The geometry of immobilizing sets of objects

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Abstract

When an object is grasped by a set of fingers, it is important to know the best positions to place them. An immobilizing set is a set of points on the object at which a firm grasp of the object is achieved, that is, where the object cannot be moved within the grasp. In this thesis a study of immobilizing sets of points for planar figures and tetrahedra is undertaken.

A new proof of Czyzowicz, Stojmenovic and Urrutia's theorem giving necessary and sufficient geometric conditions for immobilizing a triangle is obtained. The same method of proof is employed to obtain proofs of statements on immobilizing sets of polygonal planar objects.

In three dimensions, a detailed study of immobilizing sets of a tetrahedron is carried out. A 3×3 matrix A is defined for each quadruple of points, one from the interior of each face of the tetrahedron using a good choice of outward normal vectors to the faces of the tetrahedron. A necessary and sufficient condition on the quadruple of points to immobilize the tetrahedron is that the matrix A is symmetric. An analysis of the eigenvalues of symmetric matrix A leads to a new proof of Bracho, Mayer, Fetter and Montejano's theorem. This proof is adapted to give another treatment of necessary and sufficient conditions characterizing immobilizing sets of a triangle.

The set of centroids, set of circumcenters and set of orthocenters of the faces of a tetrahedron are shown to immobilize it in appropriate cases. It is shown that a set of four immobilizing points one in each face of the tetrahedron has five degrees of freedom and immobilizing sets of a tetrahedron having two fixed points have one degree of freedom. An analysis of the orientation of the tetrahedron whose vertices are the points in an immobilizing set of a given tetrahedron reveals the existence of immobilizing sets of a regular tetrahedron which are co-planar. In higher dimensions, a method of generating sets of points for which the matrix A is symmetric from another such set is presented and some geometrical properties arising from the symmetry of A are analysed.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Saul Hannington Nsubuga*)

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Chapter 1

Introduction

1.1 Background

The design of robots relies on geometric techniques because of the need to analyse motion. Various interesting geometric problems arise but rather few of them have been subjected to serious mathematical study. Nevertheless, it is important to have a firm understanding of the theoretical principles before proceeding to the more practical, algorithmic aspects of the problems. This research project is devoted to making a thorough analysis of one of the geometric issues that arise in robotics, namely the problem of the 'grasping hand'.

Grasping emerged as field in its own in the early eighties with the introduction of dextrous multi-finger robot grippers. It is concerned with characterizing and achieving conditions that will ensure that a robot gripper holds an object securely, preventing, for example, any motion due external forces. Different authors have given different types of grasping depending on the conditions a grasp is required to satisfy. The two most common types are force closed and form closed grasps, although, unfortunately, there is no agreement on terminology in the grasping literature. In [D], [MK2] and [RE] the term form closed grasp was used to mean a grasp with the property that any external *wrench* to the object can be balanced by forces and moments applied at the grasp points, yet in [MI], [MU] and [SE] force closed grasp was used for a grasp satisfying the same criteria. In [SE] a form closed grasp was defined by first considering paths (parametrized by time) an object could take in $SE(3)$, the configuration space of the object. Then a point contact of a finger on an object was assumed to stop the object from moving along the contact normal towards the finger. Then a body was said to be in a form closed grasp if the space of all *feasible velocities* it can acquire is null, that is consists of the zero velocity only.

In this thesis we study the problem of immobilization of objects, an important aspect of grasping. We assume a finger is a point where the finger touches the object. The fingers of a human or robot hand are able to get a steady hold on a body if they touch the body at a good set of points. The number of fingers (or points on the body) required for this purpose depends on the shape of the body. The set of points of contact on the body at which the fingers hold on the body in such a way that the body cannot slip from or wriggle in the grasp is called an *immobilizing set* of the body. Since a set obtained by adding points to an immobilizing set is also an immobilizing set, it is enough to consider minimal such sets. Immobilization problems were introduced by Kuperberg [KU] and were motivated by grasping problems in robotics, [MK1] and [MK2]. Their only interest is in the geometric aspects only and no account of force, torque, moment, etc. are considered.

1.2 Thesis outline

The last section of this chapter introduces general preliminary material that will be needed later in the thesis. It consists of standard definitions and theorems in mathematics.

In Chapter 2 a study of immobilization of planar objects is undertaken. A key observation in this chapter is the idea that an orthogonal line to an edge of a polygonal object divides the plane into two half planes each of whose points have different properties. This is given in the form of Lemma 2.4 and is used to obtain different proofs of results by Czyzowicz, Stojmenovic and Urrutia [C1].

Chapter 3 studies the assignment of Plücker coordinates to lines in space. The two types of planes that lie on a Klein quadric are analysed and the geometrical configurations of four lines having linearly dependent Plücker coordinates are obtained.

In Chapter 4 we undertake the problem of finding the criteria that immobilizing sets of a tetrahedron fulfil. For each set of four points, one in the interior of each face of a tetrahedron, one defines a 3×3 matrix A. It is found out that the four points immobilize the tetrahedron if and only if A is symmetric and has a property we call *almost positive definite*. The symmetry of A is referred to as the symmetry condition. The positions of the four points in the faces of the tetrahedron can be encoded using a 4×4 stochastic matrix. It turns out that if the points are interior to their faces the matrix A is almost positive definite whenever it is symmetric. The approach given here is different from that of Bracho, Fetter, Mayer and Montejano [BR] and generalizes to other dimensions. An example of this is an explicit algebraic condition on a set of three points to immobilize a triangle.

Chapter 5 uses the criteria developed in Chapter 4 to find immobilizing points of the tetrahedron, the most natural of these being the set of centroids of the faces of the tetrahedron. It is seen that any given tetrahedron has many immobilizing sets. It is also shown that if two of the points in the faces of a tetrahedron are fixed, the remaining two points that complete an immobilizing set are linearly related and, for each face, lie on a line whose direction is independent of the choice of fixed points. Vector algebra is employed to obtain the five dimensional space of solutions of the symmetry condition and a method of classifying immobilizing sets is proposed.

In the last chapter generalizations to higher dimensions are made of some of the results in chapters 4 and 5. In particular, it is shown that the set of centroids of an n simplex, $n \geq 2$, immobilizes the simplex and a method of obtaining other solutions of the symmetry condition from one solution is presented. The situation in higher dimensions is different because the symmetry of A does not imply that A is almost positive definite, an explicit example is given in dimension 4.

1.3 Preliminaries

1.3.1 Euclidean and projective spaces

The objects we seek to immobilize will be assumed to be subsets of a Euclidean space. To define coordinates of a line it will be assumed that the line lies in a real Projective space.

Definition 1.1 *A Euclidean vector space E is a finite dimensional vector space over $\mathbb{R}$, together with a positive definite symmetric and bilinear form ϕ (i.e. $\phi : E \times E \to \mathbb{R}$ is symmetric and bilinear, and $\phi(x,x) > 0$ for all $x \in E$, $x \neq 0$). We write $\phi(x,y) = \langle x, y \rangle$ and call this number the scalar product of x and y.*

The standard example of a Euclidean vector space is $E = \mathbb{R}^n$, with

$$\phi((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^{n} x_i y_i.$$

Definition 1.2 *Let E be a vector space over the field K. The projective space derived from E, denoted $P(E)$, is the quotient of $E\backslash 0$ by the equivalence relation '$x \sim y$ if and only if $y = \lambda x$ for some non-zero $\lambda \in K$'. A projective space is called real if $K = \mathbb{R}$. If $E = \mathbb{R}^{n+1}$ and $K = \mathbb{R}$, then the projective space derived from E is called the real projective space of dimension n or the real projective n-space, and is denoted $\mathbb{P}^n$.*

$\mathbb{P}^n$ will be considered as the projective extension of $\mathbb{R}^n$. A point in $\mathbb{P}^n$ is denoted by an ordered set of $n+1$ real numbers called the homogeneous coordinates of the point. Let $x \in \mathbb{P}^n$ be given by $(x_0, \ldots, x_n)$. If $x_0 \neq 0$ then x represents the point $(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in \mathbb{R}^n$ and if $x_0 = 0$ then x represents the point at infinity in the direction of the line spanned by the non-zero *vector* $(x_1, \ldots, x_n)$.

1.3.2 Rigid body motion

Since a body is said to be immobilized if it cannot execute any rigid motions, the concept of immobilization of an object is intimately related to the rigid motions the object is capable of. For this reason we briefly review the theory of rigid body motion.

The motion of a rigid body is more complicated than that of a particle. The motion of a particle can be described by giving the location of the particle at each instant of time relative to an inertial Cartesian coordinate frame. However a rigid body motion is a displacement of all particles (making up the object) such that the distance between any two particles remains fixed and the orientation of any set of particles is preserved at all times. Thus a mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ describes a rigid body motion/transformation if it satisfies the properties:

- P1: $\|g(X) - g(Y)\| = \|X - Y\|$ for all points X, $Y \in \mathbb{R}^n$.
- P2: Orientation is preserved.

Translations and rotations are good examples of rigid body transformations. The set of all translations $\mathbb{R}^n \to \mathbb{R}^n$ is a Lie group which will be denoted by T_n, it is isomorphic to the additive group $\mathbb{R}^n$. Let $\mathbf{x}' = (x'_1, \ldots, x'_n)$ be the effect of a rotation g (about the origin) on the vector $\mathbf{x} = (x_1, \ldots, x_n)$ and (g_{ik}) the matrix of g. Then $x'_i = \sum_k g_{ik} x_k$. Since a rotation does not alter lengths or angles it leaves the scalar product of any two vectors invariant. Thus if $\mathbf{y}' = g(\mathbf{y})$,

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} = \sum_k x_k y_k = \mathbf{x}' \cdot \mathbf{y}' &= \sum_k g_{ik} x_k \cdot \sum_l g_{il} y_l \\ &= \sum_{i,k,l} g_{ik} g_{il} x_k y_l \end{aligned}$$

Equating coefficients of $x_k y_l$ we obtain $\sum_i g_{ik} g_{il} = \delta_{kl}$, *i.e.* $[g^t g]_{kl} = \delta_{kl}$ hence g is an orthogonal matrix. Rotations preserve orientation therefore the matrix (g_{ik}) has positive determinant. The set of all rigid motions $\mathbb{R}^n \to \mathbb{R}^n$ fixing the origin is a Lie group denoted $SO(n)$ and called the rotation group of $\mathbb{R}^n$. It can be identified with the group of matrices:

$$SO(n) = \left\{ R \in \mathrm{GL}(n, \mathbb{R}) \,:\, R\, R^T = I, \det R \,=\, +1 \right\}.$$

Theorem 1.3 *Let $R \in SO(n)$. There exists a real skew symmetric matrix S such that*

$$R = I_n + S + \frac{1}{2!} S^2 + \frac{1}{3!} S^3 + \cdots + \frac{1}{n!} S^n + \cdots$$

where I_n is the identity $n \times n$ matrix.

Proof See [PR].

Remarks

The set of all rigid body motions $\mathbb{R}^n \to \mathbb{R}^n$ is a Lie group called the proper Euclidean group or group of proper Euclidean motions, and is denoted $SE(n)$. Let $g \in SE(n)$, then the action of g can be expressed as $g(X) = R(X) + \mathbf{t}$ where $R \in SO(n)$ and $\mathbf{t}$ is a vector in $\mathbb{R}^n$. The product of two such transformations $g_1 = (R_1, \mathbf{t}_1)$ and $g_2 = (R_2, \mathbf{t}_2)$ where

$$g_1(X) = R_1(X) + \mathbf{t}_1, \quad g_2(X) = R_2(X) + \mathbf{t}_2$$

is

$$\begin{aligned} g_2(R_1(X) + \mathbf{t}_1) &= R_2(R_1(X) + \mathbf{t}_1) + \mathbf{t}_2 \\ &= R_2 R_1(X) + R_2(\mathbf{t}_1) + \mathbf{t}_2 \end{aligned}$$

i.e. $(R_2, \mathbf{t}_2)(R_1, \mathbf{t}_1) = (R_2 R_1, R_2(\mathbf{t}_1) + \mathbf{t}_2)$. Therefore one can write $SE(n) = SO(n) \rtimes \mathbb{R}^n$. The group $SE(n)$ can be identified with the space of $n+1$ by $n+1$ matrices of the form

$$g = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}.$$

Theorem 1.4 *The dimension of $SE(n)$ is $\frac{n}{2}(n+1)$.*

Theorem 1.5 (Chasles) *Every rigid motion in three dimensions, with the exception of pure translations, can be realized by a rotation about an axis combined with a translation parallel to that axis.*

Proof See [SE]

1.3.3 Divergence theorem

Corollaries of the following Theorem will be needed in Chapters 4 and 5.

Theorem 1.6 (Divergence Theorem) *Let V be the volume bounded by a closed surface S and $\mathbf{A}$ be a vector function of position with continuous derivatives, then*

$$\iiint_V \nabla \cdot \mathbf{A}\, dV = \iint_S \mathbf{A} \cdot \mathbf{n}\, dS = \iint_S \mathbf{A} \cdot d\mathbf{S}$$

where $\mathbf{n}$ is the positive (outward) normal to S.

Proof See [SPI].

Corollary 1.7 $\iint_S \mathbf{n}\, dS = \mathbf{0}$ *for any closed surface S.*

Proof
Let $\mathbf{A} = 1\mathbf{C}$ where $\mathbf{C}$ is a constant vector. Then by the Divergence Theorem

$$\iiint_V \nabla \cdot (1\mathbf{C})\, dV = \iint_S 1\mathbf{C} \cdot \mathbf{n}\, dS,$$

that is

$$\mathbf{C} \cdot \iiint_V \nabla 1\, dV = \mathbf{C} \cdot \iint_S 1\mathbf{n}\, dS.$$

Since $\nabla 1 = 0$ and $\mathbf{C}$ is an arbitrary constant vector,

$$0 = \iint_S \mathbf{n}\, dS.$$

Corollary 1.8 *Let S be a closed surface and $\mathbf{r}$ the position vector of an arbitrary point in S, then $\iint_S \mathbf{r} \times \mathbf{n}\, dS = \mathbf{0}$.*

Proof
Let $\mathbf{A} = \mathbf{r} \times \mathbf{C}$ where $\mathbf{C}$ is a constant vector. Then by the Divergence Theorem

$$\iiint_V \nabla \cdot (\mathbf{r} \times \mathbf{C})\, dV = \iint_S (\mathbf{r} \times \mathbf{C}) \cdot \mathbf{n}\, dS,$$

that is

$$\mathbf{C} \cdot \iiint_V \nabla \times \mathbf{r}\, dV = \mathbf{C} \cdot \iint_S \mathbf{n} \times \mathbf{r}\, dS.$$

Since $\nabla \times \mathbf{r} = 0$ and $\mathbf{C}$ is an arbitrary constant vector,

$$0 = \iint_S \mathbf{n} \times \mathbf{r}\, dS.$$

1.3.4 Hodge star operator

The aim of this section is to give a brief introduction of the Hodge star operator $\star : \bigwedge^p \mathbb{R}^n \to \bigwedge^{n-p} \mathbb{R}^n$. This will be needed in Sections 4.4 and 6.2. Most of the material in this section comes from [F] and [KO].

Definition 1.9 *The space of p-vectors on $\mathbb{R}^n$, denoted $\bigwedge^p \mathbb{R}^n$, is the space consisting of all sums $\sum \alpha_i(x_{i1} \wedge \cdots \wedge x_{ip})$, where α_i are scalars and $x_{ij} \in \mathbb{R}^n$, subject to the following constraints:*

1. *For each i $\quad x_1 \wedge \cdots (\lambda x_i + \beta y_i) \wedge x_{i+1} \wedge \cdots \wedge x_p$ $= \lambda(x_1 \wedge \cdots \wedge x_i \wedge x_{i+1} \wedge \cdots \wedge x_p) + \beta(x_1 \wedge \cdots \wedge y_i \wedge x_{i+1} \wedge \cdots \wedge x_p)$, i.e. $x_1 \wedge \cdots \wedge x_p$ is linear in each variable,*

2. *$x_1 \wedge \cdots \wedge x_p = 0$ if for some pair of indices $i \neq j$, $x_i = x_j$,*

3. *$x_1 \wedge \cdots \wedge x_p$ changes sign if any two x_i are interchanged.*

One calls $x_1 \wedge \cdots \wedge x_p$ the *exterior product* of the vectors $x_1, \ldots, x_p$. If $e_1, \ldots, e_n$ denotes the standard unit basis of $\mathbb{R}^n$ then the set

$$\{e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_p} : 1 \leq \lambda_1 < \cdots < \lambda_p \leq n\}$$

is a basis of $\bigwedge^p \mathbb{R}^n$. Thus the dimension of $\bigwedge^p \mathbb{R}^n$ is $\binom{n}{p}$.

Lemma 1.10 *An inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ defines an inner product $\langle\ |\ \rangle$ on $\bigwedge^p \mathbb{R}^n$ as follows:*

$$\langle x_1 \wedge \cdots \wedge x_p \,|\, y_1 \wedge \cdots \wedge y_p \rangle = det(\langle x_i, y_j \rangle),$$

where $x_1 \wedge \cdots \wedge x_p$, $y_1 \wedge \cdots \wedge y_p \in \bigwedge^p \mathbb{R}^n$.

Proof See [ML].

Remarks

Suppose an inner product is defined on $\bigwedge^p \mathbb{R}^n$ then the length of vectors in $\bigwedge^p \mathbb{R}^n$ is defined. The magnitude $\|x_1 \wedge \cdots \wedge x_p\|$ of the p-vector $x_1 \wedge \cdots \wedge x_p$ is the volume of the 'parallelepiped' spanned by $x_1, \ldots, x_p$. If $e_1, \ldots, e_n$ denotes the standard orthonormal basis of $\mathbb{R}^n$ then the set

$$\{e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_p} : 1 \leq \lambda_1 < \cdots < \lambda_p \leq n\}$$

is an orthonormal basis of $\bigwedge^p \mathbb{R}^n$.

Definition 1.11 *The volume elements of* $\mathbb{R}^n$ *are the non-zero elements of the 1-dimensional space* $\bigwedge^n \mathbb{R}^n$*. Two volume elements* ω_1 *and* ω_2 *are said to be equivalent if there exists a* $c > 0$ *such that* $\omega_1 = c\omega_2$*. An equivalence class* $[\omega]$ *of volume elements on* $\mathbb{R}^n$ *is called an orientation on* $\mathbb{R}^n$*.*

Suppose $\mathbb{R}^n$ is given the standard inner product, a fixed orientation and $e_1, \ldots, e_n$ is an orthonormal basis of $\mathbb{R}^n$. Then an orthonormal basis $e_1 \wedge \cdots \wedge e_n$ of $\bigwedge^n \mathbb{R}^n$ is determined. Fix $z \in \bigwedge^p \mathbb{R}^n$. The map $\bigwedge^{n-p} \mathbb{R}^n \to \bigwedge^n \mathbb{R}^n$ given by $x \rightsquigarrow z \wedge x$ is a linear transformation into $\mathbb{R}$. This can be expressed as

$$z \wedge x = f_z(x) \;\; e_1 \wedge \cdots \wedge e_n$$

where f_z is a linear functional on $\bigwedge^{n-p} \mathbb{R}^n$. Therefore there exists a unique vector $\star z \in \bigwedge^{n-p} \mathbb{R}^n$ such that

$$\langle \star z \mid x \rangle e_1 \wedge \cdots \wedge e_n = z \wedge x.$$

Thus there exists a linear map $\star : \bigwedge^p \mathbb{R}^n \to \bigwedge^{n-p} \mathbb{R}^n$ called the Hodge star operator defined by $\star : z \rightsquigarrow \star z$.

Theorem 1.12 *The operator* $\star$ *is an isomorphism.*

Proof

Since $\star$ is a linear map, is onto (and $\dim \bigwedge^p \mathbb{R}^n = \binom{n}{p} = \binom{n}{n-p} = \dim \bigwedge^{n-p} \mathbb{R}^n$), $\star$ is an isomorphism of vector spaces.

Chapter 2

Immobilization in a plane

2.1 Introduction

In this chapter the problem of immobilizing objects in the plane is studied, focusing on polygons with particular emphasis on triangles. Czyzowicz, Stojmenovic and Urrutia [C1] found that three non-vertex points immobilize a triangle if the points lie in different edges of the triangle and the normal lines at them are concurrent. The proof given by these authors, in part, analyses the small distances and small angles corresponding to small rigid motions. The proof also considers the case when the normal lines meet inside the triangle separately from when they meet outside the triangle. In this chapter a different proof is given which appeals to two simple lemmas and does away with the need to locate the position of the point of concurrency of the normal lines. Hence a simpler proof of the theorem explaining the nature of immobilizing points of a convex polygon has been obtained. At the end of the chapter triples of points that immobilize general polygonal objects are described.

Definition 2.1 *A rigid motion $g \in SE(n)$ is said to be a small rigid motion if it lies in a small neighbourhood of the identity in $SE(n)$.*

Definition 2.2 *Let P be a polygon, not necessarily convex. A set of points S in the boundary of P, is said to immobilize P if any small rigid motion of P in the plane of P forces at least one point of S to penetrate the interior of P.*

Alternatively, a set S immobilizes P if there exists a neighbourhood U of the identity in $SE(2)$ such that for every $g \in U$ different from the identity, $g(S)$ intersects the interior of P (or equivalently $g^{-1}(P)$ intersects S). We use this latter notion of immobilization in Lemma 2.4 and following.

Clearly, the circular disk does not have any immobilizing sets since holding the disk at any number of points on its boundary leaves the disk still free to rotate about its centre. From the definition of an immobilizing set, one's focus should be on isometries of the plane close to the identity, that is, small translations and small rotations. It is observed that a set of points S immobilizes P if and only if, when P is held by point fingers at S, no translation or rotation of P is possible.

2.2 The case of a triangle

Lemma 2.3 *Let T be a triangle, T is immobilized with respect to translations by three points in its edges if and only if the points lie in different edges.*

The proof of the lemma is obvious.

Figure 2.1: The half-planes defined by orthogonal line N_X at point X in the interior of an edge of a polygon.

Lemma 2.4 *Let AB represent a line segment on the boundary of a polygon P (see Figure 2.1). Let X be any point in the interior of AB and let $U_X \subset P$ be a closed neighbourhood within P of the point X, sufficiently small that the interior of U_X lies entirely to one side of the line containing the segment AB. This is illustrated by the shaded region above. Let N_X denote the line orthogonal to AB at X. Then N_X divides the plane into two open half-planes. For each point Q in the region marked + (-), all sufficiently small anti-clockwise (clockwise) rotations g of P about Q are such that $g(U_X)$ does not meet AB.*

Proof The statement of the lemma is easy to verify.

Note: Let N_{X_i} (N_{X_e}) be the semi-infinite part of N_X starting at X and pointing into (away from) U_X. Then if $Q \neq X$ is a point of N_{X_i} (N_{X_e}), any (no) sufficiently small rotation g of P about Q is such that $g(X)$ penetrates the interior of $U_X \subset P$.

Theorem 2.5 *Three non-vertex points X, Y and Z immobilize a triangle T if and only if the orthogonal lines to the edges of T at X, Y and Z are concurrent.*

Proof

Suppose X, Y and Z immobilize T. Then no translation of T is possible when T is held at X, Y and Z, so by Lemma 2.3, X, Y and Z lie in different edges of T. Suppose the orthogonal lines N_X, N_Y, N_Z do not meet at a single point. Then N_X, N_Y and N_Z partition the plane into seven distinct regions and for each of the two half-planes on each side of lines N_X, N_Y and N_Z, a + or - sign can be attached depending on which side of the line contains points Q about which T may be rotated through some small anti-clockwise (+) or clockwise (-) angle without the given points X, Y and Z penetrating the edge of T containing the point. If this is done in order for N_X, N_Y, N_Z, a triple of signs (+ and/or -) is then attached to each of the seven regions (see Figure 2.2). It can be seen that

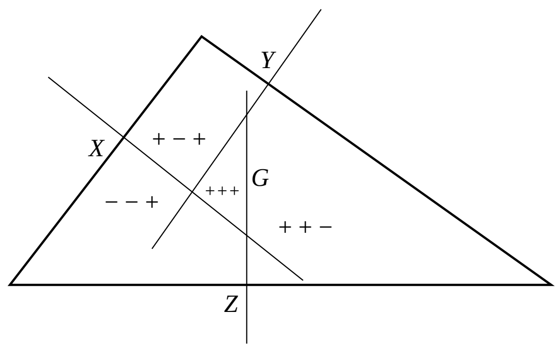


Figure 2.2: The seven regions of the plane defined by three non-concurrent lines at X, Y and Z.

the region at the centre, *i.e.* the triangle G determined by these orthogonal lines is marked +++ (in this case, but it could have been - - -), which means that for any point Q in G, there is a small anti-clockwise rotation about Q through which the triangle T may be rotated without any of X, Y or Z penetrating the interior of the triangle T.

Conversely, suppose the orthogonal lines N_X, N_Y and N_Z intersect at a point O. X, Y and Z must lie in different edges of T for this to happen. The lines N_X, N_Y and N_Z define six distinct regions of the plane. Applying Lemma 2.4 to each of the half-planes defined by the lines N_X, N_Y and N_Z in that order, we see that none of the six regions is labelled with a - - - or +++. Hence no rotation of T in the plane is possible without one of X, Y and Z penetrating one of the edges of T. In addition, by Lemma 2.3, the points X, Y and Z immobilize T with respect to translations. Therefore X, Y and Z immobilize T.

Corollary 2.6 *Let P be a plane convex figure whose boundary ∂P is a smooth curve. Suppose points X, Y and Z in ∂P immobilize P, then the tangents to ∂P*

at X, Y and Z form a triangle which contains P and the normals to ∂P at X, Y and Z are concurrent.

It is worth pointing out that the following statement [a corollary from [C2] page 186],

> *Given two points X and Y on two different sides of a triangle T, it might not be possible to find a third point Z on the remaining side of T such that X, Y and Z immobilize T. This happens only for obtuse T.*

is correct if the last sentence is omitted. Figure 2.3 shows a right angled triangle where point Z cannot be found such that the points X, Y and Z immobilize the triangle.

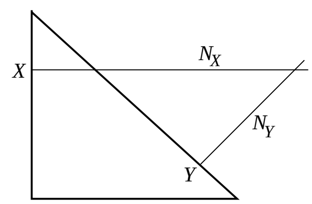


Figure 2.3: Two points of a right angled triangle that are not a subset of any immobilizing set of the triangle having three points.

2.3 The case of a polygon

Although some figures like the square require at least four points, many planar figures can be immobilized using three points. This section studies how to immobilize a polygonal object using three non-vertex points. In the triangle case immobilization with respect to translations was achieved by the requirement that no two of the three points should lie in one edge. Clearly this does not suffice where more than three edges are involved. To ensure that no translation is possible in the case of a polygon, in addition to the above requirement, no two points should lie in parallel edges. In the convex case we have the following theorem.

Theorem 2.7 *A convex polygon P can be immobilized by three non-vertex points X, Y and Z if and only if each of the points X, Y and Z belongs to a different edge of the polygon, the three lines containing the edges of P that contain the points X, Y and Z determine a triangle T that encloses P and the orthogonal lines N_X, N_Y and N_Z at X, Y and Z to the respective edges of P meet in a common point.*

Proof

If X, Y and Z do not belong to different edges of P, then they belong to one or two edges of P. Either way, P can be translated along one or both of these edges. See Figure 2.4(a). Now suppose P is convex and X, Y and Z are in different

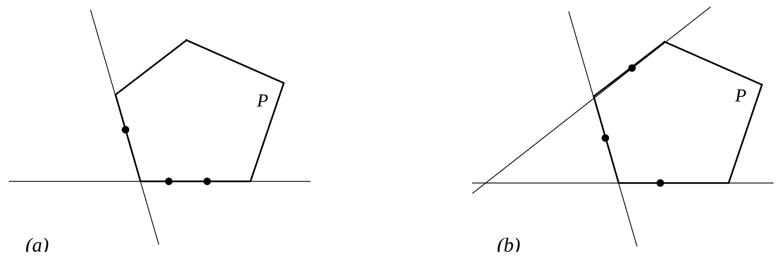


Figure 2.4: Three points of a polygon whose extended edges do not form a triangle that encloses the polygon.

edges u, v and w of P and the triangle determined by extended u, v and w does not enclose P. Then, because P is convex, that triangle is completely outside P, see Figure 2.4(b). P can then be translated along the two outer edges. If, on the other hand, X, Y and Z belong to different edges of P, the edges containing X, Y and Z when extended determine a triangle T that contains P, and the lines N_X, N_Y and N_Z do not meet in a common point, then, by Theorem 2.5, X, Y and Z do not immobilize T, hence do not immobilize P.

Conversely, suppose each of X, Y and Z belongs to a different edge of P, the three lines containing the edges of P that contain X, Y and Z determine a triangle T, P is enclosed in T and the orthogonal lines N_X, N_Y, N_Z meet in a common point. Then focusing on the triangle T, the conditions of Theorem 2.5 are satisfied, hence X, Y and Z immobilize T, and hence immobilize P.

Corollary 2.8 *Let X, Y and Z be three non-vertex points of a polygon P, not necessarily convex, and N_X, N_Y and N_Z orthogonal lines at X, Y and Z to the edges of P that contain X, Y and Z respectively. Then if N_X, N_Y and N_Z do not meet in a common point, X, Y and Z do not immobilize P.*

Next, the situation where three non-vertex points immobilize a non-convex polygon is considered. From Corollary 2.8, the concurrency of the orthogonal lines is still necessary but the lines containing the edges containing the three points need not define a triangle, and even when they do, that triangle need not enclose the polygon for the points to immobilize the polygon.

Theorem 2.9 *Let P be a polygon and X, Y and Z be three non-vertex points of P belonging to different edges of P, no two of which are parallel. Let E_X, E_Y and E_Z be the lines that contain the edges of P that contain points X, Y and Z respectively. Suppose that the orthogonal lines N_X, N_Y and N_Z to the lines E_X, E_Y and E_Z at points X, Y and Z respectively are concurrent. Then:*

(a) there exist ten ways in which lines E_X, E_Y and E_Z define a triangle; for three of these, the points X, Y and Z immobilize P.

(b) there exist six ways in which lines E_X, E_Y and E_Z are concurrent; for two of these, the points X, Y and Z immobilize P.

Proof

Suppose the orthogonal lines N_X, N_Y and N_Z are concurrent. The lines E_X, E_Y and E_Z either define a triangle or are concurrent.

