Supersymmetric Quotients of M-Theory and Supergravity Backgrounds

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Sunil Gadhia*)

To my parents

who have given so much to me.

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Abstract

In this thesis we explore discrete quotients of maximally supersymmetric supergravity backgrounds. Our main focus will be on eleven-dimensional backgrounds preserving all 32 supercharges.

We shall first consider quotients of the sphere part of the maximally supersymmetric Freund-Rubin background $\mathrm{AdS}_4 \times S^7$. Our aim will be to determine the supersymmetry preserved in the resulting background. The quotients will be by freely acting discrete subgroups G, of the isometry group of S^7. These subgroups have been classified as part of a wider classification of subgroups acting freely and properly discontinuously on the n-sphere. This classification was not easy: many partial results were obtained until Wolf's solution [37]. For each possible quotient S^7/G, called a *spherical space form*, we shall determine if it is a spin manifold and if so how much supersymmetry, $\frac{\nu}{32}$, the corresponding background $\mathrm{AdS}_4 \times (S^7/G)$ preserves. This investigation leads us to the result that spin structure and orientation dictate supersymmetry, of the quotient S^7/G, thus highlighting the importance of specifying these factors as part of the data defining a supergravity background.

The second part of this thesis looks at discrete quotients of all the maximally supersymmetric supergravity backgrounds in ten and eleven dimensions. In this case, our aim is to see if some discrete subgroup G of the four-form-preserving isometries of the background preserves a fraction $\frac{31}{32}$ of the supersymmetry. Such a background with 31 supercharges is called a preon. We shall boil down this problem to checking if some element γ, in the image of the exponential map from the Lie algebra to the symmetry group of the background, which preserves at least 30 supercharges will preserve 31. The motivation to consider such quotients

comes from [24], where it was shown that if such backgrounds exist then they are necessarily discrete quotients of maximally supersymmetric backgrounds. We shall show that ultimately no such quotients preserve $\frac{31}{32}$ supercharges, thus ruling out the existence of preons. The bulk of our work is on the eleven-dimensional case, however we shall also derive results for the ten-dimensional case which follow from our investigation in eleven dimensions.

Contents

Chapter 1

Introduction

In this thesis we shall study discrete quotients of maximally supersymmetric supergravity backgrounds. In particular we shall focus on eleven-dimensional supergravity, the low energy limit of M-Theory. To this end we review the development of M-theory and supergravity in ten and eleven dimensions in the following introduction.

1.1 A Brief History of M-Theory

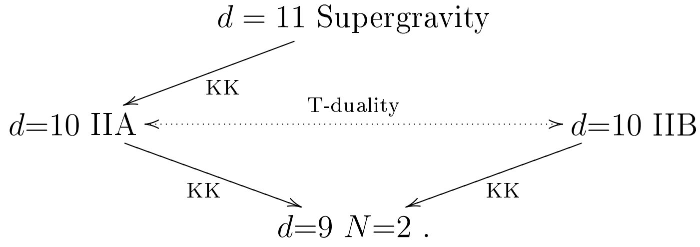
The most successful particle theory to date is the standard model. It is based on the mathematical framework of quantum field theory and was developed in the early 1970's. The standard model provides a quantum description of three of the four fundamental forces of interacting particles. These are the electromagnetic, strong nuclear and weak nuclear force. The absent force is the gravitational forces. This absence is due to serious ultra-violet divergences, in scattering amplitudes of gravitons, which cannot be renormalised in the standard way. As a consequence, there is no quantum field theory description of gravity and thus no way to incorporate gravity into the standard model. The standard model also leaves out another possible constituent of nature: supersymmetry. This symmetry relates a particle of integer spin to a particle of half-integer spin called its superpartner. Efforts have been made to incorporate supersymmetry into the standard model, the simplest possible supersymmetric model consistent with the standard model is the minimal supersymmetric standard model (however we shall not go into this further in this thesis).

Currently the most sought after theory, in theoretical physics, is one which will harmonise the classical description of gravity given by general relativity with quantum mechanics. We would like its mathematical framework to reduce to general relativity on the macroscopic scale and to quantum mechanics on the microscopic scale. The most promising theory so far incorporating these qualities is string theory. This is because at low energies string theory reduces to a gravitational theory which contains general relativity and at high energies it gives well behaved scattering amplitudes for the gravitons. Furthermore there is a natural way to incorporate supersymmetry into (super)string theory.

We know of five consistent string theories, all in ten dimensions. These are; Type I, Type IIA, Type IIB, $Spin(32)/\mathbb{Z}_2$ heterotic and $E_8 \times E_8$ heterotic string theories. For a few years it was assumed that one would turn out to be the true theory. It was then observed that some of these theories were related:

- IIA and IIB are related via T-duality,
- Heterotic $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ are also related via T-duality,
- Type I and $Spin(32)/\mathbb{Z}_2$ are related via S-duality.

These results were known for some time before it was realised in the mid 1990's that these five theories are all related by dualities, which suggested a larger framework on which they all sit. This frame-work is now known as *M-Theory* and it is hoped that it will provide a unified theory of all four interacting forces. The low energy limit of M-Theory is the unique eleven-dimensional supergravity theory, discovered in 1978 [8], which we will describe below and whose maximally supersymmetric solutions will concern us most in this thesis. It is hoped that by studying solutions of this limit we will gain a better insight into M-Theory itself. We may obtain the ten-dimensional Type IIA and Type IIB supergravity theories, which are the low energy limits of the corresponding string theories, from it. The Type IIA and IIB theories are derived via compactifications and T-duality. Each of these theories has maximally supersymmetric solutions, that is, they have solutions which preserve all 32 supercharges.



1.1.1 Eleven-Dimensional Supergravity

The bosonic field content of eleven-dimensional supergravity consists of the metric g and the four-form $F = dA^{(3)}$. In addition there is a fermionic sector with gravitino ψ. When this field is set to zero the action is given by:

$$S = \frac{1}{2}\int d^{11}x\sqrt{-g}R - \int \frac{1}{2}F \wedge *F - \int \frac{1}{6}A \wedge F \wedge F,$$

where $d^{11}x := dx^0 \wedge ..... \wedge dx^{10}$. The corresponding bosonic equations of motion are:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \frac{1}{2}\left(F^2_{\mu\nu} - \frac{1}{2}g_{\mu\nu}F^2\right)$$

$$d * F = -\frac{1}{2}F \wedge F.$$

where

$$F^2_{\mu\nu} = \frac{1}{6}F_{\mu\sigma_1\sigma_2\sigma_3}F_\nu^{\sigma_1\sigma_2\sigma_3} \text{ and } F^2 = \frac{1}{24}F_{\sigma_1\sigma_2\sigma_3\sigma_4}F^{\sigma_1\sigma_2\sigma_3\sigma_4}$$

with

$$R_{\mu\nu} = R_{\mu\alpha\nu}{}^{\alpha} \text{ and } [\nabla_\mu, \nabla_\nu]X^\sigma = R_{\mu\nu}{}^{\sigma}{}_{\rho}X^\rho.$$

∇ is the Levi-Civita connection of g, R is the corresponding curvature and $\mu, \nu, \sigma = 0, ..., 10$. The trace of $R_{\mu\nu}$ gives,

$$R = \frac{1}{6}F^2.$$

Furthermore the Bianchi identity is given by,

$$dF = d^2A = 0.$$

The theory is invariant under supersymmetry transform whose infinitesimal form is schematically:

$$\delta g \sim \varepsilon\psi,$$

$$\delta A \sim \varepsilon\psi,$$

$$\delta\psi \sim \mathcal{D}\varepsilon + \varepsilon\psi\psi,$$

where the spinor ε parameterises the variation and the connection $\mathcal{D}$ is given below. Our interest is in bosonic solutions of these equations, i.e., gravitino $\psi = 0$. Thus the equations above reduce to the supergravity Killing spinor equation,

$$\mathcal{D}_X\varepsilon = 0, \tag{1.1.1}$$

where ε is called a Killing spinor. Each Killing spinor is completely determined by its value at a point and via parallel transport at any other point. A maximally supersymmetric background contains 32 linearly independent such Killing spinors. Furthermore the supersymmetry of a background is given as a fraction $\frac{\nu}{32}$ where ν is the number of linearly independent (Killing) spinors satisfying the above equation. The connection $\mathcal{D}_\mu$ is given by:

$$\mathcal{D}_\mu = \nabla_\mu - \frac{1}{288}\left(\Gamma_\mu^{\nu_1\nu_2\nu_3\nu_4} - 8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right)F_{\nu_1\nu_2\nu_3\nu_4} \tag{1.1.2}$$

where

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{\alpha\beta}\Gamma_{\alpha\beta}.$$

1.2 Maximally Supersymmetric Solutions to 11-d Supergravity

As mentioned above, we will be interested in quotients of maximally supersymmetric backgrounds. They have been classified, along with ten-dimensional maximally supersymmetric solutions in [16], and we shall review them now. First let us briefly consider the geometries involved.

1.2.1 Minkowski Space

We shall take the eleven-dimensional Minkowski space to be $\mathbb{R}^{10,1}$ with metric,

$$\eta = \mathrm{diag}(1, 1, \dots, -1)$$

and isometry group $SO(10, 1) \ltimes \mathbb{R}^{11}$.

1.2.2 Anti-de Sitter Space

Anti-de Sitter space, AdS_p, is the *universal cover* of the quadric traced by equation:

$$-x_1^2 - x_2^2 + \sum_{i=3}^{p+1} = -R^2 \text{ where } R > 0$$

in the pseudo-euclidean space $\mathbb{R}^{2,p-1}$ with coordinates $(x_1, ...x_{p+1})$. The quadric has orientation preserving isometry group $O(2, p)$ which acts linearly on the $\mathbb{R}^{2,p-1}$ and preserves the quadric. The topology of the space is $S^1 \times \mathbb{R}^{p-2}$ and thus the fundamental group is $\mathbb{Z}$ for $p > 3$. The group of orientation preserving isometries of AdS_p is $\widetilde{SO(2, p-1)}$, which is the universal cover of $SO(2, p-1)$. $\widetilde{SO(2, p-1)}$ is obtained by the central extension of $SO(2, p-1)$ by $\mathbb{Z}$, see [18]. As a maximally supersymmetric solution to 11-dimensional supergravity, anti-de Sitter space forms the Lorentzian part of particular set of solutions for the background called Frenud-Rubin backgrounds. These backgrounds are of the form $AdS_p \times S^q$ where (p, q) is $(4, 7)$ or $(7, 4)$. Each factor comes with an orientation specified by a four-form flux or its dual. The orientation-preserving isometries of $AdS_p \times S^q$ are the groups of orientation-preserving isometries of each factor.

1.2.3 The Sphere

The sphere $S^n \subset \mathbb{R}^{n+1}$ is defined as follows:

$$S^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x^i)^2 = 1\}.$$

Thus the orientation preserving isometries of the quadratic are $SO(n+1)$. We will be most interested in this space in our investigation of supersymmetric spherical space forms.

1.2.4 The Cahen-Wallach Space

The Cahen-Wallach (CW) space is an indecomposable Lorentzian symmetric space. In light cone coordinates $x^{\pm}$, x^i for $i = 1, ..., 9$, we can write the metric as

$$ds^2 = 2dx^+dx^- + A_{ij}x^ix^jdx^-dx^- + \delta_{ij}dx^idx^j,$$

with four-form $F = \mu dx^- \wedge dx^1 \wedge dx^2 \wedge dx^3$ and $A_{ij} = diag(\frac{-\mu^2}{36}(4, 4, 4, 1, 1, 1, 1, 1, 1))$. We are interested in group actions of isometries on this space so it is instructive for us to look at the space as a quotient of a Lie group, see [15] for more on this construction. We will think of the CW space as the homogeneous space G/H where G is a simply connected Lie group and H the isotropy subgroup. They are described as follows. Let $\mathfrak{g}$ be the 20-dimensional Lie algebra with basis $(\boldsymbol{e}_\pm, \boldsymbol{e}_i, \boldsymbol{e}_i^*)$, for $i = 1, \ldots, 9$, and nonzero brackets

$$[\boldsymbol{e}_-, \boldsymbol{e}_i] = \boldsymbol{e}_i^* \qquad [\boldsymbol{e}_-, \boldsymbol{e}_i^*] = -\lambda_i^2 \boldsymbol{e}_i \qquad [\boldsymbol{e}_i^*, \boldsymbol{e}_j] = -\lambda_i^2 \delta_{ij} \boldsymbol{e}_+ \ ,$$

where

$$\lambda_i = \begin{cases} \frac{\mu}{3} \ , & i = 1, 2, 3 \\ \frac{\mu}{6} \ , & i = 4, \ldots, 9 \end{cases} \qquad \text{and } \mu \neq 0. \tag{1.2.1}$$

Let $\mathfrak{h}$ denote the abelian Lie subalgebra spanned by the $\{\boldsymbol{e}_i^*\}$ and let $H < G$ denote the corresponding Lie subgroup. The subgroup $\mathrm{SO}(3) \times \mathrm{SO}(6) < \mathrm{SO}(9)$ acts as automorphisms on $\mathfrak{g}$ preserving $\mathfrak{h}$ and hence acts as isometries on G/H. Moreover $S := G \rtimes (\mathrm{SO}(3) \times \mathrm{SO}(6))$ preserves the four-form flux, hence it is also the symmetry group of the background. It is this group which concerns us the most. We will consider the action of its discrete subgroups on the maximally supersymmetric wave in our investigation of the existence of preons.

1.2.5 Maximally Supersymmetric Solutions

The following solutions exhaust the possibilities of maximally supersymmetric backgrounds of eleven-dimensional supergravity. They have been classified in [16], along with maximally supersymmetric ten-dimensional solutions of Type IIA and IIB supergravity theories.

Theorem 1. *Let (M, g, F_4) be a maximally supersymmetric solution of eleven-dimensional supergravity. Then it is locally isometric to one of the following:*

- $\mathrm{AdS}_7(-7R) \times S^4(8R)$ *and* $F = \sqrt{6R}\,\mathrm{dvol}(S^4)$, *where* $R > 0$ *is the constant scalar curvature of* M*;*

- $\mathrm{AdS}_4(8R) \times S^7(-7R)$ *and* $F = \sqrt{-6R}\,\mathrm{dvol}(\mathrm{AdS}_4)$, *where* $R < 0$ *is again the constant scalar curvature of* M*; or*

- $\mathrm{CW}_{11}(A)$ *with* $A = -\frac{\mu^2}{36}\,\mathrm{diag}(4,4,4,1,1,1,1,1,1)$ *and* $F = \mu\, dx^- \wedge dx^1 \wedge dx^2 \wedge dx^3$. *One must distinguish between two cases:*
  - $\mu = 0$*: which recovers the flat space solution* $\mathbb{R}^{10,1}$ *with* $F = 0$*; and*
  - $\mu \neq 0$*: all these are isometric and describe an Hpp-wave.*

1.3 Clifford Algebra Conventions

Definition 2. Let $\mathbb{R}^{r,s}$ denote the real vector space with quadratic form η such that,

$$\eta(\boldsymbol{x}) := x_1^2 + ... + x_r^2 - x_{r+1}^2 - ... - x_n^2$$

where $n = r + s$ is the dimension of the vector space and (r, s) is called the signature of the quadratic pseudo-orthonormal form. The Clifford algebra denoted $Cl(r, s)$ is generated by the pseudo-orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^n$ of $\mathbb{R}^{r,s}$ such that

$$\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = 2\eta_{ij}\mathbf{1}.$$

This is given in terms of gamma matrices, which we shall use, as

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij}\mathbf{1}.$$

For our purposes $Cl(n)$ is given as;

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}\mathbf{1}.$$

Definition 3. $Spin(r,s) = \{\hat{g} \in Cl(r,s) : \hat{g} = \gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_{2k}} \mid 0 \leq i_1 < i_2 < \dots < i_{2k} \leq n \text{ and } 0 \leq k \leq n\}$.

Definition 4. $Spin(n)$ is the universal cover of $SO(n)$ for $n \geq 3$. It forms a double cover of $SO(n)$ with covering map:

$$\theta_0 : Spin(n) \to SO(n).$$

This map can be given explicitly in the following way:

$$\theta_0(\hat{g}) \cdot \boldsymbol{v} = \hat{g}\boldsymbol{v}\hat{g}^{-1}$$

for $\boldsymbol{v} \in \mathbb{R}^n$ and $\hat{g} \in Spin(n)$.

1.4 Thesis Outline

In Chapter 2 we will give the conditions for a spherical space form to be spin and determine the number of inequivalent spin structures on it. We then review some background on representation theory and polyhedral groups which will be relevent to our discussion of spherical space forms. After this we review the classification of spherical space forms as given in [37], applying it to S^7 we construct $SO(8)$ representations for the relevent groups.

In Chapter 3 we shall determine all the isomorphic lifts to $Spin(8)$ of the groups solving the spherical space form problem, in the case of S^7. The possible isomorphic lifts determine the inequivalent spin structures on the spherical space form.

In Chapter 4 we shall construct an explicit basis for the space of spinors of both irreducible representations of $Spin(8)$. Then we shall act on them with the groups obtained in Chapter 3 to determine the supersymmetry of the spherical space form S^7/G (in each representation). Since the quotient preserves all the supersymmetry on the AdS_4 factor the supersymmetry, $\frac{\nu}{32}$, of $\mathrm{AdS}_4 \times (S^7/G)$ is synonymous with the supersymmetry of S^7/G.

In Chapter 5 we shall begin our work on determining the existence of M-Theory preons. In this chapter we shall consider the 'exponential' properties of the isometry groups of the maximally supersymmetric eleven-dimensional supergravity backgrounds. Group elements, in the image of the exponential map will be used to determine the existence of preons.

In Chapter 6 we shall take discrete quotients of the maximally supersymmetric backgrounds of eleven-dimensional supergravity to see if any of them preserve 31 supercharges, hence giving preonic backgrounds. Ultimately we shall show no such backgrounds exist.

In Appendix A we shall review Bär's cone construction, as presented in [11], and relate Killing spinors on S^7 to parallel spinors on $\mathbb{R}^8$. In Appendix B we shall review the classification of one parameter subgroups of the isometry groups of $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ as presented in [18].

Throughout this thesis references [7, 32, 33, 36], have been a good source of material on differential geometry and Lie groups.

Chapter 2

Spherical Space Forms

In Chapters 2 to 4 we will consider discrete quotients of the sphere part of the maximally supersymmetric Freund-Rubin background $\mathrm{AdS}_4 \times S^7$ [23] and investigate the supersymmetry of the resulting background. The geometry of $\mathrm{AdS}_4 \times S^7$ is as follows:

$$\mathrm{AdS}_4(8R) \times S^7(-7R) \text{ with flux } F = -\sigma_S \sqrt{-6R}\, \mathrm{dvol}(\mathrm{AdS}_4)$$

where σ_S is a sign, dependent on the orientation of the sphere, the parameter $R < 0$ is the eleven-dimensional scalar curvature and the numbers in brackets are the scalar curvatures of each factor. Since the background is maximally supersymmetric, it will admit 32 Killing spinors solving the equation (1.1.1). The supersymmetry of this background boils down to the existence of geometric Killing spinors in each factor. On AdS_4 the equation (1.1.2) reduces to

$$\nabla_\mu \phi_A = \frac{1}{288} \left(8\delta_\mu^{\nu_1} \Gamma^{\nu_2\nu_3\nu_4} \right) F_{\nu_1\nu_2\nu_3\nu_4} \phi_A,$$

and on S^7 it becomes

$$\nabla_\mu \phi_S = -\frac{1}{288} \Gamma_\mu^{\nu_1\nu_2\nu_3\nu_4} F_{\nu_1\nu_2\nu_3\nu_4} \phi_S \tag{2.0.1}$$

where a Killing spinor on $\mathrm{AdS}_4 \times S^7$ is given by $\phi = \phi_A \otimes \phi_S$. Given that $F = -\sigma_S \sqrt{-6R}\,\mathrm{dvol}(\mathrm{AdS}_4)$ the equation (2.0.1) becomes

$$\nabla_\mu \phi_S = -\frac{4!\sigma_S}{288}\Gamma_\mu \sqrt{-6R}\phi_S = -\frac{\sigma_S}{12}\sqrt{-6R}\Gamma_\mu \phi_S,$$

and similarly in the AdS_4 factor. Thus we get

$$\nabla_X \phi_A = -\frac{\sigma_S}{6} f X \cdot \phi_A \text{ on } \mathrm{AdS}_4$$

and

$$\nabla_X \phi_S = \frac{\sigma_S}{12} f X \cdot \phi_S \text{ on } S^7,$$

where $f = -\sqrt{-6R}$. The Lie algebra of symmetries is $\mathfrak{so}(2,3) \oplus \mathfrak{so}(8)$ and its real 32-dimensional representation, on the Killing Spinors, is given by the tensor product representation

$$\Delta^{2,3} \otimes \Delta^{8,0}_{\sigma_S} ,$$

where $\Delta^{p,q}$ denotes the real half-spin representation of $\mathfrak{so}(p,q)$ and the subscript denotes chirality ($\sigma_S = \pm 1$). The chirality $\sigma_S = \pm 1$ has to do with the choice of irreducible 8-dimensional representation of Spin(8). This in turn corresponds to a choice of *orientation* of the sphere. Changes in the sign result in changing the sign of F hence orientation of AdS. At the moment we will leave the orientation of the sphere unfixed and consider quotients of the background in both orientations. What we shall find is that orientation is very relevant to supersymmetry and that *supersymmetry chooses orientation.*

We shall now begin our investigation of the supersymmetry of the background $\mathrm{AdS}_4 \times (S^7/G)$ where (S^7/G) is a seven-dimensional spherical space form. To this end we shall organise our discusson as follows. The rest of this chapter reviews the spherical space form problem and presents the solution to it, as given by Wolf in [37]. We then apply it to the case of S^7, determining the groups G giving seven-dimensional spherical space forms. Chapter 3 then considers the action of these groups on the spin bundle. In particular we construct $Spin(8)$ representations for the groups. Chapter 4 gives the results of our investigation, we determine the supersymmetry of the spherical space forms and highlight the importance that spin structure and orientation play on the supersymmetry.

2.1 Preliminaries

In this chapter we will give a summary account of the classification of spherical space forms, as solved by Wolf, given in [37]. [37] gives an extensive and much involved classification of spherical space forms, moreover it is concerned with spaces of constant curvature in general, to which we refer the interested reader. Spherical space forms are quotients of S^n by freely acting discrete subgroups G, of its isometry group $SO(n+1)$. The quotients are smooth and locally isometric to S^n itself and of constant positive curvature. The classification shows there are six 'types' of groups which solve the spherical space form problem. Each of these types has finite order and presentation. We shall make use of this classification in the case of S^7. Using the work of Wolf we shall construct representations for theses groups in $SO(8)$. In the following sections we shall, in spirit, draw on the notation used in [37].

Lemma 5. *Let M^n be a riemannian manifold of dimension $n \geq 2$. Then M^n is complete, connected and of unit positive curvature if and only if it is isometric to*

$$S^n/G \quad \textit{with} \quad G \subset SO(n+1).$$

Where G is a subgroup of the isometry group of the unit sphere S^n acting freely and properly discontinuously. The quotient manifold is called a spherical space form.

As mentioned we are interested in spherical space forms of S^7. We shall construct representations of the corresponding groups, which act freely and properly discontinuously, in $SO(8)$. To this end we review some background concepts in discrete group theory and representation theory that we shall need in the following work.

2.1.1 Finite Groups

Definition 6. The *generalised quaternionic group* $\mathbf{Q2^a}$, for every integer $a > 2$, is generated by A and B subject to the relations

$$\mathbf{Q2^a} : A^{2^{a-1}} = 1, B^2 = A^{2^{a-2}}, BAB^{-1} = A^{-1}.$$

Q8 is the quaternionic group.[1]

Finite Subgroups of SO(3)

Theorem 7. *Every finite subgroup of $SO(3)$ is either a cyclic (C_m), dihedral (D_m), tetrahedral (T), octahedral (O) or icosahedral (I) group. If two finite subgroups of $SO(3)$ are isomorphic, then they are conjugate in $SO(3)$.*

These groups together with their relations are given below;

$$C_m : \ A^m = 1.$$

$$D_m : \ A^m = B^2 = 1, \ BAB^{-1} = A^{-1}.$$

$$T : \ A^3 = P^2 = Q^2 = 1, \ PQ = QP, \ APA^{-1} = Q, \ AQA^{-1} = PQ.$$

$$O : \ A^3 = P^2 = Q^2 = R^2 = 1, \ PQ = QP, \ APA^{-1} = Q, \ AQA^{-1} = PQ,$$

$$RAR^{-1} = A^{-1}, \ RPR^{-1} = QP, \ RQR^{-1} = Q^{-1}.$$

$$I : X^5 = Y^3 = Z^4 = XYZ = 1.$$

These groups are called the *polyhedral groups*. Their lift to $SU(2)$, the double cover of $SO(3)$, defines the *binary polyhedral groups* as follows.

$$C_m^* : \ X^m = 1.$$

$$D_m^* : \ X^m = B^4 = 1, \ BXB^{-1} = X^{-1}.$$

$$T^* : \ X^3 = P^4 = Q^4 = 1, \ P^2 = Q^2, \ XPX^{-1} = Q, \ XQX^{-1} = PQ, \ PQP^{-1} = Q^{-1}.$$

[1]Note that the generalised quaternionic group $\mathbf{Q2^a} = D^*_{2^{a-2}}$.

$$O^* : \; X^3 = P^4 = Q^4 = R^4 = 1, \; P^2 = Q^2 = R^2, \; PQP^{-1} = Q^{-1}, \; XPX^{-1} = Q,$$

$$XQX^{-1} = PQ, \; RXR^{-1} = X^{-1}, \; RPR^{-1} = QP, \; RQR^{-1} = Q^{-1}.$$

$$I^* : X^5 = Y^3 = Z^2 = XYZ = -1.$$

In the classification of spherical spaceforms we will be interested in T^*, O^*, I^*, and the central extension of T^* and O^*. The central extension of T^* is denoted T^*_v and is given by:

$$T^*_v : \; X^{3^v} = P^4 = Q^4 = 1, \; P^2 = Q^2, \; XPX^{-1} = Q, \; XQX^{-1} = PQ, \; PQP^{-1} = Q^{-1}.$$

Note that T^*_v becomes T^* for $v = 1$. Similarly O^*_v is given by:

$$O^*_v : \; X^{3^v} = P^4 = Q^4 = R^4 = 1, \; P^2 = Q^2 = R^2, \; PQP^{-1} = Q^{-1}, \; XPX^{-1} = Q,$$

$$XQX^{-1} = PQ, \; RXR^{-1} = X^{-1}, \; RPR^{-1} = QP, \; RQR^{-1} = Q^{-1}$$

where $O^*_1 = O^*$.

2.1.2 Representations of Finite Groups

In this section we review some definitions and theorems, from representation theory that we will use in this work.

Definition 8. Let G be a group and let V be a vector space over a field F. Then a *representation* of G, on V, is a homomorphism $\pi : G \to GL_F(V)$ of G into the group of all invertible linear transformations of V. V is called the *representation space* of π.

Definition 9. π is called *faithful* if it is one to one.

Definition 10. If V is finite-dimensional, the dimension of V is called the *degree* of π. The *character* of a finite dimensional representation π, is the function $\chi_\pi : G \to F$ given by $\chi_\pi(g) = \text{trace}(\pi(g))$. Equivalent representations have the same character.

Definition 11. If U is a $\pi(G)$-invariant subspace of V, then $\pi'(g) : u \mapsto \pi(g)u$ defines a representation π' of G on U. Representations π' of this form are called *subrepresentations* of π. A subrepresentation π' of U, is proper if U is a proper subspace of V i.e. $0 \neq U \neq V$. We call π *irreducible*, if it has no proper subrepresentation. If π_i are representations on V_i over F, then the direct sum $\bigoplus_i v_i \mapsto \bigoplus_i \pi_i(g)v_i$, $v_i \in V_i$. π is *fully reducible* if it is equivalent to a direct sum of irreducible representations, i.e., if it is a direct sum of irreducible representations.

Definition 12. For a subgroup H of a group G the *index* of H, is the cardinality of the set of left cosets of H in G.

2.1.3 Induced Representations

In this section we shall look at the method of building a representation of a group from that of a subgroup, whose representation we do know.

Let π be a representation of a group G on a vector space V. If $V = \oplus_{i=1}^{n} V_i$ and if for every $g \in G$ the transformation $\pi(g)$ permutes the V_i, then π is *monomial relative* to $\{V_1, ..., V_n\}$. If each V_i has dimension one and π is monomial relative to $\{V_1, ..., V_n\}$, then π is monomial. If π is monomial relative to $\{V_1, ..., V_n\}$ and $\pi(G)$ is transitive on the V_i , then π is *monomial* and *transitive relative* to $\{V_1, ..., V_n\}$.

Let π be monomial and transitive relative to $\{V_1, ..., V_n\}$ and let H be the subgroup of G consisting of $g \in G$ such that $\pi(g)V_1 = V_1$, say. Let $\{b_1, ..., b_n\} \subset G$ then $\pi(b_i)V_1 = V_i$ after a possible reordering of the $\{b_1, ..., b_n\}$, if and only if $G = \bigcup b_i H$ with $b_1 \in H$. Thus the V_i correspond to the left cosets gH. Now let σ be the representation of H on V_1, which is a subrepresentation of the restriction of π to H. Given $g \in G$, we have a permutation $i \to k(i)$ of $\{1, ..., n\}$ such that $\pi(g)V_i = V_{k(i)}$. Then $\pi(b_{k(i)}^{-1} g b_i)$ preserves V_1, so $b_{k(i)}^{-1} g b_i \in H$. If $v_i \in V_i$, then $v_i = \pi(b_i)v$ for some $v \in V_1$, and $\pi(g)v_i = \pi(b_{k(i)})\sigma(b_{k(i)}^{-1} g b_i)v$. Thus π is *completely* determined by σ and the $\pi(b_i)|_{V_1}$.

We can view an $mn \times mn$ matrix A as an $n \times n$ matrix (A_{ij}) whose entries are $m \times m$ matrices A_{ij}. Then A is said to be in *block form* and the (A_{ij}) are the *blocks*. Now choose a basis of V_1. For each i, its $\pi(b_i)$-image is a basis of V_i, and the union of these is a basis of V. In that basis of V the matrix of $\pi(g)$ has block form with $\sigma(b_j^{-1}gb_i)$ for the (i,j)th block, where we adopt the convention that $\sigma(c) = 0$ if $c \notin H$.

Conversely let H be a subgroup of some finite index n in G and let σ be a representative of H on a vector space U. Choose coset representatives $\{b_i\}$ such that $G = \bigcup b_i H$ with $b_1 \in H$. In block form we set $\pi(g) = (\sigma(b_j^{-1}gb_i)$ with convention that $\sigma(c) = 0$ if $c \notin H$. Thus π is a representation of G on $U \oplus ... \oplus U$ (n times). If we alter the choice of b_i, this is equivalent to changing bases in the summands U thus obtaining the same representation. Now π is called the **induced representation** of σ from H to G denoted σ^G.

Lemma 13. *If σ is a representation on a vector space U of a subgroup H of finite index n in a group G, then there is a well defined induced representation σ^G of G on $V = U \oplus ... \oplus U$ given by $\sigma^G = (\sigma(b_j^{-1}gb_i))$ where $G = \bigcup_{i=1}^n b_i H$, and $\sigma(c) = 0$ for $c \notin H$.*

$V = U \oplus \sigma^G(b_2)U ... \oplus \sigma^G(b_n)U$, and σ^G is monomial and transitive relative to $\{V_i\}$, then

$$\{g \in G : \pi(g)V_1 = V_1\} = H$$

is a subgroup of index n in G and $\pi = \sigma^G$ where σ is the representation of H on V_1. In particular if π is a monomial and transitive representation of degree n of G, then π is induced from a representation of degree one of a subgroup of index n.

2.1.4 Necessary Conditions on Freely Acting Groups

Here we shall determine the properties of the groups which have fixed-point-free representations.

Definition 14. If π is a representation of a group G, and if $1 \neq g \in G$ implies that $\pi(g)$ does not have $+1$ as an eigenvalue, then π is *fixed-point-free.* Thus a fixed-point-free group is an abstract finite group which has a fixed-point-free representation.

Theorem 15. *Let G be a finite group which admits a fixed-point-free representation over a field F. If H is a subgroup of order pq in G, p and q are primes, then H is cyclic.*

Definition 16. Let G be a finite group. If, for a prime p, every subgroup of order p^2 in G is cyclic, then we say that G satisfies the *p^2-condition.* If p and q are primes such that every subgroup of order pq in G is cyclic, then we say that G satisfies the *pq-condition.*

Definition 17. Let G be a group of order N. If p is a prime, then a *p-group* is a group in which the order of every element is a power of p. Thus, if a subgroup $H \subset G$ has order p^a, then p^a divides N and H is a p-subgroup of G. If in addition p^a is the highest power of p which divides N, then H is called a *Sylow p-subgroup.*

Theorem 18. *If G is a finite group, then the following conditions are equivalent.*
1. *G satisfies the $p^2 -$ condition.*
2. *Every abelian subgroup of G is cyclic.*
3. *If p is an odd prime, then every Sylow $p -$ subgroup of G is cyclic; the Sylow 2-subgroups of G are cyclic or generalised quaternionic groups.*

The fixed-point-free groups fall into two classes according to Theorems 15 and 18. The first are all finite groups in which every Sylow subgroup is cyclic and the second is where all odd Sylow subgroups are cyclic and the Sylow 2-subgroups are generalised quaternionic.

The following theorem classifies all the finite groups, with every Sylow subgroup cyclic, which is one of the two classes of finite groups with fixed-point-free representations. These groups play a major role in our work and indeed are one of the types which solve the spherical space form problem. They will be called Type I groups in our work. We will see that they are normal subgroups, of every group in the solution of the spherical space form problem. Furthermore we will use their representation to induce representations for each of the groups solving the spherical space form problem.

Theorem 19. *Let G be a group of finite order N in which every Sylow subgroup* *is cyclic. Then G is generated by two elements A and B with defining relations,*

$$A^m = B^n = 1, \quad BAB^{-1} = A^r, \quad N = nm; \quad ((r-1)n, m) = 1, \quad r^n \equiv 1(m). \tag{2.1.1}$$

Both the commutator group $G' = \{A\}$ and the quotient G/G' are cyclic. Let d be the order of r in the multiplicative group of residues modulo m of integers prime to m. Then d divides n, and G satisfies all $pq-$conditions if and only if $n|d$ is divisible by every prime divisor of d. Conversely any group generated as above has order N and every Sylow subgroup cyclic.

Theorem 20. *Let G be a group of order N which satisfies all $pq-$conditions and has every Sylow subgroup cyclic, so G is given by $A^m = B^n = 1$ and $BA^{-1}B = A^r$ where $N = mn = mn'd$ and $((r-1)n, m) = 1$, where d is the order of r in K_m, the multiplicative group residues modulo m of integers prime to m. n' is divisible by every prime divisor of d. Suppose G is not cyclic of order 1 or 2. Let $R(\theta)$ denote the rotation matrix*

$$\begin{pmatrix} \cos(2\pi\theta) & \sin(2\pi\theta) \\ -\sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix}$$

Given integers k and l with $(k, m) = (l, n) = 1$, let $\hat{\pi}_{k,l}$ be the real representation of degree $2d$ of G defined by the 2×2 block matrices

$$\hat{\pi}_{k,l} = \begin{pmatrix} R(k/m) & & & & & \\ & R(kr/m) & & & & \\ & & R(kr^2/m) & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & R(kr^{d-1}) \end{pmatrix} \tag{2.1.2}$$

and

$$\hat{\pi}_{k,l} = \begin{pmatrix} 0 & 1 & & & & & \\ \vdots & & \ddots & & & & \\ \vdots & & & \ddots & & & \\ \vdots & & & & \ddots & & \\ \vdots & & & & & & 1 \\ R(l/n') & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}. \qquad (2.1.3)$$

Then the irreducible, faithful, fixed-point-free, representations of G *are the* $\hat{\pi}_{k,l}$. $\hat{\pi}_{k,l}$ *is equivalent to* $\hat{\pi}_{k',l'}$ *if and only if there are numbers* $e = \pm 1$ *and* $c = 0, 1, ..., d-1$ *such that* $k \equiv ekr^c (mod\, m)$ *and* $l' \equiv el(mod\, n')$.

Lemma 21. *Let* H *be a subgroup of* G *generated by* A *and* B^d. *Then* H *is a normal cyclic subgroup of* G *of order* mn/d *with generator* AB^d.

The real irreducible representation of degree 2-d is given in Theorem 20. We will use it to build representations for the groups that follow using the induced representation method.

2.2 Classification of Spherical Spaceforms

In this section we will give the six types of group which act freely and properly discontinuously on spheres. These six types exhaust all possibilities of such groups, thus solving the spherical space form problem. Since we are interested in the seven-dimensional spherical space forms we will give explicit representations, for each type of group, in $SO(8)$. We follow Wolf's notation [37], at least in spirit, in that our group types agree with his.

2.2.1 Type I Groups

Type I groups are the main *dramatis personae* of this story. They form subgroups of all the groups we will consider. Furthermore the representations of all the groups we will encounter are constructed via induction of irreducible representations of Type I groups. The Type I groups $G = G_{\mathrm{I}}(m, n, r)$ are those in

Theorem 20 and are generated by A and B subject to the relations

$$A^m = 1 \qquad B^n = 1 \qquad BAB^{-1} = A^r \ , \tag{2.2.1}$$

where $(m, n(r-1)) = 1$ and $r^n \equiv 1(m)$. This last identity is simply consistency between the second and third relations. The order of $G_{\mathrm{I}}(m,n,r)$ is thus mn.

