Operator theory on $\mathcal{C}_p$ spaces

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To me and all the other survivors of child sexual abuse who are in the process of rebuilding their lives with honesty, dignity and fierce determination after their voices were long repressed, and, most importantly, to those who sadly did not survive the healing process...

Abstract

It is known that, if T is a disjoint preserving linear mapping on $L_p(\mathcal{X}, \mathcal{F}, \mu)$, where $(\mathcal{X}, \mathcal{F}, \mu)$ is a σ-finite measure space and $1 \leq p < \infty$, there exist a positive linear mapping Φ and a measurable function h such that $Tf = h\Phi f$ for all $f \in L_p$. This positive mapping Φ is induced by a σ-endomorphism on $(\mathcal{X}, \mathcal{F}, \mu)$, has the property of sending characteristic functions to characteristic functions, and is itself disjoint preserving.

In this thesis we discuss the analogue of this result when L_p is replaced by the von-Neumann Schatten ideal $\mathcal{C}_p$, its non-commutative analogue. In this non-commutative setting we look at two notions of disjointness, namely complete-disjointness and Arazy-disjointness, introduce completely-disjoint and Arazy-dis joint preserving mappings on $\mathcal{C}_p$ spaces and prove the following: If $\mathcal{H}$ is a complex separable Hilbert space and $\mathcal{T}$ is a bounded completely-disjoint or Arazy-disjoint preserving mapping on $\mathcal{C}_p(\mathcal{H})$, $1 \leqslant p < \infty$, that is sequentially continuous with respect to the strong operator topology, then there exist a positive linear mapping Φ and a linear operator H on $\mathcal{H}$ such that $\mathcal{T}(K) = H\Phi(K)$ for all K in $\mathcal{C}_p$. The mapping Φ, which is first defined on finite rank projections and then extended to the whole of $\mathcal{C}_p$, has the property of sending projections to projections, and is completely-disjoint preserving.

Next a complete characterisation of invertible completely-disjoint and Arazy-disjoint preserving mappings on $\mathcal{C}_p(\mathcal{H})$ is obtained when $\mathcal{H}$ is finite dimensional. The same description is obtained for a certain class of completely-disjoint and Arazy-disjoint preserving mappings on $\mathcal{C}_p(\mathcal{H})$, when $\mathcal{H}$ is a separable infinite dimensional Hilbert space. We also examine how completely-disjoint/Arazy-disjoint preserving mappings and isometries on $\mathcal{C}_p$ spaces are related. We then move on to discuss briefly the characterisation of non-invertible completely-disjoint and Arazy-disjoint preserving mappings in the finite dimensional case.

A standard result on L_p spaces is that, if we complexify a bounded linear operator on a real L_p space, its norm remains the same. In Chapter 4 we give an example which demonstrates that this result does not extend to the non-commutative $\mathcal{C}_p$ setting when $p \neq 2$. In a positive direction, we prove that if $\widetilde{T}$ is the complexification of a linear operator T on $\mathcal{C}_{p,sa}$, the space of all self-

adjoint operators in $\mathcal{C}_p$, then $||\widetilde{T}||_{\mathcal{C}_p} \leq 2^{\frac{2-p}{p}}||T||_{\mathcal{C}_{p,sa}}$ whenever $1 < p < 2$ and $||\widetilde{T}||_{\mathcal{C}_p} \leq 2^{\frac{p-2}{p}}||T||_{\mathcal{C}_{p,sa}}$ whenever $2 \leq p < \infty$.

In the final chapter we examine the relationship between the Schur product of matrices of a specific form and isometries on $\mathcal{C}_p$ spaces. Some properties of Schur products of matrices of this form are also given. We finish the thesis by discussing some open problems.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

Vasiliki Lioudaki

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Introduction

This thesis begins with a literature review of certain classes $\mathcal{C}_p$ $(1 \leq p < \infty)$ of linear operators on a Hilbert space $\mathcal{H}$ which were introduced by von Neumann and Schatten [20]. It turns out that each of these classes is a two sided ideal in $\mathcal{B}(\mathcal{H})$, and consists of compact operators. When provided with a suitable norm, $\mathcal{C}_p$ becomes a Banach space with properties closely analogous to those of the sequence space l_p. The development of this theory involves difficulties arising from the non-commutativity of $\mathcal{C}_p$, which has no parallel in the case of l_p.

A Banach space X is said to be uniformly convex if to each ϵ, $0 < \epsilon \leq 2$, there corresponds a $\delta(\epsilon) > 0$ such that the conditions $||x|| = ||y|| = 1$, $||x - y|| \geq \epsilon$ imply $\|\frac{x+y}{2}\| \leq 1 - \delta(\epsilon)$.

We remark that all Euclidean spaces of all dimensions and all Hilbert spaces are uniformly convex. This follows, for example, from the identity

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2) \tag{1}$$

which is known to be characteristic of such spaces [21].

In [12] J.A.Clarkson proves that the spaces L_p, $1 < p < \infty$, are uniformly convex by exhibiting a set of inequalities for these spaces which are in close analogy with identity (1). In the first chapter we state and prove McCarthy's inequalities which are the analogue of Clarkson's inequalities in $\mathcal{C}_p$ spaces and can be used to show that $\mathcal{C}_p$ is as uniformly convex a Banach space as L_p, $1 < p < \infty$, [19]. We then consider mappings on L_p spaces which send functions with disjoint supports to functions with disjoint supports, and provide a characterisation for them found in [18]. As we shall see a **disjoint preserving mapping** T on $L_p(\mathcal{X}, \mathcal{F}, \mu)$, where $(\mathcal{X}, \mathcal{F}, \mu)$ is a σ-finite measure space and $1 \leq p < \infty$, is induced by a positive linear mapping Φ on L_p, called **the support mapping** of T, and a measurable function h [15]. More explicitly,

$$T = h\Phi.$$

This positive linear mapping Φ is the extension of a σ-**endomorphism** which is first defined on characteristic functions and then on the space of all measurable functions. It maps characteristic functions to characteristic functions and is disjoint preserving. If $\mathcal{X}$ is a finite measure space, the function h is the image of the identity function under the disjoint preserving mapping T. If $\mathcal{X}$ is an arbitrary σ-finite measure space and $\{X_i : i \geq 1\}$ is a countable decomposition of $\mathcal{X}$ into subsets of finite measure, then $h = \sum_{i \geq 1} T(\chi_{X_i})$, where χ_E is the characteristic function of the set E.

Disjoint preserving mappings on L_p spaces include L_p isometries, $1 \leq p < \infty$, $p \neq 2$ ([7], [26], [18]) as well as another category of operators found in [25]. Their general structure, in the context of L_p isometries, $p \neq 2$, was investigated by J.Lamperti in 1958, but the idea goes back to Banach.

The characterisation of disjoint preserving mappings on L_p spaces motivates us to introduce the analogous concept of 'disjoint' operators as well as 'disjoint preserving' mappings in the non-commutative setting of $\mathcal{C}_p$ spaces. In Chapter 2 we see that the notion of disjointness, which is unique in L_p spaces, is not unique in $\mathcal{C}_p$ spaces.

We first introduce **completely-disjoint operators** in $\mathcal{C}_p$ spaces and give the definition of **Arazy-disjoint operators** found in [4] and [5]. We give some different ways of describing completely-disjoint and Arazy-disjoint operators and then discuss the similarities and differences between them. After introducing **completely-disjoint** and **Arazy-disjoint preserving mappings** on $\mathcal{C}_p$ spaces some examples of such mappings are given and the analogue of **the support mapping** Φ on L_p spaces is introduced in the non-commutative $\mathcal{C}_p$ spaces. The support mapping Φ, which is first defined on finite-rank projections, is gradually extended to a mapping on $\mathcal{B}(\mathcal{H})$, the space of all bounded linear mappings on $\mathcal{H}$. Interestingly, it turns out that the support mapping Φ for a completely-disjoint or an Arazy-disjoint preserving mapping $\mathcal{T}$ on $\mathcal{C}_p$ spaces is positive, completely-disjoint preserving and has properties closely analogous to the ones the support mapping for a disjoint preserving mapping on L_p has.

In Chapter 2 we also show that when $\mathcal{H}$ is finite dimensional and $\mathcal{T}$ is a completely-disjoint/Arazy-disjoint preserving mapping on $\mathcal{C}_p(\mathcal{H})$, then

$$\mathcal{T} = \mathcal{T}(I)\Phi,$$

where Φ is the support mapping induced by $\mathcal{T}$ and I is the identity operator.

In the infinite dimensional case the situation is, of course, much more complicated, but fairly similar. More explicitly, we prove the following.

Let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space and $1 \leqslant p < \infty$. Let $\mathcal{T}$ be a bounded completely-disjoint/Arazy-disjoint preserving mapping on $\mathcal{C}_p(\mathcal{H})$ and suppose that it is sequentially continuous with respect to the strong operator topology. If $\{P_i\}_{i\geq 1}$ is a partition of the identity (i.e a family $\{P_i\}_{i\in I}$ of pairwise orthogonal projections on $\mathcal{H}$ such that $clin\left[\bigcup_i P_i(\mathcal{H})\right] = \mathcal{H}$) with $dimP_i(\mathcal{H}) < \infty$ for all i and Φ is the support mapping induced by $\mathcal{T}$, then:

$$\mathcal{T}(K) = \left(\sum_{i\geq 1} \mathcal{T}(P_i)\right)\Phi(K)$$

for all $K \in \mathcal{C}_p(\mathcal{H})$. The convergence is in the strong operator topology.

In Chapter 3 we shall see how completely-disjoint and Arazy-disjoint preserving mappings on $\mathcal{C}_p$ spaces are related. First we prove the following theorem which gives a nice characterisation of invertible Arazy-disjoint preserving mappings on $M_n(\mathbb{C})$, which is nothing else but the space $\mathcal{C}_p(\mathcal{H})$ when $\mathcal{H}$ is a finite dimensional Hilbert space over $\mathbb{C}$. This description is very explicit and does not involve the support mapping Φ defined in Chapter 2.

Let $\mathcal{T}$ be a linear mapping on $M_n(\mathbb{C})$. The following are then equivalent:
(i) $\mathcal{T}$ is an invertible mapping which maps completely-disjoint to Arazy-disjoint matrices ;
(ii) $\mathcal{T}$ is an invertible Arazy-disjoint preserving mapping ;
(iii) there are unique, up to a scalar, unitary operators U, W and a unique scalar $\lambda > 0$ such that

$$\text{either} \quad \mathcal{T}(K) = \lambda UKW \qquad (K \in M_n(\mathbb{C}))$$

$$\text{or} \quad \mathcal{T}(K) = \lambda UK^tW \qquad (K \in M_n(\mathbb{C})).$$

Such an operator on $M_n(\mathbb{C})$ is completely-disjoint preserving if and only if U is a unimodular scalar multiple of W^*.

As a corollary we then show that if $\mathcal{T}$ is an invertible completely-disjoint preserving mapping on $M_n(\mathbb{C})$, then both $\mathcal{T}$ and $\mathcal{T}^{-1}$ are Arazy-disjoint preserving. However, an Arazy-disjoint preserving mapping on $M_n(\mathbb{C})$ need not be completely-disjoint preserving.

Utilising the result we obtained in the finite dimensional case a description of a class of completely-disjoint preserving mappings on $\mathcal{C}_p(\mathcal{H})$, $1 \leq p < \infty$, when $\mathcal{H}$ is an infinite dimensional Hilbert space over the complex numbers, is obtained by considering operators on $\mathcal{H}$ as infinite dimensional matrices with respect to a fixed orthonormal basis and then approximating them by finitely supported matrices. More explicitly we prove the following theorem.
Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Let $\mathcal{T}$ be a linear mapping on $\mathcal{C}_p(\mathcal{H})$, $1 \leq p < \infty$. The following are then equivalent:
(i) $\mathcal{T}$ is an invertible element of $\mathcal{B}(\mathcal{C}_p(\mathcal{H}))$ such that $\mathcal{T}$ and $\mathcal{T}^{-1}$ map C-disjoint to A-disjoint operators and rank-one projections to rank-one projections;
(ii) $\mathcal{T}$ is an invertible element of $\mathcal{B}(\mathcal{C}_p(\mathcal{H}))$ such that $\mathcal{T}$ and $\mathcal{T}^{-1}$ are C-disjoint preserving mappings and map rank-one projections to rank-one projections;

(iii) there is a unique, up to a scalar, unitary operator W such that

$$\text{either} \quad \mathcal{T}(K) = W^*KW \qquad (K \in \mathcal{C}_p(\mathcal{H}))$$

$$\text{or} \quad \mathcal{T}(K) = W^*K^tW \qquad (K \in \mathcal{C}_p(\mathcal{H})).$$

We also demonstrate that working over **complex** Hilbert spaces does make a difference, and show how completely-disjoint/Arazy-disjoint preserving mappings and isometries on $\mathcal{C}_p$ spaces are related. We finish this chapter by proving that Arazy-disjoint preserving mappings on $M_2(\mathbb{C})$ are either invertible or identically zero. We conjecture that the same holds for such mappings on $M_n(\mathbb{C})$.

In Chapter 4 we address a different topic, namely the complexification of a linear operator first on L_p and then on $\mathcal{C}_p$ spaces. It is well known that if we complexify a bounded linear operator on a real L_p space, its norm remains the same. However, as we shall see, that is not the case in $\mathcal{C}_p$ spaces. In other words, if T is a linear operator on $\mathcal{C}_{p,sa}$, the space of all self-adjoint operators in $\mathcal{C}_p$, $X \in \mathcal{C}_p$ with $X = A + iB$, where $A = \frac{X+X^*}{2}$ and $B = \frac{X-X^*}{2i}$ and an operator $\widetilde{T} : \mathcal{C}_p \to \mathcal{C}_p$, called the complexification of T, is defined by $\widetilde{T}X = TA + iTB$, then the norm of the complexified operator need **not** be equal to the norm of the operator T we start with. However, using McCarthy's inequalities, the following estimates can be obtained:

$$||\widetilde{T}||_{\mathcal{C}_p} \leq 2^{\frac{2-p}{p}} ||T||_{\mathcal{C}_{p,sa}}, \quad when \;\; 1 < p < 2$$

and

$$||\widetilde{T}||_{\mathcal{C}_p} \leq 2^{\frac{p-2}{p}} ||T||_{\mathcal{C}_{p,sa}}, \quad when \;\; 2 \leq p < \infty \;.$$

In the last chapter of this thesis we demonstrate some properties of Schur products of two matrices when one of the matrices has a particular form and prove that the Schur product with a matrix of that form is an isometry on $\mathcal{C}_p$ spaces. We conclude by discussing some questions for future work.

Much of our notation and terminology is standard and is therefore not formally introduced in the text if the meaning is clear from the context. We have however included a list of some of the notation at the end of the thesis.

Chapter 1

Background

It is better to be at the bottom of a ladder you want to climb than be half way up one you do not.

In this chapter we shall first present some of the basic background material about von Neumann-Schatten classes of operators, or otherwise called $\mathcal{C}_p$ spaces, and then move on to state and prove Clarkson-McCarthy inequalities which shall be very useful in later chapters.

Finally we present a rather interesting result about disjoint preserving mappings on the commutative L_p spaces which we shall refer to repeatedly in this thesis.

1.1 Von Neumann-Schatten Classes

Throughout this thesis $\mathcal{H}$ will denote a complex, separable, infinite dimensional Hilbert space with inner product $\langle \cdot\ ,\ \cdot \rangle$, unless otherwise stated. We shall denote the space of all bounded linear operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. If $T \in \mathcal{B}(\mathcal{H})$, the **adjoint** operator to T is the unique operator $T^* \in \mathcal{B}(\mathcal{H})$ for which $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$. An operator T in $\mathcal{B}(\mathcal{H})$ is said to be **normal** if $T^*T = TT^*$ and **self-adjoint** or **Hermitian** if $T = T^*$.

Let us now define the von Neumann-Schatten classes of operators. The proofs and more details on the following can be found in [23], Chapter 2.

Definition 1.1.1. *When $1 \leq p < \infty$, $\mathcal{C}_p$ is the set of all operators T in $\mathcal{B}(\mathcal{H})$ which satisfy the following condition: for each orthonormal system $\{\psi_k : k \in K\}$*

in $\mathcal{H}$,

$$\sum_{k\in K} |\langle T\psi_k, \psi_k\rangle|^p < \infty.$$

We define $\mathcal{C}_\infty$ to be the whole of $\mathcal{B}(\mathcal{H})$.

Throughout this section, and indeed the rest of this thesis, $\mathcal{C}_p$ for $1 \leq p < \infty$ is considered to be a space over the complex numbers unless otherwise stated in the text.

It is well known that an operator T on $\mathcal{H}$ is compact if and only if there is a sequence (F_n) of operators on $\mathcal{H}$, for which F_n has finite rank not more than n and $\lim_{n\to\infty} ||T - F_n|| = 0$. The next theorem shows that membership of $\mathcal{C}_p$ is equivalent to a statement about the rapidity with which $||T - F_n||$ can converge to 0.

Theorem 1.1.2. *Suppose $T \in \mathcal{B}(\mathcal{H})$ and $1 \leq p < \infty$. Then $T \in \mathcal{C}_p$ if and only if there is a sequence (F_n) of operators on $\mathcal{H}$, such that F_n has finite rank not more than n and*

$$\sum_{n=1}^{\infty} ||T - F_n||^p < \infty.$$

The following theorem gives a description of the algebraic properties of $\mathcal{C}_p$ spaces.

Theorem 1.1.3. *If $1 \leq p \leq \infty$, $\mathcal{C}_p$ is a two sided ideal in $\mathcal{B}(\mathcal{H})$ which contains each operator of finite rank on $\mathcal{H}$ and the adjoint of each of its members. If $1 \leq p < \infty$, then each element of $\mathcal{C}_p$ is a compact operator.*

If $\mathcal{H}$ is an infinite dimensional Hilbert space, then there is no linear functional τ on $\mathcal{B}(\mathcal{H})$ such that $\tau(AB) = \tau(BA)$ for all A and B in $\mathcal{B}(\mathcal{H})$ and $\tau(A^*A) > 0$ if $A \neq 0$.

There is, however, a linear functional τ on the ideal $\mathcal{C}_1$ of $\mathcal{B}(\mathcal{H})$ which satisfies these conditions whenever $A \in \mathcal{C}_1$ and $B \in \mathcal{B}(\mathcal{H})$.

Definition 1.1.4. *The ideal $\mathcal{C}_1$ in $\mathcal{B}(\mathcal{H})$ is called the trace class of operators on $\mathcal{H}$. If $T \in \mathcal{C}_1$ and $\{\phi_j : j \in J\}$ is an orthonormal basis in $\mathcal{H}$, then the trace of T, denoted by $\tau(T)$, is defined by the equation $\tau(T) = \sum_{j\in J}\langle T\phi_j, \phi_j\rangle$.*

$\tau(T)$ depends only on T (not on the choice of the orthonormal basis), and it is the sum of the eigenvalues of T. One can show that a compact operator T is in $\mathcal{C}_p$ if and only if the sequence of the eigenvalues of $(T^*T)^{1/2}$ is in ℓ_p. Sometimes this is used as the definition of $\mathcal{C}_p$ spaces.

Definition 1.1.5. *If $1 \leq p < \infty$ and $T \in \mathcal{C}_p$, then $||T||_p = \left[\tau\left(\left((T^*T)^{1/2}\right)^p\right)\right]^{1/p}$. The norm on $\mathcal{C}_\infty$ is the operator norm.*

From the definition of the $\mathcal{C}_p$ norm it follows that $||T||_{\mathcal{B}(\mathcal{H})} \leq ||T||_p$. It can also be shown that the $\mathcal{C}_p$ norm of T is equal to the ℓ_p norm of the sequence of the eigenvalues of $(T^*T)^{1/2}$. The following lemma gives a nice formula for the $\mathcal{C}_p$ norm in the case of a separable Hilbert space.

Proposition 1.1.6. *If $1 \leq p < \infty$ and $T \in \mathcal{B}(\mathcal{H})$, then:*

$$T \in \mathcal{C}_p \text{ if and only if there exists } M \text{ such that } \left[\sum_{j=1}^{n} |\langle T\phi_j, \psi_j\rangle|^p\right]^{1/p} \leq M$$

for every pair $(\phi_1, ..., \phi_n)$, $(\psi_1, ..., \psi_n)$ of orthonormal systems in $\mathcal{H}$. When this is satisfied, the least such constant M is $||T||_p$.

Corollary 1.1.7. *If $\mathcal{H}$ is a separable Hilbert space and $T \in \mathcal{C}_p$, then $||T||_p = sup\{\ell_p$ norm of the diagonal of the matrix of T with respect to $(\phi_i), (\psi_i) : (\phi_i), (\psi_i)$ are orthonormal bases in $\mathcal{H}\}$.*

The next corollary gives an interesting relationship between the $\mathcal{C}_p$ norm of T and the ℓ_p norm of the sequence of its own eigenvalues.

Corollary 1.1.8. *If $1 \leq p < \infty$, $T \in \mathcal{C}_p$, and (λ_n) is the sequence of the non-zero eigenvalues of T, counted according to their multiplicities, then :*

$$||(\lambda_n)||_{\ell_p} \leq ||T||_p \ .$$

It is easy to show that if T is a normal operator in $\mathcal{C}_p$, then the $\mathcal{C}_p$ norm of T is actually equal to the ℓ_p norm of the sequence of its eigenvalues, counted according to their multiplicities.

Lemma 1.1.9. *If $1 \leq p \leq \infty$, q is the index conjugate to p, and $T \in \mathcal{B}(\mathcal{H})$, then:*

$$T \in \mathcal{C}_p \text{ if and only if } sup\{|\tau(FT)| : F \text{ has finite rank and } ||F||_q \leq 1\} < \infty.$$

When this is so, $||T||_p$ is equal to the above supremum.

Now using this lemma the following theorem can easily be derived.

Theorem 1.1.10. *If $1 \leq p < \infty$, then the equation in definition* 1.1.5 *defines a norm on $\mathcal{C}_p$, and with this norm $\mathcal{C}_p$ is a Banach space. Also the set of all finite-rank operators on $\mathcal{H}$ is a dense subspace of $\mathcal{C}_p$.*

The following generalization of Hölder's inequality holds in $\mathcal{C}_p$ spaces.

Theorem 1.1.11. *If $1 \leq r, s, t \leq \infty$, $\frac{1}{t} = \frac{1}{r} + \frac{1}{s}$, $R \in \mathcal{C}_r$ and $S \in \mathcal{C}_s$, then:*

$$RS \in \mathcal{C}_t \text{ and } ||RS||_t \leq ||R||_r||S||_s \ .$$

In particular, if $1 \leq p \leq \infty$, q is the index conjugate to p, $S \in \mathcal{C}_p$ and $R \in \mathcal{C}_q$, then $SR, RS \in \mathcal{C}_1$. Furthermore, $\tau(SR) = \tau(RS)$.

The most important cases of the previous theorem are those in which $p = 1$ and $p = \infty$ or $p = q = 2$.

It is well known that if $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then ℓ_q is isometrically isomorphic to $(\ell_p)^*$. The same holds in $\mathcal{C}_p$ spaces.

Theorem 1.1.12. *Suppose that $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for each T in $\mathcal{C}_q$ the equation $f_T(S) = \tau(ST)$ $(S \in \mathcal{C}_p)$ defines a continuous linear functional on $\mathcal{C}_p$. Furthermore, the mapping $T \to f_T$ is an isometric isomorphism from $\mathcal{C}_q$ onto the dual space $(\mathcal{C}_p)^*$.*

It follows easily that the Banach space $\mathcal{C}_p$ is reflexive if $1 < p < \infty$. For $(\mathcal{C}_p)^*$ can be identified with $\mathcal{C}_q$ and $(\mathcal{C}_q)^*$ can be identified with $\mathcal{C}_p$; these identifications give rise to a canonical isometric isomorphism from $\mathcal{C}_p$ onto $(\mathcal{C}_p)^{**}$.

Definition 1.1.13. *The ideal $\mathcal{C}_2$ in $\mathcal{B}(\mathcal{H})$ is called the Schmidt class of operators on $\mathcal{H}$.*

We know that $\mathcal{C}_2$ is a Banach space with the $\mathcal{C}_2$ norm. The following theorem says that $\mathcal{C}_2$ is in fact a Hilbert space.

Theorem 1.1.14. *The Schmidt class $\mathcal{C}_2$ is a Hilbert space with respect to the inner product $[\ ,\ ]$ defined by $[S, T] = \tau(T^*S)$ for each S and T in $\mathcal{C}_2$.The norm derived from this inner product is $||\cdot||_2$. If $\{\phi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are orthonormal bases in $\mathcal{H}$ and $F_{j,k} = \phi_j \otimes \psi_k$, then $\{F_{j,k} : j, k \in J\}$ is an orthonormal basis in $\mathcal{C}_2$.*

Theorem 1.1.15. *If $T \in \mathcal{B}(\mathcal{H})$, $\{\phi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are orthonormal bases in $\mathcal{H}$, and $F_{j,k} = \phi_j \otimes \psi_k$, then the following are equivalent:*

$$(i) \sum_{j\in J} ||T\psi_j||^2 < \infty \qquad (ii) \sum_{j,k\in J} |\langle T\psi_k, \phi_j\rangle|^2 < \infty \qquad (iii) T \in \mathcal{C}_2.$$

When these conditions are satisfied, the sums in (i) and (ii) are both equal to $||T||_2^2$, and

$$T = \sum_{j,k\in J} \langle T\psi_k, \phi_j\rangle F_{j,k}\ ,$$

the summation converging with respect to $||\cdot||_2$.

The next theorem gives us the additional information that in the Hilbert space $\mathcal{C}_2$, if the ℓ_2 norm of the sequence of the eigenvalues of T is equal to the $\mathcal{C}_2$ norm of T, then T is normal.

Theorem 1.1.16. *If $T \in \mathcal{C}_2$ and (λ_n) is the sequence of the non-zero eigenvalues of T, counted according to their multiplicities, then:*

$$||(\lambda_n)||_{\ell_2}^2 \leq ||T||_2^2.$$

Equality occurs if and only if T is normal.

1.2 Clarkson-McCarthy inequalities

We now turn to the Clarkson-McCarthy inequalities which we shall use in Chapter 2 to give a norm description for a class of operators defined therein. They will also play a vital role in Chapter 4 where we establish some estimates for the norm of the complexification of a linear operator on $\mathcal{C}_p$ spaces.

The proofs of the following two Lemmas as well as the proofs of the Theorems in this section can all be found in [19]. They are however sketched here as they are of some independent interest and so that one can see how the conditions for which these inequalities hold as equalities can be obtained.

The trace of a linear operator can be defined on all positive self-adjoint operators as finite in $\mathcal{C}_1$ and ∞ otherwise.

Lemma 1.2.1. *Suppose A, B are operators on $\mathcal{H}$ and $-A \leq B \leq A$. Then:*

$$if\ 0 < \gamma \leq 1, \quad \tau(A+B)^\gamma + \tau(A-B)^\gamma \leq 2\tau(A^\gamma);$$

$$if\ 1 \leq \gamma < \infty, \quad \tau(A+B)^\gamma + \tau(A-B)^\gamma \geqslant 2\tau(A^\gamma).$$

If $\gamma \neq 1$ and the quantities involved are finite, then equality holds if and only if $AB = 0$.

Lemma 1.2.2. *Suppose A, B are operators on $\mathcal{H}$ and $A \geqslant 0$, $B \geqslant 0$. Then:*

$$if\ 0 < \gamma \leq 1, \quad \tau(A+B)^\gamma \leq \tau(A^\gamma) + \tau(B^\gamma);$$

$$if\ 1 \leq \gamma < \infty, \quad \tau(A+B)^\gamma \geqslant \tau(A^\gamma) + \tau(B^\gamma).$$

If $\gamma \neq 1$ and the quantities involved are finite, then equality holds if and only if $AB = 0$.

Before giving some concrete results from [19] we note the following rather elementary Lemma.

Lemma 1.2.3. *Let T, S be operators on a Hilbert space $\mathcal{H}$. If $T^*TS^*S = 0$, then $|T||S| = |S||T| = 0$.*

Proof. This is a simple application of the Weierstrass approximation theorem and the Riesz Functional Calculus. □

The inequalities given in the next two Theorems are the so called Clarkson-McCarthy inequalities.

Theorem 1.2.4. *Let T, S be operators on $\mathcal{H}$. Then:*

$$if \quad 2 \leq p < \infty, \quad 2(||T||_p^p + ||S||_p^p) \leq ||T+S||_p^p + ||T-S||_p^p,$$

$$if \quad 1 \leq p \leq 2, \quad 2(||T||_p^p + ||S||_p^p) \geq ||T+S||_p^p + ||T-S||_p^p.$$

If $p \neq 2$, equality holds if and only if $TS^* = S^*T = 0$. If $p = 2$, equality always holds.

Proof. We prove this for $2 \leq p < \infty$. In the same way, with the sense of all inequalities reversed, the statement can be proved for $1 \leq p \leq 2$.

Applying Lemma 1.2.1 with $A = T^*T + S^*S$, $B = T^*S + S^*T$ and Lemma 1.2.2 with $A = T^*T$, $B = S^*S$ and $\gamma = p/2 \geqslant 1$, we have the following:

$$\begin{aligned} ||T+S||_p^p + ||T-S||_p^p &= \tau\left((T^*T + S^*S) + (T^*S + S^*T)\right)^{p/2} \\ &\quad + \tau\left((T^*T + S^*S) - (T^*S + S^*T)\right)^{p/2} \\ &\geqslant 2\tau(T^*T + S^*S)^{p/2} \\ &\geqslant 2\left(\tau(T^*T)^{p/2} + \tau(S^*S)^{p/2}\right) \\ &= 2(||T||_p^p + ||S||_p^p). \end{aligned}$$

If $p \neq 2$, equality holds for T, S if and only if $(T^*T + S^*S)(T^*S + S^*T) = 0$ and $T^*TS^*S = 0$ (see Lemma 1.2.1 and 1.2.2). So first suppose that equality holds for T, S. Then it also holds for T^*, S^*. So $T^*TS^*S = TT^*SS^* = 0$. Thus by the previous Lemma, $|T||S| = |T^*||S^*| = 0$. Polar decomposition now implies that $TS^* = S^*T = 0$. Conversely, if $TS^* = S^*T = 0$, then $(T^*T+S^*S)(T^*S+S^*T) = 0$ and $T^*TS^*S = 0$ and consequently equality holds for T, S. □

Theorem 1.2.5. *Let T, S be operators on $\mathcal{H}$. Then:*

$$if \quad 2 \leq p < \infty, \quad ||T+S||_p^p + ||T-S||_p^p \leq 2^{p-1}(||T||_p^p + ||S||_p^p),$$

$$if \quad 1 \leq p \leq 2, \quad ||T+S||_p^p + ||T-S||_p^p \geq 2^{p-1}(||T||_p^p + ||S||_p^p).$$

If $p \neq 2$, equality holds if and only if $(T+S)(T-S)^* = (T-S)^*(T+S) = 0$. If $p = 2$, equality always holds.

Proof. We prove this for $2 \leq p < \infty$. In the same way, with the sense of all inequalities reversed, the statement can be proved for $1 \leq p \leq 2$.

Applying Lemma 1.2.2 with $A = (T+S)^*(T+S) \geqslant 0$, $B = (T-S)^*(T-S) \geqslant 0$ and Lemma 1.2.1 with $\gamma = p/2 \geqslant 1$, $A = (T+S)^*(T+S) + (T-S)^*(T-S)$ and $B = (T+S)^*(T-S) + (T-S)^*(T+S)$ we have the following:

$$
\begin{aligned}
&2\left(||T+S||_p^p + ||T-S||_p^p\right) = \\
&\qquad = 2\Big(\tau\left((T+S)^*(T+S)\right)^{p/2} + \tau\left((T-S)^*(T-S)\right)^{p/2}\Big) \\
&\qquad \leq 2\tau\left((T+S)^*(T+S) + (T-S)^*(T-S)\right)^{p/2} \\
&\qquad \leq \tau\left\{((T+S)^*(T+S) + (T-S)^*(T-S))\right. \\
&\qquad\qquad\qquad \left. + ((T+S)^*(T-S) + (T-S)^*(T+S))\right\}^{p/2} \\
&\qquad\quad + \tau\left\{((T+S)^*(T+S) + (T-S)^*(T-S))\right. \\
&\qquad\qquad\qquad \left. -\ ((T+S)^*(T-S) + (T-S)^*(T+S))\right\}^{p/2} \\
&\qquad = \tau(4T^*T)^{p/2} + \tau(4S^*S)^{p/2} \\
&\qquad = 2^p\left(\tau(T^*T)^{p/2} + \tau(S^*S)^{p/2}\right) \\
&\qquad = 2^p(||T||_p^p + ||S||_p^p).
\end{aligned}
$$

If $p \neq 2$ and equality holds for T, S, then $(T+S)^*(T+S)(T-S)^*(T-S) = (T+S)(T+S)^*(T-S)(T-S)^* = 0$ by Lemma 1.2.2. Hence the same argument we used in the proof of the previous Theorem implies that $(T+S)(T-S)^* = (T-S)^*(T+S) = 0$. On the other hand, if $(T+S)(T-S)^* = (T-S)^*(T+S) = 0$, then the conditions for equality given by Lemma 1.2.1 and Lemma 1.2.2 are obviously satisfied. □

Given an operator T in $\mathcal{B}(\mathcal{H})$, by the **Cartesian decomposition of** T we mean the decomposition $T = C + iD$, where C, D are self-adjoint operators, called the real and imaginary parts of T, and defined as

$$C = \frac{T+T^*}{2}, \qquad D = \frac{T-T^*}{2i}.$$

In later chapters the real and imaginary parts of T will be denoted by ReT and ImT respectively.

The following inequalities give estimates for the Cartesian decomposition of operators.

Theorem 1.2.6. *Let C, D be self-adjoint operators on $\mathcal{H}$. Then:*

$$if \quad 2 \leq p < \infty, \quad ||C + iD||_p^p \geqslant ||C||_p^p + ||D||_p^p.$$

If $p \neq 2$, equality holds if and only if $CD = 0$.

Proof. Applying Lemma 1.2.1 with $A = C^2 + D^2$, $B = i(CD - DC)$ and Lemma 1.2.2 with $A = C^2$, $B = D^2$ and $\gamma = p/2 \geqslant 1$, we have the following:

$$\begin{aligned}
||C + iD||_p^p + ||C - iD||_p^p &= \tau\left((C + iD)^*(C + iD)\right)^{p/2} + \\
&\qquad + \tau\left((C - iD)^*(C - iD)\right)^{p/2} \\
&= \tau\left((C - iD)(C + iD)\right)^{p/2} + \\
&\qquad + \tau\left((C + iD)(C - iD)\right)^{p/2} \\
&= \tau\left(C^2 + D^2 + i(CD - DC)\right)^{p/2} + \\
&\qquad + \tau\left(C^2 + D^2 - i(CD - DC)\right)^{p/2} \\
&\geqslant 2\tau(C^2 + D^2)^{p/2} \\
&\geqslant 2\left(\tau(C^2)^{p/2} + \tau(D^2)^{p/2}\right) \\
&= 2(||C||_p^p + ||D||_p^p).
\end{aligned}$$

Since $||C + iD||_p^p = ||C - iD||_p^p$, we obtain: $||C + iD||_p^p \geqslant ||C||_p^p + ||D||_p^p$.
If $p \neq 2$, equality holds iff $(C^2 + D^2)(CD - DC) = 0$ and $C^2D^2 = 0$. However, $(C^2 + D^2)(CD - DC) = 0$ and $C^2D^2 = 0$ if and only if $CD = 0$. Indeed, if $C^2D^2 = 0$, then $|C||D| = |D||C| = 0$ and so $CD = 0$. If $CD = 0$, then $(C^2 + D^2)(CD - DC) = C^2D^2 = 0$ trivially. □

Remark 1.2.7. It is worth mentioning at this point that in the simple case of the $\mathcal{C}_p$ space $M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the set of all $n \times n$ complex matrices, one can show that the inequality $||C||_p^p + ||D||_p^p \leq ||C + iD||_p^p$ holds for p=2,4,6 by simply doing some very elementary calculations.
Indeed, let $\mathbb{M}_{n,sa}(\mathbb{C})$ be the set of all nxn self-adjoint complex matrices. In the proof of Proposition 4.2.5 in Chapter 4, we prove that for all $C, D \in \mathbb{M}_{2,sa}(\mathbb{C})$,

$||C+iD||_2^2 = ||C||_2^2 + ||D||_2^2$. Some calculations show that for all $C, D \in \mathbb{M}_{4,sa}(\mathbb{C})$,

$$||C+iD||_4^4 = ||C||_4^4 + ||D||_4^4 + \tau\left(2(CD)^*(CD) + (CD-DC)^*(CD-DC)\right)$$

and since $2(CD)^*(CD) + (CD-DC)^*(CD-DC) \geq 0$, it follows that

$$\tau\left(2(CD)^*(CD) + (CD-DC)^*(CD-DC)\right) \geq 0.$$

Hence for all C,D $\in \mathbb{M}_{4,sa}(\mathbb{C})$, $||C||_4^4 + ||D||_4^4 \leq ||C+iD||_4^4$. In a similar way, we can show that for all C,D $\in \mathbb{M}_{6,sa}(\mathbb{C})$, $||C||_6^6 + ||D||_6^6 \leq ||C+iD||_6^6$.

Corollary 1.2.8. *Let C, D be self-adjoint operators on $\mathcal{H}$. Then:*

$$if \quad 2 \leq p < \infty, \quad ||C+iD||_p^p \leq 2^{p-2}\left(||C||_p^p + ||D||_p^p\right).$$

If $p \neq 2$, equality holds if and only if $C^2 = D^2$ and $CD = -DC$.

