Global regularity of nonlinear dispersive equations and Strichartz estimates.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Evgeni Y Ovcharov*)

To my parents

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Abstract

The main part of the thesis is set to review and extend the theory of the so called Strichartz-type estimates. We present a new viewpoint on the subject according to which our primary goal is the study of the (endpoint) inhomogeneous Strichartz estimates. This is based on our result that the class of all homogeneous Strichartz estimates (understood in the wider sense of homogeneous estimates for data which might be outside the energy class) are equivalent to certain types of endpoint inhomogeneous Strichartz estimates. We present our arguments in the abstract setting but make explicit derivations for the most important dispersive equations like the Schrödinger , wave, Dirac, Klein-Gordon and their generalizations. Thus some of the explicit estimates appear for the first time although their proofs might be based on ideas that are known in other special contexts.

We present also several new advancements on well-known open problems related to the Strichartz estimates. One problem we pay a special attention is the endpoint homogeneous Strichartz estimate for the kinetic transport equation (and its generalization to estimates with vector-valued norms.) For example, this problem was considered by Keel and Tao [30], but at the time the authors were not able to resolve it. We also fall short of resolving that problem but instead we prove a weaker version of it that can be useful for applications. Moreover, we also make a conjecture and give a counterexample related to that problem which might be useful for its potential resolution. Related to the latter is the fact that we now primarily use complex interpolation in the proof of the homogeneous and the inhomogeneous Strichartz estimates, which produces more natural norms in the vector-valued and the abstract setting compared to the real method of interpolation employed in earlier works.

Another important direction of the thesis is to study the range of validity of the Strichartz estimates for the kinetic transport equation which requires a separate and more delicate approach due to its vector-valued dispersive inequality and a special invariance property. We produce an almost optimal range of estimates for that equation. It is an interesting fact that the failure of certain endpoint estimates with L^∞ or L^1-space norms can be shown on characteristics of Besicovitch sets. With regard to applications of these estimates we demonstrate for the first time in the context of a nonlinear kinetic system (the Othmer-Dunbar-Alt kinetic model of bacterial chemotaxis) that its global well-posedness for small data can be achieved via Strichartz estimates for the kinetic transport equation.

Another new development in the thesis is connected to the question of the global regularity of the Dirac-Klein-Gordon system in space dimensions above one for large initial data. That question was instigated in the 1970's by Chadam and Glassey [12, 13, 22] and although a great number of mathematicians have made contributions in the past 30 years, we, together with the independent recent preprint by Grünrock and Pecher [24], present the first global result for large data. In particular, we prove that in two space dimensions the system has spherically symmetric solutions for all time if the initial data is spherically symmetric and lies in a certain regularity class. Our result is achieved via new inhomogeneous Strichartz estimates for spherically symmetric functions that we prove in the abstract setting and in particular for the wave equation.

We make a number of other lesser improvements and generalizations in relation to the Strichartz estimates that shall be presented in the main body of this text.

Contents

Chapter 1

Introduction

1.1 Preliminaries

Strichartz estimates are a type of a-priori estimates for the solutions of a large class of linear partial differential equations whose common property is that their solutions tend to disperse over time. Originally, such estimates were proved by R. Strichartz [44] in the late 1970's for the wave equation but later researchers extended them to other dispersive equations. The original method of proof relied on the recently discovered by Stein and Tomas fundamental results on the restriction properties of the multidimensional Fourier transform. However, the techniques were based on heavy harmonic analysis and the estimates were limited to special cases. In his article [38], Pecher showed that the time and space exponents need not be equal and thus provided most of the Strichartz estimates for the homogeneous equation in the special context of the Klein-Gordon equation. The next major advancement in the method came out in Ginibre and Velo [20] who invented a simpler and more flexible proof that relied only on the duality principle in Functional Analysis. In the late 1980's, Yajima extended the method to equations with inhomogeneous terms to cover different time and space exponents. These ideas were finalized in the mid 1990's in the papers by Lindblad and Sogge [31] and Ginibre and Velo [21]. Today, the core of these techniques is known as the TT^*-method.

By the mid 1990's Strichartz estimates became a standard tool in the analysis of the Schrödinger and the wave equations and gradually became familiar to researches working outsides these two equations. For example, in 1996 came out Castella and Perthame's short article [11], where they prove some homogeneous Strichartz estimates for the kinetic transport equation.

The next breakthrough came in 1997 when Keel and Tao [30] brought a much awaited unification in the theory. The authors elucidated the fundamental property of scaling in the estimates, presented the method in the abstract level, and gave some new tools based on bilinear-form interpolation and scaling invariant decompositions which are today the core of studying the end-point estimates and the inhomogeneous estimates.

In a paper of 2005, Foschi [19] gave a further refinement of the method by introducing a dyadic Whitney decomposition which is more effective than the original one of [30] in the inhomogeneous setting.

Note that in our historical review of method we selected only the works that, as it seems to us, have contributed most to its transformation to present state. We omitted a number of original works whose results are given in special contexts and have been recovered by others. Nowadays the field is vast and continues to grow fast and many important special advances are not mentioned at all. For example, we are not saying anything about Strichartz estimates with potentials or variable coefficients, estimates over manifolds or special domains, discrete Strichartz estimates, or estimates involving angular variables or spherical symmetry. There are numerous applications of the method to other equations apart from the mentioned three. However, we shall mention other works on Strichartz estimates as we pass through our exposition, especially at places where we have been influenced by them.

After these historical remarks let us now introduce the subject from mathematical perspective. We denote by $U(t)$ the continuous linear evolution group of a linear homogeneous differential equation. The two most important properties of $U(t)$ are

- the dispersive estimate:

$$\|U(t)f\|_{L^\infty_x} \lesssim \frac{1}{|t|^\sigma} \|f\|_{L^1_x}, \quad t \in \mathbb{R},\ \forall f \in L^1(X; d\mu) \tag{1.1}$$

- the energy estimate

$$\|U(t)f\|_{L^2_x} \lesssim \|f\|_{L^2_x}, \quad t \in \mathbb{R},\ \forall f \in L^2(X; d\mu) \tag{1.2}$$

where $\sigma > 0$ is the rate of decay, f is the initial profile of the wave, and by $L^p = L^p(X; d\mu)$ we denote the Lebesgue space L^p over some measure space $(X, d\mu)$. The two inequalities above reflect the physical phenomenon that the amplitude of the wave decays over time (equation (1.1)), while its total energy remains constant (in the case of equality in equation (1.2)).

As it stands today, the whole body of Strichartz estimates is built upon the consequences of these two estimates. The homogeneous Strichartz estimates have the form

$$\|U(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2_x}, \qquad \forall f \in L^2_x.$$

To the inhomogeneous equation we associate the following operator

$$W(t)F = \int_{-\infty}^{t} U(t-s)F(s)ds. \tag{1.3}$$

Under the assumption that supp $F \subseteq [0,\infty) \times \mathbb{R}^n$, (1.3) gives the Duhamel's formula. The inhomogeneous Strichartz have the form

$$\|W(t)F\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}, \tag{1.4}$$

where by $L^q_t L^r_x$ we denote the Lebesgue space $L^q(\mathbb{R}; L^r(X; d\mu))$. We show in the sequel that the homogeneous Strichartz estimates can be identified as a special subclass of the inhomogeneous ones, see Theorem 1.3.2. From this point of view, the study of the inhomogeneous Strichartz estimates shall be our prime goal.

The Lebesgue norms in the dispersive and energy inequalities shall be suitably generalized to vector-valued Lebesgue norms and abstract Banach space norms in the subsequent chapters.

The thesis is organized as follows. We continue this introductory chapter with an example of a typical application of the Strichartz estimates to the global well-posedness of a nonlinear dispersive equation. The remainder of the chapter presents some types of equivalent Strichartz estimates, this equivalence shall be used throughout the text.

We begin in essence with studying the Strichartz estimates for the kinetic transport (KT) equation in the next chapter. This chapter is completely self-contained and the exposition follows very closely our preprint [37]. This is done with the intent to expose the subject on a concrete level first. Moreover, in the treatment of the KT equation we shall have to overcome some technical difficulties peculiar to its vector-valued setting as opposed to the scalar setting of the wave and Schrödinger equations for example. But in fact in doing so we shall then be able to generalize easily our techniques to the abstract setting, i.e. to the case of arbitrary Banach spatial norms. In Chapter 3 we give an application of the Strichartz estimates for the KT equation to the global well-posedness of a nonlinear kinetic system. In Chapter 4 we prove some new inhomogeneous Strichartz estimates with spherical symmetry and in Chapter 5 we make an application of these to the global well-posedness of the Dirac-Klein-Gordon (DKG) system in two spatial dimensions when the data is spherically symmetric. These four chapters contain all the novel ideas and the major new developments we propose in our thesis. We suggest that the remaining two chapters are regarded as an appendix.

The promised review of the method and the derivations to concrete PDE's shall be postponed till the final chapters of this thesis. We do this to help our potential readers (and the examiners as well) to extract the new ideas we propose in our work in a quick and uncomplicated manner. The repeated exposition of the method on the abstract level should then be regarded as routine and straightforward generalization. We shall be less careful to maintain an even exposition in such case and some well-known facts and techniques shall only be sketched. In Chapter 6 we present the Strichartz estimates for some of the most important dispersive equations. This is intended to be used as a reference. Their proof is given in Chapter 7.

Finally, the Appendix presents some fundamental facts from Analysis that shall be needed throughout the text.

1.2 Working example

This section provides a working example for one of the most typical applications of the Strichartz estimates in the analysis of nonlinear PDE's. Let us consider the following estimate

$$\begin{aligned}\|u(t)\|_{L_t^4 L_x^4} + \left\|D^{1/2}u(t)\right\|_{L_t^\infty L_x^2} + \left\|D^{-1/2}\partial_t u(t)\right\|_{L_t^\infty L_x^2} \lesssim \\ \|f\|_{\dot{H}^{1/2}} + \|\partial_t g\|_{\dot{H}^{-1/2}} + \|F\|_{L_t^{4/3} L_x^{4/3}}.\end{aligned} \tag{1.5}$$

for the solution of

$$\begin{aligned}\Box u(t,x) = F(t,x,u) \quad (t,x) \in \mathbb{R}^{1+3} \\ (u, \partial_t u)_{t=0} = (f,g),\end{aligned}$$

proved by Strichartz in his original paper. For the definition of the operator D^s of fractional differentiation and that of the Sobolev space $\dot{H}^s = \dot{H}^s(\mathbb{R}^n)$ see section 8.

Example 1.2.1. [41, p. 110] Let us apply the above inequality to prove global existence and uniqueness for $\Box u = u^3$ on $\mathbb{R}^{1+3}$ with data $(u, \partial_t u)_{t=0}(f,g) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$, provided that

$$E_0 = \|f\|_{\dot{H}^{1/2}} + \|g\|_{\dot{H}^{-1/2}}$$

is sufficiently small. To see this, denote by $X(u)$ the left hand side of (1.5). We now iterate in this norm. The iterates are defined inductively by $u_0 = 0$ and

$$\Box u_j = u_{j-1}^3$$

with data (f,g), for $j \in \mathbb{N}$. Then by (1.5), using the fact that

$$\|uvw\|_{L^{4/3}} \leq \|u\|_{L^4} \|v\|_{L^4} \|w\|_{L^4}$$

we have

$$X(u_j) \leq CE_0 + CX(u_{j-1})^3.$$

So if $X(u_{j-1}) \leq 2CE_0$, then so is $X(u_j)$, provided $C(2CE_0)^2 \leq 1/2$. Then, since

$$\Box(u_{j+1} - u_j) = u_j^3 - u_{j-1}^3 = (u_j - u_{j-1})u_j^2 + u_{j-1}(u_j + u_{j-1})(u_j - u_{j-1})$$

with vanishing initial data, we have

$$X(u_{j+1}-u_j) \le C'[X(u_j)+X(u_{j-1})]^2 X(u_j-u_{j-1}) \le C'(4CE_0)^2 X(u_j-u_{j-1}),$$

so $\{u_j\}$ is Cauchy provided $C'16C^2E_0^2 \le 1/2$.

1.3 Equivalent estimates

In this section we present some instances where we have equivalence between two given Strichartz estimates. To do so, let us first introduce the setting. Consider two abstract Banach spaces $\mathcal{B}_1$, $\mathcal{B}_2$. Suppose that the duality pairing $\langle \cdot, \cdot \rangle$ for these two spaces is the same and that $\mathcal{B}_1$ and ${\mathcal{B}_2}^*$ have a common dense subset $\mathcal{S}$. We define the adjoint $U^*(t) : \mathcal{S} \to {\mathcal{B}_1}^*$ to $U(t) : \mathcal{S} \to \mathcal{B}_2$ by

$$\langle U(t)f, g \rangle = \langle f, U^*(t)g \rangle \quad \forall f, g \in \mathcal{S}.$$

A typical example is $\mathcal{B}_1 = L^p$, $\mathcal{B}_2 = L^q$, which have the same duality pairing $\langle f, g \rangle = \int fg dx$, and $\mathcal{S}$ being taken as the Schwartz class on $\mathbb{R}^n$.

Lemma 1.3.1 (The Duality lemma)**.** *The following two estimates for $W(t)$ are equivalent*

$$\|W(t)F\|_{L^q(\mathbb{R};\mathcal{B}_2)} \lesssim \|F\|_{L^p(\mathbb{R};\mathcal{B}_1)}\,,$$
$$\|W(t)F\|_{L^{p'}(\mathbb{R};{\mathcal{B}_1}^*)} \lesssim \|F\|_{L^{q'}(\mathbb{R};{\mathcal{B}_2}^*)}\,,$$

for $1 \le p, q \le \infty$, whenever they are both invariant to the transformation $U(t) \leftrightarrow U(-t)$.

Proof. The proof is a straightforward generalization of the proof of Lemma 2.3.14 □

Theorem 1.3.2 (The Equivalence theorem)**. A.** *The following three estimates are equivalent*

$$\|U(t)f\|_{L^q(\mathbb{R};\mathcal{B}_2)} \lesssim \|f\|_{\mathcal{B}_1}\,, \qquad \forall f \in \mathcal{B}_1,$$
$$\|W(t)F\|_{L^q(\mathbb{R};\mathcal{B}_2)} \lesssim \|F\|_{L^1(\mathbb{R};\mathcal{B}_1)}\,, \qquad \forall F \in L^1(\mathbb{R};\mathcal{B}_1),$$
$$\|W(t)F\|_{L^\infty(\mathbb{R};{\mathcal{B}_1}^*)} \lesssim \|F\|_{L^{q'}(\mathbb{R};{\mathcal{B}_2}^*)}\,, \qquad \forall F \in L^{q'}(\mathbb{R};{\mathcal{B}_2}^*).$$

B. *Whenever $\mathcal{B}_1$ is a Hilbert space, the homogeneous estimate above is equivalent to*

$$\|W(t)F\|_{L^q(\mathbb{R};\mathcal{B}_2)} \lesssim \|F\|_{L^{q'}(\mathbb{R};{\mathcal{B}_2}^*)}\,, \qquad \forall F \in L^{q'}(\mathbb{R};{\mathcal{B}_2}^*).$$

Proof. The proof is a straightforward generalization of the proof of Theorem 2.3.15. □

Chapter 2

Strichartz Estimates for the Kinetic Transport Equation

2.1 Introduction

In this chapter we study the Strichartz estimates for the kinetic transport (KT) equation

$$\partial_t u(t,x,v) + v \cdot \nabla_x u(t,x,v) = F(t,x,v), \quad (t,x,v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \tag{2.1}$$

to which we prescribe the initial data

$$u(0,x,v) = f(x,v). \tag{2.2}$$

Strichartz estimates for the KT equation were first proved by Castella and Perthame [11] in 1996, where they derive some homogeneous estimates and also one symmetric inhomogeneous estimate. In 1998 Keel and Tao [30] extended the range of the homogeneous estimates but were unable to resolve the endpoint homogeneous Strichartz estimate. Recently, in 2007, Guo and Peng [25] demonstrated the expected failure of the endpoint homogeneous estimate in spatial dimension $n = 1$. Interestingly enough, they also showed that the estimate holds if one replaces the spatial L^∞-norm by the BMO norm.

The first application of the Strichartz estimates for (2.1) appeared in Bournaveas et al. [8] where the authors prove the existence of weak global solutions to a kinetic model of chemotaxis without uniqueness. Another exposition of the results of [11] can be found in Perthame [40] which contains a modern survey to the field of nonlinear kinetic equations from point of view of Mathematical Analysis.

So, to summarize, the situation with respect to the Strichartz estimates for the KT equation is as follows. The homogeneous estimates are understood, but without the arguably most difficult endpoint estimate. The range of the known inhomogeneous estimates is incomplete, actually void, save for one symmetric inhomogeneous estimate. The possible application of these estimates to the analysis of nonlinear kinetic equations is still rather misty as the only known such application [8] delivers the quite unsatisfactory existence of weak global solutions for small data without uniqueness.

It is quite clear that the situation can be radically improved, at least with regard to the inhomogeneous estimates, if one takes into account the recent developments that took place in the context of other equations. Moreover, our work contains several genuinely new developments. For convenience, in the following list we summarize all that is new.

1. We prove the validity of the endpoint homogeneous estimate. In fact, our method is easily generalizable to the abstract setting and can be used to prove the endpoint homogeneous Strichartz estimate for a vector-valued spatial norm. We also prove the entire range of homogeneous and inhomogeneous Strichartz estimates for admissible exponents.

2. We also consider generalized homogeneous estimates for data outside the "energy" or in this context rather the "transport" class. We show that the question of finding the entire range of these estimates is tied to the question of finding the entire range of some endpoint inhomogeneous estimates.

3. The parallel to number 2 question is that of finding the entire range of the inhomogeneous Strichartz estimates (including those for non-admissible exponents). We resolve these two questions in spatial dimension $n = 1$ and leave open some estimates in higher dimensions.

4. We devise counterexamples to show that our estimates are essentially sharp. It is quite interesting that the failure of the estimates that contain L^∞_x-norms might be shown on the characteristic functions of Besicovitch sets, we thus extend the counterexamples of Guo and Peng [25] in a new geometrical fashion.

5. We show that the Othmer-Dunbar-Alt kinetic model of bacterial chemotaxis is globally well-posed for small data via an application of the Strichartz estimates for the KT equation. We thus demonstrate the usefulness of these estimates for showing global existence and uniqueness in the context of a nonlinear kinetic equation. Our result improves and extends the previous works [8, 28] on that system.

2.1.1 Basic facts about the Strichartz estimates for the KT equation

The KT equation is an interesting model for studying Strichartz estimates that offers some peculiar advantages. First, the kinetic transport evolution group $U(t)$ has a very simple explicit form

$$U(t)f = f(x - tv, v).$$

Second, the homogeneous KT equation is invariant to the transformation

$$f \to f^{\alpha}, \quad U(t)f \to (U(t)f)^{\alpha}, \tag{2.3}$$

which allows us to derive new estimates from a known homogeneous Strichartz estimate. The exponents (in (2.6), analogously for (2.8)) transform according to the rule

$$(q, r, p, a) \to (\alpha q, \alpha r, \alpha p, \alpha a), \quad 0 < \alpha < \infty. \tag{2.4}$$

And third, the KT equation is probably the simplest model of an equation with a vector-valued dispersive inequality

$$\|U(t)f\|_{L_x^\infty L_v^1} \lesssim \frac{1}{|t|^n} \|f\|_{L_x^1 L_v^\infty}. \tag{2.5}$$

This fact makes the endpoint Strichartz estimate harder than those of the wave and the Schrödinger equations. Furthermore, the specific power invariance (2.3) to the context of the KT equation causes the study of the Strichartz estimates to take a separate approach from the other equations we have considered so far.

The homogeneous Strichartz estimates have the form

$$\|U(t)f\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L_{x,v}^a}, \tag{2.6}$$

where by $L_t^q L_x^r L_v^p$ we mean $L^q(\mathbb{R}; L^p(\mathbb{R}^n; L^r(\mathbb{R}^n)))$. Note that now the class of the initial data can be any $L^a(\mathbb{R}^{2n})$ for $a > 0$ due to the power transform (2.3). The inhomogeneous estimates have the form

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}, \tag{2.7}$$

where the inhomogeneous evolution operator $W(t)$ is defined as

$$W(t)F = \int_{-\infty}^{t} U(t-s)F(s)ds.$$

Whenever we are interested in the initial value problem (IVP) only, we shall assume that supp $F \subseteq [0, \infty)$. Thus, by the abbreviation $L_t^q L_x^r L_v^p$ we may also understand $L^q([0, \infty); L^p(\mathbb{R}^n; L^r(\mathbb{R}^n)))$.

We shall also consider generalized homogeneous estimates of the form

$$\|U(t)f\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L_x^b L_v^c}. \tag{2.8}$$

Following Keel and Tao [30], we shall call the Lebesgue exponents for which estimate (2.6) holds for every $f \in L_{x,v}^a$ *admissible*. And as in Foschi [19] we shall call the exponents for which (2.7) holds *acceptable*. Our next goal shall be to define precisely the range of admissible/acceptable exponents for the KT equation. To that end we denote by $\mathrm{HM}(p, r)$ the harmonic mean of p and r, in other words $a = \mathrm{HM}(p,r)$ if and only if

$$\frac{1}{a} = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{p}\right).$$

Definition 2.1.1. Set

$$\begin{cases} p^*(a) = \frac{na}{n+1}, & r^*(a) = \frac{na}{n-1}, \quad \text{if} \quad \frac{n+1}{n} \le a \le \infty, \\ p^*(a) = 1, & r^*(a) = \frac{a}{2-a}, \quad \text{if} \quad 1 \le a \le \frac{n+1}{n}. \end{cases} \tag{2.9}$$

When $a = 2$ we simply write $p^* = p^*(2)$, $r^* = r^*(2)$.

Definition 2.1.2. We say that the exponent triplet (q, r, p) is *KT-admissible* if

$$\frac{1}{q} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right), \quad a \overset{def}{=} \mathrm{HM}(p,r), \tag{2.10}$$

$$1 \le p, q, r \le \infty, \quad p^*(a) \le p \le a, \quad a \le r \le r^*(a), \tag{2.11}$$

except in the case $n = 1$, $(q, r, p) = (a, \infty, a/2)$.

A consequence of the above definition is the fact that if the triplet (q, r, p) is KT-admissible then $a \le q \le \infty$ and $p \le r$. Triplets of the form $(q, r, p) = (a, r^*(a), p^*(a))$, for $(n+1)/n \le a < \infty$, shall be called endpoint. When $a = 1$ the only admissible triplet is $(\infty, 1, 1)$, and similarly, when $a = \infty$ the only admissible triplet is (∞, ∞, ∞).

Note that due to the power invariance (2.3) we could have chosen the bounds $p^*(a) = \frac{na}{n+1}$ and $r^*(a) = \frac{na}{n-1}$ for any $a > 0$. Indeed, these are the correct bounds when $a = 2$ as we shall see in Theorem 2.2.1. However, the reason for having the second condition in (2.9) is to restrict the exponents q, r, p and a to the interval $[1, \infty]$ where we can use duality.

Definition 2.1.3. We say that the exponent triplet (q, r, p) is *KT-acceptable* if

$$\frac{1}{q} < n\left(\frac{1}{p} - \frac{1}{r}\right), \quad 1 \le q \le \infty, \quad 1 \le p < r \le \infty, \tag{2.12}$$

or if $q = \infty$, $1 \le p = r \le \infty$.

Note that every KT-admissible triplet is KT-acceptable too. The significance of this definition lies in the fact that if a triplet (q, r, p) is not admissible then the estimates

$$\|U(t)f\|_{L_t^q L_x^r L_v^p} < \infty, \quad \|W(t)F\|_{L_t^q L_x^r L_v^p} < \infty$$

can be violated on some $f \in C_0^\infty(\mathbb{R}^{2n})$, $F(t) \in C_0^\infty(\mathbb{R}^{2n+1})$, respectively, see section 2.6. In other words the notion of acceptability delimits the range of exponents for which the homogeneous and inhomogeneous Strichartz estimate can hold for a general class of initial data. But even more importantly, in the context of the inhomogeneous Strichartz estimates only, we shall see that there are triplets (q,r,p), $(\tilde q,\tilde r,\tilde p)$ that are not admissible but merely acceptable for which estimate (2.7) holds.

We introduce the following definition.

Definition 2.1.4. We say that the two KT-acceptable exponent triplets (q,r,p) and $(\tilde q,\tilde r,\tilde p)$ are *jointly KT-acceptable* if

$$\frac{1}{q}+\frac{1}{\tilde q}=n\left(1-\frac{1}{r}-\frac{1}{\tilde r}\right),\quad \frac{1}{q}+\frac{1}{\tilde q}\le 1, \tag{2.13}$$

$$\mathrm{HM}(p,r)=\mathrm{HM}(\tilde p',\tilde r')\overset{def}{=}a, \tag{2.14}$$

and if further the exponents satisfy the following restrictions

(i)

$$\frac{n-1}{p'}<\frac{n}{\tilde r},\quad \frac{n-1}{\tilde p'}<\frac{n}{r}, \tag{2.15}$$

for $r,\tilde r\neq\infty$.

(ii) if $r=\infty$ then the point $(1/q,1/r,1/p,1/\tilde q,1/\tilde r,1/\tilde p)\in\Sigma_1\cup B$,

$$\Sigma_1=\{(\mu,0,\kappa,\nu,1-\kappa,1)\ :\ 0<\mu,\nu<1,\ 0<\mu+\nu<1,\ \kappa=(\mu+\nu)/n\},$$
$$B=(0,0,0,0,1,1).$$

(iii) if $\tilde r=\infty$ then the point $(1/q,1/r,1/p,1/\tilde q,1/\tilde r,1/\tilde p)\in\Sigma_2\cup C$,

$$\Sigma_2=\{(\mu,1-\kappa,1,\nu,0,\kappa)\ :\ 0<\mu,\nu<1,\ 0<\mu+\nu<1,\ \kappa=(\mu+\nu)/n\},$$
$$C=(0,1,1,0,0,0).$$

Remark 2.1.5. We do not know whether condition (2.15) for dimensions $n>1$ is necessary (apparently, this condition is void for $n=1$). This parallels the Schrödinger equation (and other dispersive equations) for which the entire range of inhomogeneous Strichartz estimates is not yet known when the rate of dispersion $\sigma>1$ and a similar condition appears.

We also note that the two sets Σ_1 and Σ_2 that describe the acceptable triplets with $r=\infty$ or $\tilde r=\infty$, respectively, are almost optimal. In fact, there are only some points lying on the boundary of these sets whose corresponding inhomogeneous estimates (2.17) are still unresolved for dimensions $n>1$, see (2.88), (2.89). More precisely, these estimates correspond to points lying on the hypotenuse AB of ΔOAB in fig. 2.1 in a sense explained below.

"A picture is worth a thousand words", one Chinese proverb says. We finish our introduction with a graphic illustration of the range of validity of the inhomogeneous Strichartz estimates for the operator $W(t)$ given in Theorem 2.2.2. Remember that the Lebesgue space L^p is best seen as a "function" of $1/p$ rather than p in the context of interpolation. Therefore, the range of validity of estimate (2.17) in terms of its exponents $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})$ can be seen as a region in $\mathbb{R}^6$. Its projection over the $(1/q, 1/\tilde{q})$-plane (for $\sigma \geq 1$) is visualized on fig. 2.1.

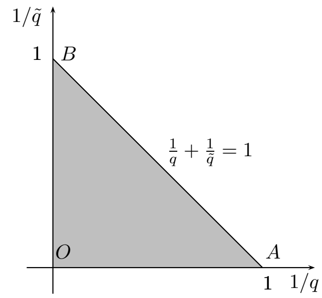

Figure 2.1: Acceptable range of $(1/q, 1/\tilde{q})$ for $\sigma \geq 1$.

The inner part of ΔOAB corresponds to the non-endpoint inhomogeneous estimates, while its three sides correspond to the endpoint inhomogeneous estimates. In the context of Theorem 2.2.2, the inner part of ΔOAB corresponds to part (i), the cathetus OA - to part (ii), the cathetus OB - to part (iii), and the hypotenuse AB - to part (iv).

Let M be the middle of the hypotenuse AB. By duality each estimate corresponding to a point P in ΔOAM is equivalent to an estimate corresponding to a point P' in ΔOMB where P' is the reflection of P along the median OM. The symmetric inhomogeneous estimates on OM are equivalent to the homogeneous estimates (2.6), see the Equivalence theorem 2.3.15, part B, and use the power invariance (2.3). Each inhomogeneous estimate on either of the two catheti OA, OB is equivalent to a generalized homogeneous estimate (2.8), a new result contained in Theorem 2.3.15. The inhomogeneous estimates can be put into three groups each having its own method of proof and in ascending order of difficulty these are: the inner part of ΔOAB, the two catheti OA and OB, and the hypotenuse AB.

2.2 Main results

Theorem 2.2.1 (Strichartz estimates for admissible exponents)**.** *Let $u(t)$ be the solution to the IVP for* (2.1), (2.2)*. Then the estimate*

$$\|u(t)\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L^a_{x,v}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}, \tag{2.16}$$

holds for all $f \in L^a(\mathbb{R}^{2n})$, $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$ if and only if (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two KT-admissible exponent triplets and $a = \mathrm{HM}(p, r) = \mathrm{HM}(\tilde{p}', \tilde{r}')$, apart from the case when $n > 1$ and (q, r, p) being an endpoint triplet, which remains open.

Theorem 2.2.2 (Global inhomogeneous estimates)**.** *Suppose that (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two jointly KT-acceptable exponent triplets that further satisfy the following conditions* *(i)* $1 < q, \tilde{q} < \infty$, $q > \tilde{q}'$, *then the estimate*

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \tag{2.17}$$

holds for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$.

(ii) $\tilde{q} = \infty$, $1 < q < \infty$, *then the estimate*

$$\|W(t)F\|_{L_t^{q,\infty} L_x^r L_v^p} \lesssim \|F\|_{L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \tag{2.18}$$

holds for all $F \in L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'}$.

(iii) $q = \infty$, $1 < \tilde{q} < \infty$, *then the estimate*

$$\|W(t)F\|_{L_t^\infty L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}',1} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}. \tag{2.19}$$

holds for all $F \in L_t^{\tilde{q}',1} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$.

(iv) $1 < q, \tilde{q} < \infty$, $q = \tilde{q}'$, *these endpoint inhomogeneous estimates are left open, although we can prove some weaker versions of* (2.17) *under the assumption of a compact velocity space, see section 2.5.4*

Conversely, in space dimension $n = 1$ *if estimate (2.17) holds for all* $F \in L_t^{q'} L_x^{r'} L_v^{p'}$ *then* (q, r, p) *and* $(\tilde{q}, \tilde{r}, \tilde{p})$ *must be two jointly KT-acceptable exponent triplets. In space dimensions* $n > 1$, *we can only show that* (q, r, p) *and* $(\tilde{q}, \tilde{r}, \tilde{p})$ *must be KT-acceptable and that conditions* (2.13), (2.14) *are necessary.*

Remark 2.2.3. The endpoint inhomogeneous estimates (2.18), (2.19), can be upgraded to contain entirely of Lebesgue norms in a narrower range. That is so for estimate (2.18) if $q \geq \tilde{p}'$ and the condition (2.15) is given as strict inequalities, and similarly for the estimate (2.19) if $\tilde{q}' \leq p$ and the condition (2.15) is given as strict inequalities, see Lemma 2.5.6.

Definition 2.2.4. Set

$$\begin{cases} r^*(b,c) = \infty & \text{if } n = 1 \\ r^*(b,c) = \frac{n}{n-1}c & \text{if } n > 1. \end{cases} \tag{2.20}$$

Theorem 2.2.5 (Generalized homogeneous estimates)**.** *Suppose that the exponent 5-tuple* (q, r, p, b, c) *satisfies the following conditions*

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{b}, \quad \mathrm{HM}(p, r) = \mathrm{HM}(b, c) \stackrel{def}{=} a, \tag{2.21}$$

$$a \leq r < r^*(b,c), \quad p \leq b \leq a \leq c \leq r. \tag{2.22}$$

Then the estimate

$$\|U(t)f\|_{L_t^{q,\infty}L_x^rL_v^p} \lesssim \|f\|_{L_x^bL_v^c} \tag{2.23}$$

holds for all $f \in L_x^bL_v^c$ *and Lebesgue exponents* $1 \leq q, r, p, b, c \leq \infty$. *Furthermore, if* $q \geq c$ *and* (q, r, p, b, c) *satisfy* (2.22) *with strict inequalities, then the estimate*

$$\|U(t)f\|_{L_t^qL_x^rL_v^p} \lesssim \|f\|_{L_x^bL_v^c} \tag{2.24}$$

holds for all $f \in L_x^bL_v^c$ *and Lebesgue exponents* $0 < q, r, p, b, c \leq \infty$, *see Lemma 2.5.5. Conversely, if estimate (2.24) holds for all* $f \in L_x^bL_v^c$ *then* (q, r, p, b, c) *must satisfy conditions* (2.21) *and* (2.22). *However, we do not have a counterexample showing the necessity of the upper bound* $r^*(b, c)$ *according to definition 2.2.4, in dimensions* $n > 1$, *and when* $b \neq c$.

Remark 2.2.6. As we have already mentioned, we do not know whether the given upper bound $r^*(b, c)$ is sharp. We, therefore, cannot discard the possibility that in the case of $n > 1$, $b \neq c$, there might be some additional estimates of the form (2.24) resulting from a bigger upper bound $r^*(b, c)$.

Remark 2.2.7. Let us recall that the appearance of Lorentz norms in some of the estimates above is not a great obstacle to applications. For example, if we restrict ourselves to finite time intervals $[0, T]$, we have the continuous embeddings

$$\begin{aligned} L^{q,r}([0,T]) &\hookrightarrow L^p([0,T]), \quad q > p,\ 1 \leq q, p, r \leq \infty, \\ L^p([0,T]) &\hookrightarrow L^{q,r}([0,T]), \quad p > q,\ 1 \leq q, p, r \leq \infty, \end{aligned}$$

see [1, p. 217]. Let us recall also the global continuous embeddings $L^{q,c}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ whenever $q \geq c$, and $L^q(\mathbb{R}) \hookrightarrow L^{q,c}(\mathbb{R})$ whenever $q \leq c$. For example, let (∞, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ be such that estimate (2.19) holds and let $1 \leq \widetilde{Q} < \tilde{q}$. Then we have the local inhomogeneous estimate

$$\|W(t)F\|_{L_t^\infty([0,T];L_x^qL_v^r)} \lesssim_T \|F\|_{L_t^{\tilde{Q}'}([0,T];L_x^{\tilde{r}'}L_v^{\tilde{p}'})}$$

for any $0 < T < \infty$ and any $F \in L_t^{\tilde{Q}'}([0, T]; L_x^{\tilde{r}'}L_v^{\tilde{p}'})$.

