is bounded for $1 \leq p \leq 2$. The aim of this section is to extend $\mathcal{F}$ continuously keeping the range space $\ell^{p'}$ fixed and to do so in a optimal way to be specified later. Our arguments imitate those in [26].

We denote by $\mathcal{B}(R)$ the class of Borel sets in $R$ and by $L^0(R)$ the space of $\mathcal{B}(R)$-measurable functions on $R$.

**Definition 6.2.1.** (i) Assume $Z \subset L^0(R)$ is a Banach function space over $(R, \mathcal{B}(R), dx)$. We say that its norm $\|\cdot\|_Z$ is *σ-order continuous* if for every sequence $(f_n)$, $f_n \in Z$, with $f_n \searrow 0$ we have that $\|f_n\|_Z \to 0$.
(ii) If $Z, Y$ are Banach function spaces over $(R, \mathcal{B}(R), dx)$, we say that $Z$ is *continuously included* in $Y$ if $Z \subseteq Y$ and there is a constant $C > 0$ such that

$$\|f\|_Y \leq C\|f\|_Z \ ,$$

for every $f \in Z$.

For Banach function spaces $Z$ over $(R, \mathcal{B}(R), dx)$ we shall always assume that if $|g| \leq |f|$ *a.e.* and $f \in Z$, then $g \in Z$ and $\|g\|_Z \leq \|f\|_Z$. We now state the main theorem of this section. We shall prove it via a series of lemmas.

**Theorem 6.2.2.** *For every $1 \leq p \leq 2$ there is a Banach function space $F^p(R) \subset L^0(R)$ with σ-order continuous norm $\|\cdot\|_{F^p(R)}$ such that:*

*(i) $L^p(R)$ is continuously included in $F^p(R)$ and the Fourier transform map $\mathcal{F} : L^p(R) \to \ell^{p'}$ has a continuous extension from $F^p(R)$ into $\ell^{p'}$.*

*(ii) If $Z$ is a Banach function space over $(R, \mathcal{B}(R), dx)$ with the same properties as $F^p(R)$ above, then $Z$ is continuously included in $F^p(R)$.*

*(iii) $F^p(R)$ is continuously included in $L^1(R)$ and the continuous extension of $\mathcal{F}$ from $L^p(R)$ to $F^p(R)$ is again the Fourier transform $f \mapsto \widehat{f}$, $f \in F^p(R)$.*

This theorem relies on the theory of vector measures and more precisely its proof is based on the integration map of a certain vector measure. We note that this approach to optimal extensions for various operators via the integration map of appropriate vector measures is very effective; see [6],[7],[8],[9],[10],[26], [28],[29],[30]. We shall now give some preliminaries concerning vector measures. The main references are [26] and [11].

Let $X$ be a complex Banach space and $\Sigma$ a $\sigma$-algebra of subsets of a non-empty set $\Omega$.

**Definition 6.2.3.** A set function $m : \Sigma \to X$ is called a *vector measure* if for every sequence $(A_n)$ of pairwise disjoint sets in $\Sigma$ we have

$$m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n) \ ,$$

where the series is norm convergent in $X$.

A set $A \in \Sigma$ is called $m$-null if $m(B) = 0$ for all $B \in \Sigma$ which are contained in $A$.

**Lemma 6.2.4.** *Let $T : L^p(R) \to X$ be a bounded linear operator, $p \geq 1$. Then the set function $m : \mathcal{B}(R) \to X$ defined by $m(A) := T(\mathbf{1}_A)$ is a vector measure.*

*Proof.* Let $(A_n)$ be a sequence of pairwise disjoint Borel sets in $R$. Since $|R| = 1$ we get

$$|\bigcup_{n=k}^{\infty} A_n| \to 0 \quad \text{as} \quad k \to \infty \ .$$

Using the fact that $m$ is finitely additive we have

$$\begin{aligned} \lim_k \|m(\bigcup_{n=1}^{\infty} A_n) - \sum_{n=1}^{k} m(A_n)\| &= \lim_k \|m(\bigcup_{n=k+1}^{\infty} A_n)\| \\ &\leq \|T\| \lim_k |\bigcup_{n=k+1}^{\infty} A_n|^{1/p} = 0 \ . \end{aligned}$$

□

We denote by $X^*$ the dual Banach space of $X$.

**Definition 6.2.5.** The *semi-variation* of a vector measure $m : \Sigma \to X$ is the set function $\|m\| : \Sigma \to [0, \infty)$ defined by

$$\|m\|(A) := \sup_{\|x^*\|=1} |\langle m, x^* \rangle|(A) \ , \tag{6.2}$$

where $\langle m, x^* \rangle$ denotes the complex measure $A \to \langle m(A), x^* \rangle$ for each $x^* \in X^*$.

It is known (see [26]) that

$$\sup_{B \in \Sigma, B \subseteq A} \|m(B)\| \leq \|m\|(A) \leq 4 \sup_{B \in \Sigma, B \subseteq A} \|m(B)\| \ , \tag{6.3}$$

for every $A \in \Sigma$.

**Definition 6.2.6.** A set function $\nu : \Sigma \to X$ is called *weakly countably additive* if for every sequence $(A_n)$ of pairwise disjoint sets in $\Sigma$ we have

$$\langle \nu(\bigcup_{n=1}^{\infty} A_n), x^* \rangle = \sum_{n=1}^{\infty} \langle \nu(A_n), x^* \rangle \ ,$$

for every $x^* \in X^*$.

Due to a theorem of Orlicz-Pettis (see [26]), a weakly countably additive set function is vector measure. We shall use this fact later.

