

The Penrose transform and its applications

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Liana David)

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Introduction.

0.1 History.

Twistor theory began as a subject in late 1960's with the appearance of Penrose's papers [33], [34] and [35]. Penrose showed that solutions of some of the most important conformally invariant equations on \mathbb{C}^4 which come from physics (like the Maxwell's equations, the Dirac equation, the wave equation) could be expressed as contour integrals of holomorphic functions over projective lines in the 3-dimensional complex projective space. In [19] it was shown that the freedom allowed in the function for a fixed solution was exactly the Čech representative of a sheaf cohomology class. The resulting isomorphism between a sheaf cohomology group on a region of the complex 3-dimensional projective space and solutions of a conformally invariant operator on a region of the space-time \mathbb{C}^4 has become known as the Penrose transform. Even if its original motivation came from physics, twistor theory has developed in many areas of pure mathematics like differential geometry, integrable systems and representation theory (see [3] for a collection of review articles showing the diversity of directions where twistor theory can be applied).

Twistor theory and differential geometry. At the heart of twistor theory in 4-dimensions is the non-linear graviton construction, which associates to a self-dual complex 4-manifold M (that is M is oriented and has an holomorphic metric with self-dual Weyl tensor) a complex 3-dimensional manifold Z (called the “twistor space” of M) with a family of projective lines (called “twistor lines”) with the normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ from which M can be recovered. This is a natural generalization of the correspondence between \mathbb{C}^4 and the 3-dimensional complex projective space with a projective line removed, which appeared in the original Penrose transform. The central idea of twistor theory in 4-dimensions is to transform problems of field theory on M into algebraic geometric problems on Z , and to use the powerful methods of geometry in the study of field theory. By imposing reality conditions a twistor theory for Riemannian, Lorentzian and ultra-hyperbolic signature in 4 dimensions has been obtained.

There is a large literature following the original paper of Atiyah ([1]) on the subject of Riemannian twistor theory, considerably generalizing the Riemannian twistor theory in 4 dimensions. A twistor theory for $4n$ -dimensional quaternionic manifolds with applications in the study of quaternionic-Kähler manifolds was developed in [37], [38], [39], and [40]. A twistor theory for Riemannian symmetric

spaces with applications in the study of minimal surfaces was developed in [13].

Twistor theories for dimensions lower than 4 have been constructed as well. The beginning of the twistor theory in 3-dimensions, or more appropriately called “mini-twistor theory” has been marked by a paper of Hitchin (see [21]) and by a paper of Jones and Tod (see [25]). Hitchin showed that there is a one-to-one correspondence between a class of 3-dimensional manifolds (called “Einstein-Weyl”), and complex 2-dimensional manifolds (called “mini-twistor spaces”) with a family of projective lines (called “mini-twistor lines”) with normal bundle $\mathcal{O}(2)$. Jones and Tod showed that the space of orbits of conformal vector fields on self-dual conformal 4-spaces inherit a natural Einstein-Weyl structure, and they interpreted the Hitchin correspondence as a reduction of the non-linear graviton construction. They used this interpretation to find new classes of Einstein-Weyl 3-spaces and self-dual 4-spaces. The Einstein-Weyl 3-dimensional spaces generalize the Einstein 3-dimensional spaces, but unlike the Einstein 3-dimensional spaces they have a very rich geometry. A treatment of Einstein-Weyl 3-dimensional spaces using the notions of density line bundles and Weyl derivatives can be found in [15].

Twistor theory and integrable systems. To come back to 4 dimensions, there is a remarkable one to one correspondence due to Ward (see [47]), which associates to a self-dual Yang-Mills field on a 4-dimensional self-dual conformal manifold M a holomorphic vector bundle on the twistor space of M trivial on the twistor lines. The anti-self-dual Yang-Mills equations play a universal role in integrable systems, since almost all the interesting integrable systems (like the Bogomolny equations in 3 dimensions, the Ernst equation and the Liouville’s equation in 2 dimensions, the Nahm’s equations and the Painleve equations in 1 dimension) can be obtained by reducing the anti-self-dual Yang-Mills equations. By reducing the Penrose-Ward transform for \mathbb{C}^4 a twistor theory has been obtained for almost all the interesting integrable systems, and constructions like the inverse scattering method have been studied using the powerful methods of complex holomorphic geometry. A systematic account of the theory of integrable systems from this point of view can be found in [29].

Penrose transforms and representation theory. The original Penrose transform for \mathbb{C}^4 has undertaken large generalizations. Independently of Penrose, in 1967 Schmid studied representations of Dolbeault cohomology groups using methods similar to the Penrose transform for \mathbb{C}^4 (see [42]). A more general holomorphic Penrose transform with the conformal group of \mathbb{C}^4 replaced by an arbitrary

complex semi-simple Lie group G was constructed in [7]. This transform maps Dolbeault cohomology groups of homogeneous vector bundles on a complex homogeneous manifold Z to solutions of invariant holomorphic differential equations on a complex homogeneous manifold X , where X is a moduli space of complex compact submanifolds of Z . Using the Bott-Borel-Weyl theorem and the Bernstein-Gelfand-Gelfand resolution an algorithm for computing in practice this transform has been given. The holomorphic Penrose transform has applications in constructing unitary representations of cohomology groups and in the study of homomorphisms between Verma modules.

Other Penrose transforms. In [50] Woodhouse constructed transforms which identified smooth solutions of differential operators defined on the Euclidean 4-dimensional space or on Cauchy surfaces of the real 4-dimensional Minkowski space with some CR-cohomology groups on the corresponding twistor space. (These transforms were obtained by restricting the original Penrose transform of Penrose to the Minkowski or Euclidean space considered as real slices in the complex 4-dimensional Minkowski space). Other Penrose type transforms from cohomology groups on complex manifolds Z to solutions of differential equations on smooth manifolds X appear in the literature as well (see [22], [41], [46]). A mechanism which unified all the Penrose transforms mentioned above, which become known in the literature as the “smooth Penrose transform” was developed in [6]. It identifies cohomology groups on a CR-manifold Z with solutions of differential equations on a smooth manifold X .

An arbitrary differential equation can have solutions which are not smooth or holomorphic. A natural problem which arose was to extend the Penrose transform to other categories of solutions. Hyper-function zero rest mass fields have been discussed in [4]. A Penrose transform for the real analytic category is informally known by the mathematicians, but, as far as we know it doesn't have a rigorous treatment in the literature.

0.2 Outline of the thesis.

0.2.1 Overview.

In this thesis, we give several extensions to the smooth Penrose transform mentioned above.

We construct a Penrose transform which maps smooth compactly supported cohomology groups on a CR-manifold Z to kernels (or cokernels) of differential operators on smooth compactly supported fields on another manifold X , where

Z and X are related in the same way as in the smooth Penrose transform. We will refer to this transform as being the “smooth compactly supported Penrose transform”. The operators which arise from the smooth compactly supported Penrose transform are the same as the operators which arise from the smooth Penrose transform, except that they are defined on smooth compactly supported fields. There is also a shift in dimension which appears at the compactly supported smooth Penrose transform. The compactly supported smooth Penrose transform is related to the smooth Penrose transform via the natural “Serre duality” pairing between compactly supported cohomology and smooth cohomology. This pairing traced through the transforms becomes an algebraic operation on fields over X followed by integration. In most of our examples the cohomology groups which pair via the Serre “duality” are identified with the kernel and the cokernel of two differential operators, one of them defined on smooth fields on X and the other one defined on smooth compactly supported fields on X . A direct calculation shows that the contraction of an element of the kernel with a representative of an element of the cokernel, followed by integration is well defined (that is, independent of the choice of the representative of the element of the cokernel). The motivation for constructing a Penrose transform for compactly supported cohomology comes from representation theory: representations in compactly supported cohomology groups are more easily unitarised.

A Penrose transform for currents (with compact and non-compact supports) which maps distribution cohomology groups on a CR-manifold Z to kernels (or cokernels) of differential operators on another manifold X is constructed as well. We notice that when Z is actually a complex manifold, distribution cohomology groups on Z are the same as smooth cohomology groups on Z . In such a case the operators which arise on X are elliptic and the Penrose transform for currents does not give more information than the smooth Penrose transform when the cohomology group on Z corresponds to the kernel of an operator on X . However, when the cohomology group on Z corresponds to a cokernel of an operator on X , the smooth Penrose transform combined with the Penrose transform for currents imply that the cokernels of the operator in the smooth and distribution categories coincide. This can have interesting consequences in practical situations. For example the twistor space of the Euclidean space \mathbb{R}^3 is a 2 dimensional complex manifold and the Penrose transform generates the Laplacian Δ on \mathbb{R}^3 . It follows that any compactly supported distribution on \mathbb{R}^3 is the sum of a compactly supported function on \mathbb{R}^3 and the image through Δ of a compactly supported distribution. An example of a manifold whose twistor space is not complex, but

only CR, is the Minkowski space \mathbb{R}^4 . The twistor space of the Minkowski space \mathbb{R}^4 is a real hypersurface \mathcal{P} in $\mathbb{C}\mathbb{P}^3$, and the Penrose transform for currents applied to the restriction of the bundle $\mathcal{O}(-2)$ from $\mathbb{C}\mathbb{P}^3$ to \mathcal{P} generates the wave operator on \mathbb{R}^4 (which is not elliptic).

In order to show the richness of the theory described above, we apply the Penrose transforms for currents and for smooth compactly supported cohomology in some practical situations which are already known in the literature (see [46] and [5]). We consider the twistor correspondence of the Euclidean 3-dimensional space, of the real Minkowski 4-dimensional space, of the odd-dimensional hyperbolic space, and we apply our extended Penrose transforms to these correspondences.

As a further application of the methods presented above, we consider a Penrose transform for \mathbb{R}^3 with a point removed. Our motivation for considering this transform is the Green function of the Laplace operator on \mathbb{R}^3 , which should have an interpretation on the twistor space of \mathbb{R}^3 . A twistor interpretation of the Green function of the Laplace operator on an arbitrary self-dual 4-dimensional Riemannian manifold has been constructed in [1].

We identify the solutions of the Laplace operator on $\mathbb{R}^3 \setminus \{0\}$ which cannot be extended over \mathbb{R}^3 in terms of a relative Dolbeault cohomology group on the twistor of \mathbb{R}^3 . As far as we know, the relative involutive cohomology for an arbitrary involutive structure has not been defined in the literature. We give a definition for the relative involutive cohomology which generalizes the relative Dolbeault and relative de Rham cohomology, and we show the excision property of the involutive relative cohomology so defined. We hope that the relative involutive cohomology will be successfully applied in the study of other Penrose-type transforms, and we present a result which we believe is a first step in this direction.

As an application of twistor theory in differential geometry we illustrate a method of constructing explicitly self-dual 4-dimensional manifolds. We do this by reducing the Atiyah-Ward A_k -Ansatz (for $k = 0, 1, 2$) from 4 dimensional self-dual spaces to 3 dimensional Einstein-Weyl spaces. The Atiyah-Ward Ansätze is a method of constructing an $SL(2, \mathbb{C})$ self-dual Yang-Mills field on a self-dual 4-manifold from a self-dual Maxwell field and a solution of an auxiliary equation. The reduced A_0 -Ansatz which we obtain on 3-dimensional Einstein-Weyl spaces gives an interpretation of the affine monopole equations, while the reduced Ansatz for $k = 1, 2$ provides a method of constructing solutions of the $SL(2, \mathbb{C})$ Einstein-Weyl Bogomolny equations on 3-dimensional Einstein-Weyl spaces from a solution of the abelian monopole equation and an auxiliary equation (obtained

by reducing the corresponding auxiliary equation from the Atiyah-Ward Ansatz in 4 dimensions). The reduced Ansätze have an interpretation in terms of extensions of line bundles on the mini-twistor space similar to the interpretation of the Atiyah-Ward Ansätze on the twistor space. In this thesis we obtain in an explicit form the reduced Ansätze on 3-dimensional Einstein-Weyl spaces, but explicit examples and applications still need to be developed.