(a) Suppose the lines E_X, E_Y, E_Z define a triangle. Let orthogonal line N_Q at point Q and point Q be represented by a line with a marked point. Consider the constellation of three concurrent lines, each line with a marked point different from the point of concurrency. The constellation represents the three concurrent diagonal lines N_X, N_Y and N_Z. There are only two essentially different cases as shown in Figure 2.5. Now consider a segment of E_X at X. The points of P in

Figure 2.5: The constellations representing three concurrent lines each having a marked point different from the point of concurrency.

the immediate neighbourhood of X lie on one side of this segment (or on one side of E_X). Using a shading to represent the side of E_X that contains points of P in the immediate neighbourhood of X, six figures are obtained for each of the constellations in Figure 2.5. However two pairs of these are the same configuration, resulting in ten configurations presented in Figure 2.6(a) to (j). In Figure 2.6 triples of signs have been attached to each of the six regions defined by the orthogonal lines according to the principle of Lemma 2.4. It is observed that Figures 2.6(d), 2.6(g) and 2.6(j) have no subregion marked - - - nor +++. This means that with these configurations there is no region in the plane at which a rotation of P can be effected without any of X, Y or Z penetrating P through their edges. So these immobilize P with respect to rotations. In Figure 2.6(a)

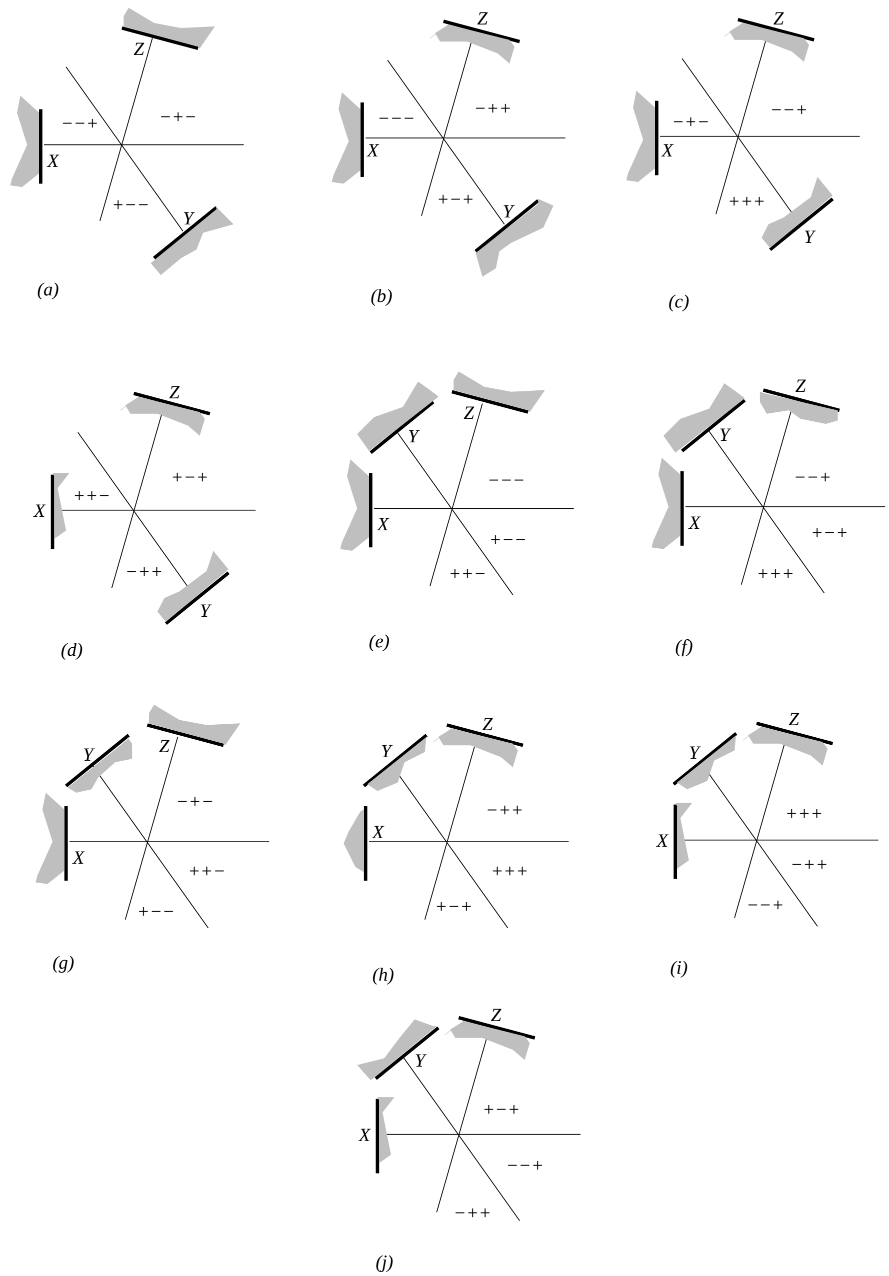


Figure 2.6: The *signed* regions of the ten different configurations that represent the case when the lines E_X, E_Y and E_Z define a triangle.

none of the six regions is marked with - - - or +++ but any rotation about the point of concurrency of lines N_X, N_Y and N_Z does not lead to any of the points X, Y, Z penetrating their respective edges.

It remains to show that the points X, Y, Z of Figure 2.6(d), 2.6(g) and 2.6(j) immobilize P with respect to translations. For each figure first consider the line E_X containing the edge containing point X. Shade out the open half-plane with boundary E_X that does *not* contain interior points in the immediate neighbourhood of X. The shading represents all the planar translations of the polygon P that would cause point X to penetrate P through the edge in E_X. The unshaded half-plane represents the directions in which P can be translated without point X penetrating P through the edge. Doing this for each of the three lines E_X, E_Y and E_Z in each figure, it is seen that the entire plane is shaded in the Figures 2.6(g) and 2.6(j). In Figure 2.6(d) the triangle defined by E_X, E_Y and E_Z is left unshaded but encloses a part of the polygon P. Just like in the convex case, this part, and hence the whole polygon, is immobilized with respect to translations by the points X, Y and Z. This means that the points X, Y and Z of Figures 2.6(d), 2.6(g) and 2.6(j), in addition to immobilizing P with respect to rotations, immobilize P with respect to translations. Therefore P is immobilized by the points X, Y and Z in these configurations.

(b) Suppose that the lines E_X, E_Y and E_Z are concurrent. The concurrent orthogonal lines N_X, N_Y and N_Z and concurrent E_X, E_Y and E_Z are represented by the constellation in Figure 2.7. Consider a segment of E_X at X. The points

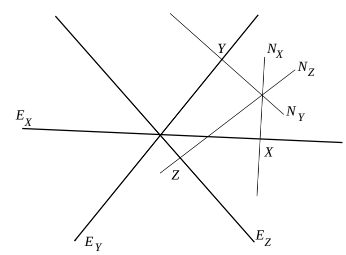


Figure 2.7: A constellation representing three concurrent lines having concurrent orthogonal lines.

of P in the immediate neighbourhood of X lie on one side of this segment. Using a shading to represent the side of this segment that contains points of P in the immediate neighbourhood of X six different configurations are obtained (see Figure 2.8 (a) to (f)). Attach triples of signs to each subregion in each of the figures as was done earlier. It is seen that the points X, Y and Z of Figures 2.8(c) and 2.8(e) immobilize P with respect to rotations.

As was done in part (a) of the proof, for each of the Figures 2.8(c) and 2.8(e), shade out, for each point, the half-planes that represent planar translations of P that would cause the point to penetrate the polygon. It is seen that the entire plane is shaded in the configuration of both figures. Therefore the points X, Y and Z of Figures 2.8(c) and 2.8(e) immobilize P.

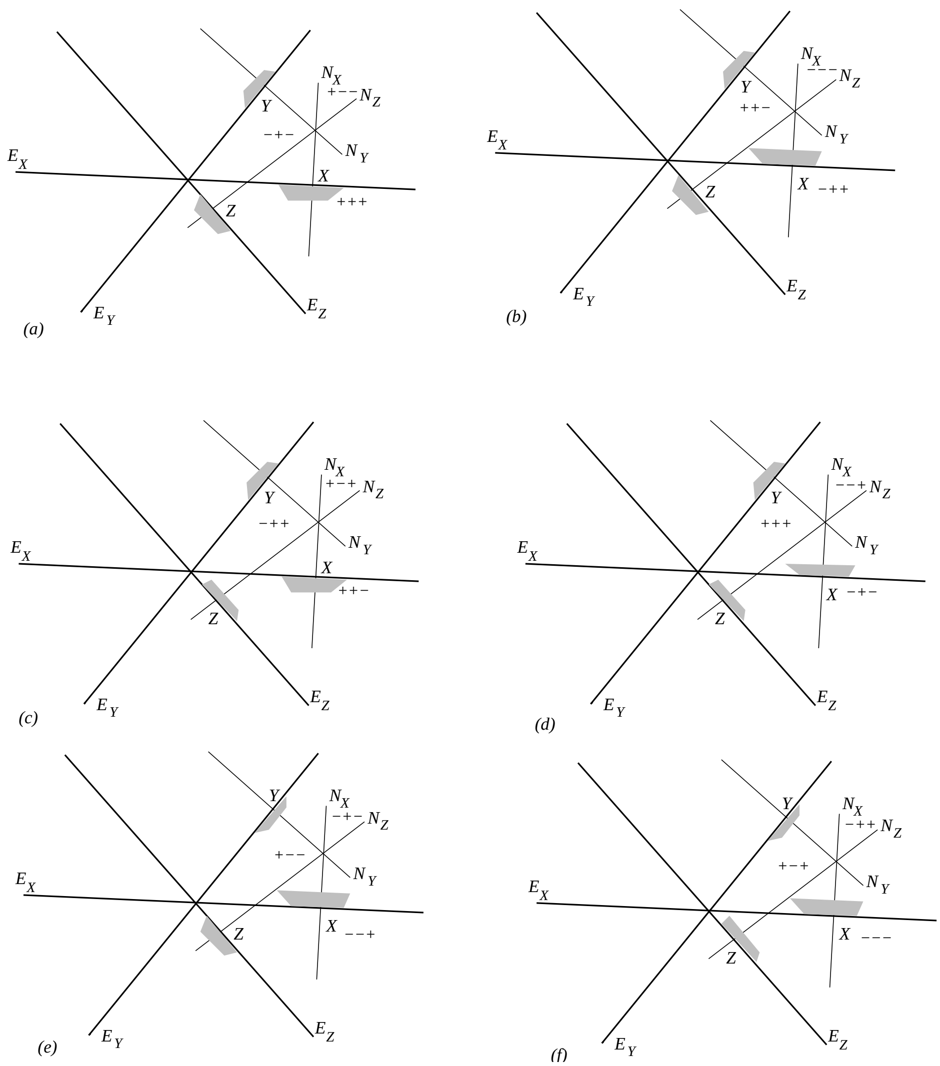


Figure 2.8: The *signed* regions of the six different configurations that represent the case when E_X, E_Y and E_Z are concurrent.

2.4 Conclusion

Let P be a general polygon, not necessarily convex and X, Y and Z three non-vertex points of P belonging to different edges, no two of which are parallel. Let E_X, E_Y and E_Z be the lines containing the edges containing the points X, Y and Z, N_X, N_Y and N_Z be the orthogonal lines to E_X, E_Y and E_Z at X, Y and Z respectively and $\mathcal{C}$ be the set of configurations of E_X, E_Y and E_Z for concurrent N_X, N_Y and N_Z in Figures 2.6(d), 2.6(g), 2.6(j), 2.8(c) and 2.8(e). Then the points X, Y and Z immobilize a polygon if and only if the configuration of the lines E_X, E_Y and E_Z containing their edges and the orthogonal lines N_X, N_Y and N_Z at them is one of the configurations in $\mathcal{C}$.

In [C2] to each edge e_X of P containing point X was assigned the halfplane containing points from the interior of P in the immediate neighbourhood of X and whose boundary contains the edge e_X. The edges e_X, e_Y and e_Z were then said to form a *triangular triple* of P if the intersection of the three halfplanes assigned to them is a triangle. It was then claimed in Theorem 3 on page 187 of [C2] that

> *A polygon P can be immobilized by three points X, Y and Z different from the vertices of P if and only if*
>
> - *the orthogonals at the points X, Y and Z to its respective edges e_X, e_Y and e_Z meet at a common point, and*
> - *e_X, e_Y and e_Z form a triangular triple of P.*

From Theorem 2.9 this is clearly wrong, as the configuration in Figure 2.6(g), for example, shows.

Chapter 3

Line geometry

3.1 Introduction

In Chapter 2 it was shown that a necessary and sufficient condition for a set of three points in the edges of a triangle to immobilize the triangle is that the normal lines to the edges at these points should be concurrent. This means that the search for a set of immobilizing points of a triangle can be construed as a search for three concurrent lines orthogonal to the edges of a triangle. Similarly, it is to be expected that the normal lines at the immobilizing points of a 3-dimensional simplex will form a special type of configuration. It is therefore necessary to study the relevant geometry of lines in space. The first problem encountered is assigning coordinates to lines in space. The discovery of these coordinates was attributed to Cayley in [BA] (see pg 56) and to Plücker in [GR] (see pg 461). We begin with the simple task of assigning coordinates to lines in $\mathbb{P}^2$.

3.2 Line coordinates in $\mathbb{P}^2$

Every linear homogeneous equation $u_0X_0 + u_1X_1 + u_2X_2 = 0$ in $\mathbb{P}^2$, where u_0, u_1, u_2 are not all zero, represents a line in $\mathbb{P}^2$, and conversely. The homogeneous line coordinates of a line having equation $u_0X_0 + u_1X_1 + u_2X_2 = 0$ are (u_0, u_1, u_2). A point $X = (X_0, X_1, X_2)$ in $\mathbb{P}^2$ lies on the line $u = (u_0, u_1, u_2)$ if and only if $u_0X_0 + u_1X_1 + u_2X_2 = 0$.

3.3 Plücker coordinates of a line in $\mathbb{P}^3$

In $\mathbb{P}^3$ every linear homogeneous equation $u_0X_0 + u_1X_1 + u_2X_2 + u_3X_3 = 0$, where u_0, u_1, u_2, u_3 are not all zero, represents a plane, and conversely. The homogeneous coordinates of a plane having equation $u_0X_0 + u_1X_1 + u_2X_2 + u_3X_3 = 0$

are (u_0, u_1, u_2, u_3). A point $X = (X_0, X_1, X_2, X_3)$ in $\mathbb{P}^3$ lies on the plane $u = (u_0, u_1, u_2, u_3)$ if and only if $u_0X_0 + u_1X_1 + u_2X_2 + u_3X_3 = 0$.

The equation of a straight line going through the points $X = (X_0, X_1, X_2, X_3)$ and $Y = (Y_0, Y_1, Y_2, Y_3)$ in $\mathbb{P}^3$ is given by

$$\frac{X_3U_0 - U_3X_0}{X_3Y_0 - Y_3X_0} = \frac{X_3U_1 - U_3X_1}{X_3Y_1 - Y_3X_1} = \frac{X_3U_2 - U_3X_2}{X_3Y_2 - Y_3X_2},$$

where $U_0, \ldots, U_3$ are the coordinates of an arbitrary point on the line. Clearly the coordinates of such a line cannot be simply read off its equation like that of a line in $\mathbb{P}^2$. The coordinates of a line in $\mathbb{P}^3$ are obtained by introducing redundant coordinates which are related by a quadratic relation. These are called Plücker coordinates and are defined from two equivalent points of view. The two dual sets of coordinates obtained were called Plücker *ray coordinates* and Plücker *axis coordinates* in [GR].

3.3.1 Plücker ray coordinates

Let $X = (X_0, \ldots, X_3)$, $Y = (Y_0, \ldots, Y_3)$ be any two distinct points on the line ℓ in $\mathbb{P}^3$. Consider the set of coordinates defined by

$$p_{ij} = X_iY_j - X_jY_i, \quad 0 \leq i \neq j \leq 3.$$

Not all p_{ij} can be zero since $X \neq Y$ but $p_{ij} = -p_{ji}$ for all i, j. The six numbers p_{01}, p_{02}, p_{03}, p_{23}, p_{31}, p_{12} constitute a set of homogeneous coordinates for ℓ. Replacing X and Y with $U = \lambda_{11}X + \lambda_{12}Y$ and $V = \lambda_{21}X + \lambda_{22}Y$ where $U \neq V$ are two new points on ℓ, the coordinates of ℓ are replaced by

$$U_i V_j - U_j V_i = (\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})\, p_{ij},$$

where $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \neq 0$ since $U \neq V$. So the homogeneous coordinates $\{p_{ij}\}$ are unchanged. The numbers $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$ are the *Plücker ray coordinates* of the line.

3.3.2 Plücker axis coordinates

A line in $\mathbb{P}^3$ is also uniquely determined by two intersecting planes. If $(u_0, \ldots, u_3)$ and $(w_0, \ldots, w_3)$ are the coordinates of different planes χ and ψ that meet in the line ℓ, not all the numbers

$$q_{ij} = u_iw_j - u_jw_i, \quad 0 \leq i \neq j \leq 3,$$

are zero and $q_{ij} = -q_{ji}$. The six numbers $q_{01}, q_{02}, q_{03}, q_{23}, q_{31}, q_{12}$ are the *Plücker axis coordinates* of ℓ. Any two distinct planes through ℓ determine coordinates proportional to q_{ij}.

Proposition 3.1 *The Plücker ray coordinates and the Plücker axis coordinates of a line are connected by the equations*

$$p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12} = q_{23} : q_{31} : q_{12} : q_{01} : q_{02} : q_{03}. \tag{3.1}$$

Proof

Two points that determine a line ℓ lie on any plane that contains ℓ. Therefore if $X = (X_0, \ldots, X_3)$, $Y = (Y_0, \ldots, Y_3)$ are two points that determine ℓ and $u = (u_0, \ldots, u_3)$, $w = (w_0, \ldots, w_3)$ are two planes that meet in ℓ, then

$$u_0X_0 + u_1X_1 + u_2X_2 + u_3X_3 = 0 \tag{3.2}$$
$$u_0Y_0 + u_1Y_1 + u_2Y_2 + u_3Y_3 = 0 \tag{3.3}$$
$$w_0X_0 + w_1X_1 + w_2X_2 + w_3X_3 = 0 \tag{3.4}$$
$$w_0Y_0 + w_1Y_1 + w_2Y_2 + w_3Y_3 = 0 \tag{3.5}$$

On multiplying Equations (3.2) and (3.4) by w_0 and u_0 respectively and subtracting the two outcomes we obtain

$$q_{01}X_1 + q_{02}X_2 + q_{03}X_3 = 0. \tag{3.6}$$

Doing the same thing with Equations (3.3) and (3.5) we obtain

$$q_{01}Y_1 + q_{02}Y_2 + q_{03}Y_3 = 0. \tag{3.7}$$

Now solve for the ratio of the q's in Equations (3.6) and (3.7) to obtain

$$\begin{aligned} q_{01} : q_{02} : q_{03} &= X_2Y_3 - X_3Y_2 : X_3Y_1 - X_1Y_3 : X_1Y_2 - X_2Y_1 \\ &= p_{23} : p_{31} : p_{12}. \end{aligned}$$

To get the remaining part of (3.1), multiply (3.2) and (3.3) by Y_0 and X_0 respectively and subtract the two outcomes to obtain

$$p_{01}u_1 + p_{02}u_2 + p_{03}u_3 = 0. \tag{3.8}$$

Do the same thing with Equations (3.4) and (3.5) to obtain

$$p_{01}w_1 + p_{02}w_2 + p_{03}w_3 = 0. \tag{3.9}$$

Solving for the ratio of the p's in Equations (3.8) and (3.9) we obtain

$$\begin{aligned} p_{01} : p_{02} : p_{03} &= u_2w_3 - u_3w_2 : u_3w_1 - u_1w_3 : u_1w_2 - u_2w_1 \\ &= q_{23} : q_{31} : q_{12}. \end{aligned}$$

For the rest of this thesis Plücker coordinates of a line will mean Plücker ray coordinates.

Lemma 3.2 *The Plücker coordinates of a line ℓ in $\mathbb{R}^3$ going through the point P with direction vector $\mathbf{n}$ are $(\mathbf{n}, P \times \mathbf{n})$.*

Proof

Let $P = (P_x, P_y, P_z)$ and $\mathbf{n} = (n_x, n_y, n_z)$. As an element of $\mathbb{P}^3$, the line ℓ goes through the points $X = (1, P_x, P_y, P_z)$ and $Y = (0, n_x, n_y, n_z)$. Therefore its Plücker coordinates are

$$(n_x, n_y, n_z, P_y n_z - P_z n_y, P_z n_x - P_x n_z, P_x n_y - P_y n_x) = (\mathbf{n}, P \times \mathbf{n}).$$

3.4 The Klein quadric

Proposition 3.3 *There is a one to one correspondence between the lines of $\mathbb{P}^3$ and the points of the quadric $\xi_0\xi_3 + \xi_1\xi_4 + \xi_2\xi_5 = 0$ in $\mathbb{P}^5$.*

Proof

First, we show that the Plücker coordinates of any line satisfy the relation

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0.$$

Suppose $X = (X_0, \ldots, X_3)$ and $Y = (Y_0, \ldots, Y_3)$ are two distinct points on the line. Then the determinant Ξ of the matrix

$$\begin{bmatrix} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \\ X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{bmatrix}$$

is zero. Expanding Ξ using 2×2 minors yields

$$\begin{aligned} \Xi &= (X_0Y_1 - X_1Y_0)(X_2Y_3 - X_3Y_2) - (X_0Y_2 - X_2Y_0)(X_1Y_3 - X_3Y_1) \\ &\quad + (X_0Y_3 - X_3Y_0)(X_1Y_2 - X_2Y_1) + (X_1Y_2 - X_2Y_1)(X_0Y_3 - X_3Y_0) \\ &\quad - (X_1Y_3 - X_3Y_1)(X_0Y_2 - X_2Y_0) + (X_2Y_3 - X_3Y_2)(X_0Y_1 - X_1Y_0) \\ &= 2\,(p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12}). \end{aligned}$$

Conversely, if $\xi = (\xi_0, \xi_1, \ldots, \xi_5) \in \mathbb{P}^5$ with $\xi_0 \neq 0$ satisfies $\xi_0\xi_3 + \xi_1\xi_4 + \xi_2\xi_5 = 0$, set $X = (\xi_0, 0, -\xi_5, \xi_4)$ and $Y = (0, \xi_0, \xi_1, \xi_2)$. The Plücker coordinates of the line going through the points X and Y are

$$\begin{aligned} &= (\xi_0\xi_0, \xi_0\xi_1, \xi_0\xi_2, -\xi_2\xi_5 - \xi_1\xi_4, \xi_0\xi_4, \xi_0\xi_5) \\ &= (\xi_0\xi_0, \xi_0\xi_1, \xi_0\xi_2, \xi_0\xi_3, \xi_0\xi_4, \xi_0\xi_5) \\ &= (\xi_0, \ldots, \xi_5). \end{aligned}$$

For the following cases the given points X and Y define the appropriate line:

(i) $\xi_0 = 0,\ \xi_1 \neq 0;\quad X = (0, 0, -\xi_1, -\xi_2),\ Y = (1, \frac{\xi_5}{\xi_1}, 0, -\frac{\xi_3}{\xi_1})$

(ii) $\xi_0 = 0,\ \xi_2 \neq 0;\quad X = (0, 0, -\xi_1, -\xi_2),\ Y = (1, -\frac{\xi_4}{\xi_2}, \frac{\xi_3}{\xi_2}, 0)$

(iii) $\xi_0 = \xi_1 = \xi_2 = 0,\ \xi_3 \neq 0;\quad X = (0, -\xi_4, \xi_3, 0),\ Y = (0, -\frac{\xi_5}{\xi_3}, 0, 1)$

(iv) $\xi_0 = \xi_1 = \xi_2 = 0,\ \xi_4 \neq 0;\quad X = (0, -\xi_4, \xi_3, 0),\ Y = (0, 0, -\frac{\xi_5}{\xi_4}, 1)$

(v) $\xi_0 = \xi_1 = \xi_2 = 0,\ \xi_5 \neq 0;\quad X = (0, 1, 0, -\frac{\xi_3}{\xi_5}),\ Y = (0, 0, \xi_5, -\xi_4)$

Hence each ξ satisfying $\xi_0\xi_3 + \xi_1\xi_4 + \xi_2\xi_5 = 0$ corresponds to a line in $\mathbb{P}^3$ whose Plücker coordinates $p_{01}, \ldots, p_{12}$ are $\xi_0, \ldots, \xi_5$. This correspondence is one-to-one.

The equation $\xi_0\xi_3 + \xi_1\xi_4 + \xi_2\xi_5 = 0$ determines a quadric Q in $\mathbb{P}^5$ whose matrix of coefficients is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since this matrix has determinant -1 quadric Q is nonsingular. This quadric is known as the *Klein quadric*.

Corollary 3.4 *The geometry in which the line is the fundamental element is four dimensional.*

Definition 3.5 *Two points $\xi = (\xi_0, \ldots, \xi_5)$ and $\eta = (\eta_0, \ldots, \eta_5)$ of $\mathbb{P}^5$ are said to be conjugate with respect to the Klein quadric if*

$$\xi_0\eta_3 + \xi_1\eta_4 + \xi_2\eta_5 + \xi_3\eta_0 + \xi_4\eta_1 + \xi_5\eta_2 \;=\; 0.$$

Proposition 3.6 *The line ℓ with Plücker coordinates $p_{01}, \ldots, p_{12}$ and line ℓ' having Plücker coordinates $p'_{01}, \ldots, p'_{12}$ intersect if and only if*

$$(\ell, \ell') \;:=\; p_{01}p'_{23} + p_{02}p'_{31} + p_{03}p'_{12} + p_{23}p'_{01} + p_{31}p'_{02} + p_{12}p'_{03} \;=\; 0.$$

Proof

Suppose X, Y are points that determine the line ℓ and X', Y' are points that determine the line ℓ'. The lines ℓ and ℓ' intersect if and only if the four points X, Y, X' and Y' lie in one plane. That is, if and only if

$$\begin{vmatrix} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \\ X'_0 & X'_1 & X'_2 & X'_3 \\ Y'_0 & Y'_1 & Y'_2 & Y'_3 \end{vmatrix} \;=\; 0,$$

but this determinant equals (ℓ, ℓ').

Therefore two lines in $\mathbb{P}^3$ intersect if and only if their corresponding points on Q are conjugate.

Proposition 3.7 *The Klein quadric Q contains two families of 2-planes.*

The proof comes from [SE] but corrections have been made to it.

Proof

Let the coordinates of a point $\xi = (\xi_0, \ldots, \xi_5) \in \mathbb{P}^5$ be given by $\xi = (\mathbf{u}, \mathbf{v})$ where $\mathbf{u} = (\xi_0, \xi_1, \xi_2)$ and $\mathbf{v} = (\xi_3, \xi_4, \xi_5)$. On performing the change of coordinates:

$$\xi_0 = X_0 + X_3 \qquad \xi_3 = X_0 - X_3$$

$$\xi_1 = X_1 + X_4 \qquad \xi_4 = X_1 - X_4$$

$$\xi_2 = X_2 + X_5 \qquad \xi_5 = X_2 - X_5,$$

the equation of Q becomes

$$X_0^2 + X_1^2 + X_2^2 - X_3^2 - X_4^2 - X_5^2 = 0.$$

Let M be a 3×3 real matrix. Consider the points in $\mathbb{P}^5$ satisfying the three homogeneous linear equations $X = MX'$ where

$$X = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} \quad \text{and} \quad X' = \begin{pmatrix} X_3 \\ X_4 \\ X_5 \end{pmatrix}.$$

The points satisfying these equations lie in a 2-plane, the 'graph' of the matrix M. If the matrix M is orthogonal then the points also lie on the Klein quadric, since $X = MX'$ implies

$$\begin{aligned} X \cdot X &= MX' \cdot MX' \\ &= X'^t M^t M X' \\ &= X'^t X' \\ &= X' \cdot X' \end{aligned}$$

and $X \cdot X = X' \cdot X'$ is the new equation for the Klein quadric after undergoing the above change of coordinates. Suppose

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

and $X = MX'$ is a plane on Q. Then

$$\begin{aligned} m_{11}X_3 + m_{12}X_4 + m_{13}X_5 &= X_0, \\ m_{21}X_3 + m_{22}X_4 + m_{23}X_5 &= X_1, \\ m_{31}X_3 + m_{32}X_4 + m_{33}X_5 &= X_2, \end{aligned}$$

which on substituting in the equation of Q yields:

$$m_{11}^2+m_{21}^2+m_{31}^2=m_{12}^2+m_{22}^2+m_{32}^2=m_{13}^2+m_{23}^2+m_{33}^2=1$$

and

$$m_{1i}m_{1j}+m_{2i}m_{2j}+m_{3i}m_{3j}=0, i,j=1,2,3, i\neq j,$$

hence M is an orthogonal matrix, thus $X=MX'$ represents a 2-plane on Q if and only if M is orthogonal.

Since $\mathbf{u}=X+X'$, $\mathbf{v}=X-X'$, the equation $MX'=X$ of a plane in $\mathbb{P}^5$ can be written as

$$(I_3-M)\mathbf{u}+(I_3+M)\mathbf{v}=\mathbf{0}.$$

Case 1. $M^tM=I_3$, $\det(M)=+1$
Suppose also that $(M+I_3)$ is nonsingular (which is the general case when $\det M=+1$), then one can write

$$\mathbf{v}=(M+I_3)^{-1}(M-I_3)\mathbf{u}.$$

However, the matrix $M_+=(M+I_3)^{-1}(M-I_3)$ is skew-symmetric because

$$\begin{aligned}
M_+^t &= (M^t-I_3)(M^t+I_3)^{-1}\\
&= (M^t-I)MM^t(M^t+I_3)^{-1}\\
&= (I-M)M^{-1}(M^t+I_3)^{-1}\\
&= (I-M)[(M^t+I_3)M]^{-1}\\
&= -(M-I_3)(M+I_3)^{-1}\\
&= -M_+
\end{aligned}$$

since $M-I_3$ and $M+I_3$ commute. Let

$$M_+=\begin{pmatrix} 0 & -\gamma & \beta\\ \gamma & 0 & -\alpha\\ -\beta & \alpha & 0\end{pmatrix}$$

and $\mathbf{m}_+$ the vector (α,β,γ) associated to M_+. Then $\mathbf{v}=\mathbf{m}_+\times\mathbf{u}$. Recall that if $(\mathbf{u},\mathbf{v})$ represents the Plücker coordinates of a line ℓ in $\mathbb{P}^3$, the vector $\mathbf{u}$ denotes the direction of ℓ and $\mathbf{v}$ is given by $P\times\mathbf{u}$ for any point P on the line. Thus when $\det(M)=+1$ and $M+I_3$ is nonsingular the points of the plane with equation $(I_3-M)\mathbf{u}+(I_3+M)\mathbf{v}=\mathbf{0}$ lie on Q and represent lines going through the point having position vector $\mathbf{m}_+$.