Let $d > 0$ be the smallest integer such that $r^d \equiv 1(m)$. Notice that d divides n, for letting $n = dp + q$ where $q < d$, we see that $r^q \equiv 1(m)$ which violates the minimality of d, unless $q = 0$. If $d = 1$ we can set $r = 1$ with no loss of generality. Furthermore the condition $(m, n(r-1)) = 1$ is replaced with $(m,n) = 1$ and $AB = BA$, whence $G = \mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ with generator AB, say, where we have used that $(m,n) = 1$ and the group is abelian.

We shall now give the representations of Type I groups, in $SO(8)$. In 8-dimensions the representation space can be decomposed as four two-dimensional spaces corresponding to $d = 1$, two four-dimensional spaces corresponding to $d = 2$ and one 8-dimensional space corresponding to $d = 4$. The corresponding representations are given below;

$d = 1$

If $d = 1$, as shown before, we have an abelian group $G = \mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ thus it is convenient for our purposes to think of the group as a cyclic group with a generator C of order $N = nm$.

In this case we may construct a fixed-point-free representation by the direct sum of four 2−dimensional irreducible representations. This just means a direct sum of four rotation matrices as follows,

$$\varrho(C) = \begin{pmatrix} R\left(\frac{k_1}{N}\right) & & & \\ & R\left(\frac{k_2}{N}\right) & & \\ & & R\left(\frac{k_3}{N}\right) & \\ & & & R\left(\frac{k_4}{N}\right) \end{pmatrix}$$

where $(k_i, N) = 1$ for all i so that the representation is faithful. Without loss of generality, we can choose the generator in such a way that $k_1 = 1$. Moreover we can then conjugate (if necessary, by O(8)) to reorder the exponents such that $1 = k_1 \le k_2 \le k_3 \le k_4 < N$. Any cyclic subgroup of SO(8) acting freely on S^7 is conjugate (perhaps in O(8)) to the cyclic group generated by $\varrho(C)$.

$d = 2$

If $d = 2$ the representation is formed via direct sum of two irreducible representations $\hat{\varrho}(g) = \hat{\varrho}_1(g) \oplus \hat{\varrho}_2(g)$ thus we have,

$$\hat{\varrho}(A) = \begin{pmatrix} R\left(\frac{k_1}{m}\right) & & & \\ & R\left(\frac{rk_1}{m}\right) & & \\ & & R\left(\frac{k_2}{m}\right) & \\ & & & R\left(\frac{rk_2}{m}\right) \end{pmatrix} \text{ and } \hat{\varrho}(B) = \begin{pmatrix} & \mathbf{1} & & \\ R\left(\frac{2\ell_1}{n}\right) & & & \\ & & & \mathbf{1} \\ & & R\left(\frac{2\ell_2}{n}\right) & \end{pmatrix}. \tag{2.2.2}$$

$d = 4$

Similarly, if $d = 4$ we have

$$\hat{\varrho}(A) = \begin{pmatrix} R\left(\frac{k}{m}\right) & & & \\ & R\left(\frac{rk}{m}\right) & & \\ & & R\left(\frac{r^2k}{m}\right) & \\ & & & R\left(\frac{r^3k}{m}\right) \end{pmatrix} \quad \text{and} \quad \hat{\varrho}(B) = \begin{pmatrix} & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \\ R\left(\frac{4\ell}{n}\right) & & & \end{pmatrix}.$$

The parameters k_i above belong to the multiplicative group of residues modulo m, of integers prime to m. Similarly for the l_i.

2.2.2 Type II Groups

A Type II group $G = G_{\mathrm{II}}(m, n, r, s, t)$ is obtained by adding a new generator S to the Type I group $G_{\mathrm{I}}(m, n, r)$ generated by A and B, where n is now even, so that m is odd. Furthermore $G_{\mathrm{I}}(m, n, r) < G_{\mathrm{II}}(m, n, r, s, t)$ and thus we can induce a representation on $G_{\mathrm{II}}(m, n, r, s, t)$ from that of $G_{\mathrm{I}}(m, n, r)$. This new generator obeys $S^2 = B^{n/2}$, whence it has order 4 , and $SAS^{-1} = A^s$ and $SBS^{-1} = B^t$, where, for consistency between the relations, we demand that

$s^2 \equiv 1(m)$, $t^2 \equiv 1(n)$ and $r^{t-1} \equiv 1(m)$. Furthermore S^2 commutes with A and clearly B. Let $n = 2^u n'$ where $(2, n') = 1$. It follows that $G_{\mathrm{II}}(m, n, r, s, t)$ has order $2mn = 2^{u+1}mn'$. The subgroup generated by $B^{n'}$ and S has order 2^{u+1} and is therefore a Sylow 2-subgroup. Wolf shows that it is a generalised quaternionic subgroup and, in particular, this implies that $u \geq 2$. Consistency between the relations then implies that $t \equiv -1(2^u)$.

We now use the induced representation method to find a representation for this group. Choose $b_1 = 1$ and $b_2 = S$, thus it is clear that $G = H \cup S \cdot H$. Furthermore the induced representation will have twice the dimension of the inducing representation giving it degree $4d$. Therefore to induce an eight-dimensional representation we must either induce from two two-dimensional representations of of $G_{\mathrm{I}}(m, n, r)$, which requires $d = 1$, or alternatively from a four-dimensional representation of $G_{\mathrm{I}}(m, n, r)$ which in turn requires $d = 2$. We treat each case in turn.

$d = 1$

We induce a representation of $G = G_{\mathrm{II}}(m, n, s, t)$ from a representation of the cyclic subgroup H generated by A and B. The corresponding matrices in the induced representation are as follows:

$$\hat{\varrho}(A) := \begin{pmatrix} \varrho(A) & \varrho(S^{-1}A) \\ \varrho(AS) & \varrho(S^{-1}AS) \end{pmatrix} = \begin{pmatrix} \varrho(A) & 0 \\ 0 & \varrho(A^s) \end{pmatrix} ,$$

$$\hat{\varrho}(B) := \begin{pmatrix} \varrho(B) & \varrho(S^{-1}B) \\ \varrho(BS) & \varrho(S^{-1}BS) \end{pmatrix} = \begin{pmatrix} \varrho(B) & 0 \\ 0 & \varrho(B^t) \end{pmatrix} ,$$

$$\hat{\varrho}(S) := \begin{pmatrix} \varrho(S) & \varrho(S^{-1}S) \\ \varrho(SS) & \varrho(S^{-1}SS^{-1}) \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \varrho(B^{\frac{n}{2}}) & 0 \end{pmatrix},$$

with

$$\varrho(A) = R\left(\tfrac{k}{m}\right) \qquad \text{and} \qquad \varrho(B) = R\left(\tfrac{\ell}{n}\right) ,$$

where $(k,m) = 1$ and $(\ell,n) = 1$. The subgroup H is normal and $G/H \cong \mathbb{Z}_2$, whence the induced representation will have real dimension 4. We choose coset representatives 1 and S, whence the induced representation is given by

$$\hat{\varrho}(A) = \begin{pmatrix} R\left(\frac{k}{m}\right) & \\ & R\left(\frac{ks}{m}\right) \end{pmatrix} \qquad \hat{\varrho}(B) = \begin{pmatrix} R\left(\frac{\ell}{n}\right) & \\ & R\left(\frac{\ell t}{n}\right) \end{pmatrix} \qquad \hat{\varrho}(S) = \begin{pmatrix} & \mathbf{1} \\ -\mathbf{1} & \end{pmatrix} ,$$

where we have used that $R\left(\frac{\ell}{2}\right) = -\mathbf{1}$ since ℓ is odd. Now we just take the direct sum of two such representations:

$$\hat{\varrho}(A) = \begin{pmatrix} R\left(\frac{k_1}{m}\right) & & & \\ & R\left(\frac{k_1 s}{m}\right) & & \\ & & R\left(\frac{k_2}{m}\right) & \\ & & & R\left(\frac{k_2 s}{m}\right) \end{pmatrix} ,$$

$$\hat{\varrho}(B) = \begin{pmatrix} R\left(\frac{\ell_1}{n}\right) & & & \\ & R\left(\frac{\ell_1 t}{n}\right) & & \\ & & R\left(\frac{\ell_2}{n}\right) & \\ & & & R\left(\frac{\ell_2 t}{n}\right) \end{pmatrix}$$

and

$$\hat{\varrho}(S) = \begin{pmatrix} & \mathbf{1} & & \\ -\mathbf{1} & & & \\ & & & \mathbf{1} \\ & & -\mathbf{1} & \end{pmatrix} .$$

$d = 2$

Now we have $r^2 \equiv 1(m)$ and we induce the eight-dimensional representation from that of a real four-dimensional representation in similar fashion as above. Where we now have,

$$\varrho(A) = \begin{pmatrix} R\left(\frac{k}{m}\right) & \\ & R\left(\frac{kr}{m}\right) \end{pmatrix} \qquad \varrho(B) = \begin{pmatrix} & 1 \\ R\left(\frac{2\ell}{n}\right) & \end{pmatrix} \qquad \text{and}$$

$$\varrho(S) = \begin{pmatrix} & \mathbf{1} \\ -\mathbf{1} & \end{pmatrix} .$$

We find that

$$\hat{\varrho}(A) = \begin{pmatrix} R\left(\frac{k}{m}\right) & & & \\ & R\left(\frac{rk}{m}\right) & & \\ & & R\left(\frac{sk}{m}\right) & \\ & & & R\left(\frac{rsk}{m}\right) \end{pmatrix}$$

and

$$\hat{\varrho}(B) = \begin{pmatrix} & \mathbf{1} & & \\ R\left(\frac{2\ell}{n}\right) & & & \\ & & & R\left(\frac{\ell(t-1)}{n}\right) \\ & & R\left(\frac{\ell(t+1)}{n}\right) & \end{pmatrix} .$$

For the extra generator, since $n \equiv 0 \pmod 4$, we have

$$\hat{\varrho}(S) = \begin{pmatrix} & & \mathbf{1} & \\ & & & \mathbf{1} \\ -\mathbf{1} & & & \\ & -\mathbf{1} & & \end{pmatrix} ,$$

where we have used that since $(\ell, n) = 1$ and n is even, ℓ is odd and hence $R(\ell/2) = -\mathbf{1}$.

2.2.3 Type III Groups

Type III groups have generators $\{A, B, I, J\}$. We note that, in the case of S^7, these groups do not give spherical space forms. However type IV groups with generators $\{A, B, S, I, J\}$, of which type III groups are normal subgroups, do. Furthermore we can find an irreducible representation of type III groups, as we shall see, and induce a representation on type IV from this. We shall now make some observations about type III groups. Type III groups have generators $\{A, B, I, J\}$ where $\{A, B\}$ generate type I groups and has *odd order* and $\{I, J\}$ generates **Q8**, the quaternions, both of which are normal subgroups of type III groups, as shown below. **Q8** is isomorphic to the group of unit quaternions $\pm 1, \pm\boldsymbol{i}, \pm\boldsymbol{j}, \pm\boldsymbol{k}$; one isomorphism being given by $I \mapsto \boldsymbol{i}$ and $J \mapsto \boldsymbol{j}$. We let I, J generate a group **Q8** with relations

$$I^2 = J^2 \qquad I^4 = 1 \qquad IJ = J^{-1}I \ .$$

The group $G_I(m,n,r)$ acts on **Q8** as follows

$$AIA^{-1} = I \qquad AJA^{-1} = J \qquad BIB^{-1} = J \qquad BJB^{-1} = IJ \ .$$

In other words, B permutes the imaginary units cyclically: $\boldsymbol{i} \to \boldsymbol{j} \to \boldsymbol{k}$. Therefore the action of B has order 3 and thus 3 must divide the order of B. This means $n \equiv 0(3)$. By splitting this group up as a direct product $H_1 \times H_2 = \left\langle A, B^{3^v} \right\rangle \times \left\langle B^{n'}, I, J \right\rangle$ where we define $v \geq 1$ and n' by $n = 3^v n'$ with $(3, n') = 1$ we may form a representation of type III groups. Note that B can be regained from this split, via Euclidean algorithm, by finding a, b such that $a3^v + bn' \equiv 1(n)$. We will work with complex representations and then at the end obtain a real representation by the usual trick of doubling and taking the real part. $\left\langle A, B^{3^v} \right\rangle$ is a type I group, whose real irreducible representations we know from previous sections. The corresponding complex irreducible representations are the equivalent rotations in the complex plane given by

$$\varrho_1(A) = \exp\left(2\pi i k/m\right) \qquad \text{and} \qquad \varrho_1(B^{3^v}) = \exp\left(2\pi i 3^v \ell/n\right) \ ,$$

where $(k, m) = 1$ and $(\ell, n) = 1$.

To work out the irreducible representations of H_2 we make the following observation. If $v = 1$, so that $n \not\equiv 0(9)$ then $H_2 \cong T^*$, the binary tetrahedral group. For $v > 1$, so that $n \equiv 0(9)$, H_2 is given by T_v^* which is a nontrivial central extension

$$1 \longrightarrow \left\langle B^{3n'} \right\rangle \longrightarrow H_2 \longrightarrow T^* \longrightarrow 1 \ ,$$

which allows us to build representations of H_2 as projective representations of T^*. In practise, if $\overline{\varrho} : T^* \to \mathrm{GL}(V)$ is a complex representation of T^*, then we obtain a complex representation $\varrho : H_2 \to \mathrm{GL}(V)$ by setting $\varrho(q) = \overline{\varrho}(q)$ for $q \in \mathbf{Q8}$ and $\varrho\left(B^{n'}\right) = \exp(2\pi i p/3^v)\overline{\varrho}\left(B^{n'}\right)$, where $(p, 3) = 1$.

Irreducible representations of T^* are one-dimensional quaternionic. Indeed, notice that we can embed T^* in the quaternions: we have seen that $I \mapsto \boldsymbol{i}$ and $J \mapsto \boldsymbol{j}$, but also $B^{n'}$ can be written as a quaternion. Let $\boldsymbol{b}$ denote the quaternion corresponding to (the action of) $B^{n'}$. Since $B^{n'} I B^{-n'} = J$ and $B^{n'} J B^{-n'} = IJ$, we have that $\boldsymbol{b}\boldsymbol{i}\boldsymbol{b}^{-1} = \boldsymbol{j}$ and $\boldsymbol{b}\boldsymbol{j}\boldsymbol{b}^{-1} = \boldsymbol{k}$. This defines $\boldsymbol{b}$ up to a real scale. Demanding that $|\boldsymbol{b}| = 1$ we find that there are two possibilities for $\boldsymbol{b}$:

$$\boldsymbol{b} = \pm \exp\left(\tfrac{\pi}{3\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})\right) \ .$$

Notice that since $(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})/\sqrt{3}$ squares to -1, then $\boldsymbol{b}^3 = \mp 1$, whence $\boldsymbol{b}^n = 1$ forces us to take the negative sign, since n is an odd multiple of 3. Therefore

$$\boldsymbol{b} = -\tfrac{1}{2}(1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \ .$$

Therefore $B^{n'}$ corresponds to the quaternion $\boldsymbol{b}$ if $n' \equiv 1(3)$ and to its quaternionic conjugate $\bar{\boldsymbol{b}} = \frac{1}{2}(-1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$ if $n' \equiv 2(3)$.

Taking a complex two-dimensional representation of the quaternions where, for example,

$$\boldsymbol{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ , \quad \boldsymbol{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and hence} \quad \boldsymbol{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ ,$$

then $B^{n'}$ is represented by

$$\overline{\varrho}\left(B^{n'}\right) := \begin{cases} -\frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} \ , & n' \equiv 1 \pmod 3 \\ \frac{1}{2}\begin{pmatrix} -1+i & 1+i \\ -1+i & -1-i \end{pmatrix} \ , & n' \equiv 2 \pmod 3 \end{cases}$$

when $v = 1$ and $\varrho\left(B^{n'}\right) = \exp(2\pi p/3^v)\overline{\varrho}\left(B^{n'}\right)$ for $v > 1$.

We may obtain a real four-dimensional representation, from this representation on $\mathbb{C}^2$, by the usual isomorphism $\exp(i\theta) \to R(\theta)$. Tensoring these two representations (over $\mathbb{C}$) $\varrho_1 \otimes \overline{\varrho}$ $(v = 1)$ and $\varrho_1 \otimes \varrho$ $(v > 1)$, we obtain a complex two-dimensional representation of H.

2.2.4 Type IV Groups

Type IV groups $G_{\mathrm{IV}}(m,n,r,s,t)$ have generators $\{A,B,S,I,J\}$, where $\{A,B,I,J\}$ generate Type III groups and $\{A,B\}$ generates a Type I group with *odd order* $N=mn$. They are constructed in the following way. We start with a Type I group $G_1=G_{\mathrm{I}}(m,n,r)$ generated by A,B with $A^m=1$, $B^n=1$ and $BAB^{-1}=A^r$ with $(n(r-1),m)=1$ and $r^n\equiv 1(m)$. Then we introduce generators $\{I,J\}$, as in the previous section. This will give us Type III groups. Finally, we introduce a new generator S of order 4 subject to the relations $S^2=I^2$ and

$$SAS^{-1}=A^s \qquad SBS^{-1}=B^t \qquad SIS^{-1}=JI \qquad SJS^{-1}=J^{-1}\ ,$$

where $s^2\equiv 1(m)$, $t^2\equiv 1(n)$, $r^{t-1}\equiv 1(m)$ and $t\equiv -1(3)$. These conditions follow for consistency between the relations. As usual we let d be the smallest positive integer such that $r^d\equiv 1(m)$. It turns out that for S^7 we require $d=1$. There is no single fixed-point-free representation for this group. Instead we decompose the group into the cases $n\not\equiv 0(9)$ and $n\equiv 0(9)$ for which we can find fixed point free representations. If $n\not\equiv 0(9)$ then we induce a representation on this group from that of Type III groups, in the case $n\not\equiv 0(9)$. When $n\equiv 0(9)$ there are two cases to consider, the first where we split the group up as a product group $\langle A,B^{3^v}\rangle\times\langle B^{n'},I,J,S\rangle$ where $\langle B^{n'},I,J,S\rangle$ is the group O_v^*. The second case is where we induce a representation from Type III groups, in the case that $n\not\equiv 0(9)$. The complex two-dimensional representations of Type III groups are

$$\varrho_1(A)\otimes\varrho(1)\ ,\quad \varrho_1(1)\otimes\varrho(I)=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix} \text{ and } \varrho_1(1)\otimes\varrho(J)=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$$

where $\varrho_1(A)$ and $\varrho_1(B^{3^v})$ are one dimensional complex (since $d=1$) and B is given by $\varrho_1(B^{3^v})\otimes\overline{\varrho}(B^{n'})$ when $n\not\equiv 0(9)$, or $\varrho_1(B^{3^v})\otimes\varrho(B^{n'})$ when $n\equiv 0(9)$.

Note $\varrho(B^{n'}) = \exp(2\pi i p/3^v)\overline{\varrho}(B^{n'})$ where

$$\overline{\varrho}\left(B^{n'}\right) := \begin{cases} -\frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} , & n' \equiv 1(3) \\ \frac{1}{2}\begin{pmatrix} -1+i & 1+i \\ -1+i & -1-i \end{pmatrix} , & n' \equiv 2(3) \ . \end{cases}$$

We use this representation to induce a representation on Type IV groups. We label the coset representatives by 1 and S. The complex four-dimensional representation, also denoted $\hat{\varrho}_{\mathbb{C}}$ in a slight abuse of notation, are defined below. Case (i): $n \not\equiv 0(9)$ where $\hat{\varrho}_{\mathbb{C}}$ is induced from a representation on type III groups, in the case that $n \not\equiv 0(9)$.

$$\hat{\varrho}_{\mathbb{C}}(A) = \begin{pmatrix} e^{\frac{i2\pi k}{m}}\mathbf{1} & \\ & e^{\frac{i2\pi sk}{m}}\mathbf{1} \end{pmatrix} \qquad \hat{\varrho}_{\mathbb{C}}(B^3) = \begin{pmatrix} e^{\frac{i2\pi 3\ell}{n}}\mathbf{1} & \\ & e^{\frac{i2\pi 3\ell t}{n}}\mathbf{1} \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(B^{n'}) = \begin{pmatrix} \overline{\varrho}\left(B^{n'}\right) & \\ & \overline{\varrho}\left(B^{n'}\right)^{\dagger} \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(I) = \begin{pmatrix} i & & & \\ & -i & & \\ & & & -i \\ & & -i & \end{pmatrix} \qquad \hat{\varrho}_{\mathbb{C}}(J) = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(S) = \begin{pmatrix} & \mathbf{1} \\ -\mathbf{1} & \end{pmatrix} ,$$

Case (ii): $n \equiv 0(9)$ where we find the representation by splitting the group up as $\left\langle A, B^{3^v}\right\rangle \times \left\langle B^{n'}, I, J, S\right\rangle$.

$$\hat{\varrho}_{\mathbb{C}}(A) = \begin{pmatrix} e^{\frac{i2\pi k}{m}}\mathbf{1} & \\ & e^{\frac{i2\pi k}{m}}\mathbf{1} \end{pmatrix} \qquad \hat{\varrho}_{\mathbb{C}}(B^{3^v}) = \begin{pmatrix} e^{\frac{i2\pi 3^v \ell}{n}}\mathbf{1} & \\ & e^{\frac{i2\pi 3^v \ell}{n}}\mathbf{1} \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(B^{n'}) = \begin{pmatrix} e^{\frac{i2\pi p}{3^v}}\overline{\varrho}\left(B^{n'}\right) & \\ & e^{\frac{-i2\pi p}{3^v}}\overline{\varrho}\left(B^{n'}\right)^{\dagger} \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(I) = \begin{pmatrix} i & & & \\ & -i & & \\ & & & -i \\ & & -i & \end{pmatrix} \qquad \hat{\varrho}_{\mathbb{C}}(J) = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(S) = \begin{pmatrix} & \mathbf{1} \\ -\mathbf{1} & \end{pmatrix} \quad ,$$

Case (iii): $n \not\equiv 0(9)$ where we induce a representation from the representation of a Type III groups, in the case $n \not\equiv 0(9)$.

$$\hat{\varrho}_{\mathbb{C}}(A) = \begin{pmatrix} e^{\frac{i2\pi k}{m}}\mathbf{1} & \\ & e^{\frac{i2\pi sk}{m}}\mathbf{1} \end{pmatrix} \qquad \hat{\varrho}_{\mathbb{C}}(B^{3^v}) = \begin{pmatrix} e^{\frac{i2\pi 3^v \ell}{n}}\mathbf{1} & \\ & e^{\frac{i2\pi 3^v \ell t}{n}}\mathbf{1} \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(B^{n'}) = \begin{pmatrix} e^{\frac{i2\pi p}{3^v}}\overline{\varrho}\left(B^{n'}\right) & \\ & e^{\frac{-i2\pi p}{3^v}}\overline{\varrho}\left(B^{n'}\right)^{\dagger} \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(I) = \begin{pmatrix} i & & & \\ & -i & & \\ & & & -i \\ & & -i & \end{pmatrix} \qquad \hat{\varrho}_{\mathbb{C}}(J) = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(S) = \begin{pmatrix} & \mathbf{1} \\ -\mathbf{1} & \end{pmatrix} \quad ,$$

In each case we have used the condition $t \equiv -1(3^v)$ in the calculation of $\hat{\varrho}_{\mathbb{C}}(B^{n'})$. This is a representation in $\mathbb{C}^4$, but we can obtain an 8-dimensional real representation by the isomorphism $\exp(i\theta) \to R(\theta)$. Below we have written the last case as an example,

$$\hat{\varrho}(A) = \begin{pmatrix} R\left(\frac{k}{m}\right) & & & \\ & R\left(\frac{k}{m}\right) & & \\ & & R\left(\frac{sk}{m}\right) & \\ & & & R\left(\frac{sk}{m}\right) \end{pmatrix}$$

$$\hat{\varrho}(B^{3^v}) = \begin{pmatrix} R\left(\frac{3^v\ell}{n}\right) & & & \\ & R\left(\frac{3^v\ell}{n}\right) & & \\ & & R\left(\frac{3^v\ell t}{n}\right) & \\ & & & R\left(\frac{3^v\ell t}{n}\right) \end{pmatrix}$$

$$\hat{\varrho}(I) = \begin{pmatrix} & 1 & & & & & & \\ -1 & & & & & & & \\ & & & -1 & & & & \\ & & 1 & & & & & \\ & & & & & & & -1 \\ & & & & & & 1 & \\ & & & & & -1 & & \\ & & & & 1 & & & \end{pmatrix}$$

$$\hat{\varrho}(J) = \begin{pmatrix} & \mathbf{1} & & \\ \mathbf{-1} & & & \\ & & & \mathbf{-1} \\ & & \mathbf{1} & \end{pmatrix} \qquad \text{and} \qquad \hat{\varrho}(S) = \begin{pmatrix} & \mathbf{1} \\ \mathbf{-1} & \end{pmatrix} \ ,$$

and where for $n' \equiv 1(3)$,

$$\hat{\varrho}(B^{n'}) = \frac{1}{2} \begin{pmatrix} R\left(\frac{p}{3^v}\right) & & & \\ & R\left(\frac{p}{3^v}\right) & & \\ & & R\left(\frac{-p}{3^v}\right) & \\ & & & R\left(\frac{-p}{3^v}\right) \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 & -1 & & & & \\ 1 & -1 & 1 & -1 & & & & \\ 1 & -1 & -1 & 1 & & & & \\ 1 & 1 & -1 & -1 & & & & \\ & & & & -1 & 1 & 1 & 1 \\ & & & & -1 & -1 & -1 & 1 \\ & & & & -1 & 1 & -1 & -1 \\ & & & & -1 & -1 & 1 & -1 \end{pmatrix}$$

whereas for $n' \equiv 2(3)$,

$$\hat{\varrho}(B^{n'}) = \frac{1}{2} \begin{pmatrix} R\left(\frac{p}{3^v}\right) & & & \\ & R\left(\frac{p}{3^v}\right) & & \\ & & R\left(\frac{-p}{3^v}\right) & \\ & & & R\left(\frac{-p}{3^v}\right) \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 & 1 & & & & \\ -1 & -1 & -1 & 1 & & & & \\ -1 & 1 & -1 & -1 & & & & \\ -1 & -1 & 1 & -1 & & & & \\ & & & & -1 & -1 & -1 & -1 \\ & & & & 1 & -1 & 1 & -1 \\ & & & & 1 & -1 & -1 & 1 \\ & & & & 1 & 1 & -1 & -1 \end{pmatrix} .$$

2.2.5 Type VI Groups

Type VI groups are of the form $G_{VI}(m,n,s,t) = \{K \times I^*, S\}$ where K is a type I group with order prime to 30 and $\{K \times I^*\}$ is a normal subgroup of $G_{VI}(m,n,s,t)$. In fact $\{K \times I^*\}$ is the Type V group, solving the general spherical spaceform problem see [37]. S is an order 4 element which normalises I^* and K. Conjugation of I^* by S is the outer automorphism φ of I^*. Thus type VI groups are made up of type I groups, together with their corresponding relations and in addition $(mn, 30) = 1$, and the following relations involving S: $S^2 = -1$ and

$$SAS^{-1} = A^s \qquad SBS^{-1} = B^t \qquad SxS^{-1} = \varphi(x) \quad \forall x \in I^* ,$$

where φ is an outer automorphism of I^* to be described below and where $s^2 \equiv 1(m)$, $t^2 \equiv 1(n)$ and $r^{t-1} \equiv 1(m)$, for consistency between the relations. In terms of quaternions the generators of I^* are;

$$X = \tfrac{1}{2}(\tau + \tau^{-1}\boldsymbol{i} + \boldsymbol{k}) \qquad Y = \tfrac{1}{2}(1 + \tau^{-1}\boldsymbol{j} + \tau\boldsymbol{k}) \qquad Z = \boldsymbol{k} .$$

One checks that indeed $X^5 = Y^3 = Z^2 = XYZ = -1$. Let $G_{\mathrm{I}}(m,n,r)$ be a Type I group with order prime to 30 and generators A and B as usual, and let I^* denote the binary icosahedral group the lift of the binary icosahedral group in $SO(3)$ to $Spin(3) \cong SU(2)$. It forms a finite subgroup of $SU(2)$ of order 120. In terms of generators and relations I^* is generated by X, Y, Z where $X^5 = Y^3 = Z^2 = XYZ$ is a central element of order 2. In terms of $SU(2)$ or quaternions this central element is -1 and we will often denote it in this way, abusing notation slightly.

To define the outer automorphism φ of I^* it is convenient to think of I^* as $SL(2, \mathbb{F}_5)$, the group of two-by-two matrices with unit determinant in the finite field $\mathbb{F}_5$. This is the field whose elements are residue classes of integers modulo 5. It can be shown that the group of outer automorphisms of $SL(2, \mathbb{F}_5)$ has order 2 and is generated, for example, by conjugation with the element

$$M = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{F}_5) , \quad \text{with inverse} \quad M^{-1} = \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix} .$$

The matrices M and M^{-1} have determinant $\pm 2 \in \mathbb{F}_5$, respectively. In order to explicitly write this outer automorphism in terms of quaternions, say, we need to choose an explicit isomorphism between I^* and $SL(2, \mathbb{F}_5)$. We do so by choosing matrices in $SL(2, \mathbb{F}_5)$ for the generators X, Y, Z and then working out the effect of conjugation by M on the generators and rewriting the result as words in the generators. After some experimentation, a possible choice is

$$X = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} .$$

One checks that $X^5 = Y^3 = Z^2 = XYZ = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Conjugating by M we find

$$\begin{aligned} MXM^{-1} &= \begin{pmatrix} -2 & -2 \\ -2 & 0 \end{pmatrix} = -X^3YX^4YX \\ MYM^{-1} &= \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} = -X^2YX^3 \\ MZM^{-1} &= \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} = X^3YX^3ZX^3YX^3 . \end{aligned} \tag{2.2.3}$$

Finally, we write this back in terms of quaternions, to find that the effect of the outer automorphism on the generators is

$$\begin{aligned} \tfrac{1}{2}(\tau + \tau^{-1}\boldsymbol{i} + \boldsymbol{k}) &\mapsto -\tfrac{1}{2}(\tau^{-1} + \tau\boldsymbol{i} + \boldsymbol{j}) \\ \tfrac{1}{2}(1 + \tau^{-1}\boldsymbol{j} + \tau\boldsymbol{k}) &\mapsto \ \ \tfrac{1}{2}(1 + \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) \\ \boldsymbol{k} &\mapsto -\tfrac{1}{2}(\tau\boldsymbol{i} - \tau^{-1}\boldsymbol{j} + \boldsymbol{k})\ . \end{aligned}$$

One can check that this is indeed an automorphism and moreover that it is not inner. This is the automorphism φ in the definition of $G_{\mathrm{VI}}(m, n, r, s, t)$.

Let H be the normal subgroup generated by A, B, X, Y, Z this is of index 2 in G. Thus we can induce a representation on G, which is twice the dimension of the representation on H, from H. H splits up as $K \times I^*$ thus the representation on H is given by tensor product of representations on K and I^*. In order to build a fixed-point-free representation of G in $SO(8)$ we are forced to take $d = 1$.

$d = 1$

We denote by $G_{\mathrm{VI}}(m, n, s, t)$ the Type VI group above with $d = 1$ and hence $r = 1$ without loss of generality. As we did for Type IV groups, we will obtain an 8-dimensional real representation of $G = G_{\mathrm{VI}}(m, n, s, t)$ by doubling a 4-dimensional complex representation induced from the following 2-dimensional complex representation of the normal subgroup $H = G_{\mathrm{I}}(m, n) \times I^*$, note when $d = 1$ the complex representation of K is one dimensional and the complex representation of I^* lies in $SU(2)$ so is two dimensional:

$$\varrho(A) = \exp(\tfrac{2\pi i k}{m}) \otimes \mathbf{1} \qquad \varrho(B) = \exp(\tfrac{2\pi i \ell}{n}) \otimes \mathbf{1}$$

$$\varrho(X) = \frac{1}{2}\begin{pmatrix} \tau + i\tau^{-1} & i \\ i & \tau - i\tau^{-1} \end{pmatrix} \qquad \varrho(Y) = \frac{1}{2}\begin{pmatrix} 1 & \tau^{-1} + i\tau \\ -\tau^{-1} + i\tau & 1 \end{pmatrix}\ ,$$

where $(k, m) = (\ell, n) = 1$.

Let us label the cosets in G/H by 1 and S^{-1}. We have

$$\begin{aligned} AS^{-1} &= S^{-1}A^s & BS^{-1} &= S^{-1}B^t \\ XS^{-1} &= S^{-1}\varphi(X) & YS^{-1} &= S^{-1}\varphi(Y) \\ SS^{-1} &= 1 & S &= S^{-1}(-1)\ , \end{aligned}$$

whence the (complex) induced representation $\hat{\varrho}_{\mathbb{C}}$ is given by

$$\hat{\varrho}_{\mathbb{C}}(A) = \begin{pmatrix} \exp(\frac{2\pi i k}{m})\mathbf{1} & \\ & \exp(\frac{2\pi i k s}{m})\mathbf{1} \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(B) = \begin{pmatrix} \exp(\frac{2\pi i \ell}{n})\mathbf{1} & \\ & \exp(\frac{2\pi i \ell t}{n})\mathbf{1} \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(X) = \frac{1}{2}\begin{pmatrix} \tau + i\tau^{-1} & i & & \\ i & \tau - i\tau^{-1} & & \\ & & -\tau^{-1} - i\tau & -1 \\ & & 1 & -\tau^{-1} + i\tau \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(Y) = \frac{1}{2}\begin{pmatrix} 1 & \tau^{-1} + i\tau & & \\ -\tau^{-1} + i\tau & 1 & & \\ & & 1+i & -1+i \\ & & 1+i & 1-i \end{pmatrix}$$

$$\hat{\varrho}_{\mathbb{C}}(S) = \begin{pmatrix} & \mathbf{1} \\ -\mathbf{1} & \end{pmatrix} .$$

Finally we let $\exp(i\theta) \to R(\theta)$, to arrive at a real 8-dimensional representation $\hat{\varrho} : G \to \mathrm{SO}(8)$:

$$\hat{\varrho}(A) = \begin{pmatrix} R(\frac{k}{m}) & & & \\ & R(\frac{k}{m}) & & \\ & & R(\frac{ks}{m}) & \\ & & & R(\frac{ks}{m}) \end{pmatrix}$$

$$\hat{\varrho}(B) = \begin{pmatrix} R(\frac{\ell}{n}) & & & \\ & R(\frac{\ell}{n}) & & \\ & & R(\frac{\ell t}{n}) & \\ & & & R(\frac{\ell t}{n}) \end{pmatrix}$$

$$\hat{\varrho}(X) = \frac{1}{2} \begin{pmatrix} \tau & \tau^{-1} & 0 & 1 & & & & \\ -\tau^{-1} & \tau & -1 & 0 & & & & \\ 0 & 1 & \tau & -\tau^{-1} & & & & \\ -1 & 0 & \tau^{-1} & \tau & & & & \\ & & & & -\tau^{-1} & -\tau & -1 & 0 \\ & & & & \tau & -\tau^{-1} & 0 & -1 \\ & & & & 1 & 0 & -\tau^{-1} & \tau \\ & & & & 0 & 1 & -\tau & -\tau^{-1} \end{pmatrix}$$

$$\hat{\varrho}(Y) = \frac{1}{2} \begin{pmatrix} 1 & 0 & \tau^{-1} & \tau & & & & \\ 0 & 1 & -\tau & \tau^{-1} & & & & \\ -\tau^{-1} & \tau & 1 & 0 & & & & \\ -\tau & -\tau^{-1} & 1 & 0 & & & & \\ & & & & 1 & 1 & -1 & 1 \\ & & & & -1 & 1 & -1 & -1 \\ & & & & 1 & 1 & -1 & -1 \\ & & & & -1 & 1 & 1 & 1 \end{pmatrix}$$

$$\hat{\varrho}(S) = \begin{pmatrix} & \mathbf{1} \\ -\mathbf{1} & \end{pmatrix} \ .$$

Chapter 3

Seven-Dimensional Supersymmetric Spherical Space Forms

In this chapter we shall consider the action of the groups solving the seven-dimensional spherical space form problem on the spin bundle. Given S^7 is a spin manifold, we would like to know when S^7/G is spin. Then for those cases where S^7/G is spin, we shall investigate how many Killing spinors it admits. The investigation is simplified considerably by using Bär's cone construction [4] and working with parallel spinors on $\mathbb{R}^8$.

In the next section we shall investigate when S^7/G is spin and how we may determine the number of linearly independent Killing spinors on the quotient via Bär's cone construction. We shall then investigate the number of inequivalent spin structures on the quotient. Indeed our results will show the importance of spin structure as a factor governing the amount of supersymmetry of the quotient.

3.1 Spin Structure on Spherical Space Forms

Let M be a Riemannian spin manifold and let $P_{SO}(M) \to M$ denote the bundle of orthonormal frames and $P_{Spin}(M) \to M$ the spin bundle on M. Since M is spin there is a bundle map $\theta : P_{Spin}(M) \to P_{SO}(M)$ which is a double covering map. This map agrees fiberwise with the double covering map $\theta_0 : Spin(n) \to SO(n)$, such that $\theta(\hat{\gamma}\hat{p}) = \theta_0(\hat{\gamma})\theta(\hat{p})$ where $\hat{\gamma} \in Spin(n)$ and $\hat{p} \in P_{Spin}(m)$. Consider a quotient of M by a discrete subgroup G of the isometry group of M which acts freely and properly discontinuously. Since M is spin the quotient M/G is again spin if and only if the action of G on the bundle $P_{SO}(M)$ lifts to an action on $P_{Spin}(M)$ in such a way that the surjection $\theta : P_{Spin}(M) \to P_{SO}(M)$ is G equivariant, i.e.,

$$G \cdot \theta(p) = \theta(G \cdot p).$$

That is provided there exists a group $\hat{G} \subset Spin(n)$, to which G lifts isomorphically. If we try to lift G to $Spin(n)$ but find $-\mathbf{1}$ in the process then G cannot be lifted to $Spin(n)$ isomorphically. Instead this lift gives a double cover of G whose action on M under the map θ will not be free. Provided G lifts to $Spin(n)$, the spin bundle $P_{Spin}(M/G)$ on the quotient is given by:

$$\begin{array}{c} P_{Spin}(M/G) := P_{Spin}(M)/G \\ \big\downarrow \theta \\ P_{SO}(M/G) := P_{SO}(M)/G \end{array}$$

which is the double cover of $P_{SO}(M/G)$ (see [18], [14] for more on this and for the corresponding argument for one-parameter groups).