For an application of the Strichartz estimates to the Othmer-Dunbar-Alt kinetic model of chemotaxis see section 3.0.3.

2.3 Preliminaries

2.3.1 Basic properties of the kinetic transport equation

Lemma 2.3.1 (The dispersive estimate [40]). *The kinetic transport evolution group $U(t)$ obeys the estimate*

$$\|U(t)f\|_{L^\infty_x L^1_v} \leq \frac{1}{|t|^n} \|f\|_{L^1_x L^\infty_v}, \tag{2.25}$$

for all $f \in L^1_x L^\infty_v$.

Proof.

$$\begin{aligned}\int_{\mathbb{R}^n} |U(t)f|\, dv &= \int_{\mathbb{R}^n} |f(x-tv,v)|\, dv \leq \int_{\mathbb{R}^n} \sup_{y\in\mathbb{R}^n} |f(x-tv,y)|\, dv \\ &\leq \frac{1}{|t|^n} \int_{\mathbb{R}^n} \sup_{y\in\mathbb{R}^n} |f(z,y)|\, dz = \frac{1}{|t|^n} \|f\|_{L^1_x L^\infty_v}.\end{aligned}$$

□

Lemma 2.3.2 (The transport estimate). *The kinetic transport evolution group $U(t)$ obeys the estimate*

$$\|U(t)f\|_{L^\infty_t L^a_x L^a_v} \leq \|f\|_{L^a_{x,v}}, \quad 0 < a \leq \infty, \tag{2.26}$$

for all $f \in L^a_{x,v}$.

Proof. Trivial. □

Corollary 2.3.3 (The decay estimate). *The kinetic transport evolution group $U(t)$ obeys the estimate*

$$\|U(t)f\|_{L^r_x L^p_v} \leq \frac{1}{|t|^{n\left(\frac{1}{p}-\frac{1}{r}\right)}} \|f\|_{L^p_x L^r_v}, \quad 1 \leq p \leq r \leq \infty, \tag{2.27}$$

for all $f \in L^p_x L^r_v$.

Proof. Complex interpolation between the dispersive estimate (2.5) and the two transport estimates (2.26) with $a = 1$ and $a = \infty$. □

Lemma 2.3.4. *The formal adjoint to $U(t)$ is the operator $U^*(t) = U(-t)$.*

Proof. We denote by $\langle\cdot,\cdot\rangle$ the scalar product on $L^2(\mathbb{R}^{2n})$. Thus,

$$\begin{aligned}\langle U(t)f, g\rangle &= \int_{-\infty}^{\infty} f(x-tv,v)\overline{g(x,v)}dxdv \\ &= \int_{-\infty}^{\infty} f(y,v)\overline{g(y+tv,v)}dydv = \langle f, U(-t)g\rangle,\end{aligned}$$

where we have made the substitution $y = x - tv$. □

Lemma 2.3.5 (Scaling properties of $U(t)$ and $W(t)$). *The evolution operators $U(t)$ and $W(t)$ enjoy the following scaling properties*

$$
\begin{aligned}
U(t)f_\lambda &= f\left(x/\lambda - tv/\lambda, v\right) = U(t/\lambda, x/\lambda, v)f, \\
&\quad \textit{where } f_\lambda(x,v) = f\left(x/\lambda, v\right), \\
U(t)f_\lambda &= f\left(x/\lambda - tv/\lambda, v/\lambda\right) = U(t, x/\lambda, v/\lambda)f, \\
&\quad \textit{where } f_\lambda(x,v) = f\left(x/\lambda, v/\lambda\right), \\
W(t)F_\lambda = \lambda \int_0^{t/\lambda} & F\left(s, x/\lambda - (t/\lambda - s)\, v, v\right) ds = \lambda W\left(t/\lambda, x/\lambda, v\right) F, \\
&\quad \textit{where } F_\lambda(t,x,v) = F(t/\lambda, x/\lambda, v), \\
W(t)F_\lambda = \int_0^{t} & F\left(s, x/\lambda - (t-s)v/\lambda, v/\lambda\right) ds = W(t, x/\lambda, v/\lambda)F, \\
&\quad \textit{where } F_\lambda(t,x,v) = F\left(t, x/\lambda, v/\lambda\right).
\end{aligned}
$$

Proof. Direct inspection. □

Lemma 2.3.6. *Whenever $f \in C_0^1(\mathbb{R}^{2n})$, the space of all continuously differentiable functions on $\mathbb{R}^{2n}$ of compact support, the kinetic transport evolution group has the following continuity property*

$$
U(t)f \in C(\mathbb{R}; L_x^r L_v^p). \tag{2.28}
$$

for all $1 \le p \le r \le \infty$.

Proof. By Hölder's inequality

$$
\|U(t_2)f - U(t_1)f\|_{L_x^r L_v^p} \lesssim \|U(t_2)f - U(t_1)f\|_{L_{x,v}^\infty}
$$

and then by Sobolev embedding we obtain that the term on the right is bounded by

$$
(t_2 - t_1) \sup_{x,v} |vD_x f(x,v)|
$$

which tends to zero as $t_2 - t_1 \to 0$. □

Lemma 2.3.7. *If $f \in L_{x,v}^a$, $1 \le a < \infty$, then $U(t)f \in C(\mathbb{R}; L_{x,v}^a)$. Suppose that $f : \mathbb{R}^n \times V \to \mathbb{R}$, where the velocity space V is a compact set in $\mathbb{R}^n$. There exist $f \in L_{x,v}^\infty$ such that $U(t)f \notin C(\mathbb{R}; L_{x,v}^\infty)$. If $f \in L_{x,v}^\infty$ and in addition if f is uniformly continuous on $\mathbb{R}^n \times V$, then $U(t)f \in C(\mathbb{R}; L_{x,v}^\infty)$, and for each t fixed, $U(t)f$ is uniformly continuous on $\mathbb{R}^n \times V$.*

Proof. The first claim is due to the following standard argument. Suppose that $\chi_Q(x,v)$ is the characteristic function of a cube Q in $\mathbb{R}^{2n}$. The claim holds for χ_Q. Then it holds for the class of simple functions on $\mathbb{R}^{2n}$ and by density for all $f \in L^a_{x,v}$. The counterexample needed for the second claim can be taken again on χ_Q. And lastly, the third claim is trivial. $\square$

2.3.2 Duality and the TT^*-principle.

Consider the operator $T : L^2_{x,v} \to L^q_t L^r_x L^{r'}_v$, given by

$$T[f](t,x,v) = f(x - tv, v).$$

Its formal adjoint $T^* : L^{q'}_t L^{r'}_x L^r_v \to L^2_{x,v}$ is the L^2-valued integral

$$T^*[F](x,v) = \int_{-\infty}^{\infty} F(s, x + sv, v) ds.$$

Then the composition of the two $TT^* : L^{q'}_t L^{r'}_x L^r_v \to L^q_t L^r_x L^{r'}_v$ has the form

$$TT^*[F](t,x,v) = \int_{-\infty}^{\infty} F(s, x - (t-s)v, v) ds.$$

By the TT^*-principle, see e.g. [41, p. 113], T and TT^* are equally bounded with $\|T\|^2 = \|TT^*\|$. Thus, the two estimates are equivalent

$$\|Tf\|_{L^q_t L^r_x L^{r'}_v} \le C \|f\|_{L^2_{x,v}}, \qquad \forall f \in L^2_{x,v}, \tag{2.29}$$

$$\|TT^* F\|_{L^q_t L^r_x L^{r'}_v} \le C^2 \|F\|_{L^{q'}_t L^{r'}_x L^r_v}, \quad \forall F \in L^{q'}_t L^{r'}_x L^{p'}_v, \tag{2.30}$$

where $C = \|T\|$.

By duality, (2.30) is equivalent to

$$\left| \int_{-\infty}^{\infty} \langle TT^*[F](t), G(t) \rangle dt \right| \le C^2 \|F\|_{L^{q'}_t L^{r'}_x L^r_v} \|G\|_{L^{q'}_t L^{r'}_x L^r_v}, \quad \forall F,\ \forall G \in L^{q'}_t L^{r'}_x L^r_v,$$

or equivalently

$$\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle U^*(s)F, U^*(t)G \rangle ds dt \right| \le C^2 \|F\|_{L^{q'}_t L^{r'}_x L^r_v} \|G\|_{L^{q'}_t L^{r'}_x L^r_v}.$$

By symmetry, the last inequality simplifies to

$$|B(F,G)| \le C^2 \|F\|_{L^{q'}_t L^{r'}_x L^r_v} \|G\|_{L^{q'}_t L^{r'}_x L^r_v}, \quad \forall F,\ \forall G \in L^{q'}_t L^{r'}_x L^r_v, \tag{2.31}$$

where the bilinear form $B(F,G)$ is defined by

$$B(F,G) = \iint_{s<t} \langle U^*(s)F, U^*(t)G \rangle ds dt$$

and $\langle\cdot,\cdot\rangle$ is the bilinear pairing on $\mathbb{R}^{2n}$, i.e.

$$\langle f,g\rangle = \int_{R^{2n}} f(x,v)g(x,v)dxdv.$$

We note here that the homogeneous Strichartz estimates of (2.29) are typically proven via the corresponding estimate for the TT^*-operator (2.30), or the corresponding bilinear estimate (2.31).

We now turn to the inhomogeneous estimates. Suppose that (q,r,p) and $(\tilde{q},\tilde{r},\tilde{p})$ are two exponent triplets such that

$$\begin{aligned}
\|Tf\|_{L^q_t L^r_x L^p_v} &\leq C\,\|f\|_{L^a_{x,v}}, && \forall f\in L^a_{x,v},\\
\|Tf\|_{L^{\tilde{q}}_t L^{\tilde{r}}_x L^{\tilde{p}}_v} &\leq C\,\|f\|_{L^{a'}_{x,v}}, && \forall f\in L^{a'}_{x,v},
\end{aligned}$$

for some $1\leq a\leq\infty$. By considering

$$L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v \xrightarrow{T^*} L^a_{x,v} \xrightarrow{T} L^q_t L^r_x L^p_v$$

we obtain the consequence

$$\|TT^*F\|_{L^q_t L^r_x L^p_v} \leq C^2\,\|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v}, \quad \forall F\in L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v. \tag{2.32}$$

Note now that the exponents in the two sides of (2.32) are not any longer symmetric. This estimate does not imply boundedness for the operator T. However, this estimate implies boundedness for the inhomogeneous operator $W(t)$.

Lemma 2.3.8 (The TT^*-lemma)**.** *The following two estimates are equivalent*

$$\|TT^*F\|_{L^q_t L^r_x L^p_v} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v}, \quad \forall F\in L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v, \tag{2.33}$$

$$\|W(t)F\|_{L^q_t L^r_x L^p_v} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v}, \quad \forall F\in L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v. \tag{2.34}$$

Proof. Estimate (2.33) implies (2.34) since we can write

$$\int_{\mathbb{R}} = \int_{-\infty}^{t} + \int_{t}^{\infty}$$

in the definition of the TT^*-operator and then make a change of variables in the third integral to transform it to an integral like the second one. The details are left to the interested reader. The converse follows by the estimate

$$|W(t)F| \leq TT^*\,|F|.$$

□

Analogously to (2.31), we have that the inhomogeneous estimate

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}, \quad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'},$$

is equivalent to the estimate

$$|B(F,G)| \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \|G\|_{L_t^{q'} L_x^{r'} L_v^{p'}}, \qquad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'},\ \forall G \in L_t^{q'} L_x^{r'} L_v^{p'}. \tag{2.35}$$

We summarize this in the following lemma below which was first proved by Keel and Tao [30] in a slightly different context.

Lemma 2.3.9. *(i) The boundedness of the operator $T : L^2_{x,v} \to L_t^q L_x^r L_v^{r'}$ of the form $Tf = U(t)f$ is equivalent to the boundedness of the bilinear mapping $B : L_t^{q'} L_x^{r'} L_v^r \times L_t^{q'} L_x^{r'} L_v^r \to \mathbb{C}$.*

(ii) The boundedness of the operator $W(t) : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \to L_t^q L_x^r L_v^p$ is equivalent to that of the bilinear mapping $B : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to \mathbb{C}$.

The bilinear formulation of the TT^*-principle of Lemma 2.3.9 together with time decompositions of the bilinear operator $B(F,G)$ that are scaling invariant is a very powerful new technique. At this stage we shall only introduce some definitions. Denote by Ω the region $\{(t,s)|s<t\}$ on the (t,s)-coordinate plane.

Definition 2.3.10. We call any positive integer that is a power of two a dyadic number. Furthermore, we call a square Q in $\mathbb{R}^2$ dyadic if its side length is a dyadic number and the coordinates of its vertices are integer multiples of dyadic numbers.

We apply Whitney's dyadic decomposition on Ω and obtain the family $\mathcal{O}$ of essentially disjoint dyadic squares Q (except for overlapping on the sides) such that the distance between any square $Q \in \mathcal{O}$ and the boundary of Ω ($\{(t,s)|t=s\}$) is approximately proportional to the diameter of Q, see figure 2.2. By $\mathcal{O}_\lambda$ we denote the collection of all squares in $\mathcal{O}$ whose side length is λ.

Thus we obtain the representations

$$\Omega = \bigcup_\lambda \bigcup_{Q \in \mathcal{O}_\lambda}, \quad B(F,G) = \sum_\lambda \sum_{Q \in \mathcal{O}_\lambda} B_Q(F,G),$$

where

$$B_Q(F,G) = \iint_Q \langle U^*(s)F(s), U^*(t)G(t)\rangle ds dt. \tag{2.36}$$

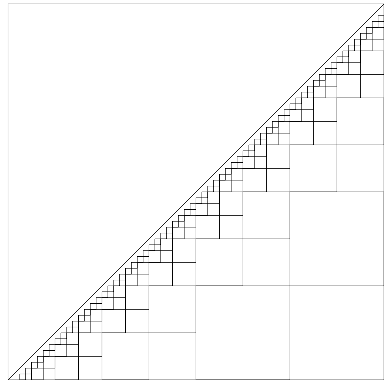

Figure 2.2: Whitney's decomposition for the region $s < t$

The advantage of the above decomposition is that whenever $Q = J \times I$ and $Q \in \mathcal{O}_\lambda$ we have

$$\lambda = |I| = |J| \sim \operatorname{dist}(\Omega, \partial\Omega) \sim \operatorname{dist}(I, J). \tag{2.37}$$

The very special property (2.37) of this decomposition allows us to obtain the following scaling invariance

$$|B_Q(F,G)| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \|F\|_{L_t^{\tilde{q}'}(J;L_x^{\tilde{r}'}L_v^{\tilde{p}'})} \|G\|_{L_t^{q'}(I;L_x^{r'}L_v^{p'})}, \tag{2.38}$$

of each dyadic piece B_Q in the bilinear form B, where

$$\beta(q,r,\tilde{q},\tilde{r}) = \frac{1}{q} + \frac{1}{\tilde{q}} - n\left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right). \tag{2.39}$$

The latter shall be proved in section 2.5.1 and in particular Lemma 2.5.4 gives the range for the ordered 6-tuple of exponents $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})$, where the local scaling (2.38) is known to be true. Another scaling invariant quantity is given by

Lemma 2.3.11. *If* $\frac{1}{q} + \frac{1}{\tilde{q}} \leq 1$, *then*

$$\sum_{Q \in \mathcal{O}_\lambda} |B_Q(F,G)| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \|F\|_{L_t^{\tilde{q}'}(\mathbb{R};L_x^{\tilde{r}'}L_v^{\tilde{p}'})} \|G\|_{L_t^{q'}(\mathbb{R};L_x^{r'}L_v^{p'})}. \tag{2.40}$$

Proof. In view of (2.38)

$$\sum_{Q \in \mathcal{O}_\lambda} |B_Q(F,G)| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \sum_{Q \in \mathcal{O}_\lambda,\, Q = J \times I} \|F\|_{L_t^{\tilde{q}'}(J;L_x^{\tilde{r}'}L_v^{\tilde{p}'})} \|G\|_{L_t^{q'}(I;L_x^{r'}L_v^{p'})}.$$

An application of Lemma 2.3.12 below concludes the proof. □

Lemma 2.3.12. *Suppose* $\frac{1}{p} + \frac{1}{\tilde{p}} \geq 1$. *Then*

$$\sum_{Q \in \mathcal{O}_\lambda,\ Q = J \times I} \|f\|_{L^{\tilde{p}}(J)} \|g\|_{L^p(I)} \leq \|f\|_{L^{\tilde{p}}(\mathbb{R})} \|g\|_{L^p(\mathbb{R})}.$$

Proof. The lemma follows directly from the inequality

$$\sum_j |a_j b_j| \leq \left(\sum_j |a_j|^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}} \left(\sum_j |b_j|^p\right)^{\frac{1}{p}},$$

which holds in the range $\frac{1}{p} + \frac{1}{\tilde{p}} \geq 1$, and the fact that for each dyadic interval I there are at most two dyadic squares in $\mathcal{O}_\lambda$ with side I. $\square$

Consider the bilinear operator $A : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to l_s^\infty$, (for a definition of l_s^∞ see page 105), defined by the formula

$$A(F,G) = \{b_\lambda\}_{\lambda \in 2^{\mathbb{Z}}} = \left\{ \sum_{Q \in \mathcal{O}_\lambda} |B_Q(F,G)| \right\}_{\lambda \in 2^{\mathbb{Z}}}.$$

The boundedness of $A : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to l^1$ implies the boundedness of $B : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to \mathbb{C}$. Thus, in view of the bilinear formulation of the TT^*-principle in Lemma 2.3.9 the estimate

$$\|\{b_\lambda\}\|_{l^1} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \|G\|_{L_t^{q'} L_x^{r'} L_v^{p'}}, \quad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}, \forall G \in L_t^q L_x^r L_v^p,$$

implies the boundedness of $W(t) : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \to L_t^q L_x^r L_v^p$. We summarize this argument in

Lemma 2.3.13. *The boundedness of the bilinear operator* $A : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to l^1$ *implies the inhomogeneous Strichartz estimate*

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}, \quad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}.$$

2.3.3 Equivalent estimates

Lemma 2.3.14 (The Duality lemma)**.** *The following two estimates for* $W(t)$ *are equivalent*

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}, \quad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'},$$
$$\|W(t)F\|_{L_t^{\tilde{q}} L_x^{\tilde{r}} L_v^{\tilde{p}}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{p'}}, \quad \forall F \in L_t^{q'} L_x^{r'} L_v^{p'},$$

for $1 \leq p, q \leq \infty$.

Proof. By duality, the first estimate is equivalent to

$$|B(F,G)| \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \|G\|_{L_t^{q'} L_x^{r'} L_v^{p'}},$$

for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$, $G \in L_t^{q'} L_x^{r'} L_v^{p'}$. We have that

$$B(F,G) = \iint_{\tau<\sigma} \langle U^*(-\sigma)F, U^*(-\tau)G \rangle d\tau d\sigma,$$

by making the substitution $\sigma = -s$, $\tau = -t$ in the definition of $B(F,G)$. The integral in the line above can be written as

$$(-1)^n \iint_{\tau<\sigma} \langle U^*(\sigma)F', U^*(\tau)G' \rangle d\tau d\sigma,$$

by making the substitution $x \to -x$ and setting $F'(t,x,v) = F(-t,-x,v)$, $G'(t,x,v) = G(-t,-x,v)$. Hence the second estimate follows. The converse follows by the same argument. $\square$

Theorem 2.3.15 (The Equivalence theorem)**. A.** *The following three estimates are equivalent*

$$\|U(t)f\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L_x^b L_v^c}, \qquad \forall f \in L_x^b L_v^c, \tag{2.41}$$

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^1 L_x^b L_v^c}, \qquad \forall F \in L_t^1 L_x^b L_v^c, \tag{2.42}$$

$$\|W(t)F\|_{L_t^\infty L_x^{b'} L_v^{c'}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{p'}}, \qquad \forall F \in L_t^{q'} L_x^{r'} L_v^{p'}. \tag{2.43}$$

B. *Whenever* $b = c = 2$ *estimate* (2.41) *is equivalent to*

$$\|W(t)F\|_{L_t^q L_x^r L_v^{r'}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{r}}, \qquad \forall F \in L_t^{q'} L_x^{r'} L_v^{r}. \tag{2.44}$$

Proof. Part A. The homogeneous estimate (2.41) trivially implies the first inhomogeneous estimate (2.42). By the Duality lemma 2.3.14, the two inhomogeneous estimates (2.42) and (2.43) are equivalent. All it remains is to show that (2.42) implies (2.41).

Formally, the proof follows if we choose an inhomogeneous term $F(t) = \delta(t)f$, where $\delta(t)$ is the delta function on $\mathbb{R}$ and $f \in L_x^b L_v^c$. Indeed, we have

$$W(t)[\delta(\cdot)f] = U(t)f, \quad \|\delta(t)f\|_{L_t^1 L_x^b L_v^c} = \|f\|_{L_x^b L_v^c},$$

which furnishes the argument. To give a rigorous proof instead of $\delta(t)$ we consider a smooth approximation of the identity $\delta_\epsilon(t)$, $\epsilon > 0$. Suppose that $f \in C_0^1(\mathbb{R}^{2n})$ and thus by Lemma 2.3.6 $U(t)f \in C(\mathbb{R}; L_x^b L_v^c)$. In view of Lemma 8.0.9,

$$\|W(t)[\delta_\epsilon f]\|_{L_x^b L_v^c} = \|\delta_\epsilon * U(t)f\|_{L_x^b L_v^c} \to \|U(t)f\|_{L_x^b L_v^c}$$

on $\mathbb{R}$ as $\epsilon \to 0$. Finally, by Fatou's Lemma 8.0.10

$$\|U(t)f\|_{L^q_t(\mathbb{R};L^b_xL^c_v)} \lesssim \liminf_{\epsilon\to 0} \|W(t)\delta_\epsilon f\|_{L^q_t(\mathbb{R};L^b_xL^c_v)} \lesssim \liminf_{\epsilon\to 0} \|\delta_\epsilon f\|_{L^1_t(\mathbb{R};L^b_xL^c_v)} \lesssim \|f\|_{L^b_xL^c_v}.$$

The general case of $f \in L^b_xL^c_v$ follows by density since $C^1_0(\mathbb{R}^{2n})$ is dense in $L^b_xL^c_v$ and $U(t)$ is linear.

Part B. This follows directly from Lemma 2.3.9. ☐

Remark 2.3.16. The Equivalence theorem still holds if instead of the Lebesgue L^q-norm in time we have the more general Lorentz $L^{q,\tilde{q}}$-norm in time. The formulation and proof in this case are straightforward and shall be omitted.

2.4 Estimates for admissible exponents

2.4.1 Non-endpoint estimates

We prove here the sufficient part of Theorem 2.2.1 in the case of non-endpoint estimates. In particular, we prove separately the homogeneous and the inhomogeneous estimate

$$\|U(t)f\|_{L^q_tL^r_xL^p_v} \lesssim \|f\|_{L^a_x}, \quad 1 \le a < \infty, \tag{2.16a}$$

$$\|W(t)F\|_{L^q_tL^r_xL^p_v} \lesssim \|F\|_{L^{\tilde{q}'}_tL^{\tilde{r}'}_xL^{\tilde{p}'}_v}, \tag{2.16b}$$

respectively, where (q,r,p) and $(\tilde{q},\tilde{r},\tilde{p})$ are two non-endpoint KT-admissible triplets subject to the scaling condition $a = \mathrm{HM}(p,r) = \mathrm{HM}(\tilde{p}',\tilde{r}')$.

Proof. It suffices to consider only estimate

$$\|U(t)f\|_{L^q_tL^r_xL^{r'}_v} \lesssim \|f\|_{L^2_x},$$

since the general case of (2.16a) will follow by the power invariance (2.3). By the TT^*-principle, the last inequality is equivalent to (2.30).

In view of the decay estimate

$$\|U(t)f\|_{L^r_xL^{r'}_v} \lesssim \frac{1}{|t|^{\beta(r)}} \|f\|_{L^{r'}_xL^r_v}, \quad 2 \le r \le \infty,$$

where $\beta(r) = n(1-2/r)$, cf. Corollary 2.3.3, we obtain the following estimate for TT^*F

$$\|TT^*F\|_{L^r_xL^{r'}_v} \lesssim \int_{-\infty}^{\infty} \|U(t-s)F(s)\|_{L^r_xL^{r'}_v}\,ds \lesssim \int_{-\infty}^{\infty} \frac{\|F(s)\|_{L^{r'}_xL^r_v}}{|t-s|^{\beta(r)}} ds.$$

We take the L^q-norm in t and in view of the Hardy-Littlewood-Sobolev (HLS) theorem of fractional integration, see [1, pp. 228-229], [41], we obtain

$$\|TT^*F\|_{L_t^q L_x^r L_v^{r'}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^r},$$

whenever $0 < \beta(r) < 1$, $1+1/q = 1/q'+\beta(r)$. The latter conditions are equivalent to $2 < r < r^*$, $1/q + n/r = n/2$. The left endpoint $r = 2$ follows trivially from the transport estimate (2.26). The right endpoint $r = r^*$ will be treated in sections 2.4.2, 2.4.3, 2.5.4.

The inhomogeneous estimate (2.16b) is a consequence of estimate (2.16a), the factorization (2.32), and the TT^*-lemma 2.3.8. □

2.4.2 The endpoint homogeneous estimate

This section is optional and presents a conjecture we have on the endpoint estimate

$$\|U(t)f\|_{L_t^2 L_x^{r^*} L_v^{r^{*\prime}}} \lesssim \|f\|_{L^2_{x,v}}, \quad \forall f \in L^2(\mathbb{R}^{2n}). \tag{2.45}$$

A direct proof of it is probably not easy and quite likely technically involved. Indeed, the known real interpolation techniques are well-adapted to the scalar L^p-norm where they produce Lorentz norms that imply the correct spatial norm in the endpoint case. In the context of the mixed $L_x^r L_v^p$-norm, however, they produce and endpoint norm which is neither weaker nor stronger than the desired endpoint norm, see Keel and Tao [30]. The biggest technical challenge to Keel and Tao's perturbative technique is the invariance $\mathrm{HM}(p,r) = \mathrm{HM}(\tilde{p},\tilde{r})$ in the local inhomogeneous estimate

$$\|W(t)F\|_{L_t^q(I;L_x^r L_v^p)} \lesssim \|F\|_{L_t^{\tilde{q}'}(J;L_x^{\tilde{r}'} L_v^{\tilde{p}'})},$$

where $|I| = |J| = 1$, $dist(I,J) = 1$, which does not allow us to perturbate the spatial exponents freely, see section 2.6.5.

We shall, therefore, seek a more indirect approach towards a resolution of the endpoint estimate (2.45). First, note that in view of the power invariance (2.3), (2.45) is equivalent to

$$\|U(t)f\|_{L_t^2 L_x^{r^*(a)} L_v^{r^*(a)\prime}} \lesssim \|f\|_{L^a_{x,v}}, \quad \forall f \in L^2(\mathbb{R}^{2n}),\ \forall a \in (0,\infty), \tag{2.46}$$

and thus from now on we shall be only interested in proving (2.45). Furthermore, by scaling and duality (2.45) is equivalent to the local estimate

$$\|T^*F\|_{L^2_{x,v}} \lesssim \|F\|_{L_t^2([0,T];L_x^{r^{*\prime}} L_v^{r^*})}, \quad \forall F \in L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*}), \tag{2.47}$$

for any time $T > 0$, where

$$T^* F(x, v) = \int_0^T F(s, x + sv, v) ds.$$

If we knew that (2.47) holds then this would have implied that $T^* : L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*}) \to L_{x,v}^2$ is bounded or equivalently continuous on these spaces. Let us first *try* proving the weaker statement that T^* is everywhere defined (but perhaps unbounded) on the same spaces. In other words we are going to show that

$$\|T^* F\|_{L_{x,v}^2} < \infty, \quad \forall F \in L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*}).$$

The above inequality is equivalent to $\|T^* F\|_{L_{x,v}^2}^2 < \infty$ which by symmetry is equivalent to $B(F, F) < \infty$. Consider the following modified bilinear operator

$$B_\epsilon(F, G) = \int_0^T \int_0^{t-\epsilon} \langle U^*(s) F(s), U^*(t) G(t) \rangle ds dt. \tag{2.48}$$

Recall again the dyadic decomposition in fig. 2.2 and see that

$$B_\epsilon(F, F) \leq \sum_Q B_Q(F, F)$$

where the sum is finite for every $\epsilon > 0$. Assume that the scaling invariance (2.38) holds for the admissible triplets $(q, r, p) = (\tilde{q}, \tilde{r}, \tilde{p}) = (2, r^*, r^{*\prime})$ for $n > 1$. In this case we have that $\beta(2, r^*, 2, r^*) = 0$. Then notice that $B_\epsilon(F, F) < \infty$ as it is bounded by a finite sum.

It is not hard to see that essentially

$$B(F, F) \leq B_\epsilon(U(-\epsilon) F, U(-\epsilon) F).$$

Thus the desired boundedness of $B(F, F)$ will follow if we can prove the following

Conjecture 2.4.1.

$$\forall F \in L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*}),\ \exists \epsilon(F) > 0 : \ \|U(-\epsilon) F\|_{L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*})} < \infty.$$

Note that it would be enough to take F with a compact support in $\mathbb{R}_x^n \times \mathbb{R}_v^n$, $n > 1$.

We shall next show that $T^* : L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*}) \to L_{x,v}^2$ is closed. We need to show that if $F_n \to F$ in $L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*})$ and $T^* F_n \to G$ in $L_{x,v}^2$ we have that $T^* F = G$. Consider the identity

$$\langle Tf, F_n \rangle = \langle f, T^* F_n \rangle$$

for $f \in \mathcal{S}(\mathbb{R}^{2n})$, the Schwartz class on $\mathbb{R}^{2n}$, and $F_n \in L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*})$, where by $\langle \cdot, \cdot \rangle$ we denote the duality pairing on $\mathbb{R}^{2n+1}$ in the left hand side and the duality pairing on $\mathbb{R}^{2n}$ in the right hand side. Since both pairings are continuous with respect to their arguments, it follows directly from the Hölder's inequality, we have that

$$\langle Tf, F\rangle = \langle f, G\rangle = \langle f, T^*F\rangle.$$

By density, $G = T^*F$.

Thus if our conjecture is correct, the proof would follow from the Closed-graph theorem, which for convenience we state below, see any standard course in functional analysis, e.g. [23, p. 45].

Theorem 2.4.2. *A closed linear operator mapping a Banach space into a Banach space is continuous.*

Furthermore, by considering the action of $U(\epsilon)$ on rectangles in $\mathbb{R}^{2n+1}$, one can see that for a fixed ϵ

$$U(-\epsilon) : L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*}) \to L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*})$$

is unbounded. But it is easy to see that this operator is closed, so again by the Closed-graph theorem it follows that

$$\forall \epsilon > 0,\ \exists F \in L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*}) :\ \|U(-\epsilon)F\|_{L_t^2([0,T]; L_x^{r^{*\prime}} L_v^{r^*})} = \infty.$$

This negative result shows that our conjecture is quite subtle, but it is, still, not unnatural. Indeed, it is in line with the spirit of the Strichartz estimates which suggests to expect less than continuity of $U(-\epsilon)$, perhaps merely integrability, with respect to ϵ on the considered spaces.

2.4.3 Inhomogeneous estimates - revisited

We revisit again the inhomogeneous estimate

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}, \tag{2.16b}$$

where now (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ can be *any* two KT-admissible triplets subject to the scaling condition $a = \mathrm{HM}(p, r) = \mathrm{HM}(\tilde{p}', \tilde{r}')$. We can show that this estimate holds for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$ unless when both (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ are endpoint. The new claim is the validity of that estimate when one of the exponent triplet is endpoint but the other is not. This claim follows from Theorem 2.2.2, which we shall prove in the next section, part (i). In the same theorem, part (iv), and Section 2.5.4, one can find a weaker version of the double endpoint inhomogeneous estimate 2.16b as well.

2.5 Estimates for acceptable exponents

2.5.1 Local inhomogeneous estimates

Following Foschi [19], we want to find the range of local estimates for $W(t)$ that are invariant to the scaling

$$\|W(t)[\chi_{\lambda J}F]\|_{L^q(\lambda I;L^r_x L^p_v)} \lesssim \lambda^{\frac{1}{q}+\frac{1}{\tilde{q}}-n\left(1-\frac{1}{r}-\frac{1}{\tilde{r}}\right)} \|F\|_{L^{\tilde{q}'}(\lambda J;L^{\tilde{r}'}_x L^{\tilde{p}'}_v)}, \quad \forall\lambda > 0, \tag{2.49}$$

where I and J are two unit intervals separated by a unit distance and $\chi_{\lambda J}$ is the characteristic of the rescaled interval λJ.

The bilinear formulation of (2.49) is

$$|B_Q(F,G)| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \|F\|_{L^{\tilde{q}'}_t(J;L^{\tilde{r}'}_x L^{\tilde{p}'}_v)} \|G\|_{L^{q'}_t(I;L^{r'}_x L^{p'}_v)}, \tag{2.50}$$

where Q is the square $I \times J$.

Lemma 2.5.1. *Estimate* (2.49) *holds for any two (non-endpoint) KT-admissible triplets* (q,r,p) *and* $(\tilde{q},\tilde{r},\tilde{p})$ *with* $a = \tilde{a}'$.

Proof. The proof follows trivially from Theorem 2.2.1 due to the fact that $\beta(q,r,\tilde{q},\tilde{r}) = 0$ under the hypothesis of the lemma. □

Lemma 2.5.2. *Estimate* (2.49) *holds with* $(q,r,p) = (\infty,r,p)$ *and* $(\tilde{q},\tilde{r},\tilde{p}) = (\infty,p',r')$, *where* $1 \le p \le r \le \infty$.