In the degenerate case suppose $\det(M)=1$ and $M+I_3$ is singular. Since $M\in SO(3)$, let $\mathbf{a}$ be the non-zero unit vector for which $M\mathbf{a}=\mathbf{a}$. Choose an orthonormal basis $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ for $\mathbb{R}^3$ such that $M\mathbf{a}=\mathbf{a}$ and in the plane $\text{Span}\{\mathbf{b},\mathbf{c}\}$, M is a rotation $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then, in the chosen basis,

$$M + I_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \cos\theta+1 & -\sin\theta \\ 0 & \sin\theta & \cos\theta+1 \end{pmatrix}$$

and $\det(M+I_3) = 2\left[(\cos\theta+1)^2 + \sin^2\theta\right], \Rightarrow \cos\theta = -1$, so $\theta=\pi$. Therefore $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, so if $\mathbf{u}$ and $\mathbf{v}$ have coordinates $(u_x,u_y,u_z)^t$ and $(v_x,v_y,v_z)^t$ with respect to basis $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ respectively, then the original equation becomes

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

giving solution $\mathbf{u} = u_x\mathbf{a}$, $\mathbf{v} = v_y\mathbf{b} + v_z\mathbf{c}$, where u_x, v_y, v_z are arbitrary, but not all vanishing together. The case $u_x \neq 0$ describes all lines parallel to $\mathbf{a}$.

Case 2. $M^tM = I_3$, $\det(M) = -1$

Then $(M-I_3)$ is generally nonsingular hence $\mathbf{u} = (M-I_3)^{-1}(M+I_3)\mathbf{v}$ where $M_- = (M-I_3)^{-1}(M+I_3)$ is skew symmetric. Therefore, like we argued in the first case, $\mathbf{u} = \mathbf{m}_- \times \mathbf{v}$ for the vector $\mathbf{m}_-$ associated to the matrix M_-. Since $\mathbf{m}_- \cdot \mathbf{u} = \mathbf{m}_- \cdot (\mathbf{m}_- \times \mathbf{v}) = 0$, the lines, having Plücker coordinates $(\mathbf{u},\mathbf{v})$, associated to the orthogonal matrix are all perpendicular to vector $\mathbf{m}_-$. If $\mathbf{v} = q \times \mathbf{u}$, where q is an arbitrary point on a line, then

$$\begin{aligned} \mathbf{u} &= \mathbf{m}_- \times \mathbf{v} \\ &= \mathbf{m}_- \times (q \times \mathbf{u}) \\ &= (\mathbf{m}_- \cdot \mathbf{u})q - (\mathbf{m}_- \cdot q)\mathbf{u} \\ &= -\mathbf{m}_- \cdot q\,\mathbf{u} \\ \Rightarrow \quad \mathbf{m}_- \cdot q &= -1 \end{aligned}$$

Hence the lines associated to M when $\det(M) = -1$ and $M - I_3$ is nonsingular are the lines lying in a 2-plane in $\mathbb{R}^3$.

In the degenerate case suppose $\det(M) = -1$ and $M - I_3$ is singular. Then there exists a unit vector $\mathbf{a}$ such that $M\mathbf{a} = \mathbf{a}$. In the complimentary subspace $\{\mathbf{a}\}^\perp$, M defines a reflection in some line through the origin. An orthonormal basis $\{\mathbf{b},\mathbf{c}\}$ for this plane may now be chosen with axes along and perpendicular to the line of reflection. Then with respect to the orthonormal basis $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$, M has matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Proceeding as above, in this basis, the original equation becomes

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

with solution $\mathbf{u} = u_x\mathbf{a} + u_y\mathbf{b}, \mathbf{v} = v_z\mathbf{c}$, where u_x, u_y, v_z are arbitrary, but not all vanishing together. The case $(u_x, u_y) \neq (0, 0)$ describes all lines lying in the plane $\text{Span}\{\mathbf{a}, \mathbf{b}\}$.

Remarks

A 2-plane on Q whose corresponding orthogonal matrix has positive determinant is called an α-plane of Q, and one whose orthogonal matrix has negative determinant is called a β-plane of Q. If $(\mathbf{u}, \mathbf{v})$ is used to represent the Plücker coordinates of a line in $\mathbb{R}^3$, $\mathbf{v} = \mathbf{0}$ means that the line passes through the origin and $\mathbf{u} = \mathbf{0}$ means that the line lies in the plane at infinity. Two distinct α-planes intersect in one point, this point representing the single line common to the two bundles (a bundle is the collection of all lines going through a point in $\mathbb{R}^3$). A particular α-plane does not in general intersect a particular β-plane. This reflects the fact that in $\mathbb{P}^3$ a generally chosen point does not lie on a generally chosen plane, hence no line lying on a plane is expected to pass through the point. However if an α-plane and a β-plane intersect, they do so in a line, which corresponds to the set of lines in a plane passing through a point. Such a configuration is called a plane pencil of lines.

Lemma 3.8 *The Klein quadric does not contain a 3-space.*

Proof

Let $\mathcal{Q}$ be the nonsingular matrix given at the end of Proposition 3.3. Then Q can be written as $\frac{1}{2}\,\xi^t\mathcal{Q}\xi = 0$. Suppose S is a $\mathbb{P}^3 \subset \mathbb{P}^5$ containing independent points $\xi_1, \ldots, \xi_4$. Then $S = \text{Span}(\sum t_i\xi_i)$ and if $S \subset Q$,

$$\begin{aligned} \left(\sum t_i\xi_i\right)^t \mathcal{Q} \left(\sum t_i\xi_i\right) &= 0 \ \forall \ t_i \\ \Leftrightarrow \ \xi_i^t\mathcal{Q}\xi_j &= 0 \ \forall \ i, j. \end{aligned}$$

Since $\mathcal{Q}$ is nonsingular this implies $\{\mathcal{Q}\xi_j\}$ are four linearly independent vectors in $\mathbb{R}^6$, each of which is perpendicular to $\xi_1, \ldots, \xi_4$.

Proposition 3.9 *Four lines, no three having linearly dependent Plücker coordinates , have linearly dependent Plücker coordinates if and only if every line which meets three of the lines intersects the fourth. Then the four lines:*

1. *belong to a bundle (assemblage of all lines in space through a point) or a plane of lines, or*
2. *intersect in pairs so that the pencils determined by the two pairs have a line in common but lie in different planes and have different vertices, or*
3. *belong to one ruling of a quadric surface.*

Proof

Suppose the lines $\ell_1,\ \ell_2,\ \ell_3,\ \ell_4,$ no three having linearly dependent Plücker coordinates, have linearly dependent Plücker coordinates $l_1,\ l_2,\ l_3,\ l_4$ where $l_i = (p_{01}^{(i)}, \ldots, p_{21}^{(i)})$. Then there exists non-zero constants $k_1,\ k_2,\ k_3$ such that

$$\begin{aligned} l_4 &= k_1 l_1 + k_2 l_2 + k_3 l_3 \\ \textit{i.e. } (p_{01}^{(4)}, \ldots, p_{21}^{(4)}) &= (k_1 p_{01}^{(1)} + k_2 p_{01}^{(2)} + k_3 p_{01}^{(3)}, \ldots, k_1 p_{21}^{(1)} + k_2 p_{21}^{(2)} + k_3 p_{21}^{(3)}). \end{aligned}$$

Now if the line ℓ with Plücker coordinates $l = (p_{01}, \ldots, p_{21})$ meets the lines $\ell_1,\ \ell_2,\ \ell_3$, then

$$\begin{aligned} (\ell, \ell_4) &= p_{01}p_{23}^{(4)} + p_{02}p_{31}^{(4)} + p_{03}p_{12}^{(4)} + p_{23}p_{01}^{(4)} + p_{31}p_{02}^{(4)} + p_{12}p_{03}^{(4)} \\ &= k_1(\ell, \ell_1) + k_2(\ell, \ell_2) + k_3(\ell, \ell_3) \\ &= 0. \end{aligned}$$

Hence ℓ meets ℓ_4 as well.

Conversely, suppose $\ell_1,\ \ell_2,\ \ell_3$ and ℓ_4 are four lines no three of which have linearly dependent Plücker coordinates. Let $\xi_1,\ \xi_2\ \xi_3$ and ξ_4 be the points on Q corresponding (see Proposition 3.3) to the lines $\ell_1,\ \ell_2,\ \ell_3$ and ℓ_4 respectively. Then no three of $\xi_1, \ldots, \xi_4$ are collinear. Suppose ℓ is another line in $\mathbb{P}^3$ whose corresponding point on Q is ξ and ℓ meets the lines $\ell_1,\ \ell_2,\ \ell_3$. Let W be the $\mathbb{P}^2$ defined by $\xi_1,\ \xi_2,\ \xi_3$ and $q = (q_0, \ldots, q_5)$ be any point on the line in $\mathbb{P}^5$ joining ξ to ξ_i for $i = 1, 2, 3$. Then there exists a $t \in \mathbb{R}$ such that

$$\begin{aligned} q &= (1-t)l + tl_i \\ &= \left((1-t)p_{01} + tp_{01}^{(i)}, \ldots, (1-t)p_{12} + tp_{12}^{(i)}\right). \end{aligned}$$

Therefore

$$\begin{aligned} q_0q_3 + q_1q_4 + q_2q_5 &= \left[(1-t)p_{01} + tp_{01}^{(i)}\right]\left[(1-t)p_{23} + tp_{23}^{(i)}\right] \\ &+ \left[(1-t)p_{02} + tp_{02}^{(i)}\right]\left[(1-t)p_{31} + tp_{31}^{(i)}\right] \\ &+ \left[(1-t)p_{03} + tp_{03}^{(i)}\right]\left[(1-t)p_{12} + tp_{12}^{(i)}\right] \\ &= \frac{1}{2}(1-t)^2(\ell,\ell) + \frac{1}{2}t^2(\ell_i,\ell_i) + t(1-t)(\ell,\ell_i) \\ &= 0 \end{aligned}$$

Hence q belongs to Q, so the lines η_i in $\mathbb{P}^5$ joining ξ to ξ_i belong to Q. If $\xi \notin W$ the collection ξ, ξ_1, ξ_2 and ξ_3 define a $\mathbb{P}^3 \subset Q$, which cannot happen according to Lemma 3.8. Therefore $\xi \in W$. By hypothesis ξ is conjugate to ξ_4, so the line η_4 joining ξ to ξ_4 lies in Q. Since Q cannot contain a $\mathbb{P}^3$ (Lemma 3.8), the point ξ_4 lies in the same $\mathbb{P}^2$ as ξ, that is $\xi_4 \in W$. Hence the lines ℓ_1, ℓ_2, ℓ_3, ℓ_4 have linearly dependent Plücker coordinates.

Now suppose four lines $\ell_1, \dots, \ell_4$, no three having linear dependent Plücker coordinates, have linearly dependent Plücker coordinates. Then the 4 by 6 matrix M of their Plücker coordinates has rank three. Then M has three linearly independent rows which define three linearly independent points of $\mathbb{P}^5$. Three such points of $\mathbb{P}^5$ define a $\mathbb{P}^2$ in $\mathbb{P}^5$, let W be this $\mathbb{P}^2$.

Case 1. When W lies completely in Q, we have either an α-plane or a β-plane of Q (see Proposition 3.7 and remarks at the end of its proof). If W is an α-plane then its points represent concurrent lines in $\mathbb{P}^3$ and so the four lines belong to a bundle. If W is a β-plane then the lines it represents in $\mathbb{P}^3$ are coplanar and then the four lines belong to a plane field of lines.

Case 2. Suppose W meets Q in a degenerate conic, that is a line pair within Q having lines m_1 and m_2. Then two of the four points on Q corresponding to the lines $\ell_1, \dots, \ell_4$ lie on m_1 and the other two on m_2 (since no three of these four points are linearly dependent). A pair of points on m_i, $i = 1, 2$ is conjugate and two points, one on m_1 and another on m_2, are not conjugate. Therefore the four lines intersect in pairs and the pencils they generate have a line in common, represented in Q by the common point to the lines.

Case 3. When W meets Q transversally in a non-degenerate conic C, then no two points of C are conjugate, otherwise each line joining conjugate points would be part of C, making it degenerate. So the lines of $\mathbb{P}^3$ represented by the conic C form a 1-dimensional family of lines which is such that any two lines are skew to each other. The four lines belong to one ruling of a quadric surface S generated by any three lines represented by points on C.

Note The subspace W defines another $\mathbb{P}^2$, called the *polar* of W, comprising all points in $\mathbb{P}^5$ conjugate to every point of W, see [SO1]. If this $\mathbb{P}^2$ is denoted W', then W' intersects Q transversally in a non-degenerate conic C'. Every point of C is conjugate to every point of C' and vice versa. The conic C' represents lines that belong to the other ruling of the quadric surface S.

The first three cases do occur.

Case 4. If W were to touch Q along a 'double' line, the four points on Q corresponding to the four lines would have to be on this double line, implying that any three of these points are linearly dependent. This would contradict the given hypothesis.

Note Four lines satisfying the conditions of Proposition 3.9 were called *semi-concurrent* in [BR] and *linearly dependent* in [GR]. We, however, will continue to say that such lines have linearly dependent Plücker coordinates. An example of such a set of lines is any four lines in one ruling of a non-degenerate quadric surface.

Chapter 4

Immobilization in space

4.1 Introduction

In this chapter, the problem of immobilizing a tetrahedron is studied. Let T be a tetrahedron having vertices $V_1, \ldots, V_4$, faces F_i, where F_i is the face of T opposite vertex V_i, and $\mathbf{n}_i$ is an outward normal vector to face F_i. The immobilizing points of a tetrahedron were first studied by Bracho, Fetter, Mayer and Montejano [BR](1995) where, for fixed points $P_1, \ldots, P_4$, with $P_i \in \text{int}F_i$, an energy function E on $SE(3)$ was defined. For an element g of $SE(3)$ near the identity the function E measures the 'total amount of penetration' caused by g at the four points. They showed that four given points immobilize the tetrahedron if the energy function defined at these points has an isolated maximum at the identity in $SE(3)$. This is one of the two main results in [BR] and was used to prove the second main result: interior points $P_1, \ldots, P_4$ immobilize a tetrahedron if and only if $\sum_{i=1}^4 P_i \times \mathbf{n_i} = \mathbf{0}$, where the $\mathbf{n}_i$ are chosen so that $\sum_{i=1}^4 \mathbf{n}_i = \mathbf{0}$. In this thesis, a different proof of the main proposition in [BR] is given leading to a new proof of the main theorem in [BR]. This proof lays the grounds for generalizations to other dimensions. The vectors in this chapter will be column vectors.

4.2 Orientation on a tetrahedron

Four distinct points in $\mathbb{R}^3$ having coordinates $V_1, \ldots, V_4$ describe a tetrahedron if no r $(2 \leq r \leq 4)$ of them lie in the same $r-2$ dimensional affine subspace of $\mathbb{R}^3$.

Definition 4.1 *A tetrahedron will be said to be positively oriented if the determinant*

$$det \begin{bmatrix} V_1 & \cdots & V_4 \\ 1 & \cdots & 1 \end{bmatrix}$$

is positive and negatively oriented if it is negative.

This definition is motivated by the definition of an *oriented affine n-simplex* in [RU] and is clearly dependent on the ordering given to the vertices. In this thesis tetrahedra will be assumed to be positively oriented. One of the effects of choosing an orientation on the tetrahedron is to fix the outwards and inward directions of normal vectors to the faces of the tetrahedron. For example, if a tetrahedron is not oriented the vector $(V_3 - V_4) \times (V_2 - V_4)$ is orthogonal to the face having vertices V_2, V_3 and V_4 and could be inward or outward pointing. However if T is positively oriented, then

$$\begin{aligned} \det \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} &= \det \begin{bmatrix} (V_1 - V_4) & (V_2 - V_4) & (V_3 - V_4) \end{bmatrix} \\ &= (V_1 - V_4) \cdot (V_2 - V_4) \times (V_3 - V_4) \\ &= (V_4 - V_1) \cdot (V_3 - V_4) \times (V_2 - V_4) \\ &> 0. \end{aligned}$$

Since V_1 is the vertex opposite face having vertices V_2, V_3 and V_4, vector $V_3 - V_4 \times V_2 - V_4$ is outward pointing. Therefore if this face is held horizontally, with vertex V_4 behind V_3, and V_2 on the right of edge V_3V_4 as shown in Figure 4.1, vertex V_1 lies below this face (see direction of arrow in Figure 4.1). By changing the face of the tetrahedron lying in the horizontal plane the chosen

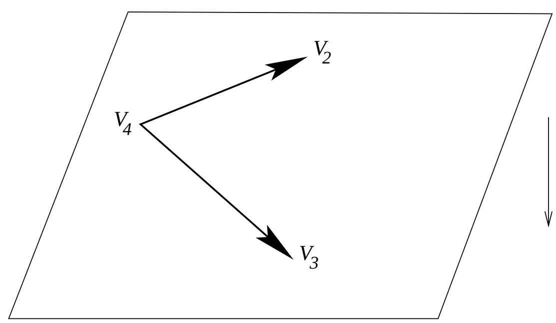


Figure 4.1: Plane containing vertices V_2, V_3 and V_4. The arrow indicates that vertex V_1 lies below this plane.

orientation can be represented by different figures. All the positive orientations can be represented geometrically by the generic $3d$ picture in Figure 4.2. This can be shown by considering the vertex V_4 in Figure 4.2 to be 'at the back' and vertex V_3 to be 'at the top'. By fixing V_4 and having V_1 then V_2, then V_3 successively at the top, 3 distinct positive orientations are obtained. Then the corresponding 3 orientations with T turned to bring each of V_1, V_2, V_3, V_4 to the back gives a total of 12 distinct positive orientations, corresponding to the 12 positive permutations of $\{V_1, V_2, V_3, V_4\}$, all of which can be brought to the generic disposition in Figure 4.2 simply by rotating T.

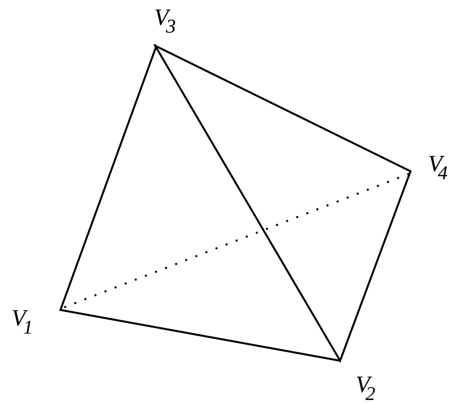


Figure 4.2: A positively oriented tetrahedron.

Lemma 4.2 *There exists a set of outward normal vectors* $\mathbf{n}_1, \ldots, \mathbf{n}_4$ *with the property that* $\sum_{i=1}^4 \mathbf{n}_i = \mathbf{0}$.

Proof

Define

$$\begin{aligned}
\mathbf{n}_1 &= (V_3 - V_4) \times (V_2 - V_4) &= (V_3 \times V_2) + (V_2 \times V_4) + (V_4 \times V_3) \\
\mathbf{n}_2 &= (V_4 - V_3) \times (V_1 - V_3) &= (V_4 \times V_1) + (V_1 \times V_3) + (V_3 \times V_4) \\
\mathbf{n}_3 &= (V_1 - V_2) \times (V_4 - V_2) &= (V_1 \times V_4) + (V_4 \times V_2) + (V_2 \times V_1) \\
\mathbf{n}_4 &= (V_2 - V_1) \times (V_3 - V_1) &= (V_2 \times V_3) + (V_3 \times V_1) + (V_1 \times V_2).
\end{aligned}$$

For a positively oriented tetrahedron each vector $\mathbf{n}_i$ points outward and they satisfy $\sum_{i=1}^4 \mathbf{n}_i = \mathbf{0}$. This set of normal vectors will be referred to as the *standard outward normals* of the tetrahedron. Any other set of outward normals $\mathbf{m}_i$ satisfying $\sum_{i=1}^4 \mathbf{m}_i = \mathbf{0}$ is simply a scalar multiple of the $\mathbf{n}_i$. For if $\mathbf{m}_i = k_i\,\mathbf{n}_i$, then

$$\begin{aligned}
\mathbf{0} &= k_1\,\mathbf{n}_1 + k_2\,\mathbf{n}_2 + k_3\,\mathbf{n}_3 + k_4\,\mathbf{n}_4 \\
&= k_1\,(-\mathbf{n}_2 - \mathbf{n}_3 - \mathbf{n}_4) + k_2\,\mathbf{n}_2 + k_3\,\mathbf{n}_3 + k_4\,\mathbf{n}_4 \\
&= (k_2 - k_1)\,\mathbf{n}_2 + (k_3 - k_1)\,\mathbf{n}_3 + (k_4 - k_1)\,\mathbf{n_4}.
\end{aligned}$$

However any three of the $\mathbf{n}_i$ are linearly independent, hence $k_1 = k_2 = k_3 = k_4$.

Observe that $|\mathbf{n}_i| = 2A_i$ where A_i is the area of face F_i. The condition that $\sum_{i=1}^4 \mathbf{n}_i = \mathbf{0}$ can also be viewed as resulting from the Divergence Theorem of Vector Calculus. Indeed if $\hat{\mathbf{n}}_i$ is the outward unit normal vector to face F_i of tetrahedron T, $\hat{\mathbf{n}}$ the outward unit normal vector to T, then

$$\sum_{i=1}^4 \mathbf{n}_i = \sum_{i=1}^4 2A_i\,\hat{\mathbf{n}}_i$$

$$
\begin{aligned}
&= 2\sum_{i=1}^{4}\int_{F_i} dS\,\hat{\mathbf{n}}_i \\
&= 2\sum_{i=1}^{4}\int_{F_i} \hat{\mathbf{n}}_i\,dS \\
&= 2\int_{\partial T} \hat{\mathbf{n}}\,dS \\
&= \mathbf{0}
\end{aligned}
$$

by Corollary 1.7.

With this choice of orientation the volume V of T is given by

$$V = \frac{1}{6}\,(V_4 - V_3 \cdot \mathbf{n}_3) = \frac{1}{6}\det\begin{bmatrix} V_1 & V_2 & V_3 & V_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Lemma 4.3 *Let $\mathbf{n}_1,\ldots,\ \mathbf{n}_4$ be the standard outward normals of a tetrahedron having vertices $V_1,\ldots,V_4$ and V its volume, then $(V_j - V_k)\cdot\mathbf{n}_k = 6V$ for all $j \neq\ k$.*

Proof

Since the magnitude of $\mathbf{n}_k$ is twice the area of face F_k and V_jV_k, is an edge of the tetrahedron not in F_k, then $(V_j - V_k)\cdot\mathbf{n}_k = \pm 6V$. The angle between $V_j - V_k$ and $\mathbf{n}_k$ is acute.

Lemma 4.4 *Let π_i be the plane of F_i, then $P_i \cdot \mathbf{n}_i$ is independent of the choice of $P_i \in \pi_i,\ 1 \le i \le 4$.*

Indeed

$$
\begin{aligned}
P_1 \cdot \mathbf{n}_1 &= V_4 \cdot (V_3 \times V_2), \\
P_2 \cdot \mathbf{n}_2 &= V_3 \cdot (V_4 \times V_1), \\
P_3 \cdot \mathbf{n}_3 &= V_2 \cdot (V_1 \times V_4), \\
P_4 \cdot \mathbf{n}_4 &= V_1 \cdot (V_2 \times V_3).
\end{aligned}
$$

4.3 Immobilizing the tetrahedron

Definition 4.5 *Let $X,\ Y \subset \mathbb{R}^3$. The $SE(3)$-motions of X in Y is the set*

$$SE(3)(X,Y) = \{g\ \in\ SE(3) : g(X)\ \subset\ Y\}$$

considered as a subset of $SE(3)$.

Definition 4.6 *Let $K \subset \mathbb{R}^3$ be a compact convex body, int(K) its interior (which is assumed to be non-empty) and $\mathcal{O}K$ its outside, i.e. $\mathcal{O}K = \mathbb{R}^3 - int(K)$. A set of points $P \subset \partial K \subset \mathcal{O}K$ is said to immobilize K if the identity map $I_3 \in SE(3)$ is an isolated point of $SE(3)(P, \mathcal{O}K)$.*

Let K be a three dimensional sphere. Since any rotation of K about its centre belongs to $SE(3)(\partial K, \partial K)$, a sphere does not have an immobilizing set.

Definition 4.7 *Let $P_1, \ldots, P_4$ be fixed interior points in the faces $F_1, \ldots, F_4$, respectively, of a tetrahedron T and $\mathbf{n}_i$ the standard outward normals of T. The extended energy function $\bar{E}: SE(3) \to \mathbb{R}$ is the function defined as:*

$$\bar{E}(g) = \sum_{i=1}^{4} [g(P_i) - P_i] \cdot \mathbf{n}_i$$

The extended energy function can be defined for a general convex body having points $P_1, \ldots, P_4$ in its boundary.

Lemma 4.8 *The extended energy function is invariant under translations.*

Proof

For any $t \in \mathbb{R}^3$ let T_t be the translation $T_t(x) = x + t$, then

$$\begin{aligned} \bar{E}(T_t \circ g) &= \sum_{i=1}^{4} [g(P_i) + t - P_i] \cdot \mathbf{n}_i \\ &= \sum_{i=1}^{4} [g(P_i) - P_i] \cdot \mathbf{n}_i + t \cdot \sum_{i=1}^{4} \mathbf{n}_i \\ &= \bar{E}(g). \end{aligned}$$

Let $g \in SE(3)$ and $\bar{g} \in SE(3)/T_3 = SO(3)$ be the coset $\bar{g} = gT_3$. Then following Lemma 4.8 we can define the *energy function* $E : SO(3) \to \mathbb{R}$ by

$$E(\bar{g}) = \sum_{i=1}^{4} [h(P_i) - P_i] \cdot \mathbf{n}_i$$

where $h \in SE(3)$ is any element of the coset $\bar{g}$.

The fact that it is enough to consider the energy function E defined on $SO(3)$ only, and not $SE(3)$, corresponds to the fact that to immobilize T using four points chosen from different faces of T one only needs to immobilize T with respect to rotations.

From here onwards P will denote the set of points $\{P_1, P_2, P_3, P_4\}$.

Lemma 4.9 *For $g \in SE(3)$, g close to I_3, $g \in SE(3)(P, \mathcal{OT})$ if and only if $[g(P_i) - P_i] \cdot \mathbf{n}_i \geq 0$ for $i = 1, 2, 3, 4$.*

Proof

Suppose $g \in SE(3)(P, \mathcal{OT})$ and g is close to I_3. Then $g(P_i) \in \mathcal{OT}$ for all i and $g(P_i)$ is near P_i for all i. Since $\mathbf{n}_i$ is an outer normal to the plane π_i at P_i, $[g(P_i) - P_i] \cdot \mathbf{n}_i \geq 0$ for all i.

Conversely, suppose $[g(P_i) - P_i] \cdot \mathbf{n}_i \geq 0$ for $i = 1, 2, 3, 4$. Since $\mathbf{n}_i$ is an outer normal to F_i, $g(P_i)$ lies in plane π_i or in the outer half-space determined by π_i. This means $g(P_i) \notin \operatorname{int} T$ and also that $g(P_i)$ is close to P_i for every i. Hence $g \in SE(3)(P, \mathcal{OT})$ and g is close to the identity I_3.

Hence if $g \in SE(3)(P, \mathcal{OT})$ and is close to the identity $\bar{E}(g) \geq 0$.

Lemma 4.10

For any small neighbourhood N of the identity I_3, $N \cap T_3 \cap SE(3)(P, \mathcal{OT}) = \{I_3\}$.

Proof

Let $g \in T_3$, then $g(x) = t + x$ for some $t \in \mathbb{R}^3$, for all $x \in \mathbb{R}^3$. By Lemma 4.9, $g \in N \cap SE(3)(P, \mathcal{OT})$ if and only if $[g(P_i) - P_i] \cdot \mathbf{n}_i \geq 0 \quad \forall \quad i$, therefore if $g \in N \cap T_3 \cap SE(3)(P, \mathcal{OT})$, $t \cdot \mathbf{n}_i \geq 0$ for all i. But $t \cdot \mathbf{n}_1 = -t \cdot \mathbf{n}_2 - t \cdot \mathbf{n}_3 - t \cdot \mathbf{n}_4 \leq 0 \Rightarrow t \cdot \mathbf{n}_1 = 0$. Similarly $t \cdot \mathbf{n}_i = 0 \quad \forall \quad i \Rightarrow t = 0$ as $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are linearly independent.

Proposition 4.11 *The points $P_1, \ldots, P_4$ immobilize T if and only if the energy function $E : SO(3) \to \mathbb{R}$ has an isolated local maximum at $\bar{I}_3 \in SO(3)$.*

Proof

From Lemma 4.8, E is well-defined on $SE(3)/T_3$, let $\pi : SE(3) \to SE(3)/T_3$ be the natural quotient map. Denote the coset of g by $\bar{g}$ and let I_3 be the identity element in $SE(3)$. Then $E(\bar{I}_3) = \bar{E}(I_3) = 0$. For $g \in SE(3)$ consider the three equations

$$[g(P_i) - P_i] \cdot \mathbf{n}_i = -\mathbf{u} \cdot \mathbf{n}_i, \quad i = 1, 2, 3,$$

where $\mathbf{u} = (u_x, u_y, u_z)$. If $\mathbf{n}_i = (n_{ix}, n_{iy}, n_{iz})$, $k_i = [P_i - g(P_i)] \cdot \mathbf{n}_i$, then the equations can be written as

$$\mathbf{u} \cdot \mathbf{n}_i = k_i, \quad i = 1, 2, 3$$

or as $\mathcal{N}\mathbf{u}^t = k$ where

$$\mathcal{N} = \begin{pmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{pmatrix}, \quad k = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}.$$

Since the matrix $\mathcal{N}$ is nonsingular the system $\mathcal{N}\mathbf{u}^t = k$ has a unique solution. Let this be denoted $\mathbf{u}_g$. Define $\hat{g} = T_{\mathbf{u}_g} \circ g$. Now suppose E does not have an isolated local maximum at $\bar{I}_3$. Then for every neighbourhood $\bar{V}$ of $\bar{I}_3$ in $SE(3)/T_3$, there exists a $\bar{g}$ close to the identity, $\bar{g} \neq \bar{I}_3$ such that $E(\bar{g}) \geq 0$ and so for each neighbourhood $\pi^{-1}(\bar{V})$ of I_3 in $SE(3)$, there exists a g, close to I_3, $g \neq I_3$, such that $\bar{E}(g) \geq 0$. Take the $\hat{g}$ corresponding to this g, then

$$\begin{aligned}
\bar{E}(\hat{g}) &= \sum_{i=1}^{4} [\hat{g}(P_i) - P_i] \cdot \mathbf{n}_i \\
&= \sum_{i=1}^{4} [g(P_i) + \mathbf{u}_g - P_i] \cdot \mathbf{n}_i \\
&= \sum_{i=1}^{4} ([g(P_i) - P_i] \cdot \mathbf{n}_i + \mathbf{u}_g \cdot \mathbf{n}_i) \\
&= [g(P_4) - P_4] \cdot \mathbf{n}_4 + \mathbf{u}_g \cdot \mathbf{n}_4 \\
&= [g(P_4) + \mathbf{u}_g - P_4] \cdot \mathbf{n}_4 \\
&= E(g) \\
&\geq 0.
\end{aligned}$$

Thus $[g(P_i) + \mathbf{u}_g - P_i] \cdot \mathbf{n}_i \geq 0$ for i = 1, 2, 3, 4. Hence, by Lemma 4.9, $\hat{g} = T_{\mathbf{u}_g} \circ g \in SE(3)(P, \mathcal{O}T)$, implying that P does not immobilize T.