The main importance of the reduced Ansätze comes from the fact that solutions of the Einstein-Weyl Bogomolny equations on 3-dimensional Einstein-Weyl spaces generate self-dual 4-manifolds. Certain 3-dimensional Einstein-Weyl spaces (called Toda) admit a shear-free twist-free geodesic congruence. On such a space the 1-form defined by the congruence satisfies the auxiliary equation from the reduced A_2 -Ansatz, and the solution of the Einstein-Weyl Bogomolny equations it generates, together with the corresponding self-dual 4-space need to be identified. Explicit examples can be obtained by considering the Einstein-Weyl spaces which are both Toda and hyper-CR, and the Einstein-Weyl spaces with axial symmetry. Another type of application refers to the canonical monopole on an Einstein-Weyl 3-space B , which corresponds to the $SL(2, \mathbb{C})$ bundle $TS(-2)$ on the mini-twistor space S of B . The canonical monopole has particular importance since it generates self-dual conformal 4-manifolds which admit an Einstein representative of non-zero scalar curvature. The canonical monopole has an explicit formula in terms of the scalar curvature and the Faraday curvature of the Einstein-Weyl connection, but it is difficult to be written down in concrete examples. One might hope that, in several cases, it can be understood via the reduced Ansätze. For example, when B admits a symmetry K with the flow moving any geodesic, the holomorphic vector field X induced by K on S is non-vanishing and defines $TS(-2)$ as an extension of $\mathcal{O}(-2)$ by $\mathcal{O}(2)$. Using the reduced A_2 -Ansatz it should be possible to write down explicitly the canonical monopole in terms of the divergence, twist and acceleration defined by K . These problems need more investigation.

0.2.2 Structure of chapters.

Chapters 1. Chapters 1 is a review chapter. Here we briefly review the basic material we need from the theory of involutive structures (relations between involutive structures, homogeneous involutive structures, compatible vector bundles, involutive cohomology). We end Chapter 1 by presenting briefly the smooth Penrose transform, in order to show that the transforms we construct in the following chapters are natural extensions of it.

Chapter 2. Chapter 2 is concerned with the theory of currents (currents on oriented and non-oriented manifolds, currents with values in vector bundles, operations with currents). This material is known in the literature (see [9] or [17]), but not in the form we shall present here. We also introduce the notion of involutive cohomology for currents (which doesn't appear in the literature), which provides the link with the material presented in Chapter 1. Our treatment of currents is particularly well-suited for the Penrose transform for currents which we shall develop in Chapter 4.

Chapter 3. In Chapter 3 we construct a Penrose transform for smooth compactly supported cohomology (and we discuss the Serre "duality" between compactly supported cohomology and ordinary cohomology, as mentioned above). We discuss a compactly supported holomorphic Penrose transform, which relates compactly supported Dolbeault cohomology groups on a complex manifold Z to kernels (or cokernels) of differential operators acting between top degree compactly supported cohomology groups on another complex manifold X . Under mild conditions (see [27]) compactly supported cohomology groups are duals of usual cohomology groups, and we could guess that the compactly supported holomorphic Penrose transform we develop is dual to the holomorphic Penrose transform. Further investigation in this direction is needed.

Chapter 4. In Chapter 4 we construct a Penrose transform for distribution cohomology for compact and non-compact supports.

Chapter 5. In Chapter 5 we apply the theory developed in the previous chapters to some concrete examples already existing in the literature.

Chapter 6. In Chapter 6 we define the relative involutive cohomology for an arbitrary involutive structure, and we prove that it has the excision property. We also look at the induced Penrose transform on \mathbb{R}^3 with the origin removed, and show how the theory of relative involutive cohomology could be used for other Penrose type transforms.

Chapter 7. In Chapter 7 we develop the reduced Atiyah-Ward Ansätze for 3-dimensional Einstein-Weyl spaces. We begin this chapter with a brief introduction in conformal geometry, followed by a conformally invariant treatment of the Atiyah-Ward Ansätze in 4-dimensions. The Atiyah-Ward Ansätze have been well understood (see [2] and [48]), but the form in which we present it here is particularly well suited for the process of reduction and does not appear in

the literature. We obtain the reduced Atiyah-Ward Ansätze on 3-dimensional Einstein-Weyl spaces in an explicit form, in the language of conformal geometry (using Weyl derivatives and density line bundles). The possible applications of this theory have already been mentioned above.

Chapter 1

Involutive structures and Penrose transforms.

1.1 Introduction.

The main tool of our treatment of Penrose transforms is the theory of involutive structures. A study of involutive structures together with their applications in partial differential equations can be found in [45]. Following [6] and [46] we present in this chapter the basic material we shall need about involutive structures and involutive cohomology, in order to fix our conventions and notations. The proofs we give will be very brief, but for more details the reader will be referred to the existing literature.

1.2 Involutive structures.

1.2.1 Involutive structures on manifolds.

For M a real manifold, we shall use the following conventions:

Notation.

1. TM , \mathcal{E}_M^k will always refer to the complexified tangent bundle and the k -exterior power of the complexified cotangent bundle of the manifold M . In particular, $\mathcal{E}_M = \mathcal{E}_M^0$ refers to the trivial complex line bundle whose sections are complex valued functions on M . We omit the “ M ” and write simply \mathcal{E}^k , etc, if there can be no confusion as which manifold is intended.
2. The space of smooth complex valued k -forms on M will be denoted $\mathcal{E}^k(M)$, and the space of smooth compactly supported complex valued k -forms on M will be denoted $\mathcal{D}^k(M)$. In particular $\mathcal{E}(M)$ will denote the space of

smooth complex valued functions on M and $\mathcal{D}(M)$ will denote the space of smooth compactly supported complex valued functions on M .

3. If V is a vector bundle over M , $\Gamma(M, V)$ will denote the space of smooth sections of V and $\Gamma_c(M, V)$ will denote the space of smooth compactly supported sections of V .

Definition 1.1. *An involutive structure on the manifold M is a complex sub-bundle $T^{0,1} \subset TM$ such that $[T^{0,1}, T^{0,1}] \subset T^{0,1}$ (meaning that the space of smooth sections of $T^{0,1}$ is closed under the Lie bracket).*

Definition 1.2. *Define the vector bundle $\mathcal{E}^{1,0}$ to be the annihilator of $T^{0,1}$ and define $\mathcal{E}^{0,1}$ by the exactness of*

$$0 \rightarrow \mathcal{E}^{1,0} \rightarrow \mathcal{E}^1 \rightarrow \mathcal{E}^{0,1} \rightarrow 0.$$

We write $\mathcal{E}^{p,q} = \wedge^p \mathcal{E}^{1,0} \otimes \wedge^q \mathcal{E}^{0,1}$.

Lemma 1.3. *The condition $[T^{0,1}, T^{0,1}] \subset T^{0,1}$ is equivalent to the condition $d(\mathcal{E}^{1,0}) \subset \mathcal{E}^{1,0} \wedge \mathcal{E}^1$.*

This lemma implies the existence of quotient complexes

$$\Gamma(M, \mathcal{E}) \xrightarrow{\bar{\partial}} \Gamma(M, \mathcal{E}^{0,1}) \xrightarrow{\bar{\partial}} \Gamma(M, \mathcal{E}^{0,2}) \xrightarrow{\bar{\partial}} \dots$$

and

$$\Gamma_c(M, \mathcal{E}) \xrightarrow{\bar{\partial}} \Gamma_c(M, \mathcal{E}^{0,1}) \xrightarrow{\bar{\partial}} \Gamma_c(M, \mathcal{E}^{0,2}) \xrightarrow{\bar{\partial}} \dots$$

obtained from the complex of differential forms on M and from the complex of compactly supported differential forms on M . Here $\mathcal{E}^{0,k}$ is identified with the quotient of \mathcal{E}^k by the ideal generated by $\mathcal{E}^{1,0}$ and the maps $\bar{\partial}$ of these quotient complexes are induced by the exterior derivatives on forms on M .

Definition 1.4. *The involutive cohomology $H^*(M, \mathcal{E})$ of the involutive manifold (M, \mathcal{E}) is the cohomology of the complex*

$$\Gamma(M, \mathcal{E}) \xrightarrow{\bar{\partial}} \Gamma(M, \mathcal{E}^{0,1}) \xrightarrow{\bar{\partial}} \Gamma(M, \mathcal{E}^{0,2}) \xrightarrow{\bar{\partial}} \dots$$

The compactly supported involutive cohomology $H^(M, \mathcal{E}_c)$ of the involutive manifold (M, \mathcal{E}) is the cohomology of the complex*

$$\Gamma_c(M, \mathcal{E}) \xrightarrow{\bar{\partial}} \Gamma_c(M, \mathcal{E}^{0,1}) \xrightarrow{\bar{\partial}} \Gamma_c(M, \mathcal{E}^{0,2}) \xrightarrow{\bar{\partial}} \dots$$

1.2.2 Involutive structures and compatible vector bundles.

Definition 1.5. A complex vector bundle $V \rightarrow M$ is compatible with the involutive structure \mathcal{E} (or \mathcal{E} -compatible) if there is defined a linear operator (called the partial connection)

$$\bar{\partial} : \Gamma(M, V) \rightarrow \Gamma(M, V \otimes \mathcal{E}^{0,1})$$

such that

$$\bar{\partial}(fs) = f\bar{\partial}(s) + (\bar{\partial}f)s, \quad \forall f \in \mathcal{E}(M), s \in \Gamma(M, V)$$

and such that the natural extension to

$$\bar{\partial} : \Gamma(M, V \otimes \mathcal{E}^{0,k}) \rightarrow \Gamma(M, V \otimes \mathcal{E}^{0,k+1})$$

satisfies $\bar{\partial}^2 = 0$.

Definition 1.6. Given an \mathcal{E} -compatible vector bundle $V \rightarrow M$, the involutive cohomology $H^*(M, \mathcal{E}(V))$ is the cohomology of the complex

$$\Gamma(M, V) \xrightarrow{\bar{\partial}} \Gamma(M, V \otimes \mathcal{E}^{0,1}) \xrightarrow{\bar{\partial}} \Gamma(M, V \otimes \mathcal{E}^{0,2}) \xrightarrow{\bar{\partial}} \dots$$

The compactly supported involutive cohomology $H^*(M, \mathcal{E}_c(V))$ is the cohomology of the complex

$$\Gamma_c(M, V) \xrightarrow{\bar{\partial}} \Gamma_c(M, V \otimes \mathcal{E}^{0,1}) \xrightarrow{\bar{\partial}} \Gamma_c(M, V \otimes \mathcal{E}^{0,2}) \xrightarrow{\bar{\partial}} \dots$$

Examples.

1. For any involutive structure \mathcal{E} the bundles $\mathcal{E}^{p,0}$ are \mathcal{E} -compatible, with the partial connection induced by the exterior derivative.
2. An involutive structure $T^{0,1} \subset TM$ is a complex structure iff $TM = T^{0,1} \oplus \overline{T^{0,1}}$ (here $\overline{T^{0,1}}$ denotes the image of $T^{0,1}$ through the natural conjugation of TM). In this case the compatible vector bundles are the holomorphic ones and the involutive cohomology is the Dolbeault cohomology.
3. If $T^{0,1} \cap \overline{T^{0,1}} = \{0\}$ then the involutive structure is a CR-structure. This is the structure acquired by a real hypersurface in a complex manifold. Our definition is wider than most since it includes “higher codimension” cases and it includes also complex manifolds. The involutive cohomology is often known in this case as ∂_b -cohomology (the “ b ” standing for boundary).

1.2.3 Homogeneous involutive structures and compatible vector bundles.

On homogeneous manifolds homogeneous involutive structures and homogeneous compatible vector bundles can be described in terms of Lie algebras and representation theory (see for example [46], page 76 for a detailed exposition, [6] or [5] for a brief exposition). For simplicity we shall restrict to the homogeneous manifolds $G/H := \{gH, g \in G\}$ with H connected. The complexified Lie algebra of G will be denoted by \mathfrak{g} and the complexified Lie algebra of H will be denoted by \mathfrak{h} .

Following [26] or [10] we recall that homogeneous vector bundles on G/H are defined by representations of H . More precisely, let

$$\rho : H \rightarrow \text{Aut}(V_0)$$

be a complex representation of H on the complex vector space V_0 . We define $V := G \times_H V_0$ to be the space of orbits of the action

$$\tilde{\rho} : H \rightarrow \text{Aut}(G \times V_0)$$

given by

$$\tilde{\rho}(h)(g, v) := (gh^{-1}, \rho(h)v)$$

where $h \in H$, $(g, v) \in G \times V_0$.