Proof. Due to the decay estimate (2.27) we have that

$$\begin{aligned}\sup_{t\in\lambda I} \|W(t)[\chi_{\lambda J}F]\|_{L^r_x L^p_v} &\lesssim \sup_{t\in\lambda I} \int_{\lambda J} \frac{\|F(\tau)\|_{L^p_x L^r_v}}{|t-\tau|^{n\left(\frac{1}{p}-\frac{1}{r}\right)}} d\tau \\ &\lesssim \lambda^{\beta(\infty,r,\infty,p')} \|F\|_{L^1(\lambda J;L^p_x L^r_v)}.\end{aligned}$$

□

Lemma 2.5.3. *Whenever* (q,r,p) *and* $(\tilde{q},\tilde{r},\tilde{p})$ *are exponent triplets for which estimate* (2.49) *holds, we have that* (2.49) *also holds with* (Q,r,p) *and* $(\tilde{Q},\tilde{r},\tilde{p})$, *where* $1 \le Q \le q$, $1 \le \tilde{Q} \le \tilde{q}$.

Proof. A trivial application of Hölder's inequality

$$\begin{aligned}&\|W(t)[\chi_{\lambda J}F]\|_{L^Q(\lambda I;L^r_x L^p_v)} \lesssim \lambda^{\frac{1}{Q}-\frac{1}{q}} \|W(t)[\chi_{\lambda J}F]\|_{L^q(\lambda I;L^r_x L^p_v)} \\ &\lesssim \lambda^{\beta(Q,r,\tilde{q},\tilde{r})} \|F\|_{L^{\tilde{q}'}(\lambda J;L^{\tilde{r}'}_x L^{\tilde{p}'}_v)} \lesssim \lambda^{\beta(Q,r,\tilde{Q},\tilde{r})} \|F\|_{L^{\tilde{Q}'}(\lambda J;L^{\tilde{r}'}_x L^{\tilde{p}'}_v)}.\end{aligned}$$

□

Let us define the range of validity of the local estimates (2.49) as the set $\mathcal{E}$ in $\mathbb{R}^6$. Each point in $\mathcal{E}$ corresponds to a 6-tuple of exponents $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})$. Below we find the convex hull $\mathcal{E}^*$ ($\mathcal{E}^* \subseteq \mathcal{E}$) of the points in $\mathbb{R}^6$ that correspond to the estimates in the three lemmas above. We shall call any point or collection of points in $\mathcal{E}$ *acceptable.*

Lemma 2.5.4 (Local inhomogeneous estimates)**.** *Estimate* (2.49) *holds whenever the exponent triplets* (q, r, p), $(\tilde{q}, \tilde{r}, \tilde{p})$ *satisfy the following conditions*

$$0 < \frac{1}{r}, \frac{1}{\tilde{r}} \le 1, \quad 0 \le \frac{1}{q}, \frac{1}{\tilde{q}}, \frac{1}{p}, \frac{1}{\tilde{p}} \le 1, \tag{2.51}$$

$$\frac{1}{r} \le \frac{1}{p}, \quad \frac{1}{\tilde{r}} \le \frac{1}{\tilde{p}}, \quad \mathrm{HM}(p, r) = \mathrm{HM}(\tilde{p}', \tilde{r}'), \tag{2.52}$$

$$\frac{1}{\tilde{r}} - \frac{1}{\tilde{p}} - \frac{1}{r} + \frac{1}{p} \le \frac{2}{nq}, \quad \frac{1}{r} - \frac{1}{p} - \frac{1}{\tilde{r}} + \frac{1}{\tilde{p}} \le \frac{2}{n\tilde{q}}, \tag{2.53}$$

$$\frac{n-1}{p'} < \frac{n}{\tilde{r}}, \quad \frac{n-1}{\tilde{p}'} < \frac{n}{r}, \tag{2.54}$$

or if the point $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})$ *lies inside one of the cubes in* $\mathbb{R}^6$ *below*

$$\begin{aligned} &(\kappa, 0, \mu, \nu, 1-\mu, 1), \qquad 0 \le \kappa, \mu, \nu \le 1, \\ &(\kappa, 1-\mu, 1, \nu, 0, \mu), \qquad 0 \le \kappa, \mu, \nu \le 1. \end{aligned} \tag{2.55}$$

Proof. We apply the Riesz-Thorin convexity theorem to interpolate between the already proven local estimates. In essence, we find the convex hull of the locally acceptable sets associated with Lemmas 2.5.1 and 2.5.2 and then expand that set by the rule given in Lemma 2.5.3.

The set of acceptability S_1 of the local estimates in Lemma 2.5.1 is given by the system

$$0 < \frac{1}{r}, \frac{1}{\tilde{r}} \le 1, \quad 0 \le \frac{1}{q}, \frac{1}{\tilde{q}}, \frac{1}{p}, \frac{1}{\tilde{p}} \le 1, \tag{2.56}$$

$$\frac{1}{q} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right), \quad \frac{1}{\tilde{q}} = \frac{n}{2}\left(\frac{1}{\tilde{p}} - \frac{1}{\tilde{r}}\right), \tag{2.57}$$

$$\frac{1}{r} + \frac{1}{p} + \frac{1}{\tilde{r}} + \frac{1}{\tilde{p}} = 2, \tag{2.58}$$

$$\frac{n-1}{p} < \frac{n+1}{r}, \quad \frac{n-1}{\tilde{p}} < \frac{n+1}{\tilde{r}}, \tag{2.59}$$

or if $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \{B, C | B = (0,0,0,0,1,1), C = (0,1,1,0,0,0)\}$.

Note that S_1 is a convex polyhedron in $\mathbb{R}^6$ and the two points B and C lie on its boundary. The set of acceptability S_2 of the local estimates in Lemma 2.5.2 is the convex hull (in fact a triangle) of the three points

$$A = (0,0,1,0,0,1), \qquad B = (0,0,0,0,1,1), \qquad C = (0,1,1,0,0,0). \tag{2.60}$$

Vertices B and C are already included in S_1, thus it would suffice to take only the vertex A.

Hence, we obtain the following set

$$\frac{1}{Q} = \frac{\theta}{q}, \qquad \frac{1}{R} = \frac{\theta}{r}, \qquad \frac{1}{P} = 1 - \theta + \frac{\theta}{p},$$
$$\frac{1}{\tilde{Q}} = \frac{\theta}{\tilde{q}}, \qquad \frac{1}{\tilde{R}} = \frac{\theta}{\tilde{r}}, \qquad \frac{1}{\tilde{P}} = 1 - \theta + \frac{\theta}{\tilde{p}}, \qquad 0 \le \theta \le 1,$$

where $(1/Q, 1/R, 1/P, 1/\tilde{Q}, 1/\tilde{R}, 1/\tilde{P})$ are the coordinates of the new set S_3 written in terms of $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})$ and θ. Of course, to S_3 we must also add the line segments $[A, B]$ and $[A, C]$. We shall treat this case separately at the end.

Finally, we apply the rule given in Lemma 2.5.3 and thus we replace the equations for Q and $\tilde{Q}$ above with the following inequalities

$$1 \ge \frac{1}{Q} \ge \frac{\theta}{q}, \qquad 1 \ge \frac{1}{\tilde{Q}} \ge \frac{\theta}{\tilde{q}},$$

plus the restrictions

$$\frac{1}{r} \le \frac{1}{p}, \quad \frac{1}{\tilde{r}} \le \frac{1}{\tilde{p}}, \tag{2.61}$$

which were implicitly assumed in (2.57).

1. We first eliminate q and $\tilde{q}$ from the system for S_1 to obtain

$$\frac{1}{Q} \ge \frac{n}{2}\left(\frac{\theta}{p} - \frac{\theta}{r}\right), \quad \Leftrightarrow \quad \frac{1}{Q} \ge \frac{n}{2}\left(\theta - 1 + \frac{1}{P} - \frac{1}{R}\right),$$
$$\Leftrightarrow \quad \theta \le \frac{1}{P'} + \frac{1}{R} + \frac{2}{nQ}.$$

Similarly,

$$\theta \le \frac{1}{\tilde{P}'} + \frac{1}{\tilde{R}} + \frac{2}{n\tilde{Q}}, \qquad \frac{1}{Q}, \frac{1}{\tilde{Q}} \le 1.$$

2. As expected, condition (2.58) is invariant

$$\frac{1}{R} + \frac{1}{P} + \frac{1}{\tilde{R}} + \frac{1}{\tilde{P}} = 2.$$

3. Reworking condition (2.59), we obtain

$$\theta < \frac{n+1}{n-1}\frac{1}{R} + \frac{1}{P'}, \qquad \theta < \frac{n+1}{n-1}\frac{1}{\tilde{R}} + \frac{1}{\tilde{P}'}.$$

4. Condition (2.61) is replaced by

$$\frac{1}{P'} + \frac{1}{R} \le \theta, \quad \frac{1}{\tilde{P}'} + \frac{1}{\tilde{R}} \le \theta.$$

5. Finally, conditions (2.56) are transformed into

$$\frac{1}{P'}, \frac{1}{\tilde{P}'}, \frac{1}{R}, \frac{1}{\tilde{R}} \le \theta, \quad 0 \le \frac{1}{Q}, \frac{1}{\tilde{Q}}, \frac{1}{P}, \frac{1}{\tilde{P}}, \frac{1}{R}, \frac{1}{\tilde{R}} \le 1.$$

6. We group all conditions obtained in the previous 5 steps according to their type

$$0, \frac{1}{P'}, \frac{1}{\tilde{P}'}, \frac{1}{R}, \frac{1}{\tilde{R}}, \frac{1}{P'} + \frac{1}{R}, \frac{1}{\tilde{P}'} + \frac{1}{\tilde{R}} \le \theta. \tag{2.62}$$

$$\theta \le \frac{1}{R} + \frac{1}{P'} + \frac{2}{nQ}, \frac{1}{\tilde{R}} + \frac{1}{\tilde{P}'} + \frac{2}{n\tilde{Q}}, \frac{n+1}{n-1}\frac{1}{R} + \frac{1}{P'}, \frac{n+1}{n-1}\frac{1}{\tilde{R}} + \frac{1}{\tilde{P}'}, 1. \tag{2.63}$$

$$0 \le \frac{1}{Q}, \frac{1}{\tilde{Q}}, \frac{1}{P}, \frac{1}{\tilde{P}}, \frac{1}{R}, \frac{1}{\tilde{R}} \le 1, \qquad \frac{1}{P} + \frac{1}{R} + \frac{1}{\tilde{R}} + \frac{1}{\tilde{P}} = 2. \tag{2.64}$$

7. We discard the redundant conditions like

$$0, \frac{1}{P'}, \frac{1}{\tilde{P}'}, \frac{1}{R}, \frac{1}{\tilde{R}} \le \theta,$$

which are all weaker than the other two in (2.62).

There exists θ solving all inequalities in (2.62), (2.63), if and only if every quantity in (2.62) is bounded from above by any quantity in (2.63). Thus we form all possible combinations between the quantities in the two types of (reduced) inequalities to obtain the following set of conditions

$$\begin{gathered}
\frac{1}{R} + \frac{1}{P'} \le \frac{1}{\tilde{R}} + \frac{1}{\tilde{P}'} + \frac{2}{n\tilde{Q}}, \quad \frac{1}{\tilde{R}} + \frac{1}{\tilde{P}'} \le \frac{1}{R} + \frac{1}{P'} + \frac{2}{nQ}, \\
\frac{1}{R} + \frac{1}{P'} < \frac{n+1}{n-1}\frac{1}{\tilde{R}} + \frac{1}{\tilde{P}'}, \quad \Leftrightarrow \quad \frac{n-1}{P'} < \frac{n}{\tilde{R}}, \\
\frac{1}{\tilde{R}} + \frac{1}{\tilde{P}'} < \frac{n+1}{n-1}\frac{1}{R} + \frac{1}{P'}, \quad \Leftrightarrow \quad \frac{n-1}{\tilde{P}'} < \frac{n}{R}, \\
\frac{1}{R} \le \frac{1}{P}, \quad \frac{1}{\tilde{R}} \le \frac{1}{\tilde{P}},
\end{gathered}$$

describing the region S_3.

8. We apply the rule given in Lemma 2.5.3 to the two line segments $[A, B]$ and $[A, C]$ to obtain the following two cubes in $\mathbb{R}^6$

$$\begin{aligned}
&(\mu, 0, \kappa, \nu, 1-\kappa, 1), \qquad 0 \le \mu, \nu, \kappa \le 1, \\
&(\mu, 1-\kappa, 1, \nu, 0, \kappa), \qquad 0 \le \mu, \nu, \kappa \le 1.
\end{aligned} \tag{2.65}$$

Hence, the computation of the set $\mathcal{E}^*$ is finished. $\square$

2.5.2 Non-endpoint global inhomogeneous estimates

Recall that under Lemma 2.3.13 in order to show the inhomogeneous Strichartz estimate

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}, \quad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'},$$

it is enough to show the estimate

$$\|\{b_\lambda\}\|_{l^1} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \|G\|_{L_t^{q'} L_x^{r'} L_v^{p'}}, \quad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}, \, \forall G \in L_t^{q'} L_x^{r'} L_v^{p'},$$

where

$$\{b_\lambda\}_{\lambda \in 2^{\mathbb{Z}}} = \left\{ \sum_{Q \in \mathcal{O}_\lambda} |B_Q(F,G)| \right\}_{\lambda \in 2^{\mathbb{Z}}}.$$

Suppose that $(1/q, 1/\tilde{q}) \in \Delta$, where $\Delta = \{1/q > 0, 1/\tilde{q} > 0, 1/q + 1/\tilde{q} < 1\}$, and that $P = (1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \mathcal{E}^*$ together with a small neighborhood of P on the $(1/q, 1/\tilde{q})$-plane. We shall denote this set by $\mathcal{E}_1$. In virtue of Corollary 2.3.11 we have the estimate

$$|b_\lambda| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \|G\|_{L_t^{q'} L_x^{r'} L_v^{p'}},$$

or in other words $\{b_\lambda\} \in l^\infty_{\beta(q,r,\tilde{q},\tilde{r})}$. Let us set

$$1/q_0 = 1/q + \epsilon, \quad 1/\tilde{q}_0 = 1/\tilde{q} + \epsilon, \quad 1/q_1 = 1/q - 3\epsilon, \quad 1/\tilde{q}_1 = 1/\tilde{q} - 3\epsilon.$$

By the assumptions, if $\epsilon > 0$ is small enough we have that the perturbations of P with temporal components equal to any of the above also belong to $\mathcal{E}_1$. Suppose also that

$$1/q + 1/\tilde{q} = n\left(1 - 1/r - 1/\tilde{r}\right). \tag{2.66}$$

Then we have that $\beta(q_0, r, \tilde{q}_0, \tilde{r}) = 2\epsilon$, and $\beta(q_1, r, \tilde{q}_0, \tilde{r}) = \beta(q_0, r, \tilde{q}_1, \tilde{r}) = -2\epsilon$. The following bilinear maps

$$\begin{aligned}
A &: L_t^{\tilde{q}_0'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q_0'} L_x^{r'} L_v^{p'} \to l^\infty_{2\epsilon}, \\
A &: L_t^{\tilde{q}_0'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q_1'} L_x^{r'} L_v^{p'} \to l^\infty_{-2\epsilon}, \\
A &: L_t^{\tilde{q}_1'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q_0'} L_x^{r'} L_v^{p'} \to l^\infty_{-2\epsilon},
\end{aligned}$$

are bounded. In virtue of Lemma 8.0.7 we have that the map

$$A : (L_t^{\tilde{q}_0'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}, L_t^{\tilde{q}_1'} L_x^{\tilde{r}'} L_v^{\tilde{p}'})_{1/4, \tilde{q}'} \times (L_t^{q_0'} L_x^{r'} L_v^{p'}, L_t^{q_1'} L_x^{r'} L_v^{p'})_{1/4, q'} \to (l^\infty_{2\epsilon}, l^\infty_{-2\epsilon})_{1/2,1}.$$

is also bounded. Finally, in view of the well-known interpolation identities of the Lorentz spaces and that of Lemma 8.0.6, this simplifies to

$$A : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to l^1.$$

Now let us recapitulate all conditions that we have imposed so far on the exponents. We have the conditions of the local estimates (set $\mathcal{E}^*$) plus the scaling condition (2.66). Note that conditions (2.53) together with (2.66) are equivalent to (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ being KT-acceptable. Let us also note that the two locally acceptable cubes in (2.65) give rise to the two globally acceptable sets (cubic cross-sections) Σ_1 and Σ_2 in definition 2.1.4. Putting all these together we obtain an explicit description of the cross-section of $\mathcal{E}_1$ with the hyperplane (2.66), which is presented as part (i) of Theorem 2.2.2.

2.5.3 Endpoint global inhomogeneous estimates, case of $\tilde{q} = \infty$

Suppose now that $(1/q, 1/\tilde{q})$ lies on either one of the two catheti of ΔOAB (without loss of generality we suppose that $1/\tilde{q} = 0$), without the two endpoints $(0, 0)$ and $(1, 0)$. Suppose that $P_2 = (1/q, 1/r, 1/p, 0, 1/\tilde{r}, 1/\tilde{p}) \in \mathcal{E}^*$ together with a small neighborhood of $1/q$ on $\mathbb{R}$. We shall denote this set by $\mathcal{E}_2$. In addition, we suppose that P_2 lies in the cross-section of $\mathcal{E}_2$ with the hyperplane (2.66). Then we have

$$A : L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q_0'} L_x^{r'} L_v^{p'} \to l_\epsilon^\infty,$$
$$A : L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q_1'} L_x^{r'} L_v^{p'} \to l_{-\epsilon}^\infty,$$

where

$$\frac{1}{q_0} = \frac{1}{q} - \frac{1}{\epsilon}, \quad \frac{1}{q_1} = \frac{1}{q} + \frac{1}{\epsilon}.$$

The real method with parameters $(\theta, q) = (1/2, 1)$ gives that

$$A : L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q',1} L_x^{r'} L_v^{p'} \to l^1.$$

By the TT^*-principle, this means that

$$\|W(t)F\|_{L_t^{q,\infty} L_x^r L_v^p} \lesssim \|F\|_{L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \tag{2.67}$$

for all $F \in L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'}$ whenever (q, r, p) and $(\infty, \tilde{r}, \tilde{p})$ satisfy the assumptions we have made so far. This is summarized as part (ii) and part (iii) of Theorem 2.2.2.

In view of the Equivalence theorem 2.3.15 and especially the remark afterwards, we also have the following homogeneous estimate

$$\|U(t)f\|_{L_t^{q,\infty} L_x^r L_v^p} \lesssim \|f\|_{L_x^b L_v^c}\,, \tag{2.68}$$

for all $f \in L_x^b L_v^c$, where we have set $b = \tilde{r}'$, $c = \tilde{p}'$, in the same range as the above inhomogeneous estimate. We can get rid of the Lorentz norm in (2.68) and replace it with the Lebesgue norm $\|\cdot\|_{L_t^q}$ if we slightly perturbate the exponents q, b, and c, and then interpolate with the real method with $(\theta, q) = (1/2, c)$, and make use of Proposition 8.0.5. Using the Equivalence theorem 2.3.15 in the other direction, we also sharpen (2.67) to a Lebesgue norm in time. To summarize, we have

Lemma 2.5.5. *The estimate*

$$\|U(t)f\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L_x^b L_v^c}\,, \tag{2.69}$$

holds for all $f \in L_x^b L_v^c$ *whenever*

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{b}, \quad p < b < a < c < r, \quad r < \frac{n}{n-1}c,$$
$$\mathrm{HM}(r,p) = \mathrm{HM}(b,c) \overset{def}{=} a, \quad 1 < q, b, c, p, r < \infty, \quad q \geq c.$$

Note that condition $q \geq c$ *can be removed if we replace the* $L_t^q -$ *norm by a* $L_t^{q,c} -$ *norm. More generally, the estimate*

$$\|U(t)f\|_{L_t^{q,\infty} L_x^r L_v^p} \lesssim \|f\|_{L_x^b L_v^c}\,, \tag{2.70}$$

holds for all $f \in L_x^b L_v^c$ *whenever*

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{b}, \quad p \leq b < a < c \leq r, \quad r \leq \frac{n}{n-1}c,$$
$$\mathrm{HM}(r,p) = \mathrm{HM}(b,c) \overset{def}{=} a, \quad 1 < q, b, c, p, r < \infty.$$

Lemma 2.5.6. *Suppose that* (q, r, p) *and* $(\infty, \tilde{r}, \tilde{p})$ *are two jointly KT-acceptable exponent triplets such that* $r < \frac{n}{n-1}\tilde{r}'$ *(*$r < \infty$ *in* $n = 1$*),* $q \geq \tilde{p}'$*, and* $1 < q, b, c, p, r < \infty$*. Then the estimate*

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'}}\,,$$

holds for all $F \in L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'}$*. Similarly, if* (∞, r, p) *and* $(\tilde{q}, \tilde{r}, \tilde{p})$ *are two jointly KT-acceptable exponent triplets such that* $\tilde{r} < \frac{n}{n-1}r'$ *(*$\tilde{r} < \infty$ *in* $n = 1$*),* $\tilde{q}' \leq p$*, and* $1 < q, b, c, p, r < \infty$*, then* *the estimate*

$$\|W(t)F\|_{L_t^\infty L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}$$

holds for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$.

Proof. The lemma follows directly from Lemma 2.5.5. Note that the condition $p < b$ which in the current notation translates to $p < \tilde{r}'$ is a consequence of the conditions

$$\frac{1}{q} + \frac{1}{\infty} = n\left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right), \quad \frac{1}{q} < n\left(\frac{1}{p} - \frac{1}{r}\right),$$

so that it is automatically kept. The next inequality $b < c$, in current notation $\tilde{p} < \tilde{r}$, can also be assumed to hold since then otherwise the triplet $(\infty, \tilde{a}, \tilde{a})$ is KT-admissible and so it is the triplet (q, r, p), a case treated in Theorem 2.2.1. □

2.5.4 Endpoint global inhomogeneous estimates, case of $q = \tilde{q}'$

We are not able to prove this type of estimates for the KT equation. The main reason for that is, especially when compared to the Schrödinger and the wave equation, that we cannot perturbate the spatial and velocity exponents $(p, r, \tilde{p}, \tilde{r})$ freely, even on the level of the local estimates, see the counterexample of section 2.6.5.

However, we can prove weaker versions of this type of endpoint estimates if we assume that the velocity L_v^p-norms are given over a compact velocity space $V \subset \mathbb{R}^n$. In fact, such an assumption is physically relevant. Consider now a 6-tuple of exponents $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})$ whose projection lies on the hypotenuse of ΔOAB, recall fig. 2.1. We also assume that $P_3 = (1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \mathcal{E}^*$ with a small neighborhood of $(r, p, \tilde{r}, \tilde{p}) \in \mathbb{R}^4$. We shall denote this set by $\mathcal{E}_3$. In addition, assume that P_3 lies in the cross-section of $\mathcal{E}_3$ with the hyperplane (2.66). Consider the small perturbations of the point P_3

$$\begin{aligned}
&\frac{1}{r_0} = \frac{1}{r} + \epsilon, \quad \frac{1}{\tilde{r}_0} = \frac{1}{\tilde{r}} + \epsilon, \quad \frac{1}{p_0} = \frac{1}{p} - \epsilon, \quad \frac{1}{\tilde{p}_0} = \frac{1}{\tilde{p}} - \epsilon,\\
&\frac{1}{r_1} = \frac{1}{r} - 3\epsilon, \quad \frac{1}{\tilde{r}_1} = \frac{1}{\tilde{r}} - 3\epsilon, \quad \frac{1}{p_1} = \frac{1}{p} + 3\epsilon, \quad \frac{1}{\tilde{p}_1} = \frac{1}{\tilde{p}} + 3\epsilon.
\end{aligned}$$

We have that $\beta(q, r_0, \tilde{q}, \tilde{r}_0) = 2n\epsilon$ and $\beta(q, r_1, \tilde{q}, \tilde{r}_0) = \beta(q, r_0, \tilde{q}, \tilde{r}_1) = -2n\epsilon$. Hence the maps

$$\begin{aligned}
A &: L_t^{\tilde{q}'} L_x^{\tilde{r}_0'} L_v^{\tilde{p}_0'} \times L_t^{q'} L_x^{r_0'} L_v^{p_0'} \to l^\infty_{-2\epsilon},\\
A &: L_t^{\tilde{q}'} L_x^{\tilde{r}_0'} L_v^{\tilde{p}_0'} \times L_t^{q'} L_x^{r_1'} L_v^{p_1'} \to l^\infty_{2\epsilon},\\
A &: L_t^{\tilde{q}'} L_x^{\tilde{r}_1'} L_v^{\tilde{p}_1'} \times L_t^{q'} L_x^{r_0'} L_v^{p_0'} \to l^\infty_{2\epsilon},
\end{aligned}$$

are bounded. In virtue of Lemma 8.0.7 and the well-known interpolation identity

$$(L^p(\mathbb{R};\mathcal{A}_0), L^p(\mathbb{R};\mathcal{A}_1))_{\theta,p} = L^p(\mathbb{R};(\mathcal{A}_0,\mathcal{A}_1)_{\theta,p}), \quad 1 < p < \infty, \tag{2.71}$$

see [2], the map

$$\begin{aligned} A : (L_t^{\tilde{q}'} L_x^{\tilde{r}_0'} L_v^{\tilde{p}_0'}, L_t^{\tilde{q}'} L_x^{\tilde{r}_1'} L_v^{\tilde{p}_1'})_{1/4,\tilde{q}'} \times \\ (L_t^{q'} L_x^{r_0'} L_v^{p_0'}, L_t^{q'} L_x^{r_1'} L_v^{p_1'})_{1/4,q'} \to (l^\infty_{2\epsilon}, l^\infty_{-2\epsilon})_{1/2,1} \end{aligned}$$

is also bounded. In view of the fact that V is compact we have that $L^{\tilde{P}'}(V) \hookrightarrow L^{\tilde{p}_0'}(V)$ and $L^{\tilde{P}}(V) \hookrightarrow L^{\tilde{p}_1'}(V)$ whenever $1 \le \tilde{P} \le min(\tilde{p}_0, \tilde{p}_1)$. Analogously, $L^{P'}(V) \hookrightarrow L^{p_0'}(V)$ and $L^{P'}(V) \hookrightarrow L^{p_1'}(V)$ whenever $1 \le P \le min(p_0, p_1)$. Thus we also have that the map

$$\begin{aligned} A : (L_t^{\tilde{q}'} L_x^{\tilde{r}_0'} L_v^{\tilde{P}'}, L_t^{\tilde{q}'} L_x^{\tilde{r}_1'} L_v^{\tilde{P}'})_{1/4,\tilde{q}'} \times \\ (L_t^{q'} L_x^{r_0'} L_v^{P'}, L_t^{q'} L_x^{r_1'} L_v^{P'})_{1/4,q'} \to (l^\infty_{2\epsilon}, l^\infty_{-2\epsilon})_{1/2,1} \end{aligned}$$

is bounded. Finally, in view of the interpolation identity (2.71), the above map simplifies to

$$A : L_t^{\tilde{q}'} L_x^{\tilde{r}',\tilde{q}'} L_v^{\tilde{P}'} \times L_t^{q'} L_x^{r',q'} L_v^{P'} \to l^1.$$

By the TT^*-principle, this implies the estimate

$$\|W(t)F\|_{L_t^q L_x^r L_v^P(V)} \lesssim_V \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{P}'}(V)}, \tag{2.72}$$

for any $P, \tilde{P}$, such that $1 \le P < p$ and $1 \le \tilde{P} < \tilde{p}$, and any two jointly KT-acceptable exponent triplets (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ whose exponents further satisfy the following conditions

$$\begin{gathered} 1 < q, \tilde{q} < \infty, \quad q = \tilde{q}', \quad q \le r, \quad \tilde{q} \le \tilde{r} \\ \frac{n-1}{p'} < \frac{n}{\tilde{r}}, \quad \frac{n-1}{\tilde{p}'} < \frac{n}{r}. \end{gathered}$$

2.6 Counterexamples and sharpness of the estimates

In this section we present some counterexamples to the Strichartz estimates (2.6), (2.8) for $U(t)$, and the inhomogeneous estimates (2.7) for $W(t)$.

2.6.1 Geometric interpretation of the Strichartz estimates for the homogeneous KT equation

Let us begin with the case $n = 1$. We denote by $A[f]$ the velocity averages of the kinetic transport evolution group $U(t)$,

$$A[f](t,x) = \int_{-\infty}^{\infty} f(x - tv, v)dv.$$

We can write this integral as a line integral

$$A[f] = \frac{1}{\sqrt{1+t^2}} \int_{\gamma_{x,t}} f(l)dl$$

along the straight line $\gamma_{x,t}$ in the (x,v)-plane that passes through the point $X = (x, 0)$ with gradient equal to $-1/t$. Let $f(x,v) = \chi_Q(x,v)$, where χ_Q is the characteristic of some measurable set Q in the plane Oxv. We can interpret $\|U(t)\chi_Q\|_{L_x^\infty L_v^1}$ geometrically. By the above,

$$\|U(t)\chi_Q\|_{L_x^\infty L_v^1} = \frac{1}{\sqrt{1+t^2}} \operatorname{ess\,sup}_{x\in\mathbb{R}} \int_{\gamma_{x,t}} \chi_Q(l)dl,$$

or in other words $\|U(t)\chi_Q\|_{L_x^\infty L_v^1}$ is the essential supremum of the (line) measure of all intersections of Q with straight lines in a fixed direction (times a factor in t).

Lemma 2.6.1. *There exist an open set Q on the plane of arbitrary small positive measure ϵ that contains in its interior a unit line segment in every direction. The sets $Q(\epsilon)$ can be chosen to be uniformly bounded with respect to ϵ.*

Proof. We repeat the construction of a Besicovitch set, see Besicovitch [3]. As we do not need to turn the line segment continuously, so we drop from the construction the joints between the triangles. This allows us to keep Q bounded regardless of ϵ. We iterate a finite number of times, thus the resulting set is a polygon, a modification of the so called Perron-Schoenberg tree. For Q we choose any open set that properly contains the latter so that the measure of their difference is small enough. We shall also call the set Q a Besicovitch set. □

It is now clear that the one-dimensional endpoint

$$\|U(t)f\|_{L_t^2 L_x^\infty L_v^1} \lesssim \|f\|_{L_{x,v}^2} \tag{2.73}$$

fails on the characteristic functions of Besicovitch sets (thus we recover the result of [25] by different means).

Lemma 2.6.2. *The Strichartz estimate*

$$\|U(t)f\|_{L_t^q L_x^\infty L_v^p} \lesssim \|f\|_{L_x^b L_v^c} \tag{2.74}$$

fails for some $f \in L^b_x L^c_v$*, whenever* $0 < q, p, b, c \leq \infty$*,* $c \neq \infty$*. Furthermore, if* $c = \infty$ *all remaining estimates of the form* (2.74) *are*

$$\|U(t)f\|_{L^q_t L^\infty_x L^{nq}_v} \lesssim \|f\|_{L^{nq}_x L^\infty_v} \tag{2.75}$$

and also fail, except in the trivial case of $q = \infty$ *which holds. Therefore, there are no non-trivial homogeneous Strichartz estimates* (2.8) *with* $r = \infty$*.*

Proof. We begin with the case $n = 1$. It is enough to consider only the case when all exponents q, r, p, c are bigger or equal to one. We set $f = \chi_Q$, where Q is a Besicovitch set on the (x, v)-plane of measure $\epsilon > 0$. If $c < \infty$, by Hölder's inequality $\|\chi_Q\|_{L^b_x L^c_v} \lesssim \|\chi_Q\|_{L^c_x L^c_v} = \epsilon^{1/c}$. Note that the assumption $c < \infty$ is essential to guarantee that $\|\chi_Q\|_{L^b_x L^c_v}$ is small if the measure of Q is small. In view of the power invariance (2.3), we can always assume that $p = 1$ in (2.74). Then we have that the right hand side of (2.74) is $O(\epsilon^{1/c})$ while the left hand side is $O(1)$. For higher dimensions $n > 1$ we repeat the same argument with the product set $Q^n = Q \times Q \times ...Q$.

The fact that the estimates (2.74) must have the form (2.75) whenever $r = c = \infty$ follows by scaling. These estimates fail because the pair $(q, r, p) = (q, \infty, nq)$ is not KT-acceptable, see (2.84). □

Lemma 2.6.3. A. *The Strichartz estimate*

$$\|W(t)F\|_{L^q_t L^\infty_x L^p_v} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v}, \tag{2.76}$$

fails for some $F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v$*, whenever* $1 \leq q, \tilde{q}, p, \tilde{p}, \tilde{r} \leq \infty$*,* $\tilde{p}' \neq \infty$*.*

B. *Moreover, if* $\tilde{p}' = \infty$ *estimate* (2.76) *holds for all* $F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^\infty_v$ *whenever* (q, r, p) *and* $(\tilde{q}, \tilde{r}, \tilde{p})$ *are two jointly KT-acceptable triplets, that is*

$$(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \Sigma_1 \cup B,$$

(recall definition 2.1.4.)

C. *Thus, the only remaining estimates of the form* (2.76) *with* $r = \tilde{p}' = \infty$ *have their exponents lying on the boundary* $\partial\Sigma_1$ *of the set* Σ_1*. They are*

$$\|W(t)F\|_{L^q_t L^\infty_x L^{nq}_v} \lesssim \|F\|_{L^1_t L^{nq}_x L^\infty_v}, \tag{2.77}$$

$$\|W(t)F\|_{L^\infty_t L^\infty_x L^{nq}_v} \lesssim \|F\|_{L^{q'}_t L^{nq}_x L^\infty_v} \tag{2.78}$$

$$\|W(t)F\|_{L^q_t L^\infty_x L^n_v} \lesssim \|F\|_{L^q_t L^n_x L^\infty_v}, \quad \text{for } 1 < q < \infty. \tag{2.79}$$

Estimates (2.77), (2.78) *fail. Also for* $n = 1$ *estimate* (2.79) *fails too. Thus, the only unresolved inhomogeneous estimates having* $r = \infty$ *are those in* (2.79) *for* $n > 1$*.*

Proof. Part A. Again, it is enough to consider in detail only the case when $n = 1$. We put the inhomogeneous term $F(t, x, v) = \chi(t \in [0, 1]; (x, v) \in Q)$, where Q is a Besicovitch set on the (x, v)-plane of measure $\epsilon > 0$. Then F is a characteristic function of a set in $\mathbb{R}^{2+1}$ with measure ϵ. Thus, as above, estimate (2.76) fails whenever $\tilde{p}' < \infty$.