A $\Sigma$-measurable function $f : \Omega \to \mathbb{C}$ is said to be $m$-integrable if

$$\int_\Omega |f| d |\langle m, x^* \rangle| < \infty \ , \quad x^* \in X^* \ , \tag{6.4}$$

and for every $A \in \Sigma$ there is a unique element of $X$ denoted by $\int_A f dm$ such that

$$\langle \int_A f dm, x^* \rangle = \int_A f d\langle m, x^* \rangle \ . \tag{6.5}$$

The space of $m$-integrable functions is denoted by $L^1(m)$ and is equipped with the seminorm

$$\|f\|_{L^1(m)} := \sup_{\|x^*\|=1} \int_\Omega |f| d |\langle m, x^* \rangle| \ . \tag{6.6}$$

A function $f$ is called $m$-null if $\|f\|_{L^1(m)} = 0$. The quotient space $L^1(m)$ modulo $m$-null functions is a Banach space which is denoted again by $L^1(m)$. We now state some of the properties of $L^1(m)$.

- The $\mathbb{C}$ valued, $\Sigma$-simple functions are dense in $L^1(m)$.
- If $|g| \leq |f|$ $m$-a.e. ,then $\|g\|_{L^1(m)} \leq \|f\|_{L^1(m)}$.
- The norm $\|\cdot\|_{L^1(m)}$ is $\sigma$-order continuous.
- If $f \in L^\infty(m)$, i.e. $|f| \leq C$ $m$-a.e. , then $f \in L^1(m)$ and from (6.2),(6.6) we have

$$\|f\|_{L^1(m)} \leq \|f\|_{L^\infty(m)} \|m\|(\Omega) \ . \tag{6.7}$$

Let $f \in L^1(m)$ and $m_f : \Sigma \to X$ be the set function

$$m_f(A) := \int_A f dm \ .$$

Using (6.5) one can see that $m_f$ is weakly countably additive according to the Definition 6.2.6. Therefore $m_f$ is a vector measure. We note that from (6.2) and (6.6) we have

$$\|f\|_{L^1(m)} = \|m_f\|(\Omega) \ .$$

Hence, it follows from (6.3) that

$$\sup_{A \in \Sigma} \| \int_A f dm \| \leq \|f\|_{L^1(m)} \leq 4 \sup_{A \in \Sigma} \| \int_A f dm \| \ . \tag{6.8}$$

Last, we define the integration map $I_m : L^1(m) \to X$ to be

$$I_m(f) := \int_\Omega f dm \ .$$

From (6.8), $I_m$ is bounded and

$$\|I_m(f)\| \le \|f\|_{L^1(m)} \ . \tag{6.9}$$

After these preliminaries concerning vector measures, we now specialize to a certain vector measure and give some lemmas which yield the proof of Theorem 6.2.2.

Let $1 \le p \le 2$. We consider the set function $m_p : \mathcal{B}(R) \to \ell^{p'}$,

$$m_p(A) := \widehat{\mathbf{1}_A} \ .$$

The Hausdorff-Young inequality and Lemma 6.2.4 imply that $m_p$ is a vector measure.

**Lemma 6.2.7.** *Let $1 \le p \le 2$.*

*(i) The vector measure $m_p$ is mutually absolutely continuous with respect to Haar measure $dx$ on $R$.*

*(ii) The space $L^1(m_p)$ is continuously included in $L^1(R)$. Moreover $L^1(m_p)$ is dense in $L^1(R)$.*

*Proof.* (i) Let $A \in \mathcal{B}(R)$ be a $m_p$-null set and $B \in \mathcal{B}(R),\ B \subseteq A$. Then $\widehat{\mathbf{1}_B}(\gamma_n) = 0$ for every $\gamma_n \in \widehat{R^+}$. In particular $\widehat{\mathbf{1}_B}(0) = 0$ and hence $|B| = 0$.

Conversely, if $A$ is a $dx$-null set, then clearly $m_p(B) = 0$ for every $B \in \mathcal{B}(R),\ B \subseteq A$.

(ii) Let $f \in L^1(m_p)$. From (6.4) we have that

$$\int_R |f| d|\langle m_p, (a_n)\rangle| < \infty \ , \tag{6.10}$$

for every $(a_n) \in \ell^p$. If we assume that $\gamma_0 = 0$ and consider

$$a_n = \mathbf{1}_{\{0\}}(n) \ ,$$

then

$$\langle m_p, (a_n)\rangle(A) = \sum \widehat{\mathbf{1}_A}(\gamma_n) a_n = |A| \ .$$

Therefore by (6.10) we get

$$\int_R |f| dx < \infty \ ,$$

and hence $f \in L^1(R)$. Moreover, since $\|\mathbf{1}_{\{0\}}(n)\|_{\ell^p} = 1$, the above argument together with (6.6) give us that

$$\|f\|_{L^1(R)} \le \|f\|_{L^1(m_p)} \ ,$$

for every $f \in L^1(m_p)$.

Last, (i) implies that the simple functions in $L^1(R)$ coincide with those in $L^1(m_p)$. Therefore $L^1(m_p)$ is dense in $L^1(R)$.

□

We now state the following theorem which shall be very useful to us. For its proof we refer to [22].