Proof. Since $||C+iD||_p = ||C-iD||_p$, we can simply use Theorem 1.2.5 with $T = C$ and $S = iD$ to obtain that $2||C+iD||_p^p \leq 2^{p-1}\left(||C||_p^p + ||D||_p^p\right)$. For $p \neq 2$ equality holds if and only if $(C+iD)(C-iD)^* = (C-iD)^*(C+iD) = 0$ or equivalently: $C^2 = D^2$ and $CD = -DC$. □

In the same way, with the sense of all inequalities reversed, using again Lemma 1.2.1, Lemma 1.2.2 and Theorem 1.2.5, we obtain the following:

Theorem 1.2.9. *Let C, D be self-adjoint operators on $\mathcal{H}$. Then:*

$$if \quad 1 \leq p \leq 2, \quad ||C||_p^p + ||D||_p^p \leq 2^{2-p}||C+iD||_p^p.$$

Theorem 1.2.10. *Let C, D be self-adjoint operators on $\mathcal{H}$. Then:*

$$if \quad 1 \leq p \leq 2, \quad ||C+iD||_p^p \leq ||C||_p^p + ||D||_p^p.$$

More on the Clarkson-McCarthy inequalities can be found in [12],[19], [16] and [24].

1.3 Disjoint preserving mappings on L_p spaces

In this section we temporarily leave $\mathcal{C}_p$ spaces to look at disjoint preserving mappings on $L_p(\mathcal{X}, \mathcal{F}, \mu)$, where $(\mathcal{X}, \mathcal{F}, \mu)$ is a σ-finite measure space and $1 \leq p < \infty$. This kind of mappings have been well-studied in the literature and were first introduced in [18].

From an intuitive point of view, these are simply operators which map functions in $L_p(\mathcal{X}, \mathcal{F}, \mu)$ which 'live' on different parts of the space $\mathcal{X}$ to functions again in $L_p(\mathcal{X}, \mathcal{F}, \mu)$ with the same property.

More formally, we shall say that two functions f, g in $L_p(\mathcal{X}, \mathcal{F}, \mu)$ have **disjoint supports** if and only if $fg = 0$ a.e with respect to the measure μ.

We are now in a position to define disjoint preserving mappings on L_p spaces.

Definition 1.3.1. *A bounded linear mapping T on an L_p space, $1 \leq p < \infty$, is called disjoint preserving if and only if $fg = 0$ implies $T(f)T(g) = 0$ for all functions $f, g \in L_p$.*

In other words, a bounded linear mapping T on an L_p space, $1 \leq p < \infty$, is disjoint preserving if and only if it maps functions with disjoint supports to functions with disjoint supports. Disjoint preserving mappings on L_p spaces are also called Lamperti operators.

In order to state the main Theorem in this section we need to explain what we mean by a **σ-endomorphism**.

Definition 1.3.2. *A σ-endomorphism Φ of the σ-finite measure space $(\mathcal{X}, \mathcal{F}, \mu)$ is an endomorphism of $\mathcal{F}$ modulo μ-null sets as a Boolean σ-algebra. This means:*

1. $\Phi : \mathcal{F} \to \mathcal{F}$,
2. $\Phi\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} \Phi E_n \ , \quad for \quad disjoint \quad E_n \in \mathcal{F} \ ,$
3. $\Phi(\mathcal{X} - E) = \Phi(\mathcal{X}) - \Phi(E) \ , \ for \ \ all \ \ E \in \mathcal{F} \ , \ and$
4. $E \in \mathcal{F} \ , \ \mu(E) = 0 \Rightarrow \mu(\Phi E) = 0.$

A σ-endomorphism Φ induces a positive linear mapping, also denoted by Φ, on the space of (finite-valued or extended) measurable functions such that $\Phi\chi_E = \chi_{\Phi E}$, where χ_E is the characteristic function of the set E. This process is described in detail in [13].

Properties of the mapping Φ

Let f, g be any measurable functions, and p any positive number. Then:

1. $\Phi(f) \ \ is \ \ \Phi(\mathcal{F}) - measurable$;
2. $supp\,(\Phi(f)) = \Phi(supp(f))$;
3. $|\Phi(f)|^p = \Phi(|f|^p)$;
4. $\Phi(f)\Phi(g) = \Phi(fg)$;

5. $f_n \to f \ \ a.e \ \ implies \ \ \Phi f_n \to \Phi f \ \ a.e$

Obviously property 4 implies that Φ separates supports.

Disjoint preserving mappings

Theorem 1.3.3. *Let T be a disjoint preserving mapping on $L_p(\mathcal{X}, \mathcal{F}, \mu)$, where $(\mathcal{X}, \mathcal{F}, \mu)$ is a σ-finite measure space and $1 \leq p < \infty$.*
If we define $\Phi_0(E) = supp\,(T(\chi_E))$, for $E \in \mathcal{F}$ with $\mu(E) < \infty$, then there exists a unique extension to a σ-endomorphism $\Phi : \mathcal{F} \to \mathcal{F}$ and for that Φ there exists a unique measurable function h such that $\nu(E) = \int_{\Phi(E)} |h|^p d\mu$ defines a measure on $(\mathcal{X}, \mathcal{F})$, $\nu \leq ||T||^p \mu$, $supp(h) = \Phi(\mathcal{X})$ and

$$Tf = h\Phi f \quad for \quad all \quad f \in L_p. \tag{1.1}$$

We conclude that T is a disjoint preserving mapping on $L_p,\ 1 \leq p < \infty$, if and only if it is induced by a σ-endomorphism Φ and a measurable function h. From now on the mapping Φ will be called **the support mapping** for the disjoint preserving mapping T. For more details on the above see [18] and [15].

Before leaving this chapter we state a Proposition which gives a norm description of two disjoint elements in L_p, $p \neq 2$ (see [11] and [12]). This automatically implies that any isometry on L_p spaces, $p \neq 2$, is always disjoint preserving.

Proposition 1.3.4. *(Norm description) If $p \neq 2$, $1 \leq p < \infty$ and $f, g \in L_p$, then*

$$||f + g||_p^p + ||f - g||_p^p = 2(||f||_p^p + ||g||_p^p) \quad \textit{if and only if} \quad fg = 0 \ a.e$$

The general structure of disjoint preserving mappings, in the context of L_p isometries, $p \neq 2$, was investigated by J.Lamperti [18], but the idea goes back to Banach [7].

Chapter 2

Completely-disjoint/Arazy-disjoint preserving mappings on C_p spaces and corresponding support mappings

One does not discover new continents without consenting to lose sight of the shore for a very long time

Motivated by the characterisation of disjoint preserving mappings on L_p spaces given in the last section of Chapter 1, we shall see that an analogue of the concept of two disjoint elements in L_p as well as of a disjoint preserving mapping on L_p can be found on $\mathcal{C}_p$ spaces. What is interesting is that the analogous concept of disjointness, which is unique in L_p spaces, is not unique in C_p spaces.

We start this chapter by introducing completely-disjoint and Arazy-disjoint operators in $\mathcal{C}_p$ as well as completely-disjoint and Arazy-disjoint preserving mappings on $\mathcal{C}_p$ spaces. The second notion of disjointness which is very similar and yet different to the first one was first introduced in [4] and [5] by Jonathan Arazy. In this chapter we shall investigate both notions of disjointness and discuss the similarities and differences between completely-disjoint and Arazy-disjoint operators. However, one of our main aims in this chapter is to introduce the support mapping of a linear operator on $\mathcal{C}_p$ and show that the support mapping Φ for a completely-disjoint/Arazy-disjoint preserving mapping $\mathcal{T}$ on $\mathcal{C}_p$ spaces has properties closely analogous to the ones the support mapping for a disjoint preserving mapping on L_p has, and it is always completely-disjoint preserving. Moreover, we prove that if $\mathcal{T}$ is a completely-disjoint/Arazy-disjoint preserving mapping on $\mathcal{C}_p$ and it is sequentially continuous with respect to the strong operator topology, then $\mathcal{T}$ is given by a formula which is the exact analogue of (1.1).

2.1 Completely-disjoint/Arazy-disjoint operators

We shall first introduce completely-disjoint and Arazy-disjoint operators in $\mathcal{C}_p$ spaces.

Geometric description (completely-disjoint)

Definition 2.1.1. *Two operators A and B in $\mathcal{C}_p(\mathcal{H})$ are said to be completely-disjoint, and we write $A \perp B$, if and only if there exists a closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that $A(\mathcal{M}) \subseteqq \mathcal{M}$, $A = 0$ on $\mathcal{M}^{\perp}$, $B(\mathcal{M}^{\perp}) \subseteqq \mathcal{M}^{\perp}$ and $B = 0$ on $\mathcal{M}$.*

From now on if two elements K, L in $\mathcal{C}_p$ are completely-disjoint we shall write $K \perp L$, and we shall often use the abbreviation C-disjoint for simplicity.
Next we give an equivalent matrix description.

Proposition 2.1.2. *Two operators A and B in $\mathcal{C}_p(\mathcal{H})$ are completely-disjoint if and only if an orthonormal basis of $\mathcal{H}$ can be chosen so that the matrices corresponding to A and B with respect to that basis will be:* $\begin{pmatrix} \overline{A} & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ 0 & \overline{B} \end{pmatrix}$, *where $\overline{A}$ and $\overline{B}$ are the restrictions of A, B to appropriate subspaces of $\mathcal{H}$.*

Geometric description (Arazy-disjoint)

Definition 2.1.3. *[4] Two operators K and L in $\mathcal{C}_p(\mathcal{H})$ are said to have disjoint supports if and only if there exist closed subspaces $\mathcal{M}, \mathcal{N}$ of $\mathcal{H}$ such that $K(\mathcal{M}) \subseteqq \mathcal{N}$, $K = 0$ on $\mathcal{M}^{\perp}$, $L(\mathcal{M}^{\perp}) \subseteqq \mathcal{N}^{\perp}$ and $L = 0$ on $\mathcal{M}$.*

From now on if two elements K, L in $\mathcal{C}_p$ have disjoint supports, we will say that they are Arazy-disjoint or A-disjoint for simplicity, and we will write $K \perp_A L$.

An equivalent matrix description of two A-disjoint operators follows.

Proposition 2.1.4. *[4] Two operators K and L on $\mathcal{H}$ are said to have disjoint supports if and only if two orthonormal bases $\{f_i\}$ and $\{g_i\}$ of $\mathcal{H}$ can be chosen so that the matrices of K , L as operators from $\mathcal{H}$ with basis $\{f_i\}$ into $\mathcal{H}$ with basis $\{g_i\}$ will be*

$$K = \begin{pmatrix} \widetilde{K} & 0 \\ \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ \\ 0 & \widetilde{L} \end{pmatrix},$$

where $\widetilde{K}$, $\widetilde{L}$ are the restrictions of K, L to appropriate subspaces of $\mathcal{H}$. The matrix corresponding to an operator A with respect to the pair $\{f_i\}$ and $\{g_i\}$ is defined by $A_{i,j} = (Af_j, g_i)$, $1 \leq i, j < \infty$.

Another concept which is relevant is that of **support**.

Definition 2.1.5. *[4] Let K be a bounded linear operator on a separable Hilbert space $\mathcal{H}$. The right support $r(K)$ of K is then the orthogonal projection of $\mathcal{H}$ onto $(KerK)^{\perp} = R(K)$ and the left support $l(K)$ of K is the orthogonal projection of $\mathcal{H}$ onto $\overline{ranK} = L(K)$. The subspaces $R(K)$ and $L(K)$ are called the right and left support subspaces of K respectively. If $K = K^*$, then $r(K) = l(K) = s(K)$ is simply called the support of K.*

Intuitively, the following proposition states that the left support $l(K)$ of K is the 'smallest projection' P of $\mathcal{H}$ such that the equation $PK = K$ is satisfied and similarly the right support $r(K)$ of K is the 'smallest projection' P of $\mathcal{H}$ such that the equation $KP = K$ is satisfied.

Proposition 2.1.6. *Let $\mathcal{H}$ be a separable Hilbert space and $K \in \mathcal{C}_p(\mathcal{H})$. Then:*

1. *If P is an orthogonal projection of $\mathcal{H}$ such that $PK = K$, then $l(K)K = K$ and $L(K) \subseteq ranP$.*

2. *If P is an orthogonal projection of $\mathcal{H}$ such that $KP = K$, then $Kr(K) = K$ and $R(K) \subseteq ranP$.*

Proof. 1. Let $h \in \mathcal{H}$. Then $l(K)Kh = Kh$ since $l(K)$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{ranK}$. Thus $l(K)$ satisfies the equation $PK = K$ trivially. Now let P be an orthogonal projection of $\mathcal{H}$ such that $PK = K$. Then

$$ran(l(K)) = \overline{ranK} = P(\overline{ranK}) \subseteq ranP.$$

2. To show $Kr(K) = K$, it suffices to show that $r(K)K^* = K^*$. So let $h \in \mathcal{H}$. Since $K^*h \in \overline{ranK^*} = (KerK)^{\perp}$ and $r(K)$ is the orthogonal projection onto

$(KerK)^{\perp}$, we have $r(K)K^*h = K^*h$. Now let P be an orthogonal projection of $\mathcal{H}$ such that $PK^* = K^*$. Then

$$ran(r(K)) = (KerK)^{\perp} = \overline{ranK^*} = P(\overline{ranK^*}) \subseteq ranP.$$

□

Algebraic description (completely-disjoint)

The following proposition gives an algebraic description of the complete-disjointness of two operators in $\mathcal{C}_p(\mathcal{H})$.

Proposition 2.1.7. *Two operators A and B in $\mathcal{C}_p(\mathcal{H})$ are completely-disjoint if and only if $AB = BA = AB^* = B^*A = 0$.*

In other words, $A \perp B$ if and only if $\mathcal{C}^*(A)\mathcal{C}^*(B) = 0$, where $\mathcal{C}^*(a)$ is the $\mathcal{C}^*$ algebra generated by a.

Proof. If $AB = BA = AB^* = B^*A = 0$, define

$$M = clin\{p(A, A^*)(\mathcal{H}),\ where\ \ p\ \ is\ \ a\ \ non-commutative\ \ polynomial\}.$$

Then, of course, $A(M) \subseteq M$ and $A^*(M) \subseteq M$. Also note that $B = 0$ on M and $B^*(M) = \{0\}$, which means that M is invariant for B and B^* and so $B(M^{\perp}) \subseteq M^{\perp}$. On the other hand, since M is invariant for A and A^*, $A(M^{\perp}) \subseteq M^{\perp}$. But $A(M^{\perp}) \subseteq M$ since $A(\mathcal{H}) \subseteq M$ trivially. Consequently $A(M^{\perp}) \subseteq M^{\perp} \cap M = \{0\}$. So A is 0 on $M^{\perp}$. The converse is rather trivial by Proposition 2.1.2. □

In the special case that P, Q are self-adjoint orthogonal projections the notation $P \perp Q$ will also mean that the ranges of the projections are orthogonal.

Algebraic description (Arazy-disjoint)

We shall now present an algebraic way of describing A-disjoint operators in $\mathcal{C}_p$.

Proposition 2.1.8. *Two operators K and L in $\mathcal{C}_p(\mathcal{H})$ are A-disjoint if and only if $KL^* = L^*K = 0$.*

Proof. Let $KL^* = L^*K = 0$. Then $\overline{ranL^*} \subseteq KerK$ and $\overline{ranL} \subseteq KerK^*$. Define $M = (ranL^*)^{\perp} = KerL$ and $N = (KerK^*)^{\perp}$. Then

$$K(M) \subseteq \overline{ranK} = (KerK^*)^{\perp} = N, \quad K_{/M^{\perp}} = K_{/\overline{ranL^*}} = 0$$

and

$$L(M^{\perp}) \subseteq \overline{ranL} \subseteq KerK^* = N^{\perp}, \quad L_{/M} = L_{/(ranL^*)^{\perp}} = L_{/KerL} = 0.$$

Thus $K \perp_A L$. The converse is trivial by Proposition 2.1.4. □

An equivalent expression for A-disjoint operators in terms of the left and right support comes next.

Proposition 2.1.9. *Let $K, L \in \mathcal{C}_p(\mathcal{H})$. Then $KL^* = L^*K = 0$ if and only if $r(K)r(L) = l(K)l(L) = 0$.*

Proof. We have the following:

$$r(K)r(L) = 0 \iff P_{(KerK)^\perp}P_{(KerL)^\perp} = 0 \iff \overline{ranL^*} \subseteq KerP_{(\overline{ranK^*})} \iff$$

$$\iff \overline{ranL^*} \subseteq KerK \iff KL^* = 0.$$

Similarly,

$$l(K)l(L) = 0 \iff P_{(\overline{ranK})}P_{(\overline{ranL})} = 0 \iff \overline{ranL} \subseteq KerK^* \iff$$

$$\iff K^*L = 0 \iff L^*K = 0.$$

□

For future reference it is worth mentioning at this point that if $K \perp_A L$, then $\overline{ranK} \perp \overline{ranL}$ in $\mathcal{H}$ trivially by Proposition 2.1.4.

Norm description (completely-disjoint)

In L_p spaces, two functions f, g have disjoint supports if and only if $||f+g||_p^p + ||f-g||_p^p = 2(||f||_p^p + ||g||_p^p)$ whenever $p \neq 2$ (see Proposition 1.3.4).
However, in $\mathcal{C}_p$ spaces, $p \neq 2$, this is not the case. In fact, although if $T, S \in \mathcal{C}_p$ and $T \perp S$, then $||T+S||_p^p + ||T-S||_p^p = 2(||T||_p^p + ||S||_p^p)$, the reverse need not hold as the following counterexample demonstrates.

Counterexample 2.1.10. Define

$$T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad and \quad S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $TS \neq 0$ and thus T, S are not completely-disjoint by the algebraic description of completely-disjoint operators. However, for $p = 4$ and $\mathcal{H} = \mathbb{C}^2$, since

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{\mathcal{C}_4}^4 = (|a|^2 + |c|^2)^2 + (|b|^2 + |d|^2)^2 + 2|a\bar{b} + c\bar{d}|^2,$$

the equation $||T+S||_p^p + ||T-S||_p^p = 2(||T||_p^p + ||S||_p^p)$ for T, S defined as above does hold.

However as in the case of L_p spaces the concept of complete-disjointness of two elements in $\mathcal{C}_p$ spaces, $p \neq 2$, does have an isometric formulation.

Proposition 2.1.11. *If $p \neq 2$, $1 \leq p < \infty$ and $T, S \in \mathcal{C}_p$, then $T \perp S$ if and only if*

$$||T+S||_p^p + ||T-S||_p^p = ||T+S^*||_p^p + ||T-S^*||_p^p = 2(||T||_p^p + ||S||_p^p).$$

Proof. This is an immediate consequence of the fact that $||T+S||_p^p + ||T-S||_p^p = 2(||T||_p^p + ||S||_p^p)$ if and only if $TS^* = S^*T = 0$ by Theorem 1.2.4 and the algebraic description of completely-disjoint operators. □

Corollary 2.1.12. *If $p \neq 2$, $1 \leq p < \infty$ and $T, S \in \mathcal{C}_p$, then $T \perp S$ if and only if*

$$||T+S||_p^p + ||T-S||_p^p + ||T+S^*||_p^p + ||T-S^*||_p^p = 4(||T||_p^p + ||S||_p^p).$$

Proof. Simply use Theorem 1.2.4 and Proposition 2.1.11. □

Proposition 2.1.11 applied to self-adjoint operators yields a nice result for the Cartesian decomposition of an arbitrary operator in $\mathcal{C}_p$.

Corollary 2.1.13. *If $p \neq 2$, $1 \leq p < \infty$ and $A, B \in \mathcal{C}_p$ with $A^* = A$ and $B^* = B$, then*

$$A \perp B \quad \textit{if and only if} \quad ||A+iB||_p^p = ||A||_p^p + ||B||_p^p.$$

At this point it is worth mentioning the following which is of some independent interest.

Theorem 2.1.14. *If $2 \leq p \leq \infty$ and $A, B \in \mathcal{C}_p$ with A, B positive operators, then $||A+iB||_p^2 \leq ||A||_p^2 + ||B||_p^2$.*

The proof of this theorem uses majorisation relations and can be found in [10].

Norm description (Arazy-disjoint)

The disjointness of the supports of two elements of C_p for $1 \leq p < \infty$, $p \neq 2$, has the following characterization in terms of norms, due to Ch.McCarthy (see [19] and [4] or Theorem 1.2.4).

Proposition 2.1.15. *Let $K, L \in \mathcal{C}_p$. Then*

$$||K+L||_p^p + ||K-L||_p^p \leq 2(||K||_p^p + ||L||_p^p), \ 1 \leq p \leq 2,$$

and

$$||K+L||_p^p + ||K-L||_p^p \geq 2(||K||_p^p + ||L||_p^p), \ 2 \leq p \leq \infty.$$

If $p = 2$ equality always holds (the parallelogram identity), while for $p \neq 2$, equality holds if and only if $K \perp_A L$.

Note that this is the exact analogue of the norm description of two disjoint functions in L_p as seen in Proposition 1.3.4.

Completely-disjoint versus Arazy-disjoint operators

The obvious question now to ask is whether there is a relationship between C-disjoint and A-disjoint operators.

This will be answered by the following proposition.

Proposition 2.1.16. *If K, L are two operators in $\mathcal{C}_p$, then $K \perp L$ if and only if $K \perp_A L$ and $K \perp_A L^*$.*

Proof. This follows immediately from Propositions 2.1.7 and 2.1.8. □

Corollary 2.1.17. *Let K, L be two operators in $\mathcal{C}_p$. If at least one of them is self adjoint, then K, L are completely-disjoint if and only if they are Arazy-disjoint.*

Approximation by sequences

It is perhaps now reasonable to try to obtain some simple results about operators in $\mathcal{C}_p$ spaces approximated by 'completely-disjoint' or 'Arazy-disjoint' sequences. Let us explain formally what we mean by this and see what property the limits of this kind of sequences have.

Proposition 2.1.18. *If $\{A_n\}$, $\{B_n\}$ are sequences in $\mathcal{C}_p$ which converge to A and B respectively in the $\mathcal{C}_p$ norm and for all n, A_n is completely-disjoint (Arazy-disjoint) from B_n, then A is completely-disjoint (Arazy-disjoint) from B.*

Proof. Suppose $A_n \perp B_n$. Then $A_nB_n = B_nA_n = A_nB_n^* = B_n^*A_n = 0$. Obviously, $A_n \to A$ in $\mathcal{C}_p$ implies $A_n \to A$ in norm. Similarly $B_n \to B$, $B_n^* \to B^*$ in norm. Now simply let $n \to \infty$. The same proof goes through in the Arazy-disjoint case. □

A rather obvious Lemma is now needed.

Lemma 2.1.19. *If $\{A_n\}$, $\{B_n\}$ are sequences in $\mathcal{C}_p(\mathcal{H})$ and $A_n \to A$ weakly and $B_n \to B$ strongly, then $A_nB_n \to AB$ weakly.*

Proof. Fix $x, y \in \mathcal{H}$. Then

$$(A_nB_nx, y) = (A_n(B_n - B)x, y) + (A_nBx, y)\,.$$

Since $(A_nBx, y) \to (ABx, y)$ and $|\,(A_n(B_n - B)x, y)\,| \leq M||((B_n - B)x||||y||$, where $M = sup||A_n|| < \infty$ by the Principle of Uniform Boundedness, we conclude that $(A_nB_nx, y) \to (ABx, y)$. □

We are now in a position to obtain the following result which concerns weak and strong limits of sequences in $\mathcal{C}_p$.

Proposition 2.1.20. *If $\{A_n\}$, $\{B_n\}$ are sequences in $\mathcal{C}_p$ such that $A_n \to A$ strongly and $B_n \to B$ weakly and for all n, A_n is completely-disjoint (Arazy-disjoint) from B_n, then A is completely-disjoint (Arazy-disjoint) from B.*

Proof. This is a simple application of the previous Lemma and the algebraic description of completely-disjoint (Arazy-disjoint) operators. □

Remark 2.1.21. *If $A_n \to A$, $B_n \to B$ weakly and for all n, A_n is completely-disjoint (Arazy-disjoint) from B_n, then A need not be completely-disjoint (Arazy-disjoint) from B.*

Proof. Let $\mathcal{H} = l_2$ and set

$$A_n = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ . & . & . & . & . & . & . \\ 1 & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ . & . & . & . & . & . & . \end{pmatrix}, \quad B_n = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ . & . & . & . & . & . & . \\ -1 & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ . & . & . & . & . & . & . \end{pmatrix}$$

and

$$A = B = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ . & . & . & . & . & . & . \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ . & . & . & . & . & . & . \end{pmatrix},$$

where A_n is the matrix which has 1 in the $(1,1), (1,n), (n,1), (n,n)$ positions and zero elsewhere, B_n is the matrix which has 1 in the $(1,1), (n,n)$ and -1 in the $(1,n), (n,1)$ positions and zero elsewhere, and A has only one non-zero entry, namely $(1,1)$, equal to 1.

We have $A_n^* = A_n$, $B_n^* = B_n$ and $A_nB_n = B_nA_n = 0$ for all n. Thus A_n is completely-disjoint (Arazy-disjoint) from B_n for all n by the algebraic description. Observe that $A_n \to A$ and $B_n \to B$ weakly, but $AB \neq 0$. Therefore A cannot be completely-disjoint (Arazy-disjoint) from B. □

Some properties

We lastly note some rather straightforward properties of completely-disjoint/Arazy-disjoint operators which shall be useful in later sections. Note that the following properties hold when $\perp$ is replaced by $\perp_A$.

1. If $A \perp B$, then $B \perp A$.
2. If $A \perp B$, then $A^* \perp B^*$.
3. If $A \perp B$, then $\overline{ranA}$ is orthogonal to $\overline{ranB}$.
4. If $A \perp B_1, B_2, ..., B_n$, then $A \perp B_1 + B_2 + ... + B_n$, for all $n \in \mathbb{N}$.

2.2 Completely-disjoint/Arazy-disjoint preserving mappings

Completely-disjoint preserving mappings

We begin this section with the definition of a completely-disjoint preserving mapping.

Definition 2.2.1. *A linear operator $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ is said to be completely-disjoint preserving if and only if $A \perp B$ implies $\mathcal{T}(A) \perp \mathcal{T}(B)$ for all $A, B \in \mathcal{C}_p(\mathcal{H})$.*

In order to understand completely-disjoint preserving mappings on $\mathcal{C}_p(\mathcal{H})$, we must first look at some examples.

Examples 2.2.2.

1. If $A \in \mathcal{C}_p(\mathcal{H})$, then define $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ by

$$\mathcal{T}(A) = \lambda U^* A U,$$

   where U is a co-isometry ($UU^* = I$) and λ is a complex scalar. Then $\mathcal{T}$ is completely-disjoint preserving.

   This follows from the algebraic description of completely-disjoint operators.

2. If $A \in \mathcal{C}_p(\mathcal{H})$, then define $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ by

$$\mathcal{T}(A) = \lambda A^t,$$

where A^t is the transpose of A relative to an arbitrary orthonormal basis of $\mathcal{H}$ and λ is a scalar.

Since $(AB)^t = B^t A^t$ and $(A^*)^t = (A^t)^*$ for all $A, B \in \mathcal{C}_p(\mathcal{H})$, the algebraic description of completely-disjoint operators implies that $\mathcal{T}$ is completely-disjoint preserving.

As a result the mapping $A \to \lambda U^* A^t U$ on $\mathcal{C}_p$, where U is a co-isometry is completely-disjoint preserving by the algebraic description.

3. The composition of any completely-disjoint preserving mappings is also completely-disjoint preserving.

Note that if $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ is completely-disjoint preserving, then so is the mapping

$$\overline{\mathcal{T}} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$$
$$\overline{\mathcal{T}}(A) = (\mathcal{T}(A^*))^*$$

as a composition of completely-disjoint preserving mappings.

Using this fact we prove the following Proposition which shall be used several times in later chapters.

Proposition 2.2.3. *Let $\mathcal{H}$ be a separable Hilbert space. Then:*

1. *A mapping of the form $\mathcal{T}(A) = AT$ with T invertible is completely-disjoint preserving if and only if $T = \lambda I$, where λ is a scalar and I is the identity operator.*

2. *A mapping of the form $\mathcal{T}(A) = SA$ with S invertible is completely-disjoint preserving if and only if $S = \lambda I$, where λ is a scalar and I is the identity operator.*

3. *A mapping of the form $\mathcal{T}(A) = SAT$ with S, T unitary is completely-disjoint preserving if and only if $S = \lambda T^{-1}$ for some scalar λ.*

Proof. 1. Let $\mathcal{T}(A) = AT$ with T invertible be completely-disjoint preserving. Let $\{e_i\}$ be an orthonormal basis of $\mathcal{H}$. Then since $e_i \otimes e_i \perp e_j \otimes e_j$ for $i \neq j$, we have $(e_j \otimes e_j)T(e_i \otimes e_i) = 0$ by the algebraic description of completely-disjoint operators. Thus for $i \neq j$, $\langle Te_i, e_j \rangle = 0$ and hence for all i, $Te_i = \lambda_i e_i$ for some scalar λ_i. Thus T is diagonal with respect to any orthonormal basis of $\mathcal{H}$. It

follows that $T = \lambda I$ for some scalar λ. The converse is rather trivial.
2. Assume $\mathcal{T}(A) = SA$ with S invertible is completely-disjoint preserving. We conclude that the mapping $\overline{\mathcal{T}}(A) = (\mathcal{T}(A^*))^* = (SA^*)^* = AS^*$ is completely-disjoint preserving. Therefore $S^* = \mu I$ for some scalar μ by 1. The converse is trivial.
3. Let $\mathcal{T}(A) = SAT$ with S, T unitary be completely-disjoint preserving. Define $\mathcal{T}_1(A) = AS$ and $\mathcal{T}_2(A) = S^{-1}A$. Then $(\mathcal{T}_1 \circ \mathcal{T} \circ \mathcal{T}_2)(A) = ATS$ and this mapping is completely-disjoint preserving by the algebraic description. Consequently $T = \lambda S^{-1}$ by 1. The converse is again trivial by the algebraic description.

□

In Chapter 3, we shall see that in the finite dimensional case, if we assume invertibility for a non-zero linear mapping, we deduce that there are no other completely-disjoint preserving mappings other than the following two mentioned already in this section:

$$A \to \lambda W^* A W$$

and

$$A \to \lambda U^* A^t U,$$

where W, U are co-isometries.

In the infinite dimensional case though we need some extra assumptions.

Arazy-disjoint preserving mappings

We start by giving the definition of an Arazy-disjoint preserving mapping.

Definition 2.2.4. *A linear operator $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ is said to be Arazy-disjoint preserving if and only if $A \perp_A B$ implies $\mathcal{T}(A) \perp_A \mathcal{T}(B)$ for all $A, B \in \mathcal{C}_p(\mathcal{H})$.*

Some examples of Arazy-disjoint preserving mappings on $\mathcal{C}_p$ spaces now follow.

Examples 2.2.5.

1. If $A \in \mathcal{C}_p(\mathcal{H})$, then define $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ by

   $$\mathcal{T}(A) = \lambda UAW,$$

   where U is an isometry ($U^*U = I$), W is a co-isometry ($WW^* = I$) and λ is a complex scalar. Then $\mathcal{T}$ is Arazy-disjoint preserving by Proposition 2.1.8.

2. If $A \in \mathcal{C}_p(\mathcal{H})$, then define $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ by

$$\mathcal{T}(A) = \lambda U A^t W,$$

   where A^t is the transpose of A relative to an arbitrary orthonormal basis of $\mathcal{H}$, U is an isometry ($U^*U = I$), W is a co-isometry ($WW^* = I$) and λ is a complex scalar. Then $\mathcal{T}$ is Arazy-disjoint preserving by the algebraic description.

3. The composition of any Arazy-disjoint preserving mappings is also Arazy-disjoint preserving.

As we shall see in Chapter 3, the two mappings just defined are effectively the only non-zero invertible Arazy-disjoint preserving mappings on $\mathcal{C}_p(\mathcal{H})$ when $\mathcal{H}$ is finite dimensional.

Completely-disjoint versus Arazy-disjoint preserving mappings

Proposition 2.2.6. *An Arazy-disjoint preserving mapping on $\mathcal{C}_p(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space over $\mathbb{C}$, need* **not** *be completely-disjoint preserving.*

Proof. To prove this define $\mathcal{T} : \mathcal{C}_p \to \mathcal{C}_p$ by $\mathcal{T}(A) = UAV$, where U, V are two unitary operators with $U \neq \lambda V^*$ for any scalar λ. Then the claim follows immediately from Proposition 2.2.3. □

Note that the completely-disjoint preserving mappings given in section 2.2 (examples 2.2.2) are also Arazy-disjoint preserving. This observation gives rise to the following question.

Question 2.2.7. *Are all completely-disjoint preserving mappings on $\mathcal{C}_p(\mathcal{H})$ Arazy-disjoint preserving?*

It is not easy to come up with an answer to this rather interesting question by using the algebraic descriptions straightaway. However, as will be demonstrated in Chapter 3, in the finite dimensional case, as far as one-to-one mappings are concerned, the situation becomes much simpler once we find an explicit characterisation for completely-disjoint and Arazy-disjoint preserving mappings on $M_n(\mathbb{C})$. We shall also see that a characterisation can be obtained in some cases when $\mathcal{H}$ is infinite dimensional, which we use once again together with the algebraic descriptions to give an answer to Question 2.2.7.

2.3 The support mapping

In section 1.3 we defined the support mapping for a disjoint preserving mapping on L_p spaces. In this section we shall introduce the analogue of the support mapping Φ and show that although the situation in the non-commutative setting is much more complicated, and rather more interesting, most of the analogous properties do hold. Most importantly the support mapping Φ of both completely-disjoint and Arazy-disjoint preserving mappings on $\mathcal{C}_p$ turns out to be a completely-disjoint preserving mapping. Note that some of the properties of Φ are proved in section 2.6.

The support mapping Φ

From now on the set of finite rank projections on $\mathcal{H}$ (i.e projections whose image is finite dimensional) will be denoted by $\mathcal{P}_0$, whereas the set of all projections on $\mathcal{H}$ will be denoted by $\mathcal{P}$.

We start by defining the support mapping on finite rank orthogonal projections which seems to be the obvious starting point, as were the characteristic functions in the case of L_p spaces.

Definition 2.3.1. *Let $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ be a linear mapping. Define a mapping*

$$\Phi : \mathcal{P}_0 \to \mathcal{P}$$

by $\Phi(P) =$ the orthogonal projection of $\mathcal{H}$ onto $(Ker\mathcal{T}(P))^{\perp}$ for every P in $\mathcal{P}_0$. We also define $\Phi(0) = 0$. This mapping Φ (which shall be extended later in this section and then further in section 2.6) will be called the support mapping of $\mathcal{T}$.

Therefore from an intuitive point of view, both $\Phi(P)$ and $\mathcal{T}(P)$ 'live' on the same part of the space.

Example 2.3.2. Let $\mathcal{T}$ be the completely-disjoint/Arazy-disjoint preserving mapping

$$\mathcal{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

$$\mathcal{T}(A) = \lambda U^{-1}AU,$$

where U is a unitary operator and $\lambda \neq 0$. Let M be a closed subspace of $\mathcal{H}$ and P be the orthogonal projection of $\mathcal{H}$ onto M. Then by the previous definition $\Phi(P) = U^{-1}PU$ is the orthogonal projection onto $U^{-1}(M)$.

The next lemma will be a 'key' tool in the theory which shall shortly follow both in the finite and the infinite dimensional case.

Lemma 2.3.3. *If $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ is completely-disjoint/Arazy-disjoint preserving and P, Q are in $\mathcal{P}_0$, then $P \perp Q$ implies $\mathcal{T}(P)\Phi(Q) = 0$, and $\mathcal{T}(P)\Phi(P) = \mathcal{T}(P)$.*

Proof. Let $P \perp Q$. Since for all h, $\Phi(Q)(h) \in \Phi(Q)(\mathcal{H}) = (Ker\mathcal{T}(Q))^{\perp} = \overline{\mathcal{T}(Q)^*(\mathcal{H})}$, and $\mathcal{T}(P)\mathcal{T}(Q)^* = 0$ by the algebraic description of A-disjoint and C-disjoint operators in $\mathcal{C}_p$ (see also Corollary 2.1.17), it follows that for all $h \in \mathcal{H}$, $\mathcal{T}(P)\Phi(Q)h = 0$. Also, for $h \in (Ker\mathcal{T}(P))^{\perp} = \Phi(P)(\mathcal{H})$, we have $\mathcal{T}(P)\Phi(P)(h) = \mathcal{T}(P)(h)$. For $h \in Ker\mathcal{T}(P) = Ker\Phi(P)$, the equation $\mathcal{T}(P)\Phi(P) = \mathcal{T}(P)$ holds trivially. □

We now state and prove some of the properties of the support mapping Φ of a completely-disjoint/Arazy-disjoint preserving mapping on $\mathcal{C}_p$.

Proposition 2.3.4. *Let Φ be the operator defined above and let P, Q be two arbitrary projections in $\mathcal{P}_0$.*

1. *If $P \perp Q$, then $\Phi(P) \perp \Phi(Q)$.*
2. *If $P \perp Q$, then $\Phi(P + Q) = \Phi(P) + \Phi(Q)$.*
3. *If $PQ = QP$, then $\Phi(P)\Phi(Q) = \Phi(PQ) = \Phi(Q)\Phi(P)$.*

Proof. 1. Let $P \perp Q$. To prove this set $M = (Ker\mathcal{T}(P))^{\perp}$. Then $\Phi(P)(M) = M$ trivially by the definition of Φ on $\mathcal{P}_0$. Since $Ker\mathcal{T}(P) = Ker\Phi(P)$, we obtain that $\Phi(P) = 0$ on $M^{\perp}$. Moreover, $\Phi(Q)(M^{\perp}) \subseteq M^{\perp}$ since $\mathcal{T}(P)\Phi(Q) = 0$ by Lemma 2.3.3. Lastly, if $h \in (Ker\mathcal{T}(P))^{\perp}$, then $(h, \Phi(Q)h) = 0$ since $\Phi(Q)h \in Ker\mathcal{T}(P)$ (Recall $\mathcal{T}(P)\Phi(Q) = 0$). Therefore $||\Phi(Q)h||^2 = (\Phi(Q)h, \Phi(Q)h) = (h, \Phi(Q)h) = 0$. Consequently $\Phi(Q) = 0$ on M. We conclude that $\Phi(P) \perp \Phi(Q)$ by definition 2.1.1.