In our case M is S^7 and the G is a freely acting finite subgroup of $SO(8)$. Therefore we shall try to find the isomorphic lift $\hat{G}$, of each group G solving the spherical space form problem. In practise we shall find the isomorphic lift $\hat{G}$ by lifting each generator of G to $Spin(8)$ (note there are two possible lifts). Then checking which choice of lift for each generator, if any, satisfies the group relations. Thus recovering isomorphic groups, in different spin structures, on the quotient. If indeed we do manage to find a lift $\hat{G}$, of the group G, then we can investigate the Killing spinors on the quotient. This investigation is made easier by Bär's cone construction [4]. It relates Killing spinors on S^7 to parallel spinors on the cone $\mathbb{R}^8$ as shown in our investigation of the cone construction in Appendix A. Moreover this correspondence is equivariant with respect to the isometry group. Therefore the Killing spinors on S^7/G are in one-to-one correspondence with $\hat{G}$-invariant parallel spinors on $\mathbb{R}^8$ hence the $\hat{G}$-invariant *constant* spinors of $Spin(8)$ in the relevant chiral representation. We must therefore check for each of the possible isomorphic lifts of G to inequivalent spin structures, whether $\hat{G}$ preserves any $Spin(8)$ spinors.

In practise what we shall do is form a basis for each of the chiral $Spin(8)$ modules denoted $\mathbb{S}_{\pm}$. Then act with the group $\hat{G}$ on each basis to see if there are any linearly independent spinors preserved. The number of such linearly independent invariant spinors corresponds to the number of Killing spinors on S^7/G hence the supersymmetry of the quotient.

3.1.1 Uniqueness of Spin Structure on S^7/G

The Stiefel-Whitney class $w_r(M)$, is a characteristic class that takes its values in $H^r(M, \mathbb{Z}_2)$ the rth cohomology group. The obstruction to orientability and spin structure are as follows [31], [28]:

Theorem 22. *Let $TM \to M$ be a tangent bundle with fibre metric, then M is orientable if and only if $w_1(M) = 0$.*

Theorem 23. *Let TM be the tangent bundle over orientable manifold M. Then there exists a spin bundle on M if and only if $w_2(M) = 0$.*

Theorem 24. *If $w_2(M) = 0$ then the distinct spin structures on M are in one-to-on correspondence with the elements of $H^1(M, \mathbb{Z}_2)$.*

For our purpose the manifold is S^7 and furthermore $H^1(S^7, \mathbb{Z}_2) = 0$ thus there is only one spin structure on S^7. The number of inequivalent spin structures on S^7/G is given by the number of elements in $H^1(S^7/G, \mathbb{Z}_2)$. Since S^7 is simply connected cover of S^7/G, the quotient has fundamental group G. Using this in the result, from the universal coefficient theorem of cohomology [26], $H^1(M, \mathbb{Z}_2) = Hom(\pi_1(M), \mathbb{Z}_2)$ we get $H^1(S^7/G, \mathbb{Z}_2) = Hom(\pi_1(S^7/G), \mathbb{Z}_2) = Hom(G, \mathbb{Z}_2)$ thus the number of elements in $Hom(G, \mathbb{Z}_2)$ determines the number of inequivalent spin structures on the quotient.

For each generator g of G there corresponds two possible lifts (elements) $\pm\hat{g} \in Spin(n)$. These two lifts differ by a sign. Any choice of signs, such that the lifts of the generators satisfy the group relations gives rise to a subgroup $\hat{G} \in Spin(n)$ isomorphic to G. An element of $Hom(G, \mathbb{Z}_2)$ maps each element of G to an element of $\mathbb{Z}_2$, this is tantamount to assigning a sign ± 1 to them. In particular it assigns a sign to each generator of G. Thus it follows that there is a one-to-one correspondence between the elements of $Hom(G, \mathbb{Z}_2)$ and the isomorphic lifts of G to $Spin(n)$ (via the signs). Therefore the number of inequivalent spin structures on S^7/G are in one-to-one correspondence with the isomorphic lifts of G to $Spin(n)$.

3.2 Lifting Elements of $SO(n)$ to $Spin(n)$

The Lie group $Spin(n) \subset Cl(n)$ is the double cover of $SO(n)$. Thus for every element $g \in SO(n)$, there are two corresponding elements $\pm\hat{g} \in Spin(n) \subset Cl(n)$. Indeed the transformation in $\mathbb{R}^n$ is implemented by elements in $Spin(n)$ as,

$$g \cdot \boldsymbol{v} = \hat{g} \cdot \boldsymbol{v} \cdot \hat{g}^{-1}.$$

We will now consider how to determine the $\hat{g}$ corresponding to a given $g \in SO(8)$. As we shall see later we may deconstruct any $g \in SO(8)$ into a block diagonal rotation matrix and permutation part. Thus we would like to know in general how to lift rotations and permutations of the basis elements.

3.2.1 Lifting Rotations in the Plane

Fix an ordered orthonormal basis $\{\boldsymbol{e}_i\}$ for $\mathbb{R}^n$. To each basis element $\boldsymbol{e}_i$ we associate a gamma matrix γ_i in the Clifford algebra $C\ell(n)$. The rotation $\exp(i\theta)$ in the complex plane (real two-plane) can be implemented in the Clifford algebra by conjugation by the element,

$$\exp\left(\frac{1}{2}\theta\gamma_i\gamma_j\right) = \exp\left(\frac{1}{2}\theta\gamma_{ij}\right),$$

on the basis $\{\gamma_i\}_{i=1}^n$. If in addition we want to rotate in another plane then we just conjugate by the corresponding element of this form.

Lifting Permutations

A permutation $\pi \in \Sigma_n$ acts on $\mathbb{R}^n$ by permuting the basis vectors: $P_\pi \boldsymbol{e}_i = \boldsymbol{e}_{\pi(i)}$. P_π is an orthogonal transformation and lies in $O(n)$. Furthermore every permutation can be written as a product of transpositions $P_{(ij)}$ where

$$P_{(ij)}\boldsymbol{e}_k = \begin{cases} \boldsymbol{e}_j \ , & k = i \\ \boldsymbol{e}_i \ , & k = j \\ \boldsymbol{e}_k \ , & \text{otherwise.} \end{cases}$$

If and only if π is an even permutation of the basis of $\mathbb{R}^n$ then $P_\pi \in \mathrm{SO}(n)$, since swapping even numbers of basis elements will preserve the sign of the volume form. Furthermore it can be written as a product of an *even number* of transpositions which we can lift to the Clifford algebra to regain the permutation on spinors. We can implement transpositions easily in the Clifford algebra as follows. Let the lift of $P_{(ij)}$ be

$$\hat{P}_{(ij)} := \tfrac{1}{\sqrt{2}}(\gamma_i - \gamma_j) \ .$$

It satisfies $\hat{P}_{(ij)}^2 = -\mathbf{1}$ whence $\hat{P}_{(ij)}^{-1} = -\hat{P}_{(ij)}$. Then we have that

$$\hat{P}_{(ij)}\gamma_k\hat{P}_{(ji)} = \begin{cases} \gamma_j \ , & k = i \\ \gamma_i \ , & k = j \\ \gamma_k \ , & \text{otherwise.} \end{cases}$$

If π is an even permutation then $P_\pi \in \mathrm{SO}(n)$ lifts to $\hat{P}_\pi \in \mathrm{Spin}(n)$ by writing π as a composition of transpositions $\pi = \sigma_1 \cdots \sigma_{2k}$ and defining (up to a sign)

$$\hat{P}_\pi = \hat{P}_{\sigma_1} \cdots \hat{P}_{\sigma_{2k}} \ ,$$

with each $\hat{P}_\sigma$ as above. One clearly sees that P_π on $\mathbb{R}^n$ is implemented by conjugation with $\pm\hat{P}_\pi$ on the space of spinors.

Lifting Quaternions

We have discussed polyhedral groups and their lifts to $Spin(3) \cong Sp(1) \cong SU(2)$ to give the binary polyhedral groups. In the real representation they form subgroups of $SO(4)$, which we will use later in our classification of spherical space forms given by groups involving the binary polyhedral groups. We will also need to lift these subgroups to $Spin(4)$ to get their action on spinors, which we shall explore in the following. We will let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ denote the quaternion units, so that a quaternion $\boldsymbol{q}$ can be written uniquely as

$$\boldsymbol{q} = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} \in \mathbb{H} \ ,$$

for a 'quaternion' of real numbers q_0, q_1, q_2, q_3. If $\boldsymbol{q}$ has unit norm, so that $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$, then it defines an element

$$\begin{pmatrix} q_0 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_0 - iq_1 \end{pmatrix} \in \mathrm{SU}(2) \ .$$

This is a linear transformation of $\mathbb{C}^2$, which induces a linear transformation in $\mathbb{R}^4$ with matrix

$$\begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \in \mathrm{SO}(4) \ .$$

Its lift to $\mathrm{Spin}(4) \subset C\ell(4)$ is given (up to a sign) by

$$\hat{q} = \Pi_- + \Pi_+\tilde{\boldsymbol{q}} = \Pi_- + \tilde{\boldsymbol{q}}\Pi_+ = \Pi_- + \Pi_+\tilde{\boldsymbol{q}}\Pi_+ \ , \tag{3.2.1}$$

where $\Pi_\pm = \frac{1}{2}(1 \pm \gamma_{1234})$ and $\tilde{\boldsymbol{q}} = q_0 - q_1\gamma_{12} - q_2\gamma_{13} + q_3\gamma_{23}$.

In other words, $\hat{q}$ acts trivially on negative-chirality spinors and essentially as $\boldsymbol{q}$ on positive-chirality spinors, since the map $\mathbb{H} \to C\ell(3)^{\text{even}}$ sending $\boldsymbol{q} \mapsto \tilde{\boldsymbol{q}}$ is an isomorphism which restricts isomorphically to the unit-norm elements $\mathbb{H} \supset \mathrm{Sp}(1) \cong Spin(3) \subset C\ell(3)$. In fact, the map $\mathbb{H} \to \mathrm{Spin}(4)$ sending $\boldsymbol{q} \mapsto \hat{q}$ is an injective homomorphism and thus, in particular, any relation which already holds in Sp(1) will automatically hold in Spin(4).

3.3 Lifting G to $Spin(8)$

In this section we will go through each class of subgroups $G \subset \mathrm{SO}(8)$ acting freely on S^7 and find the lift of the generators to $Spin(8)$. Then determine if the group lifts to $Spin(8)$ and to which spin structure, on the quotient, this lift occurs.

3.3.1 Cyclic Groups

Every cyclic subgroup $G \subset \mathrm{SO}(8)$ can be written as a group generated by the element A by choosing appropriate basis.

$$A = \begin{pmatrix} R\left(\frac{1}{n}\right) & & & \\ & R\left(\frac{a}{n}\right) & & \\ & & R\left(\frac{b}{n}\right) & \\ & & & R\left(\frac{c}{n}\right) \end{pmatrix} ,$$

where $(a, n) = (b, n) = (c, n) = 1$ this guarantees a free action. Without loss of generality we can order them so that $1 \leq a \leq b \leq c < n$. Let us denote this group as $G(n, a, b, c)$. We would like to lift this to $Spin(8)$. Since this is just composed of rotations in $\mathbb{R}^2$, the lift is easily given as:

$$\hat{A} = \varepsilon_A \exp\left(\frac{\pi}{n}\gamma_{12} + \frac{a\pi}{n}\gamma_{34} + \frac{b\pi}{n}\gamma_{56} + \frac{c\pi}{n}\gamma_{78}\right) ,$$

obeying

$$\hat{A}^n = \varepsilon_A^n (-1)^{1+a+b+c}\mathbf{1} .$$

Where ε_A is an overall sign. It accounts for the fact that we can lift A to two possible elements in spin differing by a minus sign. If $G(n, a, b, c)$ does lift to spin then $\hat{A}$ must obey the same conditions as A. Thus as we can see from the above $\varepsilon_A^n (-1)^{1+a+b+c}$ must equal one. There are two cases to consider depending on whether n is even or odd. If n is even then $\varepsilon_A^n = 1$ automatically furthermore since $(a, n) = (b, n) = (c, n) = 1$ clearly $1 + a + b + c \equiv 0(2)$ thus we get two possible lifts of $\hat{A}$ to spin. If n is odd then the we get a lift to spin only if we set

$$\varepsilon_A = (-1)^{1+a+b+c},$$

whence there is a unique spin structure in the quotient to which $\hat{A}$ lifts.

Since $\gamma_i^2 = -1$ it's eigenvalues are $\pm i$ thus the eigenvalues of $\hat{A}$ are

$$\varepsilon \exp\left(\frac{i\pi}{n}\left(\sigma_1 + a\sigma_2 + b\sigma_3 + c\sigma_4\right)\right)$$

with σ_i signs whose product $\sigma_1\sigma_2\sigma_3\sigma_4$ determines the chirality of the $Spin(8)$ representation. This eigenvalue will be plus or minus one depending on whether

$$\sigma_1 + a\sigma_2 + b\sigma_3 + c\sigma_4 \equiv 0(2n) \text{ or } 0(n).$$

The number of invariant spinors in a quotient by $G(n, a, b, c)$ will be equal to the number of ways we can set the eigenvalue of $\hat{A}$ equal to one for a given chirality. The maximum such number being eight. The results are given in Section 4.1.

3.3.2 Type I Groups with $d = 2$

In this section we consider Type I groups in $SO(8)$ with $d = 2$ in the notation of section 2.2.1. These groups actually lie in an $SO(4) \times SO(4)$ subgroup.

Let $G_{\mathrm{I}}(m, n, r; k_1, k_2, \ell_1, \ell_2)$ denote the subgroup of $SO(8)$ generated by

$$A = \begin{pmatrix} R\left(\frac{k_1}{m}\right) & & & \\ & R\left(\frac{rk_1}{m}\right) & & \\ & & R\left(\frac{k_2}{m}\right) & \\ & & & R\left(\frac{rk_2}{m}\right) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} & \mathbf{1} & & \\ R\left(\frac{2\ell_1}{n}\right) & & & \\ & & & \mathbf{1} \\ & & R\left(\frac{2\ell_2}{n}\right) & \end{pmatrix}$$

where m, n, r obey the conditions common to all Type I groups: $(n(r-1), m) = 1$, $r^n \equiv 1(m)$ and, in addition, n is even (hence m is odd), $r^2 \equiv 1(m)$, $(k_i, m) = 1$ and $(\ell_i, n) = 1$.

Spin Structures

A is a block diagonal matrix, with rotations in $\mathbb{R}^2$ forming the blocks. Each rotation $R(\theta)$ has lift, to spin, $\exp(\frac{\theta}{2}\gamma_{12})$ which has eigenvalues $\exp(\pm i\theta)$. Thus the lift of A is given by:

$$\hat{A} = \varepsilon_A \exp\left(\tfrac{\pi k_1}{m}\gamma_{12} + \tfrac{\pi r k_1}{m}\gamma_{34} + \tfrac{\pi k_2}{m}\gamma_{56} + \tfrac{\pi r k_2}{m}\gamma_{78}\right)$$

which obeys

$$\hat{A}^m = \varepsilon_A(-1)^{(k_1+k_2)(1+r)} \ ,$$

whence we choose $\varepsilon_A = (-1)^{(k_1+k_2)(1+r)}$, so that $\hat{A}^m = 1$. $\hat{A}$ has eigenvalues $\exp(\frac{i\pi}{m}(k_1\sigma_1 + k_1\sigma_2 + k_2\sigma_3 + k_2\sigma_4))$ where $\sigma_i = \pm 1$. Thus we see that there is a unique lift of A to $Spin(8)$.

To lift B we notice that it is block-diagonal and that we can write each block B_i as a product of a permutation and a rotation

$$B_i = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} \begin{pmatrix} R\left(\frac{2\ell_i}{n}\right) & \\ & \mathbf{1} \end{pmatrix} \ , \tag{3.3.1}$$

where the first matrix is the permutation (13)(24). The lift of this generator to spin is given by:

$$\hat{B} = \varepsilon_B \tfrac{1}{4}(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)(\gamma_5 - \gamma_7)(\gamma_6 - \gamma_8) \exp\left(\tfrac{2\pi\ell_1}{n}\gamma_{12} + \tfrac{2\pi\ell_2}{n}\gamma_{56}\right) \tag{3.3.2}$$

which obeys

$$\hat{B}^2 = \exp\left(\tfrac{2\pi\ell_1}{n}(\gamma_{12} + \gamma_{34}) + \tfrac{2\pi\ell_2}{n}(\gamma_{56} + \gamma_{78})\right)$$

whence, since n is even,

$$\hat{B}^n = (\hat{B}^2)^{n/2} = \exp\left(\pi\ell_1(\gamma_{12} + \gamma_{34}) + \pi\ell_2(\gamma_{56} + \gamma_{78})\right) = \mathbf{1}$$

for all choices of ε_B.

The final relation that must be checked is $\hat{B}\hat{A}\hat{B}^{-1} = \hat{A}^r$ which does not fix ε_B either since there are two ε_B appearing to give one. Hence there are either no lifts or there are two lifts, corresponding to different choices of ε_B. The condition $\hat{B}\hat{A}\hat{B}^{-1} = \hat{A}^r$ is true in $SO(8)$, whence it is satisfied up to a sign: $\hat{B}\hat{A}\hat{B}^{-1} = \delta\hat{A}^r$ for some sign δ. We now take the mth power and use that m is odd. On the one hand we have

$$\left(\hat{B}\hat{A}\hat{B}^{-1}\right)^m = \hat{B}\hat{A}^m\hat{B}^{-1} = \mathbf{1} \ ,$$

this is equal to

$$\left(\delta\hat{A}^r\right)^m = \delta\hat{A}^{rm} = \delta\mathbf{1} \ ,$$

whence $\delta = +1$ and the relation is satisfied. Therefore such quotients have two inequivalent spin structures labelled by ε_B and we are *free to choose* the spin structure.

3.3.3 Type I Groups with $d = 4$

In this section we consider Type I groups in $SO(8)$ with $d = 4$ in the notation of Section (2.2.1). Let $G_{\mathrm{I}}(m, n, r; k, \ell)$ be the subgroup of SO(8) generated by

$$A = \begin{pmatrix} R\left(\frac{k}{m}\right) & & & \\ & R\left(\frac{rk}{m}\right) & & \\ & & R\left(\frac{r^2k}{m}\right) & \\ & & & R\left(\frac{r^3k}{m}\right) \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \\ R\left(\frac{4\ell}{n}\right) & & & \end{pmatrix}$$

where in addition to the Type I group conditions: n is now divisible by 4, m is therefore odd, $r^4 \equiv 1(m)$, $(k, m) = 1$ and $(\ell, n) = 1$.

Spin Structures

The lift of A is again immediate

$$\hat{A} = \varepsilon_A \exp\left(\tfrac{\pi k}{m}\gamma_{12} + \tfrac{\pi rk}{m}\gamma_{12} + \tfrac{\pi r^2 k}{m}\gamma_{12} + \tfrac{\pi r^3 k}{m}\gamma_{12}\right) \ ,$$

which obeys

$$\hat{A}^m = \varepsilon_A(-1)^{k(1+r+r^2+r^3)} \ .$$

This fixes ε_A uniquely so that

$$\hat{A} = (-1)^{k(1+r+r^2+r^3)} \exp\left(\tfrac{\pi k}{m}\gamma_{12} + \tfrac{\pi r k}{m}\gamma_{12} + \tfrac{\pi r^2 k}{m}\gamma_{12} + \tfrac{\pi r^3 k}{m}\gamma_{12}\right) \ .$$

To lift B to Spin(8) we decompose it again into

$$B = \begin{pmatrix} & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \\ \mathbf{1} & & & \end{pmatrix} \begin{pmatrix} R\left(\frac{4\ell}{n}\right) & & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \end{pmatrix} \ .$$

The first matrix is the permutation (1357)(2468) in cycle language or (57)(37)(17) (68)(48)(28) in terms of transpositions. Its lift follows the method in Section 3.2.1:

$$\tfrac{1}{8}(\gamma_5 - \gamma_7)(\gamma_3 - \gamma_7)(\gamma_1 - \gamma_7)(\gamma_6 - \gamma_8)(\gamma_4 - \gamma_8)(\gamma_2 - \gamma_8) \ .$$

The lift of the rotation is also immediate: $\exp(\frac{4\pi\ell}{n}\gamma_{12})$, whence we have

$$\hat{B} = \varepsilon_B \tfrac{1}{8}(\gamma_5 - \gamma_7)(\gamma_3 - \gamma_7)(\gamma_1 - \gamma_7)(\gamma_6 - \gamma_8)(\gamma_4 - \gamma_8)(\gamma_2 - \gamma_8) \exp\left(\tfrac{4\pi\ell}{n}\gamma_{12}\right) \ .$$

An easy calculation shows that

$$\hat{B}^4 = \exp\left(\tfrac{4\pi\ell}{n}\gamma_{12} + \tfrac{4\pi\ell}{n}\gamma_{34} + \tfrac{4\pi\ell}{n}\gamma_{56} + \tfrac{4\pi\ell}{n}\gamma_{78}\right) \ ,$$

whence

$$\hat{B}^n = \left(\hat{B}^4\right)^{n/4} = \exp\left(\pi\ell(\gamma_{12} + \gamma_{34} + \gamma_{56} + \gamma_{78})\right) = (-1)^{4\ell} = 1 \ .$$

The same argument as in the case of $d = 2$ shows that $\hat{B}\hat{A}\hat{B}^{-1} = \hat{A}^r$. Therefore there are two inequivalent spin structures in the quotient distinguished by the choice of ε_B.

3.3.4 Type II Groups with $d = 1$

In this section we consider the lifts to $Spin(8)$ of the subgroups $G_{\mathrm{II}}(m,n,s,t;k_1,k_2,\ell_1,\ell_2)$ of $SO(8)$ defined in Section 2.2.2. As we saw there, the subgroup is generated by the matrices

$$A = \begin{pmatrix} R\left(\frac{k_1}{m}\right) & & & \\ & R\left(\frac{k_1 s}{m}\right) & & \\ & & R\left(\frac{k_2}{m}\right) & \\ & & & R\left(\frac{k_2 s}{m}\right) \end{pmatrix} , \ B = \begin{pmatrix} R\left(\frac{\ell_1}{n}\right) & & & \\ & R\left(\frac{\ell_1 t}{n}\right) & & \\ & & R\left(\frac{\ell_2}{n}\right) & \\ & & & R\left(\frac{\ell_2 t}{n}\right) \end{pmatrix}$$

and

$$S = \begin{pmatrix} & \mathbf{1} & & \\ -\mathbf{1} & & & \\ & & & \mathbf{1} \\ & & -\mathbf{1} & \end{pmatrix} .$$

Spin Structures

A and B are block diagonal and their lifts are given below:

$$\hat{A} = \varepsilon_A \exp\left(\tfrac{\pi k_1}{m}\gamma_{12} + \tfrac{\pi k_1 s}{m}\gamma_{34} + \tfrac{\pi k_2}{m}\gamma_{56} + \tfrac{\pi k_2 s}{m}\gamma_{78}\right)$$

and

$$\hat{B} = \varepsilon_B \exp\left(\tfrac{\pi \ell_1}{n}\gamma_{12} + \tfrac{\pi \ell_1 t}{n}\gamma_{34} + \tfrac{\pi \ell_2}{n}\gamma_{56} + \tfrac{\pi \ell_2 t}{n}\gamma_{78}\right) ,$$

which obey

$$\hat{A}^m = \varepsilon_A(-1)^{(k_1+k_2)(1+s)} \qquad \text{and} \qquad \hat{B}^n = (-1)^{(\ell_1+\ell_2)(1+t)} = 1 ,$$

where we have used that n is even and that ℓ_i and m are therefore odd. Therefore we choose $\varepsilon_A = (-1)^{(k_1+k_2)(1+s)}$ and we have two possible lifts of the cyclic subgroup generated by A and B depending on the sign of ε_B.

To lift S we first write it as a product of a permutation and a diagonal matrix:

$$S = \begin{pmatrix} & \mathbf{1} & & \\ \mathbf{1} & & & \\ & & & \mathbf{1} \\ & & \mathbf{1} & \end{pmatrix} \begin{pmatrix} -\mathbf{1} & & & \\ & \mathbf{1} & & \\ & & -\mathbf{1} & \\ & & & \mathbf{1} \end{pmatrix} ,$$

where the permutation P_S corresponds to (13)(24)(57)(68), which lifts to

$$\hat{P}_S = \tfrac{1}{4}(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)(\gamma_5 - \gamma_7)(\gamma_6 - \gamma_8)$$

up to a sign and obeys $\hat{P}_S^2 = 1$. The lift of S is then given by

$$\hat{S} = \varepsilon_S \hat{P}_S \exp\left(\tfrac{\pi}{2}(\gamma_{12} + \gamma_{56})\right) = \varepsilon_S \hat{P}_S \gamma_{1256} \ .$$

Squaring we have $\hat{S}^2 = \gamma_{12345678}$, whence squaring again $\hat{S}^4 = 1$.

We need to check the relations $\hat{S}^2 = \hat{B}^{n/2}$, $\hat{S}\hat{A}\hat{S}^{-1} = \hat{A}^s$ and $\hat{S}\hat{B}\hat{S}^{-1} = \hat{B}^t$. Since m is odd, we can show $\hat{S}\hat{A}\hat{S}^{-1} = \hat{A}^s$ is satisfied in the same manner as we did for the relation $\hat{B}\hat{A}\hat{B}^{-1} = \hat{A}^r$. To check the first relation we compute

$$\begin{aligned} \hat{B}^{n/2} &= \varepsilon_B^{n/2} \exp\left(\tfrac{\pi\ell_1}{2}\gamma_{12} + \tfrac{\pi\ell_1 t}{2}\gamma_{34} + \tfrac{\pi\ell_2}{2}\gamma_{56} + \tfrac{\pi\ell_2 t}{2}\gamma_{78}\right) \\ &= \varepsilon_B^{n/2} \sin\left(\tfrac{\pi\ell_1}{2}\right) \sin\left(\tfrac{\pi\ell_1 t}{2}\right) \sin\left(\tfrac{\pi\ell_2}{2}\right) \sin\left(\tfrac{\pi\ell_2 t}{2}\right) \gamma_{12345678} \ , \end{aligned}$$

which, using that $t \equiv -1(4)$, becomes

$$\hat{B}^{n/2} = \varepsilon_B^{n/2} \gamma_{12345678} = \hat{S}^2 \ .$$

This does not fix ε_B since $n \equiv 0(4)$.

Finally we check the third relation. One computes

$$\hat{S}\hat{B}\hat{S}^{-1} = \varepsilon_B \exp\left(\tfrac{\pi\ell_1 t}{n}\gamma_{12} + \tfrac{\pi\ell_1}{n}\gamma_{34} + \tfrac{\pi\ell_2 t}{n}\gamma_{56} + \tfrac{\pi\ell_2}{n}\gamma_{78}\right)$$

whereas, using that t is odd,

$$\hat{B}^t = \varepsilon_B \exp\left(\tfrac{\pi \ell_1 t}{n}\gamma_{12} + \tfrac{\pi \ell_1 t^2}{n}\gamma_{34} + \tfrac{\pi \ell_2 t}{n}\gamma_{56} + \tfrac{\pi \ell_2 t^2}{n}\gamma_{78}\right)$$

which can be rewritten as

$$\hat{B}^t = \varepsilon_B \exp\left(\tfrac{\pi \ell_1 t}{n}\gamma_{12} + \tfrac{\pi \ell_1}{n}\gamma_{34} + \tfrac{\pi \ell_2 t}{n}\gamma_{56} + \tfrac{\pi \ell_2}{n}\gamma_{78} + \tfrac{\pi (t^2-1)}{n}(\ell_1\gamma_{34} + \ell_2\gamma_{78})\right) \ .$$

Since $t^2 \equiv 1(n)$, the last term is $(-1)^{p(\ell_1+\ell_2)}$, where $t^2 - 1 = pn$. Since each ℓ_i is odd, their sum is even and this term is equal to 1 and hence the third relation is also satisfied.

In summary, we find that we can choose ε_B and ε_S but ε_A is fixed as above. Thus for a given group the quotient will admit four inequivalent spin structures labelled by the four possible sign combinations ε_B and ε_S.

3.3.5 Type II Groups with $d = 2$

Let $G_{\mathrm{II}}(m, n, r, s, t; k, l)$ denote the type II subgroup of $SO(8)$ generated by the following matrices

$$A = \begin{pmatrix} R\left(\frac{k}{m}\right) & & & \\ & R\left(\frac{rk}{m}\right) & & \\ & & R\left(\frac{sk}{m}\right) & \\ & & & R\left(\frac{rsk}{m}\right) \end{pmatrix} \qquad B = \begin{pmatrix} & \mathbf{1} & & \\ R\left(\frac{2\ell}{n}\right) & & & \\ & & & R\left(\frac{\ell(t-1)}{n}\right) \\ & & R\left(\frac{\ell(t+1)}{n}\right) & \end{pmatrix} \tag{3.3.3}$$

where $(k, m) = 1$ and $(\ell, n) = 1$, and

$$S = \begin{pmatrix} & & \mathbf{1} & \\ & & & \mathbf{1} \\ R\left(\frac{\ell}{2}\right) & & & \\ & R\left(\frac{\ell}{2}\right) & & \end{pmatrix} \ .$$

Spin Structures

The lift of A is immediate

$$\hat{A} = \varepsilon_A \exp\left(\tfrac{\pi k}{m}\gamma_{12} + \tfrac{\pi r k}{m}\gamma_{12} + \tfrac{\pi s k}{m}\gamma_{12} + \tfrac{\pi r s k}{m}\gamma_{12}\right) \ ,$$

which obeys

$$\hat{A}^m = \varepsilon_A (-1)^{k(1+r)(1+s)}$$

hence $\varepsilon_A = (-1)^{k(1+r)(1+s)}$.

To lift B we notice that we can factor it as

$$B = \begin{pmatrix} & \mathbf{1} & & \\ \mathbf{1} & & & \\ & & & \mathbf{1} \\ & & \mathbf{1} & \end{pmatrix} \begin{pmatrix} R\left(\frac{2\ell}{n}\right) & & & \\ & \mathbf{1} & & \\ & & R\left(\frac{\ell(t+1)}{n}\right) & \\ & & & R\left(\frac{\ell(t-1)}{n}\right) \end{pmatrix} \ ,$$

where the permutation matrix P_B corresponds to the permutation (13)(24)(57)(68), whence one possible lift to Spin(8) is given by

$$\hat{P}_B = \tfrac{1}{4}(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)(\gamma_5 - \gamma_7)(\gamma_6 - \gamma_8) \ ,$$

which obeys $\hat{P}_B^2 = \mathbf{1}$. The lift of B is then

$$\hat{B} = \varepsilon_B \hat{P}_B \exp\left(\tfrac{2\pi\ell}{n}\gamma_{12} + \tfrac{\pi\ell(t+1)}{n}\gamma_{56} + \tfrac{\pi\ell(t-1)}{n}\gamma_{78}\right) \ ,$$

which obeys

$$\hat{B}^2 = \exp\left(\tfrac{2\pi\ell}{n}(\gamma_{12} + \gamma_{34}) + \tfrac{2\pi\ell t}{n}(\gamma_{56} + \gamma_{78})\right) \ ,$$

whence

$$\hat{B}^n = (\hat{B}^2)^{n/2} = \mathbf{1}$$

for any value of ε_B. The relation $\hat{B}\hat{A}\hat{B}^{-1} = \hat{A}^r$ is again satisfied as in the case of Type I groups, with the same proof.

To lift S we factor it as follows

$$S = \begin{pmatrix} & & \mathbf{1} & \\ & & & \mathbf{1} \\ \mathbf{1} & & & \\ & \mathbf{1} & & \end{pmatrix} \begin{pmatrix} -\mathbf{1} & & & \\ & -\mathbf{1} & & \\ & & \mathbf{1} & \\ & & & \mathbf{1} \end{pmatrix} .$$

The permutation matrix P_S corresponds to the permutation (15)(26)(37)(48), whence we can lift it as

$$\hat{P}_S = \tfrac{1}{4}(\gamma_1 - \gamma_5)(\gamma_2 - \gamma_6)(\gamma_3 - \gamma_7)(\gamma_4 - \gamma_8) \ ,$$

which obeys $\hat{P}_S^2 = \mathbf{1}$. The lift of S is then

$$\hat{S} = \varepsilon_S \hat{P}_S \exp\left(\tfrac{\pi}{2}(\gamma_{12} + \gamma_{34})\right) \ .$$

We must now check three relations: $\hat{S}\hat{A}\hat{S}^{-1} = \hat{A}^s$, $\hat{S}\hat{B}\hat{S}^{-1} = \hat{B}^t$ and $\hat{S}^2 = \hat{B}^{n/2}$. As we have seen many times before, the first relation follows without calculation by noticing that it must be true up to a sign since it is true in $SO(8)$ and taking the mth power.

To check the second relation we compute

$$\hat{S}\hat{B}\hat{S}^{-1} = \varepsilon_B \hat{P}_B \exp\left(\tfrac{\pi\ell(t+1)}{n}\gamma_{12} + \tfrac{\pi\ell(t-1)}{n}\gamma_{34} + \tfrac{2\pi\ell}{n}\gamma_{56}\right)$$

whereas, since t is odd,

$$\hat{B}^t = \varepsilon_B \hat{P}_B \exp\left(\tfrac{\pi\ell(t+1)}{n}\gamma_{12} + \tfrac{\pi\ell(t-1)}{n}\gamma_{34} + \tfrac{\pi\ell(t^2+1)}{n}\gamma_{56} + \tfrac{\pi\ell(t^2-1)}{n}\gamma_{78}\right) \ ,$$

which can be rewritten as

$$\hat{B}^t = \varepsilon_B \hat{P}_B \exp\left(\tfrac{\pi\ell(t+1)}{n}\gamma_{12} + \tfrac{\pi\ell(t-1)}{n}\gamma_{34} + \tfrac{2\pi\ell}{n}\gamma_{56} + \tfrac{\pi\ell(t^2-1)}{n}(\gamma_{56} + \gamma_{78})\right) \ .$$

Using that $t^2 \equiv 1(n)$ we see that the last term in the exponential does not contribute and hence that the relation is satisfied as we have seen in previous sections.

Finally for the third relation we calculate

$$\hat{S}^2 = \exp\left(\tfrac{\pi}{2}(\gamma_{12} + \gamma_{34} + \gamma_{56} + \gamma_{78})\right)$$

whereas, since n is divisible by 4,

$$\hat{B}^{n/2} = \left(\hat{B}^2\right)^{n/4} = \exp\left(\tfrac{\pi}{2}(\gamma_{12} + \gamma_{34}) + \tfrac{\pi t}{2}(\gamma_{56} + \gamma_{78})\right)$$

which using the fact that t is odd, is seen to agree with $\hat{S}^2$.

In summary, there are four inequivalent spin structures in the quotient: corresponding to the choice of signs ε_S and ε_B which are left undetermined.

3.3.6 Type IV Groups with $d = 1$

In this section we consider the lifts to $Spin(8)$ of the subgroups $G_{\mathrm{IV}}(m, n, s, t; k, \ell)$ of $SO(8)$ defined in Section 2.2.4. As we saw there, the subgroup is generated by the matrices

$$A = \begin{pmatrix} R\left(\frac{k}{m}\right) & & & \\ & R\left(\frac{k}{m}\right) & & \\ & & R\left(\frac{sk}{m}\right) & \\ & & & R\left(\frac{sk}{m}\right) \end{pmatrix}$$

$$B^{3^v} = \begin{pmatrix} R\left(\frac{3^v\ell}{n}\right) & & & \\ & R\left(\frac{3^v\ell}{n}\right) & & \\ & & R\left(\frac{3^v\ell t}{n}\right) & \\ & & & R\left(\frac{3^v\ell t}{n}\right) \end{pmatrix}$$

$$B^{n'} = \frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & -1 & & & & \\ 1 & -1 & 1 & -1 & & & & \\ 1 & -1 & -1 & 1 & & & & \\ 1 & 1 & -1 & -1 & & & & \\ & & & & -1 & 1 & 1 & 1 \\ & & & & -1 & -1 & -1 & 1 \\ & & & & -1 & 1 & -1 & -1 \\ & & & & -1 & -1 & 1 & -1 \end{pmatrix} \qquad n' \equiv 1(3)$$

$$I = \begin{pmatrix} & 1 & & & & & & \\ -1 & & & & & & & \\ & & & -1 & & & & \\ & & 1 & & & & & \\ & & & & & & & -1 \\ & & & & & & 1 & \\ & & & & & -1 & & \\ & & & & 1 & & & \end{pmatrix}$$

$$J = \begin{pmatrix} & \mathbf{1} & & \\ \mathbf{-1} & & & \\ & & & \mathbf{-1} \\ & & \mathbf{1} & \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} & \mathbf{1} \\ \mathbf{-1} & \end{pmatrix} \ ,$$

where $n \equiv 3(6)$, $(n, m) = 1$, $(k, m) = 1$, $(\ell, n) = 1$, $n = 3^v n'$ with $(3, n') = 1$ and $v \geq 1$, $s^2 \equiv 1(m)$, $t^2 \equiv 1(n)$ and $t \equiv -1(3)$. As before $B^{n'}$ for $n' \equiv 2(3)$ is simply the square of $B^{n'}$ for $n' \equiv 1(3)$ given above.