**Theorem 6.2.8.** *Let $m : \Sigma \to X$ be a vector measure and $f : \Omega \to \mathbb{C}$ a complex valued function. Then $f \in L^1(m)$ if and only if there is a sequence $(s_n)$ of simple functions which converges pointwise to $f$ and for which $\left(\int_A s_n dm\right)$ is convergent for every $A \in \Sigma$. In this case*

$$\int_A f dm = \lim_n \int_A s_n dm \ .$$

**Lemma 6.2.9.** *Let $1 \leq p \leq 2$. The space $L^p(R)$ is continuously included in $L^1(m_p)$ and*

$$\int_A f dm_p = \widehat{f\mathbf{1}_A} \ , \qquad A \in \mathcal{B}(R) \ , \tag{6.11}$$

*for every $f \in L^1(m_p)$. In particular the integration map $I_{m_p}$ is a continuous extension of Fourier transform $\mathcal{F}$ from $L^p(R)$ to $L^1(m_p)$, still with values in $\ell^{p'}$.*

*Proof.* Let $f \in L^p(R)$. We can assume that $f \geq 0$. We consider a sequence of simple functions $(s_n)$ such that

$$0 \leq s_n \nearrow f \ ,$$

pointwise in $R$. Then $s_n\mathbf{1}_A \to f\mathbf{1}_A$ in $\|\cdot\|_{L^p(R)}$ for every $A \in \mathcal{B}(R)$. Therefore by the Hausdorff-Young inequality we get

$$\widehat{s_n\mathbf{1}_A} \to \widehat{f\mathbf{1}_A} \quad \text{in} \ \ \ell^{p'} \ . \tag{6.12}$$

<u>Claim</u>: If $s$ is a simple function, then

$$\int_A s dm_p = \widehat{s\mathbf{1}_A} \ , \qquad A \in \mathcal{B}(R) \ . \tag{6.13}$$

To see this, let $s = \sum_{i=1}^N a_i \mathbf{1}_{A_i}$. Then

$$\begin{aligned} \int_A s dm_p &= \sum_{i=1}^N a_i m_p(A_i \bigcap A) \\ &= \sum_{i=1}^N a_i \widehat{\mathbf{1}_{A_i \bigcap A}} \\ &= \left(\sum_{i=1}^N a_i \mathbf{1}_{A_i}\mathbf{1}_A\right)^{\wedge} \\ &= \widehat{s\mathbf{1}_A} \ . \end{aligned}$$

By (6.12) and (6.13) we have that $\left(\int_A s_n dm_p\right)_n$ is convergent in $\ell^{p'}$. Therefore Theorem 6.2.8 implies that $f \in L^1(m_p)$ and

$$\int_A f dm_p = \widehat{f\mathbf{1}_A} \ .$$

This proves (6.11) for every $f \in L^p(R)$.

Using (6.8) we have

$$\begin{aligned} \|f\|_{L^1(m_p)} &\leq 4 \sup_{A \in \mathcal{B}(R)} \|\int_A f dm_p\|_{\ell^{p'}} \\ &= 4 \sup_{A \in \mathcal{B}(R)} \|\widehat{f\mathbf{1}_A}\|_{\ell^{p'}} \\ &\leq 4 \sup_{A \in \mathcal{B}(R)} \|f\mathbf{1}_A\|_{L^p(R)} \\ &= 4\|f\|_{L^p(R)} \ , \end{aligned}$$

for every $f \in L^p(R)$. Hence $L^p(R)$ is continuously included in $L^1(m_p)$. It remains to prove (6.11) for every $f \in L^1(m_p)$.

Let $f \in L^1(m_p)$. There is a sequence $(s_n)$ of simple functions such that $s_n \to f$ in $\|\cdot\|_{L^1(m_p)}$. By one of the properties of the space of $m$-integrable functions, mentioned earlier (for a general vector measure $m$), we also have that

$$s_n\mathbf{1}_A \to f\mathbf{1}_A \quad \text{in } \|\cdot\|_{L^1(m_p)} \ . \tag{6.14}$$

Therefore by Lemma 6.2.7 we get

$$s_n\mathbf{1}_A \to f\mathbf{1}_A \quad \text{in } \|\cdot\|_{L^1(R)} \ .$$

This, by the Hausdorff-Young inequality, gives us

$$\widehat{s_n\mathbf{1}_A} \to \widehat{f\mathbf{1}_A} \quad \text{in } \|\cdot\|_{\ell^\infty} \ . \tag{6.15}$$

However, by (6.13) we have that

$$\widehat{s_n\mathbf{1}_A} = \int_A s_n dm_p \ .$$

It follows by (6.15) that

$$\int_A s_n dm_p \to \widehat{f\mathbf{1}_A} \quad \text{in } \|\cdot\|_{\ell^\infty} \ . \tag{6.16}$$

On the other hand, the continuity of the integration map $I_{m_p}$ together with (6.14) imply that

$$\int_A s_n dm_p \to \int_A f dm_p \quad \text{in } \|\cdot\|_{\ell^{p'}} \ .$$

Since $\|\cdot\|_{\ell^\infty} \leq \|\cdot\|_{\ell^{p'}}$ we finally get

$$\int_A s_n dm_p \to \int_A f dm_p \quad \text{in } \|\cdot\|_{\ell^\infty} . \tag{6.17}$$

Hence (6.11) follows from (6.16) and (6.17).

□

We now prove Theorem 6.2.2.

*Proof.* We set $F^p(R)$ to be $L^1(m_p)$ and $\|\cdot\|_{F^p(R)} := \|\cdot\|_{L^1(m_p)}$.

The parts (i) and (iii) of the theorem have already been established by Lemmas 6.2.7 and 6.2.9. It remains to prove (ii).

Let $Z$ be a Banach function space over $(R, \mathcal{B}(R), dx)$ with $\sigma$-order continuous norm $\|\cdot\|_Z$ and such that $L^p(R)$ is continuously included in $Z$ and the Fourier transform map $\mathcal{F} : L^p(R) \to \ell^{p'}$ has a continuous extension $\widetilde{\mathcal{F}} : Z \to \ell^{p'}$. We want to show that $Z$ is continuously included in $F^p(R)$.