Part B. In the case when $\tilde{p}' = \infty$, $1 < q, \tilde{q} < \infty$ and $1/q + 1/\tilde{q} < 1$, the estimate (2.76) holds in view of Theorem 2.2.2, see the case of the set Σ_1.

Part C. Suppose that $\tilde{p}' = \infty$, $1/q + 1/\tilde{q} < 1$, and either q or $\tilde{q}$ is equal to ∞. In such case we deal with estimates of the form (2.77) or (2.78). These estimates fail because the exponents are not KT-acceptable, see section 2.6.4 and in particular condition (2.86).

We next consider estimate (2.79) in dimension $n = 1$, so that $1/q + 1/\tilde{q} = 1$, $1 < q, \tilde{q} < \infty$. Take $F(t, x, v) = \phi(t)\psi(x)g(v)$, where $\phi(t) = \chi_{[0,1]}(t)$, $g(v) \equiv 1 \in L^\infty(\mathbb{R})$, and $\psi \in L^1(\mathbb{R})$, $\psi \geq 0$. We have

$$\begin{aligned}\int_{-\infty}^{\infty} W(t)F dv &= \int_0^t \int_{-\infty}^{\infty} \phi(s)\psi(x - (t - s)v) dv ds \\ &\geq \|g\|_{L^\infty_x} \|\psi\|_{L^1_x} \int_0^t \frac{1}{t - s}\phi(s) ds = \infty.\end{aligned}$$

Consequently, all estimates of the considered type fail.

Finally, the estimate (2.79) is left open for $n > 1$. □

Lemma 2.6.4. A. *The Strichartz estimate*

$$\|W(t)F\|_{L^q_t L^r_x L^p_v} \lesssim \|F\|_{L^{\tilde{q}'}_t L^1_x L^{\tilde{p}'}_v}, \tag{2.80}$$

fails for some $F \in L^{\tilde{q}'}_t L^1_x L^{\tilde{p}'}_v$, *whenever* $1 \leq q, \tilde{q}, p, \tilde{p}, \tilde{r} \leq \infty$, $p \neq 1$.

B. *Moreover, if* $p = 1$ *estimate* (2.80) *holds for all* $F \in L^{\tilde{q}'}_t L^1_x L^{\tilde{p}'}_v$ *whenever* (q, r, p) *and* $(\tilde{q}, \tilde{r}, \tilde{p})$ *are two jointly KT-acceptable triplets, that is*

$$(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \Sigma_2 \cup C,$$

(recall definition 2.1.4.)

C. *Thus, the only remaining estimates of the form* (2.80) *with* $\tilde{r}' = p = 1$ *have their exponents lying on the boundary* $\partial\Sigma_2$ *of the set* Σ_2. *They are*

$$\|W(t)F\|_{L^q_t L^p_x L^1_v} \lesssim \|F\|_{L^1_t L^1_x L^p_v}, \quad p = (nq)' \tag{2.81}$$

$$\|W(t)F\|_{L^\infty_t L^p_x L^1_v} \lesssim \|F\|_{L^{q'}_t L^1_x L^p_v}, \quad p = (nq)' \tag{2.82}$$

$$\|W(t)F\|_{L^q_t L^p_x L^1_v} \lesssim \|F\|_{L^q_t L^1_x L^p_v}, \quad p' = n,\ 1 < q < \infty. \tag{2.83}$$

Estimates (2.81), (2.82) *fail. Also for* $n = 1$ *estimate* (2.83) *fails too. Thus, the only unresolved* *inhomogeneous estimates having $\tilde{r} = \infty$ are those in (2.83) for $n > 1$.*

Proof. Follows by the preceding lemma and duality. □

Remark 2.6.5. In the preceding lemmas we showed that there are no nontrivial homogeneous Strichartz estimates with a Lebesgue exponent $r = \infty$. Furthermore, in the case of the inhomogeneous Strichartz estimates the set of admissible (acceptable) exponents with either r or $\tilde{r}$ being equal to ∞ is precisely described by the sets Σ_1 and Σ_2, respectively. As we saw, the only unresolved such inhomogeneous estimates have their exponents lying on the boundary $\partial\Sigma_1$, $\partial\Sigma_2$, respectively, for $n > 1$. We mention preemptively that due to these counterexamples all Strichartz estimates for $U(t)$ and $W(t)$ in spatial dimension $n = 1$ are now known (although some endpoint estimates, e.g. with $q = \infty$, or $\tilde{q} = \infty$, appear in a modified form in terms of a weaker Lorentz-norm in time.

In the remainder of this subsection we extend the geometric interpretation of the KT evolution operator to higher dimensions, more precisely that of $\|U(t)\chi_Q\|_{L^r_x L^1_v}$, where $1 \le r < \infty$, and $n \ge 1$. We only sketch our argument so this part can be skipped on a first reading. In fact, condition (2.84) can be derived on an elementary counterexample similar to those in the subsequent subsections.

It is not hard to see that

$$\int_{-\infty}^{\infty} \int_{\gamma_{x,t}} \chi_Q(l) dl dx = \sqrt{1+t^2} \iint_{\mathbb{R}^2} R(\pi/2 - \theta)\chi_Q(l, m) dl dm,$$

where by $R(\psi)$ we denote rotation by an angle ψ and $\cot\theta = -t$. Therefore, if we denote

$$N_r(\theta, Q) = \|R(\pi/2 - \theta)\chi_Q\|_{L^r_x L^1_v},$$

we obtain the representation

$$\begin{aligned}\|U(t)\chi_Q\|_{L^q_t L^r_x L^1_v} &= \left\| \left(\frac{1}{\sqrt{1+t^2}} \right)^{1/r'} N_r(\theta, Q) \right\|_{L^q(\mathbb{R})} \\ &= \left\| (\sin\theta)^{1/r' - 2/q} N_r(\theta, Q) \right\|_{L^q(0,\pi)}\end{aligned}$$

Note that when the exponent triplet $(q, r, 1)$ is KT-admissible in $n = 1$, that is $1/q = 1/2(1/p - 1/r)$ we have that

$$\|U(t)\chi_Q\|_{L^q_t L^r_x L^1_v} = \|N_r(\theta, Q)\|_{L^q(0,\pi)}.$$

We can generalize the above idea to any dimension n by considering the product $Q^n =$

$Q \times Q \times \ldots Q$. By repeating the argument above we obtain

$$\|U(t)\chi_{Q^n}\|_{L^q_t L^r_x L^1_v} = \left\| \left(\frac{1}{\sqrt{1+t^2}} \right)^{n/r'} N^n_r(\theta, Q) \right\|_{L^q(\mathbb{R})}.$$

Let Q be the unit circle in the plane Ox_1v_1. Then obviously $N_r(\theta, Q^n) = \text{const}$ for all $\theta \in [0, \pi]$. Consequently, the norm above is finite only if $r \geq 1$, $qn/r' > 1$. By the power invariance (2.3), we generalize to any admissible (q, r, p) and obtain the restrictions

$$r \geq a, \quad \frac{1}{q} < n\left(\frac{1}{p} - \frac{1}{r}\right), \quad \text{or } q = \infty,\ 1 \leq p = r \leq \infty, \tag{2.84}$$

to the range of validity of estimate (2.6). The restrictions above trivially imply that $p \leq r$.

2.6.2 Homogeneous estimates

By scaling, see Lemma 2.3.5, estimate

$$\|U(t)f\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^a_{x,v}}, \quad \forall f \in L^a_{x,v}, \tag{2.6}$$

can only hold if

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{a}, \quad a = \mathrm{HM}(p, r).$$

By the TT^*-principle, estimate (2.6) with $a = 2$ is equivalent to

$$\|TT^*F\|_{L^q_t L^r_x L^{r'}_v} \lesssim \|F\|_{L^{q'}_t L^{r'}_x L^r_v}, \quad \forall F \in L^{q'}_t L^r_x L^{r'}_v.$$

By translation invariance in x, see Proposition 8.0.13, $r \geq r'$, or equivalently $r \geq 2$. By the power invariance (2.3) we generalize the case $a = 2$ to any $0 < a < \infty$, the respective condition is $r \geq a$.

We now find the upper bound on r, that is the bound $r \leq r^*(a)$. To that end, we first consider again the special case $a = 2$. Note that the condition $q \geq 2$ is equivalent to the one at hand. By translation invariance in t, see Proposition 8.0.13, $q \geq q'$, or equivalently $q \geq 2$. By the power invariance (2.3) we generalize the case $a = 2$ to any $0 < a < \infty$, the respective condition is $q \geq a$, or equivalently $r \leq r^*(a)$. Thus in view of (2.84), $p^*(a) \leq p \leq a \leq r \leq r^*(a)$.

2.6.3 Homogeneous estimates for mixed L^p-data

If we consider the mixed L^p-data estimate

$$\|U(t)f\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^b_x L^c_v},, \quad \forall f \in L^b_x L^c_v \tag{2.8}$$

then most of the preceding section applies as well. By scaling, we have that the conditions

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{b}, \quad \mathrm{HM}(p,r) = \mathrm{HM}(b,c) \stackrel{def}{=} a, \tag{2.85}$$

are necessary. The conditions $p \le a \le r$ and $a \le r$ carry over from the preceding section. We do not have, however, a suitable counterexample giving that exact upper bound $r^*(b,c)$ to r in estimate (2.8) in $n > 1$. The last condition we need to verify is that $b \le c$. Estimate (2.8) is equivalent to

$$\|TT^*F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^1 L_x^b L_v^c}, \quad \forall F \in L_t^1 L_x^b L_v^c.$$

By duality, it is equivalent to

$$\|TT^*F\|_{L_t^\infty L_x^{b'} L_v^{c'}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{p'}}, \quad \forall F \in L_t^{q'} L_x^{r'} L_v^{p'}.$$

By the restrictions on the inhomogeneous estimates, see section 2.6.4, we obtain $b' \ge c'$ or equivalently $b \le c$. Thus we have that $p < b < a < c < r$ ($p < b$ follows from (2.84) and (2.85)) or $a = b = c = p = r$ (and $q = \infty$).

2.6.4 Global inhomogeneous estimates

In this section we present some counterexamples to the validity of the inhomogeneous estimates

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}. \tag{2.7}$$

By scaling, see Lemma 2.3.5, we obtain that the conditions

$$\frac{1}{q} + \frac{1}{\tilde{q}} = n\left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right), \quad \mathrm{HM}(p,r) = \mathrm{HM}(\tilde{p}', \tilde{r}') \stackrel{def}{=} a,$$

are necessary.

Consider $F(t,x,v) = \chi\,(0 \le t \le 1, |x| \le 1, |v| \le 1)$. When $t \gg 1$ we have that

$$TT^*F \approx W(t)F \approx \chi\left(\left|v - \frac{x}{t}\right| \le \frac{1}{t}, |v| \le 1\right) \approx \chi\left\{v \sim \frac{1}{t}, x \sim t\right\}.$$

Hence,

$$\|W(t)F\|_{L_x^r L_v^p} \sim t^{\frac{n}{r} - \frac{n}{p}}, \quad t \gg 1.$$

It follows that $\|W(t)F\|_{L_t^q L_x^r L_v^p} < \infty$ only if

$$\left(\frac{n}{r} - \frac{n}{p}\right) q < -1, \quad \text{or if } q = \infty, r = p. \tag{2.86}$$

By the duality Lemma 2.3.14, the dual exponents $(\tilde{q}, \tilde{r}, \tilde{p})$ must also satisfy (2.86). Thus we have that the conditions $p \leq r$ and $\tilde{p} \leq \tilde{r}$ are necessary for the validity of estimate (2.7). The same conclusion applies if we replace $W(t)$ by TT^* in (2.7).

We now show that conditions

$$\frac{1}{q} + \frac{1}{\tilde{q}} \leq 1, \quad \frac{1}{r} + \frac{1}{\tilde{r}} \leq 1,$$

are necessary for the validity of estimate (2.7). Indeed, that follows from the translation invariance of $W(t)$ in t and x and Proposition 8.0.13. We check this fact only for t. Consider $F_\tau(t) = F(t-\tau)$ and $W(t)F_\tau$. We have

$$\int_{-\infty}^{t} U(t-s)F(s-\tau)ds = \int_{-\infty}^{t-\tau} U(t-\tau-\sigma)F(\sigma)d\sigma,$$

or in other words $W(t)F_\tau = W(t-\tau)F$, QED.

Thus we have fully verified the necessity of the condition that (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ be two jointly KT-acceptable exponent triplets (apart from some boundary cases, e.g. $q = \infty$, etc.)

We do not have a suitable counterexample showing the necessity of condition

$$\frac{n-1}{p'} \leq \frac{n}{\tilde{r}}, \quad \frac{n-1}{\tilde{p}'} \leq \frac{n}{r}, \quad n > 1. \tag{2.15}$$

However, we can show that the similar condition

$$\frac{n}{p'} < \frac{1}{\tilde{q}} + \frac{n}{\tilde{r}}, \quad \frac{n}{\tilde{p}'} < \frac{1}{q} + \frac{n}{r}, \tag{2.15a}$$

is sharp. Indeed, (2.15a) is a direct consequence of the two conditions (2.12), (2.13). Condition (2.15a) implies (2.15) whenever $p' \leq \tilde{q}$ and $\tilde{p}' \leq q$. Thus, if there are some other global inhomogeneous estimates for $W(t)$ not included in Theorem 2.2.2, they must belong to the range $\tilde{q} < p'$ or $q < \tilde{p}'$.

2.6.5 Local inhomogeneous estimates

In this section we show that the condition $\mathrm{HM}(p, r) = \mathrm{HM}(\tilde{p}', \tilde{r}')$ is necessary for the validity of the local inhomogeneous estimates, for example for the estimate

$$\|W(t)F\|_{L_t^q([2,3];L_x^r L_v^p)} \lesssim \|F\|_{L_t^{\tilde{q}'}([1,2];L_x^{\tilde{r}'} L_v^{\tilde{p}'})}\,. \tag{2.87}$$

The implication of this is the fact that we cannot perturbate the exponents $p, r, \tilde{p}, \tilde{r}$ freely. Therefore, the perturbative techniques for treating the endpoint homogeneous and inhomogeneous estimates do not apply to the present context. However, we managed to circumvent that difficulty for the endpoint homogeneous estimate but not for the endpoint inhomogeneous estimates with exponents $(1/q, 1/\tilde{q})$ lying on the hypotenuse of ΔOAB in fig. 2.1 which we leave open.

Let us set $F(t,x,v) = \chi\,(t \in [0,1],\ (x,v) \in Q_R)$, where by Q_R we denote the cube of side length $2R$ with center at the origin of $\mathbb{R}^{2n}$, that is

$$Q_R = \{(x,v) : \|x\|_\infty \le R,\ \|v\|_\infty \le R\},$$

where $\|x\|_\infty = \sup_{1\le i\le n} |x_i|$, for $x = (x_1, .., x_n)$. Hence,

$$\|F\|_{L_t^{\tilde{q}'}([1,2];L_x^{\tilde{r}'}L_v^{\tilde{p}'})} \sim R^{\frac{n}{\tilde{p}'}+\frac{n}{\tilde{r}'}}.$$

We set $\tau = t - s$, and consider the set $Q_R(\tau)$ given by

$$\|x - (t-s)v\|_\infty \le R,\ \|v\|_\infty \le R.$$

Then, for $t \in [2,3]$, $s \in [0,1]$, and therefore $\tau \in [1,3]$, we have the inclusions

$$Q_{R/4} \subset Q_R(\tau) \subset Q_{4R}.$$

Hence,

$$\|W(t)F\|_{L_t^q([2,3];L_x^r L_v^p)} \sim R^{\frac{n}{p}+\frac{n}{r}}.$$

We conclude that condition

$$\frac{1}{r} + \frac{1}{p} = \frac{1}{\tilde{r}'} + \frac{1}{\tilde{p}'}$$

is necessary for the validity of the local estimates (2.87).

2.7 Some open problems

The following estimates remain unresolved.

1. In dimensions $n > 1$, inhomogeneous endpoint estimates of the form

$$\|W(t)F\|_{L_t^q L_x^\infty L_v^n} \lesssim \|F\|_{L_t^q L_x^n L_v^\infty}, \tag{2.88}$$

for $1 < q < \infty$.

2. In dimensions $n > 1$, inhomogeneous endpoint estimates of the form

$$\|W(t)F\|_{L_t^q L_x^p L_v^1} \lesssim \|F\|_{L_t^q L_x^1 L_v^p}, \tag{2.89}$$

for $1 < q < \infty$, where $p = n/(n-1)$.

3. In dimensions $n > 1$, all endpoint inhomogeneous estimates of the form

$$\|W(t)F\|_{L^q_t L^r_x L^p_v} \lesssim \|F\|_{L^q_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v} .$$

4. To remove the Lorentz norm in time for all endpoint estimates with q or $\tilde{q} = \infty$ (2.18), (2.19), and replace it by a Lebesgue norm.

5. In dimensions $n > 1$, the full range of validity of the estimate

$$\|W(t)F\|_{L^q_t L^r_x L^p_v} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v} ,$$

that is to show whether condition (2.15) is sharp or otherwise find all other estimates of the above type.

6. In dimensions $n > 1$, the analogous question in the context of the generalized homogeneous estimates, i.e. the question of what the precise upper bound $r^*(b, c)$ to r should be in the estimate

$$\|U(t)f\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^b_x L^c_v} .$$

7. In dimensions $n > 1$, if the upper bound $r^*(b, c) = nc/(n-1)$ given in (2.20) is not sharp, then find all remaining generalized homogeneous estimates.

We would also like to ask the following questions

1. Is there any improvement to the above estimates for spherically symmetric f or F?

2. It would be curious to know what happens with the estimates of Lemmas 2.6.2 and 2.6.3 that we showed to fail if one replaces the L^∞-norm in space by the BMO-norm?

3. What are the best constants in the Strichartz estimates for the KT equation and what are their maximizers? Are the characteristics to Besicovitch sets, or to any other distinguished family of sets, maximizers to certain estimates?

4. What is the full range of acceptability of the local estimates, that is the set $\mathcal{E}$? The answer to this question might help us understand the analogous question in the context of the Schrödinger equation, see Foschi [19, sect. Open questions]. One distinct advantage to working in the context of the KT equation is that we avoid the technical complications associated with oscillations characteristic to other contexts.

Chapter 3

Application to Kinetic Chemotaxis

In this chapter we consider the Othmer-Dunbar-Alt kinetic model of chemotaxis (3.1) - (3.3) and prove its global well-posedness for small data in space dimensions $n \geq 2$ by making use of the Strichartz estimates of Theorem 2.2.1. With this result we extend and improve the results of [28] and [8] and would like to encourage the use of the Strichartz estimates for the KT equation as a new tool for proving global well-posedness for small data in the context of a nonlinear kinetic equation.

Our attention to the Othmer-Dunbar-Alt kinetic model and the possibility of studying its well-posedness via Strichartz estimates was drawn by [8]. However, the authors manage to prove only weak global solutions to that system without uniqueness. It seems to us that our improvement is made only possible by the larger range of inhomogeneous Strichartz estimates that we have now at hand. Another distinction between our approach and that of [8] is the fact that we estimate the chemoattractant $S(t, x)$ in terms of spacetime estimates, while in [8] this quantity is estimated for fixed time.

3.0.1 Introduction

The kinetic model (3.1) - (3.3) describes a population of bacteria in motion in a field of a chemoattractant, see [8], [7], and the references therein. The system reads

$$\begin{aligned} \partial_t u(t,x,v) + v\cdot\nabla_x u(t,x,v) &= \int_{v'\in V} T[S](t,x,v,v')u(t,x,v')dv', \qquad (3.1)\\ - \int_{v'\in V} &T[S](t,x,v',v)u(t,x,v)dv', \quad t>0,\ x\in\mathbb{R}^n, v\in V \\ -\Delta_x S(t,x) + S(t,x) = \rho(t,x) &\overset{def}{=} \int_{v\in V} u(t,x,v)dv, \qquad (3.2)\\ u(0,x,v) = f(x,v) \geq 0, & \qquad (3.3) \end{aligned}$$

where by $u(t,x,v)$ we denote the cell density in phase space and we assume that the space of admissible velocities $V \subset \mathbb{R}^n$ is compact. The cell density in physical space is denoted by $\rho(t,x)$, where t and x are a time and a space coordinate respectively. The free transport operator $\partial_t u(t,x,v)+v\cdot\nabla_x u(t,x,v)$ describes the free runs of the bacteria which have velocity $v\in V$. The nonlinear terms in the right hand side of (3.1) denote a scattering operator that expresses the reorientation of the bacteria towards regions of high concentration in chemoattractant $S(t,x)$. Based on biologically realistic assumptions, the turning kernel $T[S](t,x,v,v')\geq 0$ is assumed to satisfy the following bound

$$\|T[S](t,\cdot,\cdot,\cdot)\|_{L^r_x L^{p_1}_v L^{p_2}_{v'}} \lesssim_{|V|,p_1,p_2} \|S(t,\cdot)\|_{L^r_x} + \|\nabla S(t,\cdot)\|_{L^r_x}, \qquad (3.4)$$

whenever $r \geq p_1$, p_2, see [8, Theorem 3].

3.0.2 Maximum principle for the transport equation

In this short section we present the maximum principle for the transport equation

$$\begin{aligned} \partial_t u(t,x,v) + v\cdot\nabla_x u(t,x,v) + g(t,x,v)u(t,x,v) = \\ \int_V h(t,x,v,v')u(t,x,v')dv' + F(t,x,v), \end{aligned} \qquad (3.5)$$

subject to the initial condition

$$u(0,x,v) = f(x,v). \qquad (3.6)$$

We assume that $(t,x,v)\in\mathbb{R}\times\mathbb{R}^n\times V$, $V\subseteq\mathbb{R}^n$, and the kernel $h(t,x,v,v')$ in (3.5) is nonnegative.

Lemma 3.0.1 (Maximum principle for the transport equation). *Suppose that $u(t)$ satisfies (3.5), (3.6) with $F(t,x,v)\geq 0$ and $f(x,v)\geq 0$. Then $u(t,x,v)\geq 0$ for all time.*

Proof. The claim follows directly from the representation of (3.5), (3.6) as an integral equation

$$
\begin{aligned}
&u(t,x,v) = f(x-tv,v)\exp\left(-\int_0^t g(s,x-(t-s)v,v)ds\right) + \\
&\int_0^t Q(s,x-(t-s)v,v)\exp\left(-\int_0^{t-s} g(\tau+s,x-(t-s-\tau)v,v)d\tau\right)ds,
\end{aligned} \tag{3.7}
$$

where

$$
Q(t,x,v) = \int_V h(t,x,v,v')u(t,x,v')dv' + F(t,x,v).
$$

The solution to (3.7) can be constructed by an iteration scheme whenever f, g, h and F are regular enough. It is easy to see that on each step we obtain nonnegative solutions. □

Corollary 3.0.2 (Comparison principle for the KT equation)**.** *Suppose that* $u_1(t)$ *solves*

$$
\partial_t u_1(t,x,v) + v\cdot\nabla_x u_1(t,x,v) = F_1(t,x,v), \quad u(0) = f_1(x,v),
$$

and that $u_2(t)$ *solves*

$$
\partial_t u_2(t,x,v) + v\cdot\nabla_x u_2(t,x,v) = F_2(t,x,v), \quad u(0) = f_2(x,v),
$$

where $F_1(t,x,v) \le F_2(t,x,v)$ *and* $f_1(x,v) \le f_2(x,v)$*. Then* $u_1(t) \le u_2(t)$ *for all time.*

Proof. Trivially follows from the maximum principle of Lemma 3.0.1. □

The following equation

$$
\begin{aligned}
&\partial_t u(t,x,v) + v\cdot\nabla_x u(t,x,v) = \\
&\qquad\qquad \int_{\mathbb{R}^n} K(v,v')u(t,x,v') - K(v',v)u(t,x,v)dv'
\end{aligned} \tag{3.8}
$$

is often called the scattering kinetic equation, see [40, pp. 226, 227]. The function $K(v,v') \ge 0$ is called a scattering kernel and my also depend on (t,x) through quantities that depend on the density $u(t)$ like the chemoattractant $S(t)$ in (3.2). In particular, we can take $T[S]$ to be the kernel K.

Lemma 3.0.3 (Maximum principle for the scattering kinetic equation)**.** *Suppose that* $u(t)$ *satisfies* (3.8) *and* $u(0) \ge 0$*. Then* $u(t) \ge 0$ *for all time.*

Proof. Set $g(t,x,v) = \int_V K(v,v')dv'$, $h(t,x,v,v') = K(v',v)$ and apply Lemma 3.0.1. □

Corollary 3.0.4. *The solution* $u(t)$ *to the IVP* (3.1) - (3.3) *satisfies the following bound*

$$
0 \le u(t,x,v) \le u_1(t,x,v) + u_2(t,x,v), \tag{3.9}
$$

where

$$\partial_t u_1(t,x,v) + v\cdot\nabla_x u_1(t,x,v) = 0, \quad u_1(0) = f,$$

and

$$\partial_t u_2(t,x,v) + v\cdot\nabla_x u_2(t,x,v) = \int_{v'\in V} T[S](t,x,v,v')u(t,x,v')dv', \quad u_2(0) = 0.$$

Proof. The first inequality in (3.9) follows from the maximum principle of Lemma 3.0.3. The second one follows from the comparison principle of Corollary 3.0.2. ☐

3.0.3 Global well-posedness to the Othmer-Dunbar-Alt kinetic model of chemotaxis

Theorem 3.0.5. *The IVP* (3.1)-(3.3) *is globally well-posed for small data in the class* $f \in L^1(\mathbb{R}^n \times V) \cap L^a(\mathbb{R}^n \times V)$ *for* $n < a < \infty$ *and* $n \geq 2$. *More specifically, there exist a fixed positive constant* M*, depending only on the space dimension* n*, so that whenever* $\|f\|_{L^n_{x,v}} < M$*, the IVP* (3.1)-(3.3) *admits a unique solution* $u(t) \in C([0,\infty); L^1(\mathbb{R}^n \times V) \cap L^a(\mathbb{R}^n \times V))$ *for which*

$$\|\rho\|_{L^n_t L_x^{n^2/(n-1)}} < \infty, \quad \|S\|_{L^n_t L^\infty_x} < \infty.$$

Remark 3.0.6. The theorem also holds for $a = \infty$ but in this case we cannot claim continuity of $u(t)$ but rather that $u(t) \in L^\infty([0,\infty); L^1(\mathbb{R}^n \times V) \cap L^a(\mathbb{R}^n \times V))$, see Lemma 2.3.7.

Proof. Due to Corollary 3.0.4 we can ignore the second integral term when we estimate the solution $u(t)$. Thus, for some admissible triplets (q,r,p) and $(\tilde{q},\tilde{r},\tilde{p})$ we have

$$\|u\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^a_{x,v}} + \left\| \int_V T[S]u'dv' \right\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v}, \tag{3.10}$$

where we have used the abbreviation $u' = u(t,x,v')$. To the integral term above we first apply Hölder's inequality to get

$$\int_V T[S](t,x,v,v')u(t,x,v')dv' \leq \|T[S](t,x,v,\cdot)\|_{L^{p'}_{v'}} \|u(t,x,\cdot)\|_{L^p_{v'}}.$$

We next take the $L^{\tilde{p}'}$-norm in v and then take the $L^{\tilde{r}'}$-norm in x using Hölder's inequality with $\frac{1}{\tilde{r}'} = \frac{1}{r} + \frac{1}{w}$ to get

$$\left\| \int_V T[S]u(t,x,v')dv' \right\|_{L^{\tilde{r}'}_x L^{\tilde{p}'}_v} \leq \|T[S](t,\cdot,\cdot,\cdot)\|_{L^w_x L^{\tilde{p}'}_v L^{p'}_{v'}} \|u(t,\cdot,\cdot)\|_{L^r_x L^p_v}.$$

Choosing $w = \infty$ guarantees that $w \geq \tilde{p}', p'$, so that we can use the bound (3.4) to get

$$\|T[S](t,\cdot,\cdot,\cdot)\|_{L^w_x L^{\tilde{p}'}_v L^{p'}_{v'}} \lesssim \|G * \rho\|_{L^w_x} + \|\nabla G * \rho\|_{L^w_x},$$

where $S = G * \rho$ and G is the Bessel potential

$$G(x) = \int_0^\infty e^{-\pi\frac{|x|^2}{4s}} s^{\frac{-n+2}{2}} \frac{ds}{s}.$$

We recall the following two estimates on G, see [28], $G \in L^b(\mathbb{R}^n)$, for $1 \leq b < \frac{n}{n-2}$, and $\nabla G(x) \in L^c(\mathbb{R}^n)$ for $1 \leq c < \frac{n}{n-1}$. Hence, an application of the Young's inequality yields the estimates on the chemoattractant

$$\|G * \rho(t)\|_{L^\infty_x} \lesssim \|G\|_{L^b} \|\rho(t)\|_{L^r},$$
$$\|\nabla G * \rho(t)\|_{L^\infty_x} \lesssim \|\nabla G\|_{L^c} \|\rho(t)\|_{L^r},$$

where $\frac{1}{b} = \frac{1}{c} = 1 - \frac{1}{r}$, $n \geq 2$, and $r > n$. Next we use that $\|\rho\|_{L^r} \lesssim_{|V|} \|u(t,\cdot,\cdot)\|_{L^r_x L^p_v}$ and choose $\tilde{q}' = q/2$ to obtain

$$\left\|\int_V T[S]u'dv'\right\|_{L^q_t L^r_x L^p_v} \lesssim \left\|\|u(t,\cdot,\cdot)\|^2_{L^r_x L^p_v}\right\|_{L^{q/2}_t} \lesssim \|u\|^2_{L^q_t L^r_x L^p_v}.$$

Hence, we obtain the following a-priori estimate

$$\|u\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^a_{x,v}} + \|u\|^2_{L^q_t L^r_x L^p_v} \tag{3.11}$$

for $u(t)$, whenever the following system of conditions

$$\begin{aligned} \mathrm{HM}(r,p) = \mathrm{HM}(\tilde{r}',\tilde{p}') &= a, \\ \frac{1}{q} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right), \quad \frac{1}{\tilde{q}} = \frac{n}{2}\left(\frac{1}{\tilde{p}} - \frac{1}{\tilde{r}}\right)&, \\ \tilde{q}' = q/2, \quad n \geq 2, \quad r > n&, \\ p^*(a) \leq p \leq a, \quad a \leq r \leq r^*(a)&, \\ p^*(a') \leq \tilde{p} \leq a', \quad a' \leq \tilde{r} \leq r^*(a')&, \end{aligned}$$

admits a solution. By a direct inspection, we see that the following sets of exponents

$$\begin{aligned} (q,r,p,a) &= \left(n, n^2/(n-1), n^2/(n+1), n\right), \\ (\tilde{q},\tilde{r},\tilde{p},a') &= \left(n/(n-2), n^2/(n^2-2n+2), n^2/(n-2), n/(n-1)\right) \end{aligned}$$

satisfy all of the above conditions.

By a standard bootstrap argument on estimate (3.11) there exist some fixed positive constant M, depending only on the space dimension n, such that if $\|f\|_{L^n_{x,v}} < M$ then

$$\|u\|_{L^n_t L^{n2/(n-1)}_x L^{n2/(n+1)}_v} < \infty.$$

This inequality immediately gives that

$$\|\rho\|_{L^n_t L^{n2/(n-1)}_x} < \infty, \quad \|S\|_{L^n_t L^\infty_x} < \infty.$$

Hence, considering again equation (3.1) and using the estimate on S above, we obtain the following Gronwall's inequality

$$\sup_{t\in[0,T]} \|u(t)\|^b_{L^a_{x,v}} \lesssim_T \|f\|^b_{L^a_{x,v}} + \int_0^T \|u(s)\|^b_{L^a_{x,v}}\, ds$$

for all $0 < T < \infty$, where $b = n/(n-1)$ and $n \leq a \leq \infty$. Hence,

$$\sup_{t\in[0,T]} \|u(t)\|_{L^a_{x,v}} < \infty, \quad \forall T \in (0,\infty). \tag{3.12}$$

We next sketch the proof of the local well-posedness of the system. Coupled with the global estimate (3.12), the full claim will then follow easily. Let us write equations (3.1) - (3.3) in the equivalent integral form

$$u(t) = U(t)f + \int_0^t U(t-s)F(u(s))ds, \tag{3.13}$$

where $F(u)$ is the right hand side of (3.1), (recall again the representation $S = G * \rho$.) Define the right hand side of (3.13) as the operator

$$\begin{aligned} K : L^\infty([0,T], L^a(\mathbb{R}^{2n})) \to & L^\infty([0,T], L^a(\mathbb{R}^{2n})), \\ & K(u) = U(t)f + \int_0^t U(t-s)F(u(s))ds. \end{aligned}$$

This nonlinear operator is bounded on the cited spaces as it can easily be seen from the estimate

$$\|S(t)\|_{L^a_{x,v}} \lesssim \|\rho(t)\|_{L^a_x}, \quad a > n.$$

Indeed,

$$\|K(u(t))\|_{L^a_{x,v}} \lesssim \|f\|_{L^a_{x,v}} + \int_0^t \|u(s)\|^2_{L^a_{x,v}}\, ds.$$

Hence, for any $0 < T < \infty$ the operator K is bounded.