Conversely, suppose E has an isolated local maximum at $\bar{I}_3 \in SE(3)/T_3$. Then there exists a neighbourhood V of $\bar{I}_3$ in $SE(3)/T_3$ such that $E(\bar{g}) < 0$ for all $\bar{g} \neq \bar{I}_3$, $\bar{g} \in V$. Then $E(\bar{g}) = \bar{E}(g) < 0$ for all $g \in \pi^{-1}(V) \setminus \pi^{-1}(\bar{I}_3)$. Moreover $I_3 \in \pi^{-1}(V)$. Using Lemma 4.9 and considering g 'near' I_3, it is seen that $g \notin SE(3)(P, \mathcal{O}T)$. Since $\pi^{-1}(\bar{I}_3) = T_3$, Lemma 4.10 implies I_3 is an isolated point of $SE(3)(P, \mathcal{O}T)$. Hence P immobilizes T.

Proposition 4.12 *For each choice of $P_i \in F_i$, $i = 1, \ldots, 4$ define a 3×3 matrix A by*

$$A = \sum_{i=1}^{4} \mathbf{n}_i \, P_i^t,$$

where P_i^t denotes the transpose of P_i. Then $E(R) = tr(R^t A) - 6V$ for $R \in SO(3)$.

Proof

$$\begin{aligned}
E(R) &= \sum_{i=1}^{4}[R(P_i) - P_i] \cdot \mathbf{n}_i \\
&= \sum_{i=1}^{4} tr([RP_i - P_i]^t \mathbf{n}_i) \\
&= \sum_{i=1}^{4} tr(P_i^t R^t \mathbf{n}_i) - \sum_{i=1}^{4} tr(P_i^t \mathbf{n}_i) \\
&= \sum_{i=1}^{4} tr(\mathbf{n}_i P_i^t R^t) - \sum_{i=1}^{4} P_i \cdot \mathbf{n}_i \\
&= \sum_{i=1}^{4} tr(R^t \mathbf{n}_i P_i^t) - \sum_{i=1}^{4} P_i \cdot \mathbf{n}_i \\
&= tr(R^t A) - 6\,V
\end{aligned}$$

since, by Lemma 4.4,

$$\begin{aligned}
\sum_{i=1}^{4} \mathbf{n}_i \cdot P_i &= V_4 \cdot (V_3 \times V_2) + V_3 \cdot (V_4 \times V_1) + V_2 \cdot (V_1 \times V_4) + V_1 \cdot (V_2 \times V_3) \\
&= (V_2 - V_1) \cdot [(V_4 - V_2) \times (V_3 - V_2)] \\
&= (V_2 - V_1) \cdot \mathbf{n}_1 \\
&= 6V.
\end{aligned}$$

Definition 4.13 *Let M be an $n \times n$ real symmetric matrix, M is said to be almost positive definite if the sum of any two of its eigenvalues is positive.*

This condition is equivalent to the condition that only one eigenvalue of M may be negative and if λ is such an eigenvalue, the magnitude of λ is less than the magnitude of any other eigenvalue of M.

Proposition 4.14 *Let A be a fixed 3×3 matrix and $g : SO(3) \to \mathbb{R}$ the function defined by $g(R) = tr(R^t A)$ for $R \in SO(3)$. The function g has a strict local maximum at $R = I_3 \in SO(3)$ if and only if A is symmetric and almost positive definite.*

Proof

Let $R \in SO(3)$, then by Theorem 1.3,

$$R = \exp(S) = I_3 + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \ldots$$

for a unique skew 3×3 matrix S. Therefore

$$\begin{aligned}
g(R) &= tr(R^t A) \\
&= tr(A^t R) \\
&= tr\left(A^t + A^t S + \frac{A^t S^2}{2!} + \frac{A^t S^3}{3!} + \dots \right) \\
&= tr(A^t) + tr(A^t S) + \frac{tr(A^t S^2)}{2!} + \frac{tr(A^t S^3)}{3!} + \dots
\end{aligned}$$

Thus g has a critical point at $R = I_3$ if and only if

$$tr(A^t S) + \frac{tr(A^t S^2)}{2!} + \frac{tr(A^t S^3)}{3!} + \cdots$$

has a critical point at $S = 0$. However the latter happens if and only if $tr(A^t S) = 0$ for every skew S. Let $S = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$, then

$$\begin{aligned}
tr(A^t S) &= \sum_{k=1}^{3} [A^t S]_{kk} \\
&= \sum_{i=i}^{3} A_{i1} S_{i1} + \sum_{i=i}^{3} A_{i2} S_{i2} + \sum_{i=i}^{3} A_{i3} S_{i3} \\
&= a(A_{12} - A_{21}) + b(A_{13} - A_{31}) + c(A_{23} - A_{32}).
\end{aligned}$$

Therefore g has a critical point at $R = I_3$ if and only if A is symmetric.

Now the critical point at $R = I_3$ is a strict local maximum if $tr(A^t S^2) < 0$ for every skew matrix $S \neq 0$, *i.e.* if $tr(A^t S^2) = tr(AS^2) < 0$. We show that $tr(AS^2) < 0$ for every skew $S \neq 0$ if and only if A is almost positive definite. Now $tr(AS^2) < 0$ if and only if $tr(SAS) < 0$ if and only if $tr(S^t AS) > 0$ for skew $S \neq 0$. Since A is a real symmetric matrix, there exists an orthogonal matrix P such that $P^t AP = D$, where D is a diagonal matrix. Then

$$P^t S^t ASP = P^t S^t P P^t A P P^t S P = S_1^t D S_1,$$

where $S_1 = P^t SP$. Hence $S^t AS$ is similar to $S_1^t DS_1$. Thus $tr(S^t AS) = tr(S_1^t DS_1)$ and $tr(S^t AS) > 0$ for skew $S \neq 0$ if and only if $tr(S_1^t DS_1) > 0$ for skew $S_1 \neq 0$, since S_1 is skew if and only if S is skew. Now suppose $tr(S_1^t DS_1) > 0$ for skew $S_1 \neq 0$. Let

$$S_1 = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

where λ, μ, ν are the eigenvalues of A. Then

$$tr(S_1^t D S_1) = a^2(\mu+\lambda) + b^2(\lambda+\nu) + c^2(\nu+\mu).$$

So $tr(S_1^t D S_1) > 0$ for every skew $S_1 \neq 0$ if and only if A is almost positive definite.

Lastly, suppose the critical point at $R = I_3$ is a strict local maximum. Then $tr(A^t S^2) \leq 0$ for every small skew matrix S. We show that the hypothesis in fact implies that $tr(A^t S^2) < 0$ for every small skew $S \neq 0$. Let R be written as $R = \exp(kS)$ where $k \in \mathbb{R}$ and S is the skew symmetric matrix associated to R. Then

$$g(R) = tr(A^t \exp(kS)) = \sum_{h=0} \frac{k^h}{h\,!} tr(A^t S^h). \tag{4.1}$$

By hypothesis $tr(A^t S) = 0$ for all skew S, therefore $tr(A^t S^h) = 0$ for all odd h since S skew implies S^h is skew when h is odd. Thus we only consider even powers of h in Equation 4.1. Let $S_1 = P^t S P$ and $D = P^t A P$ be the matrices defined above. Then Equation 4.1 can be written as

$$\begin{aligned} g(R) &= \sum_h \frac{k^h}{h\,!} tr(A^t S^h) \\ &= \sum_h \frac{k^h}{h\,!} tr(PDP^t S^h) \\ &= \sum_h \frac{k^h}{h\,!} tr(D S_1^h), \end{aligned}$$

which becomes $g(R) = \sum_{h=1} \frac{k^{2h}}{(2h)!} tr(D S_1^{2h})$ when we leave out the zeros for odd h. From the hypothesis $tr(A^t S^2) \leq 0$ for every small skew matrix S, which by an argument similar to the one above is equivalent to $\lambda + \mu \geq 0$ for each pair of eigenvalues λ, μ of A, suppose two eigenvalues of A sum up to zero. Let the eigenvalues of A be $-\lambda, \lambda, \mu$ where $0 < \lambda \leq \mu$, then with the choices

$$D = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$tr(D S_1^{2h}) = 0$ for every integer h. Thus a neighbourhood N of the identity exists on which $g(R) = 0$ for all $R \in N$, so g does not have a strict local maximum at $R = I_3$, contradicting the hypotheses. Hence A is symmetric and almost positive definite.

The following lemma will be needed in the proof of Theorem 4.16.

Lemma 4.15 *Let $p(\lambda) := \lambda^3 - c_1\lambda^2 + c_2\lambda - c_3$ be the characteristic polynomial of a 3×3 matrix M having real eigenvalues. The matrix M is positive definite if and only if c_1, c_2 and c_3 are positive.*

Proof

If $p(\lambda)$ has positive roots λ_1, λ_2, λ_3 then

$$\begin{aligned} p(\lambda) &= (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) \\ &= \lambda^3-(\lambda_1+\lambda_2+\lambda_3)\lambda^2+(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\lambda-\lambda_1\lambda_2\lambda_3. \end{aligned}$$

Conversely, if c_1, c_2, c_3 are all positive, then writing $\lambda=-\nu$ we have

$$\begin{aligned} p(\lambda)=p(-\nu) &= -\nu^3-c_1\nu^2-c_2\nu-c_3 \\ &= -(\nu^3+c_1\nu^2+c_2\nu+c_3) \\ &< 0 \quad \text{for all } \nu \geq 0, \end{aligned}$$

i.e. $p(\lambda)<0$ for all $\lambda \leq 0$. By hypothesis M has real roots thus the roots of $p(\lambda)$ are all positive.

The first proof of the next theorem is essentially that in [BR]. We follow it with an alternative proof that avoids some of the complicated algebra used in [BR].

Theorem 4.16 (Bracho, Fetter, Mayer and Montejano) *Let T be a tetrahedron and $\mathbf{n}_i$, $i=1,\ldots,4$ be the standard outward normals of T. Interior points P_1, ..., P_4 of faces F_1, ..., F_4 immobilize T if and only if $\sum_{i=1}^4 P_i \times \mathbf{n}_i = \mathbf{0}$.*

The statement $\sum_{i=1}^4 P_i \times \mathbf{n}_i = \mathbf{0}$ will be referred to as the symmetry condition - being equivalent to the symmetry of the matrix $A=\sum_{i=1}^4 \mathbf{n}_i P_i^t$

Proof 1

Taking Propositions 4.11, 4.12 and 4.14 into consideration, it is enough to prove that if $P_1,\ldots,P_4$ are interior points such that the matrix $A=\sum_{i=1}^4 \mathbf{n}_i\, P_i^t$ is symmetric then A is almost positive definite. The matrix A can be written as $A=N^tP$ where N is the 3×3 matrix whose rows are $\mathbf{n}_1^t$, $\mathbf{n}_2^t$, $\mathbf{n}_3^t$ and P is a 3×3 matrix with rows $(P_1-P_4)^t$, $(P_2-P_4)^t$, $(P_3-P_4)^t$. Since any three $\mathbf{n}_i$ are linearly independent, the matrix N is nonsingular. Therefore A is similar to the matrix $U=PN^t$. The matrix $U=(u_{ij})$ has nice properties: $u_{ij}=P_i-P_4\cdot\mathbf{n}_j$ and

- U1: $u_{ii}>0$ for $i=1$, 2, 3.
- U2: $u_{ij}<u_{jj}$ for $1\leq i,j\leq 3$ and $i\neq j$
- U3: $\sum_{j=1}^3 u_{ij}>0$ for each $1\leq i\leq 3$.

These properties can be deduced from the assumption that the points P_i are interior to their faces, *i.e.* the 12 inequalities $(P_i-P_j)\cdot\mathbf{n}_i>0$ for $1\leq i\neq j\leq 4$ hold, so

- $u_{ii} = P_i - P_4 \cdot \mathbf{n}_i > 0$,
- $u_{jj} - u_{ij} = P_j - P_4 \cdot \mathbf{n}_j - P_i - P_4 \cdot \mathbf{n}_j = P_j - P_i \cdot \mathbf{n}_j > 0$,
-

$$\begin{aligned}\sum_{j=1}^{3} u_{ij} &= P_i - P_4 \cdot \mathbf{n}_1 + P_i - P_4 \cdot \mathbf{n}_2 + P_i - P_4 \cdot \mathbf{n}_3 \\ &= P_i - P_4 \cdot \mathbf{n_1} + \mathbf{n_2} + \mathbf{n_3} \\ &= P_i - P_4 \cdot -\mathbf{n_4} \\ &= P_4 - P_i \cdot \mathbf{n}_4 > 0.\end{aligned}$$

The matrices A and U have the same characteristic polynomial (since $A = N^t P$ and $U = PN^t$), therefore U has real eigenvalues. Let the characteristic polynomial of U be $p(\lambda) \equiv \lambda^3 - a_1\lambda^2 + a_2\lambda - a_3$. Then $a_3 = \det(U) = \det(P)\det(N)$ and $a_1 = tr(U)$. Let $B = (a_1 I - U)$, where I is the identity matrix, then

$$\begin{aligned}\lambda \text{ is an eigenvalue of } B \;&\Leftrightarrow\; (a_1 - \lambda) \text{ is an eigenvalue of } U, \\ (a_1 - \mu) \text{ is an eigenvalue of } B \;&\Leftrightarrow\; \mu \text{ is an eigenvalue of } U.\end{aligned}$$

Let μ_1, μ_2, μ_3 be the eigenvalues of U, then U is almost positive definite if and only if

$$\left.\begin{array}{rcl}\mu_2 + \mu_3 &>& 0 \\ \mu_3 + \mu_1 &>& 0 \\ \mu_1 + \mu_2 &>& 0\end{array}\right\} \Leftrightarrow \left\{\begin{array}{rcl} tr(U) - \mu_1 &>& 0 \\ tr(U) - \mu_2 &>& 0 \\ tr(U) - \mu_3 &>& 0\end{array}\right. .$$

It therefore remains to prove that $B = tr(U)I - U$ is positive definite. Let $p(\tau) \equiv \tau^3 - c_1\tau^2 + c_2\tau - c_3$ be the characteristic polynomial of B, then by Lemma 4.15 it is enough to show that the constants c_1, c_2 and c_3 in $p(\tau)$ are all positive. Expanding $\det(\tau I - B)$ yields

$$\begin{aligned} c_1 &= 2(u_{11} + u_{22} + u_{33}) \\ c_2 &= b_1b_3 + b_2b_3 + b_1b_2 - u_{12}u_{21} - u_{13}u_{31} - u_{23}u_{32} \\ c_3 &= b_1b_2b_3 - u_{13}u_{31}b_2 - u_{12}u_{21}b_3 - u_{23}u_{32}b_1 - u_{32}u_{21}u_{13} - u_{23}u_{31}u_{12}\end{aligned}$$

where $b_i = (u_{11}+u_{22}+u_{22}) - u_{ii}$. From property U1 of the matrix U, the constant c_1 is obviously positive. It is left to prove that c_2 and c_3 are positive.

Now $-b_1 = -(u_{22}+u_{33}) < -(u_{22}+u_{23}) < u_{21} < u_{11}$ by application of U2, U3 and again U2 in that order. Likewise $-b_2 = -(u_{11} + u_{33}) < -(u_{11} + u_{13}) < u_{12} < u_{22}$. Since $b_1b_2 = (u_{22} + u_{33})(u_{11} + u_{33}) > u_{11}u_{22}$ by U1 and $b_1b_2 > 0$, $b_1b_2 > u_{12}u_{21}$. Similarly, $b_1b_3 > u_{13}u_{31}$ and $b_3b_2 > u_{32}u_{23}$. Therefore c_2 is positive. Lastly, from [BR], c_3 can be written as:

$$
\begin{aligned}
& [(u_{22}-u_{12})(u_{21}+u_{22}+u_{23})+(u_{33}-u_{13})(u_{31}+u_{32}+u_{33})]u_{11} \\
+\ & [(u_{33}-u_{23})(u_{31}+u_{32}+u_{33})+(u_{11}-u_{21})(u_{11}+u_{12}+u_{13})]u_{22} \\
+\ & [(u_{11}-u_{31})(u_{11}+u_{12}+u_{13})+(u_{22}-u_{32})(u_{21}+u_{22}+u_{23})]u_{33} \\
+\ & (u_{22}-u_{12})(u_{33}-u_{23})(u_{11}-u_{31})+(u_{33}-u_{13})(u_{11}-u_{21})(u_{22}-u_{32})
\end{aligned}
$$

which is a sum of positive terms and hence c_3 is positive. Therefore B is positive definite, hence U and $A = \sum_{i=1}^{4} \mathbf{n}_i\, P_i^t$ are almost positive definite.

Proof 2

It is enough to show that $A = \sum_{i=1}^{4} \mathbf{n}_i\, P_i^t$ is symmetric implies A is almost positive definite.

The matrix A can be written as $N^t P$ where P and N are the 3×3 matrices given in the first proof. Let $\mathcal{V}$ be the 3×3 matrix whose rows are $(V_1 - V_4)^t$, $(V_2 - V_4)^t$ and $(V_3 - V_4)^t$, then $\mathcal{V}\, N^t = -6\, V I_3$, where V is the volume of T and I_3 the 3×3 identity matrix. To preserve symmetry, the tetrahedron is cast into $\mathbb{R}^4$ having coordinates (x, y, z, w), such that T lies in the hyper-plane $w = 1$ of this space. Let $\mathcal{V}'$ be the 4×4 matrix having rows, $(V_1^t, 1)$, ..., $(V_4^t, 1)$, a 4×4 matrix N' is sought, where N' is a kind of extension of N, such that $\mathcal{V}'\, N'^t = -6V\, I_4$. Suppose N' has rows $(\mathbf{n}_1^t, q_1)$, ..., $(\mathbf{n}_4^t, q_4)$, then

$$
\mathcal{V}'\, N'^t = \begin{bmatrix}
V_1 \cdot \mathbf{n}_1 + q_1 & V_1 \cdot \mathbf{n}_2 + q_2 & V_1 \cdot \mathbf{n}_3 + q_3 & V_1 \cdot \mathbf{n}_4 + q_4 \\
V_2 \cdot \mathbf{n}_1 + q_1 & V_2 \cdot \mathbf{n}_2 + q_2 & V_2 \cdot \mathbf{n}_3 + q_3 & V_2 \cdot \mathbf{n}_4 + q_4 \\
V_3 \cdot \mathbf{n}_1 + q_1 & V_3 \cdot \mathbf{n}_2 + q_2 & V_3 \cdot \mathbf{n}_3 + q_3 & V_3 \cdot \mathbf{n}_4 + q_4 \\
V_4 \cdot \mathbf{n}_1 + q_1 & V_4 \cdot \mathbf{n}_2 + q_2 & V_4 \cdot \mathbf{n}_3 + q_3 & V_4 \cdot \mathbf{n}_4 + q_4
\end{bmatrix}.
$$

The choice

$$
\begin{aligned}
q_1 &= -V_2 \cdot \mathbf{n}_1, \\
q_2 &= -V_3 \cdot \mathbf{n}_2, \\
q_3 &= -V_4 \cdot \mathbf{n}_3, \\
q_4 &= -V_1 \cdot \mathbf{n}_4
\end{aligned}
$$

satisfies the requirement $\mathcal{V}'\, N'^t = -6V I_4$.

$$
\text{Let} \quad N' = \begin{bmatrix}
\mathbf{n}_1^t, -V_2 \cdot \mathbf{n}_1 \\
\mathbf{n}_2^t, -V_3 \cdot \mathbf{n}_2 \\
\mathbf{n}_3^t, -V_4 \cdot \mathbf{n}_3 \\
\mathbf{n}_4^t, -V_1 \cdot \mathbf{n}_4
\end{bmatrix} \quad \text{and} \quad P' = \begin{bmatrix}
P_1^t, 1 \\
P_2^t, 1 \\
P_3^t, 1 \\
P_4^t, 1
\end{bmatrix}.
$$

Then

$$
\hat{A} = P'^t\, N'
$$

$$= \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{n}_1^t, -V_2 \cdot \mathbf{n}_1 \\ \mathbf{n}_2^t, -V_3 \cdot \mathbf{n}_2 \\ \mathbf{n}_3^t, -V_4 \cdot \mathbf{n}_3 \\ \mathbf{n}_4^t, -V_1 \cdot \mathbf{n}_4 \end{bmatrix}$$

$$= \begin{pmatrix} & * \\ \sum_{i=1}^4 \mathbf{n}_i P_i^t & * \\ & * \\ \sum_{i=1}^4 \mathbf{n}_i^t & -6V \end{pmatrix}$$

$$= \begin{pmatrix} & * \\ A & * \\ & * \\ \mathbf{0} & -6V \end{pmatrix}.$$

where the $*$s represent numbers that we do not need to calculate. Therefore the eigenvalues of $\hat{A}$ are $-6V$ and the eigenvalues of A. It now remains to determine the eigenvalues of $\hat{A}$.

The points P_i in F_i are interior to their faces, so we can express them as convex linear combination of the vertices as:

$$\begin{aligned} P_1 &= a_8 V_2 + a_1 V_3 + (1 - a_1 - a_8) V_4 \\ P_2 &= a_3 V_1 + (1 - a_3 - a_6) V_3 + a_6 V_4 \\ P_3 &= a_5 V_1 + (1 - a_5 - a_4) V_2 + a_4 V_4 \\ P_4 &= (1 - a_2 - a_7) V_1 + a_2 V_2 + a_7 V_3 \end{aligned}$$

where $0 < a_i < 1$ for $1 \leq i \leq 8$ and $0 < a_1 + a_8 < 1$, $0 < a_3 + a_6 < 1$, $0 < a_4 + a_5 < 1$ and $0 < a_2 + a_7 < 1$. Then $P'^t = \mathcal{V}'^t \mathcal{A}$ where $\mathcal{A}$ is the special stochastic matrix

$$\begin{bmatrix} 0 & a_3 & a_5 & 1 - a_2 - a_7 \\ a_8 & 0 & 1 - a_4 - a_5 & a_2 \\ a_1 & 1 - a_3 - a_6 & 0 & a_7 \\ 1 - a_1 - a_8 & a_6 & a_4 & 0 \end{bmatrix}$$

that encodes the positions of the P_i.

The characteristic polynomial of $\hat{A}$ is

$$\begin{aligned} \det(\hat{A} - \lambda I) &= \det(P'^t N' - \lambda I) \\ &= \det(N'^t P' - \lambda I) \\ &= \det(P' N'^t - \lambda I) \\ &= \det(\mathcal{A}^t \mathcal{V}' N'^t - \lambda I) \\ &= \det(\mathcal{A}^t [-6V] - \lambda I) \\ &= 6^4 V^4 \det(\mathcal{A} + \mu I), \end{aligned}$$

where $\lambda = 6V\mu$. Since the sum of each column of $\mathcal{A}$ is 1, the sum of each column of $\mathcal{A} - I_4$ is zero, *i.e.* the sum of the rows of $\mathcal{A} - I_4$ is $\mathbf{0}$. Thus $(1, 1, 1, 1)$ is a left eigenvector of $\mathcal{A}$ with eigenvalue $-\mu = 1$ corresponding to the eigenvalue $\lambda = -6V$ of $\hat{A}$. Now suppose A is symmetric, then $\hat{A}$ has all its eigenvalues real. Let these be $-6V$, a', b' and c'. Then A is almost positive definite if $a' + b'$, $a' + c'$ and $b' + c'$ are positive. This is equivalent to saying that if $\mathcal{A}$ has eigenvalues 1, a, b and c, then $a + b$, $a + c$ and $b + c$ are negative because for λ an eigenvalue of $\hat{A}$, $\lambda = 6V\mu$, where $-\mu$ is an eigenvalue of $\mathcal{A}$. The proof of the theorem concludes with the following two lemmas.

Lemma 4.17 *Let $B = (b_{ij})$ be an $n \times n$ matrix such that $\sum_{j=1}^{n} |b_{ij}| \leq 1$ for each i, then every eigenvalue of B lies in the unit disc.*

Proof
Let $\mathbf{z} \in \mathbb{C}^n$ and $\|\mathbf{z}\| = \max_{1 \leq i \leq n}\{|z_i|\}$. The i^{th} entry of $B\mathbf{z}$ is $\sum_{j=1}^{n} b_{ij} z_j$ and so

$$|(B\mathbf{z})_i| \leq \sum_{j=1}^{n} |b_{ij}||z_j| \leq \|\mathbf{z}\|.$$

Hence $\|B\mathbf{z}\| \leq \|\mathbf{z}\|$. If $\mathbf{z}$ is an eigenvector with eigenvalue λ then

$$\|B\mathbf{z}\| \;=\; \|\lambda\mathbf{z}\| \;=\; |\lambda|\|\mathbf{z}\|.$$

But $\|B\mathbf{z}\| \leq \|\mathbf{z}\|$. Therefore $|\lambda| \leq 1$.

Lemma 4.18 *Let (b_{ij}) be a real $n \times n$ matrix such that*

1. *$b_{ii} = 0$ for all i,*

2. *$b_{ij} > 0$ for all $i \neq j$,*

3. *$\sum_j b_{ij} = 1$,*

   and $z_1, \ldots, z_n$ complex numbers such that

4. *$|z_j| \leq r$ for all j,*

5. *$|\sum_j b_{ij}\, z_j| = r$ for all i,*

then $z_1 = \cdots = z_n$.

Proof

Let K be the convex hull of $z_1, \ldots, z_n$. Then $K \subset D$, the disc of radius r in the complex plane. Since $b_{ij} > 0$ for all $i \neq j$, for each i, the convex linear combination $\sum_j b_{ij} z_j$ lies strictly in K. Suppose the z_i are not all equal, let z_k differ from all z_i with $i \neq k$. Then K cannot be a point set and since $|z_i| \leq r \ \forall \ i$, the interior of K lies in the interior of D. Now consider $\sum b_{ij} z_j$ with $i \neq k$, $\sum_j b_{ij} z_j = b_{ik} z_k + \sum_{j \neq k,i} b_{ij} z_j$ lies strictly in int$K$ since $z_k \neq z_j$ for $j \neq k$ and all the weights are positive. This implies $\sum_j b_{ij} z_j$ lies strictly inside D, contradicting $|\sum b_{ij} z_j| = r$.

Conclusion of Proof 2 of Theorem 4.16

From Lemma 4.17 it is deduced that the eigenvalues $1, a, b, c$ of $\mathcal{A}$ each has magnitude not exceeding one. And from Lemma 4.18 it is deduced that the eigenspace corresponding to any eigenvalue of magnitude equal to one has only one spanning eigenvector $(1, 1, 1, 1)$, which corresponds to the eigenvalue of 1. Hence the eigenvalues a, b and c all have magnitude strictly less than one. Since the trace of $\mathcal{A}$ is zero, $1 + a + b + c = 0$. Therefore

$$\begin{aligned} a + b &= -1 - c < 0, \\ a + c &= -1 - b < 0, \\ b + c &= -1 - a < 0, \end{aligned}$$

so A is almost positive definite.

Corollary 4.19 *Let K be a convex body and $\mathbf{n}_1, \ldots, \mathbf{n}_4$ normal outward vectors at boundary points $P_1, \ldots, P_4$ respectively, where $\sum_{i=1}^4 \mathbf{n}_i = \mathbf{0}$. If $P_1, \ldots, P_4$ immobilize K, then $\sum_{i=1}^4 P_i \times \mathbf{n}_i = \mathbf{0}$.*

In the conclusions of [C1](1991) and [C2](1999) it was speculated that $d + 1$ points immobilize a convex polytope if and only if the $(d-1)$-dimensional hyperplanes tangent to P at the points enclose P, and the lines orthogonal to the hyperplanes at the points of immobilization are concurrent. This is not quite correct as Corollary 4.20 shows.

Corollary 4.20 *Four points in the interior of faces of a tetrahedron T immobilize T if and only if the normal lines at these points either*

1. *are concurrent, or*

2. intersect in pairs, or

3. belong to one ruling of a quadric surface.

Proof

From Theorem 4.16 the points $P_1, \ldots, P_4$ immobilize the tetrahedron if and only if $\sum_{i=1}^4 P_i \times \mathbf{n}_i = \mathbf{0}$. Since $\sum_{i=1}^4 \mathbf{n}_i = \mathbf{0}$, $P_1, \ldots, P_4$ immobilize the tetrahedron if and only if $\sum_{i=1}^4 (\mathbf{n}_i, P_i \times \mathbf{n}_i) = \mathbf{0}$, which happens if and only if the lines having Plücker coordinates $(\mathbf{n}_1, P_1 \times \mathbf{n}_1), \ldots, (\mathbf{n}_4, P_4 \times \mathbf{n}_4)$ have linearly dependent Plücker coordinates. Applying Proposition 3.9 we get the result.

Remarks If the normals lines at the immobilizing points belong to one ruling of a quadric surface the equation of this surface can be computed as described on page 69 of [SPA].

Corollary 4.21 *Let P_i, $i = 1, \ldots, 4$ be points in the interior of faces F_i, $i = 1, \ldots, 4$ of tetrahedron T such that the set $P = \{P_1, \ldots, P_4\}$ immobilizes T and let l_i be the normal line at P_i. Let l_i' be the translate of l_i by a fixed vector $\mathbf{t}$ and P_i' the point of intersection of l_i' with π_i, the plane of F_i. Then if $P_i' \in F_i \ \forall\, i$, the set $\{P_1', P_2', P_3', P_4'\}$ immobilizes T.*

Proof

The new position $P_i' = P_i + \mathbf{t} + k_i\,\mathbf{n}_i$ for some scalar k_i (see Figure 4.3). Therefore

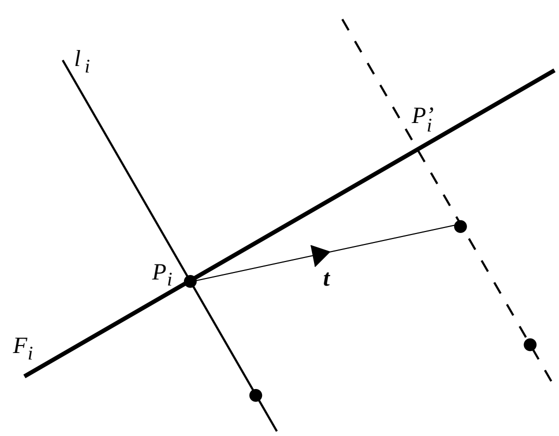


Figure 4.3: Point P_i is translated to point P_i'.

$$\begin{aligned}\sum_{i=1}^4 P_i' \times \mathbf{n}_i &= \sum_{i=1}^4 [P_i + \mathbf{t} + k_i\,\mathbf{n}_i] \times \mathbf{n}_i \\ &= \sum_{i=1}^4 P_i \times \mathbf{n}_i + \sum_{i=1}^4 \mathbf{t} \times \mathbf{n}_i + \sum_{i=1}^4 k_i\,\mathbf{n}_i \times \mathbf{n}_i \\ &= \mathbf{0}.\end{aligned}$$

4.4 The triangle case revisited

In the final section of this chapter, the meaning of the symmetry of A in the two dimensional case is investigated and compared to the results of Chapter 2. It is found out that an analogue to Theorem 4.16 holds; namely that the triangle is immobilized provided the corresponding 2×2 matrix A is symmetric and almost positive definite. In this section T will denote a triangle in $\mathbb{R}^2$ having vertices V_1, V_2, V_3. Let e_k be the edge of T opposite vertex V_k and $\mathbf{n}_k$ be an outward normal vector to edge e_k ($k = 1, 2, 3$) chosen so that $\sum_{k=1}^{3} \mathbf{n}_k = \mathbf{0}$. If P_1, P_2, P_3 are interior points in e_1, e_2, e_3 respectively we study the 2×2 matrix $A = \sum_{k=1}^{3} \mathbf{n}_k\, P_k^t$.