Let $f \in Z$. We can assume that $f \geq 0$. We consider a sequence $(s_n)$ of simple functions such that

$$0 \leq s_n \nearrow f ,$$

pointwise in $R$. Let $A \in \mathcal{B}(R)$. Since $\|\cdot\|_Z$ is $\sigma$-order continuous we get that

$$s_n \mathbf{1}_A \to f \mathbf{1}_A \quad \text{in } \|\cdot\|_Z .$$

By the continuity of $\widetilde{\mathcal{F}}$ we have that

$$\widehat{s_n \mathbf{1}_A} \to \widetilde{\mathcal{F}}(f \mathbf{1}_A) \quad \text{in } \ell^{p'} .$$

The latter together with (6.13) imply that $\left(\int_A s_n dm_p\right)_n$ is convergent. Therefore by Theorem 6.2.8 we conclude that $f \in F^p(R)$ and

$$\int_A f dm_p = \widetilde{\mathcal{F}}(f \mathbf{1}_A) , \tag{6.18}$$

for every $f \in Z$. This proves that $Z \subseteq F^p(R)$. It remains to show the continuity of this inclusion.

Let $f \in Z$. By (6.8) and (6.18) we have

$$\begin{aligned} \|f\|_{F^p(R)} &\leq 4 \sup_{A \in \mathcal{B}(R)} \left\| \int_A f dm_p \right\|_{\ell^{p'}} \\ &= 4 \sup_{A \in \mathcal{B}(R)} \|\widetilde{\mathcal{F}}(f \mathbf{1}_A)\|_{\ell^{p'}} \\ &\leq 4 \|\widetilde{\mathcal{F}}\| \sup_{A \in \mathcal{B}(R)} \|f \mathbf{1}_A\|_Z \\ &\leq 4 \|\widetilde{\mathcal{F}}\| \|f\|_Z . \end{aligned}$$

□

**Remark 6.2.10.** From Lemmas 6.2.7 and 6.2.9, for $p = 1$, we get that

$$F^1(R) = L^1(R) \ ,$$

with equivalent norms. Moreover, Lemma 6.2.9 gives us that

$$L^2(R) \subseteq F^2(R) \ .$$

We aim to verify the opposite inclusion as well. Let $f \in F^2(R)$. Then $\int f dm_2 \in \ell^2$. This together with (6.11) imply that $\widehat{f} \in \ell^2$. Hence $f \in L^2(R)$. Thus we conclude that

$$F^2(R) = L^2(R) \ .$$

We now show that the norms $\| \cdot \|_{F^2(R)}$ and $\| \cdot \|_{L^2(R)}$ are equivalent. From the proof of Lemma 6.2.9 we have that

$$\|f\|_{F^2(R)} \leq 4\|f\|_{L^2(R)} \ .$$

From Plancherel's theorem, (6.11) and (6.8) we get that

$$\|f\|_{L^2(R)} = \|\widehat{f}\|_{\ell^2} = \| \int f dm_2 \|_{\ell^2} \leq \|f\|_{F^2(R)} \ .$$

## 6.3 Concrete descriptions of $F^p(R)$

Let $1 \leq p \leq 2$. We define the space $V^p(R)$ by

$$V^p(R) = \{h \in L^{p'}(R) : \ h = \check{\phi}, \text{ for some } \phi \in \ell^p\} \ . \tag{6.19}$$

**Definition 6.3.1.** For $1 \leq p \leq 2$ we define the spaces

$$\Delta^p(R) = \{f \in L^1(R) : \int |fh| < \infty \quad \forall h \in V^p(R)\} \ , \tag{6.20}$$

$$\Phi^p(R) = \{f \in L^1(R) : \ \widehat{f\mathbf{1}_A} \in \ell^{p'} \quad \forall A \in \mathcal{B}(R)\} \ , \tag{6.21}$$

$$\Gamma^p(R) = \{f \in L^1(R) : \ \widehat{fg} \in \ell^{p'} \quad \forall g \in L^\infty(R)\} \ . \tag{6.22}$$

The aim of this section is to prove the following.

**Theorem 6.3.2.** *Let $1 \leq p \leq 2$. Each of the spaces $\Delta^p(R), \Phi^p(R), \Gamma^p(R)$ coincides with $F^p(R)$.*

We shall prove this theorem through a series of lemmas.

**Lemma 6.3.3.** *Let $\phi \in \ell^p$ and $h = \check{\phi}$. Then for every $A \in \mathcal{B}(R)$*

$$\langle m_p, \phi \rangle (A) = \int_A \tilde{h} dx \ , \tag{6.23}$$

*where $\tilde{h}(x) = h(-x)$.*

*Proof.* We note that $h \in V^p(R) \subseteq L^{p'}(R)$ and $L^{p'}(R) \subseteq L^2(R)$. Hence by Parseval's relation we have

$$\begin{aligned} \langle m_p, \phi\rangle(A) &= \langle \widehat{\mathbf{1}_A}, \widehat{h}\rangle \\ &= \langle \mathbf{1}_A, \tilde{h}\rangle \\ &= \int_A \tilde{h} dx \ . \end{aligned}$$

□

We now state two results from the general theory of vector measures.