Let $u_1(t),\, u_2(t) \in L^\infty([0,T], L^a(\mathbb{R}^{2n}))$. Similarly, we have the estimate

$$\sup_{t\in[0,T]} \|K(u_1(t) - u_2(t))\|_{L^a_{x,v}} \leq q \sup_{t\in[0,T]} \|u_1(t) - u_2(t)\|_{L^a_{x,v}} ,$$

for some $0 < q < 1$, whenever $T > 0$ is small enough and $n < a \leq \infty$. The theorem follows by standard arguments henceforth. $\square$

Chapter 4

Strichartz Estimates with Spherical Symmetry

Strichartz estimates with spherical symmetry have attracted a lot of interest recently. The gain of regularity of these estimates over the standard Strichartz estimates varies with the equation but, for example, in the context of the wave equation this gain is significant. Most attention has been dedicated to the homogeneous setting, see Sterbenz [43], Fang and Wang [18], Hidano and Kurokawa [26], Machihara et al [33], Tao [47], and Vilela [48], with only a few special inhomogeneous estimates being proved. Below, we produce a range of inhomogeneous Strichartz estimates with spherical symmetry analogous to the inhomogeneous Strichartz estimates in the standard setting. To this end we restrict the standard spaces of functions to their subspaces of spherically symmetric functions and proceed with the TT^*-argument.

4.1 Inhomogeneous Strichartz estimates with spherical symmetry

In order to present the Strichartz estimates with spherical symmetry in the abstract setting we need to introduce an appropriate set-up for that. We shall restrict ourselves to function spaces over subsets of $\mathbb{R}^n$ where the property of spherical symmetry makes immediate sense. So, let us suppose that $X \subseteq \mathbb{R}^n$, $n \geq 2$, and $(X; d\mu)$ is a measure space. We further assume that the measure $d\mu$ is absolutely continuous with respect to the Lebesgue measure dx on $\mathbb{R}^n$ and has a density function $\phi : X \to \mathbb{C}$ that is a spherically symmetric function, that is $\phi(x) = \phi(|x|)$.

Consider now a Hilbert space $\mathcal{H}$ over $(X; d\mu)$ such that the spatial rotations R act on it as an unitary operator. By that we mean the identity

$$\langle Rf, g\rangle_{\mathcal{H}} = \langle f, Rg\rangle_{\mathcal{H}}, \tag{4.1}$$

for all $f, g \in \mathcal{H}$, where we have set $Rf = f(Rx)$.

Suppose we are given a family of Strichartz estimates

$$\|U(t)f\|_{L^q(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{H}} \tag{4.2}$$

that hold for all spherically symmetric $f \in \mathcal{H}$. An exponent pair (q, θ) for which estimate (4.2) holds under the assumptions we have made shall be called radially-admissible and the full range of radially-admissible pairs shall be denoted by the set $\mathcal{A} \subseteq [1, \infty] \times [0, 1]$. We are not interested to know how the family of estimates (4.2) is obtained but a typical situation will be when $U(t)$ satisfies a dispersive and an energy inequality for non-symmetric functions. In addition, on the assumption that f is spherically symmetric one obtains a better dispersive estimate through specific properties of the evolution operator $U(t)$. However, we shall only need to assume explicitly that $U(t)$ satisfies the energy inequality

$$\|U(t)f\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}},$$

in order to define the adjoint operator through the identity

$$\langle U(t)f, g\rangle_{\mathcal{H}} = \langle f, U^*(t)g\rangle_{\mathcal{H}}, \quad \forall f, g \in \mathcal{H},\ \forall t \in \mathbb{R}.$$

Denote by $T : \mathcal{H} \to L^q(\mathbb{R}; \mathcal{B}_\theta)$ the operator $Tf = U(t)f$. This operator might not be bounded on the cited spaces as the estimate (4.2) is assumed to hold only on spherically symmetric functions. Furthermore, the standard methods of proving Strichartz estimates are based on bounded operators, e.g. duality and the TT^*-principle, the Christ-Kiselev Lemma, …etc. Therefore, we shall proceed by restricting the operator T to spaces for which T is bounded.

Denote by $\mathcal{B}^\rho$ the subspace of the Banach function space $\mathcal{B}$ that are spherically symmetric. We would like to have that $\mathcal{B}^\rho$ is closed and therefore a Banach space itself. We shall assume that in $\mathcal{B}$ convergence in norm implies convergence in measure over sets of finite measure. This in turn implies that from every convergent sequence in $\mathcal{B}$ we can select a convergent subsequence that converges pointwise μ-a.e., which guarantees the preservation of spherical symmetry under a limit. Analogously, by $\mathcal{H}^\rho$ we denote the subspace of the Hilbert space $\mathcal{H}$ of all spherically symmetric functions in $\mathcal{H}$, which itself is a Hilbert space (under the same assumption on $\mathcal{H}$.)

Suppose that $U(t)$ commutes with spatial rotations, that is $U(t)[Rf] = R[U(t)f]$. Then we have that the operator

$$T : \mathcal{H}^\rho \to L^q(\mathbb{R}; \mathcal{B}^\rho_\theta), \quad Tf = U(t)f$$

is bounded. The formal adjoint T^* is defined through the identity

$$\int_{-\infty}^{\infty} \langle U(t)f, F(t)\rangle dt = \langle f, T^*F\rangle_{\mathcal{H}^\rho},$$

for all $f \in \mathcal{H}^\rho$, $F(t) \in L^{q'}(\mathbb{R}; \mathcal{B}_\theta^{\rho *})$, where the bracket $\langle \cdot, \cdot \rangle$ on the left denotes the duality pairing on $\mathcal{B}_\theta^\rho$. Let us find T^* explicitly. A simple computation

$$\int_{-\infty}^{\infty} \langle U(t)f, F(t) \rangle dt = \int_{-\infty}^{\infty} \langle f, U^*(s)F(s) \rangle_{\mathcal{H}^\rho} = \left\langle f, \int_{-\infty}^{\infty} U^*(s)F(s)ds \right\rangle_{\mathcal{H}^\rho} \tag{4.3}$$

yields that

$$T^*F = \int_{-\infty}^{\infty} U^*(s)F(s)ds.$$

A direct consequence of the TT^*-principle is the following

Lemma 4.1.1. *The boundedness of*

$$T : \mathcal{H}^\rho \to L^q(\mathbb{R}; \mathcal{B}_\theta^\rho), \quad Tf = U(t)f$$

is equivalent to the boundedness of

$$T^* : L^{q'}(\mathbb{R}; \mathcal{B}_\theta^{\rho *}) \to \mathcal{H}^\rho, \quad T^*F = \int_{-\infty}^{\infty} U^*(s)F(s)ds.$$

The above lemma gives us conditions when T^* is bounded. But what we really want to have is a sort of a "dual" estimate to (4.2), that is the estimate

$$\left\| \int_{-\infty}^{\infty} U^*(s)F(s)ds \right\|_{\mathcal{H}} \lesssim \|F\|_{L^q(\mathbb{R}; \mathcal{B}_\theta{}^*)},$$

whenever $F \in L^{q'}(\mathbb{R}; \mathcal{B}_\theta{}^*)$ is spherically symmetric. Thus, we need to be able to identify the space $\mathcal{B}_\theta^{\rho *}$ with the subspace of spherically symmetric functions in $\mathcal{B}_\theta^*$. A useful criterion for that is the following one. Suppose that $\mathcal{B}_\theta$ and its dual $\mathcal{B}_\theta{}^*$ satisfy

$$\|f\|_{\mathcal{B}_\theta} = \sup \Big\{ \left| \int f\psi d\mu \right| : \psi \in \mathcal{B}_\theta{}^*, \; \psi \text{ is spherically symmetric, and } \|\psi\|_{\mathcal{B}_\theta{}^*} \le 1 \Big\}, \tag{4.4}$$

$$\|\phi\|_{\mathcal{B}_\theta{}^*} = \sup \Big\{ \left| \int \phi g d\mu \right| : g \in \mathcal{B}_\theta, \; g \text{ is spherically symmetric, and } \|g\|_{\mathcal{B}_\theta} \le 1 \Big\}, \tag{4.5}$$

whenever f and ϕ are spherically symmetric functions. Then, obviously, the desired property holds. An example of such spaces is given in the next

Lemma 4.1.2. $L^p(\mathbb{R}^n)$ *and* $L^{p'}(\mathbb{R}^n)$ *are associate spaces satisfying* (4.4), (4.5), *for* $1 \le p \le \infty$ *and* $n \ge 2$.

Proof. Whenever $1 \leq p < \infty$, the supremum in (4.4) is reached on

$$\psi = sign(f)\,|f|^{p-1}\,/\,\|f\|_{L_x^p}^{p-1}\,.$$

Apparently, $\psi \in L^{p'}(\mathbb{R}^n)$, $\|\psi\|_{L_x^{p'}}=1$, and ψ is spherically symmetric if f is.

In the case when $p = \infty$ the supremum generally is not reached on a concrete function in $L^1(\mathbb{R}^n)$ but on a sequence of functions that approximate the identity. They can be taken to be spherically symmetric. □

Another desirable property of T^* is given in

Lemma 4.1.3. *Suppose that $T : \mathcal{H} \to L^q(\mathbb{R}; \mathcal{B}_\theta)$ commutes with spatial rotations, then $T^* : L^{q'}(\mathbb{R}; {\mathcal{B}_\theta}^*) \to \mathcal{H}$ does too.*

Proof. For the image $RF(t)$ of $F(t)$ under a spatial rotation R we have the identity

$$\langle U(t)f, RF(t)\rangle = \int_X U(t)f \cdot RF(t)\phi(x)dx = \int_X RU(t)f{\cdot}F(t)d\mu$$
$$\langle U(t)Rf, F(t)\rangle.$$

Thus, in view of (4.3), we obtain that

$$\Big\langle f, \int_{-\infty}^{\infty} U^*(s)RF(s)ds\Big\rangle_{\mathcal{H}} = \Big\langle Rf, \int_{-\infty}^{\infty} U^*(s)F(s)ds\Big\rangle_{\mathcal{H}}.$$

The claim now follows from the assumption (4.1). □

We conclude this section by

Theorem 4.1.4 (Inhomogeneous Strichartz estimates with spherical symmetry)**.** *Suppose that the homogeneous Strichartz estimate* (4.2) *holds for all spherically symmetric $f \in \mathcal{H}$ whenever $(q, r) \in \mathcal{A}$. Then we have that the following inhomogeneous Strichartz estimate*

$$\Big\|U(t)\int_0^t U^*(s)F(s)ds\Big\|_{L^q(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};{\mathcal{B}_{\tilde{\theta}}}^*)} \tag{4.6}$$

holds for all spherically symmetric $F(t) \in L^{\tilde{q}'}(\mathbb{R}; {\mathcal{B}_{\tilde{\theta}}}^*)$, whenever (q, θ), $(\tilde{q}, \tilde{\theta}) \in \mathcal{A}$, and $q > \tilde{q}'$ or $(q, \theta) = (\tilde{q}, \tilde{\theta})$.

Proof. As usual, we consider the TT^*-operator

$$TT^* : L^{\tilde{q}'}(\mathbb{R}; {\mathcal{B}_{\tilde{\theta}}^{\rho}}^*) \to L^q(\mathbb{R}; \mathcal{B}_\theta^\rho), \quad TT^*F = U(t)\int_{-\infty}^{\infty} U^*(s)F(s)ds,$$

where (q, θ), $(\tilde{q}, \tilde{\theta}) \in \mathcal{A}$. Obviously, TT^* commutes with spatial rotations and is bounded on the cited spaces due to the preceding lemmas. In the case when $q > \tilde{q}'$ we apply the Christ-Kiselev

Lemma, otherwise we use the Equivalence Theorem 1.3.2, part B, to conclude the proof of the claim. □

4.2 Strichartz estimates for the wave equation

Define the operators

$$\widehat{U_\pm(t)f} = e^{\pm i(t|\xi|)}\hat{f}(\xi),$$
$$U_0(t)f = (U_+(t) - U_-(t))/2iD,$$
$$W_0(t)F = \int_0^t U_0(t-s)F(s)ds,$$

where the operator D has a Fourier symbol $|\xi|$. Note that D commutes with rotations and thus preserves spherical symmetry.

Then the solution to the IVP for the wave equation

$$\Box u = F(t,x), \quad t \in [0,\infty) \times \mathbb{R}^n, \tag{4.7}$$
$$u(0) = f, \quad \partial_t u(0) = g. \tag{4.8}$$

is given by the formula

$$u(t) = \partial_t U_0(t) f + U_0(t) g + W_0(t) F.$$

For simplicity, we denote by $U_0(t)[f,g] = \partial_t U_0(t)f + U_0(t)g$ the propagation of the free wave with initial data f and g.

Definition 4.2.1. We say that the exponent pair (q,r) is radially wave-admissible if

$$\frac{1}{q} + \frac{n-1}{r} < \frac{n-1}{2}, \qquad n > 1, \tag{4.9}$$

where $2 \leq q, r \leq \infty$, $(q,r) \neq (\infty,\infty)$, or if (q,r) coincides with $(\infty, 2)$.

Theorem 4.2.2 ([18], [26])**.** *The following estimate*

$$\|U_0(t)[f,g]\|_{L_t^q L_x^r} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}}, \tag{4.10}$$

holds for all spherically symmetric $f \in \dot{H}^s(\mathbb{R}^n)$, $g \in \dot{H}^{s-1}(\mathbb{R}^n)$, whenever the exponent pair (q,r) is radially wave-admissible and the Sobolev exponent s satisfies the scaling condition

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s.$$

Theorem 4.2.3. *Let $u(t)$ be the solution to the IVP for the wave equation* (4.7), (4.8), *where*

f, g, and $F(t)$ are spherically symmetric. Then the following estimate

$$\|D^{\sigma_1}u(t)\|_{L_t^qL_x^r} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|D^{\sigma_2}F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}$$

holds for all $f \in \dot{H}^s(\mathbb{R}^n)$, $g \in \dot{H}^{s-1}(\mathbb{R}^n)$, and $D^{\sigma_2}F(t) \in L_t^{\tilde{q}'}L_x^{\tilde{r}'}$ whenever (q,r), $(\tilde{q},\tilde{r})$ are two radially wave-admissible pairs[1] *and satisfy the following scaling condition*

$$\frac{1}{q} + \frac{n}{r} - \sigma_1 = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2 - \sigma_2. \tag{4.11}$$

Proof. The homogeneous Strichartz estimates of Theorem 4.2.2 hold for each of the operators $U_\pm$ separately. For simplicity let us consider $U_-(t)$ first. For $U_-(t) : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ we have that $U_-^*(t) : L^2(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ and $U_-^*(t) = D^{-2s}U_+(t)$. In view of Theorem 4.2.2, the operators $T_1 : H^s(\mathbb{R}^n) \to L_t^qL_x^r$, $T_1f = D^{\sigma_1}U_-(t)f$, and $T_2 : H^s(\mathbb{R}^n) \to L_t^{\tilde{q}}L_x^{\tilde{r}}$, $T_2f = D^{s-\beta}U_-(t)f$ are bounded on spherically symmetric data $f \in H^s(\mathbb{R}^n)$, where

$$s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q} + \sigma_1, \quad \beta = \frac{n}{2} - \frac{n}{\tilde{r}} - \frac{1}{\tilde{q}},$$

and (q,r), $(\tilde{q},\tilde{r})$ are two radially wave-admissible pairs and $q > \tilde{q}'$. Hence, in view of Theorem 4.1.4, we obtain the estimate

$$\left\| \int_0^t U_-(t-s)D^{s-\beta-2s}F(s)ds \right\|_{L_t^qL_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}.$$

Repeating the same argument for $U_+(t)$, we obtain the estimate

$$\left\| \int_0^t W_0(t-s)F(s)ds \right\|_{L_t^qL_x^r} \lesssim \left\|D^{s+\beta-1}F\right\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}.$$

Setting $\sigma_2 = s + \beta - 1$ gives condition (4.11).

The case when (q,r), $(\tilde{q},\tilde{r})$ are two radially wave-admissible pairs with $q = \tilde{q} = 2$ and $r = \tilde{r}$ follows directly from the Equivalence Theorem 1.3.2, part B.

And finally, the case when (q,r), $(\tilde{q},\tilde{r})$ are two radially wave-admissible pairs with $q = \tilde{q} = 2$ and $r \neq \tilde{r}$ is reduced to the previous one by Sobolev embedding. □

[1] Except when $q = \tilde{q} = 2$, $r \neq \tilde{r}$, and either r or $\tilde{r}$ is equal to ∞.

Chapter 5

Applications to the Dirac-Klein-Gordon system

5.1 Introduction

The central topic of this chapter is the question of the global well-posedness of the DKG system in dimensions higher than one. This is a relativistic field model that describes nuclear interactions of subatomic particles and plays an important role in the relativistic quantum electrodynamics, see [4]. The system generates a significant mathematical interest too. Mathematically, its main feature is: a system for two quantities where there is an a-priori bound for only one of the two in the L^2-class and no positive definite energy, but at the same time a presence of a special null-form structure in both nonlinearities allowing the system to be studied at very low regularities, see [22], [15] [42], [39], [32], [16].

Now that we have obtained the necessary to our goals spherically symmetric Strichartz estimates in the preceding chapter, the main challenge will be to come up with the correct definition of spherical symmetry for spinors. It is well-known that the Dirac operator does not preserve spherical symmetry, at least not in the way one expects if one takes the erroneous attitude to treat spinors as normal functions. Thus, we investigate the action of rotations on spinor-space and define spherical symmetry for spinors to be the invariance with respect to that action. However, we do not know whether this definition has been used in the physics literature before.

The basic local existence result of the DKG system is the following

Theorem 5.1.1 (D'Ancona, Foschi, Selberg [15])**.** *Consider the IVP for the DKG system* (5.1), (5.2) *for initial data in the class* $\psi_{|t=0} = \psi_0 \in L^2$, $\phi_{|t=0} = \phi_0 \in H^r$ *and* $\partial_t \phi_{|t=0} = \phi_1 \in H^{r-1}$, *where* $1/4 < r < 3/4$. *Then there exist a time* $T > 0$, *depending continuously on the*

$L^2 \times H^r \times H^{r-1}$-norm of the data, and a solution

$$\psi \in C([0,T]; L^2), \quad \phi \in C([0,T]; H^r) \cap C^1([0,T]; H^{r-1}),$$

of the DKG system (5.1), (5.2) *on* $(0,T) \times \mathbb{R}^2$*, satisfying the initial condition above. Moreover, the solution is unique in this class, and depends continuously on the data.*

5.2 Global well-posedness of spherically symmetric solutions in 2-D

The two-dimensional DKG system reads

$$(\partial_t + \sigma_1 \partial_x + \sigma_2 \partial_y + iM\sigma_3)\psi(t,x,y) = i\phi\sigma_3\psi, \quad (t,x,y) \in [0,\infty) \times \mathbb{R} \times \mathbb{R}, \tag{5.1}$$

$$(\partial_t^2 - \Delta + m^2)\phi(t,x,y) = \langle \sigma_3\psi, \psi \rangle, \tag{5.2}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.3}$$

are the Pauli spin matrices and M and m are nonnegative constants. The unknown quantities are a two-spinor $\psi(t,x,y) : [0,\infty) \times \mathbb{R}^2 \to \mathbb{C}^2$, and a real scalar field $\phi(t,x,y) : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$.

Let us recall that the system (5.1), (5.2) is form covariant with respect to Lorentzian transformations and in particular to space rotations. Suppose that the coordinate system Oxy is changed into $Ox'y'$ by a space rotation $R(\varphi)$ of an angle φ

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then we want to find a rule $\psi \to \psi'$ as $Oxy \to Ox'y'$ of the form $\psi'(t,z') = S(\varphi)\psi(t,z)$, where $S(\varphi)$ is a 2×2 matrix and z denotes (x,y), that leaves (5.1), (5.2) form invariant. Of course, for the scalar field ϕ we have $\phi'(t,z') = \phi(t,z)$. Substituting in (5.1), (5.2)

$$\psi(t,z) = S^{-1}(\varphi)\psi'(t, R(\varphi)z),$$

$$\phi(t,z) = \phi'(t, R(\varphi)z),$$

we obtain

$$(\partial_t + \sigma_1'\partial_{x'} + \sigma_2'\partial_{y'} + iM\sigma_3')\psi'(t,z') = i\phi\sigma_3'\psi', \tag{5.4}$$

$$(\partial_t^2 - \Delta + m^2)\phi'(t,z') = \langle\sigma_3'\psi',\psi'\rangle, \tag{5.5}$$

where

$$\begin{aligned}
\sigma_1' &= S(\varphi)\,(\sigma_1\cos\varphi - \sigma_2\sin\varphi)\,S^{-1}(\varphi)\\
\sigma_2' &= S(\varphi)\,(-\sigma_1\sin\varphi + \sigma_2\cos\varphi)\,S^{-1}(\varphi)\\
\sigma_3' &= \ S(\varphi)\sigma_3 S^{-1}(\varphi).
\end{aligned}$$

Thus the matrix $S(\varphi)$ must be such that $\sigma_j' = \sigma_j$, for $j = 1,2,3$. One can check that if we set

$$S(\varphi) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix}$$

all of the above conditions are satisfied. Note that the Klein-Gordan part of the system is form invariant as $\langle\sigma_3'\psi',\psi'\rangle = \langle\sigma_3\psi,\psi\rangle$ due to the fact that $S(\varphi)$ is unitary and the well-known invariance of the Laplacian Δ with respect to rotations. Thus we come with the following

Definition 5.2.1. We say that the two-spinor $\psi_0(z) : \mathbb{R}^2 \to \mathbb{C}^2$ is spherically symmetric if it satisfies

$$\psi_0(R(\varphi)z) = S(\varphi)\psi_0(z). \tag{5.6}$$

Lemma 5.2.2. *A function $\psi_0(z) : \mathbb{R}^2 \to \mathbb{C}^2$ satisfies* (5.6) *if and only if it has the form*

$$\psi_0(z) = S(\varphi)\chi(|z|),$$

where φ is the argument of the complex number $x + iy$ and $\chi(\rho) : [0,\infty) \to \mathbb{C}^2$.

Proof. Trivial. □

Remark 5.2.3. From the explicit representation above and the fact that $e^{i\varphi} = (x+iy)/|z| \in C^\infty(\mathbb{R}^2 \setminus O)$, we see that the smoothness of ψ_0 depends on the smoothness of χ and the behavior of χ around the origin.

Lemma 5.2.4. *Suppose that IVP for* (5.1), (5.2) *has a unique solution in some class of initial data. Then for a spherically symmetric data from that class the solution to* (5.1), (5.2) *remains spherically symmetric for all time.*

Proof. Trivial. □

Lemma 5.2.5. *Suppose that $u(t)$ is the solution to the IVP for the wave equation (4.7), (4.8) in space dimension $n = 2$. Suppose that the data f and g and the forcing term $F(t)$ are spherically symmetric with $f \in H^s(\mathbb{R}^2)$, $g \in H^{s-1}(\mathbb{R}^2)$, and $F(t) \in L_t^\infty L_x^1(\mathbb{R}^2)$. Then we have the estimate*

$$\|D^s u(t)\|_{L_t^\infty L_x^2} \lesssim_T \|f\|_{\dot{H}^s(\mathbb{R}^2)} + \|g\|_{\dot{H}^{s-1}(\mathbb{R}^2)} + \|F\|_{L_t^{\tilde{q}'}([0,T];L_x^1)}, \tag{5.7}$$

for $s \in [0, 1/2)$ and $1/\tilde{q} = s$.

Proof. We apply Theorem 4.2.3 with $(q, r) = (\infty, 2)$, $(\tilde{q}, \tilde{r}) = (\tilde{q}, \infty)$, $\tilde{q} > 2$, $\sigma_1 = s$, and $\sigma_2 = 0$. □

Theorem 5.2.6. *Consider the IVP for the DKG system (5.1), (5.2), with $m = 0$, for initial data in the class $\psi_{|t=0} = \psi_0 \in L^2$, $\phi_{|t=0} = \phi_0 \in H^r$ and $\partial_t \phi_{|t=0} = \phi_1 \in H^{r-1}$, where $1/4 < r < 1/2$ and ψ_0, ϕ_0, and ϕ_1 are spherically symmetric. Then there exist a spherically symmetric solution*

$$\psi \in C((0, \infty); L^2), \quad \phi \in C((0, \infty); H^r) \cap C^1((0, \infty); H^{r-1}),$$

of the DKG system (5.1), (5.2) on $(0, \infty) \times \mathbb{R}^2$, satisfying the initial condition above. Moreover, the solution is unique in this class, and depends continuously on the data.

Proof. The fundamental conserved property of the system is the charge estimate

$$\|\psi(t)\|_{L_x^2} = \|\psi_0\|_{L_x^2}.$$

Using this, the proof follows by standard arguments from Theorem 5.1.1 and Lemma 5.2.5. □

Chapter 6

Strichartz Estimates for Some Particular Dispersive Equations

6.1 The Schrödinger equation

In this section we present the Strichartz estimates for the Schrödinger equation

$$i\partial_t u + \Delta u = F(t,x), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n, \tag{6.1}$$

$$u(0,x) = f(x). \tag{6.2}$$

Let us recall that the Schrödinger evolution group has the form $\widehat{U(t)}f = e^{it|\xi|^2}\hat{f}$ in Fourier space, and

$$U(t)f = \frac{1}{(4\pi t)^{-n/2}} \int_{-\infty}^{\infty} f(y) e^{i|x-y|/(4t)} dy, \tag{6.3}$$

in physical space. These two representations immediately yield the next two fundamental estimates

(i) the energy estimate:

$$\|U(t)f\|_{L^2_x} = \|f\|_{L^2_x}, \qquad \forall f \in \mathcal{S}, \tag{6.4}$$

(ii) the dispersive estimate:

$$\|U(t)f\|_{L^\infty_x} \lesssim \frac{1}{|t|^{n/2}} \|f\|_{L^1_x}, \qquad \forall f \in \mathcal{S}, \tag{6.5}$$

where by $\mathcal{S}$ we denote the Schwartz class on $\mathbb{R}^n$. Another fundamental property of $U(t)$ that shall play a role in our arguments is

(iii) the group property:

$$U^*(t) = U(-t), \quad U(t)U^*(s) = U(t-s). \tag{6.6}$$

We next proceed with the various definitions that shall describe the range of validity of the known Strichartz estimates for (6.1), (6.2).

Definition 6.1.1. Set

$$r^* = \begin{cases} \infty & \text{if } \sigma \leq 1, \\ \frac{\sigma}{\sigma-1} & \text{if } \sigma > 1, \end{cases} \tag{6.7}$$

Definition 6.1.2 (Keel and Tao [30])**.** We say that the exponent pair (q, r) is σ-*admissible*, whenever

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}, \quad 2 \leq q \leq \infty, \quad 2 \leq r \leq r^*, \tag{6.8}$$

apart from the case $\sigma = 1$, $(q, r) = (2, \infty)$.

The next two definitions pertain to the inhomogeneous estimates.

Definition 6.1.3 (Foschi [19])**.** We say that the exponent pair (q, r) is σ-*acceptable*, whenever

$$\frac{1}{q} + \frac{2\sigma}{r} < \sigma, \quad 1 \leq q \leq \infty, \quad 2 \leq r \leq \infty, \tag{6.9}$$

or if $(q, r) = (\infty, 2)$.

We introduce the following definition.

Definition 6.1.4. We say that the two σ-acceptable exponent pairs (q, r) and $(\tilde{q}, \tilde{r})$ are *jointly* σ-*acceptable*, whenever

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \sigma\left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right), \quad \frac{1}{q} + \frac{1}{\tilde{q}} \leq 1, \tag{6.10}$$

and if further satisfy the following restrictions

(i) if $\sigma \geq 1$: then $r, \tilde{r} < \infty$,

(ii) whenever $q > \tilde{q}'$, $1 < q, \tilde{q} < \infty$: then

$$(\sigma - 1)r \leq \sigma\tilde{r}, \quad (\sigma - 1)\tilde{r} \leq \sigma r,$$

otherwise

$$(\sigma - 1)r < \sigma\tilde{r}, \quad (\sigma - 1)\tilde{r} < \sigma r.$$

Note that for $\sigma \leq 1$ condition (ii) is void. We also have the two consequences that (i) if $q = \infty$, then $r < \tilde{r}$, and (ii) if $\tilde{q} = \infty$, then $\tilde{r} < r$. They follow directly from (6.9) and (6.10).

Definition 6.1.5. In the case when $\sigma = n/2$ we shall call an exponent pair that is σ-admissible, σ-acceptable, . . . etc, Schrödinger-admissible, Schrödinger-acceptable, respectively, . . . etc.

We are now ready to formulate the Strichartz estimates for the Schrödinger equation.

Theorem 6.1.6 (Strichartz estimates for admissible exponents [30]). *Let u be the solution to the IVP for* (6.1), (6.2). *Then the estimate*

$$\|u\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{6.11}$$

holds for all $f \in L^2(\mathbb{R}^n)$, $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'}$, if and only if (q, r) and $(\tilde{q}, \tilde{r})$ are two Schrödinger-admissible exponent pairs.

Proposition 6.1.7 (Generalized homogeneous estimates). *Suppose that (q, r) is an exponent pair satisfying*

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{p}, \tag{6.12}$$

for some $p \in (1, 2]$. Then the estimate

$$\|U(t)f\|_{L_t^{q,p} L_x^r} \lesssim \|f\|_{L_x^p}, \tag{6.13}$$

holds for every $f \in L^p(\mathbb{R}^n)$ whenever the exponents r and p are in the range

- *if $n = 1$,* $\quad 1 < p \le 2, \quad p' < r \le \infty$,
- *if $n = 2$,* $\quad 1 < p \le 2, \quad p' < r < \infty$,
- *if $n \ge 3$,* $\quad 1 < p \le 2, \quad p' < r < \frac{n}{n-2} p'$,

or if $(q, r, p) = (\infty, 2, 2)$.

Remark 6.1.8. Note that for $q \ge p$ estimate (6.13) implies the estimate

$$\|U(t)f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^p}. \tag{6.14}$$

This condition always holds for $n \le 2$.

Theorem 6.1.9 (Global inhomogeneous estimates). *The estimate*

$$\|W(t)F\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{6.15}$$

holds for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'}$, if and only if (q, r) and $(\tilde{q}, \tilde{r})$ are two jointly Schrödinger-admissible exponent pairs for $n = 1, 2$. Suppose that $n \ge 3$ and that (q, r) and $(\tilde{q}, \tilde{r})$ are two jointly Schrödinger-admissible pairs with exponents in the range

(i) $1 < q, \tilde{q} < \infty$, $q > \tilde{q}'$*: then estimate* (6.15) *holds for all $F \in L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'})$,*

(ii) $1 < q, \tilde{q} < \infty$, $q = \tilde{q}'$*: then estimate*

$$\|W(t)F\|_{L_t^q L_x^{r,q}} \lesssim \|F\|_{L_t^q L_x^{\tilde{r}',q}} \tag{6.16}$$

holds for all $F \in L^q(\mathbb{R}; L^{\tilde{r}'})$

(iii) $\tilde{q} = \infty$, $1 < q < \infty$*: then estimate*

$$\|W(t)F\|_{L_t^{q,\tilde{r}'} L_x^r} \lesssim \|F\|_{L_t^1 L_x^{\tilde{r}'}} \tag{6.17}$$

holds for every $F \in L^1(\mathbb{R}; L^{\tilde{r}'})$,

(iv) $q = \infty$, $1 < \tilde{q} < \infty$*: then estimate*

$$\|W(t)F\|_{L_t^\infty L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}',r} L_x^{\tilde{r}'}} \tag{6.18}$$

holds for every $F \in L^{\tilde{q}',r}(\mathbb{R}; L^{\tilde{r}'})$.

Note that whenever $\tilde{r}' \leq q \leq r$, then estimate (6.16) implies (6.15), whenever $q \geq \tilde{r}'$ estimate (6.17) implies estimate (6.15) and similarly, whenever $\tilde{q}' \leq r$ estimate (6.18) implies estimate (6.15).

6.2 Generalized Schrödinger-type equations

In this section we shall generalize the results of the preceding section to linear operators $U(t)$ with the following properties

(i) $U(t)$ obeys the dispersive estimate:

$$\|U(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^\sigma} \|f\|_{L_x^1}, \quad \forall f \in \mathcal{S},\ t \in \mathbb{R}, \tag{6.19}$$

for any $\sigma > 0$.

(ii) $U(t)$ obeys the energy estimate:

$$\|U(t)f\|_{L_x^2} \lesssim \|f\|_{L_x^2}, \quad \forall f \in \mathcal{S},\ t \in \mathbb{R}. \tag{6.20}$$

(iii) $U(t)$ enjoys the group property:

$$U^*(t) = U(-t), \quad U(t)U^*(s) = U(t-s). \tag{6.21}$$

The next statements are a direct consequence of this definition.

Theorem 6.2.1 (Strichartz estimates for admissible exponents, [30])**.** *The estimate*

$$\|U(t)f\|_{L_t^q L_x^r} + \|W(t)F\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{6.22}$$

holds for all $f \in L^2(\mathbb{R}^n)$, $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'}$, *whenever* (q,r) *and* $(\tilde{q},\tilde{r})$ *are two* σ*-admissible exponent pairs.*

Proposition 6.2.2 (Generalized homogeneous estimates)**.** *Suppose that* (q,r) *is an exponent pair satisfying*

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{p}, \tag{6.23}$$

for some $p \in (1,2]$*. Then the estimate*

$$\|U(t)f\|_{L_t^{q,p} L_x^r} \lesssim \|f\|_{L_x^p}, \tag{6.24}$$

holds for every $f \in L^p(\mathbb{R}^n)$ *whenever the exponents* r *and* p *are in the range*

- *if* $\sigma < 1, \quad 1 < p \leq 2, \quad p' < r \leq \infty,$
- *if* $\sigma = 1, \quad 1 < p \leq 2, \quad p' < r < \infty,$
- *if* $\sigma > 1, \quad 1 < p \leq 2, \quad p' < r < \frac{\sigma}{\sigma-1}p',$

or if $(q,r,p) = (\infty,2,2)$*.*

Remark 6.2.3. Note that for $q \geq p$ estimate (6.24) implies the estimate

$$\|U(t)f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^p}. \tag{6.25}$$

This condition always holds for $\sigma \leq 1$.