There is a projection $p : V \rightarrow G/H$ induced by the canonical projection of G onto G/H . The projection p has fibers V_0 and V with the projection p is a complex vector bundle over G/H . The smooth sections of V can be identified with smooth maps

$$f : G \rightarrow V_0$$

which satisfy

$$f(gh) = \rho(h)^{-1}f(g)$$

where $g \in G$ and $h \in H$ (see [26], page 16).

Example.

1. The complexified tangent bundle of G/H is induced by the adjoint representation of H on $\mathfrak{g}/\mathfrak{h}$.

Lemma 1.7. *There is a 1-1 correspondence between homogeneous involutive structures on G/H and complex Lie algebras \mathfrak{q} such that*

$$\mathfrak{h} \subset \mathfrak{q} \subset \mathfrak{g}.$$

Proof. The involutive structure $T^{0,1}$ is defined to be the homogeneous vector subbundle of $T(G/H)$ determined by the adjoint representation of H on $\mathfrak{q}/\mathfrak{h}$. \square

Notation. An homogeneous space G/H together with an involutive structure determined by \mathbf{q} will be denoted $(G/H, \mathbf{q})$. Since \mathbf{g} is the complexification of a real Lie algebra, it has a natural conjugation. We will denote by $\bar{\mathbf{q}}$ the image of \mathbf{q} through the conjugation of \mathbf{g} .

Lemma 1.8. *The involutive manifold $(G/H, \mathbf{q})$ is a CR-manifold if $\mathbf{h} = \mathbf{q} \cap \bar{\mathbf{q}}$. It is a complex manifold if moreover $\mathbf{g} = \mathbf{q} + \bar{\mathbf{q}}$.*

We now describe homogeneous vector bundles which are compatible with an homogeneous involutive structure. For this we need the following definition.

Definition 1.9. *A (\mathbf{q}, H) -module is a complex representation ρ of H and a complex representation ρ' of \mathbf{q} such that $\rho'|_{\mathbf{h}}$ is the derivative of ρ .*

Lemma 1.10. *Homogeneous vector bundles $V = G \times_H V_0$ on G/H compatible with the involutive structure determined by \mathbf{q} are in 1-1 correspondence with (\mathbf{q}, H) -modules V_0 . The partial connection of V is the operator*

$$\bar{\partial} : \Gamma(G/H, V) \rightarrow \Gamma(G/H, (\mathbf{q}/\mathbf{h})^* \otimes V)$$

defined by the formula

$$\bar{\partial}_X(s) = X(s) + \rho'(X)s$$

where $s \in \Gamma(G/H, V)$, $X \in \mathbf{q}$ and $X(s)$ is the left invariant vector field X applied to the section s (viewed as a V_0 -valued function on G).

Proof. For a proof, see [46], page 77. Note that when $X \in \mathbf{h}$,

$$X(s) + \rho'(X)s = 0.$$

This follows by taking the derivative with respect to $h \in H$ of the equality

$$s(gh) = \rho(h^{-1})s(g)$$

where $g \in G$. □

1.2.4 Relations between involutive structures.

Let $\eta : F \rightarrow Z$ be a smooth map and \mathcal{A} an involutive structure on Z such that $\eta^*(\mathcal{A}^{1,0})$ has constant rank on F .

Lemma 1.11. *Under these conditions, the followings are true:*

1. *The bundle $\eta^*(\mathcal{A}^{1,0})$ defines an involutive structure $\eta^*(\mathcal{A})$ on F .*
2. *\mathcal{A} -compatible vector bundles on Z pull back to $\eta^*(\mathcal{A})$ -compatible vector bundles on F .*
3. *If V is a \mathcal{A} -compatible vector bundle, the pull-back on forms induces a map*

$$\eta^* : H^p(Z, \mathcal{A}(V)) \rightarrow H^p(F, \eta^*(\mathcal{A})(\eta^*V)).$$

Proof. Direct calculation. See also [6] \square

We shall often consider the particular case when $\eta : F \rightarrow Z$ is a fiber bundle and the involutive structure on Z is the trivial involutive structure: $\mathcal{A}^{1,0} := \mathcal{E}_Z^1$. Then on F we obtain the involutive structure \mathcal{C} defined by

$$\mathcal{C}^{1,0} := \eta^*(\mathcal{E}_Z^1).$$

In this special case Lemma 1.11 becomes the following

Lemma 1.12. *1. Vector bundles on Z pull-back to \mathcal{C} -compatible vector bundles on F .*

2. Suppose the fibers of η have finite dimensional de Rham cohomology. Then the k -th de Rham cohomology of the fibers of η defines a vector bundle $\mathcal{H}^k \rightarrow Z$ and the \mathcal{C} involutive cohomology in degree k is given by

$$H^k(F, \mathcal{C}(\eta^*V)) = \Gamma(Z, V \otimes \mathcal{H}^k),$$

for every vector bundle V over Z .

Proof. The first statement follows from Lemma 1.11. The second statement follows by using a ‘‘Cech-de Rham complex’’ argument. \square

The main tool of the real Penrose transform is the spectral sequence associated to two involutive structures $\mathcal{A}^{1,0} \subset \mathcal{E}^{1,0}$ on a manifold M . Suppose V is a \mathcal{A} -compatible vector bundle on M . Defining $B^1 := \mathcal{E}^{1,0}/\mathcal{A}^{1,0}$ and $B^p := \wedge^p B^1$ there is a short exact sequence

$$0 \rightarrow B^1 \rightarrow \mathcal{A}^{0,1} \rightarrow \mathcal{E}^{0,1} \rightarrow 0$$

which induces the filtration

$$K_p(\Gamma(M, V \otimes \mathcal{A}^{0,p+q})) = \Gamma(M, V \otimes B^p \otimes \mathcal{A}^{0,q})$$

of the complex

$$\Gamma(M, V) \xrightarrow{\bar{\partial}} \Gamma(M, V \otimes \mathcal{A}^{0,1}) \xrightarrow{\bar{\partial}} \Gamma(M, V \otimes \mathcal{A}^{0,2}) \rightarrow \dots$$

and the filtration

$$\tilde{K}_p(\Gamma_c(M, V \otimes \mathcal{A}^{0,p+q})) = \Gamma_c(M, V \otimes B^p \otimes \mathcal{A}^{0,q})$$

of the complex

$$\Gamma_c(M, V) \xrightarrow{\bar{\partial}} \Gamma_c(M, V \otimes \mathcal{A}^{0,1}) \xrightarrow{\bar{\partial}} \Gamma_c(M, V \otimes \mathcal{A}^{0,2}) \rightarrow \dots$$

Theorem 1.13. *Let $\mathcal{A}^{1,0} \subset \mathcal{E}^{1,0}$ be two involutive structures on the manifold M and $V \rightarrow M$ be a \mathcal{A} -compatible vector bundle. Then V is also \mathcal{E} -compatible and there are spectral sequences*

$$E_1^{p,q} = H^q(M, \mathcal{E}(B^p \otimes V)) \implies H^{p+q}(M, \mathcal{A}(V))$$

and

$$\tilde{E}_1^{p,q} = H^q(M, \mathcal{E}_c(B^p \otimes V)) \implies H^{p+q}(M, \mathcal{A}_c(V)).$$

Proof. This is just the spectral sequence of a filtered complex (see e.g.[48], page 221). See also [6]. □

1.3 The smooth Penrose transform.

We summarize here the Penrose transform as presented in [6] in a simple case. The initial data is a double fibration of smooth oriented manifolds

$$\begin{array}{ccc} & F & \\ \eta \swarrow & & \searrow \tau \\ Z & & X \end{array}$$

with the following properties:

1. There is an involutive structure \mathcal{Q} on Z which makes Z into a CR-manifold (we recall that our definition of CR-manifolds includes the complex manifolds).
2. The maps η and τ are fiber-bundle projections such that τ has compact complex fibers.
3. The map η embeds the fibers of τ as holomorphic submanifolds of Z . (A submanifold of a CR-manifold is holomorphic if the involutive structure of the CR-manifold restricts to give a complex structure on the submanifold).

Convention. We sometimes refer to Z as a “twistor space” for X .

We will consider on F the involutive structure \mathcal{E} defined by $\mathcal{E}^{1,0}$ being the annihilator of the τ -vertical vectors that are of type $(0, 1)$ with respect to the complex structure of the fibers of τ . The \mathcal{E} -cohomology is the Dolbeault cohomology of the fibers of τ parametrised over X . Let us assume that a complex \mathcal{E} -compatible vector bundle $E \rightarrow F$ is such that the dimension of the Dolbeault cohomology

$H^p(\tau^{-1}(x), E|_{\tau^{-1}(x)})$ is constant as $x \in X$ varies. Then this cohomology defines a vector bundle $\tau_*^p E \rightarrow X$.

We shall briefly recall the main steps of the smooth Penrose transform (see also [6]).

1. **Step1** Define an involutive structure \mathcal{A} on F by $\mathcal{A}^{1,0} := \eta^* \mathcal{Q}^{1,0}$. Then the \mathcal{Q} -cohomology on Z and the \mathcal{A} -cohomology on F can be related as follows. Introduce first the involutive structure $\mathcal{C}^{1,0} = \eta^* \mathcal{E}_Z^1$. Then on F we have the short exact sequence

$$0 \rightarrow \eta^* \mathcal{Q}^{0,1} \rightarrow \mathcal{A}^{0,1} \rightarrow \mathcal{C}^{0,1} \rightarrow 0$$

which induces the filtration

$$K_p(\Gamma(F, \mathcal{A}^{0,p+q} \otimes \eta^* V)) = \Gamma(F, \mathcal{A}^{0,q} \otimes \eta^*(\mathcal{Q}^{0,p} \otimes V))$$

of the complex

$$\Gamma(F, \eta^* V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \Gamma(F, \eta^* V \otimes \mathcal{A}^{0,1}) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \Gamma(F, \eta^* V \otimes \mathcal{A}^{0,2}) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \dots$$

The spectral sequence (for non-compact supports) of Theorem 1.13 gives

$$E_1^{p,q} = H^q(F, \mathcal{C}(\eta^* V \otimes \eta^* \mathcal{Q}^{0,p})) \implies H^{p+q}(F, \mathcal{A}(\eta^* V)).$$

Using Lemma 1.12 we obtain

$$E_1^{p,q} = \Gamma(Z, V \otimes \mathcal{Q}^{0,p} \otimes \mathcal{H}^q)$$

and passing to the second order of the spectral sequence we get

$$E_2^{p,q} = H^p(Z, \mathcal{Q}(V \otimes \mathcal{H}^q)).$$

When the fibers of η are contractible, the spectral sequence degenerates to the second order and it becomes a series of isomorphisms

$$H^p(Z, \mathcal{Q}(V)) \cong H^p(F, \mathcal{A}(\eta^* V))$$

with the maps induced by the pull-back of a representative form.

2. **Step2** For the second step the following lemma is essential.