First, an orientation on T is fixed. Suppose edge e_3 is lying horizontally in the plane of T and vertex V_1 is on the left of vertex V_2 as shown in Figure 4.4. Let Ω

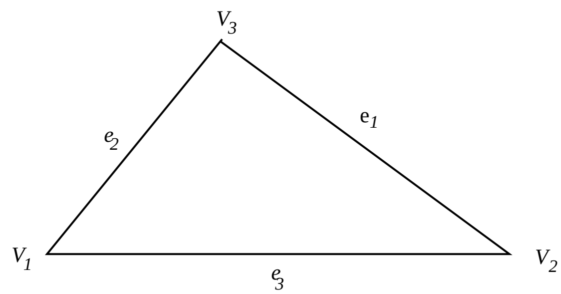


Figure 4.4: Chosen orientation on triangle

be the rotation matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Define

$$\begin{aligned} \mathbf{n}_1 &= \Omega(V_3 - V_2) \\ \mathbf{n}_2 &= \Omega(V_1 - V_3) \\ \mathbf{n}_3 &= \Omega(V_2 - V_1). \end{aligned}$$

Then $\mathbf{n}_k$ is normal to edge e_k and $\sum_{k=1}^{3} \mathbf{n}_k = \mathbf{0}$. Let $P_k = (P_{kx}, P_{ky})^t$ be an arbitrary point in e_k; then the matrix $A = \sum_{k=1}^{3} \mathbf{n}_k\, P_k^t$ is given by

$$\begin{aligned} A(1,1) &= (V_3 - V_2)_y P_{1x} + (V_1 - V_3)_y P_{2x} + (V_2 - V_1)_y P_{3x} \\ A(1,2) &= (V_3 - V_2)_y P_{1y} + (V_1 - V_3)_y P_{2y} + (V_2 - V_1)_y P_{3y} \\ A(2,1) &= (V_2 - V_3)_x P_{1x} + (V_3 - V_1)_x P_{2x} + (V_1 - V_2)_x P_{3x} \\ A(2,2) &= (V_2 - V_3)_x P_{1y} + (V_3 - V_1)_x P_{2y} + (V_1 - V_2)_x P_{3y}. \end{aligned}$$

A is symmetric if and only if

$$P_1 \cdot (V_2 - V_3) + P_2 \cdot (V_3 - V_1) + P_3 \cdot (V_1 - V_2) = 0,$$

if and only if the rows of

$$M = \begin{pmatrix} (V_3 - V_2)_x & (V_3 - V_2)_y & P_1 \cdot V_3 - V_2 \\ (V_1 - V_3)_x & (V_1 - V_3)_y & P_2 \cdot V_1 - V_3 \\ (V_2 - V_1)_x & (V_2 - V_1)_y & P_3 \cdot V_2 - V_1 \end{pmatrix}$$

are linearly dependent. But this is equivalent to the three lines with line coordinates

$$([V_3 - V_2]_x, [V_3 - V_2]_y, -P_1 \cdot (V_3 - V_2)),$$
$$([V_1 - V_3]_x, [V_1 - V_3]_y, -P_2 \cdot (V_1 - V_3)),$$
$$([V_2 - V_1]_x, [V_2 - V_1]_y, -P_3 \cdot (V_2 - V_1))$$

being concurrent.
Now the equations of lines l_1, l_2 and l_3 in Figure 4.5 are

$$[(x, y) - P_1] \cdot (V_3 - V_2) = 0,$$
$$[(x, y) - P_2] \cdot (V_1 - V_3) = 0,$$
$$[(x, y) - P_3] \cdot (V_2 - V_1) = 0$$

respectively, hence their line coordinates are

$$([V_3 - V_2]_x, [V_3 - V_2]_y, -P_1 \cdot (V_3 - V_2)),$$
$$([V_1 - V_3]_x, [V_1 - V_3]_y, -P_2 \cdot (V_1 - V_3)),$$
$$([V_2 - V_1]_x, [V_2 - V_1]_y, -P_3 \cdot (V_2 - V_1))$$

respectively. Hence the symmetry of A is equivalent to the concurrency of the

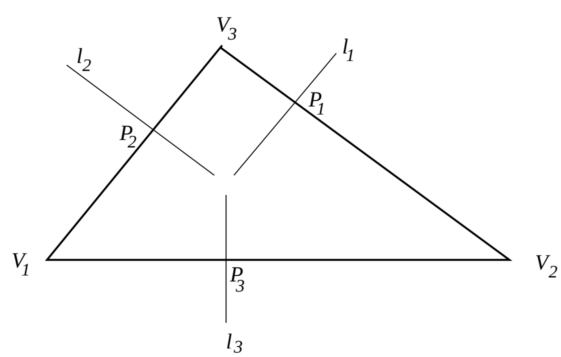


Figure 4.5: The orthogonal lines at their respective points in the edges of a triangle.

three lines at P_1, P_2, P_3.

Does A symmetric imply that A is almost positive definite? There are at least two ways this question can be answered. First, consider the stochastic matrix $\mathcal{A}$

that was introduced in the second proof of Theorem 4.16. In the two dimensional case,

$$\mathcal{A} = \begin{bmatrix} 0 & \alpha_1 & 1-\alpha_1 \\ 1-\alpha_2 & 0 & \alpha_2 \\ \alpha_3 & 1-\alpha_3 & 0 \end{bmatrix}$$

for some numbers α_1, α_2 and $\alpha_3, 0 < \alpha_i < 1$ for $1 \leq i \leq 3$. The two eigenvalues of $\mathcal{A}$ that are not equal to 1 add up to -1, since trace$(\mathcal{A}) = 0$. Therefore A is almost positive definite.

Alternatively, let $p(\lambda) = \lambda^2 - \text{trace(A)}\,\lambda + \det(\text{A})$ be the characteristic polynomial of A. Then

$$\text{tr}(A) = \star\left[P_1 \wedge (V_3 - V_2) + P_2 \wedge (V_1 - V_3) + P_3 \wedge (V_2 - V_1)\right].$$

where $X \wedge Y$ is the exterior product of vectors X and Y and $\star$ is the Hodge star map given in Subsection 1.3.4. Now

$$\begin{aligned} P_1 \wedge (V_3 - V_2) &= [\alpha_1 V_2 + (1-\alpha_1)V_3] \wedge (V_3 - V_2) \\ &= \alpha_1 V_2 \wedge V_3 + (1-\alpha_1)V_3 \wedge (-V_2) \\ &= \alpha_1 (V_2 \wedge V_3) + (1-\alpha_1)(V_2 \wedge V_3) \\ &= V_2 \wedge V_3. \end{aligned}$$

Therefore

$$\begin{aligned} \text{tr}(A) &= \star(V_2 \wedge V_3 \quad + \quad V_3 \wedge V_1 \quad + \quad V_1 \wedge V_2) \\ &= \star[(V_2 - V_1) \wedge (V_3 - V_1)] \\ &= 2(\text{Area of T}). \end{aligned}$$

Let A_T be the area of T, then

$$\begin{aligned} \det(A) &= [2A_T\, P_{2y} - 2A_T\, P_{3y}]\, P_{1x} + [2A_T\, P_{3y} - 2A_T\, P_{1y}]\, P_{2x} \\ &\quad + [2A_T\, P_{1y} - 2A_T\, P_{2y}]\, P_{3x} \\ &= 2A_T\, [P_{2y}P_{1x} - P_{3y}P_{1x} + P_{3y}P_{2x} - P_{1y}P_{2x} + P_{1y}P_{3x} - P_{2y}P_{3x}] \\ &= 2A_T\, [P_{2y}P_{1x} - P_{1y}P_{2x} + P_{3y}P_{2x} - P_{3x}P_{2y} + P_{1y}P_{3x} - P_{3y}P_{1x}] \\ &= 2A_T \star (P_1 \wedge P_2 + P_2 \wedge P_3 + P_3 \wedge P_1) \\ &= 2A_T\, A_{P_1 P_2 P_3}, \end{aligned}$$

where $A_{P_1P_2P_3}$ is the area of the triangle with vertices P_1, P_2, P_3. This triangle has the same orientation as T. Hence both the trace and determinant of A are positive thus when A is symmetric both its eigenvalues are positive showing that, provided each P_i is an interior point of e_i, the matrix A is positive definite, not merely almost positive definite.

The chapter is concluded by a theorem that unifies immobilization in 2 and 3 dimensions, thus providing an algebraic version of Theorem 2.5.

Theorem 4.22 *Let* $r = 2, 3$ *and* K *be an* r*-simplex in* $\mathbb{R}^r$ *having vertices* $V_1, \ldots, V_{r+1}$*. Suppose* F_k *denotes the face of* K *opposite vertex* V_k *and* $\mathbf{n}_k$ *an outward normal vector to* F_k *chosen so that* $\sum_{k=1}^{r+1} \mathbf{n}_k = \mathbf{0}$*. Points* $P_1, \ldots, P_{r+1}$ *in the interior of faces* $F_1, \ldots, F_{r+1}$ *respectively, immobilize* K *if and only if the matrix* $\sum_{k=1}^{r+1} \mathbf{n}_k P_k^t$ *is symmetric.*

Chapter 5

Immobilizing sets of a tetrahedron

5.1 Introduction

Although the criteria for an immobilizing set of a tetrahedron was first given in [BR], no attempt was made to find concrete sets of points that fulfilled the criteria. This chapter fills this gap. In Section 5.2 the centroids, the orthocenters and the circumcenters of faces are shown to make the matrix A symmetric. These face centers immobilize the tetrahedron if they lie in the interior of their faces, in particular, the centroids of a tetrahedron immobilize the tetrahedron. Section 5.3 investigates the situation of two fixed points being part of an immobilizing set. It is shown that if one pair of points in the faces of a tetrahedron is fixed then pairs of points exist in other faces which solve the symmetry condition. In Section 5.4 the full five dimensional solution of the symmetry condition on A is found and in the last section an analysis of the nature of immobilizing sets is undertaken.

5.2 Face centers

5.2.1 Centroids

Proposition 5.1 *Let G_i be the centroid of face F_i of a tetrahedron and V its volume. Then $\sum_{i=1}^4 \mathbf{n}_i G_i^t = 2VI_3$.*

Proof
Let $s = \frac{1}{3}(V_1 + V_2 + V_3 + V_4)$. Then $G_i = s - \frac{1}{3}V_i$. Consider

$$\sum_{i=1}^{4} G_i \mathbf{n}_i^t \quad = \quad \sum_{i=1}^{3} (G_i - G_4)\mathbf{n}_i^t$$

$$
\begin{aligned}
&= \sum_{i=1}^{3}(\frac{s}{3} - \frac{V_i}{3} - \frac{s}{3} + \frac{V_4}{3})\mathbf{n}_i^t \\
&= -\frac{1}{3}\sum_{i=1}^{3}(V_i - V_4)\mathbf{n}_i^t.
\end{aligned}
$$

Then if N and $\mathcal{V}$ are the 3×3 matrices given on Page 45 in the second proof of Theorem 4.16,

$$
\sum_{i=1}^{4} G_i \mathbf{n}_i^t = -\frac{1}{3}\sum_{i=1}^{3}(V_i - V_4)\mathbf{n}_i^t = -\frac{1}{3}\mathcal{V}^t N.
$$

However, $N\mathcal{V}^t = -6VI_3$ and $\mathcal{V}$ is nonsingular, so $N = -6VI_3{\mathcal{V}^t}^{-1}$, hence

$$
\begin{aligned}
\sum_{i=1}^{4} G_i \mathbf{n}_i^t &= -\frac{1}{3}\mathcal{V}^t N \\
&= -\frac{1}{3}\mathcal{V}^t . -6VI_3{\mathcal{V}^t}^{-1} \\
&= 2VI_3.
\end{aligned}
$$

Remarks

The equation $\sum_{i=1}^{4} \mathbf{n}_i\, G_i^t = 2\,V I_3$ may also be obtained via the Divergence Theorem of Calculus. Indeed, if $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ is the outward unit normal vector to T,

$$
\begin{aligned}
\sum_{i=1}^{4} \mathbf{n}_i\, G_i^t &= \sum_{i=1}^{4} 2A_i\, \hat{\mathbf{n}}_i\, G_i^t \\
&= 2\sum_{i=1}^{4} \hat{\mathbf{n}}_i \iint_{F_i} \mathbf{r}^t\, dS \\
&= 2\sum_{i=1}^{4} \iint_{F_i} \hat{\mathbf{n}}_i\, \mathbf{r}^t\, dS \\
&= 2\iint_{\partial T} \hat{\mathbf{n}}\, \mathbf{r}^t\, dS \\
&= 2\iint_{\partial T} \begin{pmatrix} n_x\, x & n_x\, y & n_x\, z \\ n_y\, x & n_y\, y & n_y\, z \\ n_z\, x & n_z\, y & n_z\, z \end{pmatrix} dS \\
&= 2\iint_{\partial T} \begin{pmatrix} \mathbf{B}_{11} \cdot \hat{\mathbf{n}} & \mathbf{B}_{12} \cdot \hat{\mathbf{n}} & \mathbf{B}_{13} \cdot \hat{\mathbf{n}} \\ \mathbf{B}_{21} \cdot \hat{\mathbf{n}} & \mathbf{B}_{22} \cdot \hat{\mathbf{n}} & \mathbf{B}_{23} \cdot \hat{\mathbf{n}} \\ \mathbf{B}_{31} \cdot \hat{\mathbf{n}} & \mathbf{B}_{32} \cdot \hat{\mathbf{n}} & \mathbf{B}_{33} \cdot \hat{\mathbf{n}} \end{pmatrix} dS,
\end{aligned}
$$

where

$\mathbf{B}_{11} = (x, 0, 0)$, $\mathbf{B}_{12} = (y, 0, 0)$, $\mathbf{B}_{13} = (z, 0, 0)$, $\mathbf{B}_{21} = (0, x, 0)$, $\mathbf{B}_{22} = (0, y, 0)$, $\mathbf{B}_{23} = (0, z, 0)$, $\mathbf{B}_{31} = (0, 0, x)$, $\mathbf{B}_{32} = (0, 0, y)$, $\mathbf{B}_{33} = (0, 0, z)$. By the Divergence Theorem (1.6),

$$
\iint_{\partial T} \mathbf{B}_{ii} \cdot \hat{\mathbf{n}}\; dS = \iiint_T \nabla \cdot \mathbf{B}_{ii}\; dV = \iiint_T 1\; dV = V,
$$

and for $i \neq j$,

$$\iint_{\partial T} \mathbf{B}_{ij} \cdot \hat{\mathbf{n}}\ dS = \iiint_T \nabla \cdot \mathbf{B}_{ij}\ dV = \iiint_T 0\ dV = 0.$$

Hence $\sum_{i=1}^4 \mathbf{n}_i\, G_i^t = 2\,V\,I_3$.

Corollary 5.2 *The set of centroids of faces of a tetrahedron immobilizes the tetrahedron.*

Remarks
Since the symmetry of $\sum_{i=1}^4 \mathbf{n}_i G_i^t$ is equivalent to $\sum_{i=1}^4 G_i \times \mathbf{n}_i = \mathbf{0}$, the equation $\sum_{i=1}^4 G_i \times \mathbf{n}_i = \mathbf{0}$ can be shown directly via the Divergence Theorem of Vector Calculus. If $\hat{\mathbf{n}}_i$ is the outward unit normal vector to face F_i, $\hat{\mathbf{n}}$ the outward unit normal vector to T and A_i the area of F_i, then

$$\begin{aligned} \sum_{i=1}^4 G_i \times \mathbf{n}_i &= \sum_{i=1}^4 \frac{1}{A_i} \iint_{F_i} \mathbf{r}\, dS \times 2A_i\, \hat{\mathbf{n}}_i \\ &= 2\sum_{i=1}^4 \iint_{F_i} \mathbf{r} \times \hat{\mathbf{n}}_i\, dS \\ &= 2\iint_{\partial T} \mathbf{r} \times \hat{\mathbf{n}}\, dS \\ &= \mathbf{0} \end{aligned}$$

by Corollary 1.8.

5.2.2 Orthocenters

Proposition 5.3 *Let $\mathbf{n}_1, \ldots, \mathbf{n}_4$ be the standard outward normals of a tetrahedron and H_i the orthocenter of face F_i, $i = 1, \ldots, 4$, then $\sum_{i=1}^4 H_i \times \mathbf{n}_i = \mathbf{0}$.*

Proof
Consider face F_1. See Figure 5.1.
The orthocenter H_1 satisfies the equations:

$$(H_1 - V_3) \cdot (V_4 - V_2) = 0, \tag{5.1}$$

$$(H_1 - V_4) \cdot (V_3 - V_2) = 0, \tag{5.2}$$

$$(H_1 - V_2) \cdot [(V_3 - V_2) \times (V_4 - V_2)] = 0. \tag{5.3}$$

Equations 5.1 and 5.2 can be rearranged to

$$H_1 \cdot (V_4 - V_2) = V_3 \cdot (V_4 - V_2) \tag{5.4}$$

$$H_1 \cdot (V_3 - V_2) = V_4 \cdot (V_3 - V_2) \tag{5.5}$$

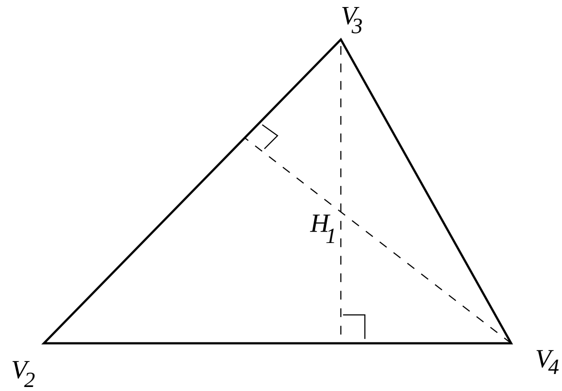


Figure 5.1: The orthocenter H_1 of face F_1.

respectively. Multiplying Equation 5.4 by $(V_3 - V_2)$ and 5.5 by $(V_4 - V_2)$ and subtracting the two results produces an equation whose left hand side

$$\begin{aligned} &= [H_1 \cdot (V_4 - V_2)](V_3 - V_2) - [H_1 \cdot (V_3 - V_2)](V_4 - V_2) \\ &= H_1 \times [(V_3 - V_2) \times (V_4 - V_2)] \\ &= H_1 \times -\mathbf{n}_1, \end{aligned}$$

and right hand side

$$\begin{aligned} &= [V_3 \cdot (V_4 - V_2)](V_3 - V_2) - [V_4 \cdot (V_3 - V_2)](V_4 - V_2) \\ &= [V_3 \cdot (V_4 - V_2)]V_3 + [V_4 \cdot (V_2 - V_3)]V_4 + [V_2 \cdot (V_3 - V_4)]V_2. \end{aligned}$$

Therefore

$$H_1 \times \mathbf{n}_1 = [V_2 \cdot (V_4 - V_3)]V_2 + [V_3 \cdot (V_2 - V_4)]V_3 + [V_4 \cdot (V_3 - V_2)]V_4.$$

By comparing the orientation of the vertices V_2, V_3, V_4 in face F_1 with the orientation of the vertices in the other faces, it is deduced that:

$$\begin{aligned} H_2 \times \mathbf{n}_2 &= [V_3 \cdot (V_4 - V_1)]V_3 + [V_4 \cdot (V_1 - V_3)]V_4 + [V_1 \cdot (V_3 - V_4)]V_1, \\ H_3 \times \mathbf{n}_3 &= [V_4 \cdot (V_2 - V_1)]V_4 + [V_1 \cdot (V_4 - V_2)]V_1 + [V_2 \cdot (V_1 - V_4)]V_2, \\ H_4 \times \mathbf{n}_4 &= [V_1 \cdot (V_2 - V_3)]V_1 + [V_2 \cdot (V_3 - V_1)]V_2 + [V_3 \cdot (V_1 - V_2)]V_3. \end{aligned}$$

Therefore $\sum_1^4 H_i \times \mathbf{n}_i = \mathbf{0}$.

Corollary 5.4 *Let T be a tetrahedron and H_1, H_2, H_3, H_4 orthocenters of its faces. If $(H_i - H_j) \cdot \mathbf{n}_j < 0$ for $i \neq j$ then the set $\{H_1, \ldots, H_4\}$ immobilizes T.*

The conditions of Corollary 5.4 merely ensure that each $H_i \in F_i$.

5.2.3 Circumcenters

Proposition 5.5 *Let* $\mathbf{n}_1, \ldots, \mathbf{n}_4$ *be the standard outward normals of a tetrahedron and* O_i *the circumcenter of face* F_i, $i = 1, \ldots, 4$, *then* $\sum_{i=1}^{4} O_i \times \mathbf{n}_i = \mathbf{0}$.

Proof Consider face F_1. See Figure 5.2. The circumcenter H_1 satisfies the equations:

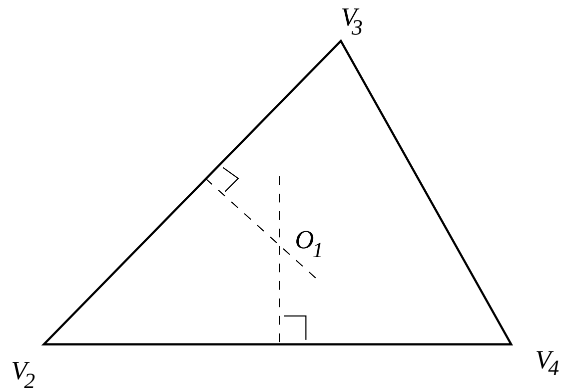


Figure 5.2: The circumcenter O_1 of face F_1.

$$\left(O_1 - \frac{V_2 + V_3}{2}\right) \cdot (V_3 - V_2) = 0, \tag{5.6}$$

$$\left(O_1 - \frac{V_2 + V_4}{2}\right) \cdot (V_4 - V_2) = 0, \tag{5.7}$$

$$(O_1 - V_2) \cdot \mathbf{n}_1 = 0. \tag{5.8}$$

Equations 5.6 and 5.7 can be rearranged to

$$O_1 \cdot (V_3 - V_2) = \frac{1}{2}(V_3 + V_2) \cdot (V_3 - V_2) \tag{5.9}$$

$$O_1 \cdot (V_4 - V_2) = \frac{1}{2}(V_4 + V_2) \cdot (V_4 - V_2) \tag{5.10}$$

respectively. Multiplying Equation 5.9 by $(V_4 - V_2)$ and 5.10 by $(V_3 - V_2)$ and subtracting the two results produces an equation whose left hand side

$$\begin{aligned} &= [O_1 \cdot (V_4 - V_2)](V_3 - V_2) - [O_1 \cdot (V_3 - V_2)](V_4 - V_2) \\ &= O_1 \times [(V_3 - V_2) \times (V_4 - V_2)] \\ &= O_1 \times -\mathbf{n}_1, \end{aligned}$$

and right hand side

$$\begin{aligned} &= \frac{1}{2}\left\{(|V_4|^2 - |V_2|^2)(V_3 - V_2) - (|V_3|^2 - |V_2|^2)(V_4 - V_2)\right\} \\ &= \frac{1}{2}\left\{(|V_3|^2 - |V_4|^2)V_2 + (|V_4|^2 - |V_2|^2)V_3 + (|V_2|^2 - |V_3|^2)V_4\right\} \end{aligned}$$

Therefore

$$O_1 \times \mathbf{n}_1 = \frac{1}{2}\left\{(|V_4|^2 - |V_3|^2)V_2 + (|V_2|^2 - |V_4|^2)V_3 + (|V_3|^2 - |V_2|^2)V_4\right\}.$$

By comparing the orientations of vertices V_2, V_3, V_4 in F_1 with the orientation of the vertices in other faces it is deduced that:

$$\begin{aligned} O_2 \times \mathbf{n}_2 &= \frac{1}{2}\left((|V_4|^2 - |V_1|^2)V_3 + (|V_1|^2 - |V_3|^2)V_4 + (|V_3|^2 - |V_4|^2)V_1\right), \\ O_3 \times \mathbf{n}_3 &= \frac{1}{2}\left((|V_2|^2 - |V_1|^2)V_4 + (|V_4|^2 - |V_2|^2)V_1 + (|V_1|^2 - |V_4|^2)V_2\right), \\ O_4 \times \mathbf{n}_4 &= \frac{1}{2}\left((|V_2|^2 - |V_3|^2)V_1 + (|V_3|^2 - |V_1|^2)V_2 + (|V_1|^2 - |V_2|^2)V_3\right). \end{aligned}$$

Thus $\sum_{i=1}^4 O_i \times \mathbf{n}_i = \mathbf{0}$.

Corollary 5.6 *Let T be a tetrahedron and O_1, O_2, O_3, O_4 circumcenters of its faces. If $(O_i - O_j) \cdot \mathbf{n}_j < 0$ for $i \neq j$, then the set $\{O_1, \ldots, O_4\}$ immobilizes T.*

The conditions of Corollary 5.6 merely ensure that each $O_i \in F_i$.

Corollary 5.7 *Let $Z_i = \alpha O_i + \beta G_i + \gamma H_i$ where α, β, γ are scalars satisfying $\alpha + \beta + \gamma = 1$. If $Z_i \in F_i$ for $1 \leq i \leq 4$, then $\{Z_1, \ldots, Z_4\}$ immobilizes T.*

Proposition 5.8 *Let T be a tetrahedron and Q any point in space. Suppose l_1, …, l_4 are lines going through Q with direction vectors $\mathbf{n}_1$, …, $\mathbf{n}_4$ respectively, and these lines intersect the faces F_1, …, F_4 of T orthogonally in P_1, …, P_4 respectively, then $\{P_1, P_2, P_3, P_4\}$ immobilize T.*

Proof

Since $\sum_{i=1}^4 P_i \times \mathbf{n}_i$ is translation invariant it can be assumed that the origin is at Q. Then $P_i = \lambda_i \mathbf{n}_i$ for scalars λ_i, $i = 1, \ldots, 4$, thus $\sum_{i=1}^4 P_i \times \mathbf{n}_i = \mathbf{0}$.

An example of such a point Q is the centroid $C = \frac{1}{4}(V_1 + V_2 + V_3 + V_4)$ of the tetrahedron. The point C lies inside the tetrahedron and the normal line $l_i = C + \lambda \mathbf{n}_i$ through C meets face F_i in an interior point of the face. To prove this, it is enough to show that $(V_j - C) \cdot \mathbf{n}_i > 0$ for all vertices V_j in face F_i of the tetrahedron. If, for example, $i = 1$ and $j = 4$, then

$$\begin{aligned} (V_4 - C) \cdot \mathbf{n}_1 &= \left[\frac{3}{4}V_4 - \frac{1}{4}(V_1 + V_2 + V_3)\right] \cdot (V_3 \times V_2 + V_2 \times V_4 + V_4 \times V_3) \\ &= -\frac{1}{4}V_1 \cdot \mathbf{n}_1 + \frac{1}{4}V_2 \cdot (V_4 \times V_3) \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{4}V_1 \cdot \mathbf{n}_1 + \frac{1}{4}V_2 \cdot \mathbf{n}_1 \\ &= \frac{1}{4}(V_2 - V_1) \cdot \mathbf{n}_1 \\ &= \frac{3}{2}V \ > \ 0. \end{aligned}$$

Similarly, all other inner products are positive, hence the result.

There are other interesting quartets of points that satisfy the symmetry condition.

i) If P_i is the foot of the altitude from V_i dropped onto face F_i, then $P_i = V_i + \alpha_i \mathbf{n}_i$ for some positive scaler α_i. Hence

$$\sum_{i=1}^{4} \mathbf{n}_i P_i^t = \sum_{i=1}^{4} \mathbf{n}_i V_i^t + \sum_{i=1}^{4} \alpha_i \mathbf{n}_i \mathbf{n}_i^t.$$

The second matrix is clearly symmetric, while, from the proof of Proposition 5.1 on Page 55,

$$\begin{aligned} \sum_{i=1}^{4} V_i \mathbf{n}_i^t &= \sum_{i=1}^{3} (V_i - V_4)\mathbf{n}_i^t \\ &= \mathcal{V}^t N \\ &= \mathcal{V}^t . - 6V I_3 \mathcal{V}^{t-1} \\ &= -6V I_3. \end{aligned}$$

The point P_i need not be interior to F_i.

ii) If A_i is the area of the face F_i and a point Q is given by

$$Q = \frac{\sum_{i=1}^{4} A_i V_i}{\sum_{i=1}^{4} A_i},$$

then Q is the centre of the inscribed sphere within T. This sphere touches the face F_i at the point $P_i = Q + r\mathbf{n}_i$ where

$$r = \frac{6V}{2\sum_{i=1}^{4} A_i}.$$

The points P_i are always interior to F_i and satisfy the conditions of Proposition 5.8, so they immobilize T.

5.3 The case of two points being fixed

When the fingers of a hand grasp an object it is usual for some of the fingers to touch the object before others. For a good grasp, the placement of the fingers that touch the object last is dependent on the positions of the fingers that touch the object first. In this section we investigate whether or not we can find an immobilizing set of a tetrahedron that contains two given finger positions.