**Theorem 6.3.4.** *([21])*

*Let $m : \Sigma \to X$ be a vector measure and $f : \Omega \to \mathbb{C}$ be a measurable function. If $X$ does not contain a subspace isomorphic to $c_0$, then $f \in L^1(m)$ if and only if $\int |f| d|\langle m_p, x^*\rangle| < \infty$ for every $x^* \in X^*$.*

**Theorem 6.3.5.** *([11])*

*Let $X$ contain no copy of $\ell^\infty$ and let $\Gamma$ be a total subset of $X^*$. Let $m : \Sigma \to X$ be a set function with the property that for every pairwise disjoint sequence $(A_n)$ of sets in $\Sigma$ and for every subsequence $(A_{n_k})$ of $(A_n)$ we have*

$$\langle m(\bigcup_k A_{n_k}), x^*\rangle = \sum_k \langle m(A_{n_k}), x^*\rangle \ ,$$

*for every $x^* \in \Gamma$. Then $m$ is a vector measure.*

**Lemma 6.3.6.** *Let $1 \leq p \leq 2$ and $f \in L^0(R)$. Then $f \in F^p(R)$ if and only if*

$$\int_R |fh| dx < \infty \ , \tag{6.24}$$

*for every $h \in V^p(R)$. That is, $F^p(R) = \Delta^p(R)$.*

*Proof.* Let $f \in F^p(R)$ and $h = \check{\phi}$, $\phi \in \ell^p$. By Lemma 6.3.3 we have

$$\langle m_p, \phi\rangle(A) = \int_A \tilde{h} dx \ ,$$

where $\tilde{h}(x) = h(-x)$. Hence the variation of $\langle m_p, \phi\rangle$ is

$$|\langle m_p, \phi\rangle|(A) = \int_A |\tilde{h}| dx \ .$$

Therefore

$$\int_R |f\tilde{h}| dx = \int_R |f| d|\langle m_p, \phi\rangle| \ ,$$

which is finite by the definition of $F^p(R)$. Since the space $V^p(R)$ has the property that $\tilde{h} \in V^p(R)$ whenever $h \in V^p(R)$, we get that

$$\int_R |fh| dx < \infty \ ,$$

for every $h \in V^p(R)$.

Conversely, let $1 < p \leq 2$ and let us assume that (6.24) holds for every $h = \check{\phi}$, $\phi \in \ell^p$. Again by Lemma 6.3.3 and the same argument as above we get that

$$\int_R |f| d|\langle m_p, \phi\rangle| < \infty \ ,$$

for every $\phi \in \ell^p$. Therefore, by Theorem 6.3.4, $f \in F^p(R)$ because the reflexive space $\ell^{p'}$ cannot contain a subspace isomorphic to $c_0$.

It remains the case $p = 1$. The constant function $h(x) = 1$ on $R$ belongs to $V^1(R)$ because

$$\widehat{h}(\gamma_n) = \int_R \chi(\gamma_n x) dx = \mathbf{1}_{\{0\}}(\gamma_n) \in \ell^1 \ ,$$

the last equality comes from (4.8) since $|\gamma_n| \geq q \ \forall \gamma_n \neq 0$. Therefore applying (6.24) for $h = 1$ we get that $f \in L^1(R)$. However, by Remark 6.2.10, $L^1(R) = F^1(R)$ and hence $f \in F^1(R)$.

□

**Lemma 6.3.7.** *Let $1 < p \leq 2$ and $f \in L^1(R)$ such that $\widehat{f\mathbf{1}_A} \in \ell^{p'}$ for every $A \in \mathcal{B}(R)$; i.e. $f \in \Phi^p(R)$ (see (6.21)). Then the set function $\nu_f : \mathcal{B}(R) \to \ell^{p'}$ defined by*

$$\nu_f(A) := \widehat{f\mathbf{1}_A} \ ,$$

*is a vector measure.*

*Proof.* Let $(A_n)$ be a sequence of pairwise disjoint sets in $\mathcal{B}(R)$ and let $(A_{n_k})_k$ be a subsequence of $(A_n)$. Then

$$\begin{aligned}
\langle \nu_f(\bigcup_k A_{n_k}), \mathbf{1}_{\{\gamma_m\}}\rangle &= \langle \widehat{f\mathbf{1}_{\bigcup A_{n_k}}}, \mathbf{1}_{\{\gamma_m\}}\rangle \\
&= \widehat{f\mathbf{1}_{\bigcup A_{n_k}}}(\gamma_m) \\
&= \sum_k \int f(x)\mathbf{1}_{A_{n_k}}(x)\chi(\gamma_m x)dx \\
&= \sum_k \widehat{f\mathbf{1}_{A_{n_k}}}(\gamma_m) \\
&= \sum_k \langle \widehat{f\mathbf{1}_{A_{n_k}}}, \mathbf{1}_{\{\gamma_m\}}\rangle \\
&= \sum_k \langle \nu_f(A_{n_k}), \mathbf{1}_{\{\gamma_m\}}\rangle \ .
\end{aligned}$$

Therefore, since the reflexive space $\ell^{p'}$ cannot contain a subspace isomorphic to $\ell^\infty$, we get by Theorem 6.3.5 that $\nu_f$ is a vector measure.

□

We now state a result from complex measure theory which we shall use later. For its proof we refer to [22].