2. Let $P \perp Q$. If $P_{\mathcal{M}}$ is the orthogonal projection of $\mathcal{H}$ onto the closed subspace $\mathcal{M}$ of $\mathcal{H}$, then $P_{\mathcal{M}_3} = P_{\mathcal{M}_2} + P_{\mathcal{M}_1}$ if and only if $\mathcal{M}_3 = \mathcal{M}_2 \oplus \mathcal{M}_1$. Since $P \perp Q$, we have $ranP \perp ranQ$ and so $P + Q$ is a projection. Thus the following two equalities are equivalent:

$$\Phi(P + Q) = \Phi(P) + \Phi(Q),$$

$$(Ker\mathcal{T}(P + Q))^{\perp} = (Ker\mathcal{T}(P))^{\perp} \oplus (Ker\mathcal{T}(Q))^{\perp}.$$

However the assumption $P \perp Q$ and the first part of this Proposition imply that the last expression is equivalent to

$$(Ker\mathcal{T}(P + Q))^{\perp} = clin\left[(Ker\mathcal{T}(P))^{\perp} \cup (Ker\mathcal{T}(Q))^{\perp}\right],$$

which in turn is obviously equivalent to

$$Ker\mathcal{T}(P+Q) = \left(clin\left[(Ker\mathcal{T}(P))^{\perp} \cup (Ker\mathcal{T}(Q))^{\perp}\right]\right)^{\perp}.$$

It thus suffices to show that

$$Ker\,(\mathcal{T}(P) + \mathcal{T}(Q)) = Ker\mathcal{T}(P) \cap Ker\mathcal{T}(Q).$$

Obviously $Ker\,(\mathcal{T}(P) + \mathcal{T}(Q)) \supseteq Ker\mathcal{T}(P) \cap Ker\mathcal{T}(Q)$. For the other inclusion note that $\mathcal{T}(P) \perp \mathcal{T}(Q)$ implies $\overline{ran\mathcal{T}(P)} \perp \overline{ran\mathcal{T}(Q)}$. If $h \in Ker\,(\mathcal{T}(P) + \mathcal{T}(Q))$, then $\mathcal{T}(P)(h) = -\mathcal{T}(Q)(h)$. Thus $\mathcal{T}(P)(h) \in cl(ran\mathcal{T}(P)) \cap cl(ran\mathcal{T}(Q)) = \{0\}$. Hence $\mathcal{T}(P)(h) = \mathcal{T}(Q)(h) = 0$.

3. Since $PQ = QP$, PQ is obviously a projection. Write $P = P - PQ + PQ$ and $Q = Q - PQ + PQ$. Then by Proposition 2.1.7 , the fact that $(P - PQ)\perp PQ$, $(Q - PQ)\perp PQ$ and $(P - PQ)\perp(Q - PQ)$ and the previous property of Φ, it follows that $\Phi(P) = \Phi(P - PQ) + \Phi(PQ)$ and $\Phi(Q) = \Phi(Q - PQ) + \Phi(PQ)$. Multiply these two equations to obtain $\Phi(P)\Phi(Q) = \Phi(PQ)$. □

Extension of the support mapping Φ to $\mathcal{C}_p(\mathcal{H})$

The next step is to extend our mapping Φ to a mapping, denoted again by Φ, to the whole of $\mathcal{C}_p(\mathcal{H})$.

First we note that if $\{Q_i\}$ is a sequence of self-adjoint pairwise orthogonal projections and $\{\lambda_i\}$ is a bounded sequence of complex numbers, then the series $\sum_{i=1}^{\infty} \lambda_i P_i$ converges strongly and unconditionally. As a result, using part 1 of Proposition 2.3.4, we obtain the following simple lemma. Let $\mathcal{C}_{p,sa}(\mathcal{H})$ be the set of all self-adjoint operators in $\mathcal{C}_p(\mathcal{H})$.

Lemma 2.3.5. *If $A \in \mathcal{C}_{p,sa}(\mathcal{H})$, $A \neq 0$ and $A = \sum_{i=1}^{\infty} \lambda_i P_i$ by the spectral theorem, where $\{\lambda_i\}$ are the non-zero distinct eigenvalues of A, then the series $\sum_{i=1}^{\infty} \lambda_i \Phi(P_i)$ converges strongly and unconditionally.*

We are now ready to extend our mapping Φ as follows.

Definition 2.3.6. *If $A \in \mathcal{C}_{p,sa}(\mathcal{H})$ with $A \neq 0$ and $A = \sum_{i=1}^{\infty} \lambda_i P_i$ by the spectral theorem, where $\{\lambda_i\}$ are the non-zero distinct eigenvalues of A, define*

$$\Phi(A) = \sum_{i=1}^{\infty} \lambda_i \Phi(P_i).$$

The convergence is in the strong operator topology.
If $K \in \mathcal{C}_p$ *with Cartesian decomposition* $K = ReK + iImK$*, define*

$$\Phi(K) = \Phi(ReK) + i\Phi(ImK).$$

Here follows an example of a support mapping Φ to illustrate this definition.

Example 2.3.7. *Let* $\mathcal{T}$ *be the completely-disjoint/Arazy-disjoint preserving mapping* $\mathcal{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *with*

$$\mathcal{T}(A) = \lambda U^{-1} A U,$$

where U *is a unitary operator and* λ *is a complex scalar. Then for all* $K \in M_n(\mathbb{C})$ *and* $\lambda \neq 0$,

$$\Phi(K) = U^{-1} K U.$$

Proof. If P is a (finite-rank) projection, then, with $\mathcal{H} = \mathbb{C}^n$,

$$\begin{aligned}\Phi(P) &= \textit{ the projection of } \mathcal{H} \textit{ onto } \left(Ker(U^{-1}PU)\right)^{\perp} \\ &= \textit{ the projection of } \mathcal{H} \textit{ onto } (U^{-1}P)(\mathbb{C}^n) \\ &= U^{-1}PU.\end{aligned}$$

Thus by the previous definition $\Phi(K) = U^{-1} K U$ for all $K \in M_n(\mathbb{C})$. □

Note that in example 2.3.7, $\mathcal{T} = \lambda\Phi$. Hence $\mathcal{T} = \Phi$ when $\lambda = 1$.

The properties of the mapping Φ

Before we prove some more properties for our support mapping Φ and make a summary of all the properies that we have shown it has so far, we make a simple observation.

Lemma 2.3.8. *If* K, L *are in* $\mathcal{C}_p$ *, then* $K \perp L$ *if and only if* $ReK \perp ReL, ImL$ *and* $ImK \perp ImL, ReL$.

Proof. To prove this simply use the algebraic description of C-disjoint operators. □

We are now in a position to state the properties of the extended support mapping.

Proposition 2.3.9. *If* $\mathcal{T}$ *is completely-disjoint/Arazy-disjoint preserving, then the extended support mapping* Φ *induced by* $\mathcal{T}$ *has the following properties: If* $A, K, L \in \mathcal{C}_p(\mathcal{H})$ *and* P, Q *are projections in* $\mathcal{P}_0$*, then*

1. *If* $A \geqslant 0$, $\Phi(A) \geqslant 0$.

2. $\Phi(K^*) = \Phi(K)^*$.

3. *If* A *is self-adjoint, then for all* $n \in \mathbb{N}$, $\Phi(A^n) = \Phi(A)^n$.

4. *If* $A \geqslant 0$, $s \geqslant 0$ *, then* $\Phi(A^s) = \Phi(A)^s$.

5. *If* $K \perp L$*, then* $\Phi(K) \perp \Phi(L)$.

6. *If* $P \perp Q$*, then* $\Phi(P+Q) = \Phi(P) + \Phi(Q)$.

7. *If* $PQ = QP$*, then* $\Phi(P)\Phi(Q) = \Phi(PQ)$.

Proof. 1. Let $A \in \mathcal{C}_p$, $A \geqslant 0$. Then by the spectral theorem, $A = \sum_{j=1}^{\infty} \lambda_j P_j$ with $\lambda_j \geqslant 0$. So $\Phi(A) = \sum_{j=1}^{\infty} \lambda_j \Phi(P_j)$ and for all $x \in \mathcal{H}$, we have $(\Phi(A)x, x) = \sum_j \lambda_j (\Phi(P_j)x, x) = \sum_j \lambda_j ||\Phi(P_j)x||^2 \geqslant 0$. Thus $\Phi(A) \geqslant 0$.
2. If K is self-adjoint, then $\Phi(K^*) = \Phi(K)^*$ trivially. This property of Φ follows then immediately by the definition of Φ for an arbitrary $K \in \mathcal{C}_p$.
3. Let $A = \sum_{j=1}^{\infty} \lambda_j P_j$ in the usual notation. Since $\Phi(P_i) \perp \Phi(P_j)$ for $i \neq j$, we have that $\Phi(A) = \sum_j \lambda_j \Phi(P_j)$. Thus for all $n \in \mathbb{N}$,

$$\Phi(A)^n = \sum_j \lambda_j^n \Phi(P_j) = \Phi(A^n).$$

4. Let $A \in \mathcal{C}_p$ with $A \geqslant 0$. Since $A = \sum_{j=1}^{\infty} \lambda_j P_j$ and $s \geq 0$, we have $A^s = \sum_{j=1}^{\infty} \lambda_j^s P_j$. The same argument that was used in the proof of 3 can now be applied again with n replaced by $s > 0$.
5. To show that $K \perp L$ implies $\Phi(K) \perp \Phi(L)$ we first show that it holds for $K, L \in \mathcal{C}_{p,sa}(\mathcal{H})$. Let $K, L \in \mathcal{C}_{p,sa}(\mathcal{H})$ with $K \perp L$. Note that if $K = \sum \lambda_i P_i$ and $L = \sum \mu_i Q_i$ by the spectral theorem (λ_i, μ_i are the distinct non-zero eigenvalues of K and L respectively), then for all i and for all j we have $P_j \perp Q_i$. In fact, since $KL = 0$, we have $P_j KL = 0$ and thus $\lambda_j P_j L = 0$. This implies $P_j L = 0$ for all j. Similarly $LP_j = 0$ for all j. Thus we have shown that if $KL = LK = 0$ and K has spectral decomposition $\sum \lambda_i P_i$, then for all j, $LP_j = P_j L = 0$. Now since $LP_j = P_j L = 0$ for all j, applying the same argument we conclude that for all i, j, $Q_i P_j = P_j Q_i = 0$. Hence for all i, j, $\Phi(P_j) \perp \Phi(Q_i)$ by Proposition 2.3.4. It follows that

$\Phi(K)\Phi(L) = \left(\sum_i \lambda_i \Phi(P_i)\right)\left(\sum_j \mu_j \Phi(Q_j)\right) = \sum_{i,j} \lambda_i \mu_j \Phi(P_i)\Phi(Q_j) = 0$. In the same way we can prove that $\Phi(L)\Phi(K) = 0$. Since $\Phi(K)$, $\Phi(L)$ are self adjoint, we have $\Phi(K) \perp \Phi(L)$ by property 2. Now let $K = ReK + iImK$ and $L = ReL + iImL$. Since $K \perp L$, the previous Lemma implies that $ReK \perp ImL, ReL$ and $ImK \perp ImL, ReL$ and so the same applies to their Φ-images $\Phi(ImK)$, $\Phi(ImL)$, $\Phi(ReK)$, $\Phi(ReL)$. Thus $\Phi(K)\Phi(L) = \Phi(L)\Phi(K) = \Phi(K^*)\Phi(L) = \Phi(L)\Phi(K^*) = 0$. Now simply use property 2 again.
Properties 6 and 7 have already been proved (see Proposition 2.3.4). □

Because of its importance and although the following is simply part of the previous Proposition we state it separately.

Proposition 2.3.10. *The support mapping Φ of a completely-disjoint/Arazy-disjoint preserving mapping $\mathcal{T}$ on $\mathcal{C}_p(\mathcal{H})$ is completely-disjoint preserving.*

The mapping Φ is proved to be a very useful tool, as we will see in the following sections.

2.4 The finite dimensional case

Note that in the case of a finite dimensional Hilbert space $\mathcal{H}$ with dimension n, the space $\mathcal{C}_p(\mathcal{H})$ can be identified with the set of all complex $n \times n$ matrices, and it is independent of p as a vector space.

Completely-disjoint/Arazy-disjoint preserving mappings

The following theorem gives us an explicit formula for completely-disjoint/Arazy-disjoint preserving mappings in the finite dimensional case.

Theorem 2.4.1. *Let $\mathcal{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a completely-disjoint/Arazy-disjoint preserving mapping. Then for all K in $M_n(\mathbb{C})$,*

$$\mathcal{T}(K) = \mathcal{T}(I)\Phi(K),$$

where Φ is the support mapping induced by $\mathcal{T}$ and I is the identity operator.

Proof. Let P_M be the projection of $\mathbb{C}^n$ onto M, where M is a (closed) subspace of $\mathbb{C}^n$. The first step is to prove that $\mathcal{T}(P_M) = \mathcal{T}(I)\Phi(P_M)$.
By the definition of Φ, $\Phi(P_M)$ is the projection of $\mathbb{C}^n$ onto $(ker\mathcal{T}(P_M))^{\perp}$. Thus $ker\Phi(P_M) = ker\mathcal{T}(P_M)$ which implies that $\mathcal{T}(I)\Phi(P_M) = \mathcal{T}(P_M)$ on $ker\mathcal{T}(P_M)$.

Consequently we only need to show that $\mathcal{T}(I)\Phi(P_M) = \mathcal{T}(P_M)$ on $(ker\mathcal{T}(P_M))^{\perp}$. So let $h \in (ker\mathcal{T}(P_M))^{\perp}$. Since $\Phi(P_M)(\mathcal{H}) = (ker\mathcal{T}(P_M))^{\perp}$ and $\mathcal{T}(P)\Phi(Q) = 0$ for $P \perp Q$ by Lemma 2.3.3, we obtain

$$\begin{aligned}\mathcal{T}(I)\Phi(P_M)h &= \mathcal{T}(I)h \\ &= \mathcal{T}(P_M)h + \mathcal{T}(P_{M^\perp})h \\ &= \mathcal{T}(P_M)h + \mathcal{T}(P_{M^\perp})\Phi(P_M)h \\ &= \mathcal{T}(P_M)h.\end{aligned}$$

The general case follows easily from the linearity of $\mathcal{T}$ and the definition of Φ. □

On invertible completely-disjoint/Arazy-disjoint preserving mappings

It turns out that if a completely-disjoint/Arazy-disjoint preserving mapping on $M_n(\mathbb{C})$ is invertible, then its support mapping Φ has an additional property. Before proving this we need the following lemma which not surprisingly holds since our Hilbert space is finite dimensional.

Lemma 2.4.2. *If $\mathcal{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is an invertible completely-disjoint/A-disjoint preserving mapping, then $\mathcal{T}(I)$ is invertible.*

Proof. Let $\{e_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Since $dim(ran\mathcal{T}(e_i \otimes e_i)) \geq 1$ by the invertibility of $\mathcal{T}$ and $ran(\mathcal{T}(e_i \otimes e_i)) \perp ran(\mathcal{T}(e_j \otimes e_j))$ for $i \neq j$, we conclude that

$$dim(ran\mathcal{T}(I)) = dim(ran\sum_{i=1}^{n}\mathcal{T}(e_i \otimes e_i)) = \sum_{i=1}^{n} dim(ran\mathcal{T}(e_i \otimes e_i)) \geq n.$$

But obviously $dim(ran\mathcal{T}(I)) \leq n$. Hence $dim(ran\mathcal{T}(I)) = n$. Consequently $\mathcal{T}(I)$ is onto and thus invertible. □

As a result we deduce the following nice property for Φ.

Proposition 2.4.3. *If $\mathcal{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is an invertible completely-disjoint/A-disjoint preserving mapping, then the mapping $\Phi : \mathcal{P}_0 \to \mathcal{P}_0$ induced by $\mathcal{T}$ maps rank-one projections to rank-one projections.*

Proof. Let $P_1 = e_1 \otimes e_1$ be a rank one projection. Extend $\{e_1\}$ to an orthonormal basis $\{e_1, ..., e_n\}$ of $\mathbb{C}^n$ and let $P_j = e_j \otimes e_j$. Since $\mathcal{T}$ is one to one, Theorem 2.4.1 implies that for all $j \in \{1, ..., n\}$, $dim(ran\Phi(P_j) \geq 1$ and so $\sum_{j=1}^{n} (dim(ran\Phi(P_j)) \geq n$. On the other hand, we have $P_i \perp P_j$ for $i \neq j$, and so $\Phi(P_i) \perp \Phi(P_j)$. Consequently $ran\Phi(P_i)$ is orthogonal to $ran\Phi(P_j)$. Therefore

$$n \geq dim\left(clin \bigcup_{j=1}^{n} ran(\Phi(P_j))\right) = \sum_{j=1}^{n} (dim(ran\Phi(P_j)).$$

Hence $dim\,(ran\Phi(P_j)) = 1$ for all $j \in \{1, ..., n\}$. $\square$

Corollary 2.4.4. *If $\mathcal{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is an invertible C-disjoint/A-disjoint preserving mapping and $\Phi : \mathcal{P}_0 \to \mathcal{P}_0$ is the mapping induced by T, then for any orthonormal basis $e = \{e_i\}$ of $\mathbb{C}^n$, we have $\Phi(e_i \otimes e_i) = U_e(e_i \otimes e_i)U_e^*$ for some unitary operator U_e.*

Proof. By the previous proposition there is a unit vector f_i such that $\Phi(e_i \otimes e_i) = f_i \otimes f_i$. Now simply define U_e by $U_e(e_j) = f_j$. Consequently $\Phi(e_i \otimes e_i) = f_i \otimes f_i = (U_e e_i) \otimes (U_e e_i) = U_e(e_i \otimes e_i)U_e^*$. $\square$

Now let $\mathcal{P}_k$ denote the set of all projections in $M_n(\mathbb{C})$ of rank k.

Proposition 2.4.5. *Let $\mathcal{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be an invertible completely-disjoint/Arazy-disjoint preserving mapping and $\Phi_{\mathcal{T}}$, $\Phi_{\mathcal{T}^{-1}}$ be the support mappings for $\mathcal{T}$, $\mathcal{T}^{-1}$ respectively. Then*

1. $\Phi_{\mathcal{T}}(\mathcal{P}_k) \subseteqq \mathcal{P}_k$ *for any* $k \geq 0$
2. $\Phi_{\mathcal{T}^{-1}}(\mathcal{P}_k) \supseteqq \mathcal{P}_k$ *for any* $k \geq 0$

Proof. 1. Since any P in $\mathcal{P}_k$ can be written as the sum of k pairwise disjoint projections and properties 5,6 of Proposition 2.3.9 hold, it suffices to have that $\Phi_{\mathcal{T}}(\mathcal{P}_1) \subseteqq \mathcal{P}_1$, which is true by Proposition 2.4.3.

2. Fix an orthonormal basis $b = \{b_1, ..., b_n\}$ of $\mathbb{C}^n$. We shall prove that there is another orthonormal basis $a = \{a_1, ..., a_n\}$ of $\mathbb{C}^n$ such that $\Phi_{\mathcal{T}^{-1}}(a_i \otimes a_i) = b_i \otimes b_i$ for all i.

Since $\mathcal{T}^{-1}$ is onto, for each j there is an $A_j \neq 0$ such that $\mathcal{T}^{-1}(A_j) = b_j \otimes b_j$. Now $\mathcal{T}$ is completely-disjoint/Arazy-disjoint preserving and for $i \neq j$, we have $b_i \otimes b_i$ is completely-disjoint/Arazy-disjoint from $b_j \otimes b_j$. Thus for $i \neq j$, A_i is completely-disjoint/Arazy-disjoint from A_j. Utilising the same argument we used in the proof of Proposition 2.4.3 (counting dimensions), we have that for all j,

$dim(ranA_j) = 1$. We conclude that $A_j = \mu_j a_j \otimes a_j$, where $\{a_1, ..., a_n\}$ is an orthonormal set and the μ_j's are non-zero scalars. Thus $b_j \otimes b_j = \mathcal{T}^{-1}(A_j) = \mathcal{T}^{-1}(\mu_j a_j \otimes a_j) = \mu_j \mathcal{T}^{-1}(a_j \otimes a_j)$. But $\Phi_{\mathcal{T}^{-1}}(a_j \otimes a_j)$ is the projection of $\mathbb{C}^n$ onto $(Ker\mathcal{T}^{-1}(a_j \otimes a_j))^{\perp}$. Hence $\Phi_{\mathcal{T}^{-1}}(a_j \otimes a_j) = b_j \otimes b_j$ for all j. Properties 5,6 of Proposition 2.3.9 once again imply that $\Phi_{\mathcal{T}^{-1}}(\mathcal{P}_k) \supseteqq \mathcal{P}_k$ for any $k \geq 0$.

□

The following simple lemma is going to be needed next.

Lemma 2.4.6. *If A is an invertible operator on $\mathcal{H}$, there is a unitary U such that UA is positive.*

Proof. By polar decomposition, $A = W|A|$, where $|A| = (A^*A)^{1/2}$ and W is a partial isometry with $KerW = KerA$ and $ranW = \overline{ranA}$. Since A is invertible, $KerW = \{0\}$ and $ranW = \mathcal{H}$. Thus W is a unitary operator on $\mathcal{H}$ and for all $x \in \mathcal{H}$, $(W^*Ax, x) = (|A|x, x) \geq 0$. Now set $U = W^*$. □

A simple, but very useful application of Proposition 2.4.5 is the following, which demonstrates that if both a given mapping $\mathcal{T}$ on $M_n(\mathbb{C})$ and its inverse are completely-disjoint preserving, then $\mathcal{T}$ is simply a scalar multiple of its support mapping. We shall also prove that if both a given mapping $\mathcal{T}$ on $M_n(\mathbb{C})$ and its inverse are A-disjoint preserving, then $\mathcal{T}$ is a scalar multiple of the composition of a unitary operator and the support mapping Φ of $\mathcal{T}$.

It is worth mentioning that in these cases $\Phi_{\mathcal{T}}(\mathcal{P}_k) = \mathcal{P}_k$ for any $k \geq 0$.

Proposition 2.4.7. *Let $\mathcal{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be an invertible linear mapping and Φ be the support mapping induced by $\mathcal{T}$.*

1. *If $\mathcal{T}, \mathcal{T}^{-1} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ are completely-disjoint preserving mappings, then there is a $\lambda \neq 0$ such that $\mathcal{T} = \lambda\Phi$. If $\mathcal{T}(I)$ is a self-adjoint operator, then $\lambda \in \mathbb{R}$.*

2. *If $\mathcal{T}, \mathcal{T}^{-1} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ are Arazy-disjoint preserving mappings, then there is a $\mu > 0$ and a unitary operator U such that $\mathcal{T} = (\mu U)\Phi$.*

Proof. 1. Let $f = \{f_1, ..., f_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Then part 2 of Proposition 2.4.5 applied to $\mathcal{T}^{-1}$ with $k = 1$ implies that there is another orthonormal basis $e = \{e_1, ..., e_n\}$ of $\mathbb{C}^n$ such that $\Phi(e_i \otimes e_i) = f_i \otimes f_i$ for all i. By Corollary 2.4.4, there is a unitary operator U_e such that for all i, $\Phi(e_i \otimes e_i) = U_e(e_i \otimes e_i)U_e^*$. Now fix i, j with $i \neq j$. We have $\mathcal{T}(e_i \otimes e_i) \perp \mathcal{T}(e_j \otimes e_j)$ and so Theorem 2.4.1 implies that $\mathcal{T}(I)U_e(e_i \otimes e_i)U_e^* \perp \mathcal{T}(I)U_e(e_j \otimes e_j)U_e^*$. We therefore deduce that

$$\mathcal{T}(I)U_e(e_j \otimes e_j)U_e^* \mathcal{T}(I)U_e(e_i \otimes e_i)U_e^* = 0$$

by the algebraic description of completely-disjoint operators. This in turn implies that $(x, e_i)\,(U_e^* \mathcal{T}(I) U_e e_i, e_j)\, \mathcal{T}(I) U_e e_j = 0$ for all $x \in \mathcal{H}$. Therefore for i, j with $i \neq j$, we have $(U_e^* \mathcal{T}(I) U_e e_i, e_j) = 0$ and so $U_e^* \mathcal{T}(I) U_e e_i = \lambda_i e_i$ for some λ_i. We conclude that $U_e^* \mathcal{T}(I) U_e = \sum_{i=1}^{n} \lambda_i e_i \otimes e_i$ which implies that $\mathcal{T}(I) = \sum_{i=1}^{n} \lambda_i \Phi(e_i \otimes e_i)$. Since $\Phi(e_i \otimes e_i) = f_i \otimes f_i$ for all i, we deduce that $\mathcal{T}(I)$ is diagonal with respect to an arbitrary orthonormal basis of $\mathbb{C}^n$. Consequently $\mathcal{T}(I) = \lambda I$, where I is the identity operator and λ is a scalar. Since $\mathcal{T} = \mathcal{T}(I)\Phi$ by Theorem 2.4.1, we conclude that $\mathcal{T} = \lambda \Phi$, which finishes the proof of 1.

2. Applying the argument we used in the proof of part 1 of this proposition we conclude that $(\mathcal{T}(I))^* \mathcal{T}(I)$ is diagonal with respect to an arbitrary orthonormal basis of $\mathbb{C}^n$. Consequently $(\mathcal{T}(I))^* \mathcal{T}(I) = \lambda I$, where I is the identity operator and λ is a non-zero scalar. Since $\mathcal{T}(I)$ is invertible by Lemma 2.4.2, there is a unitary operator W such that $W\mathcal{T}(I) \geq 0$ by Lemma 2.4.6. Now let us define $\mathcal{T}_1 : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\mathcal{T}_1 = W\mathcal{T}$. Then $\mathcal{T}_1(I) \geq 0$ and

$$\mathcal{T}_1(I) = |\mathcal{T}_1(I)| = \left((W\mathcal{T}(I))^* W\mathcal{T}(I)\right)^{1/2} = \left((\mathcal{T}(I))^* \mathcal{T}(I)\right)^{1/2} = \lambda^{1/2} I.$$

Thus $\mathcal{T}(I) = \lambda^{1/2} W^*$. We conclude that $\mathcal{T} = \mathcal{T}(I)\Phi = (\mu U)\Phi$, where $U = W^*$ and $\mu = \lambda^{1/2}$. □

The obvious question now to ask is whether the inverse (when it exists) of a completely-disjoint/Arazy-disjoint preserving mapping $\mathcal{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is also completely-disjoint/Arazy-disjoint preserving. The answer to this interesting question shall be given in Chapter 3.

2.5 The infinite dimensional case

In this section, where $\mathcal{H}$ is assumed to be a separable infinite dimensional Hilbert space over $\mathbb{C}$, we shall consider whether there is an analogue for the formula $\mathcal{T} = \mathcal{T}(I)\Phi$ (see Theorem 2.4.1) which we obtained in the finite dimensional case for an arbitrary completely-disjoint/Arazy-disjoint preserving mapping on $M_n(\mathbb{C})$.

As it might be expected, the situation is more complicated in the infinite dimensional case. However, we shall shortly see that, under some technical assumptions, a modification of the same formula holds when $\mathcal{H}$ is infinite dimensional.

In order to approach this properly, we first need to look at projections of $\mathcal{H}$ onto finite dimensional subspaces and utilise the fundamental idea used in the finite dimensional case.

On projections onto finite dimensional subspaces of $\mathcal{H}$

In Lemma 2.3.3 we demonstrated that if $\mathcal{T}$ is a completely-disjoint/Arazy-disjoint preserving mapping on $\mathcal{C}_p(\mathcal{H})$, Φ is its support mapping and P is a finite rank projection, then $\mathcal{T}(P) = \mathcal{T}(P)\Phi(P)$. It turns out that a generalisation of this involving finite dimensional subspaces of $\mathcal{H}$ holds.

Lemma 2.5.1. *Let $\mathcal{H}_0$ be a finite dimensional subspace of $\mathcal{H}$ and M be a (closed) subspace of $\mathcal{H}_0$. If $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ is C-disjoint/A-disjoint preserving and Φ is the support mapping induced by $\mathcal{T}$, then $\mathcal{T}(P_{\mathcal{H}\to M}) = \mathcal{T}(P_{\mathcal{H}\to\mathcal{H}_0})\Phi(P_{\mathcal{H}\to M})$.*

Proof. Trivially we have $\mathcal{T}(P_{\mathcal{H}\to M}) = \mathcal{T}(P_{\mathcal{H}\to\mathcal{H}_0})\Phi(P_{\mathcal{H}\to M})$ on $Ker\mathcal{T}(P_{\mathcal{H}\to M})$ since $Ker\,(\mathcal{T}(P_{\mathcal{H}\to M})) = Ker\,(\Phi(P_{\mathcal{H}\to M}))$ by the definition of Φ on $\mathcal{P}_0$. Now if $x \in (Ker\mathcal{T}(P_{\mathcal{H}\to M}))^{\perp} = \Phi(P_{\mathcal{H}\to M})(\mathcal{H})$, then the fact that $\mathcal{T}(P)\Phi(Q) = 0$ for disjoint P, Q (by Lemma 2.3.3) implies the following:

$$\begin{aligned}
\mathcal{T}(P_{\mathcal{H}\to\mathcal{H}_0})\Phi(P_{\mathcal{H}\to M})x &= \mathcal{T}(P_{\mathcal{H}\to\mathcal{H}_0})x \\
&= \mathcal{T}(P_{\mathcal{H}\to M})x + \mathcal{T}(P_{\mathcal{H}\to\mathcal{H}_0\ominus M})x \\
&= \mathcal{T}(P_{\mathcal{H}\to M})x + \mathcal{T}(P_{\mathcal{H}\to\mathcal{H}_0\ominus M})\Phi(P_{\mathcal{H}\to M})x \\
&= \mathcal{T}(P_{\mathcal{H}\to M})x.
\end{aligned}$$

□

Applying the spectral theorem we can generalise the result we have just proved for projections onto finite dimensional subspaces to self-adjoint operators in $\mathcal{C}_p$ which 'live' on a finite dimensional subspace of $\mathcal{H}$.

Proposition 2.5.2. *Let $\mathcal{T}$ be a bounded completely-disjoint/Arazy-disjoint preserving mapping on $\mathcal{C}_p(\mathcal{H})$, A be a self adjoint operator in $\mathcal{C}_p(\mathcal{H})$ such that $A = 0$ on $\mathcal{H}_0^{\perp}$, where $\mathcal{H}_0$ is a finite dimensional subspace of $\mathcal{H}$ and Φ be the support mapping induced by $\mathcal{T}$. Then*

$$\mathcal{T}(A) = \mathcal{T}(P_{\mathcal{H}\to\mathcal{H}_0})\Phi(A).$$

Proof. By the spectral theorem $A = \sum_{j=1}^{m} \lambda_j P_j$ for some m ($\lambda_j \neq 0$ for all j), where $P_iP_j = P_jP_i = 0$ for $i \neq j$, and for all i, $dimP_i(\mathcal{H}) < \infty$. Moreover, for all $j \in \{1, 2, ..., m\}$ and for all $h \in \mathcal{H}$, $P_jh = \lambda_j^{-1}AP_jh \in A(\mathcal{H}) \subseteq \mathcal{H}_0$. Therefore, for all $j \in \{1, 2, ..., m\}$, $P_j(\mathcal{H}) \subseteq \mathcal{H}_0$. Now using Lemma 2.5.1, we obtain the following:

$$\begin{aligned}\mathcal{T}(A) &= \mathcal{T}\left(\sum_{j=1}^{m} \lambda_j P_j\right) \\ &= \sum_{j=1}^{m} \lambda_j \mathcal{T}(P_j) \\ &= \sum_{j=1}^{m} \lambda_j \mathcal{T}(P_{\mathcal{H}\to\mathcal{H}_0})\Phi(P_j) \\ &= \mathcal{T}(P_{\mathcal{H}\to\mathcal{H}_0}) \sum_{j=1}^{m} \lambda_j \Phi(P_j) \\ &= \mathcal{T}(P_{\mathcal{H}\to\mathcal{H}_0})\Phi(A).\end{aligned}$$

□

Completely-disjoint/Arazy-disjoint preserving mappings on self-adjoint operators

In order to obtain a description of an arbitrary completely-disjoint/Arazy-disjoint preserving mapping we shall first try to evaluate such a mapping on self-adjoint operators.

For convenience a well known definition comes first.

Definition 2.5.3. *A partition of the identity on $\mathcal{H}$ is a family $\{P_i\}_{i\in I}$ of pairwise orthogonal projections on $\mathcal{H}$ such that clin $[\bigcup_i P_i(\mathcal{H})] = \mathcal{H}$.*

Next we state and prove the following obvious lemma.

Lemma 2.5.4. *Let $\mathcal{T}$ be a sequentially continuous with respect to the strong operator topology mapping in $\mathcal{B}(\mathcal{C}_p(\mathcal{H}))$. If $\sum_{i\geq 1} P_i = \sum_{i\geq 1} Q_i$ in the strong operator topology, where $\{P_i\}$, $\{Q_i\}$ are families of pairwise disjoint finite-rank projections, then:*

$$\sum_{i\geq 1} \mathcal{T}(P_i) = \sum_{i\geq 1} \mathcal{T}(Q_i)$$

in the same topology, whenever at least one of $\sum_{i=1}^{n} \mathcal{T}(P_i)$ and $\sum_{i=1}^{n} \mathcal{T}(Q_i)$ converges strongly.

Proof. The following hold in the strong operator topology: Since $\sum_{i=1}^{n}(P_i - Q_i) \to 0$, we have $\mathcal{T}\left(\sum_{i=1}^{n} P_i - \sum_{i=1}^{n} Q_i\right) \to 0$. So $\sum_{i=1}^{n} \mathcal{T}(P_i) - \sum_{i=1}^{n} \mathcal{T}(Q_i) \to 0$. It follows that $\sum_{i=1}^{\infty} \mathcal{T}(P_i) = \sum_{i=1}^{\infty} \mathcal{T}(Q_i)$. □

Now suppose that $A = A^* \in \mathcal{C}_p(\mathcal{H})$ and $A = \sum_{i\geq 1} \mu_i Q_i$ is the spectral decomposition of A, where $\{\mu_i\}$ are the distinct non-zero eigenvalues of A and P_0 is the projection of $\mathcal{H}$ onto $KerA$. Write $P_0 = \sum_{k\geq 1} \widetilde{Q_k}$, where $\{\widetilde{Q_k}\}_{k\geq 1}$ is a family of pairwise disjoint finite-rank projections. Rewrite A as

$$A = 0 \cdot \widetilde{Q_1} + 0 \cdot \widetilde{Q_2} + \cdots + \sum_{i\geq 1} \mu_i Q_i \tag{2.1}$$

$$= \sum_{i\geq 1} \lambda_i P_i, \tag{2.2}$$

where $\lambda_i \geq 0$ for all i, the non-zero λ_i's are distinct and $\{P_i\}_{i\geq 1}$ is a partition of the identity such that $dim P_i(\mathcal{H}) < \infty$ for all i.

Proposition 2.5.5. *Let $\mathcal{H}$ be a complex separable Hilbert space and $1 \leqslant p < \infty$. Let $\mathcal{T}$ be a bounded completely-disjoint/Arazy-disjoint preserving mapping on $\mathcal{C}_p(\mathcal{H})$ and suppose that it is sequentially continuous with respect to the strong operator topology. If $\{P_i\}_{i\geq 1}$ is a partition of the identity such that $dim P_i(\mathcal{H}) < \infty$ for all i and Φ is the support mapping induced by $\mathcal{T}$, then:*

$$\mathcal{T}(A) = \left(\sum_{i\geq 1} \mathcal{T}(P_i)\right) \Phi(A)$$

for all self-adjoint operators A. The convergence is in the strong operator topology.

Proof. Let $A = A^* \in \mathcal{C}_p(\mathcal{H})$ and write $A = \sum_{i\geq 1} \lambda_i P_i^{'}$ as in 2.2. First we show that $\sum_{i\geq 1} \mathcal{T}(P_i^{'})$ converges strongly. To do that note that $\left(\mathcal{T}(P_i^{'})(\mathcal{H})\right)_i$ are pairwise orthogonal subspaces of $\mathcal{H}$ since $\mathcal{T}$ is completely-disjoint/Arazy-disjoint preserving and $P_i^{'} \perp P_j^{'}$ for $i \neq j$, and $\mathcal{T}(P)\Phi(P) = \mathcal{T}(P)$ by Lemma 2.3.3. Thus for all

$x \in \mathcal{H}$ and $m \geq n$, we have:

$$\begin{aligned}
||\sum_{i=1}^{m} \mathcal{T}(P_i^{'})(x) - \sum_{i=1}^{n} \mathcal{T}(P_i^{'})(x)||^2 &= ||\sum_{i=n+1}^{m} \mathcal{T}(P_i^{'})(x)||^2 \\
&= \sum_{i=n+1}^{m} ||\mathcal{T}(P_i^{'})(x)||^2 \\
&= \sum_{i=n+1}^{m} ||\mathcal{T}(P_i^{'})\Phi(P_i^{'})x||^2 \\
&\leqslant \sum_{i=n+1}^{m} ||\mathcal{T}||^2 ||P_i^{'}||^2 ||\Phi(P_i^{'})x||^2 \\
&= ||\mathcal{T}||^2 \sum_{i=n+1}^{m} ||\Phi(P_i^{'})x||^2 \\
&= ||\mathcal{T}||^2 \left\| \sum_{i=n+1}^{m} \Phi(P_i^{'})x \right\|^2
\end{aligned}$$

But $\sum_{i\geq 1} \Phi(P_i^{'})$ converges strongly since $(\Phi(P_i^{'}))_i$ are pairwise completely-disjoint projections by Proposition 2.3.4. Hence $\sum_{i\geq 1} \mathcal{T}(P_i^{'})$ converges strongly. Now using Proposition 2.5.2 and Lemma 2.3.3, we have:

$$\begin{aligned}
\mathcal{T}(A) &= \sum_{i\geq 1} \lambda_i \mathcal{T}(P_i^{'}) \\
&= \sum_{i\geq 1} \mathcal{T}(P_i^{'}) \lambda_i \Phi(P_i^{'}) \\
&= \sum_{i\geq 1} \left(\sum_{j=1}^{\infty} \lambda_j \mathcal{T}(P_i^{'}) \Phi(P_j^{'}) \right) \\
&= \sum_{i\geq 1} \left(\mathcal{T}(P_i^{'}) \sum_{j=1}^{\infty} \lambda_j \Phi(P_j^{'}) \right) \\
&= \sum_{i\geq 1} \left(\mathcal{T}(P_i^{'}) \Phi(A) \right) \\
&= \left(\sum_{i\geq 1} \mathcal{T}(P_i^{'}) \right) \Phi(A)
\end{aligned}$$

in the strong operator topology. By assumption, $\sum_{i\geq 1} P_i = I$ in the strong operator topology. Thus Lemma 2.5.4 implies that $\sum_{i\geq 1} \mathcal{T}(P_i^{'}) = \sum_{i\geq 1} \mathcal{T}(P_i)$, which completes the proof. □

Completely-disjoint/Arazy-disjoint preserving mappings

Now that we have covered all of the necessary prerequisite material, we can finally state the main theorem of this chapter.