Proposition 5.9 *Let $P_1 \in F_1$, $P_2 \in F_2$ be given. There are lines l_3 in π_3 and l_4 in π_4 from which points $Q_3 \in l_3$, $Q_4 \in l_4$ can be chosen so that the points P_1, P_2, Q_3, Q_4 satisfy the symmetry condition. Each point Q_3 on l_3 corresponds to a unique point Q_4 on l_4 for which the set $\{P_1, P_2, Q_3, Q_4\}$ satisfies the symmetry condition and vice versa.*

Proof

It is desired to solve

$$\begin{aligned} P_1 \times \mathbf{n}_1 + P_2 \times \mathbf{n}_2 + Q_3 \times \mathbf{n}_3 + Q_4 \times \mathbf{n}_4 &= \mathbf{0}, && (5.11)\\ Q_3 \cdot \mathbf{n}_3 &= V_2 \cdot (V_1 \times V_4), && (5.12)\\ \text{and} \quad Q_4 \cdot \mathbf{n}_4 &= V_1 \cdot (V_2 \times V_3) && (5.13) \end{aligned}$$

for Q_3, Q_4, where Equations 5.12 and 5.13 are the conditions that $Q_3 \in \pi_3$ and $Q_4 \in \pi_4$ respectively. Writing (x_1, x_2, x_3) for Q_3 and (y_1, y_2, y_3) for Q_4 yields the system $MX = B$ where

$$M = \begin{pmatrix} 0 & -n_{3z} & n_{3y} & 0 & -n_{4z} & n_{4y} \\ n_{3z} & 0 & -n_{3x} & n_{4z} & 0 & -n_{4x} \\ -n_{3y} & n_{3x} & 0 & -n_{4y} & n_{4x} & 0 \\ n_{3x} & n_{3y} & n_{3z} & 0 & 0 & 0 \\ 0 & 0 & 0 & n_{4x} & n_{4y} & n_{4z} \end{pmatrix},$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \mathbf{n}_1 \times P_1 + \mathbf{n}_2 \times P_2 \\ V_4 \cdot V_2 \times V_1 \\ V_1 \cdot V_2 \times V_3 \end{pmatrix}.$$

The matrix M has rank 5, since three of the six 5×5 submatrices have determinants $-|\mathbf{n}_3|^2\,[\mathbf{n}_4 \times (\mathbf{n}_4 \times \mathbf{n}_3)]_x$, $|\mathbf{n}_3|^2\,[\mathbf{n}_4 \times (\mathbf{n}_4 \times \mathbf{n}_3)]_y$, $-|\mathbf{n}_3|^2\,[\mathbf{n}_4 \times (\mathbf{n}_4 \times \mathbf{n}_3)]_z$ and both $\mathbf{n}_3$ and $\mathbf{n}_4 \times (\mathbf{n}_4 \times \mathbf{n}_3) = (\mathbf{n}_4 \cdot \mathbf{n}_3)\mathbf{n}_4 - |\mathbf{n}_4|^2\mathbf{n}_3$ are non-zero. Therefore the system $MX = B$ has a one parameter family of solutions.

Reducing the augmented matrix of the system to echelon form, the solution of the system is obtained as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \end{pmatrix} = c_1 - y_3\, c_2,$$

where y_3 is arbitrary, c_1 and c_2 are 5 by 1 column vectors and c_2 is dependent on $\mathbf{n}_3$ and $\mathbf{n}_4$ only. The first three terms of c_2 simplify to the entries of the vector

$$\frac{|\mathbf{n}_4|^2\,[\mathbf{n}_3 \times (\mathbf{n}_4 \times \mathbf{n}_3)]}{|\mathbf{n}_3|^2\,[\mathbf{n}_4 \times (\mathbf{n}_4 \times \mathbf{n}_3)]_z},$$

and the last two are

$$\frac{-[\mathbf{n}_4 \times (\mathbf{n}_4 \times \mathbf{n}_3)]_x}{[\mathbf{n}_4 \times (\mathbf{n}_4 \times \mathbf{n}_3)]_z}, \quad \frac{-[\mathbf{n}_4 \times (\mathbf{n}_4 \times \mathbf{n}_3)]_y}{[\mathbf{n}_4 \times (\mathbf{n}_4 \times \mathbf{n}_3)]_z}.$$

Therefore the solution of the system is pairs of points $Q_3 \in l_3$ and $Q_4 \in l_4$ where

$$Q_3 = \hat{P}_3 + \frac{\gamma}{|\mathbf{n}_3|^2}[\mathbf{n}_3 \times (\mathbf{n}_4 \times \mathbf{n}_3)], \; Q_4 = \hat{P}_4 + \frac{\gamma}{|\mathbf{n}_4|^2}[\mathbf{n}_4 \times (\mathbf{n}_3 \times \mathbf{n}_4)],$$

where γ is a scalar and $\hat{P}_3$ and $\hat{P}_4$ are any points in π_3 and π_4 respectively such that the set $\{P_1, P_2, \hat{P}_3, \hat{P}_4\}$ satisfies the symmetry condition.

Observations

1. The direction vectors $\mathbf{N}_3 = [\mathbf{n}_3 \times (\mathbf{n}_4 \times \mathbf{n}_3)]$ and $\mathbf{N}_4 = [\mathbf{n}_4 \times (\mathbf{n}_3 \times \mathbf{n}_4)]$ of the lines l_3 and l_4 respectively are independent of the choice of the fixed points P_1 and P_2.

2. The lines l_3 and l_4 meet the line going through edge V_1V_2 at right angles, since $\mathbf{n}_3 \times \mathbf{n}_4$ lies along the line through V_1V_2 and the direction vectors $\mathbf{n}_3 \times (\mathbf{n}_4 \times \mathbf{n}_3)$ and $\mathbf{n}_4 \times (\mathbf{n}_3 \times \mathbf{n}_4)$ are both perpendicular to $\mathbf{n}_3 \times \mathbf{n}_4$.

3. If $\hat{\mathbf{N}}_3$ is the unit vector in direction $\mathbf{N}_3$ and $\hat{\mathbf{N}}_4$ is the unit vector in direction $\mathbf{N}_4$, to get a set $\{P_1, P_2, Q_3, Q_4\}$ that satisfies the symmetry condition from another such set, a displacement of $\beta\hat{\mathbf{N}}_4/|\mathbf{n}_4|$ along l_4 should be accompanied by a displacement of $\beta\hat{\mathbf{N}}_3/|\mathbf{n}_3|$ along l_3. This is because

$$\frac{|\mathbf{N}_3|}{|\mathbf{N}_4|} = \frac{|\mathbf{n}_3 \times (\mathbf{n}_4 \times \mathbf{n}_3)|}{|\mathbf{n}_4 \times (\mathbf{n}_3 \times \mathbf{n}_4)|} = \frac{|\mathbf{n}_3|}{|\mathbf{n}_4|},$$

and a displacement $\beta\hat{\mathbf{N}}_4/|\mathbf{n}_4|$ along l_4 corresponds to a value of γ given by

$$\begin{aligned} \frac{\beta\hat{\mathbf{N}}_4}{|\mathbf{n}_4|} &= \frac{\gamma}{|\mathbf{n}_4|^2}|\mathbf{N}_4|\hat{\mathbf{N}}_4 \\ \Rightarrow \gamma &= \frac{\beta|\mathbf{n}_4|}{|\mathbf{N}_4|}. \end{aligned}$$

Hence the corresponding displacement on l_3 is

$$\begin{aligned} \frac{\beta|\mathbf{n}_4|}{|\mathbf{N}_4|}.\frac{|\mathbf{N}_3|\hat{\mathbf{N}}_3}{|\mathbf{n}_3|^2} &= \beta\frac{|\mathbf{n}_3|}{|\mathbf{n}_4|}.\frac{|\mathbf{n}_4|}{|\mathbf{n}_3|^2}\hat{\mathbf{N}}_3 \\ &= \beta\frac{\hat{\mathbf{N}}_3}{|\mathbf{n}_3|}. \end{aligned}$$

The point P_3 on l_3 and its corresponding point P_4 on l_4 such that

$$P_1 \times \mathbf{n}_1 + P_2 \times \mathbf{n}_2 + P_3 \times \mathbf{n}_3 + P_4 \times \mathbf{n}_4 = \mathbf{0}$$

will be referred to as *related points*.

4. Let P_3 on l_3 and P_4 on l_4 be related points. Consider the problem of solving the symmetry condition with P_3 and P_4 fixed. Since $\{P_1, P_2, P_3, P_4\}$ satisfy the symmetry condition, the lines l_1 on F_1 and l_2 on F_2 that contain points that solve the problem go through P_1 and P_2 respectively. Moreover, their direction vectors are $\mathbf{n}_1 \times (\mathbf{n}_2 \times \mathbf{n}_1)$ and $\mathbf{n}_2 \times (\mathbf{n}_1 \times \mathbf{n}_2)$ respectively. Therefore

$$l_1 = \{x : x = P_1 + \lambda(\mathbf{n}_1 \times (\mathbf{n}_2 \times \mathbf{n}_1)\},\ l_2 = \{x : x = P_2 + \delta(\mathbf{n}_2 \times (\mathbf{n}_1 \times \mathbf{n}_2)\}.$$

For related points on these lines $\delta\,|\mathbf{n}_2|^2 = \lambda\,|\mathbf{n}_1|^2$. Hence when P_1 and P_2 are fixed, lines l_3, l_4, l_1 and l_2 are automatically fixed. We will refer to such four lines as *related lines*.

Corollary 5.10 *Let T be a tetrahedron and G_i the centroid of face F_i. There exist small neighbourhoods I and J of 0 such that for any $\lambda \in I$, $\delta \in J$, the points*

$$\begin{aligned} P_1 &= G_1 + \frac{\lambda}{|\mathbf{n}_1|^2}(\mathbf{n}_1 \times (\mathbf{n}_2 \times \mathbf{n}_1)), \\ P_2 &= G_2 + \frac{\lambda}{|\mathbf{n}_2|^2}(\mathbf{n}_2 \times (\mathbf{n}_1 \times \mathbf{n}_2)), \\ P_3 &= G_3 + \frac{\delta}{|\mathbf{n}_3|^2}(\mathbf{n}_3 \times (\mathbf{n}_4 \times \mathbf{n}_3)) \\ \textit{and} \quad P_4 &= G_4 + \frac{\delta}{|\mathbf{n}_4|^2}(\mathbf{n}_4 \times (\mathbf{n}_3 \times \mathbf{n}_4)) \end{aligned}$$

immobilize T.

Alternatively, for $s \in I'$, $t \in J'$, I', J' small neighbourhoods of 0, the points

$$\begin{aligned} P_1 &= \left(\frac{1}{3}(1 - \frac{t}{|\mathbf{n}_1|^2}) + \frac{t\,k_1}{|\mathbf{n}_1|^2}\right) V_3 + \left(\frac{1}{3}(1 - \frac{t}{|\mathbf{n}_1|^2}) + \frac{t\,(1-k_1)}{|\mathbf{n}_1|^2}\right) V_4 \\ &+ \frac{1}{3}\left(1 - \frac{t}{|\mathbf{n}_1|^2}\right) V_2, \\ P_2 &= \left(\frac{1}{3}(1 - \frac{t}{|\mathbf{n}_2|^2}) + \frac{t\,k_2}{|\mathbf{n}_2|^2}\right) V_3 + \left(\frac{1}{3}(1 - \frac{t}{|\mathbf{n}_2|^2}) + \frac{t\,(1-k_2)}{|\mathbf{n}_2|^2}\right) V_4 \\ &+ \frac{1}{3}\left(1 - \frac{t}{|\mathbf{n}_2|^2}\right) V_1, \\ P_3 &= \left(\frac{1}{3}(1 - \frac{s}{|\mathbf{n}_3|^2}) + \frac{s\,k_3}{|\mathbf{n}_3|^2}\right) V_1 + \left(\frac{1}{3}(1 - \frac{s}{|\mathbf{n}_3|^2}) + \frac{s\,(1-k_3)}{|\mathbf{n}_3|^2}\right) V_2 \\ &+ \frac{1}{3}\left(1 - \frac{s}{|\mathbf{n}_3|^2}\right) V_4, \\ P_4 &= \left(\frac{1}{3}(1 - \frac{s}{|\mathbf{n}_4|^2}) + \frac{s\,k_4}{|\mathbf{n}_4|^2}\right) V_1 + \left(\frac{1}{3}(1 - \frac{s}{|\mathbf{n}_4|^2}) + \frac{s\,(1-k_4)}{|\mathbf{n}_4|^2}\right) V_2 \\ &+ \frac{1}{3}\left(1 - \frac{s}{|\mathbf{n}_4|^2}\right) V_3, \end{aligned}$$

where

$$\begin{aligned}
k_1 &= \frac{V_3 - V_4 \cdot V_3 + V_2 - 2V_4}{3\,|V_3 - V_4|^2}, \\
k_2 &= \frac{V_3 - V_4 \cdot V_3 + V_1 - 2V_4}{3\,|V_3 - V_4|^2}, \\
k_3 &= \frac{V_1 - V_2 \cdot V_1 + V_4 - 2V_2}{3\,|V_1 - V_2|^2}, \\
k_4 &= \frac{V_1 - V_2 \cdot V_1 + V_3 - 2V_2}{3\,|V_1 - V_2|^2},
\end{aligned}$$

immobilize T.

Proof

Fix $P_1 = G_1$ and $P_2 = G_2$ and solve the symmetry condition for Q_3 and Q_4. The related lines l_1, l_2, l_3, l_4 arising out this setup are

$$\begin{aligned}
l_1 &= \left\{ x : x = G_1 + \frac{\lambda}{|\mathbf{n}_1|^2}\left[\mathbf{n}_1 \times (\mathbf{n}_2 \times \mathbf{n}_1)\right] \right\}, \\
l_2 &= \left\{ x : x = G_2 + \frac{\lambda}{|\mathbf{n}_2|^2}\left[\mathbf{n}_2 \times (\mathbf{n}_1 \times \mathbf{n}_2)\right] \right\}, \\
l_3 &= \left\{ x : x = G_3 + \frac{\delta}{|\mathbf{n}_3|^2}\left[\mathbf{n}_3 \times (\mathbf{n}_4 \times \mathbf{n}_3)\right] \right\}, \\
l_4 &= \left\{ x : x = G_4 + \frac{\delta}{|\mathbf{n}_4|^2}\left[\mathbf{n}_4 \times (\mathbf{n}_3 \times \mathbf{n}_4)\right] \right\}
\end{aligned}$$

for scalars λ and δ. Let x_i be an arbitrary member of line l_i, solving the six inequalities : $x_2 - x_1 \cdot \mathbf{n}_2 > 0$, $x_3 - x_1 \cdot \mathbf{n}_3 > 0$, $x_4 - x_1 \cdot \mathbf{n}_4 > 0$, $x_1 - x_2 \cdot \mathbf{n}_1 > 0$, $x_3 - x_2 \cdot \mathbf{n}_3 > 0$ and $x_4 - x_2 \cdot \mathbf{n}_4 > 0$ for λ, a range of values of λ for which points in both l_1 and l_2 are on their faces is obtained, *i.e.* the neighbourhood I of zero is determined. Likewise, solving the six inequalities: $x_2 - x_3 \cdot \mathbf{n}_2 > 0$, $x_1 - x_3 \cdot \mathbf{n}_1 > 0$, $x_4 - x_3 \cdot \mathbf{n}_4 > 0$, $x_1 - x_4 \cdot \mathbf{n}_1 > 0$, $x_3 - x_4 \cdot \mathbf{n}_3 > 0$ and $x_2 - x_4 \cdot \mathbf{n}_2 > 0$ for δ, the neighbourhood J of zero is determined. A choice of $\lambda \in I$ and a choice of $\delta \in J$ is a choice of related points on lines l_1, l_2 and lines l_3, l_4 respectively, hence an immobilizing set of T.

Another way of thinking about this is to write the arbitrary point x_i on line l_i as a convex linear combination of the vertices of face F_i. The lines l_1, l_2 are perpendicular to edge V_3V_4 and lines l_3, l_4 are perpendicular to edge V_1V_2. Consider line l_1 for example and suppose l_1 meets the line through edge V_3V_4 in point Z_1, then $Z_1 = k_1\, V_3 + (1 - k_1)\, V_4$, for some scalar k_1, and satisfies

$$\begin{aligned}
0 &= G_1 - Z_1 \cdot V_3 - V_4 \\
&= \frac{1}{3}\,(V_2 + V_3 + V_4) - k_1 V_3 - (1 - k_1) V_4 \cdot V_3 - V_4 \\
&= k_1 (V_4 - V_3) \cdot (V_3 - V_4) + \frac{1}{3}(V_2 + V_3 + V_4) - V_4 \cdot V_3 - V_4.
\end{aligned}$$

Hence

$$\begin{aligned} k_1 &= \frac{2V_4 - V_3 - V_2 \cdot V_3 - V_4}{3\,(V_4 - V_3 \cdot V_3 - V_4)} \\ &= \frac{V_3 - V_4 \cdot V_3 + V_2 - 2V_4}{3|V_3 - V_4|^2}. \end{aligned}$$

Since l_1 goes through G_1 and Z_1, an arbitrary point x_1 on l_1 can be written as $(1-t)\,G_1 + t\,Z_1$, *i.e.*

$$\begin{aligned} x_1 &= (1-t)\,(V_2 + V_3 + V_4) + t\,(k_1 V_3 + (1-k_1)V_4) \\ &= \left(\frac{1}{3}(1-t) + t\,k_1\right)\,V_3 + \left(\frac{1}{3}(1-t) + t\,(1-k_1)\right)\,V_4 + \frac{1}{3}\,(1-t)\,V_2 \end{aligned}$$

for t in some neighbourhood of zero. Similarly, solve for k_2, k_3 and k_4 and obtain expressions for arbitrary points x_2, x_3 and x_4 on lines l_2, l_3 and l_4 respectively as has been done above, taking care to give arbitrary points x_1 and x_2 the same moving parameter t and arbitrary points x_3 and x_4 moving parameter s. For related points on these line pairs the moving parameter of each line is divided by the corresponding $|\mathbf{n}_i|^2$ as was seen in the 'Observations' after Proposition 5.9.

In particular, if T is a regular tetrahedron then $k_1 = k_2 = k_3 = k_4 = 1/2$ and the set

$$P_1 = (1-2t)\,V_2 + tV_3 + tV_4, \tag{5.14}$$

$$P_2 = (1-2t)\,V_1 + tV_3 + tV_4, \tag{5.15}$$

$$P_3 = (1-2s)\,V_4 + sV_1 + sV_2, \tag{5.16}$$

$$P_4 = (1-2s)\,V_3 + sV_1 + sV_2 \tag{5.17}$$

immobilizes T for any choice of s and t lying between 0 and 1/2.

Corollary 5.10 assures us that every tetrahedron has many immobilizing sets. The fact that the centroids of a tetrahedron are the most natural immobilizing set of the tetrahedron gives the impression that immobilizing points are centrally located on their faces. The following corollary dispels this impression.

Corollary 5.11 *If T is a regular tetrahedron the immobilizing points of T can be chosen as close to the vertices of T as desired.*

Proof

The points $P_1, \ldots, P_4$ given by Equations 5.14, ..., 5.17 immobilize a regular tetrahedron for any choice of s and t in $(0, 1/2)$. As s and t tend towards 0, $P_1 \to V_2$, $P_2 \to V_1$, $P_3 \to V_4$ and $P_4 \to V_3$.

Proposition 5.12 *Let P_1, P_2 in F_1, F_2 be fixed. The normal line $w_1 = \{x : x = P_1 + \lambda\,\mathbf{n}_1\}$ at P_1 meets the normal line $w_2 = \{x : x = P_2 + \lambda\,\mathbf{n}_2\}$ at P_2 if and only if the line $l_1 = \{x : x = P_1 + \lambda(\mathbf{n}_1 \times (\mathbf{n}_2 \times \mathbf{n}_1)\}$ meets the line $l_2 = \{x : x = P_2 + \lambda(\mathbf{n}_2 \times (\mathbf{n}_1 \times \mathbf{n}_2)\}$ if and only if line $l_3 = \{x : x = P_3 + \lambda(\mathbf{n}_3 \times (\mathbf{n}_4 \times \mathbf{n}_3)\}$ meets line $l_4 = \{x : x = P_4 + \lambda(\mathbf{n}_4 \times (\mathbf{n}_3 \times \mathbf{n}_4)\}$ if and only if the normal line $w_3 = \{x : x = P_3 + \lambda\,\mathbf{n}_3\}$ at P_3 meets the normal line $w_4 = \{x : x = P_4 + \lambda\,\mathbf{n}_4\}$ at P_4 for a related pair of points P_3, P_4.*

Proof

Let $P_1 \in F_1$ and $P_2 \in F_2$ be fixed and P_3 and P_4 be related points in π_3 and π_4 respectively. Suppose w'_i are the Plücker coordinates of line w_i, $i = 1, \ldots, 4$ and l'_i are the Plücker coordinates of line l_i, $i = 1, \ldots, 4$ defined in the statement of the proposition. Then $w'_i = (\mathbf{n}_i, P_i \times \mathbf{n}_i)$ and

$$\begin{aligned}
l'_1 &= ((\mathbf{n}_1 \times (\mathbf{n}_2 \times \mathbf{n}_1), P_1 \times [\mathbf{n}_1 \times (\mathbf{n}_2 \times \mathbf{n}_1)]) \\
&= (\eta \times \mathbf{n}_1, P_1 \times (\eta \times \mathbf{n}_1)) \\
&= (\eta \times \mathbf{n}_1, (P_1 \cdot \mathbf{n}_1)\,\eta - (P_1 \cdot \eta)\,\mathbf{n}_1)\,, \\
l'_2 &= ((\mathbf{n}_2 \times (\mathbf{n}_1 \times \mathbf{n}_2), P_2 \times [\mathbf{n}_2 \times (\mathbf{n}_1 \times \mathbf{n}_2)]) \\
&= (\mathbf{n}_2 \times \eta, P_2 \times (\mathbf{n}_2 \times \eta)) \\
&= (\mathbf{n}_2 \times \eta, (P_2 \cdot \eta)\,\mathbf{n}_2 - (P_2 \cdot \mathbf{n}_2)\,\eta)
\end{aligned}$$

where $\eta = \mathbf{n}_1 \times \mathbf{n}_2$.

$$\begin{aligned}
l'_3 &= ((\mathbf{n}_3 \times (\mathbf{n}_4 \times \mathbf{n}_3), P_3 \times [\mathbf{n}_3 \times (\mathbf{n}_4 \times \mathbf{n}_3)]) \\
&= (\omega \times \mathbf{n}_3, P_3 \times (\omega \times \mathbf{n}_3)) \\
&= (\omega \times \mathbf{n}_3, (P_3 \cdot \mathbf{n}_3)\,\omega - (P_3 \cdot \omega)\,\mathbf{n}_3)\,, \\
l'_4 &= ((\mathbf{n}_4 \times (\mathbf{n}_3 \times \mathbf{n}_4), P_4 \times [\mathbf{n}_4 \times (\mathbf{n}_3 \times \mathbf{n}_4)]) \\
&= (\mathbf{n}_4 \times \omega, P_4 \times (\mathbf{n}_4 \times \omega)) \\
&= (\mathbf{n}_4 \times \omega, (P_4 \cdot \omega)\,\mathbf{n}_4 - (P_4 \cdot \mathbf{n}_4)\,\omega)
\end{aligned}$$

where $\omega = \mathbf{n}_3 \times \mathbf{n}_4$.

Then

$$\begin{aligned}
(l_1, l_2) &= (\eta \times \mathbf{n}_1) \cdot [(P_2 \cdot \eta)\,\mathbf{n}_2 - (P_2 \cdot \mathbf{n}_2)\,\eta] + (\mathbf{n}_2 \times \eta) \cdot [(P_1 \cdot \mathbf{n}_1)\eta - (P_1 \cdot \eta)\mathbf{n}_1] \\
&= (P_2 \cdot \eta)\,(\eta \times \mathbf{n}_1 \cdot \mathbf{n}_2) - (P_1 \cdot \eta)\,(\mathbf{n}_1 \cdot \mathbf{n}_2 \times \eta) \\
&= (P_2 \cdot \eta - P_1 \cdot \eta)\,|\eta|^2 \\
&= (P_2 - P_1) \cdot \eta\,|\eta|^2.
\end{aligned}$$

Yet

$$\begin{aligned}(w_1, w_2) &= \mathbf{n}_1 \cdot P_2 \times \mathbf{n}_2 + \mathbf{n}_2 \cdot P_1 \times \mathbf{n}_1 \\ &= P_2 \cdot (-\eta) + P_1 \cdot \eta \quad = \quad (P_1 - P_2) \cdot \eta.\end{aligned}$$

Likewise, $(l_3, l_4) = (P_4 - P_3) \cdot \omega\, |\omega|^2$ and $(w_3, w_4) = (P_3 - P_4) \cdot \omega$.
Since $(P_2 - P_1) \cdot \eta\, |\eta|^2 = 0$ if and only if $(P_2 - P_1) \cdot \eta = 0$ and $(P_4 - P_3) \cdot \omega\, |\omega|^2 = 0$ if and only if $(P_4 - P_3) \cdot \omega = 0$, w_1 meets w_2 if and only if l_1 meets l_2, and w_3 meets w_4 if and only if l_3 meets l_4.

Finally, it suffices to show that l_1 meets l_2 implies l_3 meets l_4, *i.e.* $(l_1, l_2) = 0 \Rightarrow (l_3, l_4) = 0$. Now $\sum_{i=1}^{4} \mathbf{n}_i = \mathbf{0}$ and $\sum_{i=1}^{4} P_i \times \mathbf{n}_i = \mathbf{0} \Rightarrow l_1' + l_2' + l_3' + l_4' = \mathbf{0}$ since $l_1' = (\mathbf{n}_i, P_i \times \mathbf{n}_i)$. Recall that $(l_i, l_i) = 0$ since each l_i is a line, then taking the 'products' $(l_i, l_1 + l_2 + l_3 + l_4)$ for $i = 1, 2, 3, 4$ respectively gives

$$(l_1, l_2) + (l_1, l_3) + (l_1, l_4) = 0 \tag{5.18}$$

$$(l_1, l_2) + (l_2, l_3) + (l_2, l_4) = 0 \tag{5.19}$$

$$(l_1, l_3) + (l_2, l_3) + (l_3, l_4) = 0 \tag{5.20}$$

$$(l_1, l_4) + (l_2, l_4) + (l_3, l_4) = 0. \tag{5.21}$$

Now 5.18 + 5.19 - 5.20 - 5.21 gives $(l_1, l_2) = (l_3, l_4)$. Hence l_1 meets l_2 if and only if l_3 meets l_4.

Corollary 5.13
1. Let $(P_1, P_2) = (G_1, G_2)$ where G_i is the centroid of face F_i. Then the lines l_3 and l_4 that contain points Q_3 and Q_4 that solve the symmetry condition go through the points G_3 and G_4 respectively.
2. Let $(P_1, P_2) = (H_1, H_2)$ where H_i is the orthocenter of face F_i. Then the lines l_3 and l_4 that contain points Q_3 and Q_4 that solve the symmetry condition go through H_3 and H_4 respectively.
3. Let $(P_1, P_2) = (O_1, O_2)$ where O_i is the circumcenter of face F_i. Then the lines l_3 and l_4 that contain points Q_3 and Q_4 that solve the symmetry condition go through O_3 and O_4 respectively.

To conclude this section we recount that for any two given points $P_1 \in F_1$ and $P_2 \in F_2$ there are pairs of points Q_3 and Q_4 in the planes of faces F_3 and F_4 that solve the symmetry condition. The full set $\{P_1, P_2, Q_3, Q_4\}$ is an immobilizing set of the tetrahedron if $Q_3 \in \mathrm{int}(F_3)$ and $Q_4 \in \mathrm{int}(F_4)$, that is, if the six inequalities: $(P_1 - Q_3) \cdot \mathbf{n}_1 > 0$, $(P_2 - Q_3) \cdot \mathbf{n}_2 > 0$, $(Q_4 - Q_3) \cdot \mathbf{n}_4 > 0$, $(P_1 - Q_4) \cdot \mathbf{n}_1 > 0$, $(P_2 - Q_4) \cdot \mathbf{n}_2 > 0$ and $(Q_3 - Q_4) \cdot \mathbf{n}_3 > 0$ hold. With reference to Corollary 4.20, a regular tetrahedron realises all the three types of immobilizing points. The centroids of faces belong to the first type.

Secondly, the set of solutions of the symmetry condition with fixed points $P_1 = G_1$, $P_2 = G_2$ contains $\{G_1, G_2, G_3, G_4\}$, for which the normals are concurrent, but any other solution $\{G_1, G_2, P_3, P_4\}$ is such that the normals at P_3 and P_4 are concurrent (see Proposition 5.12), hence are immobilizing points of the second type.

Lastly, let $P_1, \ldots, P_4$ be an immobilizing set of a regular tetrahedron obtained by fixing $s = s_o \neq \frac{1}{3}$ and $t = t_o \neq \frac{1}{3}$ in Equations 5.14, ..., 5.17. Then the normal line at P_1 meets the normal line at P_2 and the normal line at P_3 meets the normal line at P_4. These two are the only intersections between these four normal lines. Now fix P_1 and P_3 and solve the symmetry condition for points $Q_2 \in F_2$ and $Q_4 \in F_4$. The normals lines at a solution set $\{P_1, Q_2, P_3, Q_4\}$ where $Q_2 \neq P_2$ and $Q_4 \neq P_4$ do not intersect each other. Hence $\{P_1, Q_2, P_3, Q_4\}$ is an immobilizing set of the third type.

5.4 General immobilizing set of a tetrahedron

In this section the dimensionality of the solution space of the symmetry condition is computed by a 'brute-force' method, however a more general method will presented in Chapter 6. Let T be a given tetrahedron. Since the normal vectors $\mathbf{n}_i$ in $\sum_{i=1}^{4} P_i \times \mathbf{n}_i = \mathbf{0}$ are known, the equations $\sum_{i=1}^{4} P_i \times \mathbf{n}_i = \mathbf{0}$ are linear and can be solved for the three relations between the eight parameters that characterize four points in different faces of a tetrahedron. The result is simply a solution of $\sum_{i=1}^{4} P_i \times \mathbf{n}_i = \mathbf{0}$ and is not sufficient to ensure that each $P_i \in F_i$, therefore appropriate bounds have to be imposed on the parameters to obtain immobilizing sets of the tetrahedron.