**Lemma 6.3.8.** *Let $\mu$ be a complex valued measure on $(\Omega, \Sigma)$ and $(f_n)$ a sequence of $\mu$-integrable functions such that*

*(1)* $(f_n)$ *converges to* $f$ *pointwise on* $\Omega$*, and*
*(2)* $\left(\int_E f_n d\mu\right)_n$ *is Cauchy for every* $E \in \Sigma$*. Then* $f$ *is* $\mu$*-integrable and* $\int_E f d\mu = \lim_n \int_E f_n d\mu$ *uniformly with respect to* $E \in \Sigma$*.*

**Lemma 6.3.9.** *Let* $1 \le p \le 2$ *and* $f \in L^0(R)$*. Then* $f \in F^p(R)$ *if and only if* $\widehat{f\mathbf{1}_A} \in \ell^{p'}$ *for every* $A \in \mathcal{B}(R)$*, that is,* $F^p(R) = \Phi^p(R)$*.*

*Proof.* The inclusion $F^p(R) \subseteq \Phi^p(R)$ is obvious since if $f \in F^p(R)$, then $\int_A f dm_p \in \ell^{p'}$ which by (6.11) implies that $\widehat{f\mathbf{1}_A} \in \ell^{p'}$.

We now establish the inclusion $\Phi^p(R) \subseteq F^p(R)$.

• Let $p = 1$. From (6.21) and Remark 6.2.10 we get

$$\Phi^1(R) \subseteq L^1(R) = F^1(R) \ .$$

• Let $1 < p \le 2$ and $f \in \Phi^p(R)$.

We set $A_n = |f|^{-1}([0,n])$ and let $A \in \mathcal{B}(R)$. Clearly $(A_n \bigcap A) \nearrow A$. We also define $(f_n)$ by

$$f_n := f\mathbf{1}_{A_n \bigcap A} \ .$$

Let $h \in V^p(R)$, i.e. $h = \check{\phi}$ for some $\phi \in \ell^p$.

Since $f_n \in L^\infty(R) \subseteq L^2(R)$ and $\tilde{h} \in V^p(R) \subseteq L^2(R)$, Parseval's formula gives us

$$\langle \widehat{f_n}, \phi \rangle = \int_A f\mathbf{1}_{A_n} \tilde{h} dx \ . \tag{6.25}$$

By Lemma 6.3.7 we have that $\nu_f$ is a vector measure. Therefore

$$\begin{aligned} \langle \nu_f(A), \phi \rangle &= \lim_n \langle \nu_f(A \textstyle\bigcap A_n), \phi \rangle \\ &= \lim_n \langle \widehat{f_n}, \phi \rangle \ . \end{aligned}$$

Hence from (6.25) we get that $\left(\int_A f\mathbf{1}_{A_n} \tilde{h} dx\right)_n$ is convergent. Consequently by Lemma 6.3.8, for $\mu = \tilde{h} dx$, we have that $f$ is $\mu$-integrable, that is,

$$\int_R |f\tilde{h}| dx < \infty \ ,$$

for every $h \in V^p(R)$. Then Lemma 6.3.6 implies that $f \in F^p(R)$.

□

**Lemma 6.3.10.** *Let* $1 \le p \le 2$ *and* $f \in L^0(R)$*. Then* $f \in F^p(R)$ *if and only if* $\widehat{fg} \in \ell^{p'}$ *for every* $g \in L^\infty(R)$*. That is,* $F^p(R) = \Gamma^p(R)$*.*

*Proof.* From Lemma 6.3.9 it is clear that $\Gamma^p(R) \subseteq F^p(R)$.

Let us now prove that $F^p(R) \subseteq \Gamma^p(R)$.

• Let $p = 1$. If $f \in F^1(R) = L^1(R)$ and $g \in L^\infty(R)$, then $fg \in L^1(R)$. Therefore $\widehat{fg} \in \ell^\infty$.

• Suppose $1 < p \le 2$.

Let $f \in F^p(R)$ and $g \in L^\infty(R)$. If $h \in V^p(R)$, then

$$\int_R |fgh|dx \le \|g\|_{L^\infty(R)} \int_R |fh|dx$$

which is finite by Lemma 6.3.6. Therefore, again by Lemma 6.3.6, we have that $fg \in F^p(R)$. This means that $\int fg dm_p \in \ell^{p'}$ and so $\widehat{fg} \in \ell^{p'}$ by (6.11).

□

We note that Lemmas 6.3.6, 6.3.9 and 6.3.10 establish the proof of Theorem 6.3.2.

## 6.4 $L^p(R) \subsetneq F^p(R) \subsetneq L^1(R), \ 1 < p < 2$

In Section 6.2 we showed that the Fourier transform map $\mathcal{F} : L^p(R) \to \ell^{p'}, \ 1 \le p \le 2$, has a continuous extension from $F^p(R)$ into $\ell^{p'}$ and this extension is optimal in the sense of Theorem 6.2.2. The aim of this section is to show that this extension is proper for $1 < p < 2$. We note that the Hausdorff-Young inequality can be extended to the Lorentz space $L^{p,p'}(R) \supsetneq L^p(R)$ [36]. However, the same proof of $F^p(R) \supsetneq L^p(R)$ implies that $F^p(R) \supsetneq L^{p,p'}(R)$ as well. As we shall see, this proper inclusion is based on the $(L^p, L^2)$ restriction estimate established in Section 6.1.