Theorem 2.5.6. *Let $\mathcal{H}$ be a complex separable Hilbert space and $1 \leqslant p < \infty$. Let $\mathcal{T}$ be a bounded completely-disjoint/Arazy-disjoint preserving mapping on $\mathcal{C}_p(\mathcal{H})$ and suppose that it is sequentially continuous with respect to the strong operator topology. If $\{P_j\}_{j\geq 1}$ is a partition of the identity with $dim P_j(\mathcal{H}) < \infty$ for all j and Φ is the support mapping induced by $\mathcal{T}$, then:*

$$\mathcal{T}(K) = \left(\sum_{j\geq 1} \mathcal{T}(P_j) \right) \Phi(K)$$

for all $K \in \mathcal{C}_p(\mathcal{H})$. The convergence is in the strong operator topology.

Proof. If $K = ReK + iImK$, using Proposition 2.5.5, the linearity of $\mathcal{T}$ and the definition of Φ we have the following:

$$\begin{aligned}
\mathcal{T}(K) &= \mathcal{T}(ReK) + i\mathcal{T}(ImK) \\
&= \left(\sum_{j\geq 1} \mathcal{T}(P_j) \right) \Phi(ReK) + i \left(\sum_{j\geq 1} \mathcal{T}(P_j) \right) \Phi(ImK) \\
&= \left(\sum_{j\geq 1} \mathcal{T}(P_j) \right) \Phi(K).
\end{aligned}$$

□

In the finite dimensional case we proved that if $\mathcal{T}$ is a C-disjoint/A-disjoint preserving mapping on $M_n(\mathbb{C})$, then $\mathcal{T} = \mathcal{T}(I)\Phi$. Theorem 2.5.6 provides the infinite dimensional analogue of this. Once again it seems that $\mathcal{T}$ is essentially induced by a linear operator and a 'disjoint' preserving mapping Φ. In this case though, where $\mathcal{H}$ is infinite dimensional, the identity operator I is not in $\mathcal{C}_p(\mathcal{H})$. That is why, as just seen, an analogous expression for $\mathcal{T}(I)$ is needed, namely $\sum_{i\geq 1} \mathcal{T}(P_i)$, where $\{P_i\}_{i\geq 1}$ is a partition of the identity such that $dim P_i(\mathcal{H}) < \infty$ for all i.

It is perhaps worth pointing out at this stage that the assumption of $\mathcal{T}$ being sequentially continuous with respect to the strong operator topology in the statement of Theorem 2.5.6 **cannot** be dispensed from the rest of the hypothesis as we shall now demonstrate.

Remark 2.5.7. *A completely-disjoint/Arazy-disjoint preserving mapping $\mathcal{T}$ on $\mathcal{C}_p(\mathcal{H})$ need not be sequentially continuous with respect to the strong operator topology.*

Proof. Let $\mathcal{H} = l_2$. If $A \in \mathcal{C}_p(\mathcal{H})$ then define $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ by $\mathcal{T}(A) = JA^*J$, where $J : l_2 \to l_2$ is the conjugate linear invertible isometry with $J^2 = I$ defined by $J(x_1, x_2, ...) = (\bar{x_1}, \bar{x_2}, ...)$ with respect to the standard orthonormal basis of $\mathcal{H}$.

The conjugate linearity of J and of the mapping $A \to A^*$ imply that $\mathcal{T}$ is a linear operator. Since $(JA^*J)^* = JAJ$, the algebraic descriptions of C-disjoint/A-disjoint operators imply that $\mathcal{T}$ is a linear C-disjoint/A-disjoint preserving mapping. However, if we define a sequence $\{A_k\}$ of operators in $\mathcal{C}_p(l_2)$ by

$$A_k(x_1, x_2, ...) = (0, 0, ..., x_1, ..., x_k, 0, 0, ...),$$

where the number of zeros before the entry x_1 is k, then obviously $A_k \in \mathcal{C}_p(l_2)$ since it has finite rank and

$$A_k^*(y_1, y_2, ...) = (y_{k+1}, y_{k+2}, ..., y_{2k}, 0, 0, ...).$$

Thus $A_k^* \to 0$ strongly. However, $\mathcal{T}(A_k^*) = JA_kJ = A_k$ does not converge to 0 strongly. □

Before finishing this section we provide an example of a C-disjoint/A-disjoint preserving mapping on $\mathcal{C}_p$ which can indeed be expressed in the form given in Theorem 2.5.6.

Example 2.5.8. *If $\mathcal{H}$ is a complex separable Hilbert space and $\mathcal{T}$ is the bounded completely-disjoint/Arazy-disjoint preserving mapping $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ with*

$$\mathcal{T}(K) = U^{-1}KU,$$

where U is a unitary operator, then for all $K \in \mathcal{C}_p(\mathcal{H})$,

$$\mathcal{T}(K) = \left(\sum_{i=0}^{\infty} \mathcal{T}(P_i) \right) \Phi(K)$$

in the strong operator topology, where $\{P_i\}$ is a partition of the identity with $dim P_i(\mathcal{H}) < \infty$ for all i.

Proof. As in the case of a finite dimensional Hilbert space, for all $K \in \mathcal{C}_p(\mathcal{H})$, $\Phi(K) = \mathcal{T}(K)$ (see Example 2.3.7). For all $h \in \mathcal{H}$,

$$\sum_{i=0}^{\infty} (\mathcal{T}(P_i)h) = \sum_{i=0}^{\infty} (U^{-1}P_iUh) = U^{-1} \left(\sum_{i=0}^{\infty} P_i \right) Uh = U^{-1}Uh = h \ .$$

Hence for all $h \in \mathcal{H}$, $\left(\sum_{i=0}^{\infty} \mathcal{T}(P_i) \right) \Phi(K)h = \Phi(K)h = \mathcal{T}(K)h.$ □

2.6 Extension of the support mapping

We now turn our attention once again to the support mapping Φ induced by a completely-disjoint/Arazy-disjoint preserving mapping $\mathcal{T}$, which we have so far defined on $\mathcal{C}_p$ and used to obtain a formula for $\mathcal{T}$.

As mentioned in section 1.3, the support mapping for a disjoint preserving mapping on L_p can be extended to a mapping on the set of all measurable functions. In this section we shall show that something analogous can be done in the non-commutative setting. In other words, the support mapping Φ for a completely-disjoint/Arazy-disjoint preserving mapping $\mathcal{T}$ on $\mathcal{C}_p$ can be extended first to $\widetilde{\Phi}$ defined on the set $\mathcal{P}$ of all projections and then to the whole of $\mathcal{B}(\mathcal{H})$.

We begin by proving that such an extension does exist and it is well-defined. Then we move on to show that the extended mapping $\widetilde{\Phi}$ still has some of the rather interesting properties that the support mapping Φ we start with has.

Proposition 2.6.1. *Let $\mathcal{H}$ be a complex separable Hilbert space and $1 \leqslant p < \infty$. Let $\mathcal{T}$ be a bounded completely-disjoint/Arazy-disjoint preserving mapping on $\mathcal{C}_p(\mathcal{H})$ and suppose that $\mathcal{T}$ is sequentially continuous with respect to the strong operator topology. Then there exists an extension $\widetilde{\Phi} : \mathcal{P} \to \mathcal{P}$ of Φ such that for any arbitrary projections $P, Q \in \mathcal{P}$, the following hold:*

1. *If $P \perp Q$, then $\widetilde{\Phi}(P) \perp \widetilde{\Phi}(Q)$.*
2. *If $P \perp Q$, then $\widetilde{\Phi}(P+Q) = \widetilde{\Phi}(P) + \widetilde{\Phi}(Q)$.*

Proof. The first step is to show that such an extension exists. So fix $P \in \mathcal{P} \setminus \mathcal{P}_0$ and let $\{e_n\}_{n \geq 1}$ be a basis of $\mathcal{P}(\mathcal{H})$. If $P_n = e_n \otimes e_n \in \mathcal{P}_0$, then obviously $P = \sum_{n=1}^{\infty} P_n$ in the strong operator topology. Define $\widetilde{\Phi}(P) = \sum_{n=1}^{\infty} \Phi(P_n)$. The convergence is in the norm of $\mathcal{H}$. Note that for all $h \in \mathcal{H}$, $\sum_{n=1}^{N} \Phi(P_n) h \overset{N \to \infty}{\longrightarrow} Qh$, where Q is the projection of $\mathcal{H}$ onto $clin \left(\bigcup_{n=1}^{\infty} ran \Phi(P_n) \right)$ since $(\Phi(P_i))_i$ is a set of pairwise C-disjoint/A-disjoint projections. Thus $\widetilde{\Phi}(P) \in \mathcal{P}$. It will be shown now that $\widetilde{\Phi}(P)$ is well-defined. So let $P = \sum_{n=1}^{\infty} P_n$ and $P = \sum_{n=1}^{\infty} Q_n$ with the families $\{P_n\}$, $\{Q_n\}$ of projections in $\mathcal{P}_0$ each pairwise C-disjoint/A-disjoint. Since $\sum_{n=1}^{\infty} \Phi(P_n)$ converges strongly to the projection of $\mathcal{H}$ onto $clin \left(\bigcup_{n=1}^{\infty} ran \Phi(P_n) \right)$ and $\sum_{n=1}^{\infty} \Phi(Q_n)$ converges strongly to the projection of $\mathcal{H}$ onto $clin \left(\bigcup_{n=1}^{\infty} ran \Phi(Q_n) \right)$, it suffices to show

$$clin\left(\bigcup_{n=1}^{\infty} ran\Phi(P_n)\right) = clin\left(\bigcup_{n=1}^{\infty} ran\Phi(Q_n)\right).$$

Note first that the following are equivalent.

$$clin\left(\bigcup_{n=1}^{\infty} ran\Phi(P_n)\right) = clin\left(\bigcup_{n=1}^{\infty} ran\Phi(Q_n)\right)$$

$$\bigcap_{n=1}^{\infty} (ran\Phi(P_n))^{\perp} = \bigcap_{n=1}^{\infty} (ran\Phi(Q_n))^{\perp}$$

$$\bigcap_{n=1}^{\infty} Ker\Phi(P_n) = \bigcap_{n=1}^{\infty} Ker\Phi(Q_n)$$

$$\bigcap_{n=1}^{\infty} Ker\mathcal{T}(P_n) = \bigcap_{n=1}^{\infty} Ker\mathcal{T}(Q_n).$$

Then since $\sum_{i=1}^{n} Q_i \xrightarrow{s} P$, $\sum_{i=1}^{n} P_i \xrightarrow{s} P$ and $\mathcal{T}$ is sequentially continuous w.r.t the strong operator topology, we have $\sum_{i=1}^{\infty} \mathcal{T}(P_i) = \sum_{i=1}^{\infty} \mathcal{T}(Q_i)$. If $x \in \bigcap_{n=1}^{\infty} Ker\mathcal{T}(Q_n)$, $\sum_{n=1}^{\infty} \mathcal{T}(Q_n)x = \sum_{n=1}^{\infty} \mathcal{T}(P_n)x = 0$. So $(\mathcal{T}(P_n)x, \mathcal{T}(P_n)x) = \left(\sum_{i=1}^{\infty} \mathcal{T}(P_i)x, \mathcal{T}(P_n)x\right) = 0$ for all n since $(\mathcal{T}(P_i)(\mathcal{H}))_i$ is a set of pairwise orthogonal subspaces. Consequently $x \in \bigcap_{n=1}^{\infty} Ker\mathcal{T}(P_n)$. Similarly it can be shown that if $x \in \bigcap_{n=1}^{\infty} Ker\mathcal{T}(P_n)$, then $x \in \bigcap_{n=1}^{\infty} Ker\mathcal{T}(Q_n)$. Therefore $\widetilde{\Phi}(P)$ is well-defined.

1. Let $P \perp Q$. If $P = \sum_{n=1}^{\infty} P_n$ and $Q = \sum_{n=1}^{\infty} Q_n$ with $P_n, Q_n \in \mathcal{P}_0$ for all n and $P_n \perp P_m$, $Q_n \perp Q_m$ for $n \neq m$, then $\widetilde{\Phi}(P) = \sum_{i=1}^{\infty} \Phi(P_i)$ and $\widetilde{\Phi}(Q) = \sum_{i=1}^{\infty} \Phi(Q_i)$. Since $P \perp Q$ and $Q = \sum_{n=1}^{\infty} Q_n$ with $Q_nQ_m = Q_mQ_n = 0$ for $m \neq n$, we have $P \perp Q_n$ for all n. In fact, $QQ_n = Q_n = Q_nQ$. Therefore, for all n, $PQ_n = PQQ_n = 0$ and $Q_nP = Q_nQP = 0$ which implies that $P \perp Q_n$ for all n by the algebraic description.

Now since $P \perp Q_n$ for all n and $P = \sum_{i=1}^{\infty} P_n$, in the same way it can be shown that $P_m \perp Q_n$ for all n, m. The complete-disjointness of Φ implies now that for all

n, m, $\Phi(P_m)\perp\Phi(Q_n)$ and so the following hold in the strong operator topology:

$$\begin{aligned}
\widetilde{\Phi}(P)\widetilde{\Phi}(Q) &= \left(\sum_{n=1}^{\infty}\Phi(P_n)\right)\left(\sum_{m=1}^{\infty}\Phi(Q_m)\right)\\
&= \sum_{m=1}^{\infty}\left(\left(\sum_{n=1}^{\infty}\Phi(P_n)\right)\Phi(Q_m)\right)\\
&= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}(\Phi(P_n)\Phi(Q_m))\right)\\
&= 0.
\end{aligned}$$

Similarly, $\widetilde{\Phi}(Q)\widetilde{\Phi}(P) = 0$. It follows that $\widetilde{\Phi}(P)\perp\widetilde{\Phi}(Q)$ by the algebraic description.

2. Let $P\perp Q$. If $P = \sum_{n=1}^{\infty} P_n$ and $Q = \sum_{n=1}^{\infty} Q_n$ with $P_n, Q_n \in \mathcal{P}_0$ for all n such that $P_n\perp P_m$ and $Q_n\perp Q_m$ for $n \neq m$, then, as shown in the proof of 1, for all n, m, $P_m\perp Q_n$. That implies that for $n \neq m$, we have $P_n + Q_n\perp P_m + Q_m$. Hence the series $\sum_{n=1}^{\infty}(P_n + Q_n)$ converges strongly, and for all n, $P_n+Q_n \in \mathcal{P}_0$. Consequently $P + Q = \sum_{n=1}^{\infty} P_n + \sum_{n=1}^{\infty} Q_n = \sum_{n=1}^{\infty}(P_n + Q_n)$. The definition of $\widetilde{\Phi}$ and Proposition 2.3.9 now imply that

$$\begin{aligned}
\widetilde{\Phi}(P + Q) &= \sum_{n=1}^{\infty}\Phi(P_n + Q_n)\\
&= \sum_{n=1}^{\infty}(\Phi(P_n) + \Phi(Q_n))\\
&= \sum_{n=1}^{\infty}\Phi(P_n) + \sum_{n=1}^{\infty}\Phi(Q_n)\\
&= \widetilde{\Phi}(P) + \widetilde{\Phi}(Q).
\end{aligned}$$

□

Before proving a nice multiplicative property for $\widetilde{\Phi}$ concerning commutative projections we note the following lemmas.

Lemma 2.6.2. *Let $\mathcal{H}$ be a complex separable Hilbert space and $1 \leqslant p < \infty$. Let $\mathcal{T}$ be a bounded completely-disjoint/Arazy-disjoint preserving mapping on $\mathcal{C}_p(\mathcal{H})$ and suppose that $\mathcal{T}$ is sequentially continuous with respect to the strong operator topology. Then the extension $\widetilde{\Phi} : \mathcal{P} \to \mathcal{P}$ of Φ is also sequentially continuous with respect to the strong operator topology.*

Proof. Let $\{Q_n\}_n$ be a family of pairwise orthogonal projections in $\mathcal{P}$. We shall show that $\widetilde{\Phi}\left(\sum_{n=1}^{\infty} Q_n\right) = \sum_{n=1}^{\infty} \widetilde{\Phi}(Q_n)$. To do that write $Q_n = \sum_k P_{nk}$, where $\{P_{nk}\}_k$ is a family of pairwise orthogonal projections in $\mathcal{P}_0$. We then conclude that $\sum_n Q_n = \sum_n \sum_k P_{nk}$, where $\{P_{nk}\}_{nk}$ is a family of pairwise orthogonal projections in $\mathcal{P}_0$ since so are the families $\{Q_n\}_n$ and $\{P_{nk}\}_k$. By definition $\widetilde{\Phi}\left(\sum_n Q_n\right)$ is the projection of $\mathcal{H}$ onto $clin\left(\bigcup_{n,k} ran\Phi(P_{nk})\right)$ and $\widetilde{\Phi}(Q_n)$ is the projection of $\mathcal{H}$ onto $clin\left(\bigcup_k ran\Phi(P_{nk})\right)$. Since $clin\left(\bigcup_{n,k} ran\Phi(P_{nk})\right) = clin\bigcup_n\left(\bigcup_k ran\Phi(P_{nk})\right) = clin\bigcup_n ran\widetilde{\Phi}(Q_n)$, we conclude that $\widetilde{\Phi}\left(\sum_n Q_n\right) =$ the projection of $\mathcal{H}$ onto $clin\bigcup_n\left(ran\widetilde{\Phi}(Q_n)\right) = \sum_n \widetilde{\Phi}(Q_n)$. □

Lemma 2.6.3. *Let P in $\mathcal{P}\setminus\mathcal{P}_0$ and Q in $\mathcal{P}$. If $PQ = QP$, then we can write $P = \sum_{n=1}^{\infty} P_n$ with $P_n \in \mathcal{P}_0$ and $QP_n = P_nQ$ for all n, and $P_n \perp P_m$ for $n \neq m$.*

Proof. If $\{e_n\}$ is an orthonormal basis of $(QP)(\mathcal{H}) = (PQ)(\mathcal{H}) \subseteq P(\mathcal{H})$, it can be extended to a basis $\{f_n\}$ of $P(\mathcal{H})$. Thus $QP = \sum_{k=1}^{\infty} e_k \otimes e_k$ and $P = \sum_{k=1}^{\infty} f_k \otimes f_k$. If $P_n = f_n \otimes f_n$, then $PP_n = P_n = P_nP$. Therefore , for all n, we have

$$\begin{aligned} QP_n = QPP_n &= \left(\sum_{k=1}^{\infty} e_k \otimes e_k\right)(f_n \otimes f_n) \\ &= \sum_{k=1}^{\infty} ((e_k \otimes e_k)(f_n \otimes f_n)) \end{aligned}$$

and

$$\begin{aligned} P_nQ = P_nPQ &= (f_n \otimes f_n)\sum_{k=1}^{\infty}(e_k \otimes e_k) \\ &= \sum_{k=1}^{\infty} ((f_n \otimes f_n)(e_k \otimes e_k)) \end{aligned}$$

But $\sum_{k=1}^{\infty} ((e_k \otimes e_k)(f_n \otimes f_n)) = \sum_{k=1}^{\infty} ((f_n \otimes f_n)(e_k \otimes e_k)) = e_n \otimes e_n$ or 0. We conclude that $QP_n = P_nQ$ for all n. □

Proposition 2.6.4. *If PQ=QP, then $\widetilde{\Phi}(Q)\widetilde{\Phi}(P) = \widetilde{\Phi}(QP)$.*

Proof. Let $P, Q \in \mathcal{P}$ with $PQ = QP$. We first assume that Q is a finite-rank projection. By the previous Lemma, $P = \sum_{n=1}^{\infty} P_n$ such that $QP_n = P_nQ$ for all n, each $P_n \in \mathcal{P}_0$ and $P_n \perp P_m$ for $n \neq m$. Thus, since $\Phi(Q)$ is a bounded operator and for all $P_0, Q_0 \in \mathcal{P}_0$, $P_0Q_0 = Q_0P_0$ implies $\Phi(P_0Q_0) = \Phi(P_0)\Phi(Q_0)$ by Proposition 2.3.9, the following hold:

$$\begin{aligned}\Phi(Q)\widetilde{\Phi}(P) &= \Phi(Q)\left(\sum_{n=1}^{\infty}\Phi(P_n)\right)\\ &= \sum_{n=1}^{\infty}(\Phi(Q)\Phi(P_n))\\ &= \sum_{n=1}^{\infty}\Phi(QP_n).\end{aligned}$$

But $QP = Q\left(\sum_{n=1}^{\infty} P_n\right) = \sum_{n=1}^{\infty}(QP_n)$ with $QP_n \in \mathcal{P}_0$ for all n. Hence, by the definition of $\widetilde{\Phi}$, we obtain: $\widetilde{\Phi}(QP) = \sum_{n=1}^{\infty}\Phi(QP_n)$. It follows that

$$\Phi(Q)\widetilde{\Phi}(P) = \widetilde{\Phi}(QP) = \Phi(QP).$$

Now if $P, Q \in \mathcal{P}\setminus\mathcal{P}_0$ and $PQ = QP$, it follows that $\widetilde{\Phi}(P)\widetilde{\Phi}(Q) = \widetilde{\Phi}(QP)$. Indeed, if $Q = \sum_{n=1}^{\infty} Q_n$ with $Q_n \perp Q_m$ for $n \neq m$, each $Q_n \in \mathcal{P}_0$ and for all n, $Q_nP = PQ_n$ (see previous Lemma), then for all $N \in \mathbb{N}$,

$$\left(\sum_{n=1}^{N}\Phi(Q_n)\right)\widetilde{\Phi}(P) = \Phi(\sum_{n=1}^{N} Q_n)\widetilde{\Phi}(P) = \widetilde{\Phi}\left((\sum_{n=1}^{N} Q_n)P\right)$$

by Proposition 2.3.9 and the special case proved. However $\sum_{n=1}^{N} Q_n \xrightarrow{s} Q$ implies $(\sum_{n=1}^{N} Q_n)P \xrightarrow{s} QP$, which in turn gives us that $\widetilde{\Phi}\left((\sum_{n=1}^{N} Q_n)P\right) \xrightarrow{s} \widetilde{\Phi}(QP)$ since $\widetilde{\Phi}$ is sequentially continuous with respect to the strong operator topology by Lemma 2.6.2. Also $\sum_{n=1}^{N}\Phi(Q_n) \xrightarrow{s} \sum_{n=1}^{\infty}\Phi(Q_n) = \widetilde{\Phi}(Q)$. Therefore $\widetilde{\Phi}(Q)\widetilde{\Phi}(P) = \widetilde{\Phi}(QP)$. □

For convenience we introduce the following notation.

Definition 2.6.5. $\mathcal{B}(\mathcal{H})_{sa}$ *is defined to be the set of all self-adjoint operators in* $\mathcal{B}(\mathcal{H})$.

Now note that for all $P \in \mathcal{P}$, $\widetilde{\Phi}(P)(\mathcal{H}) \subseteq \widetilde{\Phi}(I)(\mathcal{H})$. Indeed: if $\{e_n\}$ is a basis of $P(\mathcal{H})$, it can be extended to a basis $\{f_n\}$ of the separable Hilbert space $\mathcal{H}$. Then $I = \sum_{n=1}^{\infty} f_n \otimes f_n$ and $P = \sum_{n=1}^{\infty} e_n \otimes e_n$. But $\widetilde{\Phi}(I) = \sum_{n=1}^{\infty} \Phi(f_n \otimes f_n)$, which is the projection of $\mathcal{H}$ onto $clin\left(\bigcup_{n=1}^{\infty} ran\Phi(f_n \otimes f_n)\right)$, and $\widetilde{\Phi}(P) = \sum_{n=1}^{\infty} \Phi(e_n \otimes e_n)$, which is the projection of $\mathcal{H}$ onto $clin\left(\bigcup_{n=1}^{\infty} ran\Phi(e_n \otimes e_n)\right)$ by the definition of $\widetilde{\Phi}$. Since $clin\left(\bigcup_{n=1}^{\infty} ran\Phi(e_n \otimes e_n)\right) \subseteq clin\left(\bigcup_{n=1}^{\infty} ran\Phi(f_n \otimes f_n)\right)$, we conclude that $\widetilde{\Phi}(P)(\mathcal{H}) \subseteq \widetilde{\Phi}(I)(\mathcal{H})$.

We are now in a position to define the extension $\widetilde{\Phi}$ of Φ onto $\mathcal{B}(\mathcal{H})_{sa}$ using the spectral measure for a self-adjoint operator in $\mathcal{B}(\mathcal{H})$.

Definition 2.6.6. *Let $A \in \mathcal{B}(\mathcal{H})_{sa}$ and let $E_A : \Omega \to \mathcal{B}(\mathcal{H})$ be the unique spectral measure for $(\sigma(A), \Omega, \mathcal{H})$ given by the spectral theorem, where $\sigma(A)$ is the spectrum of A and Ω is the set of all Borel subsets of $\sigma(A) \subseteq \mathbb{R}$. Define $\widetilde{E}_A : \Omega \to \mathcal{B}\left(\widetilde{\Phi}(I)(\mathcal{H})\right)$ for $(\sigma(A), \Omega, \widetilde{\Phi}(I)(\mathcal{H}))$ such that for Δ in Ω, $\widetilde{E}_A(\Delta) = \widetilde{\Phi}\left(E_A(\Delta)\right)_{/\widetilde{\Phi}(I)(\mathcal{H})}$. Here I denotes the identity operator on $\mathcal{H}$.*

We show that $\widetilde{E}_A$ is a spectral measure.

1. For Δ in Ω, $\widetilde{E}_A(\Delta) \in \mathcal{P}\left(\widetilde{\Phi}(I)(\mathcal{H})\right)$.
2. $\widetilde{E}_A(\varnothing) = \widetilde{\Phi}\left(E_A(\varnothing)\right)_{/\widetilde{\Phi}(I)(\mathcal{H})} = \widetilde{\Phi}(0)_{/\widetilde{\Phi}(I)(\mathcal{H})} = \Phi(0)_{/\widetilde{\Phi}(I)(\mathcal{H})} = 0$.
3. $\widetilde{E}_A(\sigma(A)) = \widetilde{\Phi}\left(E_A(\sigma(A))\right)_{/\widetilde{\Phi}(I)(\mathcal{H})} = \widetilde{\Phi}(I)_{/\widetilde{\Phi}(I)(\mathcal{H})} = I_{/\widetilde{\Phi}(I)(\mathcal{H})}$.
4. If $\{\Delta_n\}$ are pairwise disjoint sets in Ω, then Lemma 2.6.2 implies that

$$
\begin{aligned}
\widetilde{E}_A\left(\bigcup_{n=1}^{\infty} \Delta_n\right) &= \widetilde{\Phi}\left(E_A\left(\bigcup_{n=1}^{\infty} \Delta_n\right)\right)_{/\widetilde{\Phi}(I)(\mathcal{H})} \\
&= \widetilde{\Phi}\left(\sum_{n=1}^{\infty} E_A(\Delta_n)\right)_{/\widetilde{\Phi}(I)(\mathcal{H})} \\
&= \left(\sum_{n=1}^{\infty} \widetilde{\Phi}\left(E_A(\Delta_n)\right)\right)_{/\widetilde{\Phi}(I)(\mathcal{H})} \\
&= \sum_{n=1}^{\infty} \left(\widetilde{\Phi}\left(E_A(\Delta_n)\right)_{/\widetilde{\Phi}(I)(\mathcal{H})}\right) \\
&= \sum_{n=1}^{\infty} \widetilde{E}_A(\Delta_n).
\end{aligned}
$$

5. Since $\widetilde{\Phi} : \mathcal{P} \to \mathcal{P}$ is sequentially continuous with respect to the strong operator topology, if Δ_1, Δ_2 are two sets in Ω, then $E_A(\Delta_1)E_A(\Delta_2) = E_A(\Delta_1 \cap \Delta_2) = E_A(\Delta_2)E_A(\Delta_1)$ and Proposition 2.6.4 implies that

$$\begin{aligned}
\widetilde{E}_A(\Delta_1 \cap \Delta_2) &= \widetilde{\Phi}\left(E_A(\Delta_1 \cap \Delta_2)\right)_{/\widetilde{\Phi}(I)(\mathcal{H})} \\
&= \widetilde{\Phi}\left(E_A(\Delta_1)E_A(\Delta_2)\right)_{/\widetilde{\Phi}(I)(\mathcal{H})} \\
&= \left[\widetilde{\Phi}\left(E_A(\Delta_1)\right)\widetilde{\Phi}\left(E_A(\Delta_2)\right)\right]_{/\widetilde{\Phi}(I)(\mathcal{H})} \\
&= \widetilde{\Phi}\left(E_A(\Delta_1)\right)_{/\widetilde{\Phi}(I)(\mathcal{H})}\widetilde{\Phi}\left(E_A(\Delta_2)\right)_{/\widetilde{\Phi}(I)(\mathcal{H})} \\
&= \widetilde{E}_A(\Delta_1)\widetilde{E}_A(\Delta_2).
\end{aligned}$$

Hence $\widetilde{E}_A : \Omega \to \mathcal{B}\left(\widetilde{\Phi}(I)(\mathcal{H})\right)$ is a spectral measure for $\left(\sigma(A), \Omega, \widetilde{\Phi}(I)(\mathcal{H})\right)$.

Definition 2.6.7. *Define* $\widetilde{\Phi} : \mathcal{B}(\mathcal{H})_{sa} \to \mathcal{B}(\mathcal{H})_{sa}$ *as follows: If* $A \in \mathcal{B}(\mathcal{H})_{sa}$*, then* $\widetilde{\Phi}(A) = 0$ *on* $\left(\widetilde{\Phi}(I)(\mathcal{H})\right)^{\perp}$ *and* $\widetilde{\Phi}(A) = \int \lambda d\widetilde{E}_A(\lambda)$ *on* $\widetilde{\Phi}(I)(\mathcal{H})$*.*

Finally we can extend our mapping $\widetilde{\Phi}$ to the whole of $\mathcal{B}(\mathcal{H})$.

Definition 2.6.8. *Define* $\widetilde{\Phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *by* $\widetilde{\Phi}(K) = \widetilde{\Phi}(ReK) + i\widetilde{\Phi}(ImK)$ *for all* $K \in \mathcal{B}(\mathcal{H})$*.*

In summary, we have managed to gradually construct a mapping, namely the support mapping for a completely-disjoint/Arazy-disjoint preserving mapping on $\mathcal{C}_p$, first by defining it on finite rank-projections and then moving on step by step onto arbitrary operators in $\mathcal{B}(\mathcal{H})$.

Chapter 3

Characterisation and comparison of completely-disjoint and Arazy-disjoint preserving mappings

> *Happiness is a butterfly, which, when pursued, is always just beyond your grasp, but which, if you will sit down quietly, may alight upon you*

In this chapter, we look at both completely-disjoint and Arazy-disjoint preserving mappings on $\mathcal{C}_p$ spaces. We first obtain an explicit characterisation for invertible mappings on $M_n(\mathbb{C})$ and then go on to consider invertible bounded linear mappings on $\mathcal{C}_p(\mathcal{H})$, where $\mathcal{H}$ is infinite dimensional. We deduce that if both an invertible mapping $\mathcal{T}$ and its inverse $\mathcal{T}^{-1}$ are completely-disjoint preserving and have the additional property of sending rank-one projections to rank-one projections, then $\mathcal{T}$ can be explicitly characterised.

Finding, of course, descriptions for both kinds of 'disjoint' preserving mappings enables us to come to some conclusions about how the two kind of mappings are related. Some results on isometries on $\mathcal{C}_p$ spaces are also given. In the last section we turn our attention to Arazy-disjoint preserving mappings and prove that such mappings on $M_2(\mathbb{C})$ are either invertible or identically zero. The chapter concludes by stating a conjecture for Arazy-disjoint preserving mappings on $M_n(\mathbb{C})$.

3.1 Notation

We begin by introducing some notation.

Definition 3.1.1. *Let $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ be two $n \times n$ matrices with complex entries. The Schur or Hadamard product of A with B is defined to be the $n \times n$ matrix, denoted by $A * B$, whose entries are the pointwise product of the entries of A and B. In other words, $A * B = (\alpha_{ij}\beta_{ij})$.*

Definition 3.1.2. *Let $n \in \mathbb{N}$ and $J \subseteq \{1, 2, \dots, n\}$ with $k \leq n$ elements. Define*

$$\mathcal{F}_J : M_k(\mathbb{C}) \to M_n(\mathbb{C})$$

as follows. Given $A = (a_{ij}) \in M_k(\mathbb{C})$, $\mathcal{F}_J(A)$ is the $n \times n$ matrix with (i, j)-entry a_{ij} if $(i, j) \in J \times J$, and zero otherwise.

Define

$$\mathcal{S}_J : M_n(\mathbb{C}) \to M_k(\mathbb{C})$$

as follows. Given $A \in M_n(\mathbb{C})$, $\mathcal{S}_J(A)$ is the $J \times J$ submatrix of A, i.e the $k \times k$ matrix obtained from A by deleting the $i-th$ row if $i \notin J$, and the $j-th$ column if $j \notin J$.

Observe that

$$\mathcal{S}_J(\mathcal{F}_J(A)) = A, \quad A \in M_k(\mathbb{C}), \tag{3.1}$$

$$\mathcal{F}_J(AB) = \mathcal{F}_J(A)\mathcal{F}_J(B), \quad A, B \in M_k(\mathbb{C}). \tag{3.2}$$

Also

$$\mathcal{F}_J(\mathcal{S}_J(A)) = A, \tag{3.3}$$

$$\mathcal{S}_J(AB) = \mathcal{S}_J(A)\mathcal{S}_J(B), \tag{3.4}$$

when $A = (a_{ij}), B = (b_{ij}) \in M_n(\mathbb{C})$ with $a_{ij} = b_{ij} = 0$ for all $(i, j) \notin J \times J$.

Definition 3.1.3. *Let $\mathcal{H}$ be a Hilbert space and $x, y, z \in \mathcal{H}$. We define $z \otimes y$ on $\mathcal{H}$ by $(z \otimes y)(x) = \langle x, y \rangle z$.*

3.2 Some preliminary results

In the following section we shall find a description for invertible C-disjoint/A-disjoint mappings on $M_n(\mathbb{C})$ and prove that an invertible C-disjoint preserving mapping is also A-disjoint preserving.

In order to do this we are going to use induction on the dimension of the finite Hilbert space $\mathbb{C}^n$ and make use of mappings of the form

$$\mathcal{S}_J \circ \mathcal{T} \circ \mathcal{F}_J : M_k(\mathbb{C}) \to M_k(\mathbb{C}),$$

where $k \leq n$ and $J \subseteq \{1, 2, \ldots, n\}$ with k elements.

Our initial results in this section concern $k \times k$ matrices, $k \leq n$, which can be made into $n \times n$ matrices by being filled in with zeros, and mappings of the above form. In what follows, when a linear mapping $A : \mathbb{C}^n \to \mathbb{C}^n$ is being considered and an orthonormal basis of $\mathbb{C}^n$ has been prescribed, we shall identify A with its matrix with respect to that orthonormal basis.

Proposition 3.2.1. *Let $\{e_i\}$ be an orthonormal basis for $\mathbb{C}^n$ and $i, j \in \{1, ..., n\}$ with $i \neq j$. If $A : \mathbb{C}^n \to \mathbb{C}^n$ is linear and A-disjoint (C-disjoint) from $e_l \otimes e_l$ for all $l \neq i, j$ and $J = \{i, j\}$, then there is an $A' \in M_2(\mathbb{C})$ such that*

$$A = \mathcal{F}_J(A')$$

with respect to the basis $\{e_i\}$.

Proof. Without loss of generality we show that if $A \in M_n(\mathbb{C})$ is A-disjoint (C-disjoint) from $e_l \otimes e_l$ for all $l \neq 1, 2$ and $J = \{1, 2\}$, then

$$A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 & \cdots & 0 \\ \lambda_{21} & \lambda_{22} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.5}$$

with respect to the basis $\{e_i\}$.

Indeed, since $A = \sum_{i,j} \lambda_{ij} e_i \otimes e_j$ and $A \perp_A e_l \otimes e_l$ ($l \neq 1, 2$), we deduce that $\sum_i \lambda_{il} e_i \otimes e_l = 0$ ($l \neq 1, 2$). Thus $\lambda_{il} = 0$ for all $i \in \{1, 2, \ldots, n\}$ and for all $l \neq 1, 2$.

Thus

$$
\begin{aligned}
A &= \sum_{i,j} \lambda_{ij} e_i \otimes e_j \\
&= \sum_i \lambda_{i1} e_i \otimes e_1 + \sum_i \lambda_{i2} e_i \otimes e_2 \\
&= \begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 & \cdots & 0 \\ \lambda_{21} & \lambda_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_{n1} & \lambda_{n2} & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

with respect to the basis $\{e_i\}$. But $A \perp_A e_l \otimes e_l$ implies that $A^t \perp_A e_l \otimes e_l$, since the mapping $A \mapsto A^t$ is A-disjoint (C-disjoint) preserving. Consequently A has the form (3.5).