Spin Structures

We recall that $\{A, B\}$ is a Type I group of odd order, $N = mn$, it means that both A and B have odd order. We start by lifting A. For case (i) and (iii), where $n \not\equiv 0(9)$,

$$\hat{A} = \varepsilon_A \exp\left(\tfrac{\pi k}{m}(\gamma_{12} + \gamma_{34}) + \tfrac{\pi k s}{m}(\gamma_{56} + \gamma_{78})\right) \ ,$$

which obeys

$$\hat{A}^m = \varepsilon_A^m (-1)^{k(s+1)} = 1 \quad \text{hence} \;\; \varepsilon_A = (-1)^{k(s+1)} \; .$$

For case (ii), where $n \equiv 0(9)$,

$$\hat{A} = \varepsilon_A \exp\left(\tfrac{\pi k}{m}(\gamma_{12} + \gamma_{34} + \gamma_{56} + \gamma_{78})\right) \;\; ,$$

which obeys

$$\hat{A}^m = \varepsilon_A^m (-1)^k = 1 \;\; \text{hence} \;\; \varepsilon_A = (-1)^k \; .$$

Lifting B^3 and B^{3^v} we obtain, for case (i)

$$\hat{B}^3 = \varepsilon_B \exp\left(\tfrac{\pi 3\ell}{n}(\gamma_{12} + \gamma_{34} + \gamma_{56} + \gamma_{78})\right) \;\; ,$$

which obeys

$$(\hat{B}^3)^{n'} = \varepsilon_B (-1)^\ell = 1 \;\; \text{hence} \;\; \varepsilon_B = (-1)^\ell \; ,$$

and for case (ii) and (iii)

$$\hat{B}^{3^v} = \varepsilon_B \exp\left(\tfrac{\pi 3^v \ell}{n}(\gamma_{12} + \gamma_{34}) + \tfrac{\pi 3^v \ell t}{n}(\gamma_{56} + \gamma_{78})\right)$$

which obeys

$$\left(\hat{B}^{3^v}\right)^{n'} = \varepsilon_B (-1)^{\ell(1+t)} = 1 \;\; \text{hence} \;\; \varepsilon_B = (-1)^{\ell(1+t)}.$$

We may lift $B^{n'}$, I and J following the method outlined in Section 3.2.1 as they can be written in terms of (pairs of) quaternions in $SO(4) \times SO(4)$, which we know how to lift to $Spin(4) \times Spin(4) \subset Spin(8)$. Indeed, in terms of pairs of quaternions

$$I = \begin{pmatrix} \boldsymbol{i} & \\ & -\boldsymbol{k} \end{pmatrix} \qquad J = \begin{pmatrix} \boldsymbol{j} & \\ & -\boldsymbol{j} \end{pmatrix} \qquad B^{n'} = \begin{pmatrix} -\frac{1}{2}(1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) & \\ & \frac{1}{2}(-1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \end{pmatrix} \; ,$$

where $n' \equiv 1(3)$. Using formula (3.2.1), their lifts are

$$\begin{aligned}\hat{I} &= \varepsilon_I(\Pi_- - \Pi_+\gamma_{12})(\Pi'_- - \Pi'_+\gamma_{67})\\ \hat{J} &= \varepsilon_J(\Pi_- - \Pi_+\gamma_{13})(\Pi'_- + \Pi'_+\gamma_{57})\\ \hat{B}^{n'} &= \left(\Pi_- - \tfrac{1}{2}\Pi_+(1-\gamma_{12}-\gamma_{13}+\gamma_{23})\right)\left(\Pi'_- + \tfrac{1}{2}\Pi'_+(-1-\gamma_{56}-\gamma_{57}+\gamma_{67})\right) ,\end{aligned}$$

where $\Pi_\pm = \frac{1}{2}(1 \pm \gamma_{1234})$ and $\Pi'_\pm = \frac{1}{2}(1 \pm \gamma_{5678})$ and where $n' \equiv 1 \pmod 3$. Moreover every relation involving only these generators is automatically satisfied in Spin(8) since it is satisfied with quaternions thus $\varepsilon_I = 1$ and $\varepsilon_J = 1$. We need only check that $\hat{B}^{n'}$ and $\hat{B}^3$ or $\hat{B}^{3^v}$ commute, but this is again automatic. Indeed, the identity $\hat{B}^{n'}\hat{B}^3\hat{B}^{-n'} = \hat{B}^3$ and $\hat{B}^{n'}\hat{B}^{3^v}\hat{B}^{-n'} = \hat{B}^{3^v}$ are satisfied up to a sign. Now take the nth power and use that n is odd to deduce that the sign must be $+1$. For cases (*ii*) and (*iii*) the lift of the phase part of $B^{n'}$ is given by;

$$\phi \oplus \phi^{-1} = \exp\left(\tfrac{\pi p}{3^v}(\gamma_{12}+\gamma_{34}-\gamma_{56}-\gamma_{78})\right) ,$$

Finally the lift of S is given by

$$\hat{S} = \varepsilon_S \tfrac{1}{4}(\gamma_1-\gamma_5)(\gamma_2-\gamma_6)(\gamma_3-\gamma_7)(\gamma_4-\gamma_8)\gamma_{1234} ,$$

which obeys $\hat{S}^2 = \hat{I}^2$ and hence $\hat{S}^4 = 1$. One also checks that the relations $\hat{S}\hat{J}\hat{S}^{-1} = \hat{J}^{-1}$ and $\hat{S}\hat{I}\hat{S}^{-1} = \hat{J}\hat{I}$ are satisfied.

Whatever the lift of B, since it has odd order, the relation $\hat{S}\hat{B}\hat{S}^{-1} = \hat{B}^t$ is satisfied as we have argued above: observe that it must hold up to a sign and simply take the nth power. On the other hand, to check $\hat{S}\hat{A}\hat{S}^{-1} = \hat{A}^s$ one has to calculate:

$$\hat{S}\hat{A}\hat{S}^{-1} = \varepsilon_A \exp\left(\tfrac{\pi k}{m}(\gamma_{56}+\gamma_{78}) + \tfrac{\pi k s}{m}(\gamma_{12}+\gamma_{34})\right) ,$$

whereas

$$\hat{A}^s = \varepsilon_A^s \exp\left(\tfrac{\pi k s}{m}(\gamma_{12}+\gamma_{34}) + \tfrac{\pi k s^2}{m}(\gamma_{56}+\gamma_{78})\right) .$$

Both expressions agree if and only if

$$\varepsilon_A^{s-1}\exp\left(\tfrac{\pi k(s^2-1)}{m}(\gamma_{56}+\gamma_{78})\right)=1 \text{ hence } \varepsilon_A^{s-1}=\exp\left(\tfrac{\pi k(s^2-1)}{m}(\gamma_{56}+\gamma_{78})\right) .$$

Since $\varepsilon_A = (-1)^{k(s+1)}$, $\varepsilon_A^{(s-1)} = (-1)^{k(s^2-1)}$ which will give the same sign as $\exp\left(\tfrac{\pi k(s^2-1)}{m}(\gamma_{56}+\gamma_{78})\right)$ since $s^2 \equiv 1(m)$. Hence the relation is satisfied. Similarly for case (ii):

$$\hat{S}\hat{A}\hat{S}^{-1} = \varepsilon_A \exp\left(\tfrac{\pi k}{m}(\gamma_{56}+\gamma_{78})+(\gamma_{12}+\gamma_{34})\right) ,$$

whereas

$$\hat{A}^s = \varepsilon_A^s \exp\left(\tfrac{\pi k s}{m}(\gamma_{12}+\gamma_{34}+\gamma_{56}+\gamma_{78})\right) .$$

Both expressions agree if and only if

$$\varepsilon_A^{s-1}\exp\left(\tfrac{\pi k(s-1)}{m}(\gamma_{12}+\gamma_{34}+\gamma_{56}+\gamma_{78})\right)=1 \text{ hence } \varepsilon_A^{s-1}=\exp\left(\tfrac{\pi k(s-1)}{m}(\gamma_{56}+\gamma_{78})\right) .$$

If we set $(s-1)\equiv 0(m)$ then these expresions agree. Therefore in case(ii) we set $(s-1)\equiv 0(m)$.

In summary in each cases the quotient has two inequivalent spin structures labelled by ε_S. Signs $\varepsilon_J = 1$ and $\varepsilon_I = 1$. The signs ε_A and ε_B are fixed as shown above.

3.3.7 Type VI Groups with $d = 1$

In this section we consider the lifts to $Spin(8)$ of the subgroups $G_{\text{VI}}(m,n,s,t;k,\ell)$ of $SO(8)$ defined in Section 2.2.5. As we saw there, the subgroup is generated by the matrices

$$A = \begin{pmatrix} R\left(\frac{k}{m}\right) & & & \\ & R\left(\frac{k}{m}\right) & & \\ & & R\left(\frac{sk}{m}\right) & \\ & & & R\left(\frac{sk}{m}\right) \end{pmatrix}$$

$$B = \begin{pmatrix} R\left(\frac{\ell}{n}\right) & & & \\ & R\left(\frac{\ell}{n}\right) & & \\ & & R\left(\frac{\ell t}{n}\right) & \\ & & & R\left(\frac{\ell t}{n}\right) \end{pmatrix}$$

$$X = \frac{1}{2}\begin{pmatrix} \tau & \tau^{-1} & 0 & 1 & & & & \\ -\tau^{-1} & \tau & -1 & 0 & & & & \\ 0 & 1 & \tau & -\tau^{-1} & & & & \\ -1 & 0 & \tau^{-1} & \tau & & & & \\ & & & & \tau^{-1} & \tau & 1 & 0 \\ & & & & -\tau & \tau^{-1} & 0 & 1 \\ & & & & -1 & 0 & \tau^{-1} & -\tau \\ & & & & 0 & -1 & \tau & \tau^{-1} \end{pmatrix}$$

$$Y = \frac{1}{2}\begin{pmatrix} 1 & 0 & \tau^{-1} & \tau & & & & \\ 0 & 1 & -\tau & \tau^{-1} & & & & \\ -\tau^{-1} & \tau & 1 & 0 & & & & \\ -\tau & -\tau^{-1} & 1 & 0 & & & & \\ & & & & 1 & 1 & -1 & 1 \\ & & & & -1 & 1 & -1 & -1 \\ & & & & 1 & 1 & -1 & -1 \\ & & & & -1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} & \mathbf{1} \\ -\mathbf{1} & \end{pmatrix} ,$$

where $(n,m)=1$, $(k,m)=1$, $(\ell,n)=1$, $s^2 \equiv 1(m)$ and $t^2 \equiv 1(n)$.

Spin Structures

A and B generate K which has order prime to 30, hence it must have odd order and thus A and B each have odd order. A and B are block diagonal hence their lifts are immediate. We start by lifting A to

$$\hat{A} = \varepsilon_A \exp\left(\tfrac{\pi k}{m}(\gamma_{12}+\gamma_{34}) + \tfrac{\pi k s}{m}(\gamma_{56}+\gamma_{78})\right) ,$$

which obeys

$$\hat{A}^m = \varepsilon_A^m (-1)^{k(1+s)} \text{ hence } \varepsilon_A = (-1)^{k(1+s)} .$$

Lifting B we find

$$\hat{B} = \varepsilon_B \exp\left(\tfrac{\pi\ell}{n}(\gamma_{12}+\gamma_{34}) + \tfrac{\pi\ell t}{n}(\gamma_{56}+\gamma_{78})\right) \ ,$$

which obeys

$$\hat{B}^n = \varepsilon_B^n (-1)^{\ell(1+t)} \ \text{ hence } \ \varepsilon_B = (-1)^{\ell(t+1)} \ .$$

To lift X and Y we write them in terms of pairs of quaternions and use the method outlined in Section 3.2.1. Indeed

$$X = \begin{pmatrix} \frac{1}{2}(\tau + \tau^{-1}\boldsymbol{i} + \boldsymbol{k}) & \\ & -\frac{1}{2}(\tau^{-1} + \tau\boldsymbol{i} + \boldsymbol{j}) \end{pmatrix}$$

and

$$Y = \begin{pmatrix} \frac{1}{2}(1 + \tau^{-1}\boldsymbol{j} + \tau\boldsymbol{k}) & \\ & \frac{1}{2}(1 + \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) \end{pmatrix}$$

whence their lifts are given by repeated use of formula (3.2.1):

$$\begin{aligned} \hat{X} &= \varepsilon_X \left(\Pi_- + \tfrac{1}{2}\Pi_+(\tau - \tau^{-1}\gamma_{12} + \gamma_{23})\right)\left(\Pi'_- - \tfrac{1}{2}\Pi'_+(\tau^{-1} - \tau\gamma_{56} - \gamma_{57})\right) \\ \hat{Y} &= \varepsilon_Y \left(\Pi_- + \tfrac{1}{2}\Pi_+(1 - \tau^{-1}\gamma_{13} + \tau\gamma_{23})\right)\left(\Pi'_- + \tfrac{1}{2}\Pi'_+(1 - \gamma_{56} - \gamma_{57} + \gamma_{67})\right) \ , \end{aligned}$$

where as before $\Pi_\pm = \frac{1}{2}(1 \pm \gamma_{1234})$ and $\Pi'_\pm = \frac{1}{2}(1 \pm \gamma_{5678})$. Since the map $Sp(1) \times Sp(1) \to Spin(8)$ is a monomomorphism, the relations involving only $\hat{X}$ and $\hat{Y}$ will be automatically satisfied.

Finally we lift S:

$$\hat{S} = \varepsilon_S \tfrac{1}{4}(\gamma_1 - \gamma_5)(\gamma_2 - \gamma_6)(\gamma_3 - \gamma_7)(\gamma_4 - \gamma_8)\gamma_{1234} \ ,$$

which obeys $\hat{S}^2 = \gamma_{12345678}$ and hence $\hat{S}^4 = 1$. To check $\hat{S}\hat{A}\hat{S}^{-1} = \hat{A}^s$ one calculates:

$$\hat{S}\hat{A}\hat{S}^{-1} = \varepsilon_A \exp\left(\tfrac{\pi k}{m}(\gamma_{56}+\gamma_{78}) + \tfrac{\pi ks}{m}(\gamma_{12}+\gamma_{34})\right) \ ,$$

whereas

$$\hat{A}^s = \varepsilon_A^s \exp\left(\tfrac{\pi ks}{m}(\gamma_{12}+\gamma_{34}) + \tfrac{\pi ks^2}{m}(\gamma_{56}+\gamma_{78})\right) \ .$$

Both expressions agree if and only if

$$\varepsilon_A^{s-1} \exp\left(\tfrac{\pi k(s^2-1)}{m}(\gamma_{56}+\gamma_{78})\right) = 1 \ .$$

Since $\varepsilon_A = (-1)^{k(s+1)}$ and $s^2 \equiv 1(m)$ the sign of ε_A will be the same as the exponentials, hence the relation above is satisfied. A similar calculation shows that $\hat{S}\hat{B}\hat{S}^{-1} = \hat{B}^t$.

Finally we need to check the relations $\hat{S}\hat{X}\hat{S}^{-1} = -\hat{X}^3\hat{Y}\hat{X}^4\hat{Y}\hat{X}$ and $\hat{S}\hat{Y}\hat{S}^{-1} = -\hat{X}^2\hat{Y}\hat{X}^3$. However these are automatic. Consider the first relation. Notice that because lifting quaternions is a homomorphism, both $\hat{X}$ and $-\hat{X}^3\hat{Y}\hat{X}^4\hat{Y}\hat{X}$ are such that their fifth power is -1. We also know that the relation holds up to a sign. Taking the fifth power, we see that the sign has to be positive $\varepsilon_X = 1$. A similar argument, but taking the fifth power, shows that the second identity is satisfied $\varepsilon_Y = 1$.

In summary, both m and n are odd, and there are two inequivalent spin structures in the quotient labelled by ε_S. All other signs are fixed.

Chapter 4

Results and Conclusion

4.1 Invariant Spinors

In this chapter we shall investigate the supersymmetry of the spherical space forms. The supersymmetry of the quotient is given by the number of linearly independent group invariant Killing spinors on S^7. Using Bär's cone construction we can relate Killing spinors on S^7 to parallel spinors on $\mathbb{R}^8$. The parallel spinors on $\mathbb{R}^8$ are *constant* spinors, relative to flat coordinates (see Appendix A). Thus the invariant Killing spinors on S^7 are in one-to-one correspondence with the constant spinors of $Spin(8)$. Therefore identifying the linearly independent group invariant constant spinors of $Spin(8)$ is tantamount to determining the supersymmetry of the quotient. To determine the supersymmetry of the quotient we shall construct a basis for the space of spinors of $Spin(8)$ and act on it with each type of group. This will allow us to identify the groups preserving spinors whence determining the supersymmetric spherical space forms. The results will show two things; that in the negative chiral representation of $Spin(8)$ no spinors are preserved[1] and that the supersymmetry is dictated by spin structure.

We shall first look at the supersymmetry preserved by cyclic groups acting on the S^7. These are easier to deal than other types of group and we will not have to employ the explicit basis.

[1]Hence the correspondingly orientated sphere (background) is not supersymmetric.

4.2 Cyclic Quotients

In this section we consider quotients by the cyclic group with generator

$$\hat{A} = (-1)^{1+a+b+c} \exp\left(\frac{i\pi}{n}\left(\sigma_1 + a\sigma_2 + b\sigma_3 + c\sigma_4\right)\right),$$

these have been reported in [13]. The sign $\varepsilon_A = (-1)^{1+a+b+c}$ is fixed when n is odd, thus there is only one spin structure on the corresponding spherical space form. However when n is even we are free to choose the sign. Thus there are two inequivalent spin structures on the corresponding spherical space forms. We shall see the choice of sign, or spin structure, dictates the supersymmetry preserved. Let us begin with a few examples to determine what happens[2] :

4.2.1 $n = 2$

If $n = 2$ then $G(2,1,1,1) \cong \mathbb{Z}_2$ and as explained this lifts automatically. The geometry of the quotient is $\mathbb{RP}^7$ and $a = b = c = 1$. Therefore we require

$$\varepsilon \exp\left(\frac{i\pi}{2}(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)\right) = 1.$$

If $\varepsilon_A = 1$ then $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 0$ or ± 4. This can happen in the following ways;$\pm\pm\pm\pm$, $\pm\pm\mp\mp$, $\pm\mp\pm\mp$ and $\pm\mp\mp\pm$, for a total of 8 all of positive chirality. For the $\varepsilon_A = -1$ spin structure we require the sum to equal ± 2. This happens for $\pm\pm\pm\mp$, $\pm\pm\mp\pm$, $\pm\mp\pm\pm$ and $\pm\mp\mp\mp$ for a total of 8 all of negative chirality. We conclude that this quotient preserves *all* of the supersymmetry of the vacuum in both chiral representations.

4.2.2 $n = 3$

When $n = 3$

$$\hat{A} = (-1)^{1+a+b+c} \exp\left(\frac{i\pi}{3}\left(\sigma_1 + a\sigma_2 + b\sigma_3 + c\sigma_4\right)\right).$$

[2] Note that the basis is not required in this case. Here we only need to consider the eigenvalues of $\hat{A}$.

The possible choices for (a,b,c) are $(1,1,1)$, $(1,1,2)$ and $(1,2,2)$. The other choice $(2,2,2)$ gives the same quotient as $(1,1,2)$ since they generate the same subgroups. Let us take each case in turn:

$(1,1,1)$

We require $\sigma_1+\sigma_2+\sigma_3+\sigma_4=0$, which happens for 6 weights $\pm\pm\mp\mp$, $\pm\mp\pm\mp$ and $\pm\mp\mp\pm$, all of positive chirality.

$(1,1,2)$

Here we require $\sigma_1+\sigma_2+\sigma_3+2\sigma_4=\pm3$, which happens for 6 weights $\mp\pm\pm\pm$, $\pm\mp\pm\pm$, and $\pm\pm\mp\pm$, all of negative chirality.

$(1,2,2)$

Here we can have $\sigma_1+\sigma_2+2\sigma_3+2\sigma_4=0,\pm6$, which happens for 6 weights $\pm\pm\pm\pm$, $\pm\mp\pm\mp$ and $\pm\mp\mp\pm$, all of positive chirality.

4.2.3 $n=4$

Again there are three possible choices for (a,b,c): $(1,1,1)$, $(1,1,3)$ and $(1,3,3)$, with $(3,3,3)$ and $(1,1,3)$ generating the same subgroups.

$(1,1,1)$

Here we can have $\sigma_1+\sigma_2+\sigma_3+\sigma_4=0,\pm4$, with those weights adding up to 0 and those to ±4 in different spin structures. For the $\varepsilon_A=1$ spin structure, they must add up to zero and there are six such weights of all positive chirality: $\pm\pm\mp\mp$, $\pm\mp\pm\mp$ and $\pm\mp\mp\pm$. For the $\varepsilon_A=-1$ spin structure, they must add up to ±4 and there are two such weights $\pm\pm\pm\pm$ all of positive chirality.

$(1,1,3)$

Again we can have $\sigma_1+\sigma_2+\sigma_3+\sigma_4=0,\pm4$. For the $\varepsilon_A=1$ spin structure, the sum must give 0 which happens for two negative-chirality weights: $\pm\pm\pm\mp$.

For the $\varepsilon_A = -1$ spin structure, the sum must give ± 4 which happens for 6 negative-chirality weights: $\mp \pm \pm\pm$, $\pm \mp \pm\pm$ and $\pm \pm \mp\pm$.

$(1, 3, 3)$

Here we can have $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 0, \pm 4, \pm 8$. For the $\varepsilon_A = 1$ spin structure the sum must either be 0 or ± 8, which happens for 6 positive-chirality weights: $\pm \pm \pm\pm$, $\pm \mp \pm\mp$ and $\pm \mp \mp\pm$. For the $\varepsilon_A = -1$ spin structure, the sum must be ± 4 and this happens for two positive-chirality weights: $\pm \pm \mp\mp$.

The results above show that supersymmetry depends on spin structure, we are able to preserve $\frac{1}{4}$ or $\frac{3}{4}$ of the supersymmetry depending on spin structure.

4.2.4 $n = 4k \geq 8$

If we have $n = 4k \geq 8$ then the unique conjugacy class of groups that can be formed have weights (a, b, c): $(1, 2k-1, 2k-1)$, $(1, 2k+1, 2k+1)$, $(2k-1, 2k+1, 4k-1)$ and $(1, 2k-1, 2k+1)$. We investigate them below;

$(1, 2k-1, 2k-1)$

In this case the spinors with weights $\pm\mp\pm\mp$ and $\pm\mp\mp\pm$ are invariant relative to the positive spin structure, whereas the spinors with weights $\pm\pm\pm\pm$ are invariant relative to the negative spin structure. All have positive chirality.

$(1, 2k+1, 2k+1)$

In this case the spinors with weights $\pm\mp\pm\mp$ and $\pm\mp\mp\pm$ are invariant relative to the positive spin structure, whereas the spinors with weights $\pm\pm\mp\mp$ are invariant relative to the negative spin structure. All have positive chirality.

$(2k-1, 2k+1, 4k-1)$

In this case the spinors with weights $\pm\pm\pm\pm$ and $\pm\mp\mp\pm$ are invariant relative to the positive spin structure, whereas the spinors with weights $\pm\pm\mp\mp$ are invariant relative to the negative spin structure. All have positive chirality.

$(1, 2k-1, 2k+1)$

In this case the spinors with weights $\pm\pm\pm\mp$ are invariant relative to the positive spin structure, whereas the spinors with weights $\pm\mp\pm\pm$ and $\mp\pm\pm\pm$ are invariant relative to the negative spin structure. All have negative chirality.

Thus for the $n = 4K \geq 8$ case we conclude that either $\frac{1}{4}$ or $\frac{3}{4}$ of the supersymmetry is preserved, depending on the spin structure.[3] Moreover some experimentation suggests that these are (up to conjugation) the only cases where this phenomenon occurs.

This case of cyclic quotients highlights the importance of specifying spin structure as part of the data of a supergravity background. However the orientation of the background has no real importance. We will see for Type I groups and beyond that indeed the orientation does play a significant role and its choice, which corresponds to the chirality of the $Spin(8)$ representation, dictates the supersymmetry of the quotient.

4.3 Basis Construction

The other types of group are not as easy to deal with as the cyclic groups above. To determine the supersymmetry on their corresponding spherical space forms we shall need to employ an explicit basis. It will be convenient for us to work in the complexified $Spin(8)$ module. Thus we construct a basis for the complexified space of spinors. Each of the groups acts complex linearly on this basis. Thus for every invariant spinor in this basis there is a corresponding spinor in the real space. In analogy with quantum mechanics we define an operator $a_j = \frac{1}{2}(\Gamma_{2j-1} - i\Gamma_{2j})$ to be the annihilation operator and $a_j^\dagger = \frac{1}{2}(\Gamma_{2j-1} + i\Gamma_{2j})$ be the creation operator for $j \in \{1, 2, 3, 4\}$. The ground state is given by $|0\rangle = a_1 a_2 a_3 a_4$, then the basis is made up by acting with $a_i^\dagger$ on $|0\rangle$ as follows;

[3] We note that it was proved by Franc in [22] that the lens space $\mathbb{RP}^{4k+3}$ is the only one admitting the maximal number of Killing Spinors. It was later proved by Bär in [5] that this is still the case among all spherical space forms.

$$
\begin{aligned}
s_1 &= |0\rangle, & s_9 &= a_2^\dagger a_3^\dagger|0\rangle,\\
s_2 &= a_1^\dagger|0\rangle, & s_{10} &= a_2^\dagger a_4^\dagger|0\rangle,\\
s_3 &= a_2^\dagger|0\rangle, & s_{11} &= a_3^\dagger a_4^\dagger|0\rangle,\\
s_4 &= a_3^\dagger|0\rangle, & s_{12} &= a_1^\dagger a_2^\dagger a_3^\dagger|0\rangle,\\
s_5 &= a_4^\dagger|0\rangle, & s_{13} &= a_1^\dagger a_2^\dagger a_4^\dagger|0\rangle,\\
s_6 &= a_1^\dagger a_2^\dagger|0\rangle, & s_{14} &= a_1^\dagger a_3^\dagger a_4^\dagger|0\rangle,\\
s_7 &= a_1^\dagger a_3^\dagger|0\rangle, & s_{15} &= a_2^\dagger a_3^\dagger a_4^\dagger|0\rangle,\\
s_8 &= a_1^\dagger a_4^\dagger|0\rangle, & s_{16} &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger|0\rangle,
\end{aligned}
$$

This basis splits up to span the complexified modules of each chiral representation $\Delta^8_\pm$. The positice chiral module is spanned by $\mathcal{S}_+ = \{s_1, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{16}\}$ and the negative chiral module is spanned by $\mathcal{S}_- = \{s_2, s_3, s_4, s_5, s_{12}, s_{13}, s_{14}, s_{15}\}$. We shall now consider the action of each type of group on the basis above. By determining the linearly independent group invariant spinors on $\mathcal{S}_+$ and $\mathcal{S}_-$, we determine the supersymmetry of the correspondingly oriented spherical space form (background). It will be obvious from the following analysis that the $\mathcal{S}_-$ module does not yield invariant spinors, hence the correspondingly oriented background is not supersymmetric.

4.3.1 Methodology

Each type of group is specified by a set of generators, which depend on a set of parameters n, m, r, etc. together with a set of numerical consistency conditions which these parameters satisfy [4] . By specifying values for these parameters consistently, i.e. in a way that they obey the consistency conditions, we may identify a particular group belonging to that type. To each such group corresponds a spherical space form. We would like to know which of these spherical space forms is supersymmetric and in particular how much supersymmetry they preserve.

In the following work we shall identify classes of groups, belonging to a particular type, which preserve spinors. Hence producing supersymmetric spherical space forms. We shall do this by acting on the basis with each generator of a given

[4]For example Type I groups are generated by $\hat{A}$ which depends on k_i, r and m and $\hat{B}$ which depends on n and l_i. Together with the numerical consistency conditions $(n(r-1), m) = 1$, $(k_i, m) = 1$, $(l_i, n) = 1$ $r^d \equiv 1(m)$ and $r^n \equiv 1(m)$.

type of group. This, as we shall see, will allow us to determine it's eigenspinors. The corresponding eigenvalues will depend on the parameters associated with that generator[5]. By specifying congruence relations, involving these parameters, we shall be able to set the eigenvalue to one. We must make sure that this is done consistently. Applying this method for each generator, we may identify for it a set of eigenvalue one, or invariant, spinors. Together with a set of consistant congruence relations, for the invariance of each spinor in the set.

By comparing the sets of invariant spinors, of each generator, we will be able to identify or construct common invariant spinors. Each of these common invariant spinors will come with a set of congruence relations, derived from each generator, for its invariance. This set of congruence relations, specifies a class of groups contained in the given type which preserve that spinor. By imposing further constraints on the parameters involved, in these congruence relations, we may be able to preserve some subset of these common invariant spinors simultaneously. This is tantamount to identifying subclasses of groups, contained within the classes which preserve one of the spinors in the set, preserving multiple spinors. The method outlined above is made clear in the following section where we shall give a detailed account of finding the supersymmetric spherical space formes given by Type I groups when $d = 2$.

4.4 Type I: d=2

We shall now use the method outlined above to determine the supersymmetric spherical spaceforms given by Type I groups when $d = 2$. This case will serve as an worked example of the method above. The same method has been applied in all proceeding cases and will not be given in detail again. The generators in this case are $\hat{A}$ and $\hat{B}$ and the numerical consistency conditions are $(n(r-1), m) = 1$, $(k_i, m) = 1$, $(l_i, n) = 1$, $r^n \equiv 1(m)$ and $r^2 = 1(m)$. Since $r^2 = (r-1)(r+1) \equiv 0(m)$ and $(r-1, m) = 1$ this gives $(r+1) \equiv 0(m)$.

The action of the generators on the basis is given below where $\varepsilon_A = (-1)^{(k_1+k_2)(1+r)}$ and ε_B is not fixed (whence we have two inequivalent spin structures).

[5]For example the eigenvalues of the $\hat{A}$ eigenspinors will depend on k_i, r and m.

$$
\begin{aligned}
\hat{A}s_1 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1+k_2)(1+r)\right) s_1, & \hat{B}s_1 &= -\varepsilon_B \exp(\tfrac{2\pi i}{n}(l_1+l_2))s_1,\\
\hat{A}s_2 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1(r-1)+k_2(1+r))\right) s_2, & \hat{B}s_2 &= -\epsilon_B \exp(-\tfrac{2\pi i}{n}(l_1-l_2))s_3,\\
\hat{A}s_3 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1(1-r)+k_2(1+r))\right) s_3, & \hat{B}s_3 &= -\epsilon_B \exp(\tfrac{2\pi i}{n}(l_1+l_2))s_2,\\
\hat{A}s_4 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1(1+r)+k_2(r-1))\right) s_4, & \hat{B}s_4 &= -\epsilon_B \exp(\tfrac{2\pi i}{n}(l_1-l_2))s_5,\\
\hat{A}s_5 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1(1+r)+k_2(1-r))\right) s_5, & \hat{B}s_5 &= -\epsilon_B \exp(\tfrac{2\pi i}{n}(l_1+l_2))s_4,\\
\hat{A}s_6 &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1-k_2)(1+r)\right) s_6, & \hat{B}s_6 &= \epsilon_B \exp(-\tfrac{2\pi i}{n}(l_1-l_2))s_6,\\
\hat{A}s_7 &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1+k_2)(1-r)\right) s_7, & \hat{B}s_7 &= -\epsilon_B \exp(-\tfrac{2\pi i}{n}(l_1+l_2))s_{10},\\
\hat{A}s_8 &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1-k_2)(1-r)\right) s_8, & \hat{B}s_8 &= -\epsilon_B \exp(-\tfrac{2\pi i}{n}(l_1-l_2))s_9,\\
\hat{A}s_9 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1-k_2)(1-r)\right) s_9, & \hat{B}s_9 &= -\epsilon_B \exp(\tfrac{2\pi i}{n}(l_1-l_2))s_8,\\
\hat{A}s_{10} &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1+k_2)(r-1)\right) s_{10}, & \hat{B}s_{10} &= -\epsilon_B \exp(\tfrac{2\pi i}{n}(l_1+l_2))s_7,\\
\hat{A}s_{11} &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1-k_2)(1+r)\right) s_{11}, & \hat{B}s_{11} &= \epsilon_B \exp(\tfrac{2\pi i}{n}(l_1-l_2))s_{11},\\
\hat{A}s_{12} &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1(1+r)+k_2(1-r))\right) s_{12}, & \hat{B}s_{12} &= \epsilon_B \exp(-\tfrac{2\pi i}{n}(l_1+l_2))s_{13},\\
\hat{A}s_{13} &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1(1+r)+k_2(r-1))\right) s_{13}, & \hat{B}s_{13} &= \epsilon_B \exp(-\tfrac{2\pi i}{n}(l_1-l_2))s_{12},\\
\hat{A}s_{14} &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1(1-r)+k_2(1+r))\right) s_{14}, & \hat{B}s_{14} &= \epsilon_B \exp(-\tfrac{2\pi i}{n}(l_1+l_2))s_{15},\\
\hat{A}s_{15} &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1(r-1)+k_2(1+r))\right) s_{15}, & \hat{B}s_{15} &= \epsilon_B \exp(\tfrac{2\pi i}{n}(l_1-l_2))s_{14},\\
\hat{A}s_{16} &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1+k_2)(1+r)\right) s_{16}. & \hat{B}s_{16} &= -\epsilon_B \exp(-\tfrac{2\pi i}{n}(l_1+l_2))s_{16}.
\end{aligned}
$$

Each basis spinor of $\mathcal{S}_+$ and $\mathcal{S}_-$ is an eigenspinor of $\hat{A}$ with eigenvalue $\varepsilon_A \exp\left(\frac{i\pi\Theta_i}{m}\right)$, where Θ_i is the argument in the expression involving s_i. Setting each $\Theta_i \equiv 0(m)$ will automatically set the eigenvalue to one [6] . However this cannot always be done in a way that is consistent with the group consitency conditions. This becomes clear if we consider the numerical condition $\Theta_i \equiv 0(m)$ in each case. We first note that pairs of spinors are invariant for the same numerical condition as follows; (s_1, s_{16}) for $(k_1+k_2)(1+r) \equiv 0(m)$, (s_2, s_{15}) for $k_1(r-1)+k_2(1+r) \equiv 0(m)$, (s_3, s_{14}) for $k_1(1-r)+k_2(1+r) \equiv 0(m)$, (s_4, s_{13}) for $k_1(1+r)+k_2(r-1) \equiv 0(m)$, (s_5, s_{12}) for $k_1(1+r)+k_2(1-r) \equiv 0(m)$, (s_6, s_{11}) for $(k_1-k_2)(1+r) \equiv 0(m)$, (s_7, s_{10}) for $(k_1+k_2)(r-1) \equiv 0(m)$ and (s_8, s_9) for $(k_1-k_2)(1-r) \equiv 0(m)$. Since $(r+1) \equiv 0(m)$ s_1, s_{16}, s_6 and s_{11} are automatically $\hat{A}$ invariant. Hence the $\hat{A}$ generator in any group belonging to Type I groups with $d = 2$, will always

[6] This is because $m \equiv 1(2)$ and any given Θ_i can be turned into $(k_1+k_2)(1+r)$ via changes in sign, which numerically is a change modulo two. Thus if any Θ_i is congruent to an odd multiple of m, $(k_1+k_2)(1+r)$ will also be odd giving $\varepsilon_A = -1$. Thus the two minus one multiply to give plus one as the eigenvalue. Similarly in the case of even congruence.

preserve these four spinors. The pair (s_7, s_{10}) are invariant if $(k_1 + k_2) \equiv 0(m)$ and (s_8, s_9) are invariant if $(k_1 - k_2) \equiv 0(m)$. The remaining spinors span $\mathcal{S}_-$. None of these are invariant since the conditions above reduce to $k_i(r-1) \equiv 0(m)$, which is not allowed by the consistency conditions. Thus the set of $\hat{A}$ invariant spinors is $\{s_1, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{16}\}$.

Let us now find the $\hat{B}$ invariant spinors keeping in mind the $\mathcal{S}_-$ space is excluded by $\hat{A}$. The spinors s_1, s_6, s_{11} and s_{16} are eigenspinors of $\hat{B}$. They can all be made invariant by setting the argument Θ_i in the exponential $\exp\left(\frac{2\pi i}{n}\Theta_i\right)$ congruent to n or $\frac{n}{2}(n)$ and choosing ε_B appropriately. Importantly these congruence relations do not conflict with the group conditions. $\hat{B}$ preserves the subspaces $\{s_7,\, s_{10}\}$ on which, $\hat{B}s_7 = \varepsilon_B \lambda s_{10}$ and $\hat{B}s_{10} = \varepsilon_B \bar{\lambda} s_7$ where $\lambda = -\exp\left(-\frac{2\pi i}{n}(l_1 + l_2)\right)$ and $|\lambda| = 1$. The linearly independent $\hat{B}$ invariant spinor on the $\{s_7, s_{10}\}$ subspace is given by,

$$s_7 + \varepsilon_B \lambda s_{10}.$$

Similarly on the $\{s_8, s_9\}$ subspace,

$$s_8 + \varepsilon_B \lambda' s_9$$

is invariant where $\lambda' = -\exp\left(\frac{-2\pi i}{n}(l_1 - l_2)\right)$. Thus the $\hat{B}$ invariant spinors are $\{s_1, s_6, s_{11}, s_{16}, s_7 + \varepsilon_B \lambda s_{10}, s_8 + \varepsilon_B \lambda' s_9\}$.