Lemma 1.14. *The inclusion $\mathcal{A}^{1,0} \subset \mathcal{E}^{1,0}$ holds.*

Proof. Let $i : \tau^{-1}(x) \rightarrow F$ be the inclusion, v a tangent vector to the fiber $\tau^{-1}(x)$ which is also of type $(0, 1)$ in the complex structure of the fiber, and $\omega \in \mathcal{Q}^{1,0}$. Since the map η embeds $\tau^{-1}(x)$ as an holomorphic submanifold of (Z, \mathcal{Q}) , $(\eta \circ i)^*(\omega)$ is of type $(1, 0)$. This implies that $\eta^*(\omega)(i_*(v)) = 0$, or equivalently that $\eta^*(\omega) \in \mathcal{E}^{1,0}$. \square

Using this lemma we can define the vector bundle $B^1 \rightarrow F$ by $B^1 := \mathcal{E}^{1,0}/\mathcal{A}^{1,0}$. We obtain the short exact sequence

$$0 \rightarrow B^1 \rightarrow \mathcal{A}^{0,1} \rightarrow \mathcal{E}^{0,1} \rightarrow 0$$

which induces the filtration

$$K_p(\Gamma(F, \mathcal{A}^{0,p+q} \otimes \eta^*V)) = \Gamma(F, B^p \otimes \mathcal{A}^{0,q} \otimes \eta^*V)$$

of the complex

$$\Gamma(F, \eta^*V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \Gamma(F, \mathcal{A}^{0,1} \otimes \eta^*V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \Gamma(F, \mathcal{A}^{0,2} \otimes \eta^*V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \dots$$

The spectral sequence (for non-compact supports) of Theorem 1.13 applied to the involutive structures \mathcal{A} and \mathcal{E} is

$$E_1^{p,q} = H^q(F, \mathcal{E}(\eta^*V \otimes B^p)) \implies H^{p+q}(F, \mathcal{A}(\eta^*V)).$$

3. **Step3** We use the fact that the \mathcal{E} -cohomology is the Dolbeault cohomology along the fibers of τ . Defining

$$V_{p,q} = \tau_*^q(\eta^*V \otimes B^p)$$

we thus have

$$E_1^{p,q} = \Gamma(X, V_{p,q}) \implies H^{p+q}(F, \mathcal{A}(\eta^*V)).$$

Combining the above steps in the case when the fibers of η are contractible, we arrive at the smooth Penrose transform which is the spectral sequence

$$E_1^{p,q} = \Gamma(X, V_{p,q}) \implies H^{p+q}(Z, \mathcal{Q}(V)).$$

It translates data from Z to X .

Chapter 2

Currents on manifolds.

While the theory of currents is well-understood (see for example [9] or [17]) there does not seem to be a treatment in the literature suited to our needs. The aim of this chapter is to fix our conventions and notations for the theory of currents. The material from this chapter will be used in a later chapter where we construct a Penrose transform for currents.

2.1 Currents on oriented manifolds.

In this section all the manifolds will be considered to be oriented.

2.1.1 Definitions and properties.

Let M be an oriented real m -dimensional manifold. We recall that $\mathcal{D}^k(M)$ denotes the space of compactly supported k -forms on M . On $\mathcal{D}^k(M)$ we consider the uniform topology which is defined as follows.

Definition 2.1. *Let $f_n \in \mathcal{D}^k(M)$, $n = 1, 2, 3 \dots$ be an arbitrary sequence and $f \in \mathcal{D}^k(M)$. We say that $f_n \rightarrow f$ in the uniform topology if there is a compact subset K of M such that the support of all f_n and of f is included in K , and the coefficients of $f_n - f$ in an arbitrary local chart of M , together with all their partial derivatives converge uniformly to 0.*

Definition 2.2. *A current of degree p (or a p -current) on M is a continuous linear functional defined on the space $\mathcal{D}^{m-p}(M)$ (on which we consider the uniform topology).*

Example. Any k -dimensional oriented submanifold S of M which is a closed subset of M determines a current of integration $[S]$ on M of degree $m - k$, defined by restricting a compactly supported k -form on M to S and then integrating over S . (Because S is a closed subset of M , a compactly supported form on M

restricts to a compactly supported form on S). In particular, there is a current of integration $[M]$ of degree 0 determined by the manifold M itself.

Notation. If ω is a p -current and f is a $(m-p)$ compactly supported form, we write the pairing between ω and f as (ω, f) . The set of currents of degree p on M will be denoted $\mathcal{J}^p(M)$.

Definition 2.3. If $\omega \in \mathcal{J}^p(M)$ and $f \in \mathcal{E}^k(M)$, define $\omega \wedge f \in \mathcal{J}^{p+k}(M)$ by

$$(\omega \wedge f, g) = (\omega, f \wedge g),$$

where $g \in \mathcal{D}^{m-k-p}(M)$. In particular $\mathcal{J}^p(M)$ is a sheaf of $\mathcal{E}^0(M)$ modules.

Notation. If $\omega \in \mathcal{J}^0(M)$ and $f \in \mathcal{E}^k(M)$ then $\omega \wedge f$ will be denoted simply $f\omega$.

Lemma 2.4. Every smooth p -form g on M determines a p -current \hat{g} on M defined by

$$(\hat{g}, f) = \int_M g \wedge f,$$

where $f \in \mathcal{D}^{m-p}(M)$.

Example. The current of integration $[M]$ is determined by the constant function 1. More generally, $\hat{f} = f[M]$ for every $f \in \mathcal{E}^k(M)$.

Definition 2.5. The support of $\omega \in \mathcal{J}^p(M)$ is the smallest closed subset K of M such that

$$(\omega, g) = 0$$

when $g \in \mathcal{D}^{m-p}(M \setminus K)$.

Notation. We shall use the notation $\mathcal{J}_c^p(M)$ for the set of p -currents on M with compact support.

Remark. Elements of $\mathcal{J}_c^p(M)$ can also be seen as continuous linear functionals on $\mathcal{E}^{m-p}(M)$, where on the space $\mathcal{E}^{m-p}(M)$ we consider the topology of uniform convergence of forms together with all partial derivatives on compact subsets (see [20], page 35).

2.1.2 Operations with currents.

By duality, some operations on compactly supported forms induce operations on currents which are compatible with the inclusion of forms in currents. We will define in this way the push-forward, the pull-back and the exterior derivatives on currents.

Assumption. Unless otherwise specified, $\sigma : M \rightarrow N$ will be a smooth proper map between the oriented manifolds M and N of dimension m and n respectively. Let $m = n + s$, so that s is the fiber dimension of a fiber-bundle or minus the codimension for the inclusion of a submanifold.

Definition 2.6. *The push-forward on currents is the map*

$$\sigma_* : \mathcal{J}^p(M) \rightarrow \mathcal{J}^{p-s}(N)$$

defined by

$$(\sigma_*\omega, f) = (\omega, \sigma^*f)$$

where $f \in \mathcal{D}^{m-p}(N)$. (We put $\sigma_* = 0$ if we do not have $0 \leq p - s \leq n$).

Remark. When ω has compact support, the push forward $\sigma_*(\omega)$ is well defined even if σ is not a proper map.

Lemma 2.7. *The push-forward on currents behaves covariantly with respect to the composition of maps (that is, if σ_1 and σ_2 are smooth proper maps between two manifolds such that the composition $\sigma_1 \circ \sigma_2$ has sense, then $(\sigma_1 \circ \sigma_2)_* = (\sigma_1)_* \circ (\sigma_2)_*$) and satisfies*

$$\sigma_*(\omega \wedge \sigma^*g) = (\sigma_*\omega) \wedge g$$

for every current ω on M and form g on N .

In order to define the pull-back on currents we recall first the definition of the fiber integral (see also [11], page 61).

Definition 2.8. *Let $\sigma : M \rightarrow N$ be a fiber-bundle of fiber dimension s . The fiber integral of a form $f \in \mathcal{D}^{k+s}(M)$ is the form $\int_\sigma f \in \mathcal{D}^k(N)$ characterised by the relation*

$$\int_N \left(\left(\int_\sigma f \right) \wedge g \right) = \int_M (f \wedge \sigma^*g), \quad \forall g \in \mathcal{E}^{n-k}(N).$$

Lemma 2.9. *Let $\sigma : M \rightarrow N$ be a fiber-bundle of fiber dimension s . If $f \in \mathcal{D}^{k+s}(M)$, then*

$$d \int_\sigma f = (-1)^s \int_\sigma df$$

and

$$\int_{\sigma} (f \wedge \sigma^* g) = \left(\int_{\sigma} f \right) \wedge g.$$

Also, if $g \in \mathcal{D}^{k+s}(M)$ then

$$\sigma_* \hat{g} = \widehat{\int_{\sigma} g}.$$

Proof. The proof is easy and we will not go through all the details. The first equality is based on the definition of the fiber integral and on the Stokes' theorem. The second equality follows from the definition of the fiber integral. To prove the last equality let $h \in \mathcal{D}^{n-k}(N)$. Then

$$\begin{aligned} (\sigma_* \hat{g}, h) &= (\hat{g}, \sigma^* h) \\ &= \int_M (g \wedge \sigma^* h) \\ &= \int_N \left(\left(\int_{\sigma} g \right) \wedge h \right) \\ &= \left(\widehat{\int_{\sigma} g}, h \right) \end{aligned}$$

□

Notation. We will sometimes use the notation $\sigma_* g$ for $\int_{\sigma} g$ when $g \in \mathcal{D}^k(M)$ and $\sigma : M \rightarrow N$ is a fiber-bundle. This notation is justified by the equality $\sigma_* \hat{g} = \widehat{\int_{\sigma} g}$ which was proved in the above lemma.

Definition 2.10. Let $\sigma : M \rightarrow N$ be a fiber-bundle of fiber dimension s . The pull-back on currents is the map

$$\sigma^* : \mathcal{J}^p(N) \rightarrow \mathcal{J}^p(M)$$

defined by

$$(\sigma^* \omega, g) = (-1)^{sp} (\omega, \int_{\sigma} g).$$

where $\omega \in \mathcal{J}^p(N)$ and $g \in \mathcal{D}^{m-p}(M)$.

Lemma 2.11. The pull-back has the following properties:

1. It commutes with the inclusion of forms in currents.
2. If ω is a current on N and f is a form on N , then

$$\sigma^*(\omega \wedge f) = \sigma^* \omega \wedge \sigma^* f.$$

Proof. To prove the first statement consider $f \in \mathcal{E}^p(N)$ and $g \in \mathcal{D}^{m-p}(M)$. Then

$$\begin{aligned}
(\sigma^*(\hat{f}), g) &= (-1)^{sp} (\hat{f}, \int_{\sigma} g) \\
&= (-1)^{sp} \int_N (f \wedge \int_{\sigma} g) \\
&= (-1)^{sp+p(m-p-s)} \int_N \left(\left(\int_{\sigma} g \right) \wedge f \right) \\
&= (-1)^{p(m-p)} \int_M (g \wedge \sigma^*(f)) \\
&= \int_M (\sigma^*(f) \wedge g) \\
&= \widehat{(\sigma^*(f), g)}
\end{aligned}$$

To prove the second statement consider $\omega \in \mathcal{J}^p(N)$, $f \in \mathcal{E}^k(N)$ and $g \in \mathcal{D}^{m-(p+k)}(M)$. Then

$$\begin{aligned}
(\sigma^*\omega \wedge \sigma^*f, g) &= (\sigma^*\omega, \sigma^*f \wedge g) \\
&= (-1)^{sp} \left(\omega, \int_{\sigma} (\sigma^*f \wedge g) \right) \\
&= (-1)^{s(p+k)} \left(\omega, f \wedge \int_{\sigma} g \right) \\
&= (-1)^{s(p+k)} \left(\omega \wedge f, \int_{\sigma} g \right) \\
&= (\sigma^*(\omega \wedge f), g)
\end{aligned}$$

where we have used the equality

$$\int_{\sigma} (\sigma^*f \wedge g) = (-1)^{sk} f \wedge \int_{\sigma} g$$

□

Definition 2.12. *The exterior derivative on currents is the map*

$$d : \mathcal{J}^p(M) \rightarrow \mathcal{J}^{p+1}(M)$$

defined by

$$(d\omega, f) = (-1)^{p+1} (\omega, df),$$

where $\omega \in \mathcal{J}^p(M)$ and $f \in \mathcal{D}^{m-p-1}(M)$.

Lemma 2.13. *The exterior derivative has the following properties:*

1. *It commutes with the inclusion of forms in currents.*

2. If ω is a current on M then

$$d\sigma_*\omega = (-1)^s \sigma_*d\omega.$$

3. It commutes with the pull-back on currents.

Proof. The first statement is based on Stokes' theorem. To prove the second statement let $\omega \in \mathcal{J}^p(M)$. Then $\sigma_*(\omega) \in \mathcal{J}^{p-s}(N)$ and let $f \in \mathcal{D}^{n-p+s-1}(N)$. We obtain

$$\begin{aligned} (d\sigma_*\omega, f) &= (-1)^{p-s+1} (\sigma_*\omega, df) \\ &= (-1)^{p-s+1} (\omega, \sigma^*df) \\ &= (-1)^{p-s+1} (\omega, d\sigma^*f) \\ &= (-1)^s (d\omega, \sigma^*f) \\ &= (-1)^s (\sigma_*d\omega, f) \end{aligned}$$

The third statement follows in a similar way, using the first property of the fiber integral, given in Lemma 2.9. \square

2.1.3 Currents with values in vector bundles.

Let V be a rank k (complex) vector bundle over the m -dimensional oriented manifold M . On the space $\Gamma_c(M, \mathcal{E}^{m-p} \otimes V^*)$ we consider the uniform topology, defined as in Definition 2.1.