$\square$

Remark 3.2.2. *A generalisation of the previous Lemma holds.*

Let $n \in \mathbb{N}$ and $J \subseteq \{1, 2, \ldots, n\}$ with k elements ($k \leq n$). Let $\{e_i\}$ be an orthonormal basis for $\mathbb{C}^n$. If $A : \mathbb{C}^n \to \mathbb{C}^n$ is linear and A-disjoint (C-disjoint) from $e_l \otimes e_l$ for all $l \notin J$, then there is an $A' \in M_k(\mathbb{C})$ such that

$$A = \mathcal{F}_J(A')$$

with respect to the basis $\{e_i\}$.

We now conclude that a linear mapping $\mathcal{T}$ on $M_n(\mathbb{C})$ which maps C-disjoint to A-disjoint matrices and is such that $\mathcal{T}(e_l \otimes e_l) = \lambda_l e_l \otimes e_l$ for a fixed orthonormal basis $\{e_i\}$ of $\mathbb{C}^n$ and non-zero scalars λ_l, has the property of mapping $n \times n$ matrices which are essentially $k \times k$ matrices, $k \leq n$, filled in with zeros to matrices of exactly the same form.

Corollary 3.2.3. *Let $n \in \mathbb{N}$ and $\mathcal{T}$ be a linear mapping on $M_n(\mathbb{C})$ which maps C-disjoint to A-disjoint matrices. Suppose there are non-zero scalars λ_l such that $\mathcal{T}(e_l \otimes e_l) = \lambda_l e_l \otimes e_l$ for a fixed orthonormal basis $\{e_i\}$ of $\mathbb{C}^n$. If $J \subseteq \{1, 2, \ldots, n\}$ with k elements ($k \leq n$) and $A \in M_k(\mathbb{C})$, there is a matrix $A' \in M_k(\mathbb{C})$ such that*

$$\mathcal{T}(\mathcal{F}_J(A)) = \mathcal{F}_J(A')$$

with respect to the basis $\{e_i\}$.

Proof. Pick $i, j \in J$. Since $e_i \otimes e_j \perp e_l \otimes e_l$ for all $l \in \{1, ..., n\}$ with $l \notin J$, we have: $\mathcal{T}(e_i \otimes e_j) \perp_A \mathcal{T}(e_l \otimes e_l)$ for all $l \notin J$. But $\mathcal{T}(e_l \otimes e_l) = \lambda_l e_l \otimes e_l$ by the hypothesis. So Remark 3.2.2 implies that there is a $B \in M_k(\mathbb{C})$ such that $\mathcal{T}(e_i \otimes e_j) = \mathcal{F}_J(B)$. Since $\mathcal{T}$ is linear, we conclude that for any $A \in M_k(\mathbb{C})$ there is an $A' \in M_k(\mathbb{C})$ such that $\mathcal{T}(\mathcal{F}_J(A)) = \mathcal{F}_J(A')$. □

Mappings of the form $\mathcal{S}_J \circ \mathcal{T} \circ \mathcal{F}_J$

We are now able to prove that the mappings $\mathcal{S}_J \circ \mathcal{T} \circ \mathcal{F}_J$, where $J \subseteq \{1, 2, \ldots, n\}$ with k elements ($k \leq n$) have essentially the same properties as the linear mapping $\mathcal{T}$ we start with.

Corollary 3.2.4. *Let $n \in \mathbb{N}$ and $\mathcal{T}$ be a linear mapping on $M_n(\mathbb{C})$ which maps C-disjoint to A-disjoint matrices. Suppose there are non-zero scalars λ_l such that $\mathcal{T}(e_l \otimes e_l) = \lambda_l e_l \otimes e_l$ for a fixed orthonormal basis $\{e_i\}$ of $\mathbb{C}^n$. If $J \subseteq \{1, 2, \ldots, n\}$ with k elements ($k \leq n$), then the mapping*

$$\mathcal{S}_J \circ \mathcal{T} \circ \mathcal{F}_J : M_k(\mathbb{C}) \to M_k(\mathbb{C})$$

maps C-disjoint to A-disjoint matrices. If, moreover, $\mathcal{T}$ is one-to-one, then so is the mapping $\mathcal{S}_J \circ \mathcal{T} \circ \mathcal{F}_J$.

Proof. Let $A, B \in M_k(\mathbb{C})$ with $A \perp B$. Then the algebraic description of C-disjoint operators implies that $\mathcal{F}_J(A) \perp \mathcal{F}_J(B)$. Thus $\mathcal{T}(\mathcal{F}_J(A)) \perp_A \mathcal{T}(\mathcal{F}_J(B))$. But by Corollary 3.2.3, there exist $A', B' \in M_k(\mathbb{C})$ such that $\mathcal{T}(\mathcal{F}_J(A)) = \mathcal{F}_J(A')$ and $\mathcal{T}(\mathcal{F}_J(B)) = \mathcal{F}_J(B')$. Then $\mathcal{F}_J(A') \perp_A \mathcal{F}_J(B')$. Therefore $A' \perp_A B'$ by the algebraic description again. But $A' = \mathcal{S}_J(\mathcal{T}(\mathcal{F}_J(A)))$ and $B' = \mathcal{S}_J(\mathcal{T}(\mathcal{F}_J(B)))$. Thus $\mathcal{S}_J(\mathcal{T}(\mathcal{F}_J(A))) \perp_A \mathcal{S}_J(\mathcal{T}(\mathcal{F}_J(B)))$. Now suppose that $\mathcal{T}$ is also one-to-one and that $\mathcal{S}_J(\mathcal{T}(\mathcal{F}_J(A))) = \mathcal{S}_J(\mathcal{T}(\mathcal{F}_J(B)))$ for some $A, B \in M_k(\mathbb{C})$. By Corollary 3.2.3, there exist $A', B' \in M_k(\mathbb{C})$ such that

$$\mathcal{T}(\mathcal{F}_J(A)) = \mathcal{F}_J(A') \quad \textit{and} \quad \mathcal{T}(\mathcal{F}_J(B)) = \mathcal{F}_J(B'). \tag{3.6}$$

Thus $\mathcal{S}_J\left(\mathcal{F}_J(A')\right) = \mathcal{S}_J\left(\mathcal{F}_J(B')\right)$ and so $A' = B'$. Therefore $\mathcal{F}_J(A') = \mathcal{F}_J(B')$. Equation 3.6 now implies that $\mathcal{T}(\mathcal{F}_J(A)) = \mathcal{T}(\mathcal{F}_J(B))$ and hence $\mathcal{F}_J(A) = \mathcal{F}_J(B)$ since $\mathcal{T}$ is one-to-one. Apply $\mathcal{S}_J$ on both sides to obtain $A = B$. □

On rank-one matrices

We now state three simple lemmas concerning rank-one matrices. They will shortly be used to simplify our dealings with both kinds of 'disjoint' preserving mappings.

Lemma 3.2.5. *Let A be a rank-one formally self-adjoint matrix (i.e if $A = (a_{ij})$, then $a_{ji} = \overline{a_{ij}}$) with 1's on the diagonal. Then A has unimodular entries.*

Proof. Consider any two columns k and l, say $k < l$, of A. Then $\begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \end{pmatrix} = a_{lk} \begin{pmatrix} a_{1l} \\ a_{2l} \\ \vdots \end{pmatrix}$, i.e for all $m \in \mathbb{N}$, $a_{mk} = a_{ml}a_{lk}$ since A has 1's on the diagonal. But A is self-adjoint. Thus for all i, j, we have $|a_{ij}|^2 = \overline{a_{ij}}a_{ij} = a_{ji}a_{ij} = a_{jj} = 1$. Therefore A has unimodular entries. □

Lemma 3.2.6. *Let A be a rank-one (finite or infinite dimensional) matrix with unimodular entries. Then there are unitary matrices U, W such that $UKW^* = K * A$ for all K.*

Proof. Let $A = (a_{ij})$ with respect to $\{e_i\}$. Since A is rank-one, any two columns of it are linearly dependent. Thus there are λ_j's, $j \geq 2$ such that $a_{ij} = \lambda_j a_{i1}$ for all $i, j \geq 1$. Let $\lambda_1 = 1$. The matrix A has unimodular entries and so $|a_{ij}| = |\lambda_j| = 1$ for all i, j. Now define $U = diag\{a_{i1}, i \geq 1\}$ and $W^* = diag\{\lambda_j, j \geq 1\}$. Then U, W are obviously unitary and $UKW^* = K * A$ for all K with respect to the fixed orthonormal basis $\{e_i\}$. □

Lemma 3.2.7. *Let A be an $n \times n$ ($n \geq 3$) self-adjoint matrix with unimodular entries and 1's on the diagonal. If every 3×3 submatrix of A has rank one, then A has also rank one.*

Proof. Let $A = (a_{ij})$ and fix $1 \leq i, j \leq n$ with $i \neq j$ (say $i < j$). We will show that $\begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$ and $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}$ are linearly dependent. For any $k \neq i, j$ (without loss of generality we may assume that $j < k$), $\begin{pmatrix} a_{ii} \\ a_{ji} \\ a_{ki} \end{pmatrix}$ and $\begin{pmatrix} a_{ij} \\ a_{jj} \\ a_{kj} \end{pmatrix}$ are linearly dependent since any 3×3 submatrix of A has rank one. Therefore there is a non-zero scalar

λ_k such that $\begin{pmatrix} a_{ij} \\ a_{jj} \\ a_{kj} \end{pmatrix} = \lambda_k \begin{pmatrix} a_{ii} \\ a_{ji} \\ a_{ki} \end{pmatrix}$, as $\begin{pmatrix} a_{ii} \\ a_{ji} \\ a_{ki} \end{pmatrix} \neq 0$ (recall that $a_{ii} = 1 \neq 0$). Thus

$$\lambda_k = \lambda_k a_{ii} = a_{ij}$$

and so λ_k does not actually depend on k. We rename it λ'. Thus $a_{kj} = \lambda' a_{ki}$ for all $k \neq i, j$. If $k = j$, then $1 = a_{jj} = \lambda' a_{ji}$, and so $\lambda' = a_{ij}$ since A is self-adjoint and has unimodular entries. For $k = i$, we have $\lambda' = a_{ij}$, which, we already know, holds. Hence $a_{kj} = \lambda' a_{ki}$ for all k.

Therefore $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} = \lambda' \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$ and so A is rank-one. □

One more lemma is needed.

Lemma 3.2.8. *Let $\mathcal{H}$ be a separable Hilbert space and fix an orthonormal basis $\{e_i\}$ of $\mathcal{H}$. Let $\mathcal{T}$ be a linear mapping on $\mathcal{C}_p(\mathcal{H})$ which maps C-disjoint to A-disjoint operators. If dim $(ran\mathcal{T}(e_i \otimes e_i)) = 1$ for all i, then there is an isometry U_1 and a co-isometry W_1 and there are scalars $\lambda_i \in \mathbb{R}^+, \lambda_i \neq 0$ such that*

$$W_1 \mathcal{T}(e_i \otimes e_i) U_1 = \lambda_i e_i \otimes e_i$$

for all i.

Proof. Since $dim\,(ran\mathcal{T}(e_i \otimes e_i)) = 1$ for all i, we can write $\mathcal{T}(e_i \otimes e_i) = \lambda_i w_i \otimes u_i$, where w_i, u_i are unit vectors in $\mathcal{H}$ and $\lambda_i \in \mathbb{R}^+ \setminus \{0\}$ for all i. For $i \neq j$ we have $e_i \otimes e_i \perp e_j \otimes e_j$ and so $w_i \otimes u_i \perp_A w_j \otimes u_j$. Thus $(u_i, u_j) = (w_i, w_j) = 0$ for all i, j with $i \neq j$. Extend $\{u_i\}$ and $\{w_i\}$ to bases of $\mathcal{H}$. Define linear isometries W, U_1 by $W(e_i) = w_i$ and $U_1(e_i) = u_i$, and set $W_1 = W^*$. Then W_1 is a co-isometry and $W_1 \mathcal{T}(e_i \otimes e_i) U_1 = \lambda_i e_i \otimes e_i$ for all i. □

3.3 The finite dimensional case

In our attempt to find a characterisation for C-disjoint/A-disjoint preserving mappings on $M_n(\mathbb{C})$ using induction, we shall first need to consider mappings on $M_2(\mathbb{C})$.

Invertible completely-disjoint/Arazy-disjoint preserving mappings on $M_2(\mathbb{C})$

The following theorem is a fundamental result stating that there are only two possible formulae for invertible mappings on $M_2(\mathbb{C})$ which map C-disjoint to A-disjoint matrices.

Theorem 3.3.1. *Let $\mathcal{T}$ be a linear mapping on $M_2(\mathbb{C})$. The following are then equivalent:*
(i) $\mathcal{T}$ is an invertible mapping which maps C-disjoint to A-disjoint matrices;
(ii) $\mathcal{T}$ is an invertible A-disjoint preserving mapping;
(iii) there are unique, up to a scalar, unitary operators U, W and a unique scalar $\lambda \in \mathbb{R}^+, \lambda \neq 0$ such that

$$\text{either} \quad \mathcal{T}(K) = \lambda UKW \qquad (K \in M_2(\mathbb{C}))$$

$$\text{or} \quad \mathcal{T}(K) = \lambda UK^tW \qquad (K \in M_2(\mathbb{C})).$$

Such an operator on $M_2(\mathbb{C})$ is C-disjoint preserving if and only if U is a unit scalar multiple of W^*.

Proof. (iii) $\Longrightarrow$ (ii): To prove this implication simply use the algebraic description of A-disjoint matrices and note that $\mathcal{T}$ is obviously invertible in both cases.
(ii) $\Longrightarrow$ (i): This is trivial by the fact that $K \perp L$ implies $K \perp_A L$ for all $K, L \in M_2(\mathbb{C})$.
(i) $\Longrightarrow$ (iii): Let $\{e_i\}$ be the standard basis for $\mathbb{C}^2$. Then $dim\,(ran\,(\mathcal{T}(e_i \otimes e_i))) \geqslant 1$ for $i = 1, 2$ since $\mathcal{T}$ is invertible. Also $\mathcal{T}(e_i \otimes e_i) \perp_A \mathcal{T}(e_j \otimes e_j)$ for $i \neq j$ and consequently $ran\,(\mathcal{T}(e_i \otimes e_i)) \perp ran\,(\mathcal{T}(e_j \otimes e_j))$. We conclude that $ran\,(\mathcal{T}(e_i \otimes e_i)) = 1$, $i = 1, 2$. Lemma 3.2.8 now implies that there are isometries U_1, W_1 and non zero scalars $\lambda, \mu \in \mathbb{R}^+$ such that if we define $\widetilde{\mathcal{T}}(\cdot) = W_1\mathcal{T}(\cdot)U_1$, then $\widetilde{\mathcal{T}}(e_1 \otimes e_1) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$ and $\widetilde{\mathcal{T}}(e_2 \otimes e_2) = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}$. Suppose that $\widetilde{\mathcal{T}}(e_1 \otimes e_2) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\widetilde{\mathcal{T}}(e_2 \otimes e_1) = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ for some complex scalars $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$.
Now let $u = (Y, 1)$, $v = (1, -Y)$ in $\mathbb{C}^2$ with $Y \in \mathbb{R}$. Then

$$u \otimes u = \begin{pmatrix} Y^2 & Y \\ Y & 1 \end{pmatrix} \quad \text{and} \quad v \otimes v = \begin{pmatrix} 1 & -Y \\ -Y & Y^2 \end{pmatrix}.$$

Since $u \otimes u \perp v \otimes v$, we have $\widetilde{\mathcal{T}}(u \otimes u) \perp_A \widetilde{\mathcal{T}}(v \otimes v)$. Thus

$$\widetilde{\mathcal{T}}(u \otimes u) \left(\widetilde{\mathcal{T}}(v \otimes v)\right)^* = \left(\widetilde{\mathcal{T}}(v \otimes v)\right)^* \widetilde{\mathcal{T}}(u \otimes u) = 0.$$

But

$$\widetilde{\mathcal{T}}(u \otimes u) = \begin{pmatrix} Y^2\lambda + Y\alpha + Y\alpha' & Y\beta + Y\beta' \\ Y\gamma + Y\gamma' & \mu + Y\delta + Y\delta' \end{pmatrix}$$

and

$$\left(\widetilde{\mathcal{T}}(v \otimes v)\right)^* = \begin{pmatrix} \overline{\lambda} - Y\overline{\alpha} - Y\overline{\alpha'} & -Y\overline{\gamma} - Y\overline{\gamma'} \\ -Y\overline{\beta} - Y\overline{\beta'} & Y^2\overline{\mu} - Y\overline{\delta} - Y\overline{\delta'} \end{pmatrix}.$$

The (1,1)-entry of $\widetilde{\mathcal{T}}(u \otimes u) \left(\widetilde{\mathcal{T}}(v \otimes v)\right)^*$ is zero, so the coefficient of the polynomial in Y^3 should be zero. Thus

$$-\lambda\overline{\alpha} - \lambda\overline{\alpha'} = 0. \tag{3.7}$$

Similarly, the (2,2)-entry of $\widetilde{\mathcal{T}}(u \otimes u) \left(\widetilde{\mathcal{T}}(v \otimes v)\right)^*$ is zero, so the coefficient of the polynomial in Y^3 should be zero. Thus

$$\delta\overline{\mu} + \delta'\overline{\mu} = 0. \tag{3.8}$$

If we repeat the same argument with $u = (Y, i)$ and $v = (-i, -Y)$ in $\mathbb{C}^2$ and $Y \in \mathbb{R}$, we have:

$$u \otimes u = \begin{pmatrix} Y^2 & -Yi \\ Yi & 1 \end{pmatrix} \quad \textit{and} \quad v \otimes v = \begin{pmatrix} 1 & Yi \\ -Yi & Y^2 \end{pmatrix},$$

$$\widetilde{\mathcal{T}}(u \otimes u) \left(\widetilde{\mathcal{T}}(v \otimes v)\right)^* = \left(\widetilde{\mathcal{T}}(v \otimes v)\right)^* \widetilde{\mathcal{T}}(u \otimes u) = 0,$$

$$\widetilde{\mathcal{T}}(u \otimes u) = \begin{pmatrix} Y^2\lambda - Y\alpha i + Y\alpha' i & -Y\beta i + Y\beta' i \\ -Y\gamma i + Y\gamma' i & \mu - Y\delta i + Y\delta' i \end{pmatrix}$$

and

$$\left(\widetilde{\mathcal{T}}(v \otimes v)\right)^* = \begin{pmatrix} \overline{\lambda} - Y\overline{\alpha}i + Y\overline{\alpha'}i & -Y\overline{\gamma}i + Y\overline{\gamma'}i \\ -Y\overline{\beta}i + Y\overline{\beta'}i & Y^2\overline{\mu} - Y\overline{\delta}i + Y\overline{\delta'}i \end{pmatrix}.$$

Looking at the $(1,1)$ and $(2,2)$ entries of $\widetilde{\mathcal{T}}(u \otimes u)\left(\widetilde{\mathcal{T}}(v \otimes v)\right)^*$ and the coefficient of the polynomial in Y^3 again, we get

$$-\lambda\overline{\alpha} + \lambda\overline{\alpha'} = 0 \tag{3.9}$$

$$-\delta\overline{\mu} + \delta'\overline{\mu} = 0 \tag{3.10}$$

respectively. Combining (3.7)-(3.10) (using $\lambda, \mu \neq 0$), we get

$$\alpha = \alpha' = \delta = \delta' = 0 \tag{3.11}$$

Now using (3.11) and the fact that the $(1,1)$ and $(2,2)$ entries of the matrix $\widetilde{\mathcal{T}}(u \otimes u)\left(\widetilde{\mathcal{T}}(v \otimes v)\right)^*$ are zero when $u = (Y,1)$ and $v = (1,-Y)$ with $Y \in \mathbb{R}$, we get

$$|\lambda|^2 - |\beta|^2 - |\beta'|^2 - 2Re(\beta\overline{\beta'}) = 0 \tag{3.12}$$

$$|\mu|^2 - |\gamma|^2 - |\gamma'|^2 - 2Re(\gamma\overline{\gamma'}) = 0 \tag{3.13}$$

respectively. Using (3.11) and the fact that the $(1,1)$ and $(2,2)$ entries of the matrix $\widetilde{\mathcal{T}}(u \otimes u)\left(\widetilde{\mathcal{T}}(v \otimes v)\right)^*$ are zero when $u = (Y,i)$ and $v = (-i,-Y)$ with $Y \in \mathbb{R}$, we get

$$|\lambda|^2 - |\beta|^2 - |\beta'|^2 + 2Re(\beta\overline{\beta'}) = 0 \tag{3.14}$$

$$|\mu|^2 - |\gamma|^2 - |\gamma'|^2 + 2Re(\gamma\overline{\gamma'}) = 0 \tag{3.15}$$

respectively. Combining (3.12),(3.14) and (3.13),(3.15), we conclude that

$$|\lambda|^2 - |\beta|^2 - |\beta'|^2 = 0$$

$$Re\left(\beta\overline{\beta'}\right) = 0$$

and

$$|\mu|^2 - |\gamma|^2 - |\gamma'|^2 = 0$$

$$Re\left(\gamma\overline{\gamma'}\right) = 0.$$

By looking at the $(1,1)$ and $(2,2)$ entries of the matrix $\widetilde{\mathcal{T}}(u \otimes u)\left(\widetilde{\mathcal{T}}(v \otimes v)\right)^*$ and the coefficients of the polynomial in Y^2 when $u = (1+i, Y)$ and $v = (Y, -1+i)$ with $Y \in \mathbb{R}$, we obtain that

$$Im\left(\beta\overline{\beta'}\right) = 0$$

and

$$Im\left(\gamma\overline{\gamma'}\right) = 0.$$

Therefore

$$|\lambda|^2 - |\beta|^2 - |\beta'|^2 = 0$$

$$|\mu|^2 - |\gamma|^2 - |\gamma'|^2 = 0$$

$$\beta\beta' = 0$$

$$\gamma\gamma' = 0.$$

Now we use (3.11) and the fact that the $(1,1)$ and $(2,2)$ entries of the matrix $\left(\widetilde{\mathcal{T}}\,(v \otimes v)\right)^* \widetilde{\mathcal{T}}\,(u \otimes u)$ are also zero, when $u = (Y,1)$ and $v = (1,-Y)$ with $Y \in \mathbb{R}$ to obtain

$$|\lambda|^2 - |\gamma|^2 - |\gamma'|^2 + 2Re(\gamma\overline{\gamma'}) = 0$$

$$|\mu|^2 - |\beta|^2 - |\beta'|^2 + 2Re(\beta\overline{\beta'}) = 0.$$

The same argument applied to $u = (Y,i)$ and $v = (-i,-Y)$ with $Y \in \mathbb{R}$ yields

$$|\lambda|^2 - |\gamma|^2 - |\gamma'|^2 - 2Re(\gamma\overline{\gamma'}) = 0$$

$$|\mu|^2 - |\beta|^2 - |\beta'|^2 - 2Re(\beta\overline{\beta'}) = 0.$$

Therefore

$$|\lambda|^2 - |\gamma|^2 - |\gamma'|^2 = 0$$

and

$$|\mu|^2 - |\beta|^2 - |\beta'|^2 = 0.$$

We thus have four cases:

Case 1: $\beta = \gamma = 0$

Then $\widetilde{\mathcal{T}}(e_1 \otimes e_2) = 0$ which is a contradiction to $\widetilde{\mathcal{T}}$ being invertible.

Case 2: $\beta' = \gamma' = 0$

This case is similarly rejected.

Case 3: $\beta = \gamma' = 0$

Then $|\beta'| = |\gamma| = |\lambda| = |\mu|$. Since the (1,2)-entry of $\widetilde{\mathcal{T}}\,(u \otimes u)\left(\widetilde{\mathcal{T}}\,(v \otimes v)\right)^*$ is zero when $u = (Y,1)$, $v = (1,-Y)$ and $\alpha = \alpha' = \delta = \delta' = \beta = \gamma' = 0$, we obtain $\overline{\lambda}\gamma = \mu\overline{\beta'}$. Thus

$$\lambda\mu = \lambda\frac{\overline{\lambda}\gamma}{\overline{\beta'}} = \frac{|\lambda|^2\,\gamma}{\overline{\beta'}} = \frac{|\beta'|^2\,\gamma}{\overline{\beta'}} = \beta'\gamma.$$

We deduce that the determinant of the 2×2 matrix $\widetilde{A} = \begin{pmatrix} \lambda & \beta' \\ \gamma & \mu \end{pmatrix}$ is zero and thus $\widetilde{A}$ is rank-one. Now note that:

$$\widetilde{\mathcal{T}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad \widetilde{\mathcal{T}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix},$$

$$\widetilde{\mathcal{T}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}, \quad \widetilde{\mathcal{T}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta' \\ 0 & 0 \end{pmatrix}$$

i.e $\widetilde{\mathcal{T}}(K) = K^t * \widetilde{A}$ for all $K \in M_2(\mathbb{C})$.

Since $\widetilde{A}$ is rank-one and all its entries have the same modulus, $\widetilde{A} = \lambda' A$ for some $\lambda' \in \mathbb{R}^+, \lambda' \neq 0$, and for some rank-one 2×2 matrix A with unimodular entries. By Lemma 3.2.6, there are unitaries U_2, W_2 such that $U_2 K^t W_2^* = K^t * A$ for all $K \in M_2(\mathbb{C})$. Thus $\widetilde{\mathcal{T}}(K) = \lambda' U_2 K^t W_2^*$ for all $K \in M_2(\mathbb{C})$. Consequently,

$$\mathcal{T}(K) = \lambda' U K^t W,$$

where $U = W_1^* U_2$, $W = W_2^* U_1^*$ are unitary operators and $\lambda' \in \mathbb{R}^+, \lambda' \neq 0$.
Case 4: $\beta' = \gamma = 0$

Then $|\beta| = |\gamma'| = |\lambda| = |\mu|$ and in the same way, we can define $\widetilde{A} = \begin{pmatrix} \lambda & \beta \\ \gamma' & \mu \end{pmatrix}$ which is rank-one and all its entries have the same modulus, and for all $K \in M_2(\mathbb{C})$,

$$\widetilde{\mathcal{T}}(K) = K * \widetilde{A}.$$

Therefore for all $K \in M_2(\mathbb{C})$,

$$\mathcal{T}(K) = \lambda' U K W$$

for some unitaries U, W and some $\lambda' \in \mathbb{R}^+, \lambda' \neq 0$.

To show that such an operator is C-disjoint preserving if and only if U is a unit scalar multiple of W^* simply apply Proposition 2.2.3.

Lastly, we show that U, W are unique up to a scalar and that λ' is also unique. So suppose there are $\lambda', \lambda_1 \in \mathbb{R}^+, \lambda', \lambda_1 \neq 0$, and unitary operators U, U_1, W, W_1 such that for all K,

$$\lambda' U K W = \lambda_1 U_1 K W_1.$$

Set $K = I$, the identity operator, to get

$$\lambda' UW = \lambda_1 U_1 W_1. \tag{3.16}$$

So $\left\|\lambda' UW\right\| = \|\lambda_1 U_1 W_1\|$. Thus

$$\lambda' = \lambda_1. \tag{3.17}$$

We conclude that for all K, we have $UKW = U_1 K W_1$.
Now fix i. Then for any k, j we have:

$$\left(U(e_i \otimes e_j)W\right)(e_k) = \left(U_1(e_i \otimes e_j)W_1\right)(e_k).$$

So

$$< We_k, e_j > Ue_i =< W_1 e_k, e_j > U_1 e_i.$$

Since $W \neq 0$, there are k_0, j_0 such that $\langle We_{k_0}, e_{j_0}\rangle \neq 0$. Therefore $Ue_i = \mu U_1 e_i$ for all i, where $\mu = \frac{\langle W_1 e_{k_0}, e_{j_0}\rangle}{\langle We_{k_0}, e_{j_0}\rangle} \in \mathbb{C}, \mu \neq 0$. Consequently $U = \mu U_1$ and thus (3.16), (3.17) imply that

$$W = \frac{1}{\mu} W_1.$$

Thus U, W are unique up to a scalar.

□

The following Proposition which gives a characterisation of invertible C-disjoint preserving mappings is essentially part of the previous Theorem but we state it separately because of its independent interest.

Proposition 3.3.2. *Let $\mathcal{T}$ be a linear mapping on $M_2(\mathbb{C})$. The following are then equivalent:*
(i) $\mathcal{T}$ is an invertible completely-disjoint preserving mapping;
(ii) there is a unique, up to a scalar, unitary operator W and a unique scalar $\lambda \in \mathbb{C}$, $\lambda \neq 0$ such that

$$\textit{either} \quad \mathcal{T}(K) = \lambda W^* K W \qquad (K \in M_2(\mathbb{C}))$$

$$\textit{or} \quad \mathcal{T}(K) = \lambda W^* K^t W \qquad (K \in M_2(\mathbb{C})).$$

Remark 3.3.3. *It's worth pointing out at this stage that the two formulae appearing in the statement of Theorem 3.3.1 cannot be the same. In other words, we cannot have*

$$\lambda UKW = \lambda_1 U_1 K^t W_1$$

for all K.

Proof. To show this note that

$$K^t = \frac{\lambda}{\lambda_1} U_2 K W_2$$

for all K, where $U_2 = U_1^* U$ and $W_2 = W W_1^*$ are unitaries. Thus for all K_1, K_2 we have:

$$\frac{\lambda}{\lambda_1} U_2 K_1 K_2 W_2 = (K_1 K_2)^t = K_2^t K_1^t = \left(\frac{\lambda}{\lambda_1}\right)^2 U_2 K_2 W_2 U_2 K_1 W_2.$$

So for all K_1, K_2,

$$K_1 K_2 = \frac{\lambda}{\lambda_1} K_2 W_2 U_2 K_1.$$

Set $K_1 = e_i \otimes e_j$ and $K_2 = e_j \otimes e_i$ with $i \neq j$, where $\{e_k\}$ is an orthonormal basis for $\mathbb{C}_2$, to obtain

$$e_i \otimes e_i = \frac{\lambda}{\lambda_1} (e_j \otimes e_i) W_2 U_2 (e_i \otimes e_j).$$

Thus

$$e_i = (e_i \otimes e_i)(e_i) = \frac{\lambda}{\lambda_1} (e_j \otimes e_i) W_2 U_2 (e_i \otimes e_j)(e_i) = 0,$$

which is a contradiction. □

Invertible completely-disjoint/Arazy-disjoint preserving mappings on $M_n(\mathbb{C})$

We now procceed to investigate what happens when we consider invertible C-disjoint/A-disjoint preserving mappings on $M_n(\mathbb{C})$ for $n > 2$. It turns out that the two formulae we obtained for such mappings on $M_2(\mathbb{C})$ can also be obtained for mappings on $M_n(\mathbb{C})$ for $n > 2$.

We begin by stating a simple but very useful lemma. To prove it we shall use the fact that mappings of the form $\mathcal{S}_J \circ \mathcal{T} \circ \mathcal{F}_J$ map C-disjoint to A-disjoint matrices whenever $\mathcal{T}$ does and is defined by $\mathcal{T}(K) = K * A$ or $\mathcal{T}(K) = K^t * A$, where A is a matrix which has some specific properties.

Lemma 3.3.4. *Let A be an $n \times n$ self-adjoint matrix with unimodular entries and 1's on the diagonal. Let $\mathcal{T}$ be a linear mapping on $M_n(\mathbb{C})$ defined by one of the following two formulae*

$$(i)\ \mathcal{T}(K) = K * A,\ K \in M_n(\mathbb{C})$$

$$(ii)\ \mathcal{T}(K) = K^t * A,\ K \in M_n(\mathbb{C})$$

which maps C-disjoint to A-disjoint matrices. Then A has rank one.

Proof. Suppose $\mathcal{T}(K) = K * A$ for all $K \in M_n(\mathbb{C})$ and $A = (a_{ij})$. We will show that any 3×3 submatrix of A has rank one.

So fix $1 \leq i, j, k \leq n$, all distinct (without loss of generality we may assume that $i < j < k$). Define

$$\mathcal{T}_3 : M_3(\mathbb{C}) \to M_3(\mathbb{C})$$

$$\mathcal{T}_3 = \mathcal{S}_J \circ \mathcal{T} \circ \mathcal{F}_J,$$

where $J = \{i, j, k\}$. Then $\mathcal{T}_3$ maps C-disjoint to A-disjoint operators. This can easily be shown by noting that the Schur product of matrices preserves submatrices.

Also note that

$$(1,1,1) \otimes (1,1,1) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$(0,1,-1) \otimes (0,1,-1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

are C-disjoint self-adjoint matrices. Thus

$$\mathcal{T}_3\left((1,1,1) \otimes (1,1,1)\right) \perp_A \mathcal{T}_3\left((0,1,-1) \otimes (0,1,-1)\right).$$

Consequently, since $\mathcal{T}(K) = K * A$ for all K, we have:

$$\begin{aligned} 0 =& \mathcal{T}_3\left((1,1,1) \otimes (1,1,1)\right) \left(\mathcal{T}_3\left((0,1,-1) \otimes (0,1,-1)\right)\right)^* \\ =& \begin{pmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{jj} & -a_{jk} \\ 0 & -a_{kj} & a_{kk} \end{pmatrix}^* . \end{aligned}$$

So

$$\begin{pmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{a_{jj}} & -\overline{a_{kj}} \\ 0 & -\overline{a_{jk}} & \overline{a_{kk}} \end{pmatrix} = 0.$$

Thus the (1,3)-entry, namely $a_{ik}\overline{a_{kk}} - a_{ij}\overline{a_{kj}}$, is zero. Therefore $a_{ik}\overline{a_{kk}} = a_{ij}\overline{a_{kj}}$, which yields $a_{ik} = a_{ij}a_{jk}$.
Since

$$a_{ii} = 1 \text{ for all } i,$$

$$|a_{ij}| = 1 \text{ for all } i, j,$$

$$a_{ji} = \overline{a_{ij}} \text{ for all } i, j \text{ and}$$

$$a_{ik} = a_{ij}a_{jk} \text{ for all distinct } i, j, k,$$

we deduce that the 3×3 matrix

$$\begin{pmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{pmatrix},$$

which can now be written as

$$\begin{pmatrix} 1 & a_{ij} & a_{ik} \\ \overline{a_{ij}} & 1 & a_{jk} \\ \overline{a_{ik}} & \overline{a_{jk}} & 1 \end{pmatrix},$$

is rank one because any two columns are obviously linearly dependent. We therefore conclude that any 3×3 submatrix of A has rank one. By Lemma 3.2.7, A is a rank one matrix.
The case that $\mathcal{T}(K) = K^t * A$ is treated similarly. Alternatively, when $\mathcal{T}$ is given by (ii), consider the mapping $K \to (K^t * A)^t = K * A^t$ which maps C-disjoint to A-disjoint matrices since $\mathcal{T}$ does. Observe that A^t has exactly the same properties as A does. So (i) implies that A^t is rank one. Consequently A is also rank one.

□

Next we prove a rather technical result for a linear mapping which maps C-disjoint to A-disjoint matrices. We first work on $M_3(\mathbb{C})$ and then on $M_n(\mathbb{C})$ for an arbitrary n. As we shall see this is done so because an induction argument which will soon be used can only be started from $n = 4$. The case $n = 2$ has already been dealt with.

Lemma 3.3.5. *Let $A = (a_{ij})$ be a 3×3 self-adjoint matrix with unimodular entries and 1's on the diagonal, and fix an orthonormal basis $\{e_i\}$ for $\mathbb{C}^3$. Suppose that $\mathcal{T}$ is a linear mapping on $M_3(\mathbb{C})$ which maps C-disjoint to A-disjoint matrices and for all $i, j \in \{1, 2, 3\}$*

$$\text{either} \quad \mathcal{T}(e_i \otimes e_j) = (e_i \otimes e_j) * A \tag{3.18a}$$

$$\text{or} \quad \mathcal{T}(e_i \otimes e_j) = ((e_i \otimes e_j) * A)^t \,. \tag{3.18b}$$

Then

$$\text{either} \quad \mathcal{T}(K) = K * A, \quad K \in M_3(\mathbb{C}) \tag{3.19a}$$

$$\text{or} \quad \mathcal{T}(K) = (K * A)^t, \quad K \in M_3(\mathbb{C}) \,. \tag{3.19b}$$

Proof. We first show that if $i, j \in \{1, 2, 3\}$ and $\mathcal{T}(e_i \otimes e_j) = (e_i \otimes e_j) * A$, then $\mathcal{T}(e_j \otimes e_i) = (e_j \otimes e_i) * A$. Without loss of generality we shall prove it for $i = 1$ and $j = 2$.

So suppose that $\mathcal{T}(e_1 \otimes e_2) = (e_1 \otimes e_2) * A = \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathcal{T}(e_2 \otimes e_1) = ((e_2 \otimes e_1) * A)^t$. Then $\mathcal{T}(e_2 \otimes e_1) = \begin{pmatrix} 0 & a_{21} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \perp \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, it follows that $\begin{pmatrix} 1 & a_{12} + \overline{a_{12}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \perp_A \begin{pmatrix} 1 & -(a_{12} + \overline{a_{12}}) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This is a contradiction as can easily be seen by the algebraic description of A-disjoint matrices. Therefore $\mathcal{T}(e_2 \otimes e_1) = (e_2 \otimes e_1) * A$.

Now note that there are only three possibilities of choosing two numbers i, j, whose order is irrelevant, with $i \neq j$ from the set $\{1, 2, 3\}$, namely $\{1, 2\}$, $\{2, 3\}$ and $\{1, 3\}$. Since A has 1's on the diagonal, to prove the statement, it suffices to show that either 3.18a holds for all possibilities or 3.18b does.