Now that we know the sets of $\hat{A}$ and $\hat{B}$ invariant spinors we can compare them to identify any common invariant spinors. From the set of $\hat{A}$ and $\hat{B}$ invariant spinors we can see s_1, s_6, s_{11} and s_{16} are common to both sets. Spinors s_7, s_8, s_9 and s_{10} are $\hat{A}$ invariant but not $\hat{B}$ invariant. However $s_7 + \varepsilon_B \lambda s_{10}$ and $s_8 + \varepsilon_B \lambda' s_9$ are $\hat{A}$ invariant, hence each of these is a common invariant spinor. The table below gives each common invariant spinor along with its invariance conditions in the relative spin structure, which we label by $(\varepsilon_A, \varepsilon_B)$. In this case ε_A is fixed but ε_B is not, whence there are two inequivalent spin structures labelled by $(\varepsilon_A, 1)$ and $(\varepsilon_A, -1)$.

Spin Structure $(\varepsilon_A, \varepsilon_B)$	*Invariant Spinors*	*Invariance Conditions*
$(\varepsilon_A, 1)$	s_1	$l_1 + l_2 \equiv \frac{n}{2}(n)$
	s_6	$l_1 - l_2 \equiv 0(n)$
	$s_7 + \lambda s_{10}$	$(k_1 + k_2) \equiv 0(m)$, $(l_i, n) = 1$
	$s_8 + \lambda' s_9$	$(k_1 - k_2) \equiv 0(m)$, $(l_i, n) = 1$
	s_{11}	$l_1 - l_2 \equiv 0(n)$
	s_{16}	$l_1 + l_2 \equiv \frac{n}{2}(n)$
$(\varepsilon_A, -1)$	s_1	$l_1 + l_2 \equiv 0(n)$
	s_6	$l_1 - l_2 \equiv \frac{n}{2}(n)$
	$s_7 - \lambda s_{10}$	$(k_1 + k_2) \equiv 0(m)$, $(l_i, n) = 1$
	$s_8 - \lambda' s_9$	$(k_1 - k_2) \equiv 0(m)$, $(l_i, n) = 1$
	s_{11}	$l_1 - l_2 \equiv \frac{n}{2}(n)$
	s_{16}	$l_1 + l_2 \equiv 0(n)$

Each invariance condition in the table above specifies a class of groups which preserve the corresponding spinor. For example, the class of groups preserving s_1 in the $(\varepsilon_A, 1)$ spin structure are those for which $l_1 + l_2 \equiv \frac{n}{2}(n)$. Thus any Type I group with $d = 2$ will preserve s_1, provided we fix the generator $\hat{B}$ so that $l_1 + l_2 \equiv \frac{n}{2}(n)$. Since the condition $l_1 + l_2 \equiv \frac{n}{2}(n)$ does not appear in the $\varepsilon_B = -1$ case the corresponding class of groups specified by this condition do not preserve any spinors in the $(\varepsilon_A, -1)$ spin structure. The only classes of groups preserving spinors in both spin structures, are those specified by the conditions $(k_1 + k_2) \equiv 0(m)$ and $(k_1 - k_2) \equiv 0(m)$.

Each of the classes of groups in the table above may contain subclasses of groups preserving multiple spinors, in the table. Identifying spinors above which can be preserves simultaneously is tantamount to identifying these subclasses. For example the class of groups preserving s_1 when $\varepsilon_B = 1$ contain a subclass of groups preserving $s_7 + \lambda s_{10}$. This subclass is specified by further constraining the values k_1 and k_2 can take, in the generator $\hat{A}$, so that $(k_1 + k_2) \equiv 0(m)$.

The table below gives the spinors which can be preserved simultaneously, together with the invariance conditions which specify the subclasses preserving them.

Spin Structure: $(\varepsilon_A, 1)$	
Group Invariant Spinors	*Invariance Conditions*
$s_1,\ s_{16}$	$l_1 + l_2 \equiv \frac{n}{2}(n)$
$s_1,\ s_{16},\ s_7 + \lambda s_{10}$	$(k_1 + k_2) \equiv 0(m),\ l_1 + l_2 \equiv \frac{n}{2}(n)$
$s_1,\ s_{16},\ s_8 + \lambda' s_9$	$(k_1 - k_2) \equiv 0(m),\ l_1 + l_2 \equiv \frac{n}{2}(n)$
$s_6,\ s_{11}$	$l_1 - l_2 \equiv 0(n)$
$s_6,\ s_{11},\ s_7 + \lambda s_{10}$	$(k_1 + k_2) \equiv 0(m),\ \ l_1 - l_2 \equiv 0(n)$
$s_6,\ s_{11},\ s_8 + \lambda' s_9$	$(k_1 - k_2) \equiv 0(m),\ l_1 - l_2 \equiv 0(n)$
$s_1,\ s_6,\ s_{11},\ s_{16}$	$n = 4,\ l_i = 1$ or 3
$s_1,\ s_6,\ s_{11},\ s_{16},\ s_7 + \lambda s_{10}$	$n = 4,\ l_i = 1$ or 3, $(k_1 + k_2) \equiv 0(m)$
$s_1,\ s_6,\ s_{11},\ s_{16},\ s_8 + \lambda' s_9$	$n = 4,\ l_i = 1$ or 3, $(k_1 - k_2) \equiv 0(m)$
Spin Structure: $(\varepsilon_A, -1)$	
$s_1,\ s_{16}$	$l_1 + l_2 \equiv 0(n)$
$s_1,\ s_{16},\ s_7 - \lambda s_{10}$	$(k_1 + k_2) \equiv 0(m),\ l_1 + l_2 \equiv 0(n)$
$s_1,\ s_{16},\ s_8 - \lambda' s_9$	$(k_1 - k_2) \equiv 0(m),\ l_1 + l_2 \equiv 0(n)$
$s_6,\ s_{11}$	$l_1 - l_2 \equiv \frac{n}{2}(n)$
$s_6,\ s_{11},\ s_7 - \lambda s_{10}$	$(k_1 - k_2) \equiv 0(m),\ l_1 - l_2 \equiv \frac{n}{2}(n)$
$s_6,\ s_{11},\ s_8 - \lambda' s_9$	$(k_1 - k_2) \equiv 0(m),\ l_1 - l_2 \equiv \frac{n}{2}(n)$
$s_1,\ s_6,\ s_{11},\ s_{16}$	$n = 4,\ l_i = 1$ or 3
$s_1,\ s_6,\ s_{11},\ s_{16},\ s_7 - \lambda s_{10}$	$n = 4,\ l_i = 1$ or 3, $(k_1 + k_2) \equiv 0(m)$
$s_1,\ s_6,\ s_{11},\ s_{16},\ s_8 - \lambda' s_9$	$n = 4,\ l_i = 1$ or 3, $(k_1 - k_2) \equiv 0(m)$

The last three conditions in the table above, specify classes of groups which preserve the same amount of supersymmetry in both spin structures. This is either $\frac{1}{2}$ or $\frac{5}{8}$. None of the other conditions in the table are present in both spin structures, hence the remaining conditions specify classes of groups which preserve supersymmetry in only one spin structure. Such classes preserve either $\frac{1}{4}$ or $\frac{3}{8}$ of the supersymmetry.

4.5 Type I: d=4

This section deals with Type I groups with $d = 4$, $n \equiv 0(2)$ and $m \equiv 1(2)$. The consistency conditions are $(n(r-1), m) = 1$, $(k_i, m) = 1$, $(l_i, n) = 1$, $r^n \equiv 1(m)$ and $r^4 - 1 = (r-1)(r+1)(r^2+1) \equiv 0(m)$ hence $(r+1)(r^2+1) \equiv 0(m)$. The method for finding group invariant spinors, is the same as in the previous section. The action of $\hat{A}$ and $\hat{B}$ on the basis is as follows;

$$
\begin{aligned}
\hat{A}s_1 &= \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1+r+r^2+r^3)\right) s_1, & \hat{B}s_1 &= i\varepsilon_B \exp(i\tfrac{4\pi l}{n})s_1,\\
\hat{A}s_2 &= \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1-r-r^2-r^3)\right) s_2, & \hat{B}s_2 &= i\varepsilon_B \exp(-i\tfrac{4\pi l}{n})s_3,\\
\hat{A}s_3 &= \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1-r+r^2+r^3)\right) s_3, & \hat{B}s_3 &= i\varepsilon_B \exp(i\tfrac{4\pi l}{n})s_4,\\
\hat{A}s_4 &= \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1+r-r^2+r^3)\right) s_4, & \hat{B}s_4 &= i\varepsilon_B \exp(i\tfrac{4\pi l}{n})s_5,\\
\hat{A}s_5 &= \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1+r+r^2-r^3)\right) s_5, & \hat{B}s_5 &= i\varepsilon_B \exp(i\tfrac{4\pi l}{n})s_2,\\
\hat{A}s_6 &= \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1+r-r^2-r^3)\right) s_6, & \hat{B}s_6 &= i\varepsilon_B \exp(-i\tfrac{4\pi l}{n})s_9,\\
\hat{A}s_7 &= \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1-r+r^2-r^3)\right) s_7, & \hat{B}s_7 &= i\varepsilon_B \exp(-i\tfrac{4\pi l}{n})s_{10},\\
\hat{A}s_8 &= \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1-r-r^2+r^3)\right) s_8, & \hat{B}s_8 &= -i\varepsilon_B \exp(-i\tfrac{4\pi l}{n})s_6,\\
\hat{A}s_9 &= \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1-r-r^2+r^3)\right) s_9, & \hat{B}s_9 &= i\varepsilon_B \exp(i\tfrac{4\pi l}{n})s_{11},\\
\hat{A}s_{10} &= \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1-r+r^2-r^3)\right) s_{10}, & \hat{B}s_{10} &= -i\varepsilon_B \exp(i\tfrac{4\pi l}{n})s_7,\\
\hat{A}s_{11} &= \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1+r-r^2-r^3)\right) s_{11}, & \hat{B}s_{11} &= -i\varepsilon_B \exp(i\tfrac{4\pi l}{n})s_8,\\
\hat{A}s_{12} &= \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1+r+r^2-r^3)\right) s_{12}, & \hat{B}s_{12} &= i\varepsilon_B \exp(-i\tfrac{4\pi l}{n})s_{15},\\
\hat{A}s_{13} &= \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1+r-r^2+r^3)\right) s_{13}, & \hat{B}s_{13} &= i\varepsilon_B \exp(-i\tfrac{4\pi l}{n})s_{12},\\
\hat{A}s_{14} &= \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1-r+r^2+r^3)\right) s_{14}, & \hat{B}s_{14} &= i\varepsilon_B \exp(-i\tfrac{4\pi l}{n})s_{13},\\
\hat{A}s_{15} &= \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1-r-r^2-r^3)\right) s_{15}, & \hat{B}s_{15} &= i\varepsilon_B \exp(i\tfrac{4\pi l}{n})s_{14},\\
\hat{A}s_{16} &= \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1+r+r^2+r^3)\right) s_{16}. & \hat{B}s_{16} &= -i\varepsilon_B \exp(-i\tfrac{4\pi l}{n})s_{16}.
\end{aligned}
$$

The sign $\varepsilon_A = (-1)^{k(1+r+r^2+r^3)}$ and ε_B is not fixed, thus we have two inequivalent spin structures. $\hat{A}$ is diagonal in this basis so we can make each spinor invariant by setting the argument Θ_i in the exponential congruent to m, but again the spinors spanning $\mathcal{S}_-$ cannot be made invariant consistently. Under the action of $\hat{B}$, s_1 is invariant for $4l \equiv -\frac{n}{2}(2n)$ when $\varepsilon_B = 1$ or $4l \equiv \frac{n}{2}(2n)$ when $\varepsilon_B = -1$. Similarly s_{16} is invariant for $4l \equiv \frac{n}{2}(2n)$ when $\varepsilon_B = 1$ or $4l \equiv -\frac{n}{2}(2n)$ when $\varepsilon_B = -1$. These conditions are satisfied without violating the group conditions, provided $n = 8$ and $l \in \{1, 3, 5, 7\}$ is chosen appropriately. The $\hat{B}$ invariant spinors on the $\{s_7, s_{10}\}$ subspace are $s_7 + \lambda s_{10}$ when $\varepsilon_B = 1$ and $s_7 - \lambda s_{10}$ when $\varepsilon_B = -1$ where $\lambda = i\exp(-i\frac{4\pi l}{n})$. There are $\hat{B}$ invariant spinors on the; $\{s_2, s_3, s_4, s_5\}$, $\{s_6, s_8, s_9, s_{10}\}$ and $\{s_{12}, s_{13}, s_{14}, s_{15}\}$ subspaces but they can only be made $\hat{A}$ invariant if $r^2 \equiv 1(m)$, which violates the group condition $r^4 \equiv 1(m)$ therefore they are not allowed. The table below gives the individual invariant spinors and the specifications for the class of groups which preserve them.

Spin Structure $(\varepsilon_A, \varepsilon_B)$	*Invariant Spinors*	*Invariance Conditions*
$(\varepsilon_A, 1)$	s_1	$4l \equiv -\frac{n}{2}(2n)$
	s_{16}	$4l \equiv \frac{n}{2}(2n)$
	$s_7 + \lambda s_{10}$	$(1 + r^2) \equiv 0(m)$, $(l_i, n) = 1$
$(\varepsilon_A, -1)$	s_1	$4l \equiv \frac{n}{2}(2n)$
	s_{16}	$4l \equiv -\frac{n}{2}(2n)$
	$s_7 - \lambda s_{10}$	$(1 + r^2) \equiv 0(m)$, $(l_i, n) = 1$

From the table above we see that a given condition is present in both spin structures. Thus each corresponding class of groups, preserves the same amount of supersymmetry in both spin structures. As a consequence we expect the subclasses of groups preserving multiple spinors, to preserve the same amount of supersymmetry in each spin structures as well. The table below gives the spinors which can be preserved simultaneously and the corresponding conditions for their invariance,

Spin Structure: $(\varepsilon_A, 1)$	
Group Invariant Spinors	*Invariance Conditions*
$s_1,\ s_7 + \lambda s_{10}$	$(1 + r^2) \equiv 0(m)$, $4l \equiv -\frac{n}{2}(2n)$
$s_{16},\ s_7 + \lambda s_{10}$	$(1 + r^2) \equiv 0(m)$, $4l \equiv \frac{n}{2}(2n)$
Spin Structure: $(\varepsilon_A, -1)$	
$s_1,\ s_7 - \lambda s_{10}$	$(1 + r^2) \equiv 0(m)$, $4l \equiv \frac{n}{2}(2n)$
$s_{16},\ s_7 - \lambda s_{10}$	$(1 + r^2) \equiv 0(m)$, $4l \equiv -\frac{n}{2}(2n)$

.

We can clearly see each condition above is present in both spin structures. Thus, as expected, each subclass of groups preserves the same amount of supersymmetry in both spin structure. In conclusion there are classes of groups contained in Type I groups with $d = 4$ which preserve either $\frac{1}{8}$ or $\frac{1}{4}$ of the supersymmetry in both spin structures.

4.6 Type II: d=1

This section deals with Type II groups with $d = 1$. The consistency conditions are $s^2 \equiv 1(m)$, $t^2 \equiv 1(n)$, $(m, n) = 1$, $(k_i, m) = 1$ and $(l_i, n) = 1$. The action of $\hat{A}$, $\hat{B}$ and $\hat{S}$ is given below;

$$\begin{aligned}
\hat{A}s_1 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1 + k_2)(1+s)\right) s_1,\\
\hat{A}s_2 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(-k_1(1-s) + k_2(1+s))\right) s_2,\\
\hat{A}s_3 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1(1-s) + k_2(1+s))\right) s_3,\\
\hat{A}s_4 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1(1+s) - k_2(1-s))\right) s_4,\\
\hat{A}s_5 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1(1+s) + k_2(1-s))\right) s_5,\\
\hat{A}s_6 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(-k_1(1+s) - k_2(1+s))\right) s_6,\\
\hat{A}s_7 &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1 + k_2)(1-s)\right) s_7,\\
\hat{A}s_8 &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1 - k_2)(1-s)\right) s_8,\\
\hat{A}s_9 &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1 - k_2)(1-s)\right) s_9,\\
\hat{A}s_{10} &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1 + k_2)(1-s)\right) s_{10},\\
\hat{A}s_{11} &= \varepsilon_A \exp\left(\tfrac{i\pi}{m}(k_1 - k_2)(1+s)\right) s_{11},\\
\hat{A}s_{12} &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1(1+s) + k_2(1-s))\right) s_{12},\\
\hat{A}s_{13} &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1(1+s) - k_2(1-s))\right) s_{13},\\
\hat{A}s_{14} &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1(1-s) + k_2(1+s))\right) s_{14},\\
\hat{A}s_{15} &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(-k_1(1+s) + k_2(1+s))\right) s_{15},\\
\hat{A}s_{16} &= \varepsilon_A \exp\left(\tfrac{-i\pi}{m}(k_1 + k_2)(1+s)\right) s_{16}.
\end{aligned}$$

$$
\begin{aligned}
\hat{B}s_1 &= \varepsilon_B \exp\left(\tfrac{i\pi}{n}(l_1+l_2)(1+t)\right)s_1, & \hat{S}s_1 &= \varepsilon_S s_1,\\
\hat{B}s_2 &= \varepsilon_B \exp\left(\tfrac{i\pi}{n}(-l_1(1-t)+l_2(1+t))\right)s_2, & \hat{S}s_2 &= -\varepsilon_S s_3,\\
\hat{B}s_3 &= \varepsilon_B \exp\left(\tfrac{i\pi}{n}(l_1(1-t)+l_2(1+t))\right)s_3, & \hat{S}s_3 &= \varepsilon_S s_2,\\
\hat{B}s_4 &= \varepsilon_B \exp\left(\tfrac{i\pi}{n}(l_1(1+t)-l_2(1-t))\right)s_4, & \hat{S}s_4 &= -\varepsilon_S s_5,\\
\hat{B}s_5 &= \varepsilon_B \exp\left(\tfrac{i\pi}{n}(l_1(1+t)+l_2(1-t))\right)s_5, & \hat{S}s_5 &= \varepsilon_S s_4,\\
\hat{B}s_6 &= \varepsilon_B \exp\left(\tfrac{-i\pi}{n}(l_1-l_2)(1+t)\right)s_6, & \hat{S}s_6 &= \varepsilon_S s_6,\\
\hat{B}s_7 &= \varepsilon_B \exp\left(\tfrac{-i\pi}{n}(l_1+l_2)(1-t)\right)s_7, & \hat{S}s_7 &= \varepsilon_S s_{10},\\
\hat{B}s_8 &= \varepsilon_B \exp\left(\tfrac{-i\pi}{n}(l_1-l_2)(1-t)\right)s_8, & \hat{S}s_8 &= -\varepsilon_S s_9,\\
\hat{B}s_9 &= \varepsilon_B \exp\left(\tfrac{i\pi}{n}(l_1-l_2)(1-t)\right)s_9, & \hat{S}s_9 &= -\varepsilon_S s_8,\\
\hat{B}s_{10} &= \varepsilon_B \exp\left(\tfrac{i\pi}{n}(l_1+l_2)(1-t)\right)s_{10}, & \hat{S}s_{10} &= \varepsilon_S s_7,\\
\hat{B}s_{11} &= \varepsilon_B \exp\left(\tfrac{i\pi}{n}(l_1-l_2)(1+t)\right)s_{11}, & \hat{S}s_{11} &= \varepsilon_S s_{11},\\
\hat{B}s_{12} &= \varepsilon_B \exp\left(\tfrac{-i\pi}{n}(l_1(1+t)+l_2(1-t))\right)s_{12}, & \hat{S}s_{12} &= -\varepsilon_S s_{13},\\
\hat{B}s_{13} &= \varepsilon_B \exp\left(\tfrac{i\pi}{n}(-l_1(1+t)+l_2(1-t))\right)s_{13}, & \hat{S}s_{13} &= \varepsilon_S s_{12},\\
\hat{B}s_{14} &= \varepsilon_B \exp\left(\tfrac{-i\pi}{n}(l_1(1-t)+l_2(1-t))\right)s_{14}, & \hat{S}s_{14} &= -\varepsilon_S s_{15},\\
\hat{B}s_{15} &= \varepsilon_B \exp\left(\tfrac{i\pi}{n}(l_1(1-t)-l_2(1+t))\right)s_{15}, & \hat{S}s_{15} &= \varepsilon_S s_{14},\\
\hat{B}s_{16} &= \varepsilon_B \exp\left(\tfrac{-i\pi}{n}(l_1+l_2)(1+t)\right)s_{16}. & \hat{S}s_{16} &= \varepsilon_S s_{16}.
\end{aligned}
$$

The signs $\varepsilon_A = (-1)^{(k_1+k_2)(1+s)}$, $\varepsilon_B = (-1)^{(l_1+l_2)(1+t)}$ and ε_S is not fixed, giving two inequivalent spin structures. $\hat{A}$ and $\hat{B}$ are both diagonal in this basis, thus all basis spinors can be made $\hat{A}$ and $\hat{B}$ invariant by setting the arguments in the exponentials to be congruent to m and n respectively. However this can only be done consistently for the spinors which span $\mathcal{S}_+$. $\hat{S}$ preserves the subspaces $\{s_1\}$, $\{s_6\}$, $\{s_7, s_{10}\}$, $\{s_8, s_9\}$, $\{s_{11}\}$ and $\{s_{16}\}$. The table below gives the spinors and classes of groups under which they are preserved,

Spin Structure $(\varepsilon_A, \varepsilon_B, \varepsilon_S)$	*Invariant Spinors*	*Invariance Conditions*
$(\varepsilon_A, \varepsilon_B, 1)$	s_1	$(k_1+k_2)(1+s) \equiv 0(m)$, $(l_1+l_2)(1+t) \equiv 0(n)$
	s_6	$(k_1-k_2)(1+s) \equiv 0(m)$, $(l_1-l_2)(1+t) \equiv 0(n)$
	s_7+s_{10}	$(k_1+k_2)(1-s) \equiv 0(m)$, $(l_1+l_2)(1-t) \equiv 0(n)$
	s_{11}	$(k_1-k_2)(1+s) \equiv 0(m)$, $(l_1-l_2)(1+t) \equiv 0(n)$
	s_{16}	$(k_1+k_2)(1+s) \equiv 0(m)$, $(l_1+l_2)(1+t) \equiv 0(n)$
	s_8-s_9	$(k_1-k_2)(1-s) \equiv 0(m)$, $(l_1-l_2)(t-1) \equiv 0(n)$
$(\varepsilon_A, \varepsilon_B, -1)$	(s_7-s_{10})	$(k_1+k_2)(1-s) \equiv 0(m)$, $(l_1+l_2)(1-t) \equiv 0(n)$
	(s_8+s_9)	$(k_1-k_2)(1-s) \equiv 0(m)$, $(l_1-l_2)(1-t) \equiv 0(n)$

.

The only classes of groups preserving a spinor in both spin structures are those which preserve spinors in the $\{s_7, s_{10}\}$ and $\{s_8, s_9\}$ subspaces. The rest only preserve spinors in the $(\varepsilon_A, \varepsilon_B, 1)$ spin structure. In the $(\varepsilon_A, \varepsilon_B, -1)$ spin structure we can at most preserve one spinor consistently, therefore we need only consider which spinors can be preserved simultaneously in the $(\varepsilon_A, \varepsilon_B, 1)$ spin structure. These results are given in the table below,

Spin Structure: $(\varepsilon_A, \varepsilon_B, 1)$	
Group Invariant Spinors	*Invariance Conditions*
$s_1,\ s_{16}$	$(k_1+k_2)(1+s) \equiv 0(m)$, $(l_1+l_2)(1+t) \equiv 0(n)$
$s_6,\ s_{11}$	$(k_1-k_2)(1+s) \equiv 0(m)$, $(l_1-l_2)(1+t) \equiv 0(n)$
$s_1,\ s_{16},\ s_6,\ s_{11}$	$(k_1+k_2) \equiv 0(m)$, $(l_1+l_2) \equiv 0(n)$
$s_1,\ s_{16},\ s_8-s_9$	$k_1+sk_2 \equiv 0(m)$, $k_2+sk_1 \equiv 0(m)$,
	$l_1+tl_2 \equiv 0(n)$, $l_2+tl_1 \equiv 0(n)$
$s_1,\ s_{16},\ s_6,\ s_{11},\ s_7+s_{10}$	$(k_1+k_2) \equiv 0(m)$, $(l_1+l_2) \equiv 0(n)$
	$(1+s) \equiv 0(m)$, $(1+t) \equiv 0(n)$
$s_1,\ s_{16},\ s_6,\ s_{11},\ s_8-s_9$	$(k_1-k_2) \equiv 0(m)$, $(l_1-l_2) \equiv 0(n)$
	$(1+s) \equiv 0(m)$, $(1+t) \equiv 0(n)$

.

The first four results specify classes of groups preserving supersymmetry only in the $(\varepsilon_A, \varepsilon_B, 1)$ spin structure. These classes preserve either $\frac{1}{4}$, $\frac{3}{8}$ or $\frac{1}{2}$ of the supersymmetry. The last two classes in the table preserve $\frac{5}{8}$ of the supersymmetry in the $(\varepsilon_A, \varepsilon_B, 1)$ spin structure. In the $(\varepsilon_A, \varepsilon_B, -1)$ spin structure $\frac{1}{8}$ of the supersymmetry is preserved.

4.7 Type II: d=2

This section considers Type II groups with $d = 2$. The consistency conditions are $(n(r-1), m) = 1$, $(l, n) = 1$, $(k, m) = 1$, $t^2 \equiv 1(n)$, $s^2 \equiv 1(m)$ $r^2 \equiv 1(m)$ hence $(r + 1) \equiv 0(m)$. The action of $\hat{A}$, $\hat{B}$ and $\hat{S}$ on the basis is given below;

$$
\begin{aligned}
&\hat{A}s_1 = \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1+r)(1+s)\right) s_1, && \hat{B}s_1 = -\varepsilon_B \exp(i\theta)s_1, && \hat{S}s_1 = \varepsilon_S s_1,\\
&\hat{A}s_2 = \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(rs+s+r-1)\right) s_2, && \hat{B}s_2 = -\varepsilon_B \exp(i\phi)s_3, && \hat{S}s_2 = -\varepsilon_S s_4,\\
&\hat{A}s_3 = \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1-r+s+rs)\right) s_3, && \hat{B}s_3 = -\varepsilon_B \exp(i\theta)s_2, && \hat{S}s_3 = -\varepsilon_S s_5,\\
&\hat{A}s_4 = \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1+r-s+rs)\right) s_4, && \hat{B}s_4 = -\varepsilon_B s_5, && \hat{S}s_4 = \varepsilon_S s_2,\\
&\hat{A}s_5 = \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1+r+s-rs)\right) s_5, && \hat{B}s_5 = -\varepsilon_B \exp(i\delta)s_4, && \hat{S}s_5 = \varepsilon_S s_3,\\
&\hat{A}s_6 = \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1+r)(1-s)\right) s_6, && \hat{B}s_6 = \varepsilon_B \exp(i\phi)s_6, && \hat{S}s_6 = \varepsilon_S s_{11},\\
&\hat{A}s_7 = \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1-r)(1+s)\right) s_7, && \hat{B}s_7 = -\varepsilon_B \exp(-i\delta)s_{10}, && \hat{S}s_7 = \varepsilon_S s_7,\\
&\hat{A}s_8 = \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1-r)(1-s)\right) s_8, && \hat{B}s_8 = -\varepsilon_B s_9, && \hat{S}s_8 = \varepsilon_S s_9,\\
&\hat{A}s_9 = \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1-r)(1-s)\right) s_9, && \hat{B}s_9 = -\varepsilon_B s_8, && \hat{S}s_9 = \varepsilon_S s_8,\\
&\hat{A}s_{10} = \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1-r)(1+s)\right) s_{10}, && \hat{B}s_{10} = -\varepsilon_B \exp(i\delta)s_7, && \hat{S}s_{10} = \varepsilon_S s_{10},\\
&\hat{A}s_{11} = \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1+r)(1-s)\right) s_{11}, && \hat{B}s_{11} = \varepsilon_B \exp(i\phi)s_{11}, && \hat{S}s_{11} = \varepsilon_S s_6,\\
&\hat{A}s_{12} = \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1+r+s-rs)\right) s_{12}, && \hat{B}s_{12} = \varepsilon_B \exp(-i\delta)s_{13}, && \hat{S}s_{12} = \varepsilon_S s_{14},\\
&\hat{A}s_{13} = \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1+r-s+rs)\right) s_{13}, && \hat{B}s_{13} = \varepsilon_B s_{12}, && \hat{S}s_{13} = \varepsilon_S s_{15},\\
&\hat{A}s_{14} = \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1-r+s-+rs)\right) s_{14}, && \hat{B}s_{14} = \varepsilon_B \exp(-i\theta)s_{15}, && \hat{S}s_{14} = -\varepsilon_S s_{12},\\
&\hat{A}s_{15} = \varepsilon_A \exp\left(\tfrac{i\pi k}{m}(1-r-s-rs)\right) s_{15}, && \hat{B}s_{15} = \varepsilon_B \exp(-i\phi)s_{14}, && \hat{S}s_{15} = -\varepsilon_S s_{13},\\
&\hat{A}s_{16} = \varepsilon_A \exp\left(\tfrac{-i\pi k}{m}(1+r)(1+s)\right) s_{16}. && \hat{B}s_{16} = -\varepsilon_B \exp(i\theta)s_{16}. && \hat{S}s_{16} = \varepsilon_S s_{16}.
\end{aligned}
$$

The arguments in the exponentials are; $\theta = \frac{2(t+1)\pi l}{n}$, $\phi = \frac{2(t-1)\pi l}{n}$ and $\delta = \frac{4\pi l}{n}$. The sign $\varepsilon_A = (-1)^{k(1+r)(1+s)}$ is fixed but ε_B and ε_S are free, hence we have four inequivalent spin structures. $\hat{A}$ is diagonal hence all the spinors are its eigenspinors, but only the ones spanning $\mathcal{S}_+$ can be made $\hat{A}$ invariant consistently. $\hat{B}$ preserves the $\{s_1\}$, $\{s_6\}$, $\{s_7, s_{10}\}$, $\{s_8, s_9\}$, $\{s_{11}\}$ and $\{s_{16}\}$ subspaces of $\mathcal{S}_+$. We find that $\hat{S}$ preserves the $\{s_1\}$, $\{s_6, s_{11}\}$, $\{s_7\}$, $\{s_{10}\}$, $\{s_8, s_9\}$ and $\{s_{16}\}$ subspaces. The table below gives the common invariant spinors of $\hat{A}$, $\hat{B}$ and $\hat{S}$;

Spin Structure $(\varepsilon_A, \varepsilon_B, \varepsilon_S)$	*Invariant Spinors*	*Invariance Conditions*
$(\varepsilon_A, 1, 1)$	s_1	$(t+1)l \equiv \frac{n}{2}(n)$
	s_{16}	$(t+1)l \equiv \frac{n}{2}(n)$
	$s_8 - s_9$	$(1-s) \equiv 0(m)$
	$s_7 + \lambda s_{10}$	$(1+s) \equiv 0(m)$, $(l_i, n) = 1$
$(\varepsilon_A, -1, 1)$	s_1	$(t+1) \equiv 0(n)$
	s_{16}	$(t+1) \equiv 0(n)$
	$s_8 + s_9$	$(1-s) \equiv 0(m)$
	$s_7 - \lambda s_{10}$	$(1+s) \equiv 0(m)$, $(l_i, n) = 1$
$(\varepsilon_A, 1, -1)$	$s_8 - s_9$	$(1-s) \equiv 0(m)$

.

The table shows that none of the groups of Type II with $d = 2$ preserve spinors in the $(\varepsilon_A, -1, -1)$ spin structure. There is a class which preserves spinors on the $\{s_8, s_9\}$ subspace, in all three spin structures above and a class preserving spinors on the $\{s_7, s_{10}\}$ in two of the spin structures. The rest only preserve spinors in one of the spin structures. The table below shows the spinors which can be preserved simultaneously,

Spin Structure: $(\varepsilon_A, 1, 1)$	
Group Invariant Spinors	*Invariance Conditions*
s_1, s_{16}	$(1+s)(1+r) \equiv 0(m)$, $(t+1)l \equiv \frac{n}{2}(n)$
$s_1, s_{16},\ s_7 + \lambda s_{10}$	$(1+s) \equiv 0(m)$, $(t+1)l \equiv \frac{n}{2}(n)$
Spin Structure: $(\varepsilon_A, -1, 1)$	
s_1, s_{16}	$(1+s)(1+r) \equiv 0(m)$, $(t+1)l \equiv 0(n)$
$s_1, s_{16},\ s_7 - \lambda s_{10}$	$(1+s) \equiv 0(m)$, $(t+1)l \equiv 0(n)$

.

None of the invariance conditions above are present in both spin structures therefore the corresponding groups only presere spinors in one of the spin structures.

In summary the $(\varepsilon_A, -1, -1)$ spin structure does not contain any group invariant spinors. There is always a class of groups which preserves $\frac{1}{8}$ of the supersymmetry in the remaining spin structures, as shown in the first table above. Out of these, the class specified by $(1-s) \equiv 0(m)$ preserves $\frac{1}{8}$ supersymmetry in all three spin structures. In each of the $(\varepsilon_A, 1, 1)$ and $(\varepsilon_A, -1, 1)$ spin structures there are classes of groups which preserve $\frac{1}{4}$ and $\frac{3}{8}$ of the supersymmetry.

4.8 Type IV: d=1

This section deals with Type IV groups with $d = 1$. These groups have five generators $\{\hat{A}, \hat{B}, \hat{I}, \hat{J}, \hat{S}\}$. There are three cases to consider, the first is where the representation is induced from a normal subgroup generated by $\{\hat{A}, \hat{B}, \hat{I}, \hat{J}\}$. We decompose this group as a Type I group and T^* the binary tetrahedral group $\{\hat{A}, \hat{B}^3\} \times \{\hat{B}^{n'}, \hat{I}, \hat{S}\}$ furthermore we set $n \not\equiv 0(9)$. The second case is where the group decomposes as a Type I group and O_v^*, the extended binary octahedral group, $\{\hat{A}, \hat{B}^{3^v}\} \times \{\hat{B}^{n'}, \hat{I}, \hat{J}, \hat{S}\}$ in this case $n \equiv 0(9)$. The third case is where the representation of the group is induced from that of the normal subgroup $\{\hat{A}, \hat{B}, \hat{I}, \hat{J}\}$, who we decompose as $\{\hat{A}, \hat{B}^{3^v}\} \times \{\hat{B}^{n'}, \hat{I}, \hat{J}\}$ where $\{\hat{B}^{n'}, \hat{I}, \hat{J}\}$ is the extended binary tetrahedral group and again we have $n \equiv 0(9)$. In each of the cases three cases there are two inequivalent spin structures labelled by ε_S. All other signs are fixed as shown in Section 3.3.6.

We begin by looking at the action of $\hat{I}$, $\hat{J}$, $\hat{S}$ and $\hat{B}^{n'}$ on the spinors, since they have the same form in each of the three cases (see Section 3.3.6).

$$
\begin{array}{lll}
\hat{J}s_1 = s_1, & \hat{S}s_1 = \varepsilon_S s_1, & \hat{I}s_1 = s_1, \\
\hat{J}s_2 = -s_3, & \hat{S}s_2 = -\varepsilon_S s_4, & \hat{I}s_2 = is_2, \\
\hat{J}s_3 = s_2, & \hat{S}s_3 = -\varepsilon_S s_5, & \hat{I}s_3 = -is_3, \\
\hat{J}s_4 = s_5, & \hat{S}s_4 = \varepsilon_S s_2, & \hat{I}s_4 = -is_5, \\
\hat{J}s_5 = -s_5, & \hat{S}s_5 = \varepsilon_S s_2, & \hat{I}s_5 = -is_4, \\
\hat{J}s_6 = s_6, & \hat{S}s_6 = \varepsilon_S s_{11}, & \hat{I}s_6 = s_6, \\
\hat{J}s_7 = -s_{10}, & \hat{S}s_7 = \varepsilon_S s_7, & \hat{I}s_7 = s_8, \\
\hat{J}s_8 = s_9, & \hat{S}s_8 = \varepsilon_S s_9, & \hat{I}s_8 = s_7, \\
\hat{J}s_9 = s_8, & \hat{S}s_9 = \varepsilon_S s_8, & \hat{I}s_9 = -s_{10}, \\
\hat{J}s_{10} = -s_7, & \hat{S}s_{10} = \varepsilon_S s_{10}, & \hat{I}s_{10} = -s_9, \\
\hat{J}s_{11} = s_{11}, & \hat{S}s_{11} = \varepsilon_S s_6, & \hat{I}s_{11} = v_{11}, \\
\hat{J}s_{12} = s_{13}, & \hat{S}s_{12} = \varepsilon_S s_{14}, & \hat{I}s_{12} = -is_{13}, \\
\hat{J}s_{13} = -s_{12}, & \hat{S}s_{13} = \varepsilon_S s_{15}, & \hat{I}s_{13} = -is_{12}, \\
\hat{J}s_{14} = -s_{15}, & \hat{S}s_{14} = -\varepsilon_S s_{12}, & \hat{I}s_{14} = is_{14}, \\
\hat{J}s_{15} = s_{14}, & \hat{S}s_{15} = -\varepsilon_S s_{13}, & \hat{I}s_{15} = -is_{15}, \\
\hat{J}s_{16} = s_{16}. & \hat{S}s_{16} = \varepsilon_S s_{16}. & \hat{I}s_{16} = s_{16}.
\end{array}
$$

The invariant spinors of the above generators are as follows;
$\hat{J}$ preserves s_1, s_6, s_{11}, s_{16}, $s_7 - s_{10}$, $s_8 + s_9$ where $\varepsilon_J = 1$.
$\hat{S}$ preserves s_1, s_7, s_{10}, s_{16}, $s_6 + s_{11}$, $s_8 + s_9$ when $\varepsilon_S = 1$ and $s_6 - s_{11}$, $s_8 - s_9$ when $\varepsilon_S = -1$.
$\hat{I}$ preserves s_1, s_6, s_{11}, s_{16}, $s_7 + s_8$, $s_9 - s_{10}$ where $\varepsilon_I = 1$.
The invariant spinors of the lift of $\hat{B}^{n'}$ are s_1, s_6, s_{11}, s_{16} and $\alpha s_7 + \frac{(\alpha+i\beta)(1+i)}{2}(s_8 + s_9) + \beta s_{10}$ such that α, $\beta \in \mathbb{C}$.
We see that the invariant spinors of $\hat{I}$, $\hat{S}$, $\hat{J}$ and $\hat{B}^{n'}$ lie on the $\{s_1, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{16}\}$ subspace, so again the $\mathcal{S}_+$ module is chosen. The simultaneously invariant spinors of generators $\hat{I}$, $\hat{J}$, $\hat{S}$ and $\hat{B}^{n'}$ are s_1, $s_6 + s_{11}$, $s_7 + s_8 + s_9 - s_{10}$ and s_{16} when $\varepsilon_S = 1$ and $s_6 - s_{11}$ when $\varepsilon_S = -1$. We will now look at the action of the remaining generators on these invariant spinors case-by-case . For convenience we present the cases in the order (i), (iii) and (ii).