Definition 2.14. *A p -current on M with values in V is a continuous linear functional defined on the space $\Gamma_c(M, \mathcal{E}^{m-p} \otimes V^*)$ (on which we consider the uniform topology).*

Notation. We will denote by $\mathcal{J}^p(M, V)$ the space of p -currents on M with values in V .

Definition 2.15. *If $\omega \in \mathcal{J}^p(M)$ and $s \in \Gamma(M, V)$ define $\omega \cdot s \in \mathcal{J}^p(M, V)$ by the formula*

$$(\omega \cdot s, \gamma \otimes s^*) := (\omega, s^*(s)\gamma)$$

where $\gamma \in \mathcal{D}^{m-p}(M)$ and $s^* \in \Gamma(M, V^*)$.

Lemma 2.16. *Every $\omega \in \mathcal{J}^p(M, V)$ can be written locally in the form*

$$\sum_{i=1}^k \omega_i \cdot s_i,$$

where $\omega_i \in \mathcal{J}^p(U)$ and $\{s_1, \dots, s_k\}$ is a frame of V over the open subset U of M .

Proof. Consider $\{s_1^*, \dots, s_k^*\}$ the dual frame of $\{s_1, \dots, s_k\}$. For every $\omega \in \mathcal{J}^p(M, V)$ define $\omega_i \in \mathcal{J}^p(U)$ on U by

$$(\omega_i, \tau) := (\omega, \tau \otimes s_i^*),$$

where $\tau \in \mathcal{D}^{m-p}(U)$. It is a simple calculation to check that the equality

$$\omega = \sum_{i=1}^k \omega_i \cdot s_i$$

holds on U . □

The operations defined on currents extend to currents with values in appropriate vector bundles. We shall define the push-forward and the pull-back of currents with values in vector bundles.

Definition 2.17. *Let $\sigma : M \rightarrow N$ be a smooth proper map and W a vector bundle over N . Suppose M is of dimension m , N is of dimension n and $s := m - n$. The push-forward σ_* is the map*

$$\sigma_* : \mathcal{J}^p(M, \sigma^*W) \rightarrow \mathcal{J}^{p-s}(N, W)$$

defined by

$$(\sigma_*\omega, f) = (\omega, \sigma^*f)$$

where $f \in \Gamma_c(N, \mathcal{E}^{m-p} \otimes W^*)$. (We put $\sigma_* = 0$ if we do not have $0 \leq p - s \leq n$).

An immediate consequence of this definition is the following

Lemma 2.18. *If ω is a current on M and s is a section of W , then*

$$\sigma_*(\omega \cdot \sigma^*(s)) = \sigma_*(\omega) \cdot s.$$

In order to define the pull-back on currents with values in vector bundles we first notice that the fiber integral \int_σ (where $\sigma : M \rightarrow N$ is a fiber-bundle) can be extended in an obvious way to act on compactly supported forms on M with values in vector bundles on M which are pull-backs of vector bundles on N .

Definition 2.19. *Let $\sigma : M \rightarrow N$ be a fiber-bundle with fiber dimension s and W a vector bundle over N . The pull-back σ^* is the map*

$$\sigma^* : \mathcal{J}^p(N, W) \rightarrow \mathcal{J}^p(M, \sigma^*(W)).$$

defined by

$$(\sigma^*\omega, g) = (-1)^{sp}(\omega, \int_\sigma g).$$

where $\omega \in \mathcal{J}^p(N, W)$ and $g \in \Gamma_c(M, \mathcal{E}^{m-p} \otimes \sigma^*W^*)$.

An immediate consequence of this definition is the following

Lemma 2.20. *If ω is a current on N and s is a section of W then*

$$\sigma^*(\omega \cdot s) = \sigma^*(\omega) \cdot \sigma^*(s).$$

2.2 Currents on non-oriented manifolds.

Let M be a manifold of dimension m , not necessarily orientable. We recall that the orientation bundle of M (see [11], page 84) is the line bundle $\tilde{\mathcal{E}}_M$ on M whose transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \{+1, -1\}$$

are defined by

$$g_{\alpha\beta} = \text{sign}(\text{Jacob}(\chi_\alpha \circ \chi_\beta^{-1}))$$

where $\{(U_\alpha, \chi_\alpha)_\alpha\}$ is a trivialisaton atlas of M and

$$\text{Jacob}(\chi_\alpha \circ \chi_\beta^{-1})$$

denotes the Jacobian of the transition functions $\chi_\alpha \circ \chi_\beta^{-1}$.

When M is orientable $\tilde{\mathcal{E}}_M$ is trivial and a choice of trivialisaton is equivalent to a choice of orientation. In this subsection we will not impose any conditions on the orientability of M and we will sketch the theory of currents in this general framework.

Notation.

1. If V be a vector bundle over M the tensor product $V \otimes \tilde{\mathcal{E}}_M$ will be denoted \tilde{V} .
2. The space of compactly supported k -forms on M with values in $\tilde{\mathcal{E}}_M$ will be denoted $\tilde{\mathcal{D}}^k(M)$.

Definition 2.21. *A p -current on M with values in V is a continuous linear functional defined on $\Gamma_c(M, \mathcal{E}^{m-p} \otimes \tilde{V}^*)$ (on which we consider the uniform topology).*

Definition 2.22. *A twisted p -current on M with values in V is an element of $\mathcal{J}^p(M, \tilde{V})$. Equivalently, it is a continuous linear functional defined on the space $\Gamma_c(M, \mathcal{E}^{m-p} \otimes V^*)$.*

On a non-oriented manifold, we can still define in the same way the pull-back, the push-forward and the exterior derivative on currents. The only difference which appears is that sometimes the currents will be twisted by an orientation bundle. More precisely, we consider again $\sigma : M \rightarrow N$ to be a proper map acting between the manifolds M and N of dimension m and n respectively, but this time we do not take M or N to be oriented. The orientation bundle of σ is by definition the bundle $\tilde{\mathcal{E}}_M \otimes \sigma^* \tilde{\mathcal{E}}_N$, and will be denoted $\tilde{\mathcal{E}}_\sigma$. Let $s := m - n$ and W a vector bundle over N . The following facts are true:

1. The push-forward is defined like in the oriented case and acts between twisted currents:

$$\sigma_* : \mathcal{J}^k(M, \widetilde{\sigma^*W}) \rightarrow \mathcal{J}^{k-s}(N, \widetilde{W}).$$

Tensoring this map by $\sigma^*(\tilde{\mathcal{E}}_N)$ we obtain the push-forward

$$\sigma_* : \mathcal{J}^k(M, \tilde{\mathcal{E}}_\sigma \otimes \sigma^*W) \rightarrow \mathcal{J}^{k-s}(N, W).$$

If $\tilde{\mathcal{E}}_\sigma$ is trivial and a choice of trivialisation is fixed, then the push-forward maps (untwisted) currents to (untwisted) currents.

2. To define the pull-back on currents we first notice that there is a fiber-integral \int_σ for any fiber bundle $\sigma : M \rightarrow N$, not necessarily oriented. If s is the fiber dimension of σ then \int_σ maps (compactly supported) twisted p -forms on M to (compactly supported) twisted $(p - s)$ -forms on N .

The pull-back on currents is defined like in the oriented case and it acts between untwisted currents:

$$\sigma^* : \mathcal{J}^k(N, W) \rightarrow \mathcal{J}^k(M, \sigma^*W).$$

In terms of twisted currents, it is the map

$$\sigma^* : \mathcal{J}^k(N, \widetilde{W}) \rightarrow \mathcal{J}^k(M, \tilde{\mathcal{E}}_\sigma \otimes \widetilde{\sigma^*W})$$

given by the dual of the fiber integral of forms on M which are twisted by $\tilde{\mathcal{E}}_\sigma$.

3. The exterior derivative of twisted currents

$$d : \mathcal{J}^p(M, \tilde{\mathcal{E}}_M) \rightarrow \mathcal{J}^{p+1}(M, \tilde{\mathcal{E}}_M)$$

is defined as in Definition 2.12.

Using a locally constant trivialisation of $\tilde{\mathcal{E}}_M$ induced by an atlas of M , the exterior derivative can be defined on forms twisted by $\tilde{\mathcal{E}}_M$ (see [11], page 85). It follows that there is also an exterior derivative

$$d : \mathcal{J}^p(M) \rightarrow \mathcal{J}^{p+1}(M).$$

2.3 Currents and involutive structures.

Let M be a manifold (not necessarily orientable) of dimension m .

Lemma 2.23. *Let V and W be vector bundles over M and*

$$D : \Gamma(M, V) \rightarrow \Gamma(M, W)$$

a differential operator. There is an operator

$$D^* : \Gamma_c(M, \mathcal{E}^m \otimes \tilde{W}^*) \rightarrow \Gamma_c(M, \mathcal{E}^m \otimes \tilde{V}^*)$$

which satisfies

$$\int_M Ds \cdot \omega = \int_M s \cdot D^*\omega, \quad \forall s \in \Gamma(M, V), \quad \forall \omega \in \Gamma_c(M, \mathcal{E}^m \otimes \tilde{W}^*).$$

(Here “ \cdot ” denotes the contraction between a form with values in a vector bundle and a section of the dual bundle).

Definition 2.24. *The map*

$$(D^*)' : \mathcal{J}^0(M, V) \rightarrow \mathcal{J}^0(M, W)$$

defined by

$$((D^*)'(\omega), \gamma) := (\omega, D^*(\gamma))$$

where $\omega \in \mathcal{J}^0(M, V)$ and $\gamma \in \Gamma_c(M, \mathcal{E}^m \otimes \tilde{W}^)$ is called the differential operator induced by D at the level of 0-currents.*

Remarks.

1. A simple calculation shows that the diagram

$$\begin{array}{ccc} \Gamma(M, V) & \xrightarrow{D} & \Gamma(M, W) \\ \downarrow & & \downarrow \\ \mathcal{J}^0(M, V) & \xrightarrow{(D^*)'} & \mathcal{J}^0(M, W) \end{array}$$

(where the vertical arrows denote the natural inclusions) is commutative, so the operator

$$(D^*)' : \mathcal{J}^0(M, V) \rightarrow \mathcal{J}^0(M, W)$$

is an extension of the operator

$$D : \Gamma(M, V) \rightarrow \Gamma(M, W).$$

2. Since the operator $(D^*)'$ maps compactly supported currents to compactly supported currents there is an induced map

$$(D^*)' : \mathcal{J}_c^0(M, V) \rightarrow \mathcal{J}_c^0(M, W).$$

Notation. In order to simplify the notations, the extension of a differential operator D at the level of 0-currents will also be denoted by D .

Now we are in position to define the distribution involutive cohomology.

Definition 2.25. Let (M, \mathcal{A}) be an involutive manifold and $V \rightarrow M$ an \mathcal{A} -compatible vector bundle with the partial connection $\bar{\partial}_{\mathcal{A}}$. The complex

$$\mathcal{J}^0(M, V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \mathcal{J}^0(M, \mathcal{A}^{0,1} \otimes V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \mathcal{J}^0(M, \mathcal{A}^{0,2} \otimes V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \dots$$

induced by the complex

$$\Gamma(M, V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \Gamma(M, \mathcal{A}^{0,1} \otimes V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \Gamma(M, \mathcal{A}^{0,2} \otimes V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \dots$$

defines the \mathcal{A} -distribution cohomology groups $H^p(M, \mathcal{A}'(V))$.

The sub-complex

$$\mathcal{J}_c^0(M, V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \mathcal{J}_c^0(M, \mathcal{A}^{0,1} \otimes V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \mathcal{J}_c^0(M, \mathcal{A}^{0,2} \otimes V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \dots$$

of the complex

$$\mathcal{J}^0(M, V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \mathcal{J}^0(M, \mathcal{A}^{0,1} \otimes V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \mathcal{J}^0(M, \mathcal{A}^{0,2} \otimes V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \dots$$

defines the \mathcal{A} -compactly supported distribution cohomology groups $H^p(M, \mathcal{A}'_c(V))$.