We therefore need to reject the 'mixed' cases. However, in all 'mixed' cases either 3.18a holds for two out of the three possibilities of choosing i, j or 3.18b does. We will consider the case that 3.18a holds for two and 3.18b holds for one. The other case is treated similarly. We present the case that 3.18a holds for $\{1,2\}$ and $\{2,3\}$ and 3.18b holds for $\{1,3\}$. Since $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \perp \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$, we have:

$\mathcal{T}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \perp_A \mathcal{T}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$. Consequently $\begin{pmatrix} 1 & \beta & \overline{\gamma} \\ \overline{\beta} & 1 & \alpha \\ \gamma & \overline{\alpha} & 1 \end{pmatrix} \perp_A \begin{pmatrix} 1 & -\beta & -\overline{\gamma} \\ 0 & 0 & 0 \\ 0 & \overline{\alpha} & 0 \end{pmatrix}$, where we have written A as $\begin{pmatrix} 1 & \beta & \gamma \\ \overline{\beta} & 1 & \alpha \\ \overline{\gamma} & \overline{\alpha} & 1 \end{pmatrix}$ with $|\alpha| = |\beta| = |\gamma| = 1$. Thus the (1,3)-entry of the matrix $\begin{pmatrix} 1 & \beta & \overline{\gamma} \\ \overline{\beta} & 1 & \alpha \\ \gamma & \overline{\alpha} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -\overline{\beta} & 0 & \alpha \\ -\gamma & 0 & 0 \end{pmatrix}$ is zero by the algebraic description of A-disjoint matrices. Hence $\beta\alpha = 0$, which is a contradiction to $|\alpha| = |\beta| = 1$. The other cases can similarly be rejected. □

In what follows, if an orthonormal basis $\{e_i\}$ of $\mathbb{C}^n$ has been prescribed, by $e_i \otimes e_j$ we shall denote both the $n \times n$ matrix with 1 in the (i, j) position and zeros elsewhere, and its $m \times m$ submatrix $\mathcal{S}_J(e_i \otimes e_j)$, where $J \subseteq \{1, 2, \ldots, n\}$ with $m \leq n$ elements.

The symbol e_i will be used to denote both the standard vector in $\mathbb{C}^n$ which has 1 in the i-th position and zeros elsewhere, and the vector in $\mathbb{C}^2$ obtained from it by keeping its i-th and j-th entries. The meaning of these symbols will always be clear from the context.

Proposition 3.3.6. *Let A be a $n \times n$ self-adjoint matrix with unimodular entries and 1's on the diagonal, and fix an orthonormal basis $\{e_i\}$ for $\mathbb{C}^n$. Suppose that $\mathcal{T}$ is a linear mapping on $M_n(\mathbb{C})$ which maps C-disjoint to A-disjoint matrices* *and for all* $i, j \in \{1, 2, ..., n\}$

$$\textit{either} \quad \mathcal{T}(e_i \otimes e_j) = (e_i \otimes e_j) * A \tag{3.20a}$$

$$\textit{or} \quad \mathcal{T}(e_i \otimes e_j) = ((e_i \otimes e_j) * A)^t . \tag{3.20b}$$

Then

$$\textit{either} \quad \mathcal{T}(K) = K * A, \quad K \in M_n(\mathbb{C}) \tag{3.21a}$$

$$\textit{or} \quad \mathcal{T}(K) = (K * A)^t, \quad K \in M_n(\mathbb{C}) . \tag{3.21b}$$

Proof. If $n = 2$, the statement can easily been proved using the A-disjoint 2×2 matrices $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ as we did in the proof of Lemma 3.3.5. For $n = 3$, it has already been proved (see Lemma 3.3.5). Now we use induction for $n \geq 4$. So suppose that the statement holds for $n - 1$.
Set

$$J_1 = \{1, \ldots, n-1\} \tag{3.22}$$

$$J_2 = \{2, \ldots, n\} \tag{3.23}$$

$$J_3 = \{1, 2, n\} \tag{3.24}$$

and define, for $m = 1, 2$,

$$\mathcal{T}_m : M_{n-1}(\mathbb{C}) \to M_{n-1}(\mathbb{C}) \ ,$$

$$\mathcal{T}_m = \mathcal{S}_{J_m} \circ \mathcal{T} \circ \mathcal{F}_{J_m}$$

and

$$\mathcal{T}_3 : M_3(\mathbb{C}) \to M_3(\mathbb{C}) \ ,$$

$$\mathcal{T}_3 = \mathcal{S}_{J_3} \circ \mathcal{T} \circ \mathcal{F}_{J_3}.$$

Since $\mathcal{T}$ maps C-disjoint to A-disjoint matrices, so does $\mathcal{T}_m$ for $m = 1, 2, 3$ by Corollary 3.2.4. (Notice that $\mathcal{T}(e_l \otimes e_l) = e_l \otimes e_l$ for all $l \in \{1, , ..., n\}$ since A has 1's on the diagonal.)

Also for all $i, j \in \{1, , ..., n-1\}$

$$\text{either} \quad \mathcal{T}_m(e_i \otimes e_j) = (e_i \otimes e_j) * A_m, \; m = 1, 2 \tag{3.25a}$$

$$\text{or} \quad \mathcal{T}_m(e_i \otimes e_j) = ((e_i \otimes e_j) * A_m)^t, \; m = 1, 2 \tag{3.25b}$$

and for all $i, j \in \{1, 2, n\}$

$$\text{either} \quad \mathcal{T}_3(e_i \otimes e_j) = (e_i \otimes e_j) * A_3, \tag{3.26a}$$

$$\text{or} \quad \mathcal{T}_3(e_i \otimes e_j) = ((e_i \otimes e_j) * A_3)^t, \tag{3.26b}$$

where A_m is the $J_m \times J_m$ submatrix of A. Obviously A_m has exactly the same properties as A does.
By the induction hypothesis for $\mathcal{T}_1$ and $\mathcal{T}_2$ and Lemma 3.3.5 for $\mathcal{T}_3$, we obtain that, for $m = 1, 2, 3$,

$$\text{either} \quad \mathcal{T}_m(K) = K * A_m, \quad K \in M_n(\mathbb{C}) \tag{3.27a}$$

$$\text{or} \quad \mathcal{T}_m(K) = (K * A_m)^t, \quad K \in M_n(\mathbb{C}). \tag{3.27b}$$

Next we prove that

$$\text{either} \quad \mathcal{T}_1(K) = K * A_1 \tag{3.28}$$

$$\mathcal{T}_2(K) = K * A_2 \tag{3.29}$$

$$\mathcal{T}_3(K) = K * A_3 \tag{3.30}$$

$$\text{or} \quad \mathcal{T}_1(K) = (K * A_1)^t \tag{3.31}$$

$$\mathcal{T}_2(K) = (K * A_2)^t \tag{3.32}$$

$$\mathcal{T}_3(K) = (K * A_3)^t. \tag{3.33}$$

So suppose that (3.28) holds. If (3.32) was true, we would have a contradiction. To show this pick $K = e_i \otimes e_j \in M_n(\mathbb{C})$ with $i, j \neq 1, n$ and $i \neq j$.
Then

$$\mathcal{F}_{J_1}(\mathcal{S}_{J_1}(K)) = K \tag{3.34}$$

and

$$\mathcal{F}_{J_2}(\mathcal{S}_{J_2}(K)) = K. \tag{3.35}$$

Hence

$$\mathcal{T}_1\left(\mathcal{S}_{J_1}\left(K\right)\right) = \mathcal{S}_{J_1}\left(\mathcal{T}\left(\mathcal{F}_{J_1}\left(\mathcal{S}_{J_1}\left(K\right)\right)\right)\right) = \mathcal{S}_{J_1}\left(\mathcal{T}\left(K\right)\right) \tag{3.36}$$

and similarly

$$\mathcal{T}_2\left(\mathcal{S}_{J_2}\left(K\right)\right) = \mathcal{S}_{J_2}\left(\mathcal{T}\left(K\right)\right). \tag{3.37}$$

Apply $\mathcal{F}_{J_1}$ on both sides of (3.36) to get

$$\mathcal{F}_{J_1}\left(\mathcal{T}_1\left(\mathcal{S}_{J_1}\left(K\right)\right)\right) = \mathcal{F}_{J_1}\left(\mathcal{S}_{J_1}\left(\mathcal{T}(K)\right)\right)$$

and $\mathcal{F}_{J_2}$ on both sides of (3.37) to get

$$\mathcal{F}_{J_2}\left(\mathcal{T}_2\left(\mathcal{S}_{J_2}\left(K\right)\right)\right) = \mathcal{F}_{J_2}\left(\mathcal{S}_{J_2}\left(\mathcal{T}(K)\right)\right).$$

Now observe that $\left(\mathcal{T}(K)\right)_{lm} = 0$ for $(l,m) \notin J_1 \times J_1$ and $(l,m) \notin J_2 \times J_2$ since we have $i, j \neq 1, n$.
Therefore (3.3) implies that

$$\mathcal{F}_{J_1}\left(\mathcal{S}_{J_1}\left(\mathcal{T}(K)\right)\right) = \mathcal{T}(K)$$

and

$$\mathcal{F}_{J_2}\left(\mathcal{S}_{J_2}\left(\mathcal{T}(K)\right)\right) = \mathcal{T}(K).$$

Hence, on the one hand,

$$\mathcal{T}(K) = \mathcal{F}_{J_1}\left(\mathcal{T}_1\left(\mathcal{S}_{J_1}\left(K\right)\right)\right)$$

and on the other hand

$$\mathcal{T}(K) = \mathcal{F}_{J_2}\left(\mathcal{T}_2\left(\mathcal{S}_{J_2}\left(K\right)\right)\right).$$

Therefore, on the one hand, the (i,j)-entry of $\mathcal{T}(K)$ for this particular $K = e_i \otimes e_j$ would be

$$\begin{aligned}
\left(\mathcal{T}(K)\right)_{ij} &= \left(\mathcal{F}_{J_1}\left(\mathcal{T}_1\left(\mathcal{S}_{J_1}(K)\right)\right)\right)_{ij} \quad \text{(use (3.28))} \\
&= \left((e_i \otimes e_j) * A_1\right)_{ij} \\
&= (A_1)_{ij} \neq 0 \text{ (all entries of A are unimodular, } A_1 \text{ is a submatrix of A)}
\end{aligned}$$

and on the other hand it would be

$$\begin{aligned}
\left(\mathcal{T}(K)\right)_{ij} &= \left(\mathcal{F}_{J_2}\left(\mathcal{T}_2\left(\mathcal{S}_{J_2}(K)\right)\right)\right)_{ij} \quad \text{(use (3.32))} \\
&= \left(\left[(e_i \otimes e_j) * A_2\right]^t\right)_{ij} \\
&= \left((e_i \otimes e_j) * A_2\right)_{ji} \\
&= 0,
\end{aligned}$$

which is a contradiction.
In a similar way (by choosing an appropriate $K \in M_n(\mathbb{C})$) all the 'mixed' cases can be rejected.
We immediately conclude that $\mathcal{T}$ is given by

$$\text{either} \quad \mathcal{T}(K) = K * A, \text{ for all } K \in M_n(\mathbb{C}) \tag{3.38a}$$

$$\text{or} \quad \mathcal{T}(K) = (K * A)^t, \text{ for all } K \in M_n(\mathbb{C}). \tag{3.38b}$$

To see this simply evaluate $\mathcal{T}$ on $e_i \otimes e_j$ for $1 \leq i, j \leq n$ by choosing a suitable $\mathcal{T}_m$, $m = 1, 2, 3$ (recall the definition of A_m, $m = 1, 2, 3$). □

Let us now turn our attention to a certain class of invertible mappings which map C-disjoint to A-disjoint matrices. The following result shall shortly play a vital role in our attempt to describe invertible C-disjoint/A-disjoint mappings on $M_n(\mathbb{C})$. To obtain it we shall look at 2×2 blocks of $n \times n$ matrices and utilise the results obtained so far for mappings on $M_2(\mathbb{C})$.

Proposition 3.3.7. *Let $n \in \mathbb{N}, n \geq 2$, and $\{e_i\}$ be a fixed orthonormal basis for $\mathbb{C}^n$. If $\widetilde{\mathcal{T}} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is an invertible mapping which maps C-disjoint to A-disjoint matrices and there are λ_i's in $\mathbb{R}^+$, $\lambda_i \neq 0$, such that*

$$\widetilde{\mathcal{T}}(e_i \otimes e_i) = \lambda_i e_i \otimes e_i \quad \textit{for all } i,$$

then

$$\textit{either} \quad \widetilde{\mathcal{T}}(K) = \lambda(K * A), \textit{ for all } K \in M_n(\mathbb{C}) \tag{3.39a}$$

$$\textit{or} \quad \widetilde{\mathcal{T}}(K) = \lambda(K^t * A), \textit{ for all } K \in M_n(\mathbb{C}), \tag{3.39b}$$

where A is an $n \times n$ self-adjoint rank-one matrix with 1's on the diagonal (and hence has unimodular entries by Lemma 3.2.5) and $\lambda \in \mathbb{R}^+$, $\lambda \neq 0$.

Proof. Fix $i, j \in \{1, ..., n\}$ with $i \neq j$, set $J = \{i, j\}$ and define

$$\mathcal{T}_2^{(ij)} : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \ ,$$

$$\mathcal{T}_2^{(ij)} = \mathcal{S}_J \circ \widetilde{\mathcal{T}} \circ \mathcal{F}_J.$$

In the notation (ij) the order of i and j is irrelevant.
Corollary 3.2.4 then implies that $\mathcal{T}_2^{(ij)}$ is a one-to-one mapping on $M_2(\mathbb{C})$ which

maps C-disjoint to A-disjoint matrices.
Theorem 3.3.1 now yields that

$$\text{either} \quad \mathcal{T}_2^{(ij)}(K) = \lambda^{(ij)} U_2^{(ij)} K W_2^{(ij)}, \text{ for all } K \in M_2(\mathbb{C}) \tag{3.40a}$$

$$\text{or} \quad \mathcal{T}_2^{(ij)}(K) = \lambda^{(ij)} U_2^{(ij)} K^t W_2^{(ij)}, \text{ for all } K \in M_2(\mathbb{C}), \tag{3.40b}$$

where $U_2^{(ij)}, W_2^{(ij)}$ are unitary operators on $\mathbb{C}^2$ and $\lambda^{(ij)} \in \mathbb{R}^+$, $\lambda^{(ij)} \neq 0$.
Then

$$\begin{aligned} \mathcal{T}_2^{(ij)}(e_i \otimes e_i) &= \lambda^{(ij)} U_2^{(ij)} (e_i \otimes e_i) W_2^{(ij)} \\ &= \lambda^{(ij)} (U_2^{(ij)} e_i) \otimes \left((W_2^{(ij)})^* e_i \right). \end{aligned}$$

But on the other hand, by the definition of $\mathcal{T}_2^{(ij)}$ and the fact that $\widetilde{\mathcal{T}}(e_i \otimes e_i) = \lambda_i e_i \otimes e_i$ (hypothesis), we have:

$$\mathcal{T}_2^{(ij)}(e_i \otimes e_i) = \lambda_i e_i \otimes e_i.$$

Consequently

$$U_2^{(ij)}(e_i) = \mu_i^{(ij)} e_i, \tag{3.41a}$$

$$(W_2^{(ij)})^*(e_i) = \rho_i^{(ij)} e_i \tag{3.41b}$$

for some complex scalars $\mu_i^{(ij)}, \rho_i^{(ij)}$.
Since

$$\lambda^{(ij)} (U_2^{(ij)} e_i) \otimes \left((W_2^{(ij)})^* e_i \right) = \lambda_i e_i \otimes e_i$$

$$i.e \qquad \lambda^{(ij)} \mu_i^{(ij)} \overline{\rho_i^{(ij)}} e_i \otimes e_i = \lambda_i e_i \otimes e_i,$$

we deduce that

$$\lambda_i = \lambda^{(ij)} \mu_i^{(ij)} \overline{\rho_i^{(ij)}}.$$

Thus

$$\mu_i^{(ij)} \overline{\rho_i^{(ij)}} = \frac{\lambda_i}{\lambda^{(ij)}} \in \mathbb{R}^+ \setminus \{0\} \tag{3.42}$$

since $\lambda_i, \lambda^{(ij)} \in \mathbb{R}^+ \setminus \{0\}$ for all i, j.
Working in the same way with e_j instead of e_i, we conclude that there are complex scalars $\mu_j^{(ij)}, \rho_j^{(ij)}$ such that

$$U_2^{(ij)}(e_j) = \mu_j^{(ij)} e_j, \tag{3.43a}$$

$$(W_2^{(ij)})^*(e_j) = \rho_j^{(ij)} e_j \tag{3.43b}$$

and

$$\mu_j^{(ij)}\overline{\rho_j^{(ij)}} = \frac{\lambda_j}{\lambda^{(ij)}} \in \mathbb{R}^+ \setminus \{0\}. \tag{3.44}$$

Since $U_2^{(ij)}, W_2^{(ij)}$ are unitaries and diagonal with respect to the basis $\{e_i, e_j\}$ of $\mathbb{C}^2$ by 3.41a, 3.41b,3.43a and 3.43b, their entries $\mu_i^{(ij)}, \mu_j^{(ij)}, \rho_i^{(ij)}, \rho_j^{(ij)}$ are unimodular. Since $\mu_i^{(ij)}\overline{\rho_i^{(ij)}} > 0$ and $|\mu_i^{(ij)}\overline{\rho_i^{(ij)}}| = 1$, we obtain $\mu_i^{(ij)}\overline{\rho_i^{(ij)}} = 1$. Similarly $\mu_j^{(ij)}\overline{\rho_j^{(ij)}} = 1$.
Hence

$$\mu_i^{(ij)} = \rho_i^{(ij)} \tag{3.45a}$$

$$\mu_j^{(ij)} = \rho_j^{(ij)}. \tag{3.45b}$$

Equations 3.42 and 3.44 now imply that, for all $i \neq j$,

$$\lambda^{(ij)} = \lambda_i = \lambda_j.$$

It follows that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$.
By the definition of $\mathcal{T}_2^{(ij)}$ we have:

$$\mathcal{T}_2^{(ij)}(e_i \otimes e_j) = (\mathcal{S}_J \circ \widetilde{\mathcal{T}} \circ \mathcal{F}_J)(e_i \otimes e_j).$$

Thus, for $i \neq j$, either

$$\begin{aligned}(\mathcal{S}_J \circ \widetilde{\mathcal{T}} \circ \mathcal{F}_J)(e_i \otimes e_j) &= \lambda U_2^{(ij)}(e_i \otimes e_j)W_2^{(ij)} \\ &= \lambda(U_2^{(ij)}e_i) \otimes \left((W_2^{(ij)})^* e_j\right) \\ &= \lambda\mu_i^{(ij)}\overline{\rho_j^{(ij)}}(e_i \otimes e_j)\end{aligned}$$

or

$$\begin{aligned}(\mathcal{S}_J \circ \widetilde{\mathcal{T}} \circ \mathcal{F}_J)(e_i \otimes e_j) &= \lambda U_2^{(ij)}(e_j \otimes e_i)W_2^{(ij)} \\ &= \lambda(U_2^{(ij)}e_j) \otimes \left((W_2^{(ij)})^* e_i\right) \\ &= \lambda\mu_j^{(ij)}\overline{\rho_i^{(ij)}}(e_j \otimes e_i).\end{aligned}$$

Thus, for $i \neq j$, either

$$\widetilde{\mathcal{T}}\left(\mathcal{F}_J(e_i \otimes e_j)\right) = \lambda\mu_i^{(ij)}\overline{\rho_j^{(ij)}}\mathcal{F}_J(e_i \otimes e_j) = \lambda\mu_i^{(ij)}\overline{\rho_j^{(ij)}}(e_i \otimes e_j)$$

or

$$\widetilde{\mathcal{T}}\left(\mathcal{F}_J(e_i \otimes e_j)\right) = \lambda\mu_j^{(ij)}\overline{\rho_i^{(ij)}}\mathcal{F}_J(e_j \otimes e_i) = \lambda\mu_j^{(ij)}\overline{\rho_i^{(ij)}}(e_j \otimes e_i).$$

We have therefore shown that for any $i, j \in \{1, ..., n\}$ with $i \neq j$

$$\text{either} \quad \widetilde{\mathcal{T}}(e_i \otimes e_j) = \lambda \mu_i^{(ij)} \overline{\rho_j^{(ij)}} (e_i \otimes e_j)$$

$$\text{or} \quad \widetilde{\mathcal{T}}(e_i \otimes e_j) = \lambda \mu_j^{(ij)} \overline{\rho_i^{(ij)}} (e_i \otimes e_j)^t.$$

By hypothesis

$$\widetilde{\mathcal{T}}(e_i \otimes e_i) = \lambda e_i \otimes e_i.$$

Since $\mu_i^{(ij)} \overline{\rho_i^{(ij)}} = 1$, we conclude that for all $i, j \in \{1, ..., n\}$,

$$\text{either} \quad \widetilde{\mathcal{T}}(e_i \otimes e_j) = \lambda \mu_i^{(ij)} \overline{\rho_j^{(ij)}} (e_i \otimes e_j)$$

$$\text{or} \quad \widetilde{\mathcal{T}}(e_i \otimes e_j) = \lambda \mu_j^{(ij)} \overline{\rho_i^{(ij)}} (e_i \otimes e_j)^t.$$

Set $A = (\alpha_{ij}) \in M_n(\mathbb{C})$ with $\alpha_{ij} = \mu_i^{(ij)} \overline{\rho_j^{(ij)}}$ for $i, j \in \{1, ..., n\}$.
Then:

$$\alpha_{ii} = 1 \quad \text{for all } i,$$

$$\alpha_{ji} = \mu_j^{(ij)} \overline{\rho_i^{(ij)}} = \mu_j^{(ij)} \overline{\mu_i^{(ij)}} = \overline{\mu_i^{(ij)} \overline{\mu_j^{(ij)}}} = \overline{\alpha_{ij}} \quad \text{and}$$

$$|\alpha_{ij}| = |\mu_i^{(ij)}||\overline{\rho_j^{(ij)}}| = 1 \quad \text{for all } i, j.$$

Therefore, for all $i, j \in \{1, ..., n\}$,

$$\text{either} \quad \widetilde{\mathcal{T}}(e_i \otimes e_j) = \lambda \left[(e_i \otimes e_j) * A\right]$$

$$\text{or} \quad \widetilde{\mathcal{T}}(e_i \otimes e_j) = \lambda \left[(e_i \otimes e_j)^t * A\right],$$

where A is an $n \times n$ self-adjoint matrix with unimodular entries and 1's on the diagonal.
Proposition 3.3.6 now implies that

$$\text{either} \quad \widetilde{\mathcal{T}}(K) = \lambda(K * A), \text{ for all } K \in M_n(\mathbb{C})$$

$$\text{or} \quad \widetilde{\mathcal{T}}(K) = \lambda(K^t * A), \text{ for all } K \in M_n(\mathbb{C}),$$

where A is an $n \times n$ self-adjoint matrix with unimodular entries and 1's on the diagonal and $\lambda \in \mathbb{R}^+ \setminus \{0\}$. Now just apply Lemma 3.3.4 to prove that A is a rank-one matrix to complete the proof. □

A description of invertible C-disjoint/A-disjoint preserving mappings on $M_n(\mathbb{C})$

We are finally in a position to state and prove one of the main results in this chapter.

Theorem 3.3.8. *Let $\mathcal{T}$ be a linear mapping on $M_n(\mathbb{C})$. The following are then equivalent:*
(i) $\mathcal{T}$ is an invertible mapping which maps C-disjoint to A-disjoint matrices;
(ii) $\mathcal{T}$ is an invertible A-disjoint preserving mapping;
(iii) there are unique, up to a scalar, unitary operators U, W and a unique non-zero scalar $\lambda \in \mathbb{R}^+$ such that

$$\text{either} \qquad \mathcal{T}(K) = \lambda UKW \qquad\qquad (K \in M_n(\mathbb{C}))$$

$$\text{or} \qquad \mathcal{T}(K) = \lambda UK^tW \qquad\qquad (K \in M_n(\mathbb{C})).$$

Such an operator on $M_n(\mathbb{C})$ is C-disjoint preserving if and only if U is a unit scalar multiple of W^*.

Proof. (iii) $\Longrightarrow$ (ii): To prove this implication simply use the algebraic description of A-disjoint matrices and note that $\mathcal{T}$ is obviously invertible in both cases.
(ii) $\Longrightarrow$ (i): This is trivial by the fact that $K \perp L$ implies $K \perp_A L$ for all $K, L \in M_n(\mathbb{C})$.
(i) $\Longrightarrow$ (iii): The uniqueness statement is proved as in Theorem 3.3.1.
Now let $\{e_i\}$ be the standard basis for $\mathbb{C}^n$. Since $\mathcal{T}$ is invertible, we have $dim\,(ran\,(\mathcal{T}(e_i \otimes e_i))) \geq 1$ for all i.
Also $\mathcal{T}(e_i \otimes e_j) \perp_A \mathcal{T}(e_j \otimes e_j)$ for $i \neq j$ and so $ran\,(\mathcal{T}(e_i \otimes e_i)) \perp ran\,(\mathcal{T}(e_j \otimes e_j))$ for $i \neq j$.
We conclude that $dim\,(ran\,(\mathcal{T}(e_i \otimes e_i))) = 1$ for all i.
Lemma 3.2.8 now implies that there are isometries U_1, W_1 and scalars $\lambda_i \in \mathbb{R}^+ \setminus \{0\}$ such that if we define

$$\widetilde{\mathcal{T}}(\cdot) = W_1 \mathcal{T}(\cdot) U_1,$$

then

$$\widetilde{\mathcal{T}}(e_i \otimes e_i) = \lambda_i e_i \otimes e_i \quad \text{for all } i.$$

Since $\mathcal{T}$ is an invertible mapping which sends C-disjoint to A-disjoint operators, then so is the mapping $\widetilde{\mathcal{T}}$ by the algebraic description.

Proposition 3.3.7 can therefore be applied to get that

$$\text{either} \quad \widetilde{\mathcal{T}}(K) = \lambda(K * A) \qquad (K \in M_n(\mathbb{C}))$$

$$\text{or} \quad \widetilde{\mathcal{T}}(K) = \lambda(K^t * A) \qquad (K \in M_n(\mathbb{C})),$$

where A is an $n \times n$ self-adjoint rank-one matrix with unimodular entries and 1's on the diagonal and $\lambda \in \mathbb{R}^+, \lambda \neq 0$.
But $K * A = U_2 K W_2^*$ for all $K \in M_n(\mathbb{C})$ for some unitaries U_2, W_2 by Lemma 3.2.6.
Thus

$$\text{either} \quad \widetilde{\mathcal{T}}(K) = \lambda U_2 K W_2^* \qquad (K \in M_n(\mathbb{C}))$$

$$\text{or} \quad \widetilde{\mathcal{T}}(K) = \lambda U_2 K^t W_2^* \qquad (K \in M_n(\mathbb{C})).$$

We conclude that

$$\text{either} \quad \mathcal{T}(K) = \lambda U K W \qquad (K \in M_n(\mathbb{C}))$$

$$\text{or} \quad \mathcal{T}(K) = \lambda U K^t W \qquad (K \in M_n(\mathbb{C})),$$

where $\lambda \in \mathbb{R}^+, \lambda \neq 0$ and $U = W_1^* U_2, W = (U_1 W_2)^*$ are unitaries.
To finish the proof just apply Proposition 2.2.3. □

Corollary 3.3.9. *The inverse of an A-disjoint mapping on $M_n(\mathbb{C})$ (when it exists) is A-disjoint preserving.*

Remark 3.3.10. As just seen, to prove this last very interesting statement we had to first characterise invertible A-disjoint preserving mappings. It is worth noting that there does not seem to be a different, more straightforward way of proving this. It appears that what is rather hard is to obtain the two formulae.

Corollary 3.3.11. *Let $\mathcal{T}$ be a linear mapping on $M_n(\mathbb{C})$. The following are then equivalent:*
(i) $\mathcal{T}$ is an invertible C-disjoint preserving mapping;
(ii) there is a unique, up to a scalar, unitary operator W and a unique scalar $\mu \in \mathbb{C}$, $\mu \neq 0$ such that

$$\textit{either} \quad \mathcal{T}(K) = \mu W^* K W \qquad (K \in M_n(\mathbb{C}))$$

$$\textit{or} \quad \mathcal{T}(K) = \mu W^* K^t W \qquad (K \in M_n(\mathbb{C})).$$

Proof. (ii) $\Longrightarrow$ (i): To prove this implication simply use the algebraic description of C-disjoint matrices.
(i) $\Longrightarrow$ (ii): If $\mathcal{T}$ is an invertible C-disjoint preserving mapping, then it obviously maps C-disjoint operators to A-disjoint operators. Now simply apply 3.3.8. The uniqueness statement is proved as in 3.3.1. □

Corollary 3.3.12. *The inverse of a C-disjoint preserving mapping on $M_n(\mathbb{C})$ (when it exists) is C-disjoint preserving.*

Before finishing this section we state a very useful corollary.

Corollary 3.3.13. *Let $\mathcal{T}$ be an invertible mapping on $M_n(\mathbb{C})$. If $\mathcal{T}$ is C-disjoint preserving, then both $\mathcal{T}$ and $\mathcal{T}^{-1}$ are A-disjoint preserving.*

Proof. To prove this combine Corollary 3.3.11 and Theorem 3.3.8. □

It has therefore been shown that the answer to Question 2.2.7 is yes as far as invertible mappings on $M_n(\mathbb{C})$ are concerned.

However the situation is not the same when mappings on $\mathbb{M}_n(\mathbb{R})$ are considered instead, as we shall see in section 3.5.

3.4 The infinite dimensional case

We now move on to spaces $\mathcal{C}_p(\mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional Hilbert space. In this section we shall prove that a characterisation can be found for a class of invertible C-disjoint/A-disjoint preserving mappings which map rank-one projections to rank-one projections. In fact, we obtain exactly the same two forms we obtained in the finite dimensional case.

First we state a couple of results we proved in the previous section. However they are expressed here in the context of infinite dimensional Hilbert spaces in which form they shall be needed.

Proposition 3.4.1. *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and fix an orthonormal basis $\{e_i\}$ for $\mathcal{H}$. Let $n \in \mathbb{N}$ and set $J_n = \{1, 2, \ldots, n\}$. If $K \in \mathcal{C}_p(\mathcal{H})$, $1 \leq p < \infty$, is A-disjoint (C-disjoint) from $e_l \otimes e_l$ for all $l \notin J_n$, then there is a $K' \in M_n(\mathbb{C})$ such that*

$$K = \mathcal{F}_{J_n}(K')$$

with respect to the basis $\{e_i\}$.

Proof. Same as the proof of Proposition 3.2.1. □

Mappings of the form $\mathcal{S}_J \circ \mathcal{T} \circ \mathcal{F}_J$

Corollary 3.4.2. *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and fix an orthonormal basis $\{e_i\}$ for $\mathcal{H}$. Let $n \in \mathbb{N}$ and set $J_n = \{1, 2, \ldots, n\}$. Suppose $\mathcal{T}$ is a linear mapping on $\mathcal{C}_p(\mathcal{H})$, $1 \leq p < \infty$, which maps C-disjoint to A-disjoint operators and there are non-zero scalars λ_l such that $\mathcal{T}(e_l \otimes e_l) = \lambda_l e_l \otimes e_l$ for all l. Then the mapping*

$$\mathcal{S}_{J_n} \circ \mathcal{T} \circ \mathcal{F}_{J_n} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

maps C-disjoint to A-disjoint operators. If, moreover, $\mathcal{T}$ is one-to-one, then so is the mapping $\mathcal{S}_{J_n} \circ \mathcal{T} \circ \mathcal{F}_{J_n}$.

Proof. Same as the proof of Corollary 3.2.4. □

A description of a class of C-disjoint preserving mappings on $\mathcal{C}_p(\mathcal{H})$

In the proof of the following result we use the description we obtained in the finite dimensional case by considering operators on $\mathcal{H}$ as infinite dimensional matrices with respect to a fixed orthonormal basis and then approximating them by finitely supported matrices.

Theorem 3.4.3. *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Let $\mathcal{T}$ be a linear mapping on $\mathcal{C}_p(\mathcal{H})$, $1 \leq p < \infty$. The following are then equivalent:*
(i) $\mathcal{T}$ is an invertible element of $\mathcal{B}(\mathcal{C}_p(\mathcal{H}))$ such that $\mathcal{T}$ and $\mathcal{T}^{-1}$ map C-disjoint to A-disjoint operators and rank-one projections to rank-one projections;
(ii) $\mathcal{T}$ is an invertible element of $\mathcal{B}(\mathcal{C}_p(\mathcal{H}))$ such that $\mathcal{T}$ and $\mathcal{T}^{-1}$ are C-disjoint preserving mappings and map rank-one projections to rank-one projections;
(iii) there is a unique, up to a scalar, unitary operator W such that

$$\textit{either} \quad \mathcal{T}(K) = W^* K W \qquad (K \in \mathcal{C}_p(\mathcal{H}))$$

$$\textit{or} \quad \mathcal{T}(K) = W^* K^t W \qquad (K \in \mathcal{C}_p(\mathcal{H})).$$

Proof. (iii) $\Longrightarrow$ (ii): Use the algebraic description of C-disjoint operators to show that $\mathcal{T}$ is C-disjoint preserving. Obviously both formulae for $\mathcal{T}$ describe invertible mappings which map rank-one projections to rank-one projections. Their inverses are trivially C-disjoint preserving.

(ii) $\Longrightarrow$ (i): This is trivial by the fact that $K \perp L$ implies $K \perp_A L$ for all $K, L \in \mathcal{C}_p(\mathcal{H})$.
(i) $\Longrightarrow$ (iii): Fix an orthonormal basis $\{e_i\}$ for $\mathcal{H}$. Since $\mathcal{T}$ maps rank-one projections to rank-one projections, there are unimodular vectors u_i in $\mathcal{H}$ such that $\mathcal{T}(e_i \otimes e_i) = u_i \otimes u_i$ for all i. Define $U_1 \in \mathcal{B}(\mathcal{H})$ by $U_1(e_i) = u_i$ for all i. Then U_1 is, of course, an isometry. We shall show that U_1 is, in fact, a unitary operator. It suffices to show that $\{u_i\}$ is a basis of $\mathcal{H}$. Suppose $h \in (clin\{u_i\})^{\perp}$ with $||h|| = 1$. We have $\mathcal{T}^{-1}(h \otimes h) = f \otimes f$ for some $f \in \mathcal{H}$ with $||f|| = 1$, and $\mathcal{T}^{-1}(u_i \otimes u_i) = e_i \otimes e_i$ for all i. Since $h \otimes h \perp u_i \otimes u_i$ for all i, we obtain that $f \otimes f \perp_A e_i \otimes e_i$ for all i. Thus $f \perp e_i$ for all i by the algebraic description of A-disjoint operators. Hence $f = 0$. However $\mathcal{T}^{-1}(h \otimes h) = f \otimes f$ and $\mathcal{T}$ is invertible. We conclude that $h = 0$, which is a contradiction.
Define

$$\widetilde{\mathcal{T}}(\cdot) = U_1^* \mathcal{T}(\cdot) U_1.$$

Then

$$\widetilde{\mathcal{T}}(e_i \otimes e_i) = e_i \otimes e_i \quad \text{for all } i. \tag{3.46}$$

Now fix $n \in \mathbb{N}$, set $J_n = \{1, 2, \ldots, n\}$ and define

$$\mathcal{T}_n : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

$$\mathcal{T}_n = \mathcal{S}_{J_n} \circ \widetilde{\mathcal{T}} \circ \mathcal{F}_{J_n}.$$

Then $\mathcal{T}_n$ is a one-to-one mapping on $M_n(\mathbb{C})$ such that $\mathcal{T}_n(e_i \otimes e_i) = e_i \otimes e_i$ for all $i \in \{1, 2, \ldots, n\}$, and thus it maps C-disjoint to A-disjoint matrices by Corollary 3.4.2.
Hence Proposition 3.3.7 now implies that

$$\text{either} \quad \mathcal{T}_n(K_{(n)}) = \lambda_{(n)}(K_{(n)} * A_{(n)}) \qquad (K_{(n)} \in M_n(\mathbb{C}))$$

$$\text{or} \quad \mathcal{T}_n(K_{(n)}) = \lambda_{(n)}(K_{(n)}^t * A_{(n)}) \qquad (K_{(n)} \in M_n(\mathbb{C})),$$

where $A_{(n)} = (a_{ij}^{(n)})$ is an $n \times n$ self-adjoint rank-one matrix with unimodular entries and 1's on the diagonal and $\lambda_{(n)} \in \mathbb{R}^+ \setminus \{0\}$.
Set $K_{(n)} = e_1 \otimes e_1$. Then $\mathcal{T}_n(K_{(n)}) = \lambda_{(n)}((e_1 \otimes e_1) * A_{(n)}) = \lambda_{(n)} e_1 \otimes e_1$ since $A_{(n)}$ has 1's on the diagonal. On the other hand, $\mathcal{T}_n(K_{(n)}) = (\mathcal{S}_{J_n} \circ \widetilde{\mathcal{T}} \circ \mathcal{F}_{J_n})(e_1 \otimes e_1) = e_1 \otimes e_1$. Thus $\lambda_{(n)} = 1$.
Next we show that for all $K_{(n)} \in M_n(\mathbb{C})$,

$$\widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(K_{(n)})\right) = \mathcal{F}_{J_n}\left(\mathcal{T}_n(K_{(n)})\right).$$

Indeed, let $K_{(n)} \in M_n(\mathbb{C})$. Since $\mathcal{F}_{J_n}(K_{(n)}) \perp e_l \otimes e_l$ for all $l \notin J_n$ by the algebraic description, we have:

$$\widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(K_{(n)})\right) \perp_A \widetilde{\mathcal{T}}\left(e_l \otimes e_l\right).$$

But $\widetilde{\mathcal{T}}(e_l \otimes e_l) = e_l \otimes e_l$. So by Proposition 3.4.1, there is a $K'_{(n)} \in M_n(\mathbb{C})$ such that

$$\widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(K_{(n)})\right) = \mathcal{F}_{J_n}(K'_{(n)}).$$

Thus

$$\mathcal{S}_{J_n}\left(\widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(K_{(n)})\right)\right) = K'_{(n)}.$$

So

$$\mathcal{T}_n(K_{(n)}) = K'_{(n)}.$$

Hence

$$\widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(K_{(n)})\right) = \mathcal{F}_{J_n}(K'_{(n)}) = \mathcal{F}_{J_n}\left(\mathcal{T}_n(K_{(n)})\right).$$

We deduce that

$$\text{either} \quad \widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(K_{(n)})\right) = \mathcal{F}_{J_n}(K_{(n)}) * \mathcal{F}_{J_n}(A_{(n)}) \quad (K_{(n)} \in M_n(\mathbb{C})) \tag{3.47}$$

$$\text{or} \quad \widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(K_{(n)})\right) = \mathcal{F}_{J_n}\left((K_{(n)})^t\right) * \mathcal{F}_{J_n}(A_{(n)}) \quad (K_{(n)} \in M_n(\mathbb{C})). \tag{3.48}$$

Similarly, using $\mathcal{T}'_n$ when $n' \geq n$, we deduce that

$$\text{either} \quad \widetilde{\mathcal{T}}\left(\mathcal{F}_{J_{n'}}(K_{(n')})\right) = \mathcal{F}_{J_{n'}}(K_{(n')}) * \mathcal{F}_{J_{n'}}(A_{(n')}) \quad (K_{(n')} \in M_{n'}(\mathbb{C})) \tag{3.49}$$

$$\text{or} \quad \widetilde{\mathcal{T}}\left(\mathcal{F}_{J_{n'}}(K_{(n')})\right) = \mathcal{F}_{J_{n'}}\left((K_{(n')})^t\right) * \mathcal{F}_{J_{n'}}(A_{(n')}) \quad (K_{(n')} \in M_{n'}(\mathbb{C})), \tag{3.50}$$

where $A_{(n')} = (a_{ij}^{(n')})$ (as $A_{(n)}$) is an n'xn' self-adjoint rank-one matrix with unimodular entries and 1's on the diagonal.