4.8.1 Case (i): $n \not\equiv 0(9)$

We shall now look at $\hat{A}$ and $\hat{B}^3$ acting on $\mathcal{S}_+$. The signs below are $\varepsilon_A = (-1)^{k(s+1)}$ and $\varepsilon_B = (-1)^l$. $\hat{A}$ acts on the basis as;

$$\hat{A}s_1 = \varepsilon_A \exp\left(\tfrac{ik\pi}{m}2(1+s)\right) s_1,$$
$$\hat{A}s_6 = \varepsilon_A \exp\left(\tfrac{ik\pi}{m}2(1-s)\right) s_6,$$
$$\hat{A}s_{11} = \varepsilon_A \exp\left(\tfrac{ik\pi}{m}2(1-s)\right) s_{11},$$
$$\hat{A}s_{16} = \varepsilon_A \exp\left(\tfrac{-ik\pi}{m}2(1+s)\right) s_{16},$$
$$\hat{A}s_7 = \varepsilon_A s_7,$$
$$\hat{A}s_8 = \varepsilon_A s_8,$$
$$\hat{A}s_9 = \varepsilon_A s_9,$$
$$\hat{A}s_{10} = \varepsilon_A s_{10}.$$

$\hat{B}^3$ acts on the basis as;

$$\hat{B}^3 s_1 = \varepsilon_B \exp\left(\tfrac{i3l\pi}{n}2(1+t)\right) s_1,$$
$$\hat{B}^3 s_6 = \varepsilon_B \exp\left(\tfrac{i3l\pi}{n}2(t-1)\right) s_6,$$
$$\hat{B}^3 s_{11} = \varepsilon_B \exp\left(\tfrac{i3l\pi}{n}2(t-1)\right) s_{11},$$
$$\hat{B}^3 s_{16} = \varepsilon_B \exp\left(\tfrac{i3l\pi}{n}2(1+t)\right) s_{16},$$
$$\hat{B}^3 s_7 = \varepsilon_B s_7,$$
$$\hat{B}^3 s_8 = \varepsilon_B s_8,$$
$$\hat{B}^3 s_9 = \varepsilon_B s_9,$$
$$\hat{B}^3 s_{10} = \varepsilon_B s_{10}.$$

Let us consider the spin structure $\varepsilon_S = 1$. We would like to know the conditions under which $\hat{A}$ and $\hat{B}^3$ preserves s_1, $s_7 + s_8 + s_9 - s_{10}$, $s_6 + s_{11}$ and s_{16}. If we set $(1+s) \equiv 0(m)$, $3(t+1) \equiv 0(n)$ and choose l, k appropriately to set $\varepsilon_A = (-1)^{k(s+1)}$ and $\varepsilon_B = (-1)^{l(t+1)}$ equal to one then s_1, $s_7 + s_8 + s_9 - s_{10}$ and s_{16} are preserved simultaneously. If instead we set $(1-s) \equiv 0(m)$, $3(t-1) \equiv 0(n)$ and choose l, k appropriately to set ε_A and ε_B equal to one then $s_6 + s_{11}$ is preserved in place of s_1 and s_{16}. In the $\varepsilon_S = -1$ case, groups specified by $(1-s) \equiv 0(m)$ and $3(t-1) \equiv 0(n)$, with ε_A and ε_B set to one preserve $s_6 - s_{11}$.

In summary there is one class of groups which preserves $\frac{3}{8}$ supersymmetry in the $\varepsilon_S = 1$ spin structure and another class of groups which preserves $\frac{1}{4}$. This latter class also preserves $\frac{1}{8}$ of the supersymmetry in the $\varepsilon_S = -1$ spin structure.

None of the other spin structures admit invariant spinors.

4.8.2 Case (iii): $n \equiv 0(9)$

We now consider the action of the Type I group $\{A, B^{3^v}\}$ and phase $\phi \oplus \phi^{-1}$ on $\mathcal{S}_+$. The signs below are $\varepsilon_A = (-1)^{k(s+1)}$ and $\varepsilon_B = (-1)^{l(1+t)}$. The action of $(\phi \oplus \phi^{-1})\hat{B}^{3^v}$ on the basis of $\mathcal{S}_+$ is given below;

$$(\phi \oplus \phi^{-1})\hat{B}^{3^v} s_1 = \varepsilon_B \exp\left(\tfrac{4i\pi l 3^v}{n}(1+t)\right) s_1,$$
$$(\phi \oplus \phi^{-1})\hat{B}^{3^v} s_6 = \varepsilon_B \exp\left(-\tfrac{8\pi i p}{3^v}\right) \exp\left(\tfrac{-4i\pi l 3^v}{n}(1-t)\right) s_6,$$
$$(\phi \oplus \phi^{-1})\hat{B}^{3^v} s_7 = \varepsilon_B s_7,$$
$$(\phi \oplus \phi^{-1})\hat{B}^{3^v} s_8 = \varepsilon_B s_8,$$
$$(\phi \oplus \phi^{-1})\hat{B}^{3^v} s_9 = \varepsilon_B s_9,$$
$$(\phi \oplus \phi^{-1})\hat{B}^{3^v} s_{10} = \varepsilon_B s_{10},$$
$$(\phi \oplus \phi^{-1})\hat{B}^{3^v} s_{11} = \varepsilon_B \exp\left(\tfrac{8\pi i p}{3^v}\right) \exp\left(\tfrac{4i\pi l 3^v}{n}(1-t)\right) s_{11},$$
$$(\phi \oplus \phi^{-1})\hat{B}^{3^v} s_{16} = \varepsilon_B \exp\left(\tfrac{-4i\pi l 3^v}{n}(1+t)\right) s_{16}.$$

Let us consider the $\varepsilon_S = 1$ spin structure. The generator $(\phi \oplus \phi^{-t})\hat{B}^{3^v}$ will preserve s_1 and s_{16} if $3^v(1+t) \equiv 0(n)$ and provided l and t are chosen appropriately to set $\varepsilon_B = (-1)^{l(t+1)}$ equal to one. $\hat{A}$ is the same as in case(i) therefore will preserve s_1 and s_{16} if $(s+1) \equiv 0(m)$ and $\varepsilon_A = (-1)^{k(s+1)}$ is set to one. Furthermore these conditions automatically preserve $s_7 + s_8 + s_9 - s_{10}$. Therefore the corresponding classes of groups, specified by the relations above, preserves $\frac{3}{8}$ of the supersymmetry. There is no consistent way to preserve any other spinors.

4.8.3 Case (ii): $n \equiv 0(9)$

The action of $\hat{A}$ on the basis is as follows, where $\varepsilon_A = (-1)^k$;

$$\hat{A}s_1 = \varepsilon_A \exp(\tfrac{4i\pi k}{m}) s_1,$$
$$\hat{A}s_6 = \varepsilon_A s_6,$$
$$\hat{A}s_7 = \varepsilon_A s_7,$$
$$\hat{A}s_8 = \varepsilon_A s_8,$$
$$\hat{A}s_9 = \varepsilon_A s_9,$$
$$\hat{A}s_{10} = \varepsilon_A s_{10},$$
$$\hat{A}s_{11} = \varepsilon_A s_{11},$$

$\hat{A}s_{16} = \varepsilon_A \exp(\frac{4i\pi k}{m})s_{16}$.

The eigenvalues of s_1 and s_{16} cannot be set to one since m is odd and $(k,m)=1$. Since there is no consistant way for $(\phi \oplus \phi^{-t})\hat{B}^{3^v}$ to preserve s_6 and s_{11} these spinors are not preserved in this case either. The only spinor which can be preserved is $s_7+s_8+s_9-s_{10}$ this requires $\varepsilon_A=\varepsilon_B=\varepsilon_S=1$. Thus we must set $k \equiv 0(2)$ so that $\varepsilon_A=(-1)^k$ equals one and chooses l, t so that $\varepsilon_B=(-1)^{l(t+1)}$ equals one. Thus the corresponding classes of groups preserve $\frac{1}{8}$ of the supersymmetry in the $\varepsilon_A=\varepsilon_B=\varepsilon_S=1$ spin structure.

4.9 Type VI: d=1

This section deals with Type IV groups with $d=1$. The action of the group generators, $\{\hat{A},\hat{B},\hat{X},\hat{Y},\hat{S}\}$, on the basis is given below. We note that m and n are odd as well as coprime and that $\varepsilon_X=\varepsilon_Y=1$. The remaining signs are; $\varepsilon_A=(-1)^{k(s+1)}$, $\varepsilon_B=(-1)^{l(1+t)}$ and ε_S is unfixed, thus labelling two inequivalent spin structures. The action of all the generators on the basis is given below;

$$\begin{array}{lll}
\hat{A}s_1 = \varepsilon_A \exp\left(\frac{ik\pi}{m}2(1+s)\right)s_1, & \hat{B}s_1 = \varepsilon_B \exp\left(\frac{il\pi}{n}2(1+t)\right)s_1, & \hat{S}s_1 = \varepsilon_S s_1, \\
\hat{A}s_2 = \varepsilon_A \exp\left(\frac{2ik\pi s}{m}\right)s_2, & \hat{B}s_2 = \varepsilon_B \exp\left(\frac{2il\pi t}{n}\right)s_2, & \hat{S}s_2 = -\varepsilon_S s_4\ , \\
\hat{A}s_3 = \varepsilon_A \exp\left(\frac{2ik\pi s}{m}\right)s_3, & \hat{B}s_3 = \varepsilon_B \exp\left(\frac{2il\pi t}{n}\right)s_3, & \hat{S}s_3 = -\varepsilon_S s_5, \\
\hat{A}s_4 = \varepsilon_A \exp\left(\frac{2ik\pi s}{m}\right)s_4, & \hat{B}s_4 = \varepsilon_B \exp\left(\frac{2il\pi t}{n}\right)s_4, & \hat{S}s_4 = \varepsilon_S s_2, \\
\hat{A}s_5 = \varepsilon_A \exp\left(\frac{2ik\pi s}{m}\right)s_5, & \hat{B}s_5 = \varepsilon_B \exp\left(\frac{2il\pi t}{n}\right)s_5, & \hat{S}s_5 = \varepsilon_S s_3, \\
\hat{A}s_6 = \varepsilon_A \exp\left(\frac{ik\pi}{m}2(s-1)\right)s_6, & \hat{B}s_6 = \varepsilon_B \exp\left(\frac{il\pi}{n}2(t-1)\right)s_6, & \hat{S}s_6 = \varepsilon_S s_{11}, \\
\hat{A}s_7 = \varepsilon_A s_7, & \hat{B}s_7 = \varepsilon_B s_7, & \hat{S}s_7 = \varepsilon_S s_7, \\
\hat{A}s_8 = \varepsilon_A s_8, & \hat{B}s_8 = \varepsilon_B s_8, & \hat{S}s_8 = \varepsilon_S s_9\ , \\
\hat{A}s_9 = \varepsilon_A s_9, & \hat{B}s_9 = \varepsilon_B s_9, & \hat{S}s_9 = \varepsilon_S s_8, \\
\hat{A}s_{10} = \varepsilon_A s_{10}, & \hat{B}s_{10} = \varepsilon_B s_{10}, & \hat{S}s_{10} = \varepsilon_S s_{10}, \\
\hat{A}s_{11} = \varepsilon_A \exp\left(\frac{-ik\pi}{m}2(s-1)\right)s_{11}, & \hat{B}s_{11} = \varepsilon_B \exp\left(\frac{-il\pi}{n}2(t-1)\right)s_{11}, & \hat{S}s_{11} = \varepsilon_S s_6, \\
\hat{A}s_{12} = \varepsilon_A \exp\left(\frac{-2ik\pi s}{m}\right)s_{12}, & \hat{B}s_{12} = \varepsilon_B \exp\left(\frac{-2il\pi t}{n}\right)s_{12}, & \hat{S}s_{12} = \varepsilon_S s_{14}\ , \\
\hat{A}s_{13} = \varepsilon_A \exp\left(\frac{-2ik\pi s}{m}\right)s_{13}, & \hat{B}s_{13} = \varepsilon_B \exp\left(\frac{-2il\pi t}{n}\right)s_{13}, & \hat{S}s_{13} = \varepsilon_S s_{15}\ , \\
\hat{A}s_{14} = \varepsilon_A \exp\left(\frac{-2ik\pi s}{m}\right)s_{14}, & \hat{B}s_{14} = \varepsilon_B \exp\left(\frac{-2il\pi t}{n}\right)s_{14}, & \hat{S}s_{14} = -\varepsilon_S s_{12}, \\
\hat{A}s_{15} = \varepsilon_A \exp\left(\frac{-2ik\pi s}{m}\right)s_{15}, & \hat{B}s_{15} = \varepsilon_B \exp\left(\frac{-2il\pi t}{n}\right)s_{15}, & \hat{S}s_{15} = -\varepsilon_S s_{13}, \\
\hat{A}s_{16} = \varepsilon_A \exp\left(\frac{-ik\pi}{m}2(1+s)\right)s_{16}. & \hat{B}s_{16} = \varepsilon_B \exp\left(\frac{-il\pi}{n}2(1+t)\right)s_{16}. & \hat{S}s_{16} = \varepsilon_S s_{16}.
\end{array}$$

$\hat{X}s_1 = s_1,$

$\hat{X}s_2 = \frac{i+t^2}{2t}s_2 + \frac{i}{2}s_3,$

$\hat{X}s_3 = \frac{i}{2}s_2 + \frac{-i+t^2}{2t}s_3,$

$\hat{X}s_4 = \frac{-(1+it^2)}{2t}s_4 + \frac{1}{2}s_5,$

$\hat{X}s_5 = \frac{-1}{2}s_4 + \frac{(it^2-1)}{2t}s_5,$

$\hat{X}s_6 = s_6,$

$\hat{X}s_7 = \frac{-i(1+t^4)}{4t^2}s_7 + \frac{t^2+i}{4t}s_8 + \frac{i}{4}s_{10} + \frac{t^2-i}{4t}s_9,$

$\hat{X}s_8 = \frac{-(t^2+i)}{4t}s_7 + \frac{it^4-i-2t^2}{4t^2}s_8 + \frac{-i}{4}s_9 + \frac{-(t^2+i)}{4t}s_{10},$

$\hat{X}s_9 = \frac{t^2-i}{4t}s_7 + \frac{i}{4}s_8 + \frac{t^2-1}{4t}s_{10} + \frac{i-it^4-2t^2}{4t^2}s_9,$

$\hat{X}s_{10} = \frac{-i}{4}s_7 - \frac{i+t^2}{4t}s_8 + \frac{i(1+t^4)}{4t^2}s_{10} + \frac{i-t^2}{4t}s_9,$

$\hat{X}s_{11} = s_{11},$

$\hat{X}s_{12} = \frac{1+it^2}{2t}s_{12} + \frac{1}{2}s_{13},$

$\hat{X}s_{13} = -\frac{1}{2}s_{12} + \frac{-1+it^2}{2t}s_{13},$

$\hat{X}s_{14} = \frac{i+t^2}{2t}s_{14} + \frac{i}{2}s_{15},$

$\hat{X}s_{15} = \frac{i}{2}s_{14} + \frac{t^2-i}{2t}s_{15},$

$\hat{X}s_{16} = s_{16}.$

$\hat{Y}s_1 = s_1,$

$\hat{Y}s_2 = \frac{1}{2}s_2 + \frac{it^2-1}{2t}s_3,$

$\hat{Y}s_3 = \frac{it^2+1}{2t}s_2 + \frac{1}{2}s_3,$

$\hat{Y}s_4 = \frac{i+1}{2}s_4 + \frac{i-1}{2}s_5,$

$\hat{Y}s_5 = \frac{i+1}{2}s_4 - \frac{i-1}{2}s_5,$

$\hat{Y}s_6 = s_6,$

$\hat{Y}s_7 = \frac{i+1}{4}s_7 + \frac{i-1}{4}s_8 - \frac{(i+t^2)(1-i)}{4t}s_9 - \frac{(i+t^2)(i+1)}{4t}s_{10},$

$\hat{Y}s_8 = \frac{i+1}{4}s_7 + \frac{1-i}{4}s_8 + \frac{(i+1)(it^2-1)}{4t}s_9 + \frac{(i+1)(i+t^2)}{4t}s_{10},$

$\hat{Y}s_9 = \frac{(i-t^2)(1-i)}{4t}s_7 + \frac{(i-t^2)(i+1)}{4t}s_8 + \frac{i+1}{4}s_9 + \frac{i-1}{4}s_{10},$

$\hat{Y}s_{10} = \frac{(i-t^2)(1-i)}{4t}s_7 + \frac{(1+it^2)(1-i)}{4t}s_8 + \frac{i+1}{4}s_9 + \frac{1-i}{4}s_{10},$

$\hat{Y}s_{11} = s_{11},$

$\hat{Y}s_{12} = \frac{i+1}{2}s_{12} + \frac{i-1}{2}s_{13},$

$\hat{Y}s_{13} = \frac{1+i}{2}s_{12} + \frac{1-i}{2}s_{13},$

$\hat{Y}s_{14} = \frac{1}{3}s_{14} + \frac{it^2-1}{2t}s_{15},$

$\hat{Y}s_{15} = \frac{1+it^2}{2t}s_{14} + \frac{1}{2}s_{15},$

$\hat{Y}s_{16} = s_{16}.$

Generators $\hat{A}$ and $\hat{B}$ are diagonal in this basis thus each basis spinor is an eigen-spinor for them. The common $\hat{S}$, $\hat{X}$ and $\hat{Y}$ invariant spinors on $\mathbb{S}_-$ cannot be made $\hat{A}$ and $\hat{B}$ without violating the group conditions $s^2 \equiv 1(m)$ and $t^2 \equiv 1(n)$.

Therefore the $\mathcal{S}_-$ module does not admit invariant spinors.

On the positive chiral space $\mathcal{S}_+$, $\hat{X}$ and $\hat{Y}$ preserve s_1, s_6, s_{11} and s_{16} automatically, furthermore these four spinors are their only common invariant spinors. As a consequence the subspace $\{s_7, s_8, s_9, s_{10}\}$ does not yield group invariant spinors so we will not consider it further.

$\hat{S}$ preserves s_1, s_{16} and $s_6 + s_{11}$ when $\varepsilon_S = 1$ and $s_6 - s_{11}$ when $\varepsilon_S = -1$. $\hat{A}$ preserves s_1 and s_{16} if we set $(1+s) \equiv 0(m)$ and choose values of k and s so that $\varepsilon_A = (-1)^{k(1+s)}$ equals one. $\hat{B}$ will also preserve s_1 and s_{16} if we set $(1+t) \equiv 0(n)$ and choose the values of l and t so that $\varepsilon_B = (-1)^{l(t+1)}$ equals one.

Similarly $\hat{A}$ and $\hat{B}$ will preserve s_6 and s_{11} if we set $(s-1) \equiv 0(m)$, $(t-1) \equiv 0(n)$ and we set ε_A and ε_B equal to one. If either ε_A or ε_B are negative, there is no consistent way to preserve any of the spinors s_1, s_6, s_{11} or s_{16}. Therefore the results are as follows; in the $\varepsilon_S = 1$ spin structure s_1 and s_{16} are preserved under groups specified by $(1+s) \equiv 0(m)$, $(t+1) \equiv 0(n)$ with parameters chosen so that $\varepsilon_A = \varepsilon_B = 1$. Alternatively $s_6 + s_{11}$ is preserved under the groups specified by $(s-1) \equiv 0(m)$, $(t-1) \equiv 0(n)$ with parameters chosen so that $\varepsilon_A = \varepsilon_B = 1$. This latter class also preserves $s_6 - s_{11}$ in the $\varepsilon_S = -1$ spin structure again we must have $\varepsilon_A = \varepsilon_B = 1$.

4.10 Conclusions

We shall now summarise the results obtained above.

- Cyclic Groups

  For $n \equiv 8(4)$ cyclic groups preserve $\frac{1}{2}$ or $\frac{1}{4}$ of the supersymmetry, depending on spin structure.

- Type I: d=1

  There are two inequivalent spin structure in this case, in each of the two there is always a class of groups which preserves $\frac{1}{8}$ of the supersymmetry. In each spin structure we find a classes preserving $\frac{1}{4}$ or $\frac{3}{8}$ of the supersymmetry, which do not preserve any supersymmetry in the other spin structure. There are also classes preserving $\frac{1}{2}$ or $\frac{5}{8}$ supersymmetry in both spinstructures.

- Type I: d=4

  There are two inequivalent spin structure in this case. Any class of group preserving supersymmetry, preserves the same amount in each of the two spin structures. These classes either preserve $\frac{1}{8}$ or $\frac{1}{4}$ of the supersymmetry.

- Type II: d=1

  There are two inequivalent spin structure in this case. There are classes of groups preserving $\frac{1}{4}$, $\frac{3}{8}$ or $\frac{1}{2}$ of the supersymmetry in the $(\varepsilon_A, \varepsilon_B, 1)$ spin structure, but none in the $(\varepsilon_A, \varepsilon_B, -1)$ spin structure. The are classes of groups preserving $\frac{5}{8}$ of the supersymmetry in the $(\varepsilon_A, \varepsilon_B, 1)$ spin structure which only preserve $\frac{1}{8}$ in the $(\varepsilon_A, \varepsilon_B, -1)$ spin structure.

- Type II: d=2

  There are four inequivalent spin structures in this case labelled by ε_B and ε_S. The $(\varepsilon_A, -1, -1)$ spin structure does not contain any group invariant spinors. There is always a class of groups which preserves $\frac{1}{8}$ of the supersymmetry in the remaining spin structures, out of these one of them preserves $\frac{1}{8}$ in all three spin structures. No other class preserves supersymmetry in multiple spin structures. In the $(\varepsilon_A, 1, 1)$ and $(\varepsilon_A, -1, 1)$ spin structures, there is a class of groups which preserve $\frac{1}{4}$ and $\frac{3}{8}$ supersymmetry.

- Type IV: d=1

  Case(i): There is one class of groups which preserves $\frac{3}{8}$ supersymmetry in the $\varepsilon_A = \varepsilon_B = \varepsilon_S = 1$ spin structure and another class of groups which preserves $\frac{1}{4}$. This latter class also preserves $\frac{1}{8}$ of the supersymmetry in the $\varepsilon_A = \varepsilon_B = -\varepsilon_S = 1$ spin structure. No other spin structure admits invariant spinors.

  Case(ii): Only the $\varepsilon_A = \varepsilon_B = \varepsilon_S = 1$ spin structure admits invariant spinors, the corresponding class of groups preserves $\frac{1}{8}$ of the supersymmetry.

  Case(iii): Only the $\varepsilon_A = \varepsilon_B = \varepsilon_S = 1$ spin structurer admits invariant spinors, the corresponding class of groups preserves $\frac{3}{8}$ of the supersymmetry.

- Type VI: d=1

  There is one class of groups which preserves $\frac{1}{4}$ and another which preserves

$\frac{1}{8}$ of the supersymmetry, in the $\varepsilon_A = \varepsilon_B = \varepsilon_S = 1$ spin structure. This latter class also preserves $\frac{1}{8}$ of the supersymmetry in the $\varepsilon_A = \varepsilon_B = -\varepsilon_S = 1$ spin structure.

Chapter 5

Supergravity Preons

In Chapters 5 and 6 we shall consider quotients of all the maximally supersymmetric supergravity backgrounds in ten and eleven dimensions. In this case our aim is to see if some discrete subgroup G of the four-form-preserving isometries of the background preserves a fraction $\frac{31}{32}$ of the supersymmetry. This work has been reported in [12]. Such a background with 31 supercharges is called a preon. We shall boil this problem down to checking if some element $\gamma \in G$, in the image of the exponential map, which preserves at least 30 supercharges will preserve 31. The conjugacy classes of such elements has been classified in the Minkowski case in [17] and in the Freund-Rubin case in [18]. We shall do the analysis of such elements in the case of the Kowalski-Glickman space, or maximally supersymmetric wave, in Chapter 6. The motivation to consider such quotients comes from [24] where it was shown that if such backgrounds exist then they are necessarily quotients of maximally supersymmetric backgrounds. We shall show that ultimately no such quotients preserve $\frac{31}{32}$ supercharges thus ruling out the existence of preons in any supergravity theory with 32 supercharges. The bulk of our work is on the eleven-dimensional case but we will also derive results for the ten-dimensional case.

To this end we shall organise the discussion as follows. In Section 5.1 we shall outline the method we use to investigate preonic solutions. In Section 5.2 we shall consider the exponential properties of the isometry groups of the backgrounds. In Chapter 6 we shall take discrete quotients of all eleven-dimensional maximally supersymmetric backgrounds and prove preons do not exist as quotients of these backgrounds. In Section 6.3 we shall explore preonic solutions in ten-dimensional theories, the results for which follow from our discussion of the eleven-dimensional case.

5.1 Methodology

Preons are solutions to supergravity theories, preserving a fraction $\frac{31}{32}$ of the supersymmetry. It was shown in [24] that if $N = 31$ solutions exist for eleven-dimensional supergravity backgrounds, then locally they admit an additional Killing spinor. They do this by assuming an $N = 31$ solution i.e.

$$\mathcal{D}_X \epsilon^r = 0 \quad \text{for } r = 1, ..., 31,$$

hence

$$\mathcal{R}_{XY} \epsilon^r = 0.$$

They consider a spinor, normal to the 31-plane spanned by the ϵ^r and show that this is also a Killing spinor. Whence $\mathcal{R} = 0$ locally on M. This means M is locally isometric to a maximally supersymmetric background. Hence if $N = 31$ solutions exist they are necessarily obtained by quotienting a maximally supersymmetric background by a discrete subgroup of the isometry group. We shall investigate such quotients here.

There are four maximally supersymmetric backgrounds to consider; the Minkowski background $\mathbb{R}^{10,1}$, the Freund-Rubin $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds and the Kowalski-Glickman background (maximally supersymmetric wave). We shall show that if a quotient by discrete subgroups of the isometry group preserves a fraction $\nu \geq \frac{31}{32}$ of the supersymmetry then it will in fact preserve all of the supersymmetry, thus proving that preonic backgrounds do not exist. Therefore given some maximally supersymmetric background (M, g, F) we want to quotient by some discrete subgroup $\Gamma \subset G$ where G is the group of F-preserving isometries of M. The problem is that the symmetry groups G of the background are non-compact Lie groups so we do not know their discrete subgroups Γ. To get around this problem we shall show there does not exist any $\gamma \in G$ which preserves $\frac{31}{32}$ Killing spinors.

We denote the space of Killing spinors on M by K. Let K^Γ be the subspace of Γ invariant Killing spinors and K^γ the space of γ invariant Killing spinors. We will show if $\dim K^\Gamma \geq 31$ then $\dim K^\Gamma = 32$. Suppose Γ is such that $\dim K^\Gamma = 31$. Then for some $\gamma \in \Gamma$ $\dim K^\gamma = 31$. Indeed were this not the case then it would mean that either for all $\gamma \in \Gamma$ $\dim K^\gamma = 32$ in which case $\dim K^\Gamma = 32$; or else for some $\gamma \in \Gamma$ $\dim K^\gamma \leq 30$ whence also $\dim K \leq 30$. So we just need to show there is no γ for which $\dim K^\gamma = 31$. As we shall see below we will be able to reduce this problem to just considering elements γ in the image E_G of the exponential map, i.e., $\gamma = \exp(X)$ such that $X \in \mathfrak{g}$ and show that if $\dim K^\gamma > 30$ then $\dim K^\gamma = 32$ for every $\gamma \in G$, whence in the spin representation $R(\gamma) = 1$. Thus if $\dim K^\gamma \geq 31$ then γ acts as the identity on spinors.

5.2 Exponential Properties of the Isometry Groups

Let us consider the exponential properties of the isometry groups G. In general G is not connected or compact and the exponential map is not surjective. However we will be able to get around this problem of surjectivity as follows.

There are two cases to consider depending on G being connected or not. Let us begin with G connected. G can be multiplicatively generated by the elements in some open neighbourhood of the identity. The exponential map provides a diffeomorphism, from an open neighbourhood of the identity in $\mathfrak{g}$ to an open neighbourhood of the identity in G. Thus any element $\gamma \in \Gamma \subset G$ can be written as a product of exponentials. However we cannot in general write γ as $\exp(X)$ for $X \in \mathfrak{g}$ since the exponential map is not surjective. But we can get around this problem, and still use elements in the image of the exponential map to determine the existence of preons, as follows. Let $R : \mathfrak{g} \to \mathrm{End}(K)$ be the action of $\mathfrak{g}$ on K, it will follow by inspection of all the groups that $\mathrm{tr}R(X) = 0$ for all $X \in \mathfrak{g}$, whence

$$\det(R(\exp(X))) = \det(e^{R(X)}) = e^{\mathrm{tr}R(X)} = 1.$$

Therefore $R : G \to SL(K)$. Now suppose G is not connected. Let $\Gamma_0 = \Gamma \cap G_0$. Then, assuming that G has finitely many components, Γ/Γ^0 is a finite group and the representation $R : \Gamma \to \mathrm{GL}(K)$ factors through $\overline{R} : \Gamma/\Gamma^0 \to \mathrm{GL}(K)$. Since K is a real representation, $\det \overline{R}(\gamma) = \pm 1$. If the determinant is 1, then we can again conclude $\overline{R}(\gamma) \in SL(K)$.

To prove that $\overline{R} : \Gamma/\Gamma_0 \to \mathrm{SL}(K)$ we can argue as follows. We note that G acts transitively on connected M. Thus any two points in M can be linked by some $\gamma \in G$. The corresponding path in M connecting the points is given by the continuous group action on M. Thus the corresponding connected path in G lies in G_0. Hence G_0 acts transitively on M. With this in mind consider $\gamma \in \Gamma \setminus \Gamma_0$. Because G_0 acts transitively there is some element $\gamma_p \in G_0$ such that $h := \gamma_p^{-1}\gamma$ fixes a point, say, $p \in M$. Namely, let $q = \gamma \cdot p$ and choose $\gamma_p \in G_0$ such that $q = \gamma_p \cdot p$. Now the tangent map $h_* : T_pM \to T_pM$ defines an orthogonal transformation on T_pM which lifts to an action on the Killing spinors, and which is induced by restriction from the action of the Pin group. If the spin lift of h_* acts with unit determinant, then from the fact that γ_p does so as well, it follows that so will γ. It is then a matter of verifying that the Pin group acts with unit determinant on the relevant spinor representation. This has been done explicitly and indeed Pin does act with unit determinant for the groups we are considering.

Now that we know each γ can be taken in $SL(K)$ we argue as follows. Assuming that γ will preserve some fraction $\nu \geq \frac{30}{32}$ of the supersymmetry, thus γ must preserve at least 31 Killing spinors. We can thus write it, in some basis, as

$$\begin{pmatrix} I_{31} & \boldsymbol{v} \\ 0 & 1 \end{pmatrix},$$

where I_{31} is the 31×31 identity matrix and $\boldsymbol{v} \in \mathbb{R}^{31}$. Matrices of this form generate a subgroup in $SL(K)$ preserving 31 Killing spinors. Now if we can show that for some power k, γ^k lies in the image of the exponential map, hence preseves all the spinors (as we will show in the next section) , then

$$R(\gamma^k) = R(\gamma)^k = \begin{pmatrix} I_{31} & k\boldsymbol{v} \\ \mathbf{0}^t & 1 \end{pmatrix} = 1 \iff \boldsymbol{v} = \mathbf{0} \ ,$$

hence that in fact $R(\gamma) = 1$.

The question is thus whether given $\gamma \in G$ some power of γ will lie in the image E_G of the exponential map. The answer is yes as follows.

Definition 25. Given $\gamma \in G$, we define its **index (of exponentiality)** by

$$ind(\gamma) = \begin{cases} min\left\{k \in \mathbb{N}\,|\gamma^k \in E_G\right\} \ , & \text{should this exist} \\ \infty & \text{otherwise.} \end{cases}$$

Definition 26. A Lie group is said to be exponential if the exponential map is surjective.

In all maximally supersymmetric backgrounds except for the wave, it is known that every element of G has finite index. Let us consider the exponential properties of these backgrounds.

The flat background $\mathbb{R}^{1,d}$ has symmetry group $SO(1, d-1) \ltimes \mathbb{R}^{1,d}$. The translations act trivially on spinors, thus it is only $Spin(1, d-1)$ that concerns us. The results in [10] show that $Spin(1, 2n)$ for $n \geq 2$ and $Spin(1, 2m-1)$ for $m \geq 2$ are indeed exponential.

The Freund-Rubin backgrounds have symmetry groups $\widetilde{SO(2,p)} \times SO(q)$ for various values of p and q, where $\widetilde{SO(2,p)}$ is the universal covering group. As explained in [18] the groups acting effectively on the Killing Spinors are $Spin(2,p) \times Spin(q)$. $Spin(q)$ is compact and connected thus exponential. However $Spin(2,p)$ is not exponential but as reviewed in [9] it follows from [35] that the square of every element is in the exponential map.

Thus we are left with the plane wave. This needs to be examined more closely and we shall do so now. We will base our discussion of the maximally supersymmetric wave [27] on the paper [15]. In particular, the geometry is that of a Lorentzian symmetric space G/H (Cahen-Wallach space), where the transvection group G and the isotropy subgroup H are described as follows. Let $\mathfrak{g}$ be the 20-dimensional Lie algebra with basis $(\boldsymbol{e}_\pm, \boldsymbol{e}_i, \boldsymbol{e}_i^*)$, for $i = 1, \ldots, 9$, and nonzero brackets

$$[\boldsymbol{e}_-, \boldsymbol{e}_i] = \boldsymbol{e}_i^* \qquad [\boldsymbol{e}_-, \boldsymbol{e}_i^*] = -\lambda_i^2 \boldsymbol{e}_i \qquad [\boldsymbol{e}_i^*, \boldsymbol{e}_j] = -\lambda_i^2 \delta_{ij} \boldsymbol{e}_+ \ ,$$

where

$$\lambda_i = \begin{cases} \frac{\mu}{3} \ , & i = 1,2,3 \\ \frac{\mu}{6} \ , & i = 4, \ldots, 9 \end{cases} \qquad \text{and } \mu \neq 0. \tag{5.2.1}$$

Let $\mathfrak{h}$ denote the abelian Lie subalgebra spanned by the $\{\boldsymbol{e}_i^*\}$ and let $H < G$ denote the corresponding Lie subgroup. The subgroup $SO(3) \times SO(6) < SO(9)$ acts as automorphisms on $\mathfrak{g}$ preserving $\mathfrak{h}$ and hence acts as isometries on G/H. Moreover $S := G \rtimes (SO(3) \times SO(6))$ preserves the four-form flux, hence it is also the symmetry group of the background. We would like to determine the exponential properties of the the group G in $S := G \rtimes (SO(3) \times SO(6))$.

The (simply-connected) universal covering group $\widetilde{G}$ typically has elements with infinite index. However, $\widetilde{G}$ has no finite-dimensional faithful representations and hence the finite-dimensional representation $R : \widetilde{G} \to SL(K)$ on Killing spinors will factor through a $\mathbb{Z}$-quotient $\widehat{G}$ of $\widetilde{G}$ for which it will be possible to prove that every element has finite index.

We will find the following lemma useful in our discussion of exponentiality of the symmetry group of the maximally supersymmetric wave.

Lemma 27. *Let $\pi : \widehat{G} \to G$ be a finite cover; that is, a surjective homomorphism with finite kernel, with $\widehat{G}$ (and hence G) connected. Then if every element of G has finite index (of exponentiality), so does every element of $\widehat{G}$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of both G and $\widehat{G}$ and let $\exp : \mathfrak{g} \to G$ and $\widehat{\exp} : \mathfrak{g} \to \widehat{G}$ denote the corresponding exponential maps, related by the following commutative diagram:

$$\begin{array}{ccc} & & \widehat{G} \\ & \overset{\widehat{\exp}}{\nearrow} & \downarrow \pi \\ \mathfrak{g} & \xrightarrow{\exp} & G \end{array}$$

Since $\pi : \widehat{G} \to G$ is a finite cover, $Z = \ker \pi$ is a finite subgroup of the centre of $\widehat{G}$. (This is because a normal discrete subgroup of a connected Lie group is central.) Now let $\widehat{\gamma} \in \widehat{G}$. Since $\gamma = \pi(\widehat{\gamma})$ has finite index, there exists some positive integer N such that $\gamma^N = \exp(X)$ for some $X \in \mathfrak{g}$. Since

$$\pi(\widehat{\gamma}^N) = \pi(\widehat{\gamma})^N = \gamma^N = \exp(X) = \pi(\widehat{\exp}(X)) \ ,$$

it follows that there is some $z \in Z$ for which $\widehat{\gamma}^N = z\widehat{\exp}(X)$. Since Z is finite z has finite order, say $|z|$, whence

$$\widehat{\gamma}^{N|z|} = \widehat{\exp}(X)^{|z|} = \widehat{\exp}(|z|X) \ ,$$

and $\widehat{\gamma}$ also has finite index. □

By virtue of this lemma it is sufficient to exhibit a finite quotient (by a central subgroup), of the groups under consideration for which every element has finite index.