2.4 A duality formula.

The following result will be used at the Penrose transform for compactly supported currents.

Theorem 2.26. Let V be a \mathcal{A} -compatible vector bundle on the m -dimensional manifold M . Suppose the rank of $\mathcal{A}^{1,0}$ is n . Then there is an isomorphism

$$T_k : \mathcal{E}^m \otimes (\mathcal{A}^{0,k} \otimes V)^* \rightarrow \mathcal{A}^{n, m-n-k} \otimes V^*$$

which satisfies

$$\omega \wedge T_k(\gamma \otimes \tau) = \tau(\omega)\gamma,$$

$\forall \tau \in (\mathcal{A}^{0,k} \otimes V)^*$, $\omega \in \mathcal{A}^{0,k} \otimes V$ and $\gamma \in \mathcal{E}^m$.

Via this isomorphism the partial connection at the level of currents

$$\bar{\partial}_{\mathcal{A}} : \mathcal{J}^0(M, \mathcal{A}^{0, k-1} \otimes V) \rightarrow \mathcal{J}^0(M, \mathcal{A}^{0, k} \otimes V)$$

is the dual of the operator

$$(-1)^k \bar{\partial}_{\mathcal{A}} : \Gamma_c(M, \mathcal{A}^{n, m-n-k} \otimes \tilde{V}^*) \rightarrow \Gamma_c(M, \mathcal{A}^{n, m-n-k+1} \otimes \tilde{V}^*)$$

where $\bar{\partial}_{\mathcal{A}}$ is the partial connection which makes the bundle $\mathcal{A}^{n,0} \otimes \tilde{V}^*$ compatible with the involutive structure \mathcal{A} .

Proof. For the first statement we notice that

$$\begin{aligned}
\mathcal{E}^m \otimes (\mathcal{A}^{0,k})^* &= \mathcal{A}^{n,m-n} \otimes (\mathcal{A}^{0,k})^* \\
&= \mathcal{A}^{n,0} \otimes \mathcal{A}^{0,m-n} \otimes (\mathcal{A}^{0,k})^* \\
&= \mathcal{A}^{n,0} \otimes \mathcal{A}^{0,m-n-k} \\
&= \mathcal{A}^{n,m-n-k}
\end{aligned}$$

The second statement also follows from a simple calculation. \square

Remark. Consider \mathcal{A} an involutive structure on the m -dimensional manifold M . Theorem 2.26 provides another way of understanding the currents with values in $\mathcal{A}^{0,k}$. If the rank of $\mathcal{A}^{1,0}$ is n , then there is a canonical identification

$$\mathcal{A}^{n,m-n-k} \cong \mathcal{A}^{n,0} \wedge \mathcal{E}^{m-n-k}$$

where by $\mathcal{A}^{n,0} \wedge \mathcal{E}^{m-n-k}$ we mean the ideal generated by $\mathcal{A}^{n,0}$ in the space of $(m-k)$ -forms. Elements of $\mathcal{J}^0(M, \mathcal{A}^{0,k})$ become elements of $(\Gamma_c(M, \mathcal{A}^{n,0} \wedge \mathcal{E}^{m-n-k}))'$. On the other hand, using the Hahn-Banach theorem (see [12], page 111) every element of the space $\Gamma_c(M, \mathcal{A}^{n,0} \wedge \mathcal{E}^{m-n-k})'$ can be (continuously) extended to an element of the space $(\mathcal{D}^{m-k}(M))'$, that is to a k -current on M . We deduce that 0-currents on M with values in $\mathcal{A}^{0,k}$ are restrictions to $\Gamma_c(M, \mathcal{A}^{n,0} \wedge \mathcal{E}^{m-n-k})$ of k -currents on M .

Examples.

1. Consider \mathcal{A} the trivial involutive structure on M defined by $\mathcal{A}^{1,0} := 0$. Then $\mathcal{A}^{0,k} = \mathcal{E}^k$ and the partial connection $\bar{\partial}_{\mathcal{A}}$ is just the exterior derivative d . Using Theorem 2.26 we see that

$$\mathcal{J}^0(M, \mathcal{E}^k) \cong \mathcal{J}^k(M)$$

and that under this isomorphism the extension

$$d : \mathcal{J}^0(M, \mathcal{E}^k) \rightarrow \mathcal{J}^0(M, \mathcal{E}^{k+1})$$

of the exterior derivative

$$d : \mathcal{E}^k(M) \rightarrow \mathcal{E}^{k+1}(M)$$

is the exterior derivative

$$d : \mathcal{J}^k(M) \rightarrow \mathcal{J}^{k+1}(M)$$

as defined in Definition 2.12.

2. Consider the fiber-bundle $\eta : F \rightarrow Z$ and the involutive structure $\mathcal{C}^{1,0} = \eta^*\mathcal{E}_Z^1$ as in the pull-back step of the smooth Penrose transform (see Chapter 1, Section 1.3). Suppose that the dimension of Z is m and that the dimension of the typical fiber of η is d . Using Theorem 2.26 there is an isomorphism

$$\mathcal{J}^0(F, \mathcal{C}^{0,k}) \cong (\Gamma_c(F, \mathcal{C}^{m,d-k}))'$$

and the partial connection

$$\bar{\partial}_c : \Gamma(F, \mathcal{C}^{0,k-1}) \rightarrow \Gamma(F, \mathcal{C}^{0,k})$$

extends to the level of currents to the operator

$$\bar{\partial}_c : \mathcal{J}^0(F, \mathcal{C}^{0,k-1}) \rightarrow \mathcal{J}^0(F, \mathcal{C}^{0,k})$$

defined by

$$(\bar{\partial}_c(\omega), \gamma) = (-1)^k(\omega, \bar{\partial}_c(\gamma))$$

where $\omega \in \Gamma_c(F, \mathcal{C}^{m,d-k})$. Thus we see that the operator induced by $\bar{\partial}_c$ at the level of currents is essentially the dual of the exterior derivative along the fibers of η acting on compactly supported forms of top degree in the base direction.

3. Similar considerations hold for the fiber-bundle $\tau : F \rightarrow X$ and the involutive structure \mathcal{E} from the push-forward step of the smooth Penrose transform (see Chapter 1, Section 1.3). Recall that τ has a complex structure on each fiber which varies smoothly with respect to the base, and that the involutive structure $\mathcal{E}^{1,0}$ on F is defined to be the annihilator of the τ -vertical vectors which are $(0,1)$ vectors along the fibers of τ . Using Theorem 2.26 we see that the operator at the level of currents induced by the partial connection $\bar{\partial}_c$ becomes essentially the dual of the $\bar{\partial}$ operator along the fibers of τ , applied to compactly supported forms of top degree in the base variable and of top degree in the holomorphic fiber variable.

Chapter 3

The Penrose transform for compactly supported cohomology

3.1 Introduction.

In this chapter we construct a Penrose transform which maps compactly supported cohomology groups on a manifold Z to kernels (or cokernels) of differential operators defined on compactly supported fields on another manifold X . The manifolds Z and X are related by a double fibration

$$\begin{array}{ccc} & F & \\ \eta \swarrow & & \searrow \tau \\ Z & & X \end{array}$$

satisfying conditions similar to those from Chapter 1, Section 1.3.

We then look at the Poincaré “duality” pairing between the ordinary cohomology and the compactly supported cohomology on Z , and how it translates into the Penrose transforms: it becomes an algebraic operation on fields over X followed by integration.

We end this chapter by constructing a compactly supported Penrose transform for the holomorphic category. We haven’t studied this transform in detail and we haven’t investigated yet its applications. Since under certain conditions (see [27]) compactly supported Dolbeault cohomology groups are dual to the ordinary Dolbeault cohomology groups, the holomorphic Penrose transform for the compactly supported cohomology should be in a certain sense dual to the ordinary holomorphic Penrose transform. However, these statements need further investigation.

The compactly supported Penrose transform is of interest to representation theorists since the Penrose transforms provide intertwining operators between representations, and representations in compactly supported cohomology are more easily unitarised.

3.2 The compactly supported transform.

We consider the double fibration

$$\begin{array}{ccc} & F & \\ \eta \swarrow & & \searrow \tau \\ Z & & X \end{array}$$

with the same properties as the double fibration from Chapter 1, Section 1.3. We suppose that the fibers of η have finite dimensional compactly supported de Rham cohomology, that Z is of dimension m , X is of dimension b , the dimension of the typical fiber of η is d and that the complex dimension of the typical fiber of τ is n . We proceed by analogy with the smooth Penrose transform.

Assumption. For simplicity we first assume that the manifolds Z , F and X are oriented. The orientation of Z and F induces an orientation of the fiber bundle η .

3.2.1 The pull-back step.

We need to identify the compactly supported cohomology of the involutive structure \mathcal{C} on F (see Chapter 1).

Lemma 3.1. *Consider the involutive structure \mathcal{C} on F defined by $\mathcal{C}^{1,0} = \eta^*(\mathcal{E}_Z^1)$ (as in Step 1 for the smooth Penrose transform). For V a vector bundle over Z and $k \geq 0$,*

$$H^k(F, \mathcal{C}_c(\eta^*V)) \cong \Gamma_c(Z, \mathcal{H}_c^k \otimes V)$$

where \mathcal{H}_c^k is the bundle whose fiber over $x \in Z$ is the k -compactly supported de Rham cohomology group of $\eta^{-1}(x)$.

Proof. We define the compact vertical supports: a form on F has compact vertical support if its support is compact when restricted to $\eta^{-1}(K)$, for all compact subsets K of Z . We use the subscript “cv” to denote compact vertical support. The proof has three steps.

1. First we prove that $\forall k \geq 0$,

$$H^k(F, \mathcal{C}_{cv}) \cong \Gamma(Z, \mathcal{H}_c^k).$$

This identification is local in the base, so it is enough to prove it for the trivial bundles $\mathbb{R}^m \times F_0$ where F_0 is the typical fiber of η . When F_0 is \mathbb{R}^d the proof follows by applying fiber by fiber the homotopy formula for compact

supports (see [11], page 38). When F_0 is not \mathbb{R}^d we shall use a ‘‘Cech-de Rham complex’’ argument to reduce the problem to the case when F_0 is \mathbb{R}^d . For this let $\mathcal{U} = \{U_\alpha\}_\alpha$ be a good cover (see [11], page 42) of the fiber F_0 . Write $U_{\alpha\beta}$ for $U_\alpha \cap U_\beta$, etc and $\hat{U}_{\alpha\dots\gamma}$ for $\mathbb{R}^m \times U_{\alpha\dots\gamma}$. Also $\mathcal{C}^{0,k}(\hat{U}_{\alpha\dots\gamma})$ will denote the space of smooth sections of $\mathcal{C}^{0,k}$ on $\hat{U}_{\alpha\dots\gamma}$. Consider the double complex $K^{p,q}$

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \vdots \\
& & \uparrow & & \uparrow & & \uparrow \\
\rightarrow & \oplus \mathcal{C}_{\text{cv}}^{0,2}(\hat{U}_{\alpha\beta\gamma}) & \rightarrow & \oplus \mathcal{C}_{\text{cv}}^{0,2}(\hat{U}_{\alpha\beta}) & \rightarrow & \oplus \mathcal{C}_{\text{cv}}^{0,2}(\hat{U}_\alpha) & \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
\rightarrow & \oplus \mathcal{C}_{\text{cv}}^{0,1}(\hat{U}_{\alpha\beta\gamma}) & \rightarrow & \oplus \mathcal{C}_{\text{cv}}^{0,1}(\hat{U}_{\alpha\beta}) & \rightarrow & \oplus \mathcal{C}_{\text{cv}}^{0,1}(\hat{U}_\alpha) & \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
\rightarrow & \oplus \mathcal{C}_{\text{cv}}^{0,0}(\hat{U}_{\alpha\beta\gamma}) & \rightarrow & \oplus \mathcal{C}_{\text{cv}}^{0,0}(\hat{U}_{\alpha\beta}) & \rightarrow & \oplus \mathcal{C}_{\text{cv}}^{0,0}(\hat{U}_\alpha) & \rightarrow 0 \\
\hline
& & -2 & & -1 & & 0 & & 1
\end{array}$$