Fix $i, j \in \{1, ..., n\}$ with $i > j$ and $n, n' \in \mathbb{N}$ with $n' \geq n$. Set $K_{(n)} = e_i \otimes e_j \in M_n(\mathbb{C})$ and $K_{(n')} = \begin{pmatrix} K_{(n)} & 0 \\ 0 & 0 \end{pmatrix} \in M_{n'}(\mathbb{C})$. Suppose (3.47) holds for n. Then $\widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(e_i \otimes e_j)\right) = a_{ij}^{(n)}(e_i \otimes e_j)$. Thus the $(i,j)-$entry of $\widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(e_i \otimes e_j)\right)$ is $a_{ij}^{(n)}$. Suppose that $\widetilde{\mathcal{T}}\left(\mathcal{F}_{J_{n'}}(K_{(n')})\right) = \mathcal{F}_{J_{n'}}\left((K_{(n')})^t\right) * \mathcal{F}_{J_{n'}}(A_{(n')})$ by (3.50). Since $\mathcal{F}_{J_n}(K_{(n)}) = \mathcal{F}_{J_{n'}}(K_{(n')}) \in \mathcal{C}_p(\mathcal{H})$, we would then have that $\widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(e_i \otimes e_j)\right) = a_{ji}^{(n')}(e_j \otimes e_i)$ and so the $(i,j)-$entry of $\widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(e_i \otimes e_j)\right)$ would also be equal to 0, which is a contradiction since $|a_{ij}^{(n)}| = 1$. We conclude that if (3.47) holds for n, then (3.49) holds for n'. Similarly, when (3.48) holds for n, then (3.50) holds for n'. Moreover, using the same $K_{(n)}$ and $K_{(n')}$, we can deduce that the $(i,j)-$entry of $A_{(n')}$ is exactly the same as the $(i,j)-$entry of $A_{(n)}$ whenever $1 \leq i,j \leq n$ and $n \leq n'$ for all $n, n' \in \mathbb{N}$ by (3.47) and (3.49) and the fact that both matrices have 1's on their diagonals.

Now define an infinite dimensional matrix $A = (a_{ij})$ as follows: For any $1 \leq i,j \leq n$, set a_{ij} to be the $(i,j)-$entry of $A_{(n)}$, where $n = max\{i,j\}$.

It follows trivially that

$$\text{either} \quad \widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(K_{(n)})\right) = \mathcal{F}_{J_n}(K_{(n)}) * A \ \text{ for all } \ n \in \mathbb{N}, \quad (K_{(n)} \in M_n(\mathbb{C}))$$

$$\text{or} \quad \widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(K_{(n)})\right) = \mathcal{F}_{J_n}\left((K_{(n)})^t\right) * A \ \text{ for all } \ n \in \mathbb{N} \ \ (K_{(n)} \in M_n(\mathbb{C})),$$

where $J_n = \{1, ..., n\}$, A is obviously an infinite dimensional self-adjoint matrix with unimodular entries and 1's on the diagonal by construction.

The matrix A is also rank-one.

Indeed, to prove this consider any two columns k and l, say $k < l$, of A. Then

$$a_{lk}\begin{pmatrix} a_{1l} \\ a_{2l} \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \end{pmatrix}$$

since for all $m \in \mathbb{N}$, $a_{ml}a_{lk} = a_{mk}$.

To see this simply recall that for any $n \in \mathbb{N}$, $A_{(n)}$ is rank-one and has 1's on the diagonal and thus $a_{lk}\begin{pmatrix} a_{1l} \\ \vdots \\ a_{nl} \end{pmatrix} = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix}$.

Since A has rank one and unimodular entries, Lemma 3.2.6 implies that there are unitaries U_2, W_2 such that

$$U_2 K W_2^* = K * A \quad \text{for all K.}$$

Thus

$$\text{either} \quad \widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(K_{(n)})\right) = U_2\mathcal{F}_{J_n}(K_{(n)})W_2^* \ \text{ for all } \ n \in \mathbb{N}, \qquad (K_{(n)} \in M_n(\mathbb{C}))$$

$$\text{or} \quad \widetilde{\mathcal{T}}\left(\mathcal{F}_{J_n}(K_{(n)})\right) = U_2\mathcal{F}_{J_n}(K_{(n)}^t)W_2^* \ \text{ for all } \ n \in \mathbb{N} \qquad (K_{(n)} \in M_n(\mathbb{C})).$$

Therefore

$$\text{either} \quad \mathcal{T}\left(\mathcal{F}_{J_n}(K_{(n)})\right) = U\mathcal{F}_{J_n}(K_{(n)})W \ \text{ for all } \ n \in \mathbb{N}, \qquad (K_{(n)} \in M_n(\mathbb{C}))$$

$$\text{or} \quad \mathcal{T}\left(\mathcal{F}_{J_n}(K_{(n)})\right) = U\mathcal{F}_{J_n}(K_{(n)}^t)W \ \text{ for all } \ n \in \mathbb{N} \qquad (K_{(n)} \in M_n(\mathbb{C})),$$

where $U = U_1U_2$, $W = (U_1W_2)^*$ are unitaries.
Now let $K \in \mathcal{C}_p(\mathcal{H})$ such that $K = (k_{ij})$ with respect to the orthonormal basis $\{e_i\}$. Let P_n be the projection of $\mathcal{H}$ onto $clin\{e_1, ..., e_n\}$.
Then for any $n \in \mathbb{N}$, $K_n = P_nKP_n$ is a finitely supported matrix with respect to $\{e_i\}$ and $K_n \to K$ in $\mathcal{C}_p(\mathcal{H})$.
i.e

$$\mathcal{F}_{J_n}\left((k_{ij})_{i,j=1}^n\right) \to K \quad \text{in } \mathcal{C}_p(\mathcal{H}).$$

Since $\mathcal{T} \in \mathcal{B}(\mathcal{C}_p(\mathcal{H}))$ and the mappings

$$K \to UKW$$

$$K \to UK^tW$$

are obviously bounded, we conclude that

$$\text{either} \quad \mathcal{T}(K) = UKW \qquad (K \in \mathcal{C}_p(\mathcal{H}))$$

$$\text{or} \quad \mathcal{T}(K) = UK^tW \qquad (K \in \mathcal{C}_p(\mathcal{H})).$$

Since $\mathcal{T}$ maps rank-one projections to rank-one projections, for any rank-one projection P, we have $UPW = W^*PU^*$ and so

$$PW = U^*W^*PU^*. \tag{3.51}$$

Moreover, $UPWUPW = UPW$ and so

$$PWUP = P. \tag{3.52}$$

Combining equations (3.51) and (3.52) we have $U^*W^*P = P$. Set $P = e_i \otimes e_i$ to obtain $U = W^*$. The uniqueness statement is proved as in Theorem 3.3.1. □

Coming back to Question 2.2.7, we conclude that, in some specific cases, a C-disjoint preserving mapping on $\mathcal{C}_p(\mathcal{H})$ is also A-disjoint preserving when $\mathcal{H}$ is an infinite dimensional Hilbert space.

Corollary 3.4.4. *If $\mathcal{T}$ is an invertible element of $\mathcal{B}(\mathcal{C}_p(\mathcal{H}))$ such that $\mathcal{T}$ and $\mathcal{T}^{-1}$ are C-disjoint preserving mappings and map rank-one projections to rank-one projections, then both $\mathcal{T}$ and $\mathcal{T}^{-1}$ are A-disjoint preserving.*

Proof. This follows immediately from Theorem 3.4.3. □

The more general case of an invertible mapping $\mathcal{T}$ on $\mathcal{C}_p(\mathcal{H})$ whose inverse is not necessarily C-disjoint preserving appears to be more complicated and will not be dealt with in this thesis.

3.5 Real Hilbert spaces

As we have already mentioned it **does** matter if we work over real or complex Hilbert spaces. The following demonstrates that the result stated in Corollary 3.3.13 does not necessarily hold for C-disjoint preserving mappings on $\mathcal{C}_p(\mathcal{H})$ when $\mathcal{H}$ is a real Hilbert space.

Proposition 3.5.1. *The inverse of an invertible C-disjoint preserving mapping $\mathcal{T}$ on $\mathbb{M}_n(\mathbb{R})$ need not be A-disjoint preserving.*

Proof. Define a linear mapping $\mathcal{T}$ on $\mathbb{M}_2(\mathbb{R})$ by

$$\mathcal{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \frac{1}{2}(b+c) \\ c & d \end{pmatrix}$$

Then $\mathcal{T}$ is C-disjoint preserving. To see this let $A, B \in \mathbb{M}_2(\mathbb{R})$ with $A \perp B$. Note that if either A or B is a rank-two matrix and $A \perp B$, then $\mathcal{T}(A) \perp \mathcal{T}(B)$ trivially. Let A, B be two self-adjoint rank-one matrices with $A \perp B$. Then $\mathcal{T}(A) = A \perp B = \mathcal{T}(B)$. Since any arbitrary rank-one matrix in $\mathbb{M}_2(\mathbb{R})$ is a scalar multiple of a self-adjoint rank-one matrix, we can easily conclude that $\mathcal{T}$ is C-disjoint preserving by the linearity of $\mathcal{T}$. Now note that $\mathcal{T}^{-1}$ exists and is given by

$$\mathcal{T}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 2b-c \\ c & d \end{pmatrix}.$$

However, $\mathcal{T}^{-1}$ is not A-disjoint preserving. Indeed, let $A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then $A \perp_A B$, but $\mathcal{T}^{-1}(A) = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$ is not A-disjoint from $\mathcal{T}^{-1}(B) = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ as can be seen by the algebraic description of A-disjoint matrices. □

Therefore Theorem 3.3.8 **cannot** be proved by using real matrices only. Consequently making use of **complex** C-disjoint/A-disjoint matrices in its proof was of vital importance.

3.6 Isometries on $\mathcal{C}_p$ spaces

Before finishing this chapter it is worth noting that surjective isometries on $\mathcal{C}_p(\mathcal{H})$, for $1 < p < \infty$ and $p \neq 2$, have exactly the same form as all invertible A-disjoint preserving mappings on $M_n(\mathbb{C})$.

Theorem 3.6.1. *[4] Let $1 < p < \infty$, $p \neq 2$ and let $\mathcal{T} : \mathcal{C}_p(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H})$ be a surjective isometry. Then*

$$\text{either} \quad \mathcal{T}(A) = UAV \tag{3.53a}$$

$$\text{or} \quad \mathcal{T}(A) = UA^tV, \tag{3.53b}$$

where U, V are unitary operators and A^t is the transpose of A with respect to some fixed orthonormal basis of the separable Hilbert space $\mathcal{H}$.

Corollary 3.6.2. *If $1 < p < \infty$, $p \neq 2$, then all surjective isometries on $\mathcal{C}_p$ are A-disjoint preserving.*

Let us now see, using the descriptions we have obtained for invertible A-disjoint/C-disjoint preserving mappings, how these kind of mappings and isometries can be better related.

The finite dimensional case

Theorem 3.6.3. *Let $\mathcal{T}$ be a linear mapping on $M_n(\mathbb{C})$. The following are then equivalent:*
(i) $\mathcal{T}$ is an invertible A-disjoint preserving mapping;
(ii) $\mathcal{T}$ is isometric, up to a positive scalar, relative to the $\mathcal{C}_p$ norm for some $p \neq 2$;
(iii) $\mathcal{T}$ is isometric, up to a positive scalar, relative to the $\mathcal{C}_p$ norm for all $1 < p < \infty$, $p \neq 2$.

Proof. (iii) $\Longrightarrow$ (ii): This implication is obvious.
(ii) $\Longrightarrow$ (i): Simply use the norm description for A-disjoint matrices (see Proposition 2.1.15).
(i) $\Longrightarrow$ (iii): Apply Theorem 3.3.8 and Corollary 3.3.11. □

The infinite dimensional case

Theorem 3.6.4. *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and fix an orthonormal basis $\{e_i\}$ for $\mathcal{H}$. Let $1 \leq p < \infty$, $p \neq 2$ and let $\mathcal{T}$ be an invertible linear mapping on $\mathcal{C}_p(\mathcal{H})$. Suppose both $\mathcal{T}$ and $\mathcal{T}^{-1}$ map rank-one projections to rank-one projections. The following are then equivalent:*
(i) $\mathcal{T}$ is an element of $\mathcal{B}(\mathcal{C}_p(\mathcal{H}))$ such that $\mathcal{T}$ and $\mathcal{T}^{-1}$ map C-disjoint to A-disjoint operators;
(ii) $\mathcal{T}$ is an element of $\mathcal{B}(\mathcal{C}_p(\mathcal{H}))$ such that $\mathcal{T}$ and $\mathcal{T}^{-1}$ are C-disjoint preserving mappings;
(iii) $\mathcal{T}$ and $\mathcal{T}^{-1}$ are isometric relative to the $\mathcal{C}_p$ norm.

Proof. (iii) $\Longrightarrow$ (ii): Simply use the norm description for C-disjoint matrices (see Proposition 2.1.15).
(ii) $\Longrightarrow$ (i): This is trivial since $K \perp L$ implies $K \perp_A L$ for all $K, L \in \mathcal{C}_p(\mathcal{H})$.
(i) $\Longrightarrow$ (iii): Apply Theorem 3.4.3. □

For more on isometries on $\mathcal{C}_p$ spaces see [5].

3.7 Further results

In the last section of this chapter we obtain a characterisation for non-zero A-disjoint preserving mappings on $M_2(\mathbb{C})$ by simply using a few pairs of A-disjoint 2×2 matrices.

Non-zero A-disjoint preserving mapping on $M_2(\mathbb{C})$

Proposition 3.7.1. *Let $\mathcal{T}$ be a linear mapping on $M_2(\mathbb{C})$. The following are then equivalent:*
(i) $\mathcal{T}$ is a non-zero A-disjoint preserving mapping;
(ii) there are unique unitary operators U, W and a unique non-zero scalar $\lambda \in \mathbb{R}^+$ such that

$$\text{either} \quad \mathcal{T}(K) = \lambda UKW \qquad (K \in M_2(\mathbb{C}))$$

$$\text{or} \quad \mathcal{T}(K) = \lambda UK^tW \qquad (K \in M_2(\mathbb{C})).$$

Such an operator on $M_2(\mathbb{C})$ is C-disjoint preserving if and only if U is a unit scalar multiple of W^*.

Proof. (i) $\Longrightarrow$ (ii): Let $\{e_i\}$ be the standard basis for $\mathbb{C}^2$. We first show that $dim\,(ran\,(\mathcal{T}(e_i \otimes e_i))) = 1$ for $i = 1, 2$.
Note that $e_1 \otimes e_2 \perp_A e_2 \otimes e_1$ and so

$$\mathcal{T}(e_1 \otimes e_2) \perp_A \mathcal{T}(e_2 \otimes e_1). \tag{3.54}$$

Case 1: $dim\,(ran\,(\mathcal{T}(e_1 \otimes e_1))) = dim\,(ran\,(\mathcal{T}(e_2 \otimes e_2))) = 0$.
Since $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \perp_A \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, we conclude that

$$\mathcal{T}(e_1 \otimes e_2) + \mathcal{T}(e_2 \otimes e_1) \perp_A -(\mathcal{T}(e_1 \otimes e_2) + \mathcal{T}(e_2 \otimes e_1))$$

and thus

$$\mathcal{T}(e_1 \otimes e_2) + \mathcal{T}(e_2 \otimes e_1) = 0. \tag{3.55}$$

Equations 3.54 and 3.55 now imply that $\mathcal{T}(e_1 \otimes e_2) = \mathcal{T}(e_2 \otimes e_1) = 0$. But obviously $\mathcal{T}(e_1 \otimes e_1) = \mathcal{T}(e_2 \otimes e_2) = 0$. Hence $\mathcal{T} = 0$ which is a contradiction.
Case 2: $dim\,(ran\,(\mathcal{T}(e_1 \otimes e_1))) = 0$, $dim\,(ran\,(\mathcal{T}(e_2 \otimes e_2))) = 1$ or 2.
The matrices $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ are A-disjoint and so

$$\mathcal{T}(e_1 \otimes e_2) \perp_A \mathcal{T}(e_2 \otimes e_1) + \mathcal{T}(e_2 \otimes e_2). \tag{3.56}$$

Similarly using the matrices $\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ we obtain:

$$\mathcal{T}(e_2 \otimes e_1) \perp_A \mathcal{T}(e_1 \otimes e_2) + \mathcal{T}(e_2 \otimes e_2). \tag{3.57}$$

Equations 3.54 and 3.56 yield $\mathcal{T}(e_1 \otimes e_2) \perp_A \mathcal{T}(e_2 \otimes e_2)$ and equations 3.54 and 3.57 yield $\mathcal{T}(e_2 \otimes e_1) \perp_A \mathcal{T}(e_2 \otimes e_2)$.
So $\mathcal{T}(e_1 \otimes e_2), \mathcal{T}(e_2 \otimes e_1)$ and $\mathcal{T}(e_2 \otimes e_2)$ are mutually A-disjoint. Thus the subspaces $ran\,(\mathcal{T}(e_1 \otimes e_2))\,, ran\,(\mathcal{T}(e_2 \otimes e_1))$ and $ran\,(\mathcal{T}(e_2 \otimes e_2))$ of $\mathbb{C}^2$ are mutually orthogonal. Since $\mathcal{T}(e_2 \otimes e_2) \neq 0$, we conclude that $\mathcal{T}(e_1 \otimes e_2) = 0$ or $\mathcal{T}(e_2 \otimes e_1) = 0$.
So suppose that $\mathcal{T}(e_2 \otimes e_1) = 0$. Then

$$\mathcal{T}(e_1 \otimes e_2) \perp_A \mathcal{T}(e_2 \otimes e_2) \tag{3.58}$$

by 3.56. But $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \perp_A \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Thus

$$\mathcal{T}(e_2 \otimes e_2) + \mathcal{T}(e_1 \otimes e_2) \perp_A \mathcal{T}(e_2 \otimes e_2) - \mathcal{T}(e_1 \otimes e_2). \tag{3.59}$$

Combining 3.58 and 3.59 and using the algebraic description we obtain

$$\mathcal{T}(e_2 \otimes e_2)\,(\mathcal{T}(e_2 \otimes e_2))^* = \mathcal{T}(e_1 \otimes e_2)\,(\mathcal{T}(e_1 \otimes e_2))^*.$$

Since $\mathcal{T}(e_2 \otimes e_2) \neq 0$, we have $\mathcal{T}(e_1 \otimes e_2) \neq 0$. If $dim\,(ran\,(\mathcal{T}(e_2 \otimes e_2))) = 2$, then 3.58 would yield $\mathcal{T}(e_1 \otimes e_2) = 0$. If $dim\,(ran\,(\mathcal{T}(e_2 \otimes e_2))) = 1$, then in view of 3.58 again and since $\mathcal{T}(e_1 \otimes e_2) \neq 0$, we have $dim\,(ran\,(\mathcal{T}(e_1 \otimes e_2) + \mathcal{T}(e_2 \otimes e_2))) = dim\,(ran\,(\mathcal{T}(e_1 \otimes e_2))) + dim\,(ran\,(\mathcal{T}(e_2 \otimes e_2))) = 2$. So $\mathcal{T}(e_1 \otimes e_2) + \mathcal{T}(e_2 \otimes e_2)$ is invertible. Thus 3.59 yields that $\mathcal{T}(e_2 \otimes e_2) = \mathcal{T}(e_1 \otimes e_2)$ $(\neq 0)$ which is a contradiction by 3.58.
Similarly we show that $\mathcal{T}(e_1 \otimes e_2) = 0$ leads to a contradiction.
Case 3: $dim\,(ran\,(\mathcal{T}(e_2 \otimes e_2))) = 0$, $dim\,(ran\,(\mathcal{T}(e_1 \otimes e_1))) = 1$ or 2.
As in case 2, using the same pairs of A-disjoint matrices we get a contradiction.
There are unitaries U_1, W_1 such that $U_1\mathcal{T}(e_1 \otimes e_1)W_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}$ for some non-zero complex scalar λ_1.
Since $\mathcal{T}(e_2 \otimes e_2) \perp_A \mathcal{T}(e_1 \otimes e_1)$, we conclude that $U_1\mathcal{T}(e_2 \otimes e_2)W_1$ should be of the form $\begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ for some other non-zero complex scalar λ_2.
Thus if we define

$$\widetilde{\mathcal{T}}(\cdot) = U_1\mathcal{T}(\cdot)W_1,$$

then

$$\widetilde{\mathcal{T}}(e_1 \otimes e_1) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\widetilde{\mathcal{T}}(e_2 \otimes e_2) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Set $\widetilde{\mathcal{T}}(e_1 \otimes e_2) = X$ and $\widetilde{\mathcal{T}}(e_2 \otimes e_1) = Y$. In the same way, as in case 2, using the following pairs of A-disjoint matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

we show that $\mathcal{T}(e_1 \otimes e_2), \mathcal{T}(e_2 \otimes e_1) \neq 0$. Therefore X, Y are also non-zero matrices. Since 3.54 holds, we have $ranX \perp ranY$ with $X, Y \neq 0$. So $X = a \otimes b$ and $Y = c \otimes d$ for some non-zero $a, b, c, d \in \mathbb{C}^2$. But $XY^* = Y^*X = 0$ and thus $(a, c) = (b, d) = 0$. If $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then

$$X = \begin{pmatrix} a_1\overline{b_1} & a_1\overline{b_2} \\ a_2\overline{b_1} & a_2\overline{b_2} \end{pmatrix}.$$

But $(a, c) = (b, d) = 0$ implies that $d = \mu_2 \begin{pmatrix} \overline{b_2} \\ -\overline{b_1} \end{pmatrix}$ and $c = \mu_1 \begin{pmatrix} \overline{a_2} \\ -\overline{a_1} \end{pmatrix}$ for some non-zero scalars μ_1, μ_2. Thus

$$Y = \mu \begin{pmatrix} \overline{a_2}b_2 & -\overline{a_2}b_1 \\ -\overline{a_1}b_2 & \overline{a_1}b_1 \end{pmatrix}$$

for some scalar $\mu \in \mathbb{C}$, $\mu \neq 0$.

We have $\widetilde{\mathcal{T}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \perp_A \widetilde{\mathcal{T}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$. Thus $\left(X + \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}\right)\left(Y^* + \begin{pmatrix} 0 & 0 \\ 0 & -\overline{\lambda_2} \end{pmatrix}\right) = 0,$ which implies $\begin{pmatrix} \overline{\mu}\lambda_1 a_2\overline{b_2} & -\overline{\mu}\lambda_1 a_1\overline{b_2} - \overline{\lambda_2}a_1\overline{b_2} \\ 0 & -\overline{\lambda_2}a_2\overline{b_2} \end{pmatrix} = 0$. Since $\lambda_1, \lambda_2, \mu \neq 0$, we have $a_2b_2 = 0$ and $(\overline{\mu}\lambda_1 + \overline{\lambda_2})a_1\overline{b_2} = 0$.

Also $\left(Y^* + \begin{pmatrix} 0 & 0 \\ 0 & -\overline{\lambda_2} \end{pmatrix}\right)\left(X + \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0$. Thus $\begin{pmatrix} \overline{\mu}\lambda_1 a_2 \overline{b_2} & 0 \\ -\overline{\mu}\lambda_1 a_2 \overline{b_1} - \overline{\lambda_2} a_2 \overline{b_1} & -\overline{\lambda_2} a_2 \overline{b_2} \end{pmatrix} = 0$. Therefore $(\overline{\mu}\lambda_1 + \overline{\lambda_2}) a_2 \overline{b_1} = 0$ (and $a_2 b_2 = 0$).

Similarly, since $\widetilde{\mathcal{T}}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \perp_A \widetilde{\mathcal{T}}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, we have $\left(Y + \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}\right)\left(X^* + \begin{pmatrix} 0 & 0 \\ 0 & -\overline{\lambda_2} \end{pmatrix}\right) = 0$, which implies $\begin{pmatrix} \lambda_1 \overline{a_1} b_1 & \lambda_1 \overline{a_2} b_1 + \mu \overline{\lambda_2}\, \overline{a_2} b_1 \\ 0 & -\mu \overline{\lambda_2}\, \overline{a_1} b_1 \end{pmatrix} = 0$. We deduce that $a_1 b_1 = 0$ and $(\lambda_1 + \mu\overline{\lambda_2})\overline{a_2} b_1 = 0$.

Also $\left(X^* + \begin{pmatrix} 0 & 0 \\ 0 & -\overline{\lambda_2} \end{pmatrix}\right)\left(Y + \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0$.Thus $\begin{pmatrix} \lambda_1 \overline{a_1} b_1 & 0 \\ \lambda_1 \overline{a_1} b_2 + \mu \overline{\lambda_2}\, \overline{a_1} b_2 & -\mu \overline{\lambda_2}\, \overline{a_1} b_1 \end{pmatrix} = 0$. Therefore $(\lambda_1 + \mu\overline{\lambda_2})\overline{a_1} b_2 = 0$ (and $a_1 b_1 = 0$).

We conclude that

$$a_1 b_1 = 0 \tag{3.60}$$

$$a_2 b_2 = 0 \tag{3.61}$$

$$(\overline{\mu}\lambda_1 + \overline{\lambda_2}) a_2 \overline{b_1} = 0 \tag{3.62}$$

$$(\lambda_1 + \mu\overline{\lambda_2}) \overline{a_2} b_1 = 0 \tag{3.63}$$

$$(\lambda_1 + \mu\overline{\lambda_2}) \overline{a_1} b_2 = 0 \tag{3.64}$$

$$(\overline{\mu}\lambda_1 + \overline{\lambda_2}) a_1 \overline{b_2} = 0 \tag{3.65}$$

and thus there are only two cases:

Case 1: $a_1 = 0$.

Then $a_2 \neq 0$ because $a \neq 0$. Thus $b_2 = 0$ by (3.61) and so $b_1 \neq 0$ because $b \neq 0$. Since both a_2 and b_1 are non-zero, we have $\mu = \frac{-\lambda_1}{\overline{\lambda_2}} = \frac{-\lambda_2}{\overline{\lambda_1}}$ by (3.63) and (3.62). We therefore have

$$\widetilde{\mathcal{T}}(e_1 \otimes e_1) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\widetilde{\mathcal{T}}(e_2 \otimes e_2) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\widetilde{\mathcal{T}}(e_1 \otimes e_2) = \begin{pmatrix} 0 & 0 \\ a_2\overline{b_1} & 0 \end{pmatrix}$$

and

$$\widetilde{\mathcal{T}}(e_2 \otimes e_1) = \begin{pmatrix} 0 & -\mu\overline{a_2}b_1 \\ 0 & 0 \end{pmatrix}$$

Hence $\widetilde{\mathcal{T}}(K) = K^t * \widetilde{A}$ for all K, where $\widetilde{A} = \begin{pmatrix} \lambda_1 & \frac{\lambda_2}{\lambda_1}\overline{a_2}b_1 \\ a_2\overline{b_1} & \lambda_2 \end{pmatrix}$ with $|\lambda_1| = |\lambda_2|$.

Since $\widetilde{\mathcal{T}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \perp_A \widetilde{\mathcal{T}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, we have $\begin{pmatrix} \lambda_1 & \frac{\lambda_2}{\lambda_1}\overline{a_2}b_1 \\ a_2\overline{b_1} & \lambda_2 \end{pmatrix} \perp_A \begin{pmatrix} \lambda_1 & -\frac{\lambda_2}{\lambda_1}\overline{a_2}b_1 \\ -a_2\overline{b_1} & \lambda_2 \end{pmatrix}$.

Thus $\begin{pmatrix} \lambda_1 & \frac{\lambda_2}{\lambda_1}\overline{a_2}b_1 \\ a_2\overline{b_1} & \lambda_2 \end{pmatrix}\begin{pmatrix} \overline{\lambda_1} & -\overline{a_2}b_1 \\ -\overline{\frac{\lambda_2}{\lambda_1}}a_2\overline{b_1} & \overline{\lambda_2} \end{pmatrix} = 0$. Hence $|\lambda_2| = |a_2||b_1| = |\lambda_1|$.

We deduce that all the entries of the matrix $\widetilde{A}$ have the same modulus. Moreover, $\widetilde{A}$ has determinant zero. Thus it is a rank-one matrix. Write $\widetilde{A} = \lambda A$ such that $\lambda \in \mathbb{R}^+ \setminus \{0\}$ and A has unimodular entries.

By Lemma 3.2.6, there are unitary operators U_2, W_2 such that $U_2 K^t W_2^* = K^t * A$ for all $K \in M_2(\mathbb{C})$. Thus $\widetilde{\mathcal{T}}(K) = \lambda U_2 K^t W_2^*$.

Hence for all $K \in M_2(\mathbb{C})$,

$$\mathcal{T}(K) = \lambda U K^t W,$$

where $U = U_1^* U_2$, $W = W_2^* W_1^*$ are unitary operators and $\lambda \in \mathbb{R}^+ \setminus \{0\}$.

Case 2: $a_1 \neq 0$.

Then $b_1 = 0$ by (3.60). Thus $b_2 \neq 0$ since $b \neq 0$. Hence $a_2 = 0$ by (3.61). Since both a_1 and b_2 are non-zero, we have $\mu = \frac{-\lambda_1}{\lambda_2} = \frac{-\lambda_2}{\lambda_1}$ by (3.64) and (3.65). We therefore have

$$\widetilde{\mathcal{T}}(e_1 \otimes e_1) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\widetilde{\mathcal{T}}(e_2 \otimes e_2) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\widetilde{\mathcal{T}}(e_1 \otimes e_2) = \begin{pmatrix} 0 & a_1\overline{b_2} \\ 0 & 0 \end{pmatrix}$$

and

$$\widetilde{\mathcal{T}}(e_2 \otimes e_1) = \begin{pmatrix} 0 & 0 \\ -\mu\overline{a_1}b_2 & 0 \end{pmatrix}$$

Hence $\widetilde{\mathcal{T}}(K) = K * \widetilde{A}$ for all K, where $\widetilde{A} = \begin{pmatrix} \lambda_1 & a_1\overline{b_2} \\ -\frac{\lambda_2}{\lambda_1}\overline{a_1}b_2 & \lambda_2 \end{pmatrix}$ with $|\lambda_1| = |\lambda_2|$.

Using again that $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \perp_A \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and working as in the previous case, we conclude that for all $K \in M_2(\mathbb{C})$,

$$\mathcal{T}(K) = \lambda UKW,$$

where U, W are unitary operators and $\lambda \in \mathbb{R}^+ \setminus \{0\}$.

(ii) $\Longrightarrow$ (i): This is trivial by the algebraic description of A-disjoint matrices.

To finish the proof simply apply Proposition 2.2.3. □

As a result we conclude the following.

Corollary 3.7.2. *A-disjoint preserving mappings on $M_2(\mathbb{C})$ are either invertible or identically zero.*

Non-zero A-disjoint preserving mapping on $M_n(\mathbb{C})$

We finish this chapter by stating a conjecture concerning non-zero A-disjoint preserving mapping on $M_n(\mathbb{C})$.

Conjecture 3.7.3. *Let $\mathcal{T}$ be a linear mapping on $M_n(\mathbb{C})$. The following are then equivalent:*

(i) $\mathcal{T}$ is a non-zero A-disjoint preserving mapping;

(ii) there are unique unitary operators U, W *and a unique scalar* $\lambda \in \mathbb{R}^+ \setminus \{0\}$ *such that*

$$\text{either} \quad \mathcal{T}(K) = \lambda UKW \qquad (K \in M_n(\mathbb{C}))$$

$$\text{or} \quad \mathcal{T}(K) = \lambda UK^tW \qquad (K \in M_n(\mathbb{C})).$$

Such an operator on $M_n(\mathbb{C})$ *is C-disjoint preserving if and only if* U *is a unit scalar multiple of* W^*.

Chapter 4

Complexification of a linear operator

The best place to find a helping hand is at the end of your own arm

In this chapter, we leave our study of C-disjoint/Arazy-disjoint preserving mappings to look at **the complexification of a linear operator** first on L_p and then on $\mathcal{C}_p$ spaces. As we shall shortly see, complexifying a linear operator in the non-commutative setting has undoubtedly a different effect on the norm of the operators concerned. For notational convenience we shall use $L^p_{\mathbb{R}}(\mu)$ to denote the space of all measurable functions $f : \mathcal{X} \to \mathbb{R}$ with $\int_{\mathcal{X}} |f(x)|^p d\mu(x) < \infty$ and $L^p_{\mathbb{C}}(\mu)$ to denote the space of all measurable functions $f : \mathcal{X} \to \mathbb{C}$ with $\int_{\mathcal{X}} |f(x)|^p d\mu(x) < \infty$, whenever $(\mathcal{X}, \mathcal{M}, \mu)$ is a measure space. In the following we consider $1 \leq p < \infty$.

4.1 Complexification on L^p spaces

Definition 4.1.1. *Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space, $1 \leq p < \infty$, and suppose $T : L^p_{\mathbb{R}}(\mu) \to L^p_{\mathbb{R}}(\mu)$ is a linear operator. Define an operator $\widetilde{T} : L^p_{\mathbb{C}}(\mu) \to L^p_{\mathbb{C}}(\mu)$, called the complexification of T, by $\widetilde{T}f = Tg + iTh$, where $f \in L^p_{\mathbb{C}}(\mu)$ with $f = g + ih$ and $g, h \in L^p_{\mathbb{R}}(\mu)$.*

The proof of the following standard result on L_p spaces follows the lines of the proof of Theorem 4.2.7 which can be found in [14], pp 203.

Theorem 4.1.2. *Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, $1 \leq p < \infty$, and suppose that $T : L^p_{\mathbb{R}}(\mu) \to L^p_{\mathbb{R}}(\mu)$ is a linear operator. If $\widetilde{T} : L^p_{\mathbb{C}}(\mu) \to L^p_{\mathbb{C}}(\mu)$ is the complexification of T, then*

$$||\widetilde{T}||_{L^p_{\mathbb{C}}(\mu)} = ||T||_{L^p_{\mathbb{R}}(\mu)}.$$

Proof. This theorem can be proved by applying the following linearisation formula twice: $\displaystyle\int_0^{2\pi} |\gamma cos\theta + \delta sin\theta|^p d\theta = (\gamma^2 + \delta^2)^{p/2}\alpha_p$, where $\alpha_p = \int_0^{2\pi} |cos\theta|^p d\theta$ is independent of $\gamma, \delta \in \mathbb{R}$. In other words:

$$\int_0^{2\pi} \left|(cos\theta, sin\theta) \cdot \left(\frac{\gamma}{(\gamma^2+\delta^2)^{1/2}}, \frac{\delta}{(\gamma^2+\delta^2)^{1/2}}\right)\right|^p d\theta =$$
$$= \int_0^{2\pi} |(cos\theta, sin\theta) \cdot (1,0)|^p \, d\theta = \alpha_p \quad (4.1)$$

by rotation invariance (special case of (4.3)). If $f = g + ih$, where $g, h \in L^p_{\mathbb{R}}(\mu)$, then Fubini's theorem yields:

$$\begin{aligned}
\|\widetilde{T}f\|^p_{L^p_{\mathbb{C}}(\mu)} &= \int_\Omega |(Tg)(x) + i(Th)(x)|^p d\mu(x) \\
&= \int_\Omega \left\{((Tg)(x))^2 + ((Th)(x))^2\right\}^{p/2} d\mu(x) \\
&= \frac{1}{\alpha_p}\int_\Omega \int_0^{2\pi} |(Tg)(x)cos\theta + (Th)(x)sin\theta|^p d\theta d\mu(x) \\
&= \frac{1}{\alpha_p}\int_0^{2\pi} \left\{\int_\Omega |\left(T(gcos\theta + hsin\theta)\right)(x)|^p d\mu(x)\right\} d\theta \\
&= \frac{1}{\alpha_p}\int_0^{2\pi} \left\{||T(gcos\theta + hsin\theta)||^p_{L^p_{\mathbb{R}}(\mu)}\right\} d\theta \\
&\leq \frac{1}{\alpha_p}\int_0^{2\pi} \left\{||T||^p_{L^p_{\mathbb{R}}(\mu)}||gcos\theta + hsin\theta||^p_{L^p_{\mathbb{R}}(\mu)}\right\} d\theta
\end{aligned}$$

Hence

$$\begin{aligned}
\|\widetilde{T}f\|^p_{L^p_{\mathbb{C}}(\mu)} &\leq \frac{1}{\alpha_p}||T||^p_{L^p_{\mathbb{R}}(\mu)} \int_0^{2\pi} \left\{\int_\Omega |g(x)cos\theta + h(x)sin\theta|^p d\mu(x)\right\} d\theta \\
&= \frac{1}{\alpha_p}||T||^p_{L^p_{\mathbb{R}}(\mu)} \int_\Omega \left\{\int_0^{2\pi} |g(x)cos\theta + h(x)sin\theta|^p d\theta\right\} d\mu(x) \\
&= \frac{1}{\alpha_p}||T||^p_{L^p_{\mathbb{R}}(\mu)} \int_\Omega \alpha_p \left\{(g(x))^2 + (h(x))^2\right\}^{p/2} d\mu(x) \\
&= ||T||^p_{L^p_{\mathbb{R}}(\mu)} \int_\Omega |g(x) + ih(x)|^p d\mu(x) \\
&= ||T||^p_{L^p_{\mathbb{R}}(\mu)}||f||^p.
\end{aligned}$$

Therefore

$$||\widetilde{T}||_{L^p_{\mathbb{C}}(\mu)} \le ||T||_{L^p_{\mathbb{R}}(\mu)}.$$

On the other hand, we trivially have:

$$||T||_{L^p_{\mathbb{R}}(\mu)} \le ||\widetilde{T}||_{L^p_{\mathbb{C}}(\mu)}.$$

We conclude that $||\widetilde{T}||_{L^p_{\mathbb{C}}(\mu)} = ||T||_{L^p_{\mathbb{R}}(\mu)}$. □

We, therefore, come to the conclusion that if we complexify a bounded linear operator on L_p, its norm remains the same, which is essentially a special case of the Marcinkiewicz-Zygmund Theorem (see section 4.2).