G has Lie algebra

$$[\boldsymbol{e}_-, \boldsymbol{e}_i] = \boldsymbol{e}_i^* \qquad [\boldsymbol{e}_-, \boldsymbol{e}_i^*] = A_{ij}\boldsymbol{e}_j \qquad [\boldsymbol{e}_i^*, \boldsymbol{e}_j] = A_{ij}\boldsymbol{e}_+ \ ,$$

for some non-degenerate symmetric matrix A_{ij}. We would like to know when G is exponential. To understand G better we consider a toy model as follows. Let $\mathfrak{g}$ denote the four-dimensional Lie algebra with basis $(\boldsymbol{e}_\pm, \boldsymbol{e}, \boldsymbol{e}^*)$ and nonzero brackets

$$[\boldsymbol{e}_-, \boldsymbol{e}] = \boldsymbol{e}^* \qquad [\boldsymbol{e}_-, \boldsymbol{e}^*] = -\boldsymbol{e} \qquad [\boldsymbol{e}, \boldsymbol{e}^*] = \boldsymbol{e}_+ \ .$$

We would like to know it's exponential properties. This algebra is the extension of the Heisenberg subalgebra, spanned by $(\boldsymbol{e}, \boldsymbol{e}^*, \boldsymbol{e}_+)$, by the outer derivation $\boldsymbol{e}_-$ which acts by infinitesimal rotations in the $(\boldsymbol{e}, \boldsymbol{e}^*)$ plane. It is a Lie subalgebra of $\mathfrak{gl}(4,\mathbb{R})$ and can be embedded as follows:

$$x\boldsymbol{e}^* + y\boldsymbol{e} + z\boldsymbol{e}_+ + t\boldsymbol{e}_- \mapsto \begin{pmatrix} 0 & t & x & 0 \\ -t & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ -y & x & -2z & 0 \end{pmatrix}, \tag{5.2.2}$$

where $x, y, z, t \in \mathbb{R}$. The corresponding Lie subgroup $G < GL(4,\mathbb{R})$ is given by

$$G = \left\{ \begin{pmatrix} \cos t & \sin t & x' & 0 \\ -\sin t & \cos t & y' & 0 \\ 0 & 0 & 1 & 0 \\ -y'\cos t - x'\sin t & x'\cos t - y'\sin t & -2z & 1 \end{pmatrix} | x', y', z, t \in \mathbb{R} \right\}. \tag{5.2.3}$$

This group is periodic in the $\boldsymbol{e}_-$ direction and G is topologically $S^1 \times \mathbb{R}^3$. Which has fundamental group $\mathbb{Z}$, i.e., for universal cover $\widetilde{G}$ we have $G \cong \widetilde{G}/\mathbb{Z}$ for some $\mathbb{Z}$-action. We know that two connected Lie groups having the same Lie algebra are obtained via quotienting the universal cover by discrete subgroups of its centre. These will be covered by some finite cover $\widehat{G}$ of them. Where $\widehat{G}$ is some discrete quotient of $\widetilde{G}$.

$$\begin{array}{ccc} & \widetilde{G} & \\ & \downarrow & \\ & \widehat{G} & \\ \swarrow & & \searrow \\ G & & G' \end{array}$$

Now if the group G is exponential then $\widehat{G}$ has finite index of exponentiality, by the lemma, thus G' which is also covered by $\widehat{G}$ will have finite index of exponentiality. In our case G' is the group acting on the Killing spinors.

We can unwind S^1 by removing the periodicity of t to obtain $\widetilde{G}$ the universal cover of G. $\widetilde{G}$ is diffeomorphic to $\mathbb{R}^4$. The matrix above gives the general form of any element in G. It is convenient, as in [21, §3.1] but using a different notation, to introduce the complex variable $w = x + iy$ in terms of which the group multiplication on $\widetilde{G}$ is given explicitly by

$$g(w_1, z_1, t_1)g(w_2, z_2, t_2) = g(w', z', t') \ ,$$

where

$$\begin{aligned} t' &= t_1 + t_2 \\ w' &= w_1 + e^{-it_1} w_2 \\ z' &= z_1 + z_2 - \tfrac{1}{2} \operatorname{Im}\left(\overline{w}_1 e^{-it_1} w_2\right) \ . \end{aligned}$$

Clearly from these we can see that $g^{-1}(w, z, t) = g(-e^{it}w, -z, -t)$ and that $g(0, 0, 0)$ is the identity. We see that elements of the form $g(0, 0, 2\pi n)$ belong to the centre of $\widetilde{G}$. The elements $\exp(t\boldsymbol{e}_-)$, generated by $\boldsymbol{e}_-$, in G act with periodicity 2π and generate the action of the fundamental group, $\mathbb{Z}$, of G on $\widetilde{G}$. Any representation of $\widetilde{G}$ in which $\exp(t\boldsymbol{e}_-)$ acts with period $2\pi n$, for some n, will factor through an n-fold cover, $\widehat{G}$, of G where $\widehat{G}$ is obtained by quotienting $\widetilde{G}$ by the infinite central subgroup generated by $g(0, 0, 2\pi n)$. We will see that for, G', the group acting effectively on the Killing spinors $\exp(t\boldsymbol{e}_-)$ acts with period $2\pi n$. Thus G' is obtained from quoteienting $\widehat{G}$. It follows from the Lemma and the fact, to be proven shortly, that if G is exponential, then every element of $\widehat{G}$ has finite index thus so will G'.

We would like to know when is the image, under the exponential map, of $\mathfrak{g}$ surjective on G. The exponential of the matrix in (5.2.2) exponentiates inside $GL(4, \mathbb{R})$ to

$$\begin{pmatrix} \cos t & \sin t & \frac{x \sin t - y(\cos t - 1)}{t} & 0 \\ -\sin t & \cos t & \frac{x(\cos t - 1) + y \sin t}{t} & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{-y \sin t + x(\cos t - 1)}{t} & \frac{x \sin t + y(\cos t - 1)}{t} & -2z + \frac{\sin t - t}{t^2}(x^2 + y^2) & 1 \end{pmatrix} . \qquad (5.2.4)$$

We would like to know if this is a surjective map. That is for every element of the form (5.2.3) there is a corresponding matrix of the form (5.2.4). Consider entries $(1,3)$ and $(2,3)$ in both matrices. These must coincide for surjectivity:

$$x' = \frac{x\sin t - y(\cos t - 1)}{t}, \ y' = \frac{x(\cos t - 1) + y\sin t}{t}.$$

We can always find x and y to satisfy this, except at special points given by $t \in 2\pi\mathbb{Z}$ where the above vanish. Thus we can say the map is surjective provided that the linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{x\sin t - y(\cos t - 1)}{t} \\ \frac{x(\cos t - 1) + y\sin t}{t} \end{pmatrix} = \frac{1}{t}\begin{pmatrix} \sin t & 1 - \cos t \\ \cos t - 1 & \sin t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

is nonsingular. When $t \in 2\pi\mathbb{Z}$, but $t \neq 0$, the element in (5.2.3) takes the form,

$$\begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ -y & x & -2z & 1 \end{pmatrix}$$

we can recover elements of this form by exponentiating elements of the form (5.2.2) with $t = 0$,

$$\exp\begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ -y & x & -2z & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x' & 0 \\ 0 & 1 & y' & 0 \\ 0 & 0 & 1 & 0 \\ -y' & x' & -2z & 1 \end{pmatrix}.$$

Therefore we conclude that the group G is exponential. Let us now consider the actual group G in the symmetry group $S := G \rtimes (SO(3) \times SO(6))$ of the Cahen-Wallach space. We would like to know the exponential properties of G. We proceed as in the case of the toy model. We exhibit this Lie algebra as a subalgebra of $\mathfrak{gl}(2d-2,\mathbb{R})$ via the following embedding

$$\sum_{i=1}^{d-2}(x_i \boldsymbol{e}_i^* + y_i \boldsymbol{e}_i) + t\boldsymbol{e}_- + z\boldsymbol{e}_+ \mapsto \begin{pmatrix} 0 & \lambda_1 t & & & & x_1 & 0 \\ -\lambda_1 t & 0 & & & & y_1 & 0 \\ & & \ddots & & & \vdots & \vdots \\ & & & 0 & \lambda_{d-2} t & x_{d-2} & 0 \\ & & & -\lambda_{d-2} t & 0 & y_{d-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -y_1 & x_1 & \cdots & -y_{d-2} & x_{d-2} & -2z & 0 \end{pmatrix} . \tag{5.2.5}$$

The Lie subgroup $G < GL(2d-2,\mathbb{R})$ with this Lie algebra consists of matrices of the form

$$\begin{pmatrix} \cos\lambda_1 t & \sin\lambda_1 t & & & & x_1 & 0 \\ -\sin\lambda_1 t & \cos\lambda_1 t & & & & y_1 & 0 \\ & & \ddots & & & \vdots & \vdots \\ & & & \cos\lambda_{d-2} t & \sin\lambda_{d-2} t & x_{d-2} & 0 \\ & & & -\sin\lambda_{d-2} t & \cos\lambda_{d-2} t & y_{d-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ u_1 & v_1 & \cdots & u_{d-2} & v_{d-2} & -2z & 1 \end{pmatrix}$$

where

$$u_i = -y_i \cos\lambda_i t - x_i \sin\lambda_i t \qquad v_i = x_i \cos\lambda_i t - y_i \sin\lambda_i t \ .$$

The exponential of the matrix in equation (5.2.5) is given by

$$\begin{pmatrix} \cos\lambda_1 t & \sin\lambda_1 t & & & & X_1 & 0 \\ -\sin\lambda_1 t & \cos\lambda_1 t & & & & Y_1 & 0 \\ & & \ddots & & & \vdots & \vdots \\ & & & \cos\lambda_{d-2} t & \sin\lambda_{d-2} t & X_{d-2} & 0 \\ & & & -\sin\lambda_{d-2} t & \cos\lambda_{d-2} t & Y_{d-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ U_1 & V_1 & \cdots & U_{d-2} & V_{d-2} & -2Z & 1 \end{pmatrix}$$

where

$$\begin{aligned}
X_i &= \frac{x_i \sin \lambda_i t + y_i(1 - \cos \lambda_i t)}{\lambda_i t} \\
Y_i &= \frac{y_i \sin \lambda_i t - x_i(1 - \cos \lambda_i t)}{\lambda_i t} \\
U_i &= -Y_i \cos \lambda_i t - X_i \sin \lambda_i t \\
V_i &= X_i \cos \lambda_i t - Y_i \sin \lambda_i t \\
Z &= z - \tfrac{1}{2} \sum_{i=1}^{d-2} (x_i^2 + y_i^2) \frac{(\lambda_i t - \sin \lambda_i t)}{\lambda_i^2 t^2} \ .
\end{aligned} \tag{5.2.6}$$

We will now specialise further to the geometry of interest, where the λ_i are given by equation (5.2.1). It is important to observe that the ratios of the λ_i are rational — in fact, integral. This means that whereas the group is not exponential, as we will now see, nevertheless the square of every element lies in the image of the exponential map.

We will now take the λ_i given by equation (5.2.1). The surjectivity of the exponential map is only in question when the linear map from (x_i, y_i) to (X_i, Y_i) in equation (5.2.6) fails to be an isomorphism. This happens whenever $\lambda_i t \in 2\pi\mathbb{Z}$ and $t \neq 0$. For the λ_i under consideration, this happens whenever $\mu t \in 6\pi\mathbb{Z}$, but $\mu t \neq 0$. Let $\mu t = 6\pi n$ and $n \neq 0$. Then the group elements with such values of t are given by

$$\begin{pmatrix}
1 & 0 & & & & x_1 & 0 \\
0 & 1 & & & & y_1 & 0 \\
 & & \ddots & & & \vdots & \vdots \\
 & & & (-1)^n & 0 & x_9 & 0 \\
 & & & 0 & (-1)^n & y_9 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
-y_1 & x_1 & \cdots & -(-1)^n y_9 & (-1)^n x_9 & -2z & 1
\end{pmatrix} ,$$

whence we see that if n is even, this is the same as if $t = 0$ for which the exponential map is surjective, whereas if n is odd, then this is not in the image of the exponential map, but its square is again of the form of the matrices with $t = 0$ and hence in the image of the exponential map. In other words, for every

$g \in G$, $g^2 \in E_G$.

Finally we shall observe that the action of $\boldsymbol{e}_-$ on the Killing spinors (see (6.2.8)) given by $\exp(t\boldsymbol{e}_-)$ is periodic, whence G' the group acting effectively on Killing spinors is obtained from an n-fold cover $\widehat{G}$. Thus it follows that G' has finite index of exponentiality.

Chapter 6

Discrete Quotients of N=32 Backgrounds

In this section we shall consider quotients of the four maximally supersymmetric 11-d backgrounds by the action of elements of the form $\gamma = \exp(X)$ where $X = \mathfrak{g}$ the Lie algebra of F preserving isometries of the background. We will impose the condition that γ preserves > 30 Killing spinors throughout and show that for each background this results in γ preserving all 32 Killing spinors. Hence proving preons do not exist as discrete quotients of maximally supersymmetric backgrounds of 11-dimensional supergravity. In Section 6.3 we will show they do not exist for Type IIA or Type IIB theories either.

For each background we shall consider the action of each representative X, of an equivalence class of elements in $\mathfrak{g}$, on the Killing spinors. These equivalence classes have been classified in [17] for the Minkowski background and in [18] for AdS backgrounds (see Appendix B). The classification involves studying the orbit decomposition of $\mathfrak{g}$ under conjugation of G to find a representative of a family of elements such that $X \sim gXg^{-1}$ $g \in G$. The method of classification is given in a series of papers [17, 18, 20, 19]. The only ones that have not been classified are those of the maximally supersymmetric wave, which we will classify here. We begin by quotienting $\mathbb{R}^{10,1}$ then $\mathrm{AdS}_4 \times S^7$, $\mathrm{AdS}_7 \times S^4$ and then the maximally supersymmetric wave.

6.1 Minkowski Background

The symmetry group for the Minkowski background is $SO(10,1) \ltimes \mathbb{R}^{11}$. The $\mathbb{R}^{11}$ denotes the translations, but they act trivially on spinors. Thus the amount of supersymmetry preserved is governed by the projection of the element $\gamma \in Spin(10,1) \ltimes \mathbb{R}^{11}$ to $Spin(10,1)$. Thus we just consider the action of $\gamma \in Spin(10,1)$, $\gamma = \exp(X)$ where $X \in \mathfrak{so}(10,1)$, on the background. As classified in [17] there are three maximal conjugacy classes of $X \in \mathfrak{so}(10,1)$. These are:

1. $X = \theta_1 \boldsymbol{e}_{12} + \theta_2 \boldsymbol{e}_{34} + \theta_3 \boldsymbol{e}_{56} + \theta_4 \boldsymbol{e}_{78} + \theta_5 \boldsymbol{e}_{9\natural}$,

2. $X = \theta_1 \boldsymbol{e}_{12} + \theta_2 \boldsymbol{e}_{34} + \theta_3 \boldsymbol{e}_{56} + \theta_4 \boldsymbol{e}_{78} + \beta \boldsymbol{e}_{09}$, and

3. $X = \theta_1 \boldsymbol{e}_{12} + \theta_2 \boldsymbol{e}_{34} + \theta_3 \boldsymbol{e}_{56} + \theta_4 \boldsymbol{e}_{78} + \boldsymbol{e}_{+9}$,

where $\boldsymbol{e}_{ij} := \boldsymbol{e}_i \wedge \boldsymbol{e}_j \in \Lambda^2 \mathbb{R}^{10,1} \cong \mathfrak{so}(10,1)$, with $\boldsymbol{e}_i$ a pseudo-orthonormal basis for $\mathbb{R}^{10,1}$, and $\boldsymbol{e}_+ := \boldsymbol{e}_0 + \boldsymbol{e}_\natural$, where as usual $\natural$ stands for 10. These basis vectors lift to γ_i in the Clifford algebra. Before we consider the quotients we shall go over the following idea, which will be very useful throughout the discussion.

6.1.1 The even-multiplicity argument

Let I_a, for $a = 1, \ldots, N$, be commuting real , $I_a^2 = 1$, and complex, $I_a^2 = -1$, structures. Consider $R(\gamma) := \exp(\sum_a \frac{1}{2}\theta_a I_a)$ acting on a complexified space of spinors W of dimension 2^N. Since the I_a's commute we can diagonalise them simultaneously. Then dealing with the complexified space of spinors W we may decompose it as

$$W = \bigoplus_{(\sigma_1,\ldots,\sigma_N)\in\mathbb{Z}_2^N} W_{\sigma_1\ldots\sigma_N}$$

where on each one-dimensional $W_{\sigma_1\ldots\sigma_N}$, γ has eigenvalue $e^{\sum_a \varepsilon_a \sigma_a \theta_a/2}$, where

$$\varepsilon_a = \begin{cases} 1 \ , & \text{if } I_a \text{ is a real structure,} \\ i \ , & \text{if } I_a \text{ is a complex structure.} \end{cases}$$

$\sigma_a = \pm 1$ and denotes a sign where $\bar{\sigma}_a = -\sigma_a$. Thus if $\exp(\sum_a \varepsilon_a \sigma_a \theta_a/2) = 1$ for some $(\sigma_a, \epsilon_a, \theta_a)$ then it will clearly be one for $(\bar{\sigma}_a, \epsilon_a, \theta_a)$. Therefore if γ preserves some $W_{\sigma_1 \ldots \sigma_N}$, it also acts as the identity on $W_{\bar{\sigma}_1 \ldots \bar{\sigma}_N}$ hence they come in pairs. This means that the subspace W^γ of γ-invariants has even complex dimension. The complex dimension of the invariant space of spinors is the same dimension as the real invariant space. Therefore it cannot be odd-dimensional. In summary elements of the form

$$\exp\left(\frac{1}{2}\Sigma_a I_a \theta_a\right),$$

preserve an even dimensional subspace.

The Killing spinors of the Minkowski background are isomorphic, as a representation of $Spin(10,1)$, with the spinor module $\Delta^{10,1}$, which is real and 32-dimensional. Since we are dealing with complex eigenvalues we will find it convenient to work in the Clifford algebra $Cl(10,1)$. This contains $Spin(10,1)$ and as an associative algebra, $Cl(10,1) \cong \mathrm{Mat}_{32}(\mathbb{C})$, whence it has a unique irreducible module W, which is complex and 32-dimensional. As a representation of $Spin(10,1)$ it is the complexification of the spinor representation.

Cases 1 and 2

In these two cases, W is the complexification of $\Delta^{10,1}$ the I_a are the images in the Clifford algebra of the infinitesimal rotations $\boldsymbol{e}_{12}, \boldsymbol{e}_{34}, \boldsymbol{e}_{56}, \boldsymbol{e}_{78}, \boldsymbol{e}_{9\natural}$ or the infinitesimal boost $\boldsymbol{e}_{09}$ in $\mathfrak{so}(10,1)$. Applying the even-multiplicity argument we see that the γ-invariant subspace is even-dimensional and hence if its dimension is > 30, it must be 32, whence preons do not exist.

Case 3

In this case, the group element is $R(\gamma) = \exp(N + \sum_a \frac{1}{2}\theta_a I_a)$, where N is the image of the infinitesimal null rotation $N = \gamma_{09} - \gamma_{9\natural}$ under the spin representation. It follows that $N^2 = 0$ in the Clifford algebra, whence $\exp(N) = 1 + N$ in the spin group. We are after the dimension of the subspace of W consisting of $\psi \in W$

satisfying

$$R(\gamma)\psi = \exp(\sum_a \tfrac{1}{2}\theta_a I_a)(1+N)\psi = \psi \ . \tag{6.1.1}$$

Let us break up $\psi = \psi_+ + \psi_-$ according to

$$V = V_+ \oplus V_- \ ,$$

where $V_\pm = \ker(\gamma_0 \pm \gamma_\natural) = \ker \gamma_\pm$. Clearly, $\ker N = \operatorname{Im} N = V_+$. Equation (6.1.1) becomes

$$\exp(\sum_a \tfrac{1}{2}\theta_a I_a)(\psi_+ + \psi_- + N\psi_-) = \psi_+ + \psi_- \ ,$$

which in turn breaks up into two equations

$$\exp(\sum_a \tfrac{1}{2}\theta_a I_a)\psi_- = \psi_- \qquad \text{and} \qquad \exp(\sum_a \tfrac{1}{2}\theta_a I_a)(\psi_+ + N\psi_-) = \psi_+ \ .$$

The dimension of each of $V_\pm$ is 16, we need to preserve 31 Killing spinors for a preonic background thus the exponential in the first equation must act as the identity on ψ_-. Then the second equation will reduce to $N\psi_- = 0$ which means that all ψ_- are zero on V_-. Thus all the ψ_+ are preserved and hence the maximal supersymmetry is $\frac{1}{2}$ whence preons do not exist in this case either.

6.2 Freund-Rubin Backgrounds

The one-parameter subgroups of the isometry group of the Freund-Rubin backgrounds $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ have been classified in [18] and the results are given in Appendix B. We shall make use of this classification in the following work on discrete quotients. The equivalence classes of elements $X \in \mathfrak{g}$ is given as follows:

$$X \sim gXg^{-1} \text{ for } g \in G.$$

Such elements will leave the same number of Killing spinors invariant. We now move on to quotients of $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ making use of the classification in [18] of the orbits above. [1]

[1] We note that the causally regular quotients of these backgrounds by one-parameter sub-

6.2.1 $\mathrm{AdS}_4 \times S^7$

The Lie algebra of symmetries is $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)$ and its action on the Killing spinors is given by the tensor product $\Delta^{3,2} \otimes \Delta^8_-$, where $\Delta^{3,2}$ is the real 4-dimensional spin representation of $\mathfrak{so}(3,2)$ and Δ^8_- is the real 8-dimensional half-spin representation of $\mathfrak{so}(8)$ consisting of negative chirality spinors. Every element of $\mathfrak{g}$ can be written as $X_A + X_S$ with $X_A \in \mathfrak{so}(3,2)$ and $X_S \in \mathfrak{so}(8)$. We may always bring $X_S \in SO(8)$ into the form $\theta_1 R_{12} + \theta_2 R_{34} + \theta_3 R_{56} + \theta_4 R_{78}$ by conjugation by elements of the maximal torus, where R_{ij} is the element of $\mathfrak{so}(8) = \mathfrak{so}(\mathbb{R}^8)$ which generates rotations in the ij-plane. The classification in [18] (see Appendix B) gives 15 possible choices for X_A. Some of these cases are similar and we will treat them together as below. We will find six are ruled out by the even multiplicity argument these are case 1, 2, 4, 10, 11, and 12. Four cases 3,5,14 and 15 are the same as the case in the previous section where we split the space up as $V = V_- \oplus V_+$ and are ruled out in the same way. The rest 6, 7, 8, 9 and 13 will be examined below.

We will work in Clifford algebra $Cl(11,2) \supset Spin(3,2) \times Spin(8)$. As an associative algebra, $Cl(11,2) \cong \mathrm{Mat}_{64}(\mathbb{C})$ and hence has a unique irreducible module W, which is 64-dimensional and complex and which decomposes under G into the direct sum of 32-dimensional complex subrepresentations (with a real structure) corresponding to $\Delta^{3,2} \otimes \Delta^8_+$ and $\Delta^{3,2} \otimes \Delta^8_-$. We are interested in the negative chiral representation $\Delta^{3,2} \otimes \Delta^8_-$.

We notice that in many of the cases below we will be able to apply the even-multiplicity argument. There is only one subtlety and that is that we are interested not in W but on a subspace V determined by some chirality condition so that we get the Δ^8_- part of the representation . In the notation of section (6.1.1) we have $N = 6$ and

$$V = \bigoplus_{\substack{(\sigma_1,\ldots,\sigma_6)\in\mathbb{Z}_2^6 \\ \sigma_3\sigma_4\sigma_5\sigma_6=-1}} W_{\sigma_1\ldots\sigma_6} \ ,$$

groups has been discussed in [18] and [29] and by discrete subgroups in [14]. However we shall not restrict ourselves to causally regular quotients in this work.

where the constraints on the signs $\sigma_3\sigma_4\sigma_5\sigma_6 = -1$ come from the chirality condition for the $\mathfrak{so}(8)$ representation. To apply the even-multiplicity argument we need to check that if $W_{\sigma_1\ldots\sigma_6} \subset V$, then also $W_{\bar\sigma_1\ldots\bar\sigma_6} \subset V$, for $\bar\sigma_a = -\sigma_a$. This is clear, though, by definition of the constraints defining V since changing the signs on each σ_a to $\bar\sigma_a$ will give the same sign in the constraint $\bar\sigma_3\bar\sigma_4\bar\sigma_5\bar\sigma_6 = -1$.

Cases 6, 7 and 8

These cases are very similar and are defined by the $\mathfrak{so}(3,2)$ component, which can take one of the following forms

- $X_A^{(6)} = -\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34}$,
- $X_A^{(7)} = -\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23})$, and
- $X_A^{(8)} = -\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \theta(\boldsymbol{e}_{12} + \boldsymbol{e}_{34})$.

We will focus on $X_A^{(7)}$ given that $X_A^{(8)}$ is very similar and that $X_A^{(6)}$ is just obtained by setting the parameters above to zero. Let $N + \beta T$ denote the image of $X_A^{(7)}$ in the Clifford algebra with $N = (\gamma_2 + \gamma_3)(\gamma_1 + \gamma_4)$. It follows that $NT = TN = 0$ and that $N^2 = 0$. Therefore the group element is given by

$$\exp(N + \beta T) = \exp(\beta T)(1 + N) \ .$$

We find it convenient to decompose V into four eight-dimensional subspaces

$$V = V_{++} \oplus V_{+-} \oplus V_{-+} \oplus V_{--} \ ,$$

where $V_{\pm\pm} = \ker(\gamma_2 \pm \gamma_3) \cap \ker(\gamma_1 \pm \gamma_4)$ with uncorrelated signs. Then N acts on V_{--} where it defines a map $N : V_{--} \to V_{++}$ but kills everything else. Let us write the invariance condition as

$$s \exp(N + T)\psi = \psi \ ,$$

where $s = \exp(X_S) \in Spin(8)$ and X_S is given above. Expanding the exponentials, using that T and N commute and that $N^2 = 0$, we arrive at

$$s \exp(T)(1 + N)\psi = s \exp(T)\psi + s \exp(T)N\psi = \psi \ .$$

The term in the image of N is in V_{++} and because $NT = 0$ this means $\operatorname{Im} T \cap V_{--} = \varnothing$, whence focusing on the V_{--} part of this equation, we find that

$$s\psi_{--} = \psi_{--} \ .$$

Now s takes the form $\exp(\Sigma\theta_a I_a)$ thus preserving an even multiple of spinors. We want $\dim K^{\gamma} > 30$ thus it forces us to preserve all the spinors on S^7 hence s acts as the identity. In this case, since Δ^8_- is eight dimensional, s will preserve a multiple of eight Killing spinors. Thus there is no integer multiple of eight to give thirty one. Hence we cannot have a preon solution via this quotient. Similarly in cases 6 and 8.

Case 9

In this case $X_A = \varphi(\boldsymbol{e}_{12} - \boldsymbol{e}_{34}) + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23}) \in \mathfrak{so}(3,2)$. The corresponding element $\exp(X_A)$ in $Spin(3,2) \subset Cl(3,2)$ is given by $\exp(\frac{1}{2}\varphi A + \frac{1}{2}\beta B)$ where A and B are the images of $\boldsymbol{e}_{12} - \boldsymbol{e}_{34}$ and $\boldsymbol{e}_{14} - \boldsymbol{e}_{23}$, respectively, in the Clifford algebra. It is easy to check that $AB = BA = 0$, whereas $A^2 = -P_+$, $A^3 = -A$, and similarly $B^2 = P_-$ and $B^3 = B$, where $P_\pm = \frac{1}{2}(1 \pm \gamma_{1234})$. Decompose $V = V_+ \oplus V_-$ where $V_\pm = \operatorname{Im} P_\pm$. Letting $s = \exp(X_S) \in Spin(8)$ we want to determine the dimension of the subspace of spinors ψ satisfying

$$s \exp(\tfrac{1}{2}\varphi A) \exp(\tfrac{1}{2}\beta B)\psi = \psi \ . \tag{6.2.1}$$

Decomposing $\psi = \psi_+ + \psi_-$, with $\psi_\pm = P_\pm\psi$, we find that the invariance equation (6.2.1) breaks up into two equations

$$s \exp(\tfrac{1}{2}\varphi A)\psi_+ = s(\cos(\tfrac{1}{2}\varphi)P_+ + \sin(\tfrac{1}{2}\varphi)A)\psi_+ \text{ and}$$

$$s\exp(\tfrac{1}{2}\beta B)\psi_- = s(\cosh(\tfrac{1}{2}\beta)P_- + \sinh(\tfrac{1}{2}\beta)B)\psi_- \ .$$

This latter equation implies that $\beta = 0$ for more than half of the supersymmetry to be preserved. If we want to preserve at least 30 Killing spinors it forces $s = 1$ since s acts with even multiplicities. This in turn implies, just like the previous case, that the dimension of the invariant subspace is a multiple of 8 and therefore cannot be equal to 31 and thus if we preserve at least 30 we must preserve 32.

Case 13

Finally, we consider the case where $X_A = \boldsymbol{e}_{12} + \boldsymbol{e}_{13} + \boldsymbol{e}_{15} - \boldsymbol{e}_{24} - \boldsymbol{e}_{34} - \boldsymbol{e}_{45}$. A calculation in the Clifford algebra shows that

$$\exp(\hat{X}_A) = 1+(\gamma_1+\gamma_4)(\gamma_2+\gamma_3)-\gamma_5(\gamma_1-\gamma_4)-2\gamma_{145}(\gamma_2+\gamma_3)-\tfrac{2}{3}(\gamma_1-\gamma_4)(\gamma_2+\gamma_3) \ . \tag{6.2.2}$$

Letting $V_{\pm\pm} = \ker(\gamma_1 \pm \gamma_4) \cap \ker(\gamma_2 \pm \gamma_3) = \ker \gamma_\pm \hat{\gamma}_\pm$ with uncorrelated signs V decomposes as the direct sum

$$V = V_{++} \oplus V_{+-} \oplus V_{-+} \oplus V_{--} \ .$$

We want to find the dimension of the subspace of ψ satisfying the equation

$$s\exp(X_A)\psi = (1 + \gamma_+\hat{\gamma}_+ - \gamma_5\gamma_- - 2\gamma_{145}\hat{\gamma}_+ - \tfrac{2}{3}\gamma_-\hat{\gamma}_+)\psi = \psi \ , \tag{6.2.3}$$

where we have let $s = \exp(X_S) \in Spin(8)$. Inspection of equation (6.2.3) reveals that the V_{+-} component is simply

$$s\psi_{+-} = \psi_{+-} \ ,$$

which if we are to preserve > 30 Killing spinors requires $s = 1$. This then implies that any element of the group $\psi = \psi_A \otimes \psi_S$ becomes $\psi_A \otimes \mathbf{1}$ thus acts with multiplicity 8 and in particular that the dimension of the subspace of invariants is a multiple of 8 and cannot therefore be equal to 31.

6.2.2 $\mathrm{AdS}_7 \times S^4$

The isometry group is $SO(6,2)\times SO(5)$ with isometry Lie algebra $\mathfrak{so}(6,2)\oplus\mathfrak{so}(5)$ and its action on the Killing spinors is given by the underlying real representation of the tensor product representation $\Delta_-^{6,2}\otimes\Delta^5$, where $\Delta_-^{6,2}$ is the quaternionic representation of $\mathfrak{so}(6,2)$ consisting of negative chirality spinors and having complex dimension 8, and Δ^5 is the quaternionic spin representation of $\mathfrak{so}(5)$ which has complex dimension four. The typical element $X \in \mathfrak{g}$ again decomposes as $X_A + X_S$, with $X_A \in \mathfrak{so}(6,2)$ and $X_S \in \mathfrak{so}(5)$. As before, every element $X_S \in \mathfrak{so}(5)$may be brought to the form $\theta_1 R_{12} + \theta_2 R_{34}$ via freedom of the maximal torus. In contrast now there are 39 possible choices for X_A, which are listed in [18] and Appendix B. It is again natural for the present purposes to treat some of these cases together, which explains the following subdivision.

Cases 1, 2, 4, 10, 11, 12, 16, 24, 25, 26, 30, 38 and 39 in Appendix B are ruled out by the even multiplicity argument. Many of the remaining cases already appeared in our discussion of $\mathrm{AdS}_4\times S^7$ and we will not repeat the arguments here for they are virtually identical to the ones above. These are cases 3, 5, 14, 15, 17, 28, 29 and 31, (which are similar to cases 3, 5, 14 and 15 above); cases 6, 7, 8, 19, 20, 21, 33, 34 and 35 (similar to cases 6,7 and 8 above); cases 9, 22 and 36 (similar to case 9 above); and cases 13 and 27 (similar to case 13 above). The remaining cases can be subdivided as follows.

Cases 23 and 37

This case involves a double null rotation. The corresponding $\hat{X}_A = N_1 + N_2 + \theta\gamma_{78}$ where N_1 is the lift of $(\boldsymbol{e}_1 + \boldsymbol{e}_4)\boldsymbol{e}_3$ and N_2 is the lift of $(\boldsymbol{e}_2 + \boldsymbol{e}_6)\boldsymbol{e}_5$. In the Clifford algebra $N_1^2 = N_2^2 = 0$ and that $N_1 N_2 = N_2 N_1$, whence $\exp(N_1 + N_2) = (1 + N_1)(1 + N_2)$. We decompose the space V of complexified Killing spinors as

$$V = V_{++} \oplus V_{+-} \oplus V_{-+} \oplus V_{--} \ ,$$

where $V_{\pm\pm} = \ker(\gamma_1 \pm \Gamma_4) \cap \ker(\gamma_2 \pm \gamma_6)$. Thus expanding things out a spinor $\psi \in V$ is invariant if it obeys

$$s(1 + N_1)(1 + N_2)\psi = \psi \ ,$$

where $s = R\left(\exp(\theta \boldsymbol{e}_{78}) \exp(X_S)\right)$. The V_{--} component of this equation is

$$s\psi_{--} = \psi_{--} \ ,$$

whence if we want 31 Killing spinors preserved it forces $s = 1$. Furthermore this means that X_s, the $Spin(5)$ part of γ, acts trivially and so γ acts with multiplicity 4, whence the dimension of the space of invariants is a multiple of 4, which if > 30 must therefore be equal to 32.

Cases 18 and 32

In this case,

$$X_A = \boldsymbol{e}_{15} - \boldsymbol{e}_{35} + \boldsymbol{e}_{26} - \boldsymbol{e}_{46} + \varphi(-\boldsymbol{e}_{12} + \boldsymbol{e}_{34} + \boldsymbol{e}_{56}) + \theta \boldsymbol{e}_{78} \ .$$

The lift to spin is

$$A + \varphi B = \gamma_{15} - \gamma_{35} + \gamma_{26} - \gamma_{46} + \varphi(-\gamma_{12} + \gamma_{34} + \gamma_{56}) + \theta\gamma_{78} \ .$$

X_S takes the form $\theta_1 R_{12} + \theta_2 R_{34}$. In the Clifford algebra, we find that $AB = BA$, whence in the spin group, $\exp(A + \varphi B) = \exp(A)\exp(\varphi B)$. Therefore we have

$$\exp(X) = s \exp(A) \exp(\varphi B + \theta\gamma_{78}) \ ,$$

where s is the lift of $\exp(X_s)$ to spin. The element s and $\exp(\varphi B + \theta\gamma_{78})$ combine to form an element of the form $\exp(\sum_a \theta_a I_a)$ for commuting complex structures I_a and therefore acts with even multiplicities. Let $W = \Delta^{6,2} \otimes \Delta^5$ be the complex 64-dimensional irreducible module of the Clifford algebra $Cl(11, 2)$. Under the action of the six complex structures I_a it decomposes into a direct sum of one-dimensional subspaces

$$W = \bigoplus_{(\sigma_1,\ldots,\sigma_6)\in\mathbb{Z}_2^N} W_{\sigma_1\ldots\sigma_6} \ ,$$

whereas the subspace $V = \Delta_-^{6,2} \otimes \Delta^5$ of complexified Killing spinors decomposes as

$$V = \bigoplus_{\substack{(\sigma_1,\ldots,\sigma_6)\in\mathbb{Z}_2^6 \\ \sigma_1\sigma_2\sigma_3\sigma_4=-1}} W_{\sigma_1\ldots\sigma_6} \ ,$$

where the constraints on the signs comes from the chirality choice of our representation for $\mathfrak{so}(6,2)$. The element $s' = s\exp(\varphi B + \theta\gamma_{78})$ preserves each subspace $W_{\sigma_1\ldots\sigma_6} \subset V$ acting on it with eigenvalue

$$e^{i(\varphi(-\sigma_1+\sigma_2+\sigma_3)+\theta\sigma_4+\theta_5\sigma_5+\theta_6\sigma_6)/2} \ .$$

We want the negative chirality representation, thus require $\sigma_1\sigma_2\sigma_3\sigma_4 = -1$. The corresponding eigenvalues and multiplicities are as follows:

- $e^{\pm i(\varphi-\theta+\sigma_5\theta_5+\sigma_6\theta_6)/2}$ —a total of 16;
- $e^{\pm i(\varphi+\theta+\sigma_5\theta_5+\sigma_6\theta_6)/2}$ —a total of 8; and
- $e^{\pm i(3\varphi+\theta+\sigma_5\theta_5+\sigma_6\theta_6)/2}$ —a total of 8.