The horizontal maps are induced from the maps of the generalized Mayer-Vietoris sequence for compact supports (see [11], page 139) taken fiber by fiber. In the vertical direction the maps are induced by the maps of the complex $(\mathcal{C}^{0,*}, \bar{\partial}_{\mathcal{C}})$. Taking the cohomology of the rows and using the exactness of the (parametrised over \mathbb{R}^m) Mayer-Vietoris sequence for compact supports (see [11], page 139), we get a double complex $H_\delta^*(K)$ which has just one nontrivial column:

$$\begin{array}{c}
\vdots \\
\uparrow \\
\mathcal{C}_{\text{cv}}^{0,2}(\mathbb{R}^m \times F_0) \\
\uparrow \\
\mathcal{C}_{\text{cv}}^{0,1}(\mathbb{R}^m \times F_0) \\
\uparrow \\
\mathcal{C}_{\text{cv}}^\infty(\mathbb{R}^m \times F_0) \\
\hline
-2 \quad -1 \quad 0
\end{array}$$

Taking the cohomology of this column we see that the total complex has cohomology $H^*(\mathbb{R}^m \times F_0, \mathcal{C}_{\text{cv}})$. On the other hand, the columns compute the compact vertical \mathcal{C} -cohomology of the spaces $\hat{U}_{\alpha\dots\gamma}$. By our previous observations, this vanishes except in dimension $d = \dim(F_0)$. Taking the cohomology of the columns we get trivial rows except the d -row:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\oplus_{\alpha<\beta<\gamma} \mathcal{E}(\mathbb{R}^m) & \rightarrow & \oplus_{\alpha<\beta} \mathcal{E}(\mathbb{R}^m) & \rightarrow & \oplus_\alpha \mathcal{E}(\mathbb{R}^m) & \rightarrow & 0 \\
0 & & 0 & & 0 & & 0 \\
0 & & 0 & & 0 & & 0 \\
\hline
-2 & & -1 & & 0 & & 1
\end{array}$$

This row computes the Cech homology (see [11], page 110) for the cover \mathcal{U} with coefficients in $\mathcal{E}(\mathbb{R}^m)$. Using the Poincare duality (see [11], page 141) for oriented manifolds and the universal-coefficient theorem for homology (see [44], page 219) we get

$$H_p(\mathcal{U}, \mathcal{E}(\mathbb{R}^m)) = \mathcal{E}(\mathbb{R}^m) \otimes_{\mathbb{C}} H_p(\mathcal{U}, \mathbb{C}) \Gamma(\mathbb{R}^m, \mathcal{H}_c^{d-p}).$$

We obtain

$$H_d^{-p,d}(K) = \Gamma(\mathbb{R}^m, \mathcal{H}_c^{d-p})$$

and for $q \neq d$, $H_d^{-p,q}(K) = 0$. This implies that the cohomology of the total complex in degree k is also isomorphic to

$$H_d^{k-d,d}(K) \cong \Gamma(\mathbb{R}^m, \mathcal{H}_c^k).$$

We have proved that the natural map $I : H^k(F, \mathcal{C}_{cv}) \rightarrow \Gamma(Z, \mathcal{H}_c^k)$ is an isomorphism and the claim from the first step of the proof follows.

2. Next we prove that the natural map

$$T : H^k(F, \mathcal{C}_c) \rightarrow H^k(F, \mathcal{C}_{cv}).$$

composed with

$$I : H^k(F, \mathcal{C}_{cv}) \rightarrow \Gamma(Z, \mathcal{H}_c^k)$$

is an isomorphism onto $\Gamma_c(Z, \mathcal{H}_c^k)$. To prove the injectivity of T consider $[\gamma_1]$ and $[\gamma_2]$ from $H^k(F, \mathcal{C}_c)$ such that $T([\gamma_1]) = T([\gamma_2])$. This is equivalent to $[\gamma_1] = [\gamma_2]$ in $H^k(F, \mathcal{C}_{cv})$ and it follows that there is $\tau \in \Gamma_{cv}(F, \mathcal{C}^{0,k-1})$ such that $\gamma_1 - \gamma_2 = \bar{\partial}_c(\tau)$. Since γ_1 and γ_2 have compact support in F , there is $f \in \eta^* \mathcal{D}(Z)$ such that $f\gamma_1 = \gamma_1$ and $f\gamma_2 = \gamma_2$. It follows that $\gamma_1 - \gamma_2 = \bar{\partial}_c(f\tau)$, and since $f\tau$ has compact support, we deduce that $[\gamma_1] = [\gamma_2]$ in $H^k(F, \mathcal{C}_c)$. The injectivity of T follows and since I is an isomorphism, $I \circ T$ is injective as well.

We now show that the image of $I \circ T$ is $\Gamma_c(F, \mathcal{H}_c^k)$. It is obvious that the image of $I \circ T$ is included in $\Gamma_c(Z, \mathcal{H}_c^k)$, since the representatives of cohomology classes in $H^k(F, \mathcal{C}_c)$ are compactly supported. To show that the space $\Gamma_c(F, \mathcal{H}_c^k)$ is included in the image of $I \circ T$, consider s an arbitrary element of $\Gamma_c(Z, \mathcal{H}_c^k)$. Since s is compactly supported, there is $f \in \mathcal{D}(Z)$ such that $fs = s$. Using the surjectivity of I we can find $[\omega] \in H^k(F, \mathcal{C}_{cv})$ such that $I([\omega]) = s$. Then $I([\eta^*(f)\omega]) = fs = s$ and the injectivity of I implies that $[\omega] = [\eta^*(f)\omega]$ in $H^k(F, \mathcal{C}_{cv})$. This implies that $[\omega] \in H^k(F, \mathcal{C}_{cv})$ admits a compactly supported representative, namely $\eta^*(f)\omega$. Then the class $[\eta^*(f)\omega]$ in $H^k(F, \mathcal{C}_c)$ has the property that $(I \circ T)([\eta^*(f)\omega]) = s$. It follows that the image of the map $I \circ T$ is $\Gamma_c(Z, \mathcal{H}_c^k)$.

3. We have shown that there is an isomorphism

$$H^k(F, \mathcal{C}_c) \cong \Gamma_c(Z, \mathcal{H}_c^k)$$

for every $k \geq 0$. When V is an arbitrary vector bundle on Z , there is an isomorphism

$$H^k(F, \mathcal{C}_c(\eta^*V)) \cong \Gamma_c(Z, \mathcal{H}_c^k \otimes V)$$

obtained by tensoring the above isomorphism by η^*V .

□

We define the involutive structure \mathcal{A} by $\mathcal{A}^{1,0} = \eta^* \mathcal{Q}^{1,0}$. The \mathcal{A} compactly supported involutive cohomology can be calculated using the following

Lemma 3.2. *There is a spectral sequence*

$$E_2^{p,q} = H^p(Z, \mathcal{Q}_c(\mathcal{H}_c^q \otimes V)) \implies H^{p+q}(F, \mathcal{A}_c(\eta^*V)).$$

Proof. The short exact sequence from Step 1 of the smooth Penrose transform (see Chapter 1, Section 1.3) induces the filtration

$$K_p(\Gamma_c(F, \mathcal{A}^{0,p+q} \otimes \eta^*V)) = \Gamma_c(F, \mathcal{A}^{0,q} \otimes \eta^*(\mathcal{Q}^{0,p} \otimes V))$$

of the complex

$$\Gamma_c(F, \eta^*V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \Gamma_c(F, \eta^*V \otimes \mathcal{A}^{0,1}) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \Gamma_c(F, \eta^*V \otimes \mathcal{A}^{0,2}) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \dots$$

From Theorem 1.13 (for compact supports) applied to the involutive structures \mathcal{A} and \mathcal{C} on F we obtain a spectral sequence

$$E_1^{p,q} = H^q(F, \mathcal{C}_c(\eta^*(\mathcal{Q}^{0,p} \otimes V))) \implies H^{p+q}(F, \mathcal{A}_c(\eta^*V)).$$

Using our identification of the \mathcal{C} compactly supported involutive cohomology, we obtain

$$E_1^{p,q} = \Gamma_c(Z, \mathcal{Q}^{0,p} \otimes \mathcal{H}_c^q \otimes V)$$

and passing to the second order the claim follows.

□

If the fibers of η are d -dimensional and contractible, this spectral sequence degenerates to the second order, since $E_2^{p,q} = 0, \forall q \neq d$. Consequently, we obtain the following

Corollary 3.3. *Let $\eta : F \rightarrow Z$ be a fiber bundle with d -dimensional contractible fibers. If \mathcal{Q} is an involutive structure on Z and V is a \mathcal{Q} -compatible vector bundle over Z , then*

$$H^k(Z, \mathcal{Q}_c(V)) \cong H^{k+d}(F, \mathcal{A}_c(\eta^*V))$$

for every $k \geq 0$.

Choose a d -form $\rho \in \Gamma_{\text{cv}}(F, \mathcal{C}^{0,d})$ which represents “1” in the compactly supported top-degree cohomology of each fiber. The above isomorphisms are induced by pull-back of forms followed by wedge product with ρ .

3.2.2 Involutive cohomologies on F .

Recall that the involutive structure \mathcal{E} is defined by the condition that $\mathcal{E}^{1,0}$ is the annihilator of the tangent vectors to the fibers of τ which are $(0, 1)$ vectors in the complex structure of the fibers. Also recall that $\mathcal{A}^{1,0} \subset \mathcal{E}^{1,0}$ and that we have defined the bundle B^1 by $B^1 := \mathcal{E}^{1,0}/\mathcal{A}^{1,0}$ and the bundle B^p by $B^p := \wedge^p B^1$.

Lemma 3.4. *There is a spectral sequence*

$$E_1^{p,q} = H^q(F, \mathcal{E}_c(B^p \otimes \eta^*V)) \implies H^{p+q}(F, \mathcal{A}_c(\eta^*V)).$$

Proof. The same short exact sequence as in Step 2 of the smooth Penrose transform (see Chapter 1, Section 1.3) induces the filtration

$$K_p(\Gamma_c(F, \mathcal{A}^{0,p+q} \otimes \eta^*V)) := \Gamma_c(F, B^p \otimes \mathcal{A}^{0,q} \otimes \eta^*V)$$

of the complex

$$\Gamma_c(F, \eta^*V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \Gamma_c(F, \mathcal{A}^{0,1} \otimes \eta^*V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \Gamma_c(F, \mathcal{A}^{0,2} \otimes \eta^*V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \dots$$

The lemma follows by applying Theorem 1.13 (for compact supports) to the involutive structures \mathcal{A} and \mathcal{E} . \square

3.2.3 The push-down step.

Lemma 3.5. *There are isomorphism*

$$H^q(F, \mathcal{E}_c(B^p \otimes \eta^*V)) \cong \Gamma_c(X, V_{p,q})$$

where $V_{p,q}$ are exactly the same bundles that arise in the smooth Penrose transform (see Chapter 1, Section 1.3).

Proof. There is a natural map

$$T : H^k(F, \mathcal{E}_c(B^p \otimes \eta^*V)) \rightarrow H^k(F, \mathcal{E}(B^p \otimes \eta^*V)).$$

Using the isomorphisms

$$I : H^q(F, \mathcal{E}(B^p \otimes \eta^*V)) \cong \Gamma(X, V_{p,q})$$

(provided by the smooth Penrose transform) we obtain a map

$$I \circ T : H^k(F, \mathcal{E}_c(B^p \otimes \eta^*V)) \rightarrow \Gamma(X, V_{p,q}).$$

An argument similar to that one used in Step 2 of the proof of Lemma 3.1 shows that $I \circ T$ is injective and its image is $\Gamma_c(X, V_{p,q})$. \square

3.2.4 The compactly supported transform.

Combining these steps in the case when the typical fiber of η is contractible we obtain the following

Theorem 3.6. *Consider the oriented manifolds Z , F and X related by the double fibration from Section 3.2 of this chapter, with η having d dimensional contractible fibers. If \mathcal{Q} is an involutive structure on Z and V is a \mathcal{Q} -compatible vector bundle over Z then there is a spectral sequence*

$$E_1^{p,q} = \Gamma_c(X, V_{p,q}) \implies H^{p+q-d}(Z, \mathcal{Q}_c(V))$$

where $V_{p,q} := \tau_*^q(\eta^*(V) \otimes B^p)$

This spectral sequence is the Penrose transform for compactly supported cohomology.