**Theorem 6.4.1.** *(i) $F^p(R) \subsetneq L^1(R)$ for every $1 < p \le 2$.*
*(ii) $L^p(R) \subsetneq F^p(R)$ for every $1 < p < 2$.*

*Proof.* (i) We consider $f(x) = \frac{1}{|x|} \frac{1}{\log^2 |x|} \mathbf{1}_{P \setminus \{0\}}$ where $P = \mathring{B}_1(0)$. Then using the fact that $P$ equals the disjoint union of $\{x : |x| = q^{-j}\}, \ j \in \mathbb{N}$, we have

$$\begin{aligned} \int_R |f(x)|dx &= \frac{1}{\log^2 q} \sum_{j=1}^{\infty} \frac{q^j}{j^2} \int_{|x|=q^{-j}} dx \\ &= \frac{1 - q^{-1}}{\log^2 q} \sum_{j=1}^{\infty} \frac{1}{j^2} \ . \end{aligned}$$

Therefore $f \in L^1(R)$. We now assume that $S = \{c_0, \dots, c_{q-1}\}$ with $c_0 = 0$ (see Section 4.2 about $S$). Let also $(\gamma_n)_{n=0}^{\infty}$ be the enumeration of

$$\{\sum_{j=-m}^{-1} d_j \pi^j : d_j \in S, m \in \mathbb{N}\}$$

such that $\gamma_0 = 0$ and if $n = n_k q^{k-1} + \cdots + n_2 q + n_1$ with $0 \le n_j \le q-1$, then

$$\gamma_n = \sum_{j=1}^{k} c_{n_j} \pi^{-j} \ .$$

Using once more the fact that $\{x : |x| = q^{-j}\} = \bar{B}_{q^{-j}}(0) \setminus \bar{B}_{q^{-j-1}}(0)$ and (4.8), we have

$$\begin{aligned}
\widehat{f}(\gamma_n) &= \frac{1}{\log^2 q} \sum_{j=1}^{\infty} \frac{q^j}{j^2} \int_{|x|=q^{-j}} \chi(\gamma_n x) dx \\
&= \frac{1-q^{-1}}{\log^2 q} \left( \sum_{j \geq \frac{\log|\gamma_n|}{\log q}} \frac{1}{j^2} \right) - \frac{q^{-1}}{\log^2(|\gamma_n| q^{-1})} \\
&\geq C \frac{1-q^{-1}}{\log^2 q} \frac{\log q}{\log|\gamma_n|} - \frac{q^{-1}}{\log^2(|\gamma_n| q^{-1})} \\
&\geq \frac{C'}{\log|\gamma_n|} ,
\end{aligned}$$

for some positive constant $C'$, provided that $|\gamma_n|$ is big enough, say $|\gamma_n| \geq q^L$. We consider the smallest $m_0 \in \mathbb{N}$ such that $|\gamma_{m_0}| = q^L$ (actually $m_0 = q^{L-1}$). Then for every $n \geq m_0$ we have $|\gamma_n| \geq q^L$. We also note that

$$\#\{n : |\gamma_n| = q^l\} = (q-1) q^{l-1} \; \forall \, l \in \mathbb{N} .$$

Hence, for every $1 < p \leq 2$ we have

$$\begin{aligned}
\|\widehat{f}\|_{\ell^{p'}}^{p'} &\geq C \sum_{n \geq m_0} \frac{1}{\log^{p'} |\gamma_n|} \\
&= C \sum_{l=L}^{\infty} \frac{(q-1)q^{l-1}}{\log^{p'} q^l} \\
&= C \frac{(q-1)q^{-1}}{\log^{p'} q} \sum_{l=L}^{\infty} \frac{q^l}{l^{p'}} = \infty .
\end{aligned}$$

Therefore $f \notin F^p(R)$ for every $1 < p \leq 2$.

(ii) For $0 < \alpha < 1$ and $\epsilon > 0$ we consider $E \subset R$ and $\mu_\epsilon$ as in Theorem 5.2.3.

For $0 < \beta < 1$ we set

$$r_\beta(x) = \frac{1}{|x|^\beta} \mathbf{1}_{R \setminus \{0\}}(x) .$$

Clearly $r_\beta \in L^1(R)$. We also define

$$I_{\beta,\epsilon}(x) = (r_\beta * \mu_\epsilon)(x) = \int \frac{1}{|x-y|^\beta} d\mu_\epsilon(y) .$$

We note that $I_{\beta,\epsilon}$ depends on $\alpha$ since $\mu_\epsilon$ does so. A simple change of variables shows that

$$\int_R \frac{1}{|x-y|^\beta} dx = \int_R \frac{1}{|x|^\beta} dx ,$$

for every $y \in R$. Therefore, since $\mu_\epsilon$ is finite, Fubini's theorem gives us that

$I_{\beta,\epsilon} \in L^1(R)$ for every $\beta < 1$.

**Claim 1:** $I_{\beta,\epsilon} \in L^{\frac{1-\alpha}{\beta-\alpha}}(R)$ $\forall$ $\beta$ with $\alpha < \beta < 1$.

*proof:* Using the regularity property $\mu_\epsilon(\bar{B}_r(x)) \leq C_\delta r^{\alpha(1-\delta)}$ (Lemma 6.1.1) we have that

$$\begin{aligned} I_{\beta,\epsilon}(x) &= \sum_{j=0}^{\infty} \int_{|x-y|=q^{-j}} \frac{1}{|x-y|^\beta} d\mu_\epsilon(y) \\ &= \sum_{j=0}^{\infty} q^{j\beta} \left(\mu_\epsilon(\bar{B}_{q^{-j}}(x)) - \mu_\epsilon(\bar{B}_{q^{-j-1}}(x))\right) \\ &\leq C_\delta(1 + q^{-\alpha(1-\delta)}) \sum_{j=0}^{\infty} q^{j(\beta-\alpha(1-\delta))}. \end{aligned}$$

For $\beta < \alpha$, choosing $\delta$ sufficiently small, the last series is convergent. Therefore for every $\beta$ such that $0 < \beta < \alpha$ we have that $I_{\beta,\epsilon} \in L^\infty(R)$. On the other hand $I_{\beta,\epsilon} \in L^1(R)$ for every $0 < \beta < 1$. Hence, by convexity, we get that $I_{\beta,\epsilon} \in L^{\frac{1-\alpha}{\beta-\alpha}}(R)$ for every $\alpha < \beta < 1$ and the claim is proved.