Theorem 6.2.4 (Global inhomogeneous estimates)**.** *Suppose that* (q,r) *and* $(\tilde{q},\tilde{r})$ *are two jointly* σ*-admissible exponent pairs and that* $\sigma \leq 1$*. Then the estimate*

$$\|W(t)F\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{6.26}$$

holds for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'}$*.For* $\sigma > 1$*, we consider the following cases*

(i) $1 < q, \tilde{q} < \infty$*,* $q > \tilde{q}'$*: then estimate* (6.26) *holds for all* $F \in L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'})$*,*

(ii) $1 < q, \tilde{q} < \infty$*,* $q = \tilde{q}'$*: then the estimate*

$$\|W(t)F\|_{L_t^q L_x^{r,q}} \lesssim \|F\|_{L_t^q L_x^{\tilde{r}',q}} \tag{6.27}$$

holds for all $F \in L^q(\mathbb{R}; L^{\tilde{r}'})$

(iii) $\tilde{q} = \infty$*,* $1 < q < \infty$*: then the estimate*

$$\|W(t)F\|_{L_t^{q,\tilde{r}'} L_x^r} \lesssim \|F\|_{L_t^1 L_x^{\tilde{r}'}} \tag{6.28}$$

holds for every $F \in L^1(\mathbb{R}; L^{\tilde{r}'})$*,* *(iv)* $q = \infty$, $1 < \tilde{q} < \infty$*: then the estimate*

$$\|W(t)F\|_{L_t^\infty L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}',r} L_x^{\tilde{r}'}} \tag{6.29}$$

holds for every $F \in L^{\tilde{q}',r}(\mathbb{R}; L^{\tilde{r}'})$.

Note that whenever $\tilde{r}' \leq q \leq r$, estimate (6.27) implies (6.26), whenever $q \geq \tilde{r}'$, estimate (6.28) implies estimate (6.26) and similarly, whenever $\tilde{q}' \leq r$, estimate (6.29) implies estimate (6.26).

Remark 6.2.5. Parts (i) and (ii) are originally proven by Foschi [19] and independently in the context of the Schrödinger equation by Vilela [49]. There is an earlier result by Kato [29] in the context of the Schrödinger equation that contains estimates similar to parts (i) - (iv) but, however, in more restricted range and based on a less sophisticated method.

6.3 The wave equation

The IVP for the wave equation reads

$$\Box u(t,x) = F(t,x), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n, \tag{6.30}$$

$$u(0) = f, \quad \partial_t u(0) = g. \tag{6.31}$$

Then the solution to the homogeneous wave equation is given by the formula

$$\frac{U(t)+U(-t)}{2} f + \frac{U(t)-U(-t)}{2iD} g,$$

where D is the operator of fractional differentiation with symbol $|\xi|$.

The inhomogeneous operator $W(t)$ is defined in the usual way, see (1.3), and thus the solution $w(t)$ to the inhomogeneous wave equation with zero initial conditions is given by

$$w(t) = \frac{W(t)-W(-t)}{2iD} F,$$

provided that supp $F \subseteq [0,\infty) \times \mathbb{R}^n$.

Typically, the dispersive estimate for the wave equation is given by

$$\|U(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^\sigma} \|f\|_{L_x^1}, \quad \sigma = (n-1)/2, \tag{6.32}$$

where f is a frequency localized initial data away from the origin, e.g. supp $\hat{f} \subseteq \{1 \leq |\xi| \leq 2\}$. Let $\{\phi_k\}_{-\infty}^{\infty}$ be a homogeneous Littlewood-Paley dyadic decomposition on $\mathbb{R}^n$. By standard scaling arguments, see [41], estimate (6.32) can be sharpened to

$$\|U(t)\phi_k * f\|_{L^\infty_x} \lesssim \frac{2^{(n-\sigma)k}}{|t|^\sigma} \|\phi_k * f\|_{L^1_x}, \tag{6.33}$$

for all $k \in \mathbb{Z}$ and any $f \in L^1(\mathbb{R}^n)$. We further rework the dispersive estimate by multiplying (6.33) by $2^{-(n-\sigma)k/2}$ and take the l^2-norm to obtain the Besov norm formulation of dispersive estimate

$$\|U(t)f\|_{\dot{B}^{-\beta}_{\infty,2}} \lesssim \frac{1}{|t|^\sigma} \|f\|_{\dot{B}^{\beta}_{1,2}}, \tag{6.34}$$

where $\beta = (n+1)/4$, $\sigma = (n-1)/2$, and $f \in \dot{B}^{\beta}_{1,2}$.

It is not hard to see that $U(t)$ obeys the energy estimate $\|U(t)f\|_{L^2_x} = \|f\|_{L^2_x}$, enjoys the group property $U^*(t) = U(-t)$, $U(t)U^*(s) = U(t-s)$, and that $U(t)$ commutes with fractional differentiation, i.e. $U(t)D^\alpha f = D^\alpha U(t)f$.

Definition 6.3.1. We say that the exponent pair (q,r) is *nonsharply σ-admissible* whenever

$$\frac{1}{q} + \frac{\sigma}{r} \le \frac{\sigma}{2}, \quad 2 \le q \le \infty, \quad 2 \le r \le \infty, \tag{6.35}$$

apart from

- $\sigma \le 1$, $(q,r) = (2/\sigma, \infty)$,
- $\sigma > 1$, $(q,r) = (2, \infty)$,
- $(q,r) = (\infty, \infty)$.

Remark 6.3.2. Note that definition 6.3.1 generally allows $r = \infty$ apart from the three endpoint cases given above. The estimates with $r = \infty$ are proven by making use of an Gargliano-type interpolation inequality that first appeared in [17], see Proposition 7.4.2.

Definition 6.3.3. We say that the two jointly σ-acceptable exponent pairs (q,r) and $(\tilde{q},\tilde{r})$ are *nonsharply jointly σ-acceptable*, whenever

$$\frac{1}{q} + \frac{1}{\tilde{q}} \le \sigma \left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right). \tag{6.36}$$

Definition 6.3.4. We shall call an exponent pair that is σ-admissible, σ-acceptable, ...etc, wave-admissible, wave-acceptable, respectively, ...etc, in the case when $\sigma = (n-1)/2$.

Theorem 6.3.5 (Strichartz estimates for wave-admissible exponents [30])**.** *The estimate*

$$\|u(t)\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|D^\rho F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x},$$

holds for all $f \in \dot{H}^s(\mathbb{R}^n)$, $g \in \dot{H}^{s-1}(\mathbb{R}^n)$ and $D^\rho F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x$, if and only if (q,r) and $(\tilde{q},\tilde{r})$ are two nonsharply wave-admissible exponent pairs and the Sobolev exponents s and ρ fulfill condition

$$\frac{1}{q}+\frac{n}{r}=\frac{n}{2}-s=\frac{1}{\tilde{q}'}+\frac{n}{\tilde{r}'}-2-\rho. \tag{6.37}$$

Proposition 6.3.6 (Generalized homogeneous estimates)**.** *Let $u(t)$ be the solution to the IVP for the wave equation* (6.30)*,* (6.31)*. The estimate*

$$\|u(t)\|_{L_t^{q,p}L_x^r} \lesssim \|D^s f\|_{L_x^p} + \left\|D^{s-1} g\right\|_{L_x^p}, \tag{6.38}$$

holds for all f and g such that $D^s f \in L^p(\mathbb{R}^n)$, $D^{s-1} g \in L^p(\mathbb{R}^n)$ and $n > 1$, whenever the Lebesgue exponent q, r and p are such that

$$\frac{1}{q}+\frac{n-1}{2r}=\frac{n-1}{2p},$$

and according to the dimension n, lie in the range

- $n = 2, \quad 1 < p \leq 2, \quad p' < r \leq \infty,$
- $n = 3, \quad 1 < p \leq 2, \quad p' < r < \infty,$
- $n > 3, \quad 1 < p \leq 2, \quad p' < r < \frac{n-1}{n-3}p',$

and the Sobolev exponent s satisfies

$$s=\frac{n}{p}-\frac{1}{q}-\frac{n}{r}.$$

Theorem 6.3.7 (Global inhomogeneous estimates)**.** *Suppose that (q,r) and $(\tilde{q},\tilde{r})$ are two nonsharply jointly wave-acceptable exponent pairs with $r,\tilde{r} < \infty$, and $n = 2,3$ (i.e. $\sigma \leq 1$). Then the solution $w(t)$ to the IVP for the wave equation* (6.30)*,* (6.31)*, with $f = g = 0$, enjoys the estimate*

$$\|w(t)\|_{L_t^q L_x^r} \lesssim \|D^\rho F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

for all F such that $D^\rho F \in L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'})$, whenever the Sobolev exponent ρ fulfills the dimensional condition

$$\frac{1}{q}+\frac{n}{r}=\frac{1}{\tilde{q}'}+\frac{n}{\tilde{r}'}-2-\rho. \tag{6.39}$$

The interpolation argument that allowed us to include $r,\tilde{r} = \infty$ in the context of Theorem 6.3.5 cannot be reproduced for pairs that are not nonsharply admissible, hence the restriction $r,\tilde{r} < \infty$.

The inhomogeneous Strichartz estimates for the wave equation of Theorem 6.3.7 are formulated only in the small dimensions $n = 2,3$, where they can be stated especially simply. For the corresponding estimates in higher dimensions see section 6.6.

6.4 The Klein-Gordon equation

The IVP for the Klein-Gordon (KG) equation reads

$$\Box u(t,x) + u(t,x) = F(t,x), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n, \tag{6.40}$$

$$u(0) = f, \quad \partial_t u(0) = g. \tag{6.41}$$

Equation (6.40) deserves a special attention since it has a stronger decay than the wave equation. However, this does not necessarily mean that the Strichartz estimates for the Klein-Gordon equation are better than those for the wave equation in a sense of a bigger gain of regularity. The reason for that lies in the fact that the gain in decay rate is payed for a greater regularity assumptions on the initial data. This might not be always desirable, especially if one is interested in solutions of low regularity. However, the dispersive estimate for the KG equation offers a flexibility to trade between the rate of decay at large times and the initial regularity of the data. We shall base the Strichartz estimates for the KG equation on that ground and obtain a whole family of estimates for a given space dimension. Originally, this fact was exploited by Machihara et al [34] to circumvent the lack of an L^2_t-type estimate in $\mathbb{R}^3$ (when $\sigma = 1$ for the wave equation) that had obstructed the study of a nonlinear Dirac equation at almost critical regularity. The estimates of this section extend the estimates of [34] to non-admissible exponents.

We define the Klein-Gordon evolution group U by $\widehat{U}(t)f = e^{it\langle\xi\rangle}\hat{f}$ and by Λ^α we denote the inhomogeneous operator of fractional differentiation with symbol $\langle\xi\rangle^\alpha$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. It is not hard to see that $U(t)$ has the group property $U^*(t) = U(-t)$, $U(t)U^*(s) = U(t-s)$, and that $U(t)$ commutes with fractional differentiation, i.e. $U(t)\Lambda^\alpha f = \Lambda^\alpha U(t) f$.

We now recall the following dispersive estimate

$$\|U(t)\phi_k * f\|_{L^\infty_x} \lesssim \frac{2^{2\beta(\theta)k}}{|t|^{2\beta(\theta)-1}} \|\phi_k * f\|_{L^1_x}, \tag{6.42}$$

for the Klein-Gordon equation from [34] and the references therein, where

$$\beta(\theta) = \frac{n+1+\theta}{4}, \quad 0 \le \theta \le 1,$$

and $\{\phi_k\}_0^\infty$ is an inhomogeneous Littlewood-Paley dyadic decomposition on $\mathbb{R}^n$. As in the case with the wave equation we multiply both sides by $2^{-\beta(\theta)}$ and take the l^2-norm to get the dispersive estimate in terms of Besov norms

$$\|U(t)f\|_{B^{-\beta(\theta)}_{\infty,2}} \lesssim \frac{1}{|t|^{2\beta(\theta)-1}} \|f\|_{B^{\beta(\theta)}_{1,2}}. \tag{6.43}$$

Thus we can vary the rate of dispersion

$$\sigma(\theta) = \frac{n-1+\theta}{2} =: \sigma_\theta \tag{6.44}$$

in (6.43) with $\theta \in [0,1]$. Note that in the dispersive estimate (6.43) the difference between the decay rate $\sigma_\theta = 2\beta(\theta) - 1$ and the regularity of the initial data equal to $2\beta(\theta)$, if measured in terms of generalized derivatives in the Besov space $\dot{B}^0_{1,2}$, remains constant with θ. However, observe also that as the forbidden L^2_t-type estimate occurs for $\sigma_\theta = 1$ it is not anymore fixed to the spatial dimension $n = 3$.

Theorem 6.4.1 (Strichartz estimates for admissible exponents)**.** *Let $u(t)$ be the solution to the IVP for the Klein-Gordon equation* (6.40), (6.41)*. The estimate*

$$\|u\|_{L^q_t L^r_x} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}} + \|\Lambda^\rho F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x},$$

holds for all $f \in H^s(\mathbb{R}^n)$, $g \in H^{s-1}(\mathbb{R}^n)$ and $\Lambda^\rho F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x$, whenever (q,r) and $(\tilde{q},\tilde{r})$ are two nonsharply σ_θ-admissible exponent pairs ($\sigma_\theta > 0$) and the Sobolev exponents s and ρ fulfill the dimensional condition

$$\frac{1}{q} + \frac{n}{r} - \frac{\theta}{\sigma_\theta q} = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - \frac{\theta}{\sigma_\theta \tilde{q}'} + \frac{\theta}{\sigma_\theta} - 2 - \rho. \tag{6.45}$$

Proposition 6.4.2 (Generalized homogeneous estimates)**.** *Let $u(t)$ be the solution to the IVP for the KG equation* (6.40), (6.41)*. The estimate*

$$\|U(t)f\|_{L^{q,p}_t L^r_x} \lesssim \|\Lambda^s f\|_{L^p_x} + \left\|\Lambda^{s-1} g\right\|_{L^p_x}, \tag{6.46}$$

holds for all f and g such that $\Lambda^s f \in L^p(\mathbb{R}^n)$, $\Lambda^{s-1} g \in L^p(\mathbb{R}^n)$, whenever the Lebesgue exponent q, r and p are such that

$$\frac{1}{q} + \frac{\sigma_\theta}{r} = \frac{\sigma_\theta}{p},$$

and according to $\sigma_\theta > 0$, lie in the range

- $\sigma_\theta < 1, \quad 1 < p \le 2, \quad p' < r \le \infty,$
- $\sigma_\theta = 1, \quad 1 < p \le 2, \quad p' < r < \infty,$
- $\sigma_\theta > 1, \quad 1 < p \le 2, \quad p' < r < \frac{\sigma_\theta}{\sigma_\theta - 1} p',$

and the Sobolev exponent s satisfies

$$s = \frac{n}{p} - \frac{1}{q} - \frac{n}{r} + \frac{\theta}{\sigma_\theta q}.$$

Theorem 6.4.3 (Global inhomogeneous estimates). *Suppose that (q, r) and $(\tilde{q}, \tilde{r})$ are two nonsharply jointly σ_θ-acceptable exponent pairs with $r, \tilde{r} < \infty$, and $0 < \sigma_\theta \leq 1$. Then the solution $w(t)$ to the IVP for the Klein-Gordon equation* (6.40), (6.41), *with $f = g = 0$, enjoys the estimate*

$$\|w(t)\|_{L_t^q L_x^r} \lesssim \|\Lambda^\rho F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

for all F such that $\Lambda^\rho F \in L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'})$, whenever the Sobolev exponent ρ fulfills the dimensional condition

$$\frac{1}{q} + \frac{n}{r} - \frac{\theta}{\sigma_\theta q} = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - \frac{\theta}{\sigma_\theta \tilde{q}'} + \frac{\theta}{\sigma_\theta} - 2 - \rho. \tag{6.47}$$

6.5 The Dirac equation

The Dirac equation is a first-order wave equation with matrix-valued coefficients. We shall distinguish the following two cases

- massless Dirac

$$i\partial_t \psi + i\alpha \cdot \nabla \psi = 0, \tag{6.48}$$

- massive Dirac

$$i\partial_t \psi + i\alpha \cdot \nabla \psi + \beta \psi = 0. \tag{6.49}$$

In both equations the spinor field $\psi(t, x) : \mathbb{R}^{1+n} \to \mathbb{C}^N$ maps $\mathbb{R}^{1+n}$ to a column vector $\psi(t, x) = (\psi_1(t, x), \ldots, \psi_N(t, x))^t$ in $\mathbb{C}^N$, where $N = 2^{\lfloor \frac{n+1}{2} \rfloor}$. We use the abbreviations

$$\alpha \cdot \nabla = \alpha_1 \partial_1 + \ldots + \alpha_n \partial_n, \quad \partial_j = \partial/\partial x_j.$$

The matrices $\alpha_j \in M_N(\mathbb{C})$, $j = 1 \ldots n$ and $\beta \in M_N(\mathbb{C})$ are the well-known Dirac matrices. If we multiply the Dirac equations (6.48), (6.49) by β we obtain a new set of coefficients γ^μ, $\mu = 0 \ldots n$, where $\gamma^0 = \beta$ and $\gamma^j = \beta\alpha_j$, $j = 1 \ldots n$. The commutation properties of the gamma matrices give the identities

$$(\gamma^\mu \partial_\mu)^2 = \Box I_N, \quad (\gamma^\mu \partial_\mu + I_N)^2 = (\Box + 1) I_N. \tag{6.50}$$

where I_N is the identity in $M_N(\mathbb{C})$ and Einstein's summation convention is used. One can use the identities (6.50) to define the gamma matrices and through them α and β but there are more than one (equivalent) sets of matrix representations that satisfy (6.50).

The unknown $\psi(t, x)$ is a spinor, which loosely speaking means that ψ changes under change of coordinates by a specific rule. For more details about the nature of that rule see Bjorken and Drell [4] or chapter 5. In a fixed coordinate system, however, and with a fixed representation of the Dirac matrices ψ can be regarded as an ordinary vector-valued function.

Due to the identities (6.50) a free wave to (6.48) satisfies the homogeneous wave equation componentwise and a free wave to (6.49) satisfies the homogeneous Klein-Gordon equation componentwise. Therefore, the Strichartz estimates to the wave and the Klein-Gordon equation from the previous subsections also apply to the Dirac equation after some minor adjustment.

Let us for the sake of concreteness consider the IVP for the massive Dirac equation

$$i\partial_t\psi + i\alpha\cdot\nabla\psi + \beta\psi = F(t,x), \qquad (t,x)\in[0,\infty)\times\mathbb{R}^n, \tag{6.51}$$

$$\psi(0) = \psi_0. \tag{6.52}$$

Note that the rate of dispersion σ_θ of the massive Dirac equation is given by (6.44).

Theorem 6.5.1 (Strichartz estimates for admissible exponents)**.** *Let $\psi(t)$ be the solution to the IVP for the massive Dirac equation* (6.51), (6.52)*. The estimate*

$$\|\psi\|_{L_t^q L_x^r} \lesssim \|\psi_0\|_{H^s} + \|\Lambda^\rho F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

holds for all $\psi_0\in H^s(\mathbb{R}^n)$ and $\Lambda^\rho F\in L_t^{\tilde{q}'}L_x^{\tilde{r}'}$, whenever (q,r) and $(\tilde{q},\tilde{r})$ are two nonsharply σ_θ-admissible exponent pairs ($\sigma_\theta>0$) and the Sobolev exponents s and ρ fulfill condition

$$\frac{1}{q}+\frac{n}{r}-\frac{\theta}{\sigma_\theta q} = \frac{n}{2}-s = \frac{1}{\tilde{q}'}+\frac{n}{\tilde{r}'}-\frac{\theta}{\sigma_\theta \tilde{q}'}+\frac{\theta}{\sigma_\theta}-1-\rho. \tag{6.53}$$

Proposition 6.5.2 (Generalized homogeneous estimates)**.** *Let $\psi(t)$ be the solution to the IVP for the massive Dirac equation* (6.51), (6.52)*. The estimate*

$$\|\psi(t)\|_{L_t^{q,p} L_x^r} \lesssim \|\Lambda^s\psi_0\|_{L_x^p}, \tag{6.54}$$

holds for all ψ_0 such that $\Lambda^s\psi_0\in L^p(\mathbb{R}^n)$, whenever the Lebesgue exponent q, r and p are such that

$$\frac{1}{q}+\frac{\sigma_\theta}{r} = \frac{\sigma_\theta}{p},$$

and according to $\sigma_\theta>0$, lie in the range

- $\sigma_\theta<1, \quad 1<p\le 2, \quad p'<r\le\infty,$
- $\sigma_\theta=1, \quad 1<p\le 2, \quad p'<r<\infty,$
- $\sigma_\theta>1, \quad 1<p\le 2, \quad p'<r<\frac{\sigma_\theta}{\sigma_\theta-1}p',$

and the Sobolev exponent s satisfies

$$s = \frac{n}{p}-\frac{1}{q}-\frac{n}{r}+\frac{\theta}{\sigma_\theta q}.$$

Theorem 6.5.3 (Global inhomogeneous estimates)**.** *Suppose that* (q,r) *and* $(\tilde{q},\tilde{r})$ *are two nonsharply jointly* σ_θ*-acceptable exponent pairs with* $r,\tilde{r} < \infty$*, and* $0 < \sigma_\theta \le 1$*. Then the solution* $\chi(t)$ *to the IVP for the massive Dirac equation* (6.51), (6.52)*, with* $\psi_0 = 0$*, enjoys the estimate*

$$\|\chi(t)\|_{L^q_t L^r_x} \lesssim \|\Lambda^\rho F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x},$$

for all F *such that* $\Lambda^\rho F \in L^{\tilde{q}'} L^{\tilde{r}'}$*, whenever the Sobolev exponent* ρ *fulfills condition*

$$\frac{1}{q} + \frac{n}{r} - \frac{\theta}{\sigma_\theta q} = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - \frac{\theta}{\sigma_\theta \tilde{q}'} + \frac{\theta}{\sigma_\theta} - 1 - \rho. \tag{6.55}$$

6.6 Generalized wave-type equations

In this section we shall generalize the results of the preceding sections to abstract linear operators $U(t)$ with the following properties

(i) $U(t)$ obeys the dispersive estimate

$$\|U(t)f\|_{B^{-\beta}_{\infty,2}} \lesssim \frac{1}{|t|^\sigma} \|f\|_{B^{\beta}_{1,2}}, \tag{6.56}$$

where $\sigma > 0,\ 0 \le \beta < \frac{n}{2}$, and $f \in \mathcal{S}$, the Schwartz class of rapidly decaying functions on $\mathbb{R}^n$.

(ii) $U(t)$ obeys the energy estimate

$$\|U(t)f\|_{L^2_x} \lesssim \|f\|_{L^2_x}, \quad \forall f \in \mathcal{S},\ t \in \mathbb{R}. \tag{6.57}$$

(iii) $U(t)$ enjoys the group property

$$U^*(t) = U(-t), \quad U(t)U^*(s) = U(t-s). \tag{6.58}$$

(iv) $U(t)$ commutes with fractional differentiation

$$U(t)\Lambda^\alpha = \Lambda^\alpha U(t). \tag{6.59}$$

Again, for the sake of concreteness we formulate the estimates in terms of inhomogeneous Besov norms. The case of homogeneous norms is completely analogous and can be treated by replacing everywhere the inhomogeneous Besov norms with homogeneous ones.

Theorem 6.6.1 (Strichartz estimates for admissible exponents)**.** *The estimate*

$$\|U(t)f\|_{L^q(\mathbb{R};B^{-\rho}_{r,2})} + \|W(t)F\|_{L^q(\mathbb{R};B^{-\rho}_{r,2})} \lesssim \|f\|_{L^2_x} + \|F\|_{L^{\tilde{q}'}(\mathbb{R};B^{\tilde{\rho}}_{\tilde{r}',2})}, \tag{6.60}$$

holds for all $f \in L^2$, $F \in L^{\tilde{q}'} B^{\tilde{\rho}}_{\tilde{r}',2}$, *whenever* (q,r) *and* $(\tilde{q},\tilde{r})$ *are two* σ*-admissible exponent pairs and the smoothness exponents* $\rho(r)$ *and* $\tilde{\rho}(\tilde{r})$ *fulfill condition*

$$\rho(r) = 2\beta\left(\frac{1}{2} - \frac{1}{r}\right). \tag{6.61}$$

Corollary 6.6.2. *The estimate*

$$\|U(t)f\|_{L^q_t L^r_x} + \|W(t)F\|_{L^q_t L^r_x} \lesssim \|f\|_{H^s} + \|\Lambda^\alpha F\|_{L^{\tilde{q}'} L^{\tilde{r}'}},$$

holds for all $f \in H^s$, $\Lambda^\alpha F \in L^{\tilde{q}'} L^{\tilde{r}'}$, *whenever* (q,r) *and* $(\tilde{q},\tilde{r})$ *are two nonsharply* σ*-admissible exponent pairs and the Sobolev exponents* α *and* s *fulfill condition*

$$\frac{n-2\beta}{\sigma q} + \frac{n}{r} = \frac{n}{2} - s = \frac{n-2\beta}{\sigma \tilde{q}'} + \frac{n}{\tilde{r}'} - \frac{n-2\beta}{\sigma} - \alpha. \tag{6.62}$$

Proposition 6.6.3 (Generalized homogeneous estimates)**.** *The estimate*

$$\|U(t)f\|_{L^{q,p}_t L^r_x} \lesssim \|\Lambda^s f\|_{L^p_x},$$

holds for all f *such that* $\Lambda^s f \in L^p(\mathbb{R}^n)$, *whenever the Lebesgue exponent* q, r *and* p *are such that*

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{p},$$

and according to $\sigma > 0$, *lie in the range*

- $\sigma < 1, \quad 1 < p < 2, \quad p' < r \leq \infty,$
- $\sigma = 1, \quad 1 < p < 2, \quad p' < r < \infty,$
- $\sigma > 1, \; r^{*\prime} < p < 2, \quad p' < r < \frac{\sigma}{\sigma - 1} p',$

and the Sobolev exponent s *satisfies*

$$s = \frac{n}{p} - \frac{n}{r} + \frac{n-2\beta}{\sigma q}.$$

Remark 6.6.4. In particular, for $\sigma \leq 1$ we have that

$$\|U(t)f\|_{L^q_t L^r_x} \lesssim \|\Lambda^s f\|_{L^p_x}$$

since then the inequality $p \leq q$ always holds.

Theorem 6.6.5 (Global inhomogeneous estimates)**.** *Suppose that* (q,r) *and* $(\tilde{q},\tilde{r})$ *are two jointly* σ*-acceptable exponent pairs and the smoothness exponents* $\rho = \rho(r)$ *and* $\tilde{\rho} = \rho(\tilde{r})$ *fulfill* *condition (6.61). Then the operator $W(t)$ obeys the estimate*

$$\|W(t)F\|_{L^q(\mathbb{R};B^{-\rho}_{r,2})} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};B^{\tilde{\rho}}_{\tilde{r}',2})}\,, \tag{6.63}$$

for all $F \in L^{\tilde{q}'}(\mathbb{R}; B^{\tilde{\rho}}_{\tilde{r}',2})$, whenever $\sigma \le 1$. If $\sigma > 1$ we consider different cases

(i) The point $(1/q, 1/\tilde{q})$ lies inside $\triangle OAB$ in fig. 2.1, that is $1 < q, \tilde{q} < \infty$ and $q > \tilde{q}'$: then $W(t)$ satisfies (6.63) for all $F \in L^{\tilde{q}'}(\mathbb{R}; B^{\tilde{\rho}}_{\tilde{r}',2})$.

(ii) The point $(1/q, 1/\tilde{q})$ lies on the hypotenuse AB, that is $1 < q, \tilde{q} < \infty$ and $q = \tilde{q}'$: then $W(t)$ satisfies (6.63) *if $q \le 2$, or otherwise*

$$\|W(t)F\|_{L^q(\mathbb{R};B^{-\rho}_{r,q})} \lesssim \|F\|_{L^q(\mathbb{R};B^{\tilde{\rho}}_{\tilde{r}',q})}$$

for all $F \in L^q(\mathbb{R}; B^{\tilde{\rho}}_{\tilde{r}',2})$.

(iii) The point $(1/q, 1/\tilde{q})$ lies on the side OA, that is $1 < q < \infty$, $\tilde{q} = \infty$: then $W(t)$ satisfies

$$\|W(t)F\|_{L^{q,\tilde{r}'}(\mathbb{R};B^{-\rho}_{r,2})} \lesssim \|F\|_{L^1(\mathbb{R};B^{\tilde{\rho}}_{\tilde{r}',2})}$$

for all $F \in L^{\tilde{q}'}(\mathbb{R}; B^{\tilde{\rho}}_{\tilde{r}',2})$.

(iv) The point $(1/q, 1/\tilde{q})$ lies on the side OB, that is $1 < \tilde{q} < \infty$, $q = \infty$, then $W(t)$ satisfies

$$\|W(t)F\|_{L^\infty(\mathbb{R};B^{-\rho}_{r,2})} \lesssim \|F\|_{L^{\tilde{q}',r}(\mathbb{R};B^{\tilde{\rho}}_{\tilde{r}',2})}$$

for all $F \in L^{\tilde{q}',r}(\mathbb{R}; B^{\tilde{\rho}}_{\tilde{r}',2})$.

Corollary 6.6.6. *By the usual embeddings between the Besov and Sobolev spaces, estimate* (6.63) *implies estimate*

$$\|W(t)F\|_{L^q H^{-\rho}_r} \lesssim \|F\|_{L^{\tilde{q}'} H^{\tilde{\rho}}_{\tilde{r}'}}\,, \tag{6.64}$$

whenever $2 \le r, \tilde{r} < \infty$.

6.7 Abstract vector-valued equations

In this section we shall present the Strichartz estimates in the abstract setting. We shall work with Banach spaces $\mathcal{B}$ whose norms are denoted by $\|\cdot\|_{\mathcal{B}}$. Thus the evolution operator $U(t) : \mathcal{S} \to \mathcal{B}$ maps for any fixed time $t \in \mathbb{R}$ a space $\mathcal{S}$ of some nice functions to an element of the Banach space $\mathcal{B}$, meaning that the images are assumed to be smooth enough and to decay rapidly enough so that all formal operations in the sequel are justified. In the context of the initial value problem (IVP) for a partial differential equation (PDE) $\mathcal{S}$ is typically considered to be a dense subset of the space of the initial data f since we have that $U(0)f = f$. However, the latter property of $U(t)$ has no bearing on what follows.

Consider a Banach spaces $\mathcal{B}_0$ and a Hilbert space $\mathcal{B}_1$ with the properties: (i) they are compatible for interpolation (see [2, 1]), (ii) ${\mathcal{B}_0}^*$ and ${\mathcal{B}_1}^*$ have a common dense subset $\mathcal{S}$ of nice functions in a sense described earlier, and (iii) they have the same duality pairing $\langle \cdot, \cdot \rangle$. Let us explain what is meant by (iii) on a concrete example. Suppose that $\mathcal{B}_0$ and $\mathcal{B}_1$ are spaces of measurable functions over some measure space $(X, d\mu)$. Then the duality pairing for both spaces has the very simple form

$$\langle f, g \rangle = \int_X f g d\mu.$$

In fact, in all applications to concrete equations we shall deal with such spaces only.

By $\mathcal{B}_\theta$ we shall denote the complex interpolation space $(\mathcal{B}_0, \mathcal{B}_1)_\theta$, for $\theta \in [0,1]$, and by $\mathcal{B}_{\theta,q}$ we shall denote the real interpolation space $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}$, for $\theta \in [0,1]$, $1 \le q \le \infty$.

The Strichartz estimates that we shall present in this work follow from the following assumptions on the operator $U(t) : \mathcal{S} \to \mathcal{B}_0 \cup \mathcal{B}_1$:

(i) $U(t)$ obeys the dispersive estimate

$$\|U(t)f\|_{\mathcal{B}_0} \lesssim \frac{1}{|t|^\sigma} \|f\|_{{\mathcal{B}_0}^*}, \quad \sigma > 0, \tag{6.65}$$

(ii) $U(t)$ obeys the energy estimate

$$\|U(t)f\|_{\mathcal{B}_1} \lesssim \|f\|_{\mathcal{B}_1}, \quad \forall t \in \mathbb{R}. \tag{6.66}$$

(iii) $U(t)$ enjoys the group property

$$U^*(t) = U(-t), \quad U(t)U^*(s) = U(t-s), \tag{6.67}$$

(iv) $U(t)$ enjoys the following regularity property

$$U(t)f \in C(\mathbb{R}; {\mathcal{B}_\theta}^*) \tag{6.68}$$

for all $\theta \in [0,1]$, $f \in \mathcal{S}$.

To formulate the Strichartz estimates in the abstract setting in what follows we shall often use norms of the form

$$\|G\|_{L^q(\mathbb{R};\mathcal{B})} = \left(\int_{-\infty}^{\infty} \|G(t)\|_{\mathcal{B}}^q \, dt \right)^{1/q},$$

where $G : \mathbb{R} \to \mathcal{B}$ maps $\mathbb{R}$ to some Banach space $\mathcal{B}$.

Before we begin we would like to remark that, for example, in Keel and Tao [30] the energy estimate (6.66) is given a slightly more general formulation

$$\|U(t)f\|_{\mathcal{B}_1} \lesssim \|f\|_{\mathcal{H}}, \quad \forall f \in \mathcal{S},\ t \in \mathbb{R}, \tag{6.69}$$

for some Hilbert space $\mathcal{H}$ and some Banach space $\mathcal{B}_1$ with $\mathcal{H} \neq \mathcal{B}_1$ in general. However, this causes some difficulties not present in formulation (6.66). One difficulty is the fact that estimate (6.69) implies the inhomogeneous estimate

$$\|W(t)F\|_{L^\infty(\mathbb{R};\mathcal{B}_1)} \lesssim \|F\|_{L^1(\mathbb{R};\mathcal{H})}\,, \tag{6.70}$$

which is of different nature compared to the standard family of inhomogeneous Strichartz estimates

$$\|W(t)F\|_{L^q(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}\,. \tag{6.71}$$

Then, the description of the full set of inhomogeneous estimates would become more involved as it would include estimates that are interpolations between (6.70) and (6.71). What is more, from the viewpoint of applications, formulation (6.66) is quite natural as it reflects the principle in physics of energy conservation more closely than (6.69).