Since A commutes with s', $\exp(A)$ preserves each of these eigenspaces. Since $A^3 = 0$ $\det A = 0$ thus $|\exp(A) - \lambda| = |(1-\lambda) + A + \frac{1}{2}A^2|$ this is zero when $\lambda = 1$ thus the only possible eigenvalue of e^A is 1. Given a spinor $\psi = \psi_A \otimes \psi_{s'}$, since $\exp(A)$ has eigenvalue one always, we need the s' component to have eigenvalue one to make ψ invariant. But s' will preserves even multiples of the above subspaces. Thus if we try to preserve more than 30 Killing spinors we end up preserving 32. Hence no preons exist.

6.2.3 Maximally supersymmetric wave

The cyclic quotients of the maximally supersymmetric wave have not been worked out before and we do so here. We will base our discussion of the maximally supersymmetric wave [27] on the paper [15]. In particular, the geometry is that of a Lorentzian symmetric space G/H, where the transvection group G and the isotropy subgroup H are described as follows. Let $\mathfrak{g}$ be the 20-dimensional Lie algebra with basis $(\boldsymbol{e}_\pm, \boldsymbol{e}_i, \boldsymbol{e}_i^*)$, for $i = 1, \ldots, 9$, and nonzero brackets

$$[\boldsymbol{e}_-, \boldsymbol{e}_i] = \boldsymbol{e}_i^* \qquad [\boldsymbol{e}_-, \boldsymbol{e}_i^*] = -\lambda_i^2 \boldsymbol{e}_i \qquad [\boldsymbol{e}_i^*, \boldsymbol{e}_j] = -\lambda_i^2 \delta_{ij} \boldsymbol{e}_+ \ ,$$

where

$$\lambda_i = \begin{cases} \frac{\mu}{3} \ , & i = 1, 2, 3 \\ \frac{\mu}{6} \ , & i = 4, \ldots, 9 \end{cases} \qquad \text{and } \mu \neq 0. \tag{6.2.4}$$

Let $\mathfrak{h}$ denote the abelian Lie subalgebra spanned by the $\{\boldsymbol{e}_i^*\}$ and let $H < G$ denote the corresponding Lie subgroup. The subgroup $SO(3) \times SO(6) < SO(9)$ acts as automorphisms on $\mathfrak{g}$ preserving $\mathfrak{h}$ and hence acts as isometries on G/H. Moreover $S := G \rtimes (SO(3) \times SO(6))$ preserves the four-form flux, hence it is also the symmetry group of the background.

Let $\mathfrak{s} = \mathfrak{g} \rtimes (\mathfrak{so}(3) \oplus \mathfrak{so}(6))$ denote the Lie algebra of S. Let us examine the possibility of bringing an element of the Lie algebra $X \in \mathfrak{s}$ to a normal form via the adjoint action of S. The adjoint action of the lie algebra is as follows:

$$\begin{aligned}
\mathrm{Ad}\left(e^{t\boldsymbol{e}_-}\right) \boldsymbol{e}_i &= \cos(\lambda_i t)\boldsymbol{e}_i + \frac{\sin(\lambda_i t)}{\lambda_i}\boldsymbol{e}_i^* & \mathrm{Ad}\left(e^{t\boldsymbol{e}_i}\right) \boldsymbol{e}_j^* &= \boldsymbol{e}_j^* - t\lambda_i^2\delta_{ij}\boldsymbol{e}_+ \\
\mathrm{Ad}\left(e^{t\boldsymbol{e}_-}\right) \boldsymbol{e}_i^* &= \cos(\lambda_i t)\boldsymbol{e}_i^* + \lambda_i \sin(\lambda_i t)\boldsymbol{e}_i & \mathrm{Ad}\left(e^{t\boldsymbol{e}_i^*}\right) \boldsymbol{e}_- &= \boldsymbol{e}_- + \lambda_i^2 t\boldsymbol{e}_i - \tfrac{1}{2}t^2\lambda_i^4\boldsymbol{e}_+ \\
\mathrm{Ad}\left(e^{t\boldsymbol{e}_i}\right) \boldsymbol{e}_- &= \boldsymbol{e}_- - t\boldsymbol{e}_i^* - \tfrac{1}{2}t^2\lambda_i^2\boldsymbol{e}_+ & \mathrm{Ad}\left(e^{t\boldsymbol{e}_i^*}\right) \boldsymbol{e}_j &= \boldsymbol{e}_j - t\lambda_i^2\delta_{ij}\boldsymbol{e}_+ \ , \\
\mathrm{Ad}\left(e^{tM_{ij}}\right) \boldsymbol{e}_k^* &= -\delta_{ik}\boldsymbol{e}_j^* + \delta_{jk}\boldsymbol{e}_i^* & \mathrm{Ad}\left(e^{tM_{ij}}\right) \boldsymbol{e}_k &= -\delta_{ik}\boldsymbol{e}_j + \delta_{jk}\boldsymbol{e}_i \ ,
\end{aligned} \tag{6.2.5}$$

where M_{ij} for $i, j = 1, 2, 3$ generates $\mathfrak{so}(3)$ and M_{ij} for $i, j = 4, 5, 6, 7, 8, 9$ generates $\mathfrak{so}(6)$. Without loss of generality we may assume that the $(\mathfrak{so}(3) \oplus \mathfrak{so}(6))$-component of X lies in the Cartan subalgebra spanned by $\{\boldsymbol{M}_{12}, \boldsymbol{M}_{45}, \boldsymbol{M}_{67}, \boldsymbol{M}_{89}\}$, while still retaining the freedom of acting with the associated maximal torus T, say.

We shall now use the equations in (6.2.5) to reduce the general element $X \in \mathfrak{s}$ to some normal form. The general form of X is:

$$X = v^+\boldsymbol{e}_+ + v^-\boldsymbol{e}_- + \Sigma_i v^i \boldsymbol{e}_i^* + \Sigma_i w^i \boldsymbol{e}_i + \theta^1 \boldsymbol{M}_{12} + \theta^2 \boldsymbol{M}_{45} + \theta^3 \boldsymbol{M}_{67} + \theta^4 \boldsymbol{M}_{89} \ .$$

There are two cases to consider in our reduction of X. This is depending on whether the component of X along $\boldsymbol{e}_-$ does or does not vanish. If it does not vanish, then from equation (6.2.5) it follows that we may set the $\boldsymbol{e}_i$-components of X equal to zero by acting with $\mathrm{Ad}\left(e^{t\boldsymbol{e}_i^*}\right)$. Acting with $\mathrm{Ad}\left(e^{t\boldsymbol{e}_3}\right)$ we may shift the $\boldsymbol{e}_3^*$-component to zero. Acting further with T we may rotate in the $(\boldsymbol{e}_1^*, \boldsymbol{e}_2^*)$, $(\boldsymbol{e}_4^*, \boldsymbol{e}_5^*)$, $(\boldsymbol{e}_6^*, \boldsymbol{e}_7^*)$ and $(\boldsymbol{e}_8^*, \boldsymbol{e}_9^*)$ planes to set the $\boldsymbol{e}_i^*$-components to zero for $i = 2, 5, 7, 9$. This brings X to the following form

$$X = v^+\boldsymbol{e}_+ + v^-\boldsymbol{e}_- + v^1\boldsymbol{e}_1^* + v^4\boldsymbol{e}_4^* + v^6\boldsymbol{e}_6^* + v^8\boldsymbol{e}_8^* + \theta^1\boldsymbol{M}_{12} + \theta^2\boldsymbol{M}_{45} + \theta^3\boldsymbol{M}_{67} + \theta^4\boldsymbol{M}_{89} \ , \tag{6.2.6}$$

where $v^- \neq 0$.

If the $\boldsymbol{e}_-$-component of X vanishes then we may set the components along some of the $\boldsymbol{e}_i$ to zero but not much else. This brings X to the form

$$\begin{aligned} X = v^+\boldsymbol{e}_+ + \sum_i v^i\boldsymbol{e}_i^* + w^1\boldsymbol{e}_1 + w^4\boldsymbol{e}_4 + w^6\boldsymbol{e}_6 + w^8\boldsymbol{e}_8 \\ + \theta^1\boldsymbol{M}_{12} + \theta^2\boldsymbol{M}_{45} + \theta^3\boldsymbol{M}_{67} + \theta^4\boldsymbol{M}_{89} \ . \end{aligned} \tag{6.2.7}$$

Thus the action on the space of Killing Spinors K is as follows. Let $R : \mathfrak{s} \to \mathrm{End}(K)$ denote the representation we find, using results from [15, §6],

$$\begin{aligned} R(\boldsymbol{e}_i) &= -\tfrac{1}{2}\lambda_i I\gamma_i\gamma_+ & R(\boldsymbol{e}_-) &= -\frac{\mu}{4}I\Pi_+ - \frac{\mu}{12}I\Pi_- \\ R(\boldsymbol{e}_i^*) &= -\tfrac{1}{2}\lambda_i^2\gamma_i\gamma_+ & R(\boldsymbol{M}_{ij}) &= \tfrac{1}{2}\gamma_{ij} \ , \end{aligned} \tag{6.2.8}$$

where $\{\gamma_+, \gamma_-, \gamma_i\}$ are the $Cl(1,9)$ gamma matrices in a Witt (light cone) basis, $I = \gamma_{123}$ and $\Pi_\pm = \frac{1}{2}\gamma_\pm\gamma_\mp$ are the projectors onto $\ker\gamma_\pm$ along $\ker\gamma_\mp$. We follow the conventions in [15], so that $\gamma_+\gamma_- + \gamma_-\gamma_+ = 2\mathbf{1}$ and $\gamma_i^2 = \mathbf{1}$. In particular, whilst $\{R(\boldsymbol{e}_-), R(\boldsymbol{M}_{ij})\}$ are semisimple, $\{R(\boldsymbol{e}_i), R(\boldsymbol{e}_i^*)\}$ are nilpotent. This means that for $X \in \mathfrak{s}$, we may decompose $R(X) = R(X)_S + R(X)_N$ into semisimple and nilpotent parts. Exponentiating and using the Baker-Campbell-Hausdorff formula, we find

$$R(\gamma) := e^{R(X)} = e^{R(X)_S + R(X)_N} = g_S g_N = g_N' g_S \ ,$$

where $g_S = e^{R(X)s}$ and g_N and g'_N are exponentials of nilpotent endomorphisms.

$$g_S = \exp(-\frac{\mu v^-}{4} I\Pi_+ e^{-\frac{\mu v^-}{12} I\Pi_-} e^{\frac{\theta^1}{2}\gamma_{12}} e^{\frac{\theta^2}{2}\gamma_{45}} e^{\frac{\theta^3}{2}\gamma_{67}} e^{\frac{\theta^4}{2}\gamma_{89}}) \ ,$$

$$g_N = \exp(-\tfrac{1}{2}(\sum_i v^i \lambda_i^2 \gamma_i + \sum_{j\in\{1,4,6,8\}} w^j \lambda_j I \gamma_j)\gamma_+).$$

Before specialising to a particular form of X let us make some general remarks about the amount of supersymmetry preserved by $R(\gamma)$. The space K of Killing spinors decomposes as $K = K_+ \oplus K_-$, where $K_\pm = K \cap \ker \gamma_\pm$. Let $\psi = \psi_+ + \psi_-$, with $\psi_\pm \in K_\pm$, be a γ-invariant Killing spinor, so that $R(\gamma)\psi = g_S g_N \psi = \psi$. Decomposing this equation, and using $g_N = \mathbf{1} + \alpha\gamma_+$, we find

$$g_S g_N(\psi_+ + \psi_-) = g_S(\psi_+ + \psi_- + \alpha\gamma_+\psi_-) = \psi_+ + \psi_- \ .$$

Since g_S respects the decomposition $K_+ \oplus K_-$ we see that, in particular, $g_S\psi_- = \psi_-$. We would like to estimate how big a subspace of K_- this is.

Let $K^0 \subset K$ denote the subspace of $R(\gamma)$-invariant Killing spinors and let $K^0_\pm = K^0 \cap K_\pm$. Then letting $K^0 + K_-$ denote the subspace of K generated by K^0 and K_-, we have the fundamental identity

$$\dim(K^0 + K_-) - \dim K^0 = \dim K_- - \dim(K^0 \cap K_-) \ .$$

Since $\dim(K^0 + K_-) - \dim K^0 \leq \operatorname{codim}(K^0 \subset K)$, we arrive at

$$\operatorname{codim}(K^0_- \subset K_-) \leq \operatorname{codim}(K^0 \subset K) \ .$$

If $R(\gamma)$ is to preserve at least $\frac{31}{32}$ of the supersymmetry, then $\operatorname{codim}(K^0 \subset K) \leq 1$, whence $\operatorname{codim}(K^0_- \subset K_-) \leq 1$. Now let $\psi_- \in K^0_-$. We have that $g_S\psi_- = \psi_-$ and $g_N\psi_- = \psi_-$. In particular, the space of g_S-invariants in K_- must have codimension at most 1: it is either 15- or 16-dimensional. We claim that this means that $g_S = 1$. Indeed, g_S is obtained by exponentiating the semisimple part of $R(X)$:

$$\begin{aligned} g_S &= \exp\left(v^- R(\boldsymbol{e}_-) + \theta^1 R(\boldsymbol{M}_{12}) + \theta^2 R(\boldsymbol{M}_{45}) + \theta^3 R(\boldsymbol{M}_{67}) + \theta^4 R(\boldsymbol{M}_{89})\right) \\ &= e^{-\frac{\mu v^-}{4} I\Pi_+} e^{-\frac{\mu v^-}{12} I\Pi_-} e^{\frac{\theta^1}{2}\gamma_{12}} e^{\frac{\theta^2}{2}\gamma_{45}} e^{\frac{\theta^3}{2}\gamma_{67}} e^{\frac{\theta^4}{2}\gamma_{89}} \ , \end{aligned}$$

whose action on $\psi_- \in K_-$ is given by

$$g_S \psi_- = e^{-\frac{\mu v^-}{12} I} e^{\frac{\theta^1}{2}\gamma_{12}} e^{\frac{\theta^2}{2}\gamma_{45}} e^{\frac{\theta^3}{2}\gamma_{67}} e^{\frac{\theta^4}{2}\gamma_{89}} \psi_- \ .$$

But now notice that each of the factors in g_S is of the form $e^{\frac{1}{2}\theta^k J_k}$ for commuting complex structures J_k which can be simultaneously diagonalised upon complexifying K_-. This is precisely the set-up in Section 6.1.1 with $W = K_- \otimes_{\mathbb{R}} \mathbb{C}$ and $N = 5$. Therefore we may apply the even-multiplicity argument to conclude that the space of such ψ_- is always divisible by 2, whence g_S cannot preserve exactly 15 thus since we still want to preserve more than 30 Killing spinors in total it forces us to preserve all 16 spinors on K_-. This means that $g_S = 1$, whence $\theta^i \in 4\pi\mathbb{Z}$ and $\mu v^- \in 24\pi\mathbb{Z}$. As a result of this the semisimple part of the group element $\gamma \in S$ is given by

$$\gamma_S = \exp\left(\frac{24\pi k}{\mu}\boldsymbol{e}_-\right) \qquad \text{for some } k \in \mathbb{Z},$$

which acts trivially on the Killing spinors. In addition, it follows from equation (6.2.5) that this element belongs to the kernel of the adjoint representation and hence to the centre of S.

Now we shall consider the nilpotent part of γ. We show it cannot preserve precisely a fraction $\frac{31}{32}$ of the supersymmetry. Since γ_S is central, we have that $g_N = \mathbf{1} - \frac{1}{2}\alpha\gamma_+$, for an endomorphism α given by

$$\alpha = \sum_{i=1}^{9} \left(\lambda_i^2 v^i \gamma_i + \lambda_i w^i I \gamma_i\right) \ , \tag{6.2.9}$$

where the coefficients v^i and w^i are the ones appearing in the expression for $X \in \mathfrak{g}$ in equations (6.2.6) and (6.2.7). It is clear from the form of g_N that it acts like the identity on K_+ and that the equation $g_N\psi = \psi$ becomes

$$(\mathbf{1} - \tfrac{1}{2}\alpha\gamma_+)(\psi_+ + \psi_-) = \psi_+ + \psi_- \implies \alpha\gamma_+\psi_- = 0 \ .$$

Since γ_+ has no kernel on K_-, it follows that we must investigate the kernel of α on K_+ or defining $\breve{\alpha}$ by $\alpha\gamma_+ = \gamma_+\breve{\alpha}$, the kernel of

$$\breve{\alpha} = \sum_{i=1}^{9} \left(\lambda_i^2 v^i \gamma_i - \lambda_i w^i I \gamma_i\right)$$

on K_-. As discussed above there are two cases we must consider corresponding to the forms (6.2.6) and (6.2.7) for X.

For X given by equation (6.2.6), the coefficients $w^i = 0$ and hence $\breve{\alpha} = \sum_{i=1}^{9} \lambda_i^2 v^i \gamma_i$ is given by the Clifford product by a vector with components $(\lambda_i^2 v_i)$ and by the Clifford relations, provided the vector is nonzero, this endomorphism has trivial kernel. If the vector is zero, so that $v^i = 0$, then we preserve all the supersymmetry. In summary, in this case we may quotient by a subgroup of the centre generated by $\exp(v^+ \boldsymbol{e}_+)$, for some v^+, while preserving all of the supersymmetry.

For X given by equation (6.2.7) we can argue as follows. First of all, since we are already in the situation when $g_S = 1$, we have some more freedom in choosing the normal form. In particular, we may conjugate by $SO(3) \times SO(6)$ to set all $w^i = 0$ except for w^1 and w^4. This still leaves an $SO(2) \times SO(5)$ which can be used to set $v^{3,6,7,8,9} = 0$. Finally we may conjugate by $e^{t\boldsymbol{e}_-}$ to set $w^4 = 0$ which then gives the further freedom under $SO(6)$ to set $v^5 = 0$. In summary, and after relabelling, we remain with

$$X = v^+ \boldsymbol{e}_+ + v^1 \boldsymbol{e}_1^* + v^2 \boldsymbol{e}_2^* + v^4 \boldsymbol{e}_4^* + w^1 \boldsymbol{e}_1 \ .$$

A calculation (performed on computer) shows that the characteristic polynomial of the endomorphism $\breve{\alpha}$ has the form

$$\chi_{\breve{\alpha}}(t) = (t^4 + 2At^2 + B)^4 = \mu_{\breve{\alpha}}(t)^4 \ ,$$

where the notation is such that $\mu_{\breve{\alpha}}$ is the minimal polynomial, and A and B are given in terms of $\boldsymbol{z} = (v^1, v^2, v^4, w^1)^t$ and $\boldsymbol{z}^2 = ((v_1)^2, (v^2)^2, (v^4)^2, (w^1)^2)^t$ as

$$A = |\boldsymbol{z}|^2 \qquad \text{and} \qquad B = Q(\boldsymbol{z}^2) \ ,$$

where Q is the quadratic form defined by the matrix

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} .$$

This matrix is not positive-definite: it has eigenvalues $0, 2, 1 \pm \sqrt{5}$ with respective eigenvectors

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2}(1 \mp \sqrt{5}) \\ 1 \\ -\frac{1}{2}(1 \mp \sqrt{5}) \\ 1 \end{pmatrix} .$$

The only way to get zero eigenvalues, i.e. $t = 0$, of $\breve{\alpha}$ is to set $B = 0$ which means $Q(\boldsymbol{z}^2) = 0$. This will ensure $\breve{\alpha}$ has nontrivial kernel. $Q(\boldsymbol{z}^2) = 0$ can be achieved, for example, by setting v^2 and v^4 to zero and $(v^1)^2 = (w^1)^2$. The characteristic polynomial again satisfies $\chi_{\breve{\alpha}}(t) = \mu_{\breve{\alpha}}(t)^4$, and the minimal polynomial now becomes

$$\mu_{\breve{\alpha}}(t) = t^2(t^2 + 4(w^1)^2) \ .$$

The dimension of the kernel of $\breve{\alpha}$ will be less than the algebraic multiplicity of 0 as a root of the characteristic polynomial. Therefore for $w^1 \neq 0$ the dimension of the kernel will be at most 8 (and, in fact, it is exactly 8). For $w^1 = 0$ all eigenvalues are zero, and the endomorphism $\breve{\alpha} \equiv 0$ and the dimension of the kernel is precisely 16.

In summary, we have shown that if an element in the image of the exponential map of the symmetry groups of the maximally supersymmetric vacua preserves at least 31 supersymmetries, it must in fact preserve all 32.

6.3 Conclusions and Type IIA and IIB Theories

6.3.1 Eleven-Dimensional Supergravity

As we have shown above, no quotient of maximally supersymmetric eleven-dimensional supergravity preserves 31 killing spinors. Thus proving conclusively that preons do not exist.

6.3.2 Type IIA

The results for type IIA follow nicely from the above work. IIA is a quotient of eleven-dimensional supergravity, by a one-parameter subgroup of G and the Killing spinors in the IIA background are the invariant Killing spinors of the eleven-dimensional background. But as we have shown no element in G preserves 31 Killing spinors, indeed if more than 30 are preserved then all of them will be.

6.3.3 Type IIB

The absence of preons in Type IIB is discussed in [25] and [1]. [25] shows any solution to IIB preserving 31 Killing spinors, is locally maximally supersymmetric. We cannot obtain preons via quotients in this case because the group acts complex linearly on Killing spinors, which can be taken to be complex chiral spinors. Thus the invariants of the quotient form a complex subspace of spinors and thus has even dimension.

If alternatively we obtain any maximally supersummetric theory via dimensional reduction of eleven-dimensional supergravity or type IIB, we still cannot obtain preons. So there is no supergravity theory with a preonic solution.

Appendix A

Cone Construction

A.1 Cone Geometry

The cone $\tilde{M}$ of M is the warped product $M \times_{r^2} \mathbb{R}_+$ with metric

$$\tilde{g} = dr^2 + r^2 g,$$

where g is the metric on M and $r > 0$ parameterises $\mathbb{R}_+$. We can visualise this by thinking of a cone with each slice of it giving some dilation of M. We can recover M at the $r = 1$ slice of the cone. The vector $\xi = r\partial_r$ on $\tilde{M}$ generates infinitesimal orientation preserving transformations. Any vector field, V, on M can be lifted isomorphically to a vector field, $\tilde{V}$, on $\tilde{M}$. This vector field will be orthogonal to ξ. Notice for vectors V, W on M,

$$\tilde{g}(V, W) = r^2 g(V, W).$$

Let $\tilde{\nabla}$ denote the Riemannian connection on the cone $\tilde{M}$. Then we have the following for the warped product $\tilde{M} = M \times_{r^2} \mathbb{R}_+$, see [4], and [11]:

$$\tilde{\nabla}_\xi \xi = \xi \ , \quad \tilde{\nabla}_\xi V = V \quad \text{and} \quad \tilde{\nabla}_V W = \nabla_V W - g(V, W)\xi \ , \tag{A.1.1}$$

for all vectors V, W tangent to M.

Let $\{E_i\}$ be a local orthonormal frame for M, and $\{\tilde{E}_I\} = \{\tilde{E}_i \equiv \frac{1}{r} E_i, \tilde{E}_r \equiv$

$\partial_r\}$ the induced local orthonormal frame for $\tilde{M}$. Now consider the connection coefficients of $\tilde{\nabla}$, $\tilde{\omega}_I{}^J{}_K$ by

$$\tilde{\nabla}_{\tilde{E}_I} \tilde{E}_J = \tilde{\omega}_I{}^J{}_K \, \tilde{E}_K \ . \tag{A.1.2}$$

A calculation shows that

$$\tilde{\omega}_r{}^J{}_K = 0 \ , \quad \tilde{\omega}_i{}^j{}_r = \frac{1}{r}\delta_i{}^j \ , \quad \tilde{\omega}_i{}^r{}_j = -\frac{1}{r}g_{ij} \quad \text{and} \quad \tilde{\omega}_i{}^j{}_k = \frac{1}{r}\omega_i{}^j{}_k \ . \tag{A.1.3}$$

Now we study the connection $\tilde{\nabla}$ on the spin bundle. Suppose that Ψ is a parallel spinor on $\tilde{M}$ then we know:

$$\tilde{\nabla}_{\tilde{E}_I}\Psi = \tilde{E}^\mu_I \partial_\mu \Psi - \tfrac{1}{4}\tilde{\omega}_I{}^{JK}\Gamma_{JK} \cdot \Psi = 0 \ . \tag{A.1.4}$$

where Γ_{JK} are in the Clifford algebra on $\tilde{M}$. Note that in the case that $\tilde{M}$ is flat, say $\mathbb{R}^n$, and relative to flat coordinates the connection coefficients $\tilde{\omega}$ vanish and thus the parallel spinors are in one-to-one correspondence with constant spinors in the spin bundle. In terms of the explicit expression (A.1.3), we have that

$$\tilde{\nabla}_{\tilde{E}_r}\Psi = \partial_r \Psi = 0 \quad \text{and} \quad \tilde{\nabla}_{\tilde{E}_i}\Psi = \frac{1}{r}\left(\nabla_{E_i}\Psi - \tfrac{1}{2}\Gamma_i\Gamma_r \cdot \Psi\right) = 0 \ . \tag{A.1.5}$$

In order to relate this equation to the Killing spinor equation on M we need to know how $\Gamma_i\Gamma_r$, the elements of the Clifford bundle on $\tilde{M}$ are related to the Clifford bundle on M. We shall do this below.

Let $Cl(TM)$ be the Clifford bundle on M, which is a bundle of Clifford algebras isomorphic to $Cl(n)$, the euclidean Clifford algebra in n-dimensions. $Cl(n)$ is generated by $\{\gamma_i\}$ subject to

$$\{\gamma_i, \gamma_j\} = -2\,\delta_{ij} \ . \tag{A.1.6}$$

On the other hand, the Clifford bundle $Cl(\tilde{M})$ on the cone is locally modelled on $Cl(n+1)$, which is generated by $\{\Gamma_I\}$ say with $I = (i, r \equiv n+1)$ with i running from 1 to n, subject to

$$\{\Gamma_I, \Gamma_J\} = -2\,\delta_{IJ} \ . \tag{A.1.7}$$

We know that $Cl(n) \cong Cl(n+1)^{even}$. The embedding is given by

$$\gamma_i \mapsto \varepsilon \Gamma_i \Gamma_r \ , \tag{A.1.8}$$

where $\varepsilon^2 = 1$. Notice that under this map $\frac{1}{2}\gamma_{ij} \mapsto \frac{1}{2}\Gamma_{ij}$, so that it induces the natural embedding of the Spin groups.

Using this we can rewrite equation (A.1.5) as

$$\partial_r \Psi = 0 \quad \text{and} \quad \nabla_i \Psi = \tfrac{1}{2}\varepsilon\gamma_i \cdot \Psi \ . \tag{A.1.9}$$

Therefore we deduce that there is a one-to-one correspondence between Killing spinors on X and parallel spinors on the cone $\tilde{X}$: a parallel spinor Ψ on the cone restricts (at $r = 1$) to a Killing spinor on X, and conversely, given a Killing spinor on X we can extend this to a parallel spinor on the cone by demanding that it does not depend on r.

A.2 Chirality of parallel spinors on the cone

Consider how the volume element gets embedded (A.1.8). Notice that it has the following additional property:

$$\gamma_1 \cdots \gamma_n \mapsto \begin{cases} \varepsilon \Gamma_1 \cdots \Gamma_r \ , & \text{if } n \text{ is odd; and} \\ \Gamma_1 \cdots \Gamma_n \ , & \text{if } n \text{ is even.} \end{cases} \tag{A.2.1}$$

Let n be odd. In an irreducible representations of $Cl(n)$, the volume element $\gamma_1 \cdots \gamma_n$ is a scalar multiple of the identity. According to the above equation, it gets mapped to ε times the volume element of $Cl(n+1)$. This means that ε is fixed in terms of the chirality of the spinor Ψ: the nature of the correspondence will depend on which irreducible representation we have chosen to work with in $Cl(n)$ equivalently, the orientation of M. On the other hand, if n is even, ε is not fixed. Therefore for each parallel spinor on $\tilde{M}$, we get one parallel spinor with $\varepsilon = 1$ and one with $\varepsilon = -1$ simply by choosing one of the two inequivalent irreducible representations of $Cl(n+1)$.

From the above analysis if we let $M = S^n$ then the cone $\tilde{M} = \mathbb{R}^{n+1}$. The Killing spinors on S^n are in one-to-one correspondence with parallel spinors on $\mathbb{R}^{n+1}$. Thus parallel spinors in $\mathbb{R}^{n+1}$ are in one-to-one correspondence with the *constant* spinors, of the relevant chiral representation of $Spin(n)$. In our case we have S^7 and thus the cone is $\mathbb{R}^8$. Thus parallel spinors are the constant spinors of $Spin(8)$, in the relevant chiral representation.

Appendix B

Classification of Discrete Isometries of Anti-de Sitter Space.

B.1 Methodology

In [18] the one-parameter subgroups of isometries of AdS_{p+1} were classified. This was done via classifying equivalence classes of the Lie algebra elements of the symmetry group G. The paper studies the orbit decomposition of the symmetry Lie algebra to find representatives up to conjugation:

$$X \sim gXg^{-1} \text{ for } g \in G.$$

For our work on preons we will be interested in discrete quotients of AdS_4 and AdS_7 and will make use of this work. It will be shown in Chapter 5 that it is enough to act with the elements of the form $\exp(X_A + X_S)$ on $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$, to determine the existence of preons on them. Where for $\mathrm{AdS}_4 \times S^7$, $X_A \in \mathfrak{so}(2,3)$ and $X_S \in \mathfrak{so}(8)$, similarly for $\mathrm{AdS}_7 \times S^4$, $X_A \in \mathfrak{so}(2,6)$ and $X_A \in \mathfrak{so}(4)$. Thus we list the equivalence classes of these elements below. More on the quotients by the corresponding one parameter subgroups, and the details of how the elements below were classified, can be found in [18] to which we refer the interested reader.

The elements below are written in a basis where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are timelike and $\boldsymbol{e}_{ij} = \boldsymbol{e}_i \wedge \boldsymbol{e}_j$. The parameters φ and β are real.

B.1.1 Classification of $\mathfrak{so}(3,2)$ elements

The list below gives the equivalence classes of elements in $\mathfrak{so}(3,2)$ the parameters in each case are all non-zero and real:

1. $\varphi \boldsymbol{e}_{12}$;
2. $\beta \boldsymbol{e}_{13}$;
3. $\boldsymbol{e}_{12} - \boldsymbol{e}_{23}$;
4. $\varphi \boldsymbol{e}_{34}$;
5. $\boldsymbol{e}_{13} - \boldsymbol{e}_{34}$;
6. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34}$
7. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23})$;
8. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \varphi(\boldsymbol{e}_{12} + \boldsymbol{e}_{34})$;
9. $\varphi(\boldsymbol{e}_{12} - \boldsymbol{e}_{34}) + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23})$;
10. $\varphi_1 \boldsymbol{e}_{12} + \varphi_2 \boldsymbol{e}_{34}$;
11. $\beta_1 \boldsymbol{e}_{13} + \beta_2 \boldsymbol{e}_{24}$;
12. $\beta \boldsymbol{e}_{13} + \varphi \boldsymbol{e}_{45}$;
13. $\boldsymbol{e}_{12} + \boldsymbol{e}_{13} + \boldsymbol{e}_{15} - \boldsymbol{e}_{24} - \boldsymbol{e}_{34} - \boldsymbol{e}_{45}$;
14. $\boldsymbol{e}_{12} - \boldsymbol{e}_{23} + \varphi \boldsymbol{e}_{45}$;
15. $\boldsymbol{e}_{13} - \boldsymbol{e}_{34} + \beta \boldsymbol{e}_{25}$;

B.1.2 Classification of $\mathfrak{so}(6,2)$ elements

The list below gives the equivalence classes of elements in $\mathfrak{so}(6,2)$:

1. $\varphi \boldsymbol{e}_{12}$;
2. $\beta \boldsymbol{e}_{13}$;

3. $\boldsymbol{e}_{12} - \boldsymbol{e}_{23}$;

4. $\varphi \boldsymbol{e}_{34}$;

5. $\boldsymbol{e}_{13} - \boldsymbol{e}_{34}$;

6. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34}$;

7. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23})$;

8. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \varphi(\boldsymbol{e}_{12} + \boldsymbol{e}_{34})$;

9. $\varphi(\boldsymbol{e}_{12} - \boldsymbol{e}_{34}) + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23})$;

10. $\varphi_1 \boldsymbol{e}_{12} + \varphi_2 \boldsymbol{e}_{34}$;

11. $\beta_1 \boldsymbol{e}_{13} + \beta_2 \boldsymbol{e}_{24}$;

12. $\beta \boldsymbol{e}_{13} + \varphi \boldsymbol{e}_{45}$;

13. $\boldsymbol{e}_{12} + \boldsymbol{e}_{13} + \boldsymbol{e}_{15} - \boldsymbol{e}_{24} - \boldsymbol{e}_{34} - \boldsymbol{e}_{45}$;

14. $\boldsymbol{e}_{12} - \boldsymbol{e}_{23} + \varphi \boldsymbol{e}_{45}$;

15. $\boldsymbol{e}_{13} - \boldsymbol{e}_{34} + \beta \boldsymbol{e}_{25}$;

16. $\varphi_1 \boldsymbol{e}_{34} + \varphi_2 \boldsymbol{e}_{56}$;

17. $\boldsymbol{e}_{13} - \boldsymbol{e}_{34} + \varphi \boldsymbol{e}_{56}$;

18. $\varphi(-\boldsymbol{e}_{12} + \boldsymbol{e}_{34} + \boldsymbol{e}_{56}) + \boldsymbol{e}_{15} - \boldsymbol{e}_{35} + \boldsymbol{e}_{26} - \boldsymbol{e}_{46}$;

19. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23}) + \varphi \boldsymbol{e}_{56}$;

20. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \varphi_1(\boldsymbol{e}_{12} + \boldsymbol{e}_{34}) + \varphi_2 \boldsymbol{e}_{56}$;

21. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \varphi \boldsymbol{e}_{56}$;

22. $\varphi_1(\boldsymbol{e}_{12} - \boldsymbol{e}_{34}) + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23}) + \varphi_2 \boldsymbol{e}_{56}$;

23. $\boldsymbol{e}_{13} - \boldsymbol{e}_{34} + \boldsymbol{e}_{25} - \boldsymbol{e}_{56}$;

24. $\varphi_1 \boldsymbol{e}_{12} + \varphi_2 \boldsymbol{e}_{34} + \varphi_3 \boldsymbol{e}_{56}$;

25. $\beta_1 \boldsymbol{e}_{13} + \beta_2 \boldsymbol{e}_{24} + \varphi \boldsymbol{e}_{56}$;

26. $\beta \boldsymbol{e}_{13} + \varphi_1 \boldsymbol{e}_{56} + \varphi_2 \boldsymbol{e}_{78}$;

27. $\boldsymbol{e}_{12} + \boldsymbol{e}_{13} + \boldsymbol{e}_{15} - \boldsymbol{e}_{24} - \boldsymbol{e}_{34} - \boldsymbol{e}_{45} + \varphi \boldsymbol{e}_{78}$;

28. $\boldsymbol{e}_{12} - \boldsymbol{e}_{23} + \varphi_1 \boldsymbol{e}_{45} + \varphi_2 \boldsymbol{e}_{67}$;

29. $\boldsymbol{e}_{13} - \boldsymbol{e}_{34} + \beta \boldsymbol{e}_{25} + \varphi \boldsymbol{e}_{67}$;

30. $\varphi_1 \boldsymbol{e}_{34} + \varphi_2 \boldsymbol{e}_{56} + \varphi_3 \boldsymbol{e}_{78}$;

31. $\boldsymbol{e}_{13} - \boldsymbol{e}_{34} + \varphi_1 \boldsymbol{e}_{56} + \varphi_2 \boldsymbol{e}_{78}$;

32. $\varphi_1(-\boldsymbol{e}_{12} + \boldsymbol{e}_{34} + \boldsymbol{e}_{56}) + \boldsymbol{e}_{15} - \boldsymbol{e}_{35} + \boldsymbol{e}_{26} - \boldsymbol{e}_{46} + \varphi_2 \boldsymbol{e}_{78}$;

33. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \varphi_1 \boldsymbol{e}_{56} + \varphi_2 \boldsymbol{e}_{78}$;

34. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23}) + \varphi_1 \boldsymbol{e}_{56} + \varphi_2 \boldsymbol{e}_{78}$;

35. $-\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \varphi_1(\boldsymbol{e}_{12} + \boldsymbol{e}_{34}) + \varphi_2 \boldsymbol{e}_{56} + \varphi_3 \boldsymbol{e}_{78}$;

36. $\varphi_1(\boldsymbol{e}_{12} - \boldsymbol{e}_{34}) + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23}) + \varphi_2 \boldsymbol{e}_{56} + \varphi_3 \boldsymbol{e}_{78}$;

37. $\boldsymbol{e}_{13} - \boldsymbol{e}_{34} + \boldsymbol{e}_{25} - \boldsymbol{e}_{56} + \varphi \boldsymbol{e}_{78}$;

38. $\varphi_1 \boldsymbol{e}_{12} + \varphi_2 \boldsymbol{e}_{34} + \varphi_3 \boldsymbol{e}_{56} + \varphi_4 \boldsymbol{e}_{78}$;

39. $\beta_1 \boldsymbol{e}_{13} + \beta_2 \boldsymbol{e}_{24} + \varphi_1 \boldsymbol{e}_{56} + \varphi_2 \boldsymbol{e}_{78}$;

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