We now prove the sharpness of Claim 1 in the following sense:

**Claim 2:** If $\beta_0$ is such that $\frac{1+\alpha}{2} < \beta_0 < 1$ and $I_{\beta_0,\epsilon} \in L^{\frac{1-\alpha+\gamma}{\beta_0-\alpha}}(R)$, then $\gamma = 0$.

*proof:* For $\frac{1}{2} < \beta < 1$, using Plancherel's theorem and Lemma 4.3.4 we have

$$\begin{aligned} \int |I_{\beta,\epsilon}(x)|^2 dx &= \int_K |\widehat{I_{\beta,\epsilon}}(x)|^2 dx \\ &\geq \int_{|x|>1} |\widehat{r_\beta}(x)\widehat{\mu_\epsilon}(x)|^2 dx \\ &= C \int_{|x|>1} \frac{|\widehat{\mu_\epsilon}(x)|^2}{|x|^{1-(2\beta-1)}} dx \\ &= C' + C\mathbb{E}_{2\beta-1}(\mu_\epsilon) \ , \end{aligned}$$

where in the last equality we used the fact that $\widehat{\mu_\epsilon} \in L^\infty$ and $1/|x|^{2-2\beta} \in L^1(R)$ for $\beta > \frac{1}{2}$ . Therefore, from Proposition 4.3.3 we get that,

$$\text{if} \quad \frac{1}{2} < \beta < 1 \quad \text{and} \quad I_{\beta,\epsilon} \in L^2(R), \quad \text{then} \quad \beta \leq \frac{1+\alpha}{2} \ . \tag{6.26}$$

Let us now assume that there is $\beta_0$ with $\frac{1+\alpha}{2} < \beta_0 < 1$ such that $I_{\beta_0,\epsilon} \in L^{\frac{1-\alpha+\gamma}{\beta_0-\alpha}}(R)$ for some $\gamma \geq 0$. Then, since $I_{\beta,\epsilon} \in L^\infty(R)$ $\forall$ $0 < \beta < \alpha$ (see the proof of Claim 1), we get by convexity that $I_{\beta,\epsilon} \in L^2(R)$ $\forall$ $0 < \beta < \frac{1+\alpha+\gamma}{2}$. Hence (6.26) yields $\gamma = 0$ and Claim 2 is proved.

Let $p \in (1, 2)$. We pick $\alpha$ such that

$$\frac{4-2\alpha}{4-3\alpha} > p \ .$$

From Theorem 6.1.2 and the fact that the restriction phenomenon is translation invariant we get that there is an $\epsilon > 0$ such that

$$\int |\widehat{f}(x)|^2 d\mu_\epsilon(x+y) \leq C_\epsilon \left( \int_K |f(x)|^p dx \right)^{2/p} \quad \forall\, y \in R. \tag{6.27}$$

Multiplying (6.27) by $r_\beta(y)$ and then integrating we get

$$\int_R \int |\widehat{f}(x-y)|^2 r_\beta(y) d\mu_\epsilon(x) dy \leq C'_\epsilon \|f\|_{L^p}^2 \ .$$

Applying first Fubini's theorem, then the change of variables $y \to x-y$, and then again Fubini's theorem we get

$$\int_R |\widehat{f}(y)|^2 I_{\beta,\epsilon}(y) dy \leq C'_\epsilon \|f\|_{L^p}^2 \ . \tag{6.28}$$

Now let $a = (a_n) \in \ell^p$ and

$$f(x) = \sum_{n=0}^{\infty} a_n \mathbf{1}_{\bar{B}_1(\gamma_n)}(x) \ .$$

Then

$$\|f\|_{L^p}^p = \sum_{n=0}^{\infty} \int_{\bar{B}_1(\gamma_n)} |a_n|^p dx = \|a\|_{\ell^p}^p \ .$$

On the other hand

$$\widehat{f}(x) = \sum_{n=0}^{\infty} \int_{\bar{B}_1(\gamma_n)} f(y)\chi(xy) dy = \sum_{n=0}^{\infty} a_n \chi(\gamma_n x) \mathbf{1}_R(x).$$

Hence (6.28) gives us

$$\int |\sum_{n=0}^{\infty} a_n \chi(\gamma_n x)|^2 I_{\beta,\epsilon}(x) dx \leq C'_\epsilon \left( \sum_{n=0}^{\infty} |a_n|^p \right)^{2/p} \ .$$

Equivalently, in the dual formulation we have

$$\left( \sum_{n=0}^{\infty} |\widehat{I_{\beta,\epsilon} g}(\gamma_n)|^{p'} \right)^{1/p'} \leq C'_\epsilon \left( \int_R |g(x)|^2 I_{\beta,\epsilon}(x) dx \right)^{1/2} \quad \forall\, g \in L^2(R, I_{\beta,\epsilon}(x)dx).$$

Hence, since $I_{\beta,\epsilon} \in L^1(R)$ we have that $\widehat{I_{\beta,\epsilon} g} \in \ell^{p'}$ for every $g \in L^\infty(R)$. Therefore Theorem 6.3.2 implies that $I_{\beta,\epsilon} \in F^p(R)$ $\forall\ 0 < \beta < 1$. Our goal is to find $\beta$ such that $I_{\beta,\epsilon} \notin L^p(R)$.

To this end, we choose $\beta$ such that

$$\beta > \frac{1+\alpha}{2} \text{ and } \frac{1-\alpha}{\beta-\alpha} < p \ .$$

Then, from Claim 2 we see that $I_{\beta,\epsilon} \notin L^p(R)$, concluding the proof of the Theorem.

$\square$

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