The Strichartz estimates for $U(t)$ have the form

$$\|U(t)f\|_{L^q(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_1}\,, \tag{6.72}$$

and more generally

$$\|U(t)f\|_{L^q(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta}}{}^*}\,. \tag{6.73}$$

For comparison estimate (6.72) correspond to estimate (6.14) with $p = 2$ and estimate (6.73) corresponds to estimate (6.14) with $1 < p \leq 2$ in the context of the Schrödinger equation. The inhomogeneous estimates have the form

$$\|W(t)F\|_{L^q(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}\,. \tag{6.74}$$

The global in time estimates (6.73), (6.74) have local counterparts

$$\|U(t)f\|_{L^q([0,T];\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_1}\,, \tag{6.75}$$

$$\|W(t)F\|_{L^q([0,T];\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}'}([0,T];\mathcal{B}_{\tilde{\theta}}{}^*)}\,, \tag{6.76}$$

for any $0 < T \leq \infty$. This follows immediately if we consider the localized operator $U(t)\chi_{[0,T]}$, where $\chi_{[0,T]}$ is the characteristic function of the interval $[0,T]$, and see that it as well as $U(t)$ satisfies conditions (6.65) - (6.68).

We continue with the necessary definitions that describe the range of validity of the homogeneous and inhomogeneous Strichartz estimates (6.72), (6.73) and (6.74).

Remark 6.7.1 (Mnemonic rule). The abstract definitions below can be easily remembered from the more familiar definitions for the Schrödinger equation if one replaces $2/r$ by θ.

Definition 6.7.2. Set

$$\begin{cases} \theta^* = 0, & \text{if} \quad \sigma \leq 1, \\ \theta^* = (\sigma - 1)/\sigma, & \text{if} \quad \sigma > 1. \end{cases} \tag{6.77}$$

Definition 6.7.3 ([30])**.** We say that the exponent pair (q, θ) is σ-*admissible*, whenever

$$\frac{1}{q} = \frac{\sigma}{2}(1 - \theta), \quad 2 \leq q \leq \infty, \quad \theta^* \leq \theta \leq 1, \tag{6.78}$$

apart from the case $\sigma = 1$, $(q, \theta) = (2, 0)$.

We shall call a pair (q, θ) endpoint if $\sigma \geq 1$ and $(q, \theta) = (2, \theta^*)$. Note that definition 6.7.3 forbids the endpoint $\sigma = 1$, $(q, \theta) = (2, 0)$ but allows all higher-dimensional endpoints for $\sigma > 1$. The following definitions pertain to the inhomogeneous estimates.

Definition 6.7.4 ([19])**.** We say that the exponent pair (q, θ) is σ-*acceptable*, whenever

$$\frac{1}{q} < \sigma(1 - \theta), \quad 1 \leq q \leq \infty, \quad \theta \in [0, 1), \tag{6.79}$$

or if $(q, \theta) = (\infty, 1)$.

We introduce the following definition.

Definition 6.7.5. We say that the two σ-acceptable exponent pairs (q, θ) and $(\tilde{q}, \tilde{\theta})$ are *jointly σ-acceptable*, whenever

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{\sigma}{2}\left(2 - \theta - \tilde{\theta}\right), \tag{6.80}$$

and if further satisfy the following restrictions

(i) if $\sigma \geq 1$: then $0 < \theta, \tilde{\theta}$,

(ii) whenever $q > \tilde{q}'$, $1 < q, \tilde{q} < \infty$: then

$$(\sigma - 1)\theta \leq \sigma\tilde{\theta}, \quad (\sigma - 1)\tilde{\theta} \leq \sigma\theta,$$

otherwise

$$(\sigma - 1)\theta < \sigma\tilde{\theta}, \quad (\sigma - 1)\tilde{\theta} < \sigma\theta.$$

Note that for $\sigma \leq 1$ condition (ii) is void. We also have the two consequences: (i) if $q = \infty$, then $\tilde{\theta} < \theta$, and (ii) if $\tilde{q} = \infty$, then $\theta < \tilde{\theta}$. They follow directly from (6.79) and (6.80). We shall call an inhomogeneous Strichartz estimate with exponent pairs (q,θ), $(\tilde q,\tilde\theta)$ *endpoint* if (i) $q=\tilde q'$, which can only happen if $\sigma\geq 1$, (ii) if $q=\infty$, and (iii) if $\tilde q=\infty$.

We next formulate the Strichartz estimates in the abstract setting.

Theorem 6.7.6 (Estimates for admissible exponents)**.** *The estimate*

$$\|U(t)f\|_{L^q(\mathbb{R};\mathcal{B}_\theta)}+\|W(t)F\|_{L^q(\mathbb{R};\mathcal{B}_\theta)}\lesssim\|f\|_{\mathcal{B}_1}+\|F\|_{L^{\tilde q'}(\mathbb{R};\mathcal{B}_{\tilde\theta}{}^*)}\,, \tag{6.81}$$

holds for all $f\in\mathcal{B}_1$, $F\in L^{\tilde q'}(\mathbb{R};\mathcal{B}_\theta{}^*)$, whenever (q,θ) and $(\tilde q,\tilde\theta)$ are two σ-admissible exponent pairs, and (q,θ) is not an endpoint pair.

Proposition 6.7.7 (Generalized homogeneous estimates)**.** *Suppose that (q,θ) is an exponent pair satisfying*

$$\frac{1}{q}=\frac{\sigma}{2}\left(2-\theta-\tilde\theta\right),$$

for some $\tilde\theta\in(0,1]$. Then the estimate

$$\|U(t)f\|_{L^{q,\infty}(\mathbb{R};\mathcal{B}_\theta)}\lesssim\|f\|_{\mathcal{B}_{\tilde\theta}{}^*}\,, \tag{6.82}$$

holds for every $f\in\mathcal{B}^*_{\tilde\theta}$ whenever the exponents θ and $\tilde\theta$ are in the range

- $\sigma<1,\qquad 0<\tilde\theta\leq 1,\qquad 0\leq\theta<\tilde\theta,$
- $\sigma=1,\qquad 0<\tilde\theta\leq 1,\qquad 0<\theta<\tilde\theta,$
- $\sigma>1,\qquad 0<\tilde\theta\leq 1,\qquad \frac{\sigma-1}{\sigma}\tilde\theta\leq\theta<\tilde\theta,$

or if $(q,\theta,\tilde\theta)=(\infty,1,1)$.

Theorem 6.7.8 (Global inhomogeneous estimates)**.** *Suppose that (q,θ) and $(\tilde q,\tilde\theta)$ are two jointly σ-acceptable exponent pairs. Then the operator $W(t)$ obeys the estimate*

$$\|W(t)F\|_{L^q(\mathbb{R};\mathcal{B}_\theta)}\lesssim\|F\|_{L^{\tilde q'}(\mathbb{R};\mathcal{B}_{\tilde\theta}{}^*)} \tag{6.83}$$

for all $F\in L^{\tilde q'}(\mathbb{R};\mathcal{B}_{\tilde\theta}{}^*)$ whenever $\sigma\leq 1$. If $\sigma>1$ we consider different cases

(i) the point $(1/q,1/\tilde q)$ lies inside $\triangle OAB$ in fig. 2.1, that is $1<q,\tilde q<\infty$ and $q>\tilde q'$: then $W(t)$ satisfies (6.83) for all $F\in L^{\tilde q'}(\mathbb{R};\mathcal{B}_{\tilde\theta}{}^*)$.

(ii) The point $(1/q,1/\tilde q)$ lies on the hypotenuse AB, that is $1<q,\tilde q<\infty$ and $q=\tilde q'$: then $W(t)$ satisfies

$$\|W(t)F\|_{L^q(\mathbb{R};\mathcal{B}_{\theta,q})}\lesssim\|F\|_{L^q(\mathbb{R};\mathcal{B}_{\tilde\theta,\tilde q}{}^*)}$$

for all $F\in L^q(\mathbb{R};\mathcal{B}_{\tilde\theta,\tilde q}{}^*)$.

(iii) The point $(1/q, 1/\tilde{q})$ lies on the side OA, that is $1 < q < \infty$, $\tilde{q} = \infty$: then $W(t)$ satisfies

$$\|W(t)F\|_{L^{q,\infty}(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^1(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}\,, \tag{6.84}$$

for all $F \in L^1(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)$.

(iv) The point $(1/q, 1/\tilde{q})$ lies on the side OB, that is $1 < \tilde{q} < \infty$, $q = \infty$: then $W(t)$ satisfies

$$\|W(t)F\|_{L^\infty(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}',1}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)} \tag{6.85}$$

for all $F \in L^{\tilde{q}',1}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)$.

Remark 6.7.9. The appearance of Lorentz norms in some of the estimates above is not a great obstacle to applications. Indeed, if we restrict to finite time intervals $[0,T]$, we have the continuous embeddings

$$L^{q,r}([0,T]) \hookrightarrow L^p([0,T]), \quad q > p,\ 1 \le q,p,r \le \infty,$$
$$L^p([0,T]) \hookrightarrow L^{q,r}([0,T]), \quad p > q,\ 1 \le q,p,r \le \infty,$$

see [1, p. 217]. For example, let q, θ and $\tilde{\theta}$ be such that estimate (6.82) holds and let $1 \le Q < q$. Then we have the local homogeneous estimate

$$\|U(t)f\|_{L^Q([0,T];\mathcal{B}_\theta)} \lesssim_T \|f\|_{\mathcal{B}_{\tilde{\theta}}{}^*}\,.$$

Similarly, if for example $\tilde{q}$, θ and $\tilde{\theta}$ are such that estimate (6.85) holds and $1 \le \tilde{Q} < \tilde{q}$, then we have the local inhomogeneous estimate

$$\|W(t)F\|_{L^\infty([0,T];\mathcal{B}_\theta)} \lesssim_T \|F\|_{L^{\tilde{Q}'}([0,T];\mathcal{B}_{\tilde{\theta}}{}^*)}\,.$$

Chapter 7

Proofs

In this chapter we shall prove the estimates we presented in the sections of the preceding chapter. We shall give complete and rigorous proofs only in the abstract formulation of the estimates. The estimates for the concrete equations we considered so far shall be derived as consequences of the abstract estimates.

We begin with a revision of the TT^*-principle in the abstract setting.

7.1 The TT^*-principle

Lemma 7.1.1. *[21, p. 56], [41, p. 113] Let T be a linear operator, $\mathcal{B}$ be a Banach space, and $\mathcal{H}$ be a Hilbert space. The following statements are equivalent:*

(i) $T : \mathcal{H} \to \mathcal{B}$ is bounded,

(ii) $T^* : \mathcal{B}^* \to \mathcal{H}$ is bounded,

(iii) $TT^* : \mathcal{B}^* \to \mathcal{B}$ is bounded.

Furthermore, we have the following equality of operator norms $\|T\|^2 = \|T^*\|^2 = \|TT^*\|$.

A few remarks are due. The second source [41] contains the proof of this lemma in the important and technically uncomplicated setting of $\mathcal{B} = L^p$, for $1 \leq p \leq \infty$, $\mathcal{H} = L^2$. The general case is presented in the first source [21] and the references therein. The general proof is word for word the same as the L^p-case if one replaces the L^p-symbol with that of the Banach space $\mathcal{B}$. We notice, however, that in the context of the Lebesgue spaces the lemma holds with $\mathcal{B} = L^\infty$ but instead of the dual space to L^∞ we can use its associate L^1.

We next present an important consequence of the TT^*-principle that shall play a role in the proof of the inhomogeneous Strichartz estimates. Suppose that the bounded linear operator $T : \mathcal{B}_1 \to L^q(\mathbb{R}; \mathcal{B}_\theta)$ is of the form $Tf = U(t)f$. Then its formal adjoint $T^* : L^{q'}(\mathbb{R}; {\mathcal{B}_\theta}^*) \to \mathcal{B}_1$ is a bounded linear operator of the form $\int_{\mathbb{R}} U^*(t)F(t)dt$. We shall the call exponent pair (q, θ)

admissible. Suppose that T is bounded for two admissible pairs (q,θ) and $(\tilde{q},\tilde{\theta})$. Then the composition of $T:\mathcal{B}_1\to L^q(\mathbb{R};\mathcal{B}_\theta)$ with $T^*:L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)\to\mathcal{B}_1$ is the bounded operator

$$TT^*:L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)\to L^q(\mathbb{R};\mathcal{B}_\theta),\quad TT^*F=\int_{\mathbb{R}}U(t-s)F(s)ds.$$

Notice the similarity between TT^*F and

$$W(t)F=\int_{-\infty}^{t}U(t-s)F(s)ds.$$

The boundedness of the operator $W(t):L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)\to L^q(\mathbb{R};\mathcal{B}_\theta)$ easily implies that of the $TT^*:L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)\to L^q(\mathbb{R};\mathcal{B}_\theta)$ under some minor assumptions on $U(t)$. The proof of that statement follows if we consider that

$$\int_{\mathbb{R}}=\int_{-\infty}^{t}+\int_{t}^{\infty}$$

in the definition of the TT^*-operator and then make a change of variables in the second integral on the right to transform it to an integral like the first one on the right. The details are left as an exercise and we now address the more important question of when the boundedness of the TT^*-operator implies that of $W(t)$. In general this implication holds whenever $q>\tilde{q}'$ and there are known counterexamples to the limiting case $q=\tilde{q}'$. This is due to the celebrated Lemma 8.0.14 of Christ-Kiselev. This combination of the TT^*-principle with the Christ-Kiselev Lemma is the standard way of obtaining Strichartz estimates for $W(t)$ from the estimates for $U(t)$. In passing we remark that in the symmetric case $(q,\theta)=(\tilde{q},\tilde{\theta})$ the opposite is also true which is an easy consequence of Lemma 7.1.2.

The TT^*-principle for Strichartz estimates can be recast in a bilinear formulation which is more effective.

Lemma 7.1.2 (Keel and Tao [30]). *Consider the bilinear form*

$$B(F,G)=\iint_{s<t}\langle U^*(s)F,U^*(t)G\rangle dsdt. \tag{7.1}$$

(i) The boundedness of the operator $T:\mathcal{H}\to L^q(\mathbb{R};\mathcal{B}_\theta)$ of the form $Tf=U(t)f$ is equivalent to the boundedness of the bilinear mapping $B:L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*)\times L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*)\to\mathbb{C}$.

(ii) The boundedness of the operator $W(t):L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)\to L^q(\mathbb{R};\mathcal{B}_\theta)$ is equivalent to that of the bilinear mapping $B:L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*)\times L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)\to\mathbb{C}$.

As with the KT equation we consider the decompositions

$$\Omega=\bigcup_{\lambda}\bigcup_{Q\in\mathcal{O}_\lambda},\quad B(F,G)=\sum_{\lambda}\sum_{Q\in\mathcal{O}_\lambda}B_Q(F,G),$$

where

$$B_Q(F,G) = \iint_Q \langle U^*(s)F(s), U^*(t)G(t)\rangle ds dt.$$

The advantage of the above decomposition is that whenever $Q = J \times I$ and $Q \in \mathcal{O}_\lambda$ we have

$$\lambda = |I| = |J| \sim \operatorname{dist}(\Omega, \partial\Omega) \sim \operatorname{dist}(I, J). \tag{7.2}$$

The very special property (7.2) of this decomposition allows us to obtain the following scaling invariance

$$|B_Q(F,G)| \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \|F\|_{L^{\tilde{q}'}(J;\mathcal{B}_{\tilde{\theta}}{}^*)} \|G\|_{L^{q'}(I;\mathcal{B}_\theta{}^*)}, \tag{7.3}$$

of each dyadic piece B_Q in the bilinear form B. The latter shall be proved in section 7.3.1 and in particular Lemma 7.3.4 gives a certain range for the ordered 4-tuple of exponents (q,θ), $(\tilde{q},\tilde{\theta})$, where the local scaling (7.3) holds true. Another scaling invariant quantity is given by

Lemma 7.1.3. *If* $\frac{1}{q} + \frac{1}{\tilde{q}} \leq 1$, *then*

$$\sum_{Q\in\mathcal{O}_\lambda} |B_Q(F,G)| \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*)}.$$

Proof. In view of (7.3)

$$\sum_{Q\in\mathcal{O}_\lambda} |B_Q(F,G)| \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \sum_{Q\in\mathcal{O}_\lambda,\, Q=J\times I} \|F\|_{L^{\tilde{q}'}(J;\mathcal{B}_{\tilde{\theta}}{}^*)} \|G\|_{L^{q'}(I;\mathcal{B}_\theta{}^*)}.$$

An application of Lemma 7.1.4 below completes the proof. □

Lemma 7.1.4. *Suppose* $\frac{1}{p} + \frac{1}{\tilde{p}} \geq 1$. *Then*

$$\sum_{Q\in\mathcal{O}_\lambda,\ Q=J\times I} \|f\|_{L^{\tilde{p}}(J)} \|g\|_{L^p(I)} \leq \|f\|_{L^{\tilde{p}}(\mathbb{R})} \|g\|_{L^p(\mathbb{R})}.$$

Proof. The lemma follows directly from the inequality

$$\sum_j |a_j b_j| \leq \left(\sum_j |a_j|^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}} \left(\sum_j |b_j|^p\right)^{\frac{1}{p}},$$

which holds in the range $\frac{1}{p} + \frac{1}{\tilde{p}} \geq 1$, and the fact that for each dyadic interval I there are at most two dyadic squares in $\mathcal{O}_\lambda$ with side I. □

Consider the bilinear operator $A : L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*) \to l_s^\infty$, defined by the formula

$$A(F,G) = \{b_\lambda\}_{\lambda\in 2^{\mathbb{Z}}} = \left\{ \sum_{Q\in\mathcal{O}_\lambda} |B_Q(F,G)| \right\}_{\lambda\in 2^{\mathbb{Z}}}.$$

Thus, in view of the bilinear formulation of the TT^* in Lemma 2.3.9, the estimate

$$\|\{b_\lambda\}\|_{l_0^1} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*)}, \quad \forall F \in L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*),\ \forall G \in L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*),$$

implies the boundedness of $W(t) : L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*) \to L^q(\mathbb{R};\mathcal{B}_\theta)$.

7.2 Estimates for admissible exponents

7.2.1 The basic case

Let us recall

Theorem 7.2.1. *The estimate*

$$\|U(t)f\|_{L^q(\mathbb{R};\mathcal{B}_\theta)} + \|W(t)F\|_{L^q(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_1} + \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}, \tag{7.4}$$

holds for all $f \in \mathcal{B}_1$, $F \in L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_\theta{}^*)$, whenever (q,θ) and $(\tilde{q},\tilde{\theta})$ are two σ-admissible exponent pairs. In the double endpoint case $(q,\theta) = (\tilde{q},\tilde{\theta}) = (2,\theta^*)$, though, the norm of the space $L^2(\mathbb{R};\mathcal{B}_{\theta^*})$ has to be replaced by that of the space $L^2(\mathbb{R};\mathcal{B}_{\theta^*,2})$.

Proof. In view of the TT^*-principle, to prove the boundedness of the operator $U(t) : \mathcal{B}_1 \to L^q(\mathbb{R};\mathcal{B}_\theta)$ whenever (q,θ) is σ-admissible it is enough to prove the boundedness of the operator $TT^* : L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*) \to L^q(\mathbb{R};\mathcal{B}_\theta)$. Note that TT^*-operator is a convolution operator for which we can apply some standard techniques from Analysis. We begin with complex interpolation between the dispersive estimate (6.56) and the energy estimate (6.57) to obtain the following decay estimate

$$\|U(t)f\|_{\mathcal{B}_\theta} \lesssim \frac{1}{|t|^{\sigma(1-\theta)}} \|f\|_{\mathcal{B}_{\theta^*}}, \quad \theta \in [0,1].$$

Using this, we obtain

$$\|TT^*F\|_{\mathcal{B}_\theta} \lesssim \int_{-\infty}^{\infty} \|U(t-s)F(s)\|_{\mathcal{B}_\theta}\, ds \lesssim \int_{-\infty}^{\infty} \frac{\|F(s)\|_{\mathcal{B}_\theta{}^*}}{|t-s|^{\sigma(1-\theta)}} ds. \tag{7.5}$$

We now take the L^q-norm in t in both sides of the above inequality. To estimate the right hand side (RHS), we apply the Hardy-Littlewood-Sobolev theorem of fractional integration, see [1, pp. 228-229], [41]. Thus we obtain

$$\|TT^*F\|_{L^q(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*)},$$

whenever $0 < \sigma(1-\theta) < 1$, $1 + 1/q = 1/q' + \sigma(1-\theta)$. The latter conditions are equivalent to $\theta^* < \theta < 1$, $2/q = \sigma(1-\theta)$. Remember that the exponent $\sigma(1-\theta) = \alpha$ must be in $(0,1)$ in order to apply the above argument. However, the left endpoint ($\alpha = 0$), i.e. $\theta = 1$, coincides with the energy estimate (6.66). The right endpoint ($\alpha = 1$), i.e. $\theta = \theta^*$, (recall definition (6.77)), is too delicate to be resolved by this argument. The corresponding estimate is the endpoint homogeneous Strichartz estimate

$$\|U(t)f\|_{L^2(\mathbb{R};\mathcal{B}_{\theta^*})} \lesssim \|f\|_{\mathcal{B}_1}, \quad \forall f \in \mathcal{B}_1. \tag{7.6}$$

This estimate has been proved false for many concrete equations when $\sigma = 1$. In higher dimensions, when $\sigma > 1$, Keel and Tao showed that the modified estimate

$$\|U(t)f\|_{L^2(\mathbb{R};\mathcal{B}_{\theta^*,2})} \lesssim \|f\|_{\mathcal{B}_1}, \quad \forall f \in \mathcal{B}_1$$

always holds. Of course, in the special case when $\mathcal{B}_\theta$ are Lebesgue spaces L^r with $r \geq 2$, this estimate implies the original one.

In view of the Christ-Kiselev Lemma 8.0.14, the above implies the inhomogeneous estimate

$$\|W(t)f\|_{L^q(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}, \tag{7.7}$$

whenever (q,θ), $(\tilde{q},\tilde{\theta})$ are two σ-admissible exponent pairs with $q > \tilde{q}'$. The double endpoint case $(q,\theta) = (\tilde{q},\tilde{\theta}) = (2,\theta^*)$ follows from the Equivalence Theorem 1.3.2, part B. □

7.2.2 Generalized global homogeneous estimates

Proposition 7.2.2. *Suppose that (q,θ) is an exponent pair satisfying*

$$\frac{1}{q} = \frac{\sigma}{2}\left(2 - \theta - \tilde{\theta}\right),$$

for some $\tilde{\theta} \in (0,1]$. Then the estimate

$$\|U(t)f\|_{L^{q,c}(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta},c'}{}^*}, \quad 1 \leq c \leq \infty, \tag{7.8}$$

holds for every $f \in \mathcal{B}_{\tilde{\theta},c'}{}'$ whenever the exponents θ and $\tilde{\theta}$ are in the range

- $\sigma < 1, \qquad 0 < \tilde{\theta} \leq 1, \qquad 0 \leq \theta < \tilde{\theta},$
- $\sigma = 1, \qquad 0 < \tilde{\theta} \leq 1, \qquad 0 < \theta < \tilde{\theta},$
- $\sigma > 1, \qquad 0 < \tilde{\theta} \leq 1, \qquad \theta^* \leq \theta < \tilde{\theta},$

or if $(q,\theta,\tilde{\theta}) = (\infty, 1, 1)$.

Proof. Suppose at first that $\sigma \neq 1$. We interpolate with the real method with parameters η, c, for $0 < \eta < 1$, $1 \le c \le \infty$, in the two inequalities below

$$\|U(t)f\|_{L^{\xi(\theta),\infty}(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\theta^*}}, \quad \theta^* \le \theta \le 1, \tag{7.9}$$

$$\|U(t)f\|_{L^{q}(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{1^*}}, \quad \theta^* \le \theta \le 1, \tag{7.10}$$

where $1/\xi(\theta) = \sigma(1-\theta)$. In view of the reiteration theorem, see [1], we obtain the estimate

$$\|U(t)f\|_{L^{Q,c}(\mathbb{R};\mathcal{B}_{[\theta]})} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta},c'}{}^*},$$

where

$$\frac{1}{Q} = \frac{1-\eta}{\xi} + \frac{\eta}{q}, \quad 0 < \eta < 1,$$
$$\tilde{\theta} = \theta(1-\eta) + \eta.$$

Expressing ξ and q in terms of θ and eliminating η from these equations, we obtain the equivalent conditions

$$\frac{1}{Q} = \frac{\sigma}{2}(2-\theta-\tilde{\theta}), \quad \theta < \tilde{\theta} < 1, \quad \theta^* \le \theta \le 1.$$

Relabeling Q by q and reformulating the inequalities above as $\theta^* < \tilde{\theta} < 1$, $\theta^* \le \theta < \tilde{\theta}$, we finish the proof in the case $\sigma \ne 1$.

The case $\sigma = 1$ is treated in exactly the same way but this time estimates (7.9), (7.10) are valid only in the range $\theta^* < \theta \le 1$. □

Note that instead of real interpolation we can use the complex method, which yields the alternative estimate

$$\|U(t)f\|_{L^{q,\infty}(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta}^*}}. \tag{7.11}$$

The argument of this section is a generalization of an argument of Kato [29], originally presented in the specific context of the Schrödinger equation. We shall further extend the range of these estimates for $\sigma > 1$ in section 7.3.4.

7.3 Estimates for acceptable exponents

This section is dedicated to the proof of the global inhomogeneous Strichartz estimates of Theorem 6.7.8 which is to be done considering several different case. We begin with the proof of some local estimates that shall be crucial in the sequel.

7.3.1 Local inhomogeneous estimates

Following Foschi [19], we want to find the range of local estimates for $W(t)$ that are invariant to the scaling

$$\|W(t)[\chi_{\lambda J}F]\|_{L^q(\lambda I;\mathcal{B}_\theta)} \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \|F\|_{L^{\tilde{q}'}(\lambda J;\mathcal{B}_{\tilde{\theta}}{}^*)}, \quad \forall\lambda > 0, \tag{7.12}$$

where I and J are two unit intervals separated by a unit distance and $\chi_{\lambda J}$ is the characteristic of the rescaled interval λJ and

$$\beta(q,\theta,\tilde{q},\tilde{\theta}) = \frac{1}{q} + \frac{1}{\tilde{q}} - \frac{\sigma}{2}(2-\theta-\tilde{\theta}). \tag{7.13}$$

The bilinear formulation of (7.12) is

$$|B_Q(F,G)| \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \|F\|_{L^{\tilde{q}'}(J;\mathcal{B}_{\tilde{\theta}}{}^*)} \|G\|_{L^{q'}(I;\mathcal{B}_\theta{}^*)}, \tag{7.14}$$

where Q is the square $I \times J$.

Lemma 7.3.1. *Estimate* (7.12) *holds for any two σ-admissible pairs* (q,θ) *and* $(\tilde{q},\tilde{\theta})$*.*

Proof. The proof follows trivially from Theorem 6.1.6 due to the fact that $\beta(q,\theta,\tilde{q},\tilde{\theta}) = 0$ under the hypothesis of the lemma. □

Lemma 7.3.2. *Estimate* (7.12) *holds with* $(q,\theta) = (\tilde{q},\tilde{\theta}) = (\infty, 0)$*.*

Proof. By the dispersive estimate (6.65) we have that

$$\begin{aligned}\sup_{t\in\lambda I} \|W(t)[\chi_{\lambda J}F]\|_{\mathcal{B}_0} &\lesssim \sup_{t\in\lambda I} \int_{\lambda J} \frac{\|F(\tau)\|_{\mathcal{B}_0{}^*}}{|t-\tau|^\sigma} d\tau \\ &\lesssim \lambda^{\beta(\infty,0,\infty,0)} \|F\|_{L^1(\lambda I;\mathcal{B}_0{}^*)}.\end{aligned}$$

□

Lemma 7.3.3. *Whenever* (q,θ) *and* $(\tilde{q},\tilde{\theta})$ *are exponent pairs for which estimate* (7.12) *holds, we have that* (7.12) *also holds with* (Q,θ) *and* $(\tilde{Q},\tilde{\theta})$*, where* $1 \le Q \le q$, $1 \le \tilde{Q} \le \tilde{q}$*.*

Proof. A trivial application of Hölder's inequality

$$\begin{aligned}\|W(t)[\chi_{\lambda J}F]\|_{L^Q(\lambda I;\mathcal{B}_\theta)} &\lesssim \lambda^{\frac{1}{Q}-\frac{1}{q}} \|W(t)[\chi_{\lambda J}F]\|_{L^q(\lambda I;\mathcal{B}_\theta)} \\ &\lesssim \lambda^{\beta(Q,r,\tilde{q},\tilde{r})} \|F\|_{L^{\tilde{q}'}(\lambda J;\mathcal{B}_{\tilde{\theta}}{}^*)} \lesssim \lambda^{\beta(Q,r,\tilde{Q},\tilde{r})} \|F\|_{L^{\tilde{q}'}(\lambda J;\mathcal{B}_{\tilde{\theta}}{}^*)}.\end{aligned}$$ □

Let us define the range of validity of the local estimates (7.12) as the set $\mathcal{E}$ in $\mathbb{R}^4$. Each point in $\mathcal{E}$ corresponds to a 4-tuple of exponents $(1/q,\theta,1/\tilde{q},\tilde{\theta})$. Below we find the convex hull $\mathcal{E}^*$ ($\mathcal{E}^* \subseteq \mathcal{E}$) of the points in $\mathbb{R}^4$ that correspond to the estimates in the three lemmas above. We shall call any point or collection of points in $\mathcal{E}$ *acceptable.*

Lemma 7.3.4 (Local inhomogeneous estimates)**.** *Estimate* (7.12)*, or equivalently* (7.14)*, holds whenever the exponent pairs* (q,θ), $(\tilde{q},\tilde{\theta}) \in \mathcal{E}^*$ *given explicitly by the following conditions*

$$0 \le \frac{1}{q}, \frac{1}{\tilde{q}} \le 1, \quad 0 \le \theta, \tilde{\theta} \le 1, \tag{7.15}$$

$$\frac{\sigma}{2}(\tilde{\theta} - \theta) \le \frac{1}{q}, \quad \frac{\sigma}{2}(\theta - \tilde{\theta}) \le \frac{1}{\tilde{q}}, \tag{7.16}$$

$$(\sigma - 1)\tilde{\theta} \le \sigma\theta, \quad (\sigma - 1)\theta \le \sigma\tilde{\theta}. \tag{7.17}$$

If $\sigma \ge 1$ *then also* $\theta, \tilde{\theta} > 0$.

Remark 7.3.5. Condition (7.17) is void when $\sigma \le 1$.

Proof. We apply the Riesz-Thorin convexity theorem to interpolate between the already proven local estimates. In essence, we find the convex hull of the locally acceptable sets associated with Lemmas 7.3.1 and 7.3.2 and then expand that set by the rule given in Lemma 7.3.3.

When $\sigma \neq 1$ the set of acceptability S_1 of the local estimates in Lemma 7.3.1 is given by the system

$$S_1 = \begin{cases} \dfrac{1}{q} = \dfrac{\sigma}{2}(1-\theta), \quad \dfrac{1}{\tilde{q}} = \dfrac{\sigma}{2}(1-\tilde{\theta}), \\ \quad \theta^* \le \theta \le 1, \quad \theta^* < \tilde{\theta} \le 1. \end{cases} \tag{7.18}$$

Interpolating with $O = (0,0,0,0)$ we get

$$S_2 = \begin{cases} \dfrac{1}{Q} = \dfrac{\eta}{q}, \quad \dfrac{1}{\tilde{Q}} = \dfrac{\eta}{\tilde{q}}, \\ \Theta = \eta\theta \quad \tilde{\Theta} = \eta\tilde{\theta}, \quad 0 \le \eta \le 1. \end{cases}$$

And lastly, applying to S_2 the rule of Lemma 7.3.3 we get

$$S_3 = \begin{cases} 1 \ge \dfrac{1}{Q} \ge \dfrac{\eta}{q}, \quad 1 \ge \dfrac{1}{\tilde{Q}} \ge \dfrac{\eta}{\tilde{q}}, \\ \quad \Theta = \eta\theta \quad \tilde{\Theta} = \eta\tilde{\theta}, \qquad 0 \le \eta \le 1. \end{cases} \tag{7.19}$$

We need to eliminate from the definition of S_3 the following variables $q, \tilde{q}, \theta, \tilde{\theta}$ and η. By expressing q and $\tilde{q}$ in terms of θ and $\tilde{\theta}$ respectively, see (7.18), we simplify the four inequalities in (7.19) to

$$0 \le \frac{1}{Q}\frac{1}{\tilde{Q}} \le 1,$$

$$\eta \le \frac{2}{\sigma Q} + \Theta, \quad \eta \le \frac{2}{\sigma\tilde{Q}} + \tilde{\Theta}.$$

The two equalities in (7.19) are simplified to

$$\eta\theta^* < \Theta \le \eta, \quad \eta\theta^* < \tilde{\Theta} \le \eta, \quad 0 \le \eta \le 1.$$

Let us now group all similar inequalities for η

$$0, \Theta, \tilde{\Theta} \le \eta, \tag{7.20}$$

$$\eta < \frac{\Theta}{\theta^*}, \frac{\tilde{\Theta}}{\theta^*}, \tag{7.21}$$

$$\eta \le \frac{2}{\sigma Q} + \Theta, \frac{2}{\sigma \tilde{Q}} + \tilde{\Theta}, 1. \tag{7.22}$$

There is $\eta \in [0,1]$ solving the inequalities in (7.20), (7.21), (7.22), if and only if any quantity on the left is bounded by any quantity on the right in these inequalities. This gives the lemma, i.e. that $S_3 = \mathcal{E}^*$. □

7.3.2 Non-endpoint global inhomogeneous estimates

Our goal in this subsection shall be to show the boundedness of

$$A : L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta}{}^*) \to l^1 \tag{7.23}$$

whenever the ordered 4-tuple (q,θ), $(\tilde{q},\tilde{\theta})$ is non-endpoint in a certain sense, recall the notation introduced at the end of section 7.1.