An arbitrary point P_i in F_i can be expressed in more than one way. The following choice of expression is made because it is considered to be more symmetrical than the rest. Let

$$\begin{aligned}
P_1 &= (\frac{1}{3} + \alpha_1)\, V_2 + (\frac{1}{3} + \beta_1)\, V_3 + (\frac{1}{3} - \alpha_1 - \beta_1)\, V_4 \\
P_2 &= (\frac{1}{3} + \alpha_2)\, V_1 + (\frac{1}{3} + \beta_2)\, V_4 + (\frac{1}{3} - \alpha_2 - \beta_2)\, V_3 \\
P_3 &= (\frac{1}{3} + \alpha_3)\, V_4 + (\frac{1}{3} + \beta_3)\, V_1 + (\frac{1}{3} - \alpha_3 - \beta_3)\, V_2 \\
P_4 &= (\frac{1}{3} + \alpha_4)\, V_3 + (\frac{1}{3} + \beta_4)\, V_2 + (\frac{1}{3} - \alpha_4 - \beta_4)\, V_1
\end{aligned}$$

where α_i, β_i are scalars that are required to lie in the interval $(-\frac{1}{3}, \frac{2}{3})$ if the points $P_1, \ldots, P_4$ are to be in the interior of their faces. For each i, $P_i \times \mathbf{n}_i$ is computed individually to obtain

$$\begin{aligned}
P_1 \times \mathbf{n}_1 &= [P_1 \cdot V_2 - V_4]\, V_3 + [P_1 \cdot V_4 - V_3]\, V_2 + [P_1 \cdot V_3 - V_2]\, V_4, \\
P_2 \times \mathbf{n}_2 &= [P_2 \cdot V_1 - V_3]\, V_4 + [P_2 \cdot V_3 - V_4]\, V_1 + [P_2 \cdot V_4 - V_1]\, V_3, \\
P_3 \times \mathbf{n}_3 &= [P_3 \cdot V_4 - V_2]\, V_1 + [P_3 \cdot V_2 - V_1]\, V_4 + [P_3 \cdot V_1 - V_4]\, V_2, \\
P_4 \times \mathbf{n}_4 &= [P_4 \cdot V_3 - V_1]\, V_2 + [P_4 \cdot V_1 - V_2]\, V_3 + [P_4 \cdot V_2 - V_3]\, V_1,
\end{aligned}$$

where the P_i on the right hand side of the expressions are written in the above given form as convex linear combinations of their faces, but for shortage of space we have not used that form. Thus the right hand side of the expressions $P_i \times \mathbf{n}_i$ contain the parameters α_i and β_i. Arrange the three equations $\sum_{i=1}^{4} P_i \times \mathbf{n}_i = \mathbf{0}$ into system $M\,X = \mathbf{0}$ where M is a 3 by 8 matrix and X is the column vector $(\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ \alpha_3,\ \beta_3,\ \alpha_4,\ \beta_4)^t$. Then M is the matrix whose columns are the eight vectors

$$\begin{aligned}
\mathbf{a}_1 &= [V_2 - V_4 \cdot V_4 - V_3]\, V_2 + [V_2 - V_4 \cdot V_3 - V_2]\, V_4 + [V_2 - V_4 \cdot V_2 - V_4]\, V_3 \\
\mathbf{b}_1 &= [V_3 - V_4 \cdot V_4 - V_3]\, V_2 + [V_3 - V_4 \cdot V_3 - V_2]\, V_4 + [V_3 - V_4 \cdot V_2 - V_4]\, V_3 \\
\mathbf{a}_2 &= [V_1 - V_3 \cdot V_4 - V_1]\, V_3 + [V_1 - V_3 \cdot V_1 - V_3]\, V_4 + [V_1 - V_3 \cdot V_3 - V_4]\, V_1 \\
\mathbf{b}_2 &= [V_4 - V_3 \cdot V_4 - V_1]\, V_3 + [V_4 - V_3 \cdot V_1 - V_3]\, V_4 + [V_4 - V_3 \cdot V_3 - V_4]\, V_1 \\
\mathbf{a}_3 &= [V_4 - V_2 \cdot V_2 - V_1]\, V_4 + [V_4 - V_2 \cdot V_1 - V_4]\, V_2 + [V_4 - V_2 \cdot V_4 - V_2]\, V_1 \\
\mathbf{b}_3 &= [V_1 - V_2 \cdot V_2 - V_1]\, V_4 + [V_1 - V_2 \cdot V_1 - V_4]\, V_2 + [V_1 - V_2 \cdot V_4 - V_2]\, V_1 \\
\mathbf{a}_4 &= [V_3 - V_1 \cdot V_2 - V_3]\, V_1 + [V_3 - V_1 \cdot V_3 - V_1]\, V_2 + [V_3 - V_1 \cdot V_1 - V_2]\, V_3 \\
\mathbf{b}_4 &= [V_2 - V_1 \cdot V_2 - V_3]\, V_1 + [V_2 - V_1 \cdot V_3 - V_1]\, V_2 + [V_2 - V_1 \cdot V_1 - V_2]\, V_3.
\end{aligned}$$

This system of equations has a solution if a non-zero triple product can be found from the eight vectors $\mathbf{a}_1, \ldots, \mathbf{b}_4$. First, the eight vectors $\mathbf{a}_1, \ldots, \mathbf{b}_4$ are translation invariant because the coefficients in each vector have sum zero. Then consider the crossproduct $\mathbf{a}_1 \times \mathbf{b}_1$. For simplicity write

$$\begin{aligned}
\mathbf{a}_1 &= r_2 V_2 + r_4 V_4 + r_3 V_3 \\
\mathbf{b}_1 &= s_2 V_2 + s_4 V_4 + s_3 V_3,
\end{aligned}$$

then

$$\mathbf{a}_1 \times \mathbf{b}_1 = (r_2 s_4 - r_4 s_2)\, V_2 \times V_4 + (r_4 s_3 - r_3 s_4)\, V_4 \times V_3 + (r_3 s_2 - r_2 s_3)\, V_3 \times V_2.$$

Let $V_2 - V_4 = \mathbf{u}$ and $V_3 - V_4 = \mathbf{w}$, then $V_2 - V_3 = (\mathbf{u} - \mathbf{w})$, hence

$$(r_2 s_4 - r_4 s_2) = [V_2 - V_4 \cdot V_4 - V_3]\, [V_3 - V_4 \cdot V_3 - V_2]$$

$$
\begin{aligned}
& - \; [V_2 - V_4 \cdot V_3 - V_2]\,[V_3 - V_4 \cdot V_4 - V_3] \\
&= (\mathbf{u} \cdot -\mathbf{w})\,(\mathbf{w} \cdot \mathbf{w} - \mathbf{u}) \quad - \quad (\mathbf{u} \cdot \mathbf{w} - \mathbf{u})\,(\mathbf{w} \cdot -\mathbf{w}) \\
&= (-\mathbf{u} \cdot \mathbf{w})\,(|\mathbf{w}|^2 - \mathbf{u} \cdot \mathbf{w}) \quad + \quad (\mathbf{u} \cdot \mathbf{w} - |\mathbf{u}|^2)\,|\mathbf{w}|^2 \\
&= (\mathbf{u} \cdot \mathbf{w})^2 - |\mathbf{u}|^2\,|\mathbf{w}|^2 \\
&= -|\mathbf{u}|^2\,|\mathbf{w}|^2 \sin^2\theta,
\end{aligned}
$$

$$
\begin{aligned}
(r_4 s_3 - r_3 s_4) &= [V_2 - V_4 \cdot V_3 \; - \; V_2]\,[V_3 - V_4 \cdot V_2 - V_4] \\
& \quad - \; [V_2 - V_4 \cdot V_2 - V_4]\,[V_3 - V_4 \cdot V_3 - V_2] \\
&= (\mathbf{u} \cdot \mathbf{w} - \mathbf{u})\,(\mathbf{w} \cdot \mathbf{u}) \quad - \quad (\mathbf{u} \cdot \mathbf{u})\,(\mathbf{w} \cdot \mathbf{w} - \mathbf{u}) \\
&= (\mathbf{u} \cdot \mathbf{w} - |\mathbf{u}|^2)\,(\mathbf{u} \cdot \mathbf{w}) \quad - \quad |\mathbf{u}|^2(|\mathbf{w}|^2 - \mathbf{u} \cdot \mathbf{w}) \\
&= -|\mathbf{u}|^2\,|\mathbf{w}|^2 \quad + \quad (\mathbf{u} \cdot \mathbf{w})^2 \\
&= -|\mathbf{u}|^2\,|\mathbf{w}|^2 \sin^2\theta,
\end{aligned}
$$

$$
\begin{aligned}
(r_3 s_2 - r_2 s_3) &= [V_2 - V_4 \cdot V_2 - V_4]\,[V_3 - V_4 \cdot V_4 - V_3] \\
& \quad - \; [V_2 - V_4 \cdot V_4 - V_3]\,[V_3 - V_4 \cdot V_2 - V_4] \\
&= (\mathbf{u} \cdot \mathbf{u})\,(\mathbf{w} \cdot -\mathbf{w}) \quad - \quad (\mathbf{u} \cdot -\mathbf{w})\,(\mathbf{w} \cdot \mathbf{u}) \\
&= -|\mathbf{u}|^2\,|\mathbf{w}|^2 + (\mathbf{u} \cdot \mathbf{w})\,(\mathbf{u} \cdot \mathbf{w}) \\
&= -|\mathbf{u}|^2\,|\mathbf{w}|^2 \sin^2\theta
\end{aligned}
$$

where θ is the non-zero angle between $\mathbf{u} = V_2 - V_4$ and $\mathbf{w} = V_3 - V_4$.
Hence

$$
\begin{aligned}
\mathbf{a}_1 \times \mathbf{b}_1 &= (|\mathbf{u}|^2\,|\mathbf{w}|^2 \sin^2\theta)\,[V_2 \times V_3 \; + \; V_3 \times V_4 + V_4 \times V_2] \\
&= |V_2 - V_4|^2\,|V_3 - V_4|^2 \sin^2\theta\,(V_2 - V_4 \quad \times \quad V_3 - V_4) \\
&= -|V_2 - V_4|^2\,|V_3 - V_4|^2 \sin^2\theta\,\mathbf{n}_1.
\end{aligned}
$$

Similarly, similar expressions can be written down for the vectors $\mathbf{a}_2 \times \mathbf{b}_2$, $\mathbf{a}_3 \times \mathbf{b}_3$ and $\mathbf{a}_4 \times \mathbf{b}_4$.

Now consider the dot product $\mathbf{a}_3 \cdot \mathbf{a}_1 \times \mathbf{b}_1$. Set V_1 to $(0, 0, 0)$ in $\mathbf{a}_3$, then

$$
\begin{aligned}
\mathbf{a}_3 \cdot \mathbf{a}_1 \times \mathbf{b}_1 &= k_4\,V_4 + k_2\,V_2 \;\cdot\; (|\mathbf{u}|^2\,|\mathbf{w}|^2 \sin^2\theta \;\; [V_2 \times V_3 + V_3 \times V_4 + V_4 \times V_2]) \\
&= |\mathbf{u}|^2\,|\mathbf{w}|^2 \sin^2\theta (k_2 + k_4)\,V_2 \cdot V_3 \times V_4
\end{aligned}
$$

where $k_2 = [V_4 - V_2 \cdot V_1 - V_4]$ and $k_4 = [V_4 - V_2 \cdot V_2 - V_1]$.
Hence

$$
\begin{aligned}
\mathbf{a}_3 \cdot \mathbf{a}_1 \times \mathbf{b}_1 &= |\mathbf{u}|^2\,|\mathbf{w}|^2 \sin^2\theta \;[V_4 - V_2 \cdot V_2 - V_4]\,V_2 \cdot V_3 \times V_4 \\
&= -\,|\mathbf{u}|^4\,|\mathbf{w}|^2 \sin^2\theta \; V_2 \cdot V_3 \times V_4 \\
&\neq 0.
\end{aligned}
$$

Therefore the system $M\,X = \mathbf{0}$ has a solution, namely

$$\begin{aligned}\alpha_1 &= \frac{\omega \cdot \mathbf{a}_3 \times \mathbf{b}_1}{\mathbf{a}_1 \cdot \mathbf{b}_1 \times \mathbf{a}_3},\\ \beta_1 &= \frac{\omega \cdot \mathbf{a}_1 \times \mathbf{a}_3}{\mathbf{b}_1 \cdot \mathbf{a}_3 \times \mathbf{a}_1},\\ \alpha_3 &= \frac{\omega \cdot \mathbf{b}_1 \times \mathbf{a}_1}{\mathbf{a}_3 \cdot \mathbf{a}_1 \times \mathbf{b}_1},\end{aligned}$$

where $\omega = \alpha_2\,\mathbf{a}_2 + \beta_2\,\mathbf{b}_2 + \beta_3\,\mathbf{b}_3 + \alpha_4\,\mathbf{a}_4 + \beta_4\,\mathbf{b}_4$. Therefore a general solution for the symmetry condition is

$$\begin{aligned}P_1 &= (\frac{1}{3} + \left[\frac{\omega \cdot \mathbf{a}_3 \times \mathbf{b}_1}{\mathbf{a}_1 \cdot \mathbf{b}_1 \times \mathbf{a}_3}\right])\,V_2 + (\frac{1}{3} + \left[\frac{\omega \cdot \mathbf{a}_1 \times \mathbf{a}_3}{\mathbf{b}_1 \cdot \mathbf{a}_3 \times \mathbf{a}_1}\right])\,V_3\\ &+ (\frac{1}{3} + \left[\frac{\omega \cdot \mathbf{a}_3 \times (\mathbf{a}_1 - \mathbf{b}_1)}{\mathbf{a}_3 \cdot \mathbf{a}_1 \times \mathbf{b}_1}\right])\,V_4,\\ P_2 &= (\frac{1}{3} + \alpha_2)\,V_1 + (\frac{1}{3} + \beta_2)\,V_4 + (\frac{1}{3} - \alpha_2 - \beta_2)\,V_3,\\ P_3 &= (\frac{1}{3} + \left[\frac{\omega \cdot \mathbf{b}_1 \times \mathbf{a}_1}{\mathbf{a}_3 \cdot \mathbf{a}_1 \times \mathbf{b}_1}\right])\,V_4 + (\frac{1}{3} + \beta_3)\,V_1 + (\frac{1}{3} - \left[\frac{\omega \cdot \mathbf{b}_1 \times \mathbf{a}_1}{\mathbf{a}_3 \cdot \mathbf{a}_1 \times \mathbf{b}_1}\right] - \beta_3)\,V_2,\\ P_4 &= (\frac{1}{3} + \alpha_4)\,V_3 + (\frac{1}{3} + \beta_4)\,V_2 + (\frac{1}{3} - \alpha_4 - \beta_4)\,V_1,\end{aligned}$$

where the scalar α_i, β_i, $i = 1, \ldots, 4$ should lie in the interval $(-\frac{1}{3}, \frac{2}{3})$ for the points $P_1, \ldots, P_4$ to be an immobilizing set of the tetrahedron.

Corollary 5.14 *The solution set of the symmetry condition is 5-dimensional.*

5.5 Orientation of an immobilizing set

The following question naturally arises: How different is one immobilizing set of a tetrahedron from another? This question is partly answered by Corollary 4.20 where it is seen that immobilizing sets of a tetrahedron can be classified geometrically into three types. In this section, the same question is handled algebraically and three different classes are found.

Definition 5.15 *Let T be an n-simplex having vertices $V_1, \ldots, V_{n+1}$ and $P_1, \ldots,$ P_{n+1} points in the interior of faces $F_1, \ldots, F_{n+1}$ respectively. We will say that the set $\{P_1, \ldots, P_{n+1}\}$ has the same orientation as T if the n-simplex τ having vertices $P_1, \ldots, P_{n+1}$ has the same orientation as T, i.e., if*

$$det\begin{bmatrix} P_1 & \cdots & P_{n+1} \\ 1 & \cdots & 1 \end{bmatrix} \textit{ has the same sign as } det\begin{bmatrix} V_1 & \cdots & V_{n+1} \\ 1 & \cdots & 1 \end{bmatrix}.$$

In two dimensions any choice of points P_1, P_2, P_3 has the same orientation as the triangle, since, if V_1, V_2, V_3 are the vertices of the triangle and $P_i \in \text{int}(F_i)$ for $i = 1, 2, 3$, then

$$\begin{aligned} P_1 &= \alpha V_2 + (1-\alpha)V_3 \\ P_2 &= \beta V_3 + (1-\beta)V_1 \\ P_3 &= \gamma V_1 + (1-\gamma)V_2 \end{aligned}$$

for $0 \le \alpha, \beta, \gamma \le 1$. Thus if $\star : \bigwedge^2(\mathbb{R}^2) \to \bigwedge^0(\mathbb{R}^2)$,

$$\begin{aligned} \det\begin{bmatrix} P_1 & P_2 & P_3 \\ 1 & 1 & 1 \end{bmatrix} &= \det\left(\begin{bmatrix} V_1 & V_2 & V_3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1-\beta & \gamma \\ \alpha & 0 & 1-\gamma \\ 1-\alpha & \beta & 0 \end{bmatrix}\right) \\ &= [(1-\alpha)(1-\beta)(1-\gamma) + \alpha\beta\gamma] \det\begin{bmatrix} V_1 & V_2 & V_3 \\ 1 & 1 & 1 \end{bmatrix}, \end{aligned}$$

and $[(1-\alpha)(1-\beta)(1-\gamma) + \alpha\beta\gamma] > 0$.

However, a regular tetrahedron does have immobilizing points of both orientations. The set of centroids has a different orientation to that of the tetrahedron since, if $s = \frac{1}{3}(V_1 + V_2 + V_3 + V_4)$,

$$\begin{aligned} \det\begin{bmatrix} G_1 & \cdots & G_4 \\ 1 & \cdots & 1 \end{bmatrix} &= \det\begin{bmatrix} (s - \frac{1}{3}V_1) & \cdots & (s - \frac{1}{3}V_4) \\ 1 & \cdots & 1 \end{bmatrix} \\ &= \det\begin{bmatrix} -\frac{1}{3}V_1 & \cdots & -\frac{1}{3}V_4 \\ 1 & \cdots & 1 \end{bmatrix} \\ &= -1/27 \det\begin{bmatrix} V_1 & \cdots & V_4 \\ 1 & \cdots & 1 \end{bmatrix}. \end{aligned}$$

For an immobilizing set having the same orientation as the tetrahedron, recall, from the proof of Corollary 5.10, that the points

$$\begin{aligned} P_1 &= (1-2t)V_2 + tV_3 + tV_4 \\ P_2 &= (1-2t)V_1 + tV_3 + tV_4 \\ P_3 &= (1-2s)V_4 + sV_1 + sV_2 \\ P_1 &= (1-2s)V_3 + sV_1 + sV_2 \end{aligned}$$

where $0 < s, t < 1/2$, immobilize the tetrahedron. As both s and t tend towards 0, the points $P_1 \to V_2$, $P_2 \to V_1$, $P_3 \to V_4$, $P_4 \to V_3$. Thus

$$\begin{aligned} \det\begin{bmatrix} P_1 & P_2 & P_3 & P_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} &\to \det\begin{bmatrix} V_2 & V_1 & V_4 & V_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \det\begin{bmatrix} V_1 & V_2 & V_3 & V_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Therefore a regular tetrahedron has immobilizing points of both orientations. For any tetrahedron, the transition from the centroids to a set of immobilizing points having the same orientation as T, if one exists, goes through a set which does not have any of the two orientations. This is when

$$\det \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 0,$$

i.e. when the immobilizing points lie in a plane. It can be concluded therefore that

1. a regular tetrahedron has an immobilizing set which lies in a plane (see example below),

2. depending on orientation, three types of immobilizing sets of a tetrahedron exist.

Example.
Let $V_1 = (-3,-3,-3)$, $V_2 = (5,-1,-1)$, $V_3 = (-1,5,-1)$ and $V_4 = (-1,-1,5)$. The vertices $V_1,\ldots,V_4$ describe a regular tetrahedron having edges of length $2\sqrt{6}$. The points $P_1 = (1,1,1)$, $P_2 = (-\frac{5}{3},\frac{1}{3},\frac{1}{3})$, $P_3 = (-\frac{1}{3},-\frac{4}{3},\frac{8}{3})$ and $P_4 = (-\frac{1}{3},\frac{8}{3},-\frac{4}{3})$ are interior to the faces of the tetrahedron and the matrix

$$\sum_{i=1}^{4} \mathbf{n}_i P_i^t = \begin{pmatrix} 128 & 32 & 32 \\ 32 & 152 & -136 \\ 32 & -136 & 152 \end{pmatrix},$$

moreover $P_1,\ldots,P_4$ lie in the plane $x - 2y - 2z + 3 = 0$.

Observe that the orientation of the set $\{P_1,\ldots,P_4\}$ is related to the sign of the term a_3 in the characteristic polynomial $p(\lambda) = \lambda^3 - a_1\lambda^2 + a_2\lambda - a_3$ of the associated symmetric matrix A. For

$$\begin{aligned} a_3 &= \det(A) \\ &= \det(PN^t) \\ &= \det \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{n}_1 \cdot \mathbf{n}_2 \times \mathbf{n}_3 \\ &= \det \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{n}_1 \cdot 6V(V_4 - V_1) \\ &= 36V^2 \det \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Chapter 6

Higher dimensional results

6.1 Introduction

In Chapter 4 a matrix A was defined of each quadruple of points one in the plane of each face of the tetrahedron. It was seen that necessary and sufficient conditions of immobilization on a quadruple of points in the faces of the tetrahedron was that A should be symmetric. In the current chapter immobilization of an n-simplex, $n \geq 2$, is defined and a necessary and sufficient condition for immobilization is obtained. The set of centroids of faces is shown to immobilize the simplex and a method of obtaining other solutions of the symmetry condition from one solution is presented. The chapter begins with finding a way of defining good normal vectors to the faces of an n-simplex.

6.2 Normals to an n-simplex

Let $n \geq 2$, $n+1$ distinct points in $\mathbb{R}^n$ having coordinates $V_1, \ldots, V_{n+1}$ describe an n-simplex if no r $(2 \leq r \leq n+1)$ of them lie in the same $(r-2)-$dimensional affine subspace of $\mathbb{R}^n$. Let T be an n-simplex having vertices $V_1, \ldots, V_{n+1}$. Label the vertices $V_1, \ldots, V_{n+1}$ to be positively oriented. Then if $e_1, \ldots, e_n$ denotes the standard unit basis of $\mathbb{R}^n$,

$$\begin{aligned}(V_1 - V_{n+1}) \wedge \cdots \wedge (V_n - V_{n+1}) &= \det \begin{bmatrix} V_1 & \cdots & V_{n+1} \\ 1 & \cdots & 1 \end{bmatrix} \\ &= n!\, V e_1 \wedge \cdots \wedge e_n\end{aligned}$$

where V is the 'n-volume' of T. Let F_i denote the i^{th} face of T, *i.e.* the $(n-1)$-simplex of T opposite vertex V_i and $\mathbf{n}_i$ an outward normal vector to F_i. For $i = 1, \ldots, n$ define $\mathbf{n}_i$ by

$$(-1)^{i-1} \star [(V_1 - V_{n+1}) \wedge \cdots \wedge (V_{i-1} - V_{n+1}) \wedge (V_{i+1} - V_{n+1}) \wedge \cdots \wedge (V_n - V_{n+1})]\,,$$

and

$$\mathbf{n}_{n+1} = (-1)^n \star [(V_1 - V_n) \wedge \cdots \wedge (V_{n-1} - V_n)] \,.$$

Then for $i = 1, \ldots, n$, $\langle V_i - V_{n+1}, \mathbf{n}_i \rangle$ is equal to

$$\begin{aligned}
& (-1)^{i-1} \langle (V_i - V_{n+1}), \star [(V_1 - V_{n+1}) \wedge \cdots \wedge (V_{i-1} - V_{n+1}) \wedge (V_{i+1} - V_{n+1}) \\
& \qquad \wedge \cdots \wedge (V_n - V_{n+1})] \rangle \\
= \; & (-1)^{i-1} \star [(V_i - V_{n+1}) \wedge (V_1 - V_{n+1}) \wedge \cdots \wedge (V_{i-1} - V_{n+1}) \wedge (V_{i+1} - V_{n+1}) \\
& \qquad \wedge \cdots \wedge (V_n - V_{n+1})] \\
= \; & \star [(V_1 - V_{n+1}) \wedge \cdots \wedge (V_n - V_{n+1})] \\
= \; & n! \, V
\end{aligned}$$

and

$$\begin{aligned}
\langle V_{n+1} - V_n, \mathbf{n}_{n+1} \rangle &= (-1)^n \langle (V_{n+1} - V_n), \star [(V_1 - V_n) \wedge \cdots \wedge (V_{n-1} - V_n)] \rangle \\
&= (-1)^n \star [(V_{n+1} - V_n) \wedge (V_1 - V_n \wedge \cdots \wedge (V_{n-1} - V_n)] \\
&= (-1)^1 \star [(V_1 - V_n) \wedge \cdots \wedge (V_{n-1} - V_n) \wedge (V_{n+1} - V_n)] \\
&= n! \, V
\end{aligned}$$

Lemma 6.1

$$\sum_{i=1}^{n+1} \mathbf{n}_i = \mathbf{0}.$$

Proof

Expand the wedge products and add, remember $\star$ is a linear map.

The normal vectors defined above will be called the standard outward normals of the simplex. From here onwards we will assume that the vectors $\mathbf{n}_1, \ldots, \mathbf{n}_{n+1}$ are the standard outward normal vectors to the simplex. One has the following results as in the case $n = 3$.

Lemma 6.2

$|\mathbf{n}_i| = (n-1)! \, A_i$, *where A_i in the 'n-area' of face F_i of the simplex.*

Lemma 6.3

For $i \neq j$ $(V_i - V_j) \cdot \mathbf{n}_j = n! \, V$, where V is the 'n-volume' of the simplex.

6.3 Immobilizing the n-simplex

In this section we propose a generalization of Bracho, Fetter, Mayer and Montejano's Theorem (4.16).

Definition 6.4 *Let $P_1, \ldots, P_{n+1}$ be a set of points in $\mathbb{R}^n$. The energy function (associated to $P_1, \ldots, P_{n+1}$) is the map $E : SO(n) \to \mathbb{R}^n$ defined by*

$$E(g) = \sum_{i=1}^{n+1} [g(P_i) - P_i] \cdot \mathbf{n}_i$$

for $g \in SO(n)$.

If each of the points $P_1, \ldots, P_{n+1}$ belongs to the boundary of a convex body K and has a unique normal vector at them, then the energy function E measure the 'total amount of penetration' into K a small rotation g causes at the points $P_1, \ldots, P_{n+1}$.

Definition 6.5 *The points $P_1, \ldots, P_{n+1}$ in the interior of faces $F_1, \ldots, F_{n+1}$ respectively immobilize the simplex if the energy function (associated to $P_1, \ldots, P_{n+1}$) has an isolated maximum at $I_n \in SO(n)$.*

This definition is suggested by Proposition 4.11

Proposition 6.6 *Let E be the energy function on $SO(n)$ and A the $n \times n$ matrix defined by*

$$A = \sum_{i=1}^{n+1} \mathbf{n}_i P_i^t$$

where $\{P_1, \ldots, P_{n+1}\}$ is the set associated to E. Then for $R \in SO(n)$,

$$E(R) = tr(R^t A) - n!\, V.$$

Proof
The proof is similar to that of Proposition 4.12.

Proposition 6.7 *Let A be a fixed $n \times n$ matrix and $g : SO(n) \to \mathbb{R}$ the function defined by $g(R) = tr(R^t A)$ for $R \in SO(n)$. The function g has a strict local maximum at $R = I_n \in SO(n)$ if and only if A is symmetric and almost positive definite.*

Proof
The proof is similar to that of Proposition 4.14

Definition 6.8 *A set of points $P_1, \ldots, P_{n+1}$ in $\mathbb{R}^n$ will be said to satisfy the symmetry condition (with respect to a particular n-simplex) if the matrix $\sum_{i=1}^{n+1} \mathbf{n}_i P_i^t$ is symmetric.*

Theorem 6.9 *Let T be an n-simplex and $\mathbf{n}_1, \ldots, \mathbf{n}_{n+1}$ its standard outward normals. Interior points $P_1, \ldots, P_{n+1}$ of faces $F_1, \ldots, F_{n+1}$ immobilize T if and only if the matrix $A = \sum_{i=1}^{n+1} \mathbf{n}_i P_i^t$ is both symmetric and almost positive definite.*

Proof

Definition 6.5 and Proposition 6.7

It will be recalled that in the 3-dimensional case, provided that each $P_i \in F_i$, the symmetry of the matrix A implies that A is almost positive definite. This is not the case in higher dimensions. The difficulty one encounters when trying to generalize the second proof of Theorem 4.16 is that the bound of 1 on the magnitude of the eigenvalues of the higher dimensional stochastic matrix $\mathcal{A}$ (see Page 46) does not infer anything on sums of pairs of its eigenvalues. For example when $n = 4$, we would have eigenvalues 1, a, b, c and d of $\mathcal{A}$ satisfying $1 + a + b + c + d = 0$. Then, for example, $a + c = -1 - (c + d)$ which is not useful. The following example in 4-dimensions demonstrates that A can be symmetric without being almost positive definite. I must thank Tony Gilbert for providing this example.

Example

Consider the 4-simplex having vertices $V_1 = (\frac{-5}{12}, -1, 0, -3)$, $V_2 = (\frac{-83}{36}, 0, 0, 1)$, $V_3 = (1, 1, 0, -3)$, $V_4 = (\frac{35}{18}, 0, -1, 1)$ and $V_5 = (\frac{35}{18}, 0, 1, 1)$. The standard outward normal vectors of the simplex are $\mathbf{n}_1 = (0, 34, 0, \frac{17}{2})$, $\mathbf{n}_2 = (16, \frac{-34}{3}, 0, \frac{-119}{18})$, $\mathbf{n}_3 = (0, -34, 0, \frac{17}{2})$, $\mathbf{n}_4 = (-8, \frac{17}{3}, 34, \frac{-187}{36})$ and $\mathbf{n}_5 = (-8, \frac{17}{3}, -34, \frac{-187}{36})$. The points

$$\begin{aligned}
P_1 &= \frac{3}{10}V_2 + \frac{2}{5}V_3 + \frac{3}{20}V_4 + \frac{3}{20}V_5 \\
P_2 &= \frac{1}{10}V_1 + \frac{1}{10}V_3 + \frac{2}{5}V_4 + \frac{2}{5}V_5 \\
P_3 &= \frac{2}{5}V_1 + \frac{2}{5}V_2 + \frac{1}{10}V_4 + \frac{1}{10}V_5 \\
P_4 &= \frac{1}{10}V_1 + \frac{7}{10}V_2 + \frac{1}{10}V_3 + \frac{1}{10}V_5 \\
P_5 &= \frac{1}{10}V_1 + \frac{7}{10}V_2 + \frac{1}{10}V_3 + \frac{1}{10}V_4
\end{aligned}$$

are interior to their faces and satisfy the symmetry condition since

$$\sum_{i=1}^{5} \mathbf{n}_i P_i^t = \begin{bmatrix} \frac{238}{5} & 0 & 0 & 0 \\ 0 & \frac{136}{5} & 0 & 0 \\ 0 & 0 & \frac{34}{5} & 0 \\ 0 & 0 & 0 & \frac{-68}{5} \end{bmatrix}.$$

However, a pair of eigenvalues of this matrix has a negative sum.