4.2 Complexification on $\mathcal{C}_p$ spaces

The situation becomes rather more difficult when we work on $\mathcal{C}_p$ spaces. We shall now examine the analogous setting in the non-commutative case. We begin by giving the definition of $\mathcal{C}_{p,sa}$ which is the analogue of $L^p_{\mathbb{R}}(\mu)$.

Definition 4.2.1. *We define $\mathcal{C}_{p,sa}$ to be the real subspace of $\mathcal{C}_p$ consisting of all the self-adjoint operators in $\mathcal{C}_p$.*

Note that throughout this section, and indeed the rest of this chapter, $\mathcal{C}_p$ for $1 \le p < \infty$ is considered to be a vector space over $\mathbb{C}$.

We complexify a linear operator on $\mathcal{C}_{p,sa}$ spaces in the following way.

Definition 4.2.2. *Let $T : \mathcal{C}_{p,sa} \to \mathcal{C}_{p,sa}$ be a linear operator. Define an operator $\widetilde{T} : \mathcal{C}_p \to \mathcal{C}_p$, called the complexification of T, by $\widetilde{T}X = TA + iTB$, where $X \in \mathcal{C}_p$ with $X = A + iB$ and $A = \frac{X+X^*}{2}, B = \frac{X-X^*}{2i} \in \mathcal{C}_{p,sa}$.*

Using the same notation as in this definition, we have the following:

$$\begin{aligned} ||\widetilde{T}X||_p &= ||TA + iTB||_p \\ &\le ||TA||_p + ||TB||_p \\ &\le ||T||_{\mathcal{C}_{p,sa}}(||A||_p + ||B||_p) \\ &\le 2||T||_{\mathcal{C}_{p,sa}}||X||_p \ . \end{aligned}$$

Hence

$$||\widetilde{T}||_{\mathcal{C}_p} \le 2||T||_{\mathcal{C}_{p,sa}} \ . \tag{4.2}$$

Now we give the main result in this chapter which very much improves inequality (4.2).

Theorem 4.2.3. *Let $T : \mathcal{C}_{p,sa} \to \mathcal{C}_{p,sa}$ be a linear operator, and let $\widetilde{T} : \mathcal{C}_p \to \mathcal{C}_p$ be the complexification of T. Then:*

$$||\widetilde{T}||_{\mathcal{C}_p} \leq 2^{\frac{2-p}{p}} ||T||_{\mathcal{C}_{p,sa}}, \quad when \quad 1 \leq p < 2$$

and

$$||\widetilde{T}||_{\mathcal{C}_p} \leq 2^{\frac{p-2}{p}} ||T||_{\mathcal{C}_{p,sa}}, \quad when \quad 2 \leq p < \infty \ .$$

Proof. If $X = ReX + iImX$ is the Cartesian decomposition of an arbitrary X in $\mathcal{C}_p$, from Theorem 1.2.6 and Corollary 1.2.8 we deduce $||ReX||_p^p + ||ImX||_p^p \leq ||X||_p^p$ $(2 \leq p < \infty)$ and $||X||_p^p \leq 2^{p-2}(||ReX||_p^p + ||ImX||_p^p)$ $(2 \leq p < \infty)$. For $2 \leq p < \infty$,

$$\begin{aligned} ||\widetilde{T}X||_p^p &= ||T(ReX) + iT(ImX)||_p^p \\ &\leq 2^{p-2}(||T(ReX)||_p^p + ||T(ImX)||_p^p) \\ &\leq 2^{p-2}||T||_p^p(||ReX||_p^p + ||ImX||_p^p) \\ &\leq 2^{p-2}||T||_p^p||X||_p^p \ . \end{aligned}$$

Thus $||\widetilde{T}||_{\mathcal{C}_p} \leq 2^{\frac{p-2}{p}} ||T||_{\mathcal{C}_{p,sa}}$. Theorem 1.2.10 and Theorem 1.2.9 imply:

$$||X||_p^p \leq ||ReX||_p^p + ||ImX||_p^p \qquad (1 \leq p \leq 2)$$

and

$$||ReX||_p^p + ||ImX||_p^p \leq 2^{2-p}||X||_p^p \qquad (1 \leq p \leq 2).$$

Hence for $1 \leq p \leq 2$,

$$\begin{aligned} ||\widetilde{T}X||_p^p &= ||T(ReX) + iT(ImX)||_p^p \\ &\leq ||T(ReX)||_p^p + ||T(ImX)||_p^p \\ &\leq ||T||_p^p(||ReX||_p^p + ||ImX||_p^p) \\ &\leq 2^{2-p}||T||_p^p||X||_p^p \ . \end{aligned}$$

Thus $||\widetilde{T}||_{\mathcal{C}_p} \leq 2^{\frac{2-p}{p}} ||T||_{\mathcal{C}_{p,sa}}$. Note that this estimate for the norm of the complexified operator can also be deduced by duality using the estimate found for this operator in the case $2 \leq p < \infty$. □

Although the estimates given in Thorem 4.2.3 may not be sharp, we shall now give an example which shows that if $p \neq 2$, the norm of the complexified operator $\widetilde{T}$ need **not** be equal to the norm of T in the non-commutative case.

We shall denote the space of all self-adjoint $n \times n$ complex matrices by $\mathbb{M}_{n,sa}(\mathbb{C})$. In the example which follows the calculations are omitted, but all one really needs to know is that

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{\mathcal{C}_4}^4 = (|a|^2 + |c|^2)^2 + (|b|^2 + |d|^2)^2 + 2|a\bar{b} + c\bar{d}|^2$$

and (hence)

$$\left\| \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \right\|_{\mathcal{C}_{4,sa}}^4 = (a^2 + |b|^2)^2 + (d^2 + |b|^2)^2 + 2|(a+d)b|^2.$$

Example 4.2.4. Define

$$T : \mathbb{M}_{2,sa}(\mathbb{C}) \longrightarrow \mathbb{M}_{2,sa}(\mathbb{C})$$

$$T \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} = \begin{pmatrix} d & a \\ a & -d \end{pmatrix}.$$

Then we have $||T||_{\mathcal{C}_{4,sa}}^4 = 4$. Indeed,

$$\begin{aligned} ||T||_{\mathcal{C}_{4,sa}}^4 &= sup \frac{2(a^2 + d^2)^2}{(a^2 + |b|^2)^2 + (d^2 + |b|^2)^2 + 2|(a+d)b|^2} \\ &\leq sup \frac{2a^4 + 2d^4 + 4a^2d^2}{a^4 + d^4} \\ &= 2 + sup \frac{4a^2d^2}{a^4 + d^4} \\ &\leq 2 + 2 = 4. \end{aligned}$$

To show that $||T||_{\mathcal{C}_{4,sa}}^4 \geq 4$ simply choose a self adjoint matrix such that $b = 0$ and $a^2 = d^2$.

The complexification of T is given by

$$\widetilde{T} : M_2(\mathbb{C}) \longrightarrow M_2(\mathbb{C})$$

$$\widetilde{T} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & a \\ a & -d \end{pmatrix}$$

and so $||\widetilde{T}||^4_{\mathcal{C}_4} \geq 8$. In fact, since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{C})$, we have

$$\begin{aligned}
||\widetilde{T}||^4_{\mathcal{C}_4} &= sup\frac{2(|x_1|^2+|x_4|^2)^2+2|x_4\overline{x_1}+x_1(-\overline{x_4})|^2}{(|x_1|^2+|x_3|^2)^2+(|x_2|^2+|x_4|^2)^2+2|x_1\overline{x_2}+x_3\overline{x_4}|^2} \\
&= 2sup\frac{(|x_1|^2+|x_4|^2)^2+2^2|Im(\overline{x_1}x_4)|^2}{(|x_1|^2+|x_3|^2)^2+(|x_2|^2+|x_4|^2)^2+2|x_1\overline{x_2}+x_3\overline{x_4}|^2} \\
&\geq 2\frac{2^2+2^2}{2} = 8.
\end{aligned}$$

Hence $||\widetilde{T}||_{\mathcal{C}_4} \geq 2^{\frac{1}{4}}||T||_{\mathcal{C}_{4,sa}}$.

The following proposition however states that complexifing an operator from $\mathcal{C}_{2,sa}$ to $\mathcal{C}_2$ does **not** affect its norm.

Proposition 4.2.5. *Let $T : \mathcal{C}_{2,sa} \to \mathcal{C}_{2,sa}$ be a linear operator, and $\widetilde{T} : \mathcal{C}_2 \to \mathcal{C}_2$ be the complexification of T. Then:*

$$||\widetilde{T}||_{\mathcal{C}_2} = ||T||_{\mathcal{C}_{2,sa}}.$$

Proof. If A and B are two operators in $\mathcal{C}_{2,sa}$, then:

$$\begin{aligned}
||A+iB||_2^2 &= \tau\left((A-iB)(A+iB)\right) \\
&= \tau(A^2)+\tau(B^2)+i\tau(AB)-i\tau(BA) \\
&= ||A||_2^2+||B||_2^2\ .
\end{aligned}$$

Thus, if $X \in \mathcal{C}_2$, then:

$$\begin{aligned}
||\widetilde{T}X||_2^2 &= ||TA+iTB||_2^2 \\
&= ||TA||_2^2+||TB||_2^2 \\
&\leq ||T||^2(||A||_2^2+||B||_2^2) \\
&= ||T||^2||X||_2^2\ .
\end{aligned}$$

Consequently $||\widetilde{T}||_{\mathcal{C}_2} \leq ||T||_{\mathcal{C}_{2,sa}}$. Obviously $||T||_{\mathcal{C}_{2,sa}} \leq ||\widetilde{T}||_{\mathcal{C}_2}$. Hence

$$||\widetilde{T}||_{\mathcal{C}_2} = ||T||_{\mathcal{C}_{2,sa}}.$$

□

The Marcinkiewicz-Zygmund Theorem on the space $L^p_{\ell_2}(\mu)$

It is interesting now to ask whether one can further extend an operator from $L^p_{\mathbb{C}}(\mu)$ to $L^p_{\ell^2}(\mu)$ spaces which are defined below and obtain the same estimate as the one given in Theorem 4.1.2.

Definition 4.2.6. *If $(\mathcal{X}, \mathcal{M}, \mu)$ is a measure space and $1 \leq p < \infty$, we define $L^p_{\ell^2}(\mu)$ to be the set of all measurable [1] functions $f : \mathcal{X} \to \ell^2$ such that $\int_{\mathcal{X}} ||f(x)||^p_{\ell^2} d\mu(x) < \infty$. If $f \in L^p_{\ell^2}(\mu)$, define $||f||_{L^p_{\ell^2}(\mu)} = \left(\int_{\mathcal{X}} ||f(x)||^p_{\ell^2} d\mu(x)\right)^{1/p}$.*

Our next theorem proved by Marcinkiewicz and Zygmund (see [14]) shows that we can indeed do so and the same estimate is obtained as a simple corollary.

Marcinkiewicz-Zygmund

Theorem 4.2.7. *Let $(\mathcal{X}, \mathcal{M}, \mu)$ and $(\mathcal{Y}, \mathcal{N}, \nu)$ be measure spaces, and assume $0 < p < \infty$. If S is a vector subspace of $L^p_{\mathbb{C}}(\mathcal{X}, \mathcal{M}, \mu)$ and T is a linear mapping from S into $L^p_{\mathbb{C}}(\mathcal{Y}, \mathcal{N}, \nu)$ such that*

$$||Tf||_p \leq M||f||_p$$

for all f in S, then, for every positive integer N and every N-tuple $(f_1, f_2, ..., f_N)$ of elements of S, we have:

$$\left\| \left(\sum_{i=1}^{N} |Tf_i|^2 \right)^{\frac{1}{2}} \right\|_p \leq M \left\| \left(\sum_{i=1}^{N} |f_i|^2 \right)^{\frac{1}{2}} \right\|_p$$

The proof of this Theorem uses the following fact:
If σ is the normalised surface measure on the unit sphere Σ in $\mathbb{C}^N$, then for all $\omega_1, \omega_2 \in \Sigma$,

$$\int_{\Sigma} |(s, \omega_2)|^p d\sigma(s) = \int_{\Sigma} |(s, \omega_1)|^p d\sigma(s)$$

i.e for all $\omega \in \Sigma$,

$$\int_{\Sigma} |(s, \omega)|^p d\sigma(s) = \alpha_p, \tag{4.3}$$

where α_p is a constant independent of ω.

Remark 4.2.8. Formula 4.1 is just a special case of 4.3.

The above theorem trivially yields the following.

[1] componentwise

Theorem 4.2.9. *Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space, $T : L^p_{\mathbb{C}}(\mu) \to L^p_{\mathbb{C}}(\mu)$ be a bounded linear operator and $1 \leq p < \infty$. We extend T to $\widetilde{T} : L^p_{\ell_2}(\mu) \to L^p_{\ell_2}(\mu)$ as follows: Let $f \in L^p_{\ell_2}(\mu)$. If for all $x \in \mathcal{X}$, $f(x) = (f_1(x), f_2(x), ...)$, where for all $j \in \mathbb{N}$, $f_j \in L^p_{\mathbb{C}}(\mu)$, then for all $x \in \mathcal{X}$, we define $\widetilde{T}f(x) = (Tf_1(x), Tf_2(x), ...)$. Then:*

$$||\widetilde{T}||_{L^p_{\ell_2}(\mu)} = ||T||_{L^p_{\mathbb{C}}(\mu)}.$$

It is not clear whether an analogue for the space $L^p_{\ell_2}(\mu)$ can be found in the non-commutative setting. However even if an analogue could be found, the norm of the extended operator $\widetilde{T}$ on that space would not be equal to the $\mathcal{C}_p$ norm of T as example 4.2.4 demonstrates.

Chapter 5

Schur products

What one can really learn is that there is always more to learn.

In this last chapter we turn our attention to Schur products on $\mathcal{C}_p$ spaces and prove some properties of Schur products of two finite or infinite matrices when one of the two matrices is rank one, self-adjoint with 1's on the diagonal (matrices of that form appeared in Chapter 3). We then demonstrate that the Schur product with a matrix of this form is an isometry on $\mathcal{C}_p$ spaces. We conclude by discussing some questions and open problems for future work.

A brief introduction

In Chapter 3 we noted that if A is a rank-one (finite or infinite dimensional) matrix with unimodular entries, then there are unitary matrices U, W such that $UKW^* = K * A$ for all K (see Lemma 3.2.6). We can therefore obtain the following Proposition as a simple corollary by using Lemma 3.2.5.

Proposition 5.0.10. *Let $\mathcal{H}$ be a separable Hilbert space and $\{e_k\}$ a fixed orthonormal basis for $\mathcal{H}$. Let $K \in \mathcal{C}_p(\mathcal{H})$, $1 \leq p \leq \infty$, and $A = (a_{ij})$ be a finite or infinite dimensional formally self-adjoint (i.e $a_{ji} = \overline{a_{ij}}$ for all i, j) rank-one matrix with 1's on the diagonal (no boundedness restrictions on A!). Then $K * A \in \mathcal{C}_p(\mathcal{H})$ and*

$$\|K * A\|_p = \|K\|_p . \tag{5.1}$$

In this chapter we shall see that there is another way of approaching this topic which highlights the underlying algebraic structure of Schur products of this form. The same result and some nice algebraic properties of this kind of Schur products are obtained.

5.1 The finite dimensional case

Let $A \in M_n(\mathbb{C})$ be a rank one matrix with 1's on the diagonal. Let us define $h : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $h(K) = K * A$. The following result states that this mapping h is a homomorphism on $M_n(\mathbb{C})$ and as a result we have that for all $m \in \mathbb{N}$,

$$(h(K))^m = h(K^m). \tag{5.2}$$

Proposition 5.1.1. *If $A = (a_{ij}) \in M_n(\mathbb{C})$ has rank one and $a_{ii} = 1$, $i = 1, \ldots, n$, then for all $K, L \in M_n(\mathbb{C})$,*

$$(A * K)(A * L) = A * (KL), \tag{5.3}$$

$$(A * K)^m = A * K^m \quad , \quad m \in \mathbb{N}. \tag{5.4}$$

Proof. Since A has rank one, any two columns are linearly dependent. So there are scalars λ_{ji} such that $\begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix} = \lambda_{ji} \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}$. However, $a_{jj} = 1$. Therefore we have $a_{ji} = \lambda_{ji} a_{jj} = \lambda_{ji}$. It follows that

$$a_{mi} = a_{mj} a_{ji} \tag{5.5}$$

for all i, j, m. Then $(A * K)(A * L) = A * (KL)$ for any $K = (k_{ij}), L = (l_{ij}) \in M_n(\mathbb{C})$. Indeed,

$$\begin{aligned}
((A * K)(A * L))_{ij} &= \sum_m (A * K)_{im} (A * L)_{mj} \\
&= \sum_m a_{im} k_{im} a_{mj} l_{mj} \\
&= \sum_m a_{im} a_{mj} k_{im} l_{mj} \\
&= a_{ij} \sum_m k_{im} l_{mj} \quad \text{(use (5.5))} \\
&= a_{ij} (KL)_{ij} \\
&= (A * (KL))_{ij} .
\end{aligned}$$

This proves (5.3). Then (5.4) is proved by induction on m.

□

Next a very simple lemma is noted.

Lemma 5.1.2. *If X is a positive self-adjoint matrix in $M_n(\mathbb{C})$ then the mapping*

$$(0, \infty) \to M_n(\mathbb{C})$$

$$p \mapsto X^p$$

is continuous.

Proof. This is an immediate consequence of the diagonalization of X. □

The following comes from [9], p.23.

Lemma 5.1.3. *(***Schur's Theorem***) Let $K, L \in M_n(\mathbb{C})$. Then*

1. *$\|K * L\| \le \|K\| \, \|L\|$,*
2. *If K, L are positive, then so is $K * L$.*

Here $\|\cdot\|$ denotes the operator norm.

If we now assume that A is also self-adjoint, then property 5.2 of h holds with K replaced by K^*K and any $m \in (0, \infty)$.

Lemma 5.1.4. *Let A be a $k \times k$ self-adjoint rank-one matrix with 1's on the diagonal. Then for all $K \in M_k(\mathbb{C})$ and all $p \in (0, \infty)$,*

$$((K^*K) * A)^p = (K^*K)^p * A. \tag{5.6}$$

Proof. Since A has rank one, $A = \lambda P$, where P is a rank-one projection and λ is the only non-zero eigenvalue of A (spectral theorem). The eigenvalue λ has multiplicity one. Thus $0 + \lambda = trA = k$ because $0, \lambda$ are the only eigenvalues of A. So $\lambda = k \ge 0$, hence A is positive.

Now we show that

$$((K^*K) * A)^{\frac{m}{n}} = (K^*K)^{\frac{m}{n}} * A \quad , \quad m, n \in \mathbb{N}. \tag{5.7}$$

Since

$$(K^*K)^{\frac{1}{n}} \ge 0 \ , \ n \in \mathbb{N},$$

Schur's Theorem 5.1.3 implies that

$$(K^*K)^{\frac{1}{n}} * A \ge 0 \ , \ n \in \mathbb{N}$$

and so

$$0 \leq \left((K^*K)^{\frac{1}{n}} * A\right)^m = (K^*K)^{\frac{m}{n}} * A \quad , \quad n, m \in \mathbb{N} \tag{5.8}$$

by (5.4) of Proposition 5.1.1. Since both sides of (5.8) are positive, taking the m-th root we obtain

$$(K^*K)^{\frac{1}{n}} * A = \left((K^*K)^{\frac{m}{n}} * A\right)^{\frac{1}{m}} \quad , \quad n, m \in \mathbb{N}. \tag{5.9}$$

As a special case ($m = n$) we have

$$(K^*K)^{\frac{1}{n}} * A = ((K^*K) * A)^{\frac{1}{n}} \quad , \quad n \in \mathbb{N}. \tag{5.10}$$

Hence for all $m, n \in \mathbb{N}$,

$$\begin{aligned} ((K^*K) * A)^{\frac{m}{n}} &= \left(((K^*K) * A)^{\frac{1}{n}}\right)^m \\ &= \left((K^*K)^{\frac{1}{n}} * A\right)^m \quad \text{(by (5.10))} \\ &= \left((K^*K)^{\frac{1}{n}}\right)^m * A \quad \text{(by (5.4))} \\ &= (K^*K)^{\frac{m}{n}} * A. \end{aligned}$$

Now fix $p \in (0, \infty)$ and let $\{p_j\}$ be a sequence in $\mathbb{Q}^+$ such that $p_j \to p$. Since K^*K and $(K^*K) * A$ are both positive, Lemma 5.1.2 yields

$$(K^*K)^{p_j} \to (K^*K)^p \,,$$

$$((K^*K) * A)^{p_j} \to ((K^*K) * A)^p \,.$$

The Schur product is a continuous mapping and therefore

$$(K^*K)^{p_j} * A \to (K^*K)^p * A.$$

But for all j,

$$((K^*K) * A)^{p_j} = (K^*K)^{p_j} * A$$

by (5.7). Let $j \to \infty$ to finish the proof. □

The following proposition shows that the homomorphism h on $M_n(\mathbb{C})$ defined at the start of this section is in fact an isometry with respect to the $\mathcal{C}_p$ norm.

Proposition 5.1.5. *Let $K \in M_n(\mathbb{C})$ and let $A \in M_n(\mathbb{C})$ be a self-adjoint rank-one matrix with 1's on the diagonal. Then for all $1 \leq p < \infty$, we have:*

$$\|K * A\|_{\mathcal{C}_p(\mathbb{C}^n)} = \|K\|_{\mathcal{C}_p(\mathbb{C}^n)} \,. \tag{5.11}$$

Proof. We have

$$\begin{aligned}
\|K * A\|_{\mathcal{C}_p(\mathbb{C}^n)}^p &= tr\left((K*A)^* \ (K*A)\right)^{\frac{p}{2}} \\
&= tr\left((K^* * A) \ (K*A)\right)^{\frac{p}{2}} && (A^* = A) \\
&= tr\left((K^*K) * A\right)^{\frac{p}{2}} && \text{(Proposition 5.1.1)} \\
&= tr\left((K^*K)^{\frac{p}{2}} * A\right) && \text{(Lemma 5.1.4)} \\
&= tr\left(K^*K\right)^{\frac{p}{2}} && (A \text{ has 1's on the diagonal}) \\
&= \|K\|_{\mathcal{C}_p(\mathbb{C}^n)}^p .
\end{aligned}$$

□

5.2 The infinite dimensional case

We begin by giving a modification of some notation introduced in Chapter 3.

Definition 5.2.1. *Let $J \subseteq \mathbb{N}$ with n elements. If $M_\infty(\mathbb{C})$ denotes the space of all infinite dimensional matrices with complex entries, define,*

$$\mathcal{F}_J : M_n(\mathbb{C}) \to M_\infty(\mathbb{C})$$

as follows. Given $A = (a_{ij}) \in M_n(\mathbb{C})$, $\mathcal{F}_J(A) \in M_\infty(\mathbb{C})$ is the matrix with $(i,j)-$entry a_{ij} if $(i,j) \in J \times J$, and zero otherwise. Also define

$$\mathcal{S}_J : M_\infty(\mathbb{C}) \to M_n(\mathbb{C})$$

as follows. Given $A \in M_\infty(\mathbb{C})$, $\mathcal{S}_J(A)$ is the $J \times J$ submatrix of A, i.e the $n \times n$ matrix obtained form A by deleting the $i-$th row if $i \notin J$ and the $j-$th column if $j \notin J$.

Observe that

$$\mathcal{S}_J(\mathcal{F}_J(A)) = A \qquad (A \in M_n(\mathbb{C})), \tag{5.12}$$

$$\mathcal{F}_J(AB) = \mathcal{F}_J(A)\mathcal{F}_J(B) \qquad (A, B \in M_n(\mathbb{C})), \tag{5.13}$$

$$\|\mathcal{F}_J(A)\|_{\mathcal{C}_p(\mathcal{H})} = \|A\|_{C_p(\mathbb{C}^n)} \qquad (A \in M_n(\mathbb{C})). \tag{5.14}$$

Also

$$\mathcal{F}_J\left(\mathcal{S}_J\left(A\right)\right) = A \qquad\qquad (A \in M_\infty\left(\mathbb{C}\right)), \tag{5.15}$$

$$\mathcal{S}_J\left(AB\right) = \mathcal{S}_J(A)\mathcal{S}_J(B) \quad (A, B \in M_\infty\left(\mathbb{C}\right)), \tag{5.16}$$

when $A = (a_{ij}), B = (b_{ij}) \in M_\infty\left(\mathbb{C}\right)$ with $a_{ij} = b_{ij} = 0$ for all $(i, j) \notin J \times J$.

Let us consider the mapping h we defined in the finite dimensional case as a mapping on $M_\infty\left(\mathbb{C}\right)$. Then h is still a homomorhism.

Proposition 5.2.2. *If $A = (a_{ij}) \in M_\infty\left(\mathbb{C}\right)$ (no boundedness restrictions on A!) has rank-one (in the sense that the span of the columns of A is one dimensional) and $a_{ii} = 1$ for all $i \in \mathbb{N}$, then for all $K, L \in M_\infty\left(\mathbb{C}\right)$,*

$$\left(A * K\right)\left(A * L\right) = A * \left(KL\right). \tag{5.17}$$

Proof. This can be proved as in Proposition 5.1.1. □

We are in a position now to prove that this mapping h is a $\mathcal{C}_p$ isometry.

Proposition 5.2.3. *Let $\mathcal{H}$ be a separable Hilbert space and $\{e_k\}$ a fixed orthonormal basis for $\mathcal{H}$. Let $K \in \mathcal{C}_p\left(\mathcal{H}\right)$, $1 \leq p < \infty$, and $A = (a_{ij})$ be an infinite dimensional formally self-adjoint (i.e $a_{ji} = \overline{a_{ij}}$ for all i, j) rank-one matrix with 1's on the diagonal (no boundedness restrictions on A!). Then $K * A \in \mathcal{C}_p\left(\mathcal{H}\right)$ and*

$$\left\|K * A\right\|_p = \left\|K\right\|_p. \tag{5.18}$$

Proof. Consider first the special case of $K \in \mathcal{C}_p\left(\mathcal{H}\right)$, $1 \leq p < \infty$, which has a finitely supported matrix $K = (K_{ij})$ with respect to (e_k). Then $K = \mathcal{F}_J(K_n)$ for some $K_n \in M_n\left(\mathbb{C}\right)$, where $J = \{1, \ldots, n\}$. Then obviously $K * A$ has finite rank and so $K * A \in \mathcal{C}_p\left(\mathcal{H}\right)$. Define $A_n = \mathcal{S}_J(A)$. Then A_n is an $n \times n$ self-adjoint rank-one matrix with 1's on the diagonal. Then, for all $1 \leq p < \infty$, we have $((K_n^* K_n) * A_n)^p = (K_n^* K_n)^p * A_n$ by Lemma 5.1.4. Hence

$$\begin{aligned}
\|K * A\|^p_{\mathcal{C}_p(\mathcal{H})} &= \|\mathcal{F}_J(K_n) * A\|^p_{\mathcal{C}_p(\mathcal{H})} \\
&= \|\mathcal{F}_J(K_n * A_n)\|^p_{\mathcal{C}_p(\mathcal{H})} && \text{(by (5.14))} \\
&= \|K_n * A_n\|^p_{\mathcal{C}_p(\mathbb{C}^n)} && \text{(by Proposition 5.1.5)} \\
&= \|K_n\|^p_{\mathcal{C}_p(\mathbb{C}^n)} = \|\mathcal{F}_J(K_n)\|^p_{\mathcal{C}_p(\mathcal{H})} = \|K\|^p_{\mathcal{C}_p(\mathcal{H})}.
\end{aligned}$$

Thus, if K is a finitely supported matrix with respect to the basis $\{e_k\}$, then $\|K * A\|_p = \|K\|_p$, $1 \leq p < \infty$.

Now let $K \in \mathcal{C}_p(\mathcal{H})$ with matrix $K = (K_{ij})$ with respect to $\{e_k\}$. Let P_n be the projection of $\mathcal{H}$ onto $clin\{e_1, \ldots, e_n\}$. Then, for any n, $K_n := P_n K P_n$ is a finitely supported matrix with respect to $\{e_k\}$, and $K_n \to K$ in $\mathcal{C}_p(\mathcal{H})$. So for any $n, m \in \mathbb{N}$,

$$\|A * K_n - A * K_m\|_p = \|A * (K_n - K_m)\|_p$$

$$= \|K_n - K_m\|_p \qquad \text{(by special case)}$$

Since $\mathcal{C}_p(\mathcal{H})$, $1 \leq p < \infty$ is complete, there is an $L \in \mathcal{C}_p(\mathcal{H})$ such that $A * K_n \to L$ and so $(A * K_n)_{ij} \to L_{ij}$ for all i, j. Also $K_n \to K$ in $\mathcal{C}_p(\mathcal{H})$ implies $(K_n)_{ij} \to K_{ij}$ for all i, j and therefore $(A * K_n)_{ij} \to (A * K)_{ij}$ for all i, j. So $A * K = L \in \mathcal{C}_p$. Thus $A * K_n \to A * K$ in $\mathcal{C}_p(\mathcal{H})$. We deduce that $\|A * K_n\|_p \to \|A * K\|_p$. On the other hand,

$$\|A * K_n\|_p = \|K_n\|_p \quad \text{(by the result of the special case)}$$

$$\to \|K\|_p .$$

Consequently $\|A * K\|_p = \|K\|_p$. □

5.3 Future work

In this last section we shall present some open problems which are of interest for possible future work.

On positivity preserving and disjoint preserving mappings

The following results concern the inverse of a disjoint preserving mapping on L_p spaces, $1 \leq p < \infty$ and its characterisation in terms of positivity preserving mappings.

For more details on the following see [[15], Theorem 3.1] and [[8] Scholium 2.1 and Scholium 2.3].

Proposition 5.3.1. *[8] Suppose that $(\Omega, \mathcal{M}, \mu)$ is an arbitrary measure space and let $1 \leq p < \infty$. A bounded linear mapping T of $L_p(\mu)$ into $L_p(\mu)$ is disjoint preserving if and only if there is a bounded positivity-preserving linear mapping $|T|$ of $L_p(\mu)$ into $L_p(\mu)$ satisfying the following condition:*

$$for\ every\ f \in L_p(\mu),\ |Tf| = |T|(|f|),\ \mu - a.e.\ on\ \Omega. \tag{5.19}$$

If this is the case, then the above condition uniquely characterizes $|T|$ among the bounded linear mappings of $L_p(\mu)$ into $L_p(\mu)$ (and $|T|$ is called the linear modulus of T). Moreover, $|T|$ has the property that

$$for\ every\ f \in L_p(\mu),\ |Tf| = ||T|(f)|,\ \mu - a.e.\ on\ \Omega.$$

Notice that if T is both disjoint preserving and positivity preserving, then $T = |T|$.

Proposition 5.3.1 gives rise to the following question.

Question 5.3.2. *1. Can we define a positivity preserving mapping on $\mathcal{C}_p$ spaces? 2. Is there a characterisation of C-disjoint or A-disjoint preserving mappings on $\mathcal{C}_p$ spaces in terms of a positivity preserving mapping?*

The well-known general relationships between invertibility and the disjoint preserving property on L_p spaces can be stated as follows.

Proposition 5.3.3. *[8] Suppose $(\Omega, \mathcal{M}, \mu)$ is an arbitrary measure space, $1 \leq p < \infty$ and T is a bounded invertible disjoint preserving mapping of $L_p(\mu)$ onto $L_p(\mu)$. Then T^{-1} is disjoint preserving, $|T|$ is an invertible linear mapping of $L_p(\mu)$ onto $L_p(\mu)$, and*

$$|T|^{-1} = |T^{-1}|.$$

Proof. Let $f, g \in L_p(\mu)$ with $fg = 0$ $\mu-$a.e. Put $F = T^{-1}f$, $G = T^{-1}g$ and $h = \min\{|F|, |G|\}$. Since $0 \leq h \leq |F|, |G|$ and $|T|$ is positivity-preserving by Proposition 5.3.1, we have

$$0 \leq |T|(h) \leq \min\{|T|(|F|), |T|(|G|)\}.$$

But

$$|T|(|F|) = |T(F)| = \left|TT^{-1}f\right| = |f|$$

and $|T|(|G|) = |g|$. So

$$0 \leq |T|(h) \leq \min\{|f|, |g|\}.$$

But $fg = 0$ $\mu-$a.e. So $\min\{|f|, |g|\} = 0$. Hence $0 = |T|(h) = |T(h)|$ $\mu-$a.e. Since T is injective, this shows that $h = 0$ $\mu-$a.e. So $FG = 0$ $\mu-$a.e. Hence T^{-1} is separation-preserving.

Now let $f \in L_p(\mu)$ with $f \geq 0$. Then Proposition 5.3.1 yields that

$$f = \left|T^{-1}(Tf)\right| = \left|T^{-1}\right|(|T(f)|) = \left|T^{-1}\right| |T|(|f|) = \left|T^{-1}\right| |T|(f),$$

$$f = \left|T\left(T^{-1}f\right)\right| = |T|\left(\left|T^{-1}(f)\right|\right) = |T| \left|T^{-1}\right|(|f|) = |T| \left|T^{-1}\right|(f).$$

Thus if $f \geq 0$,

$$|T| \left|T^{-1}\right| (f) = \left|T^{-1}\right| |T| (f) = f \quad , \quad \mu - \text{a.e.}$$

By linearity

$$|T| \left|T^{-1}\right| = \left|T^{-1}\right| |T| = I \quad , \quad \mu - \text{a.e.}$$

Thus $|T|$ is invertible and $|T|^{-1} = |T^{-1}|$.

□

Question 5.3.4. *Is the inverse of a bounded C-disjoint/A-disjoint preserving mapping on $\mathcal{C}_p$ spaces C-disjoint/A-disjoint preserving?*

We finish this thesis by noting that in the commutative case (see proof of Proposition 5.3.3) the lattice structure of L_p spaces (i.e the existence of the minimum and maximum of two functions) played a vital role. In $\mathcal{C}_p$ spaces though this structure is non-existent and the same idea cannot be applied. However, Corollaries 3.3.9, 3.3.13 in Chapter 3 give a partial answer to this question. Recall that these results were obtained by first finding a characterisation for such mappings and then using the algebraic description for C-disjoint/A-disjoint operators.

List of notation

The following notation is used throughout this thesis:

$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
$\mathbb{N}$	the natural numbers
$\mathbb{Z}$	the integer numbers
$\mathbb{Q}$	the rational numbers
$\overline{E}$	the closure of a set E
$\mathcal{H}$	a Hilbert space
$\mathbb{Z}^+$	the set $\{0, 1, 2, 3, ...\}$
$\mathcal{P}_0$	the set of all finite rank self adjoint projections
$\mathcal{P}$	the set of all self adjoint projections
$\mathbb{R}^+$	the set of (strictly) positive real numbers
X^*	the Banach space of continuous linear functionals on X
$\mathcal{M}^\perp$	the orthogonal complement in $\mathcal{H}$ of the subspace $\mathcal{M}$
$\|T\|$	the modulus of T
$M_\infty(\mathbb{C})$	the set of infinite dimensional matrices with complex entries
$K \perp L$	K C-disjoint from L
$K \perp_A L$	K A-disjoint from L
$K * L$	the Schur product of K, L
A^t	the transpose of A

$ranT$	the image of the operator T
$x \otimes y$	the tensor product of $x, y \in \mathcal{H}$
$\langle x, y\rangle$	the inner product of $x, y \in \mathcal{H}$
$clin(M)$	the closed linear span of the subspace m
χ_E	characteristic function of a set$E, i.e.\ \chi_E(x) = 1$ if $x \in E,$ and 0 otherwise
T^*	the adjoint of T
$\|\cdot\|$	the operator norm
$\|\cdot\|_p$	the $\mathcal{C}_p$ norm
$\sigma(T)$	the spectrum of an operator T
$\tau(T)$	the trace of T
$\mathcal{B}(X)$	the space of all bounded linear operators on X
$\mathcal{B}(X)_{sa}$	the space of all self adjoint bounded linear operators on X
l_p	the space of all p-convergent complex sequences
$\mathbb{C}^n$	the n-cartesian product of $\mathbb{C}$
$\mathcal{C}_p, \mathcal{C}_p(\mathcal{H})$	the space of von Neumann-Schatten class of compact operators on $\mathcal{H}$
$\mathcal{C}_p(\mathcal{H})_{sa}$	the set of self adjoint operators in $\mathcal{C}_p$
$M_n(\mathbb{C})$	the set of $n\times$n complex matrices
$L_p, L_p(\mathcal{X}, \Omega, \mu)$	the space of equivalence classes of p-integrable Ω-measurable functions
$L^p_{\mathbb{R}}(\mu)$	the space of p-integrable Ω-measurable real-valued functions
$L^p_{\mathbb{C}}(\mu)$	the space of p-integrable Ω-measurable complex-valued functions

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