Suppose that $(1/q, 1/\tilde{q}) \in \Delta_0$, where $\Delta_0 = \{1/q > 0, 1/\tilde{q} > 0, 1/q + 1/\tilde{q} < 1\}$, and that the 4-tuple (q,θ), $(\tilde{q},\tilde{\theta}) \in \mathcal{E}^*$, together with a neighborhood of small perturbations in $(1/q, 1/\tilde{q})$. Then in virtue of Corollary 7.1.3 we have the estimate

$$|b_\lambda| \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_{\theta}{}^*)},$$

or in other words $\{b_\lambda\} \in l^\infty_{\beta(q,\theta,\tilde{q},\tilde{\theta})}$. Since Δ_0 is an open set (triangle) on the $(1/q, 1/\tilde{q})$-coordinate plane, we can always find a small enough open neighborhood of points in Δ_0 around $(1/q, 1/\tilde{q})$. Let us set

$$1/q_0 = 1/q + \epsilon, \quad 1/\tilde{q}_0 = 1/\tilde{q} + \epsilon, \quad 1/q_1 = 1/q - 3\epsilon, \quad 1/\tilde{q}_1 = 1/\tilde{q} - 3\epsilon.$$

Suppose that $\epsilon > 0$ is small enough so that $(1/q_0, 1/\tilde{q}_0), (1/q_1, 1/\tilde{q}_1) \in \Delta \cap \mathcal{E}^*$. Suppose also that, cf. (7.13), $\beta(q,\theta,\tilde{q},\tilde{\theta}) = 0$. Then we have that $\beta(q_0,\theta,\tilde{q}_0,\tilde{\theta}) = 2\epsilon$, and $\beta(q_1,\theta,\tilde{q}_0,\tilde{\theta}) =$

$\beta(q_0, \theta, \tilde{q}_1, \tilde{\theta}) = -2\epsilon$. Thus we obtain the maps

$$A : L^{\tilde{q}_0'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q_0'}(\mathbb{R}; \mathcal{B}_{\theta}{}^*) \to l^\infty_{2\epsilon},$$
$$A : L^{\tilde{q}_0'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q_1'}(\mathbb{R}; \mathcal{B}_{\theta}{}^*) \to l^\infty_{-2\epsilon},$$
$$A : L^{\tilde{q}_1'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q_0'}(\mathbb{R}; \mathcal{B}_{\theta}{}^*) \to l^\infty_{-2\epsilon},$$

are bounded. In virtue of Lemma 8.0.7 we have that the map

$$A : (L^{\tilde{q}_0'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*), L^{\tilde{q}_1'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*))_{1/4,\tilde{q}'} \times (L^{q_0'}(\mathbb{R}; \mathcal{B}_{\theta}{}^*), L^{q_1'}(\mathbb{R}; \mathcal{B}_{\theta}{}^*))_{1/4,q'} \to (l^\infty_{2\epsilon}, l^\infty_{-2\epsilon})_{1/2,1}$$

is also bounded. Finally, in view of the well-known interpolation identities of the Lorentz spaces and that of Lemma 8.0.6, this implies (7.23)

Now let us recapitulate all conditions we have imposed so far on the exponents. We have the conditions of the local estimates (set $\mathcal{E}^*$) plus the scaling condition $\beta(q, \theta, \tilde{q}, \tilde{\theta}) = 0$. Remember that all inequalities in the definition of $\mathcal{E}^*$ that contain q or $\tilde{q}$ must be rewritten as strict inequalities to allow perturbation in these quantities. Also note that conditions (7.16) together with $\beta = 0$ are equivalent to (q, θ) and $(\tilde{q}, \tilde{\theta})$ being KT-acceptable.

7.3.3 Endpoint global inhomogeneous estimates, case of $q = \tilde{q}'$

We now proceed with the proof of the endpoint estimates with exponents that lie on the hypotenuse on ΔOAB, see figure 2.1. To that end we shall need the well-known interpolation identities

$$(L^p(\mathbb{R}; \mathcal{A}_0), L^p(\mathbb{R}; \mathcal{A}_1))_{\theta,p} = L^p(\mathbb{R}; (\mathcal{A}_0, \mathcal{A}_1)_{\theta,p}), \quad 1 < p < \infty, \tag{7.24}$$

see [2]. We fix an exponent 4-tuple $(1/q, \theta, 1/\tilde{q}, \tilde{\theta}) \in \mathcal{E}^*$ such that $\beta(q, \theta, \tilde{q}, \tilde{\theta}) = 0$. We perturbate the exponents in estimate (7.3) by finding two 4-tuples $(1/q, \theta_0, 1/\tilde{q}, \tilde{\theta}_0), (1/q, \theta_1, 1/\tilde{q}, \tilde{\theta}_1) \in \mathcal{E}^*$ subject to

$$\theta_0 = \theta + \epsilon, \quad \tilde{\theta}_0 = \tilde{\theta} + \epsilon, \quad \theta_1 = 1/q - 3\epsilon, \quad \tilde{\theta}_1 = \tilde{\theta} - 3\epsilon.$$

Thus $\beta(q, \theta_0, \tilde{q}, \tilde{\theta}_0) = 2\sigma\epsilon$ and $\beta(q, \theta_1, \tilde{q}, \tilde{\theta}_0) = \beta(q, \theta_0, \tilde{q}, \tilde{\theta}_1) = -2\sigma\epsilon$. Hence the maps

$$A : L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}_0}{}^*) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_0}{}^*) \to l^\infty_{2\epsilon},$$
$$A : L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}_0}{}^*) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_1}{}^*) \to l^\infty_{-2\epsilon},$$
$$A : L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}_1}{}^*) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_0}{}^*) \to l^\infty_{-2\epsilon},$$

are bounded. In virtue of Lemma 8.0.7 we have that the map

$$A : (L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}_0}{}^*), L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}_1}{}^*))_{1/4,\tilde{q}'} \times (L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_0}{}^*), L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_1}{}^*))_{1/4,q'} \\ \to (l^\infty_{2\epsilon}, l^\infty_{-2\epsilon})_{1/2,1}$$

is also bounded. Finally, in view of the interpolation identity (7.24), the above simplifies to

$$A : L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta},\tilde{q}}{}^*) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta,q}{}^*) \to l^1,$$

where by $\mathcal{B}_{\theta,q}$ we denote the real interpolation space $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}$ for $\theta \in [0,1]$, $1 \leq q \leq \infty$.

The set of validity of these estimates is determined in the same way as in the previous section. Except that now all inequalities in the definition of $\mathcal{E}^*$ that contain θ or $\tilde{\theta}$ must be rewritten as strict inequalities as we perturbate with respect to these quantities.

7.3.4 Endpoint global inhomogeneous estimates, case of $\tilde{q} = \infty$

Suppose now that $(1/q, 1/\tilde{q})$ lies on either one of the two catheti of ΔOAB in figure 2.1, for the sake of concreteness let us suppose that $1/\tilde{q} = 0$. We also exclude the two endpoints $O = (0,0)$ and $A = (1,0)$ (if $\sigma \geq 1$, otherwise $A = (\sigma, 0)$), so that we can perturbate with respect to q. We also suppose that (q,θ) belongs to $\mathcal{E}^*$ together with a neighborhood of small perturbations in q as well as the 4-tuple (q,θ), $(\tilde{q},\tilde{\theta})$. Assuming that the scaling condition $\beta(q,\theta,\tilde{q},\tilde{\theta}) = 0$ holds we have that

$$A : L^1(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q_0'}(\mathbb{R}; \mathcal{B}_\theta{}^*) \to l^\infty_\epsilon,$$
$$A : L^1(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q_1'}(\mathbb{R}; \mathcal{B}_\theta{}^*) \to l^\infty_{-\epsilon},$$

where

$$\frac{1}{q_0} = \frac{1}{q} - \frac{1}{\epsilon}, \quad \frac{1}{q_1} = \frac{1}{q} + \frac{1}{\epsilon}.$$

The real method with parameters $(\theta, q) = (1/2, 1)$ gives that

$$A : L^1(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q',1}(\mathbb{R}; \mathcal{B}_\theta{}^*) \to l^1_0.$$

By the TT^*-principle, this means that

$$\|W(t)F\|_{L^{q,\infty}(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^1(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)} \tag{7.25}$$

for all $F \in L^1(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*)$ whenever (q,θ) and $(\infty, \tilde{\theta})$ satisfy the assumptions we have made so far.

In view of the Equivalence Theorem 1.3.2, we also have the following homogeneous estimate

$$\|U(t)f\|_{L^{q,\infty}(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta}}{}^*}, \tag{7.26}$$

for all $f \in \mathcal{B}^*_{\tilde{\theta}}$, in the same range as the inhomogeneous estimate (7.25).

To summarize, we have

Proposition 7.3.6. *Suppose that (q, θ) and $(\tilde{q}, \tilde{\theta})$ are two jointly-acceptable exponent pairs. If $\tilde{q} = \infty$ then the estimate*

$$\|W(t)F\|_{L^{q,\infty}(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^1(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}, \tag{7.27}$$

holds for every $F \in L^1(\mathbb{R}; \mathcal{B}_{\tilde{\theta}})$. If analogously $q = \infty$, then by duality the estimate

$$\|W(t)F\|_{L^\infty(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}',1}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}, \tag{7.28}$$

holds for every $F \in L^{\tilde{q}',1}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*)$.

Proposition 7.3.7. *Suppose that (q, θ) is an exponent pair satisfying*

$$\frac{1}{q} = \frac{\sigma}{2}\left(2 - \theta - \tilde{\theta}\right),$$

for some $\tilde{\theta} \in (0, 1]$. Then the estimate

$$\|U(t)f\|_{L^{q,\infty}(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta}}{}^*}, \tag{7.29}$$

holds for every $f \in \mathcal{B}_{\tilde{\theta},c'}{}'$ whenever the exponents θ and $\tilde{\theta}$ are in the range

- $\sigma < 1, \qquad 0 < \tilde{\theta} \le 1, \qquad 0 \le \theta < \tilde{\theta},$
- $\sigma = 1, \qquad 0 < \tilde{\theta} \le 1, \qquad 0 < \theta < \tilde{\theta},$
- $\sigma > 1, \qquad 0 < \tilde{\theta} \le 1, \qquad \frac{\sigma-1}{\sigma}\tilde{\theta} \le \theta < \tilde{\theta},$

or if $(q, \theta, \tilde{\theta}) = (\infty, 1, 1)$.

7.4 Derivations of Strichartz estimates for concrete equations

The derivation of the Strichartz estimates from the abstract setting to each of the concrete equations in Chapter 6 is completely straightforward. In a number of cases, though, we shall need to make some extra computations. That is sketched in the context of the generalized

Schrödinger and wave equations and can be found in the two subsections that follow. The cases of the Klein-Gordon and Dirac equations are completely analogous to that of the wave equation and shall not be considered separately.

Note that in the careful exposition of Keel and Tao [30], one can find the derivation of the estimates for admissible exponents in the context of the Schrödinger and the wave equation, that is Theorem 6.1.6 and Theorem 6.3.5. The sharpness of these theorems is also given in [30]. Similarly, in Foschi [19] one can find a derivation of the inhomogeneous Strichartz estimates for the Schrödinger equation, that is Theorem 6.1.9. Foschi [19] and Vilela [49] present a number of counterexamples for the sharpness of Theorem 6.1.9. In Taggart [45] one can find a derivation of the inhomogeneous Strichartz estimates for the wave equation that are similar to those given in Theorem 6.3.7.

7.4.1 Generalized Schrödinger-type equations

The Strichartz estimates for the Schrödinger and the generalized Schrödinger-type equations follow from the abstract Strichartz estimates by the identification

$$\mathcal{B}_\theta = L^r, \quad \theta = \frac{2}{r}, \quad \mathcal{B}_{\tilde\theta} = L^{\tilde r}, \quad \tilde\theta = \frac{2}{\tilde r}, \quad p = \tilde r', \tag{7.30}$$

for $\theta, \tilde\theta \in [0,1]$.

The only additional computation is that of the generalized homogeneous estimates of Proposition 6.2.2 where one needs to upgrade an $L_t^{q,\infty}$-norm to a $L_t^{q,p}$-norm, where L^p is the class of the initial data. For that see next subsection where this matter is discussed in the context of the wave equation.

7.4.2 Generalized wave-type equations

The Strichartz for the generalized wave-type equations are contained as a special case in that of the abstract Strichartz estimates by the identification

$$\mathcal{B}_\theta = B^{-\rho}_{r,2}, \quad \theta = \frac{2}{r}, \quad \mathcal{B}_{\tilde\theta} = B^{-\tilde\rho}_{\tilde r,2}, \quad \tilde\theta = \frac{2}{\tilde r}, \quad p = \tilde r', \tag{7.31}$$

for $\theta, \tilde\theta \in [0,1]$.

In this section we suppose that $U(t)$ is a generalized wave evolution group satisfying conditions (6.56)-(6.59).

Let us for example prove the generalized homogeneous estimates of Proposition 6.6.3. In view of the abstract estimates of Proposition 6.7.7 and the identification above we have the estimate

$$\|U(t)f\|_{L^{q,\infty}(\mathbb{R};B^{-\rho}_{r,2})} \lesssim \|f\|_{B^{\tilde\rho}_{\tilde r',2}},$$

or equivalently

$$\left\|D^{-\rho-\tilde\rho}U(t)f\right\|_{L^{q,\infty}(\mathbb{R};B^{0}_{r,2})} \lesssim \|f\|_{B^{0}_{\tilde r',2}},$$

in the same range for the exponents as in Proposition 6.7.7. In fact, this estimate implies

$$\left\|D^{-\rho-\tilde\rho}U(t)f\right\|_{L^{q,\infty}_t L^r_x} \lesssim \|f\|_{L^p_x}, \tag{7.32}$$

by the usual embeddings and the fact that $r \geq 2$ and $p \leq 2$. Now we use a standard argument that shall be repeated in similar situations. We perturbate slightly the exponents q and p and use real interpolation with (θ, p). Basically, in this way we sharpen (7.32) *inside* its range of validity to

$$\left\|D^{-\rho-\tilde\rho}U(t)f\right\|_{L^{q,p}_t L^r_x} \lesssim \|f\|_{L^p_x}. \tag{7.33}$$

The final result is summarized in Proposition 6.6.3.

In the next two propositions we assume (without loss of generality) that (6.56)-(6.59) are given in terms of homogeneous Besov norms.

Proposition 7.4.1. *Suppose that (q,r) is a nonsharply σ-admissible exponent pair. Then we have the estimate*

$$\|U(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot H^\alpha}, \quad \alpha = \frac{n}{2} - \frac{n-2\beta}{\sigma q} - \frac{n}{r}$$

for all $f \in \dot H^\alpha$. Analogously, if (q,r) and $(\tilde q, \tilde r)$ are two nonsharply σ-admissible exponent pairs, then we have the estimate

$$\|W(t)F\|_{L^q_t L^r_x} \lesssim \|D^\alpha F\|_{L^{\tilde q'}_t L^{\tilde r'}_x} \tag{7.34}$$

for all $D^\alpha F \in L^{\tilde q'}(\mathbb{R}; L^{\tilde r'})$, where α is subject to (7.36).

Proof. If the pair (q,r) is nonsharply σ-admissible then we can always find an exponent R, $2 \leq R \leq r$, such that the pair (q,R) is σ-admissible. If $r < \infty$ we use the Sobolev embedding

$$\|U(t)f\|_{L^q_t L^r_x} \lesssim \left\|D^{\frac{n}{R}-\frac{n}{r}}U(t)f\right\|_{L^q_t L^R_x}$$

so that we can apply the Strichartz estimates of Theorem 6.6.1 to the right hand side. In view of 6.61, we obtain

$$\alpha = 2\beta\left(\frac{1}{2} - \left(\frac{1}{2} - \frac{1}{\sigma q}\right)\right) + n\left(\frac{1}{2} - \frac{1}{\sigma q}\right) - \frac{n}{r}.$$

If $r = \infty$ we use Proposition 7.4.2.

By the TT^*-principle, we have already proven that the inhomogeneous estimate (7.34) applies to the TT^*-operator. If $q > 2$ we use the Christ-Kiselev Lemma 8.0.14. If $q = \tilde{q} = 2$ we use Sobolev embedding and the Equivalence Theorem 1.3.2, part B. □

Proposition 7.4.2. *The estimate*

$$\|U(t)f\|_{L_t^q L_x^\infty} \lesssim \|f\|_{\dot{H}^\alpha}, \quad \alpha = \frac{n}{2} - \frac{n-2\beta}{\sigma q}$$

holds for every $f \in \dot{H}^\alpha$ whenever the exponent pair (q, ∞) is nonsharply σ-admissible.

Proof. Consider the formula

$$\|U(t)f\|_{L_x^\infty} \lesssim \left\|D^\alpha U(t)f\right\|_{L^2}^{1-\theta} \left\|D^\gamma U(t)f\right\|_{L^R}^{\theta}, \tag{7.35}$$

which is a special case of the interpolation inequality of Proposition 8.0.8, for some $R < \infty$ fixed and big enough, and some θ, α, and γ, which we need to determine. To determine α we consult formula (6.62) with $r = \infty$ which suggests

$$\alpha = \frac{n}{2} - \frac{n-2\beta}{\sigma q} < \frac{n}{2}, \quad \gamma + \left(\frac{n}{2} - \frac{n-2\beta}{\sigma q\theta} - \frac{n}{R}\right) = \alpha.$$

Thus, $\gamma = -k + k/\theta + n/r$, $k = (n-2\beta)/\sigma q$, and $\gamma > n/r$ if we can choose $\theta < 1$. Substituting in (8.4) $\mu = \alpha$, $\lambda = \gamma$, we obtain the identity

$$(1-\theta)k + (1-\theta)(-k) = 0,$$

so that we are free to choose any $\theta < 1$. In particular, we choose θ so that the pair $(q\theta, R)$ is σ-admissible. Thus we first apply Proposition 8.0.8 to the term $\|U(t)f\|_{L_t^q L_x^\infty}$ to obtain (7.35). To each term on the right hand side of (7.35) we then apply Proposition 7.4.1 with the admissible pairs $(\infty, 2)$, $(q\theta, R)$, respectively, to conclude the proof. Note that this argument only works if q is away from the two endpoints $q = 2/\sigma$ ($q = 2$ when $\sigma > 1$), $q = \infty$.

As a final remark we note that since $\alpha > 0$ we can always replace the homogeneous norm $\dot{H}^\alpha$ with the inhomogeneous norm H^α in case that our estimates are based on inhomogeneous norms. □

We next give a corollary to Theorem 6.6.5 in the case when $\sigma \leq 1$.

Corollary 7.4.3. *Suppose that (q, r) and $(\tilde{q}, \tilde{r})$ are two nonsharply jointly σ-acceptable exponent* *pairs with* $\sigma \le 1$*. Then the operator* $W(t)$ *obeys the estimate*

$$\|W(t)F\|_{L^q_t L^r_x} \lesssim \|D^\alpha F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}, \quad r, \tilde{r} < \infty,$$

for all F*,* $D^\alpha F \in L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'})$*, whenever the Sobolev exponent* α *fulfills condition*

$$\frac{n-2\beta}{\sigma q} + \frac{n}{r} = \frac{n-2\beta}{\sigma \tilde{q}'} + \frac{n}{\tilde{r}'} - \frac{n-2\beta}{\sigma} - \alpha. \tag{7.36}$$

Proof. Consider two nonsharply jointly σ-acceptable pairs (q, r) and $(\tilde{q}, \tilde{r})$. We can always find R and $\tilde{R}$ such that $2 \le R \le r$ and $2 \le \tilde{R} \le \tilde{r}$ and (q, R) and (q, R) are jointly σ-acceptable. Then as we did in the preceding propositions we use Sobolev embedding. The Sobolev exponent α in (7.36) is computed from

$$\alpha = \rho + \tilde{\rho} + \gamma + \tilde{\gamma},$$

where

$$\gamma = n/R - n/r, \quad \tilde{\gamma} = n/\tilde{R} - n/\tilde{r}.$$

□

Chapter 8

Appendix

Here we collect some standard facts and definitions from analysis that shall be used frequently in our arguments throughout the entire body of this work.

Let D^s be an operator of fractional differentiation that has symbol $|\xi|^s$ and analogously, let Λ^s be the corresponding inhomogeneous operator of symbol $(1+|\xi|^2)^{s/2}$. We define the homogeneous Sobolev space $\dot{H}^s_r(\mathbb{R}^n)$ on $\mathbb{R}^n$ by

$$\dot{H}^s_r(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \setminus \mathcal{P}(\mathbb{R}^n) : \|D^s u\|_{L^r(\mathbb{R}^n)} < \infty\}$$

for $1 < r < \infty$, $s \in \mathbb{R}$, that is the set of the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ on $\mathbb{R}^n$, factorized by all polynomials $\mathcal{P}(\mathbb{R}^n)$ on $\mathbb{R}^n$, whose Sobolev norms $\|D^s\cdot\|_{L^r(\mathbb{R}^n)}$ are finite. When $r = 2$, instead of $\dot{H}^s_2(\mathbb{R}^n)$, we simply write $\dot{H}^s(\mathbb{R}^n)$. Analogously, the inhomogeneous Sobolev space $H^s_r(\mathbb{R}^n)$ on $\mathbb{R}^n$ is defined by

$$H^s_r(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \|\Lambda^s u\|_{L^r(\mathbb{R}^n)} < \infty\}.$$

The homogeneous Besov space $\dot{B}^s_{r,q}(\mathbb{R}^n)$ on $\mathbb{R}^n$ is defined by

$$\dot{B}^s_{r,q}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n) \setminus \mathcal{P}(\mathbb{R}^n) : \left\| \{2^{sj} \|\phi_j * u\|_{L^r(\mathbb{R}^n)}\}_{j\in\mathbb{Z}} \right\|_{l^q} < \infty \right\},$$

where $\{\phi_j\}_{j\in\mathbb{Z}}$ is a homogeneous Littlewood-Paley dyadic decomposition on $\mathbb{R}^n$. Analogously, the inhomogeneous Besov space $B^s_{r,q}(\mathbb{R}^n)$ on $\mathbb{R}^n$ is defined by

$$B^s_{r,q}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n) : \left\| \{2^{sj} \|\phi_j * u\|_{L^r(\mathbb{R}^n)}\}_{j=0}^{\infty} \right\|_{l^q} < \infty \right\},$$

where $\{\phi_j\}_{j=0}^{\infty}$ is a inhomogeneous Littlewood-Paley dyadic decomposition on $\mathbb{R}^n$.

Recall the well-known continuous embeddings between the Besov spaces $\dot{B}^s_{r,2}$, and the

Sobolev spaces $\dot{H}^s_r$

$$\dot{H}^s_r \hookrightarrow \dot{B}^s_{r,2}, \quad 1 < r \le 2, \qquad \dot{B}^s_{r,2} \hookrightarrow \dot{H}^s_r, \quad 2 \le r < \infty, \tag{8.1}$$

see [2, p. 152]. Analogous embeddings are valid for the inhomogeneous Besov and Sobolev spaces too.

Let us recall several standard facts from real interpolation that shall be used often in our arguments. By $L^p = L^p(X;\mathcal{B})$ and $L^{p,q} = L^{p,q}(X;\mathcal{B})$ we denote the Lebesgue space and the Lorentz space, respectively, of vector-valued functions that map a fixed measure space $(X, d\mu)$ to a fixed Banach space $\mathcal{B}$.

Proposition 8.0.4 (see [2, p. 113])**.** *Suppose that* $0 < p_0,\, p_1,\, q_0,\, q_1 \le \infty$*,* $0 < \theta < 1$*, and* $p_0 \neq p_1$*. Then*

$$\left(L^{p_0,q_0}, L^{p_1,q_1}\right)_{\theta,q} = L^{p,q},$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$*.*

Suppose that $\mathcal{B}_0$ and $\mathcal{B}_1$ two Banach spaces that are compatible for interpolation.

Proposition 8.0.5 (see the Appendix of [14])**.** *For every* $1 \le p_0, p_1 < \infty$*,* $0 < \theta < 1$*,* $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $p \le q$ *we have*

$$L^p(X; (\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}) \hookrightarrow (L^{p_0}(X;\mathcal{B}_0), L^{p_1}(X;\mathcal{B}_1))_{\theta,q}.$$

Denote by l^p_s the space of number sequences with a norm

$$\|\{a\}_{j\in\mathbb{Z}}\|_{l^p_s} = \left(2^{js}\,|a_j|^p\right)^{1/p}, \quad 1 \le p < \infty,$$
$$\|\{a\}_{j\in\mathbb{Z}}\|_{l^\infty_s} = \sup_{j\in\mathbb{Z}} 2^{js}\,|a_j|\,, \qquad p = \infty.$$

Lemma 8.0.6 (See Theorem 5.6.1 in [2])**.** *We have the identity*

$$\left(l^\infty_{s_0}, l^\infty_{s_1}\right)_{\theta,1} = l^1_s,$$

where $s_0, s_1 \in \mathbb{R}$*,* $s_0 \neq s_1$ *and* $s = (1-\theta)s_0 + \theta s_1$*.*

Lemma 8.0.7 (See pp. 76-77 in [2])**.** *Suppose that* $(\mathcal{A}_0, \mathcal{A}_1)$*,* $(\mathcal{B}_0, \mathcal{B}_1)$*,* $(\mathcal{C}_0, \mathcal{C}_1)$ *are interpolation couples and that the bilinear operator* T *acts as a bounded transformation as indicated below:*

$$T : \mathcal{A}_0 \times \mathcal{B}_0 \to \mathcal{C}_0,$$
$$T : \mathcal{A}_0 \times \mathcal{B}_1 \to \mathcal{C}_1,$$
$$T : \mathcal{A}_1 \times \mathcal{B}_0 \to \mathcal{C}_1.$$

If $\theta_0, \theta_1 \in (0,1)$ and $p, q, r \in [1, \infty]$ such that $1/p + 1/q \geq 1$, then T also acts as a bounded transformation in the following way:

$$T : (\mathcal{A}_0, \mathcal{A}_1)_{\theta_0, pr} \times (\mathcal{B}_0, \mathcal{B}_1)_{\theta_1, qr} \to (\mathcal{C}_0, \mathcal{C}_1)_{\theta_0+\theta_1, r}.$$

Proposition 8.0.8 (Interpolation inequality, [35]). *Let $\lambda, \mu, p, q, r, \theta$ satisfy $\lambda, \mu \in \mathbb{R}$, $1 \leq p, q \leq r \leq \infty$, $0 < \theta < 1$,*

$$\lambda > \frac{n}{p} - \frac{n}{r}, \tag{8.2}$$

$$\mu < \frac{n}{q} - \frac{n}{r}, \tag{8.3}$$

$$\theta \left(\lambda - \frac{n}{p} + \frac{n}{r} \right) + (1-\theta) \left(\mu - \frac{n}{q} + \frac{n}{r} \right) = 0. \tag{8.4}$$

Then there exists a constant $C > 0$ such that

$$\|f\|_{L^r_x} \leq C \, \|f\|^{\theta}_{\dot{H}^{\lambda}_p} \, \|f\|^{1-\theta}_{\dot{H}^{\mu}_q} \tag{8.5}$$

for all $f \in \dot{H}^{\lambda}_p \cap \dot{H}^{\mu}_q$.

Our next goal shall be to generalize to the abstract setting some basic facts about approximations of the identity. Denote by $L(\mathcal{B}, \mathcal{B})$ the space of all linear continuous operators on the Banach space $\mathcal{B}$ and let $K(t) : \mathbb{R} \to L(\mathcal{B}, \mathcal{B})$ be an operator-valued function that maps $\mathbb{R}$ into $L(\mathcal{B}, \mathcal{B})$. For each $t \in \mathbb{R}$, let $m(t) : \mathbb{R} \to [0, \infty]$ denote the operator norm of $K(t)$. Suppose that $m(t) \in L^{\infty} \cap L^1$ and

$$\int_{-\infty}^{\infty} m(t) dt = 1.$$

We set

$$K_{\epsilon}(t) = \frac{1}{\epsilon} K\left(\frac{t}{\epsilon}\right), \quad F_{\epsilon} = F * K_{\epsilon} = \int_{-\infty}^{\infty} K_{\epsilon}(t-s) F(s) ds,$$

where $F : \mathbb{R} \to \mathcal{B}$ is a vector-valued function. In the next lemma we specify conditions on F and K under which $F_{\epsilon} \to F$, in the sense that $\|F(t) - F_{\epsilon}(t)\|_{\mathcal{B}} \to 0$ as $\epsilon \to 0$. We shall call any such family of kernels K_{ϵ}, for $\epsilon > 0$, an approximation of the identity.

Lemma 8.0.9. *Suppose that $F : \mathbb{R} \to \mathcal{B}$ belongs to $L^1_{loc}(\mathbb{R}; \mathcal{B})$, the space of all locally integrable $\mathcal{B}$-valued functions on $\mathbb{R}$, and $m(t) = O(|t|^{-1})$ as $|t| \to +\infty$. Then $F_{\epsilon} \to F$ at each point of continuity of F.*

In the classical setting of $\mathcal{B} = \mathbb{R}$ (or $\mathbb{C}$) the proof of this theorem can be found in a standard course of Real Analysis like e.g. [50, p. 152]. The generalization to the vector-valued setting is straightforward. The lemma shall be used under the same assumptions on the kernel to show that $\|F_{\epsilon}(t)\|_{\mathcal{B}} \to \|F(t)\|_{\mathcal{B}}$ on $\mathbb{R}$, as $\epsilon \to 0$, whenever $F \in C(\mathbb{R}; \mathcal{B})$

In the same spirit we generalize

Lemma 8.0.10 (Fatou's lemma)**.** *Suppose that* $F_k \to F$ *a.e. on* $\mathbb{R}$*, then*

$$\|F\|_{L^{p,q}(\mathbb{R};\mathcal{B})} \leq \liminf_{k\to\infty} \|F_k\|_{L^{p,q}(\mathbb{R};\mathcal{B})}\,,$$

where $p = q = 1$*,* $p = q = \infty$*, or* $1 < p < \infty$ *and* $1 \leq q \leq \infty$*.*

Remark 8.0.11. The classical Fatou's lemma is originally stated in the case of $L^{p,q}(\mathbb{R};\mathcal{B}) = L^1(\mathbb{R})$, i.e. for $p = q = 1$ and $\mathcal{B} = \mathbb{R}$.

Proof. The limit $F_k \to F$ a.e. on $\mathbb{R}$ means that $\|F(t) - F_k(t)\|_{\mathcal{B}} \to 0$ for almost all $t \in \mathbb{R}$. This implies the limit $\|F_k\|_{\mathcal{B}} \to \|F\|_{\mathcal{B}}$ a.e. on $\mathbb{R}$. By considering $f_k(t) : \mathbb{R} \to [0,\infty)$, $f_k(t) = \|F_k(t)\|_{\mathcal{B}}$, and $f(t) : \mathbb{R} \to [0,\infty)$, $f(t) = \|F(t)\|_{\mathcal{B}}$ it will be enough to show the claim only in the scalar case. However, the latter is a direct consequence of the Monotone convergence theorem for the Lorentz space $L^{p,q}(\mathbb{R})$ stated below. Indeed, let $g_k = \inf\{f_k, f_{k+1}, \dots\}$. Then $g_k \nearrow f$ and $0 \leq g_k \leq f_k$. Thus,

$$\|f\|_{L^{p,q}} = \lim \|g_k\|_{L^{p,q}} \leq \liminf \|f_k\|_{L^{p,q}}$$

and the claim follows. □

Theorem 8.0.12 (Monotone convergence theorem for Lorentz spaces)**.** *Let* (X, Σ, μ) *be a measure space and let* $\{f_k\}$ *be a sequence of measurable functions on* X*. If* $0 \leq f_k \nearrow f$ μ*-a.e. on* X*, then*

$$\|f_k\|_{L^{p,q}(X;d\mu)} \to \|f\|_{L^{p,q}(X;d\mu)}\,,$$

where $p = q = 1$*,* $p = q = \infty$*, or* $1 < p < \infty$ *and* $1 \leq q \leq \infty$*.*

Proof. In the case when $p = q = 1$ the proof can be found in [50, p. 172]. The case when $p = q = \infty$ is trivial. The rest follows from the special representation of the Lorentz norm

$$\|f\|^q_{L^{p,q}(X;d\mu)} = \int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}, \quad q < \infty,$$
$$\|f\|_{L^{p,q}(X;d\mu)} = \sup_{0<t<\infty} \{t^{1/p} f^*(t)\}, \quad q = \infty,$$

see [1, p. 216]. Indeed, the claim follows from the property

$$|f_k| \nearrow |f| \ \ \mu\text{-a.e.} \quad \Rightarrow \quad f_k^* \nearrow f^*,$$

see [1, p. 41], where by f^* we have denoted the decreasing rearrangement of f, for a definition see [1, p. 39]. □

The next lemma shall be useful to us when we are seeking to determine the boundaries of the range of validity of the Strichartz estimates for the KT equation, in section 2.6.

Lemma 8.0.13 (Hörmander [27])**.** *Whenever a (non-trivial) linear and bounded operator maps a vector-valued L^p-space to another vector-valued L^q-space, $1 \leq p, q \leq \infty$, and additionally this operator is translation invariant, then we must have that $p \leq q$.*

And finally, the next lemma is useful in the proof of the inhomogeneous estimates.

Lemma 8.0.14 (Christ-Kiselev, see Lemma 3.1 of [47], or [46])**.** *Suppose that the integral operator*

$$T[F](t) = \int_{-\infty}^{\infty} K(t,s)F(s)ds \tag{8.6}$$

is bounded from $L^p(\mathbb{R}; \mathcal{B}_1)$ to $L^q(\mathbb{R}; \mathcal{B}_2)$ for some Banach spaces $\mathcal{B}_1$, $\mathcal{B}_2$ and $1 \leq p < q \leq \infty$. The operator-valued kernel $K(t,s) : \mathbb{R}^2 \to L(\mathcal{B}_1, \mathcal{B}_2)$ maps $\mathbb{R}^2$ to the space of all bounded linear operators from $\mathcal{B}_1$ to $\mathcal{B}_2$. Assume also that the kernel K is regular enough to ensure that (8.6) is well-defined as a $\mathcal{B}_2$-valued Bochner integral for almost all $t \in \mathbb{R}$. Then the operator

$$\tilde{T}[F](t) = \int_{-\infty}^{t} K(t,s)F(s)ds$$

is also bounded on the same spaces.

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