6.4 Immobilizing sets of an n-simplex

For a given n-simplex, it is desired to find the points $P_1, \ldots, P_{n+1}$ that immobilize the simplex, that is points $P_i \in \mathrm{int}(F_i)$, $i = 1, \ldots, n+1$, such that the matrix

$$A = \sum_{i=1}^{n+1} \mathbf{n}_i \, P_i^t$$

is both symmetric and almost positive definite. Clearly, one has to first, find the points $P_1, \ldots, P_{n+1}$ that satisfy the symmetry condition and second, check which of those points make matrix A almost positive definite. This section will deal with the first problem. Observe that A can be expressed as $N^t P$ where N is the $n+1$ by n matrix whose i^{th} row is the vector $\mathbf{n}_i$ and P is the $n+1$ by n matrix whose i^{th} row is P_i. We begin by showing that the set of centroids of faces immobilizes the simplex.

Proposition 6.10 *Let G_i be the centroid of face F_i of n-simplex T, then the set $G = \{G_1, \ldots, G_{n+1}\}$ immobilizes T.*

Proof

It is enough to show that the matrix $\sum_{i=1}^{n+1} \mathbf{n}_i G_i^t$ is a positive multiple of the identity matrix I_n. Let $\hat{\mathbf{n}}_i$ be the outward unit normal vector to face F_i, $\hat{\mathbf{n}}$ be outward unit normal vector to T and A_i be the 'n-area' of face F_i, then

$$\begin{aligned} \sum_{i=1}^{n+1} \mathbf{n}_i G_i^t &= \sum_{i=1}^{n+1} (n-1)! \, A_i \hat{\mathbf{n}}_i \, G_i^t \\ &= (n-1)! \sum_{i=1}^{n+1} \hat{\mathbf{n}}_i A_i \, G_i^t \\ &= (n-1)! \sum_{i=1}^{n+1} \hat{\mathbf{n}}_i \int_{F_i} \mathbf{r}^t dS \\ &= (n-1)! \sum_{i=1}^{n+1} \int_{F_i} \hat{\mathbf{n}}_i \mathbf{r}^t dS \\ &= (n-1)! \int_{\partial T} \hat{\mathbf{n}} \mathbf{r}^t dS. \end{aligned}$$

Let $\hat{\mathbf{n}} = (n_1, \ldots, n_n)$ and $\mathbf{r} = r_1 \mathbf{e}_1 + \cdots + r_n \mathbf{e}_n$ where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the usual basis in $\mathbb{R}^n$, then the ij^{th} entry of matrix $\hat{\mathbf{n}}\mathbf{r}^t$ is $[\hat{\mathbf{n}}\mathbf{r}^t]_{ij} = n_i r_j$. Let $\mathbf{\Omega}_{ij}$ be the $(n-1)$

form $\mathbf{\Omega}_{ij} = n_i r_j d\mathbf{S}$, then, by Stokes' Theorem,

$$\int_{\partial T} \mathbf{\Omega}_{ij} = \int_T d\mathbf{\Omega}_{ij} = \begin{cases} V & i = j \\ 0 & i \neq j \end{cases}$$

Thus $\sum_{i=1}^{n+1} \mathbf{n}_i G_i^t = (n-1)!\, V\, I_n$.

Note. Let $s = \frac{1}{n}(V_1 + \cdots + V_{n+1})$, then

$$\begin{aligned}
\det \begin{bmatrix} G_1 & \cdots & G_{n+1} \\ 1 & \cdots & 1 \end{bmatrix} &= \det \begin{bmatrix} (s - \frac{1}{n}V_1) & \cdots & (s - \frac{1}{n}V_{n+1}) \\ 1 & \cdots & 1 \end{bmatrix} \\
&= \det \begin{bmatrix} -\frac{1}{n}V_1 & \cdots & -\frac{1}{n}V_{n+1} \\ 1 & \cdots & 1 \end{bmatrix} \\
&= \left(-\frac{1}{n}\right)^n \det \begin{bmatrix} V_1 & \cdots & V_{n+1} \\ 1 & \cdots & 1 \end{bmatrix}.
\end{aligned}$$

Thus the centroids have the same orientation as the simplex when n is even but a different one when n is odd.

6.4.1 The case of some points being fixed

In Section 5.3 it was shown that for any pair of points in different faces of a tetrahedron there exists pairs of points in the planes of the remaining faces that solve the symmetry condition. This is so because the number of parameters that characterize four arbitrary points in different faces of a tetrahedron is eight, the number of conditions satisfied by two fixed points in two faces is four and the symmetry condition involved three equations. Thus solving the symmetry condition in that case meant solving three equations for four unknowns. In the n-dimensional case, the number of parameters that characterize $n+1$ points in different faces of an n simplex is $(n+1)(n-1)$, the number of conditions satisfied by k fixed points in k different faces is $k(n-1)$ and the number of equations involved in the symmetry of the $n \times n$ matrix A is $n(n-1)/2$. Therefore the $(n+1-k)$ points that solve the symmetry of A exist, in general, if

$$\begin{aligned}
\frac{n}{2}(n-1) &\leq (n+1-k)(n-1) \\
k &\leq \frac{n}{2} + 1,
\end{aligned}$$

i.e. a maximum of $\frac{n+2}{2}$ points can be fixed for an n-simplex. Thus when $n = 3$ a maximum of 2 points could be fixed. However, more than $\frac{n+2}{2}$ points can be fixed if the points are known to be members of a set that satisfies the symmetry condition. This is explained in the following proposition where the number of fixed points r, $1 < r < n+1$, will be assumed, without loss of generality, to be in the faces $F_1, \ldots, F_r$.

Proposition 6.11 *Let $P_1 \in F_1, \ldots, P_{n+1} \in F_{n+1}$ be points of a set that satisfies the symmetry condition. Suppose points $P_1 \ldots, P_r$, $1 < r < n+1$, are held fixed, $\mathbf{s} = \mathbf{n}_{r+1} + \cdots + \mathbf{n}_{n+1}$ and*

$$\mathbf{N}_i = \mathbf{s} - \mathbf{n}_i - \frac{\langle \mathbf{n}_i, \mathbf{s} - \mathbf{n}_i \rangle}{\langle \mathbf{n}_i, \mathbf{n}_i \rangle} \mathbf{n}_i \quad \textit{for } r+1 \leq i \leq n+1,$$

then

1. *the points $Q_i = P_i + \gamma_i \mathbf{N}_i$ lie in π_i, the $n-1$-dimensional subspace of $\mathbb{R}^n$ containing F_i, for $r+1 \leq i \leq n+1$ and for any scalar γ_i,*
2. *if $\gamma_r = \cdots = \gamma_{n+1}$, the set $\{P_1, \ldots, P_r, Q_{r+1}, \ldots, Q_{n+1}\}$ satisfies the symmetry condition.*

Proof

For the first statement, it is enough to show that the vector $\mathbf{N}_i$ is parallel to π_i for $r+1 \leq i \leq n+1$. The calculation is:

$$\begin{aligned} \langle \mathbf{n}_i, \mathbf{N}_i \rangle &= \left\langle \mathbf{n}_i, \mathbf{s} - \mathbf{n}_i - \frac{\langle \mathbf{n}_i, \mathbf{s} - \mathbf{n}_i \rangle}{\langle \mathbf{n}_i, \mathbf{n}_i \rangle} \mathbf{n}_i \right\rangle \\ &= \langle \mathbf{n}_i, \mathbf{s} - \mathbf{n}_i \rangle - \langle \mathbf{n}_i, \mathbf{s} - \mathbf{n}_i \rangle \\ &= 0. \end{aligned}$$

For the second statement, let $\gamma_{r+1} = \cdots = \gamma_{n+1} = \gamma$, then the matrix A can now be written as:

$$\begin{aligned} A &= \sum_{i=1}^{r} \mathbf{n}_i P_i^t + \sum_{i=r+1}^{n+1} \mathbf{n}_i Q_i^t \\ &= \sum_{i=1}^{n+1} \mathbf{n}_i P_i^t + \gamma \sum_{i=r+1}^{n+1} \mathbf{n}_i \mathbf{N}_i^t \\ &= \sum_{i=1}^{n+1} \mathbf{n}_i P_i^t + \gamma \sum_{i=r+1}^{n+1} \mathbf{n}_i \left[\mathbf{s} - \mathbf{n}_i - \frac{\langle \mathbf{n}_i, \mathbf{s} - \mathbf{n}_i \rangle}{\langle \mathbf{n}_i, \mathbf{n}_i \rangle} \mathbf{n}_i \right]^t \\ &= \sum_{i=1}^{n+1} \mathbf{n_i} P_i^t + \gamma \sum_{i=r+1}^{n+1} \mathbf{n}_i \mathbf{s}^t - \gamma \sum_{i=r+1}^{n+1} \mathbf{n}_i \mathbf{n}_i^t - \gamma \sum_{i=r+1}^{n+1} \frac{\langle \mathbf{n}_i, \mathbf{s} - \mathbf{n}_i \rangle}{\langle \mathbf{n}_i, \mathbf{n}_i \rangle} \mathbf{n}_i \mathbf{n}_i^t \\ &= \sum_{i=1}^{n+1} \mathbf{n_i} P_i^t + \gamma \mathbf{s}\mathbf{s}^t - \gamma \sum_{i=r+1}^{n+1} \mathbf{n}_i \mathbf{n}_i^t - \gamma \sum_{i=r+1}^{n+1} \frac{\langle \mathbf{n}_i, \mathbf{s} - \mathbf{n}_i \rangle}{\langle \mathbf{n}_i, \mathbf{n}_i \rangle} \mathbf{n}_i \mathbf{n}_i^t \end{aligned}$$

which is a sum of symmetric matrices, hence A is symmetric.

We can therefore get solutions of the symmetry condition starting from any given solution.

6.4.2 The case of $n-1$ points being fixed

Now suppose the points $P_1 \in F_1, \ldots, P_{n+1} \in F_{n+1}$ satisfy the symmetry condition and all but two of the P_i are fixed. In addition, suppose the remaining two points lie in faces F_h and F_k, where $h < k$. By Proposition 6.11 the points

$$P_1, \ldots, P_{h-1}, P_h + \gamma(\mathbf{n}_k - \frac{\langle \mathbf{n}_h, \mathbf{n}_k \rangle}{\langle \mathbf{n}_h, \mathbf{n}_h \rangle}\mathbf{n}_h), P_{h+1}, \ldots,$$

$$P_{k-1}, P_k + \gamma(\mathbf{n}_h - \frac{\langle \mathbf{n}_k, \mathbf{n}_h \rangle}{\langle \mathbf{n}_k, \mathbf{n}_k \rangle}\mathbf{n}_k), P_{k+1}, \ldots, P_{n+1}$$

satisfy the symmetry condition for any scalar γ. Considering all the possible combinations there are $\binom{n+1}{2}$ such sets of solutions of the symmetry condition These solutions are special in the sense that their displacements from the set $\{P_1, \ldots, P_{n+1}\}$ span the set of all possible displacements from a solution of the symmetry condition to another (- this is shown later). Suppose $P_1', \ldots, P_{n+1}'$ is another set of points such that $P_i' \in F_i$ for $1 \le i \le n+1$. If $P_i' = P_i + \mathbf{d}_i$, then $\mathbf{d}_i$ must satisfy $\langle \mathbf{d}_i, \mathbf{n}_i \rangle = 0$ in order that the displacement $P_i' - P_i$ lies in the face F_i. Additionally the symmetry requirement reduces to $\sum_{i=1}^{n+1} \mathbf{n}_i \mathbf{d}_i^t$ is symmetric. For $1 \le h < k \le n+1$ the vectors

$$\mathbf{n}_k - \frac{\langle \mathbf{n}_h, \mathbf{n}_k \rangle}{\langle \mathbf{n}_h, \mathbf{n}_h \rangle}\mathbf{n}_h \quad \text{and} \quad \mathbf{n}_h - \frac{\langle \mathbf{n}_k, \mathbf{n}_h \rangle}{\langle \mathbf{n}_k, \mathbf{n}_k \rangle}\mathbf{n}_k$$

will be denoted by $\mathbf{d}_h^{hk}$ and $\mathbf{d}_k^{hk}$ respectively. Clearly $\mathbf{d}_h^{hk} = \mathbf{d}_h^{kh}$ and $\mathbf{d}_k^{hk} = \mathbf{d}_k^{kh}$ and $\langle \mathbf{d}_h^{hk}, \mathbf{n}_h \rangle = 0$ and $\langle \mathbf{d}_k^{hk}, \mathbf{n}_k \rangle = 0$. The matrix $\sum_{i=1}^{n+1} \mathbf{n}_i \mathbf{d}_i^t$ can be expressed as $N^t D$ where N is the $n+1$ by n matrix whose i^{th} row is the vector $\mathbf{n}_i$ and D is the $n+1$ by n matrix whose i^{th} row $\mathbf{d}_i$ satisfies $\langle \mathbf{d}_i, \mathbf{n}_i \rangle = 0$. Then the problem of finding displacements $\mathbf{d}_1, \ldots, \mathbf{d}_{n+1}$ that transforms one solution of the symmetry condition into another is equivalent to the problem of finding a matrix D with the property that $N^t D$ is symmetric.

Proposition 6.12 *Let D_{hk} for $1 \le h < k \le n+1$ be the $n+1$ by n matrix whose i^{th} row $\mathbf{d}_i = \mathbf{0}$, $i \ne h$, $i \ne k$ and $\mathbf{d}_h = \mathbf{d}_h^{hk}$, $\mathbf{d}_k = \mathbf{d}_k^{hk}$. Then the matrix $N^t D_{hk}$ is symmetric.*

Proof

Let $\mathbf{n}_h = (n_{h1}, \ldots, n_{hn}), \mathbf{n}_k = (n_{k1}, \ldots, n_{kn})$, then

$$\begin{aligned} \left[N^t D_{hk}\right]_{ij} &= \sum_{m=1}^{n+1} n_{mi} [D_{hk}]_{mj} \\ &= n_{hi}\left(n_{kj} - \frac{\langle \mathbf{n}_h, \mathbf{n}_k \rangle}{\langle \mathbf{n}_h, \mathbf{n}_h \rangle} n_{hj}\right) + n_{ki}\left(n_{hj} - \frac{\langle \mathbf{n}_h, \mathbf{n}_k \rangle}{\langle \mathbf{n}_k, \mathbf{n}_k \rangle} n_{kj}\right), \end{aligned}$$

$$\begin{aligned}\left[N^t D_{hk}\right]_{ji} &= \sum_{m=1}^{n+1} n_{mj}[D_{hk}]_{mi} \\ &= n_{hj}\left(n_{ki} - \frac{\langle \mathbf{n}_h, \mathbf{n}_k\rangle}{\langle \mathbf{n}_h, \mathbf{n}_h\rangle} n_{hi}\right) + n_{kj}\left(n_{hi} - \frac{\langle \mathbf{n}_h, \mathbf{n}_k\rangle}{\langle \mathbf{n}_k, \mathbf{n}_k\rangle} n_{ki}\right) \\ &= \left[N^t D_{hk}\right]_{ij}.\end{aligned}$$

Let $\mathcal{S} = \{D : D$ is an $n+1$ by n matrix whose i^{th} row $\mathbf{d}_i$ satisfies $\langle \mathbf{d}_i, \mathbf{n}_i\rangle = 0$ for $1 \leq i \leq n+1$ and $N^t D$ is symmetric $\}$. $\mathcal{S}$ is a linear space. Each matrix D_{hk}, $1 \leq h < k \leq n+1$ is a member of $\mathcal{S}$. An element of $\mathcal{S}$ will be referred to as a *symmetric displacement* and D_{hk} will be called a *special symmetric displacement.*

Lemma 6.13 *The matrices D_{hk} where $1 \leq h < k \leq n+1$ are linearly dependent.*

Proof

Rearrange the entries of each D_{hk} into a row w_{hk} having $n(n+1)$ entries, $n(n-1)$ of which are zeros. Form the $\binom{n+1}{2}$ by $n(n+1)$ matrix W whose rows are the w_{hk}, $1 \leq h < k \leq n+1$ defined above. The row w_{hk} comprises the displacements of all points $P_i \in \pi_i$ (hyperplane of F_i) under the special symmetric displacement D_{hk} The sum of all the columns of W is zero. To see this, consider the columns of W in groups of size n. There are $n+1$ such groups. Let c_j be the j^{th} group of columns of W. The sum of the columns in c_j is

$$\begin{aligned}\sum_{i\neq j} \mathbf{n}_i - \sum_{i\neq j} \frac{\langle \mathbf{n}_i, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle} \mathbf{n}_j &= \sum_{i\neq j} \mathbf{n}_i - \frac{\langle \sum_{i\neq j} \mathbf{n}_i, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle} \mathbf{n}_j \\ &= \sum_{i=1}^{n+1} \mathbf{n}_i \\ &= \mathbf{0}.\end{aligned}$$

Example When $n = 3$ the matrix W is given by

$$\begin{bmatrix} \mathbf{n}_2 - \frac{\langle \mathbf{n}_1,\mathbf{n}_2\rangle}{\langle \mathbf{n}_1,\mathbf{n}_1\rangle}\mathbf{n}_1 & \mathbf{n}_1 - \frac{\langle \mathbf{n}_2,\mathbf{n}_1\rangle}{\langle \mathbf{n}_2,\mathbf{n}_2\rangle}\mathbf{n}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{n}_3 - \frac{\langle \mathbf{n}_1,\mathbf{n}_3\rangle}{\langle \mathbf{n}_1,\mathbf{n}_1\rangle}\mathbf{n}_1 & \mathbf{0} & \mathbf{n}_1 - \frac{\langle \mathbf{n}_3,\mathbf{n}_1\rangle}{\langle \mathbf{n}_3,\mathbf{n}_3\rangle}\mathbf{n}_3 & \mathbf{0} \\ \mathbf{n}_4 - \frac{\langle \mathbf{n}_1,\mathbf{n}_4\rangle}{\langle \mathbf{n}_1,\mathbf{n}_1\rangle}\mathbf{n}_1 & \mathbf{0} & \mathbf{0} & \mathbf{n}_1 - \frac{\langle \mathbf{n}_4,\mathbf{n}_1\rangle}{\langle \mathbf{n}_4,\mathbf{n}_4\rangle}\mathbf{n}_4 \\ \mathbf{0} & \mathbf{n}_3 - \frac{\langle \mathbf{n}_2,\mathbf{n}_3\rangle}{\langle \mathbf{n}_2,\mathbf{n}_2\rangle}\mathbf{n}_2 & \mathbf{n}_2 - \frac{\langle \mathbf{n}_3,\mathbf{n}_2\rangle}{\langle \mathbf{n}_3,\mathbf{n}_3\rangle}\mathbf{n}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{n}_4 - \frac{\langle \mathbf{n}_2,\mathbf{n}_4\rangle}{\langle \mathbf{n}_2,\mathbf{n}_2\rangle}\mathbf{n}_2 & \mathbf{0} & \mathbf{n}_2 - \frac{\langle \mathbf{n}_4,\mathbf{n}_2\rangle}{\langle \mathbf{n}_4,\mathbf{n}_4\rangle}\mathbf{n}_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{n}_4 - \frac{\langle \mathbf{n}_3,\mathbf{n}_4\rangle}{\langle \mathbf{n}_3,\mathbf{n}_3\rangle}\mathbf{n}_3 & \mathbf{n}_3 - \frac{\langle \mathbf{n}_4,\mathbf{n}_3\rangle}{\langle \mathbf{n}_4,\mathbf{n}_4\rangle}\mathbf{n}_4 \end{bmatrix},$$

where $\mathbf{0} = (0, 0, 0)$. It is easy to see that each of the four columns in the matrix W has sum $\mathbf{0}$.

Lemma 6.14 *The relation*

$$\sum_{h<k} D_{hk} = 0$$

is the only dependence among $D_{hk},\ 1 \le h < k \le n + 1$.

Proof

Consider the matrix W introduced in the proof of Lemma 6.13. It is enough to show that any $\binom{n+1}{2}$ - 1 rows of W are linearly independent. Let $\bar{W}$ be the matrix obtained from W by deleting row w_{hk}. Consider groups of columns of $\bar{W}$ of size n. Column group c_r now has $n - 1$ non-zero sub-rows spanned by $\mathbf{n}_1, \ldots, \mathbf{n}_{k-1}, \mathbf{n}_{k+1}, \ldots, \mathbf{n}_{n+1}$. Likewise column group c_k has $n - 1$ non-zero sub-rows spanned by $\mathbf{n}_1, \ldots, \mathbf{n}_{h-1}, \mathbf{n}_{h+1}, \ldots, \mathbf{n}_{n+1}$. Suppose the linear combination

$$\sum_{ij \neq rk} \alpha^{ij} w_{ij} \quad \text{is zero.}$$

Since the sub-rows of column group c_h and c_k are spanned by n linearly independent vectors, $\alpha^{hj} = 0$ for $j \neq k$ and $\alpha^{ik} = 0$ for $i \neq h$. Let c_j be any other column group $j \neq h,\ \ j \neq k$. c_j has n non-zero sub-rows and one of these sub-rows has got a coefficient in $\sum_{ij \neq hk} \alpha^{ij} w_{ij}$ which has already been deduced to be zero. Hence c_j is also spanned by n linearly independent vectors. Therefore each $\alpha^{ij} = 0$ if $\sum_{ij \neq rk} \alpha^{ij} w_{ij}$ is zero.

Proposition 6.15 *The special symmetric displacements* $D_{hk},\ 1 \le h < k \le n+1$ *span the space* $\mathcal{S} = \{D : D$ *is an* $n + 1$ *by* n *matrix whose rows* $\mathbf{d}_i$ *satisfy* $\langle \mathbf{d}_i, \mathbf{n}_i \rangle = 0$ *for* $1 \le i \le n + 1$ *and* $N^t D$ *is symmetric* $\}$.

Proof

Let $\mathcal{D}$ be the linear span of all the $D_{hk}, 1 \leq h < k \leq n+1$. It is desired to show that $\mathcal{D} = \mathcal{S}$. From Proposition 6.12 we have $\mathcal{D} \subset \mathcal{S}$. To show the opposite inclusion suppose X is a symmetric displacement that is not in $\mathcal{D}$. Then, because X is an element of a linear space, it can be assumed that X is orthogonal to all D_{hk}, where the inner product between two matrices A and B is given by $tr(A^t B)$. Now if

$$W_h = \frac{\langle \mathbf{n}_h, \mathbf{n}_k \rangle}{\langle \mathbf{n}_h, \mathbf{n}_h \rangle} \quad \text{and} \quad W_k = \frac{\langle \mathbf{n}_k, \mathbf{n}_h \rangle}{\langle \mathbf{n}_k, \mathbf{n}_k \rangle},$$

then

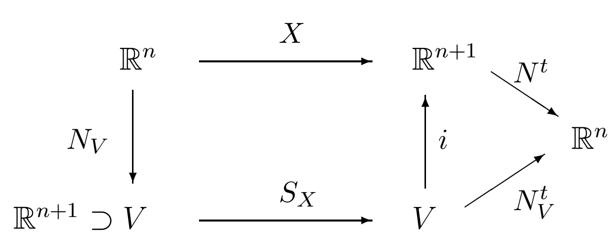
$$\begin{aligned} tr(D_{hk}^t X) &= \sum_{i=1}^{n} \sum_{j=1}^{n+1} [D_{hk}]_{ji} X_{ji} \\ &= \sum_{i=1}^{n} (n_{ki} - W_h\, n_{hi}) X_{hi} + (n_{hi} - W_k\, n_{ki}) X_{ki} \\ &= \sum_{i=1}^{n} n_{ki} X_{hi} + n_{hi} X_{ki} - W_h \sum_{i=1}^{n} n_{hi} X_{hi} - W_k \sum_{i=1}^{n} n_{ki} X_{ki} \\ &= \sum_{i=1}^{n} n_{ki} X_{hi} + n_{hi} X_{ki} \\ &= [N X^t]_{kh} + [N X^t]_{hk}. \end{aligned}$$

Therefore the assumption that X is a symmetric displacement that does not belong to $\mathcal{D}$ implies that $N^t X$ is symmetric and $N X^t$ is skew symmetric. The following two Lemmas complete the proof.

Lemma 6.16 *Every linear map $X : \mathbb{R}^n \to \mathbb{R}^{n+1}$ such that $N^t X$ is symmetric may be expressed in the form $X = SN$ where S is self-adjoint.*

Proof

Regard the matrices N, X as defining linear maps $\mathbb{R}^n \to \mathbb{R}^{n+1}$. The matrix N^t defines a linear map $\mathbb{R}^{n+1} \to \mathbb{R}^n$ whose kernel is the subspace $\{\mathbf{1}\}$ spanned by the vector $\mathbf{1}$ whose components are all equal to 1. Let $Im(N) =$ V, then $V = \{\mathbf{1}\}^\perp$. Let N_V^t be the restriction of N^t to V and $N_V : \mathbb{R}^n \xrightarrow{N} V$. The maps N_V^t and N_V are adjoints and both are isomorphisms. Now suppose $N^t X$ is self-adjoint. Consider the diagram



Since N_V and N_V^t are isomorphisms, there exists a self-adjoint map S_X such that

$$N^t X = N_V^t S_X N_V = N^t i S_X N_V,$$

where i is the inclusion map $V \to \mathbb{R}^{n+1}$. Let S be the extension of iS_X to $S : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ which is zero on $\ker N^t$. Then

$$N^t X = N^t S N_V = N^t S N$$

and S is self-adjoint. Hence $X = SN$.

Lemma 6.17 *Let NN^tS be skew symmetric where S is a symmetric matrix having a zero in its (1,1) entry. Then $S = 0$.*

Proof

First, the specified zero in the (1,1) position of S corresponds to the subspace $\ker N^t$ on which the map S obtained in Lemma 6.16 is zero. The matrix NN^t is positive on $\ker(N^t)^\perp = V$, so one can change coordinates orthogonally. Let P be the orthogonal matrix such that $PNN^tP^t = \bar{D}$, where $\bar{D}$ is a diagonal matrix with a zero in its (1,1) position and all other entries positive. Let U be the symmetric matrix PSP^t. On checking individual entries of $\bar{D}U$ using skewness, it is established that $U = 0$, hence $S = 0$.

Corollary 6.18 *The dimension of $\mathcal{S}$ is $\binom{n+1}{2} - 1$.*

This corollary is in agreement with the number obtained from the following approach: The number of parameters that characterize $n+1$ points, each of which is in a different face of an n-simplex T is $(n+1)(n-1)$ and the number of conditions involved in the symmetry of $n \times n$ matrix A is $n(n-1)/2$. Therefore the dimension of the solution to:

$$P_i \in F_i \ \text{ for } 1 \le i \le n+1 \ \text{ and } \sum_{i=1}^{n+1} \mathbf{n}_i P_i^t \ \text{ is symmetric}$$

is

$$(n+1)(n-1) - \frac{n}{2}(n-1) = \binom{n+1}{2} - 1.$$

6.4.3 Geometrical property of immobilizing sets

Theorem 2.5 and Corollary 4.20 give the geometrical property of the normal lines at points of an immobilizing set of a 2-simplex and a 3-simplex respectively. The aim of this subsection is to find a generalization of these results in higher dimensions. First, we obtain a method of assigning coordinates to lines in $\mathbb{P}^n$, $n \geq 4$. This is a generalization of the work in Chapter 3 and can be found in both [HO] and [SO2].

A k-dimensional subspace of $\mathbb{P}^n$ ($k \leq n$) is determined by $k+1$ independent points, *i.e.* by $k+1$ points no r ($r \leq k+1$) of which lie in an $(r-2)$-dimensional subspace. Let S_k be the subspace determined by the points $X_0, \dots, X_k$ with coordinates

$$\begin{array}{cccc} X_{00}, & X_{01}, & \cdots, & X_{0n} \\ X_{10}, & X_{11}, & \cdots, & X_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ X_{k0}, & X_{k1}, & \cdots, & X_{kn}. \end{array}$$

The Plücker coordinates of S_k are the $\binom{n+1}{k+1}$ $k+1$ by $k+1$ determinants of the matrix

$$\begin{bmatrix} X_{00} & X_{01} & \cdots & X_{0n} \\ X_{10} & X_{11} & \cdots & X_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ X_{k0} & X_{k1} & \cdots & X_{kn} \end{bmatrix}$$

given in a specified order. We will denote the coordinate obtained by choosing the i^{th}, j^{th}, k^{th}, ... columns, $1 \leq i < j < k < \cdots \leq n$, by $p_{ijk\dots}$. As was the case for lines in $\mathbb{P}^3$, there is a dual set of Plücker coordinates on S_k. This is obtained by considering S_k as the intersection of $n-k$ hyperspaces ($(n-1)$-dimensional subspaces) of $\mathbb{P}^n$.

Lemma 6.19 *The Plücker coordinates of a line ℓ in $\mathbb{R}^n$ going through the point $P = (P_1, \dots, P_n)$ with direction vector $\mathbf{n} = (n_1, \dots, n_n)$ are $(\mathbf{n}, \varphi(P \wedge \mathbf{n}))$, where φ is the function $\bigwedge^2 \mathbb{R}^n \to \mathbb{R}^{\binom{n}{2}}$ that assigns to a 2-vector its coefficients.*

Proof

As an element of $\mathbb{P}^n$ the line ℓ goes through the points $X = (1, P_1, \dots, P_n)$ and $Y = (0, n_1, \dots, n_n)$. Its Plücker coordinates are the 2×2 determinants of the matrix

$$\begin{bmatrix} 1 & P_1 & P_2 & \cdots & P_n \\ 0 & n_1 & n_2 & \cdots & n_n \end{bmatrix},$$

that is

$$(n_1, \dots, n_n, P_1 n_2 - P_2 n_1, P_1 n_3 - P_3 n_1, \dots, P_{n-1} n_n - P_n n_{n-1}).$$

Theorem 6.20 *If the points $P_1, \ldots, P_{n+1}$ immobilize an n-simplex then the normal lines to the simplex at these points have linearly dependent Plücker coordinates.*

Proof

Suppose the set of points $\{P_1 \ldots, P_{n+1}\}$ immobilizes an n-simplex. Then the matrix $A = \sum_{i=1}^{n+1} \mathbf{n}_i P_i^t$ is symmetric, that is

$$\sum_{k=1}^{n+1} n_{ki} P_{kj} - n_{kj} P_{ki} = 0 \quad \text{for all} \quad i \neq j.$$

However, according to Lemma 6.19 $n_{ki}P_{kj} - n_{kj}P_{ki}$ are the last $\binom{n}{2}$ entries of the Plücker coordinates of the line in $\mathbb{R}^n$ going through the point $P_k = (P_{k1}, \ldots, P_{kn})$ with direction vector $\mathbf{n}_k = (n_{k1}, \ldots, n_{kn})$. Since $\sum_{i=1}^{n+1} \mathbf{n}_i = \mathbf{0}$ (see Lemma 6.1) the symmetry of A implies

$$\sum_{k=1}^{n+1} (n_{k1}, \ldots, n_{kn}, P_{k1}n_{k2} - P_{k2}n_{k1}, \ldots, P_{k(n-1)}n_{kn} - P_{kn}n_{k(n-1)}) = \mathbf{0},$$

that is A is symmetric if and only if the Plücker coordinates of the normal lines at $P_k, 1 \leq k \leq n+1$, are linearly dependent.

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