3.3 The natural bilinear pairing.

Returning to the picture from the beginning of Section 3.2 of this chapter, suppose that the bundle $\mathcal{Q}^{1,0}$ over Z has rank g . Then for every p and t such that $p + t = m - g$ there is a canonical isomorphism

$$\mathcal{Q}^{0,p+t} \otimes \mathcal{Q}^{g,0} \cong \mathcal{E}_Z^m$$

because m is the dimension of the manifold Z . For every vector bundle V over Z which is \mathcal{Q} -compatible there is a bilinear pairing

$$\int_Z : H^p(Z, \mathcal{Q}_c(V)) \times H^t(Z, \mathcal{Q}(\mathcal{Q}^{g,0} \otimes V^*)) \rightarrow \mathbb{C}$$

induced by integration \int_Z over Z (if $\gamma \in \Gamma_c(Z, \mathcal{Q}^{0,p-1} \otimes V)$ and $\mu \in \Gamma(Z, \mathcal{Q}^{0,t} \otimes \mathcal{Q}^{g,0} \otimes V^*)$), then

$$\int_Z \bar{\partial}_{\mathcal{Q}}(\gamma) \wedge \mu = (-1)^p \int_Z \gamma \wedge \bar{\partial}_{\mathcal{Q}}(\mu)$$

and this last term vanishes when $\bar{\partial}_{\mathcal{Q}}(\mu) = 0$.

Notation. We shall use the notation $\mathcal{K}_{\mathcal{Q}}$ for $\mathcal{Q}^{g,0}$, in analogy with the usual notation of the canonical bundle of a complex manifold.

Remark. Consider the particular case when Z is a complex manifold and \mathcal{Q} is its complex structure. If Z is also compact, the above bilinear form is the Serre duality map and identifies each of these cohomology groups as the dual of the other. When Z is not compact this identification still works under special conditions: one can put natural topologies on the cohomology groups and when these topologies are Hausdorff the bilinear form becomes a duality pairing (see [27]).

On the space F there is a similar bilinear pairing

$$\int_F : H^p(F, \mathcal{A}_c(\eta^*(V))) \times H^t(F, \mathcal{A}(\mathcal{K}_F \otimes \eta^*(V^*))) \rightarrow \mathbb{C}$$

defined whenever $p + t = \text{rank}(\mathcal{A}^{0,1})$ induced by integration \int_F over F .

Consider the involutive structures \mathcal{A} and \mathcal{E} on F . We recall that they satisfy $\mathcal{A}^{1,0} \subset \mathcal{E}^{1,0}$ and that defining $B^1 := \mathcal{E}^{1,0}/\mathcal{A}^{1,0}$ and $B^p := \wedge^p B^1$ there are spectral sequences

$$\begin{aligned} E_1^{p,q} &= H^q(F, \mathcal{E}_c(B^p \otimes \eta^*V)) \implies H^{p+q}(F, \mathcal{A}_c(\eta^*(V))) \\ \tilde{E}_1^{p,q} &= H^q(F, \mathcal{E}(B^p \otimes \eta^*(V^* \otimes \mathcal{K}_{\mathcal{Q}}))) \implies H^{p+q}(F, \mathcal{A}(\eta^*(V^* \otimes \mathcal{K}_{\mathcal{Q}}))). \end{aligned}$$

Lemma 3.7. *The map \int_F induces maps*

$$E_r^{p,q} \times \tilde{E}_r^{s,t} \rightarrow \mathbb{C}$$

whenever $p + s = \text{rank}(B^1)$, $q + t = \text{rank}(\mathcal{E}^{0,1})$ and r is arbitrary.

Proof. We will check that \int_F descends to the corresponding terms of the spectral sequences. We recall that the spectral sequences $E^{p,q}$ and $\tilde{E}^{p,q}$ are induced by the short exact sequence

$$0 \rightarrow B^1 \rightarrow \mathcal{A}^{0,1} \rightarrow \mathcal{E}^{0,1} \rightarrow 0$$

which determines the filtration

$$K_p(\Gamma_c(F, \mathcal{A}^{0,p+q} \otimes \eta^*V)) = \Gamma_c(F, B^p \otimes \mathcal{A}^{0,q} \otimes \eta^*V)$$

and the filtration

$$\tilde{K}_p(\Gamma(F, \mathcal{A}^{0,p+q} \otimes \eta^*(V^* \otimes \mathcal{K}_Q)) = \Gamma(F, B^p \otimes \mathcal{A}^{0,q} \otimes \eta^*(V^* \otimes \mathcal{K}_Q))$$

of the complex $\{K^*\}$

$$\Gamma_c(F, \eta^*V) \xrightarrow{\bar{\partial}_A} \Gamma_c(F, \mathcal{A}^{0,1} \otimes \eta^*V) \xrightarrow{\bar{\partial}_A} \Gamma_c(F, \mathcal{A}^{0,2} \otimes \eta^*V) \xrightarrow{\bar{\partial}_A}$$

and of the complex $\{\tilde{K}^*\}$

$$\Gamma(F, \eta^*(V^* \otimes \mathcal{K}_Q)) \xrightarrow{\bar{\partial}_A} \Gamma(F, \mathcal{A}^{0,1} \otimes \eta^*(V^* \otimes \mathcal{K}_Q)) \xrightarrow{\bar{\partial}_A} \Gamma(F, \mathcal{A}^{0,2} \otimes \eta^*(V^* \otimes \mathcal{K}_Q)) \xrightarrow{\bar{\partial}_A}$$

respectively. Following the notations from [48] (page 223), $E_r^{p,q}$ is given in terms of the filtration $\{K_*\}$ by the formula

$$E_r^{p,q} = Z_r^{p,q}/B_{r-1}^{p,q} + Z_{r-1}^{p+1,q-1}$$

where

$$\begin{aligned} B_r^p &= K_p \cap \bar{\partial}_A(K_{p-r}) \\ Z_r^p &= \{x \in K_p \mid \bar{\partial}_A(x) \in K_{p+r}\} \\ Z_r^{p,q} &= Z_r^p \cap K^{p+q} \\ B_r^{p,q} &= B_r^p \cap K^{p+q}. \end{aligned}$$

Similar formulas hold for the terms $\tilde{E}_r^{p,q}$ of the second spectral sequence. Since $p + s = \text{rank}(B^1)$ and $q + t = \text{rank}(\mathcal{E}^{0,1})$, there is an isomorphism

$$B^{p+s} \otimes \mathcal{E}^{0,q+t} \otimes \eta^*(\mathcal{K}_Q) \cong \mathcal{E}_F^f$$

where $f := d + m$ is the dimension of the manifold F . Also, since the rank of B^1 is $b + s$ there is an isomorphism

$$B^{p+s} \otimes \mathcal{A}^{0,q+t} \cong B^{p+s} \otimes \mathcal{E}^{0,q+t}.$$

It follows that there is an isomorphism

$$B^{p+s} \otimes \mathcal{A}^{0,q+t} \otimes \eta^*(\mathcal{K}_Q) \cong \mathcal{E}_F^f.$$

The integration over F induces a bilinear map

$$\int_F : Z_r^{p,q} \times \tilde{Z}_r^{s,t} \rightarrow \mathbb{C}$$

since $Z_r^{p,q} \subset \Gamma_c(F, B^p \otimes \mathcal{A}^{0,q} \otimes \eta^*V)$ and $\tilde{Z}_r^{s,t} \subset \Gamma(F, B^s \otimes \mathcal{A}^{0,t} \otimes \eta^*(V^* \otimes \mathcal{K}_Q))$.

We need to show that the map \int_F descends to a map

$$E_r^{p,q} \times \tilde{E}_r^{s,t} \rightarrow \mathbb{C}.$$

For this it is enough to show that $B_{r-1}^{p,q} \times \tilde{Z}_r^{s,t}$ and $Z_{r-1}^{p+1,q-1} \times \tilde{Z}_r^{s,t}$ map to 0 under the integration over F . (A similar argument would show that $Z_r^{p,q} \times \tilde{B}_{r-1}^{s,t}$ and $Z_r^{p,q} \times \tilde{Z}_{r-1}^{s+1,t-1}$ map to 0 as well). Writing down the definition of $B_{r-1}^{p,q}$ and $\tilde{Z}_r^{s,t}$ in terms of the corresponding filtrations, after an integration by parts we see that the restriction of the map \int_F to $B_{r-1}^{p,q} \times \tilde{Z}_r^{s,t}$ amounts to wedging γ_1 by $\bar{\partial}_A(\gamma_2)$ and then integrating the result over F , where $\gamma_1 \in K_{p-r+1}$ and $\bar{\partial}_A(\gamma_2) \in \tilde{K}_{r+s}$ such that $\gamma_1 \wedge \bar{\partial}_A(\gamma_2)$ has compact support and is of top degree on F . But since $p + s = \text{rank}(B^1)$, this top form is 0. We have proved that the set $B_{r-1}^{p,q} \times \tilde{Z}_r^{s,t}$ maps to 0 under the map \int . On the other hand, if we wedge a form in $Z_{r-1}^{p+1,q-1}$ by a form in $\tilde{Z}_r^{s,t}$ we get 0 (again, since $p + s$ is equal to the rank of B^1). This implies that also the set $Z_{r-1}^{p+1,q-1} \times \tilde{Z}_r^{s,t}$ maps to 0 under \int_F and the claim follows. \square

We recall the identifications from Step 3 of the smooth Penrose transform (see Section 1.3, Chapter 1)

$$H^q(F, \mathcal{E}_c(B^p \otimes \eta^*V)) \cong \Gamma_c(X, V_{p,q})$$

and

$$H^q(F, \mathcal{E}(B^p \otimes \eta^*(V^* \otimes \mathcal{K}_Q))) \cong \Gamma(X, (V^* \otimes \mathcal{K}_Q)_{p,q}).$$

Using these identifications, the above lemma has the following equivalent formulation.

Lemma 3.8. *The bilinear pairing from Lemma 3.7 is also induced by the Serre duality S_D along the fibers of τ followed by integration over X . More precisely, under the identifications*

$$\begin{aligned} E_1^{p,q} &\cong \Gamma_c(X, V_{p,q}) \\ \tilde{E}_1^{p,q} &\cong \Gamma(X, (V^* \otimes \mathcal{K}_Q)_{p,q}) \end{aligned}$$

the bilinear map

$$E_1^{p,q} \times \tilde{E}_1^{s,t} \rightarrow \mathbb{C}$$

defined whenever $p + s = \text{rank}(B^1)$ and $q + t = \text{rank}(\mathcal{E}^{0,1})$ becomes the map

$$\Gamma_c(X, V_{p,q}) \times \Gamma(X, (V^* \otimes \mathcal{K}_Q)_{s,t}) \rightarrow \mathbb{C}$$

given by

$$(\sigma, \mu) \rightarrow \int_X S_D(\sigma, \mu).$$

Proof. Let $\tilde{V}_{s,t} = \tau_*^t(B^p \otimes \eta^*(V^* \otimes \mathcal{K}_{\mathcal{Q}}))$. We shall prove that

$$V_{p,q} = \tilde{V}_{s,t}^* \otimes \mathcal{E}_X^b$$

where $\tilde{V}_{s,t}^*$ is the dual of $\tilde{V}_{s,t}$ via the Serre duality along the fibers of τ and b is the dimension of the manifold X . (The proof would then follow by splitting the integration over F as integration along the fibers of τ followed by integration over X .) We first notice that

$$B^{p+s} \otimes \mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{E}}$$

which follows from the exactness of

$$0 \rightarrow \mathcal{A}^{1,0} \rightarrow \mathcal{E}^{1,0} \rightarrow B^1 \rightarrow 0$$

and also that

$$\mathcal{K}_{\mathcal{E}} = \tau^*(\mathcal{E}_X^b) \otimes \mathcal{K}_{\tau}$$

(where \mathcal{K}_{τ} is the line bundle on F which restricts to each fiber to the canonical bundle of that fiber) which follows from the exactness of

$$0 \rightarrow \tau^*(\mathcal{E}_X^1) \rightarrow \mathcal{E}^{1,0} \rightarrow \mathcal{E}_{\tau}^{1,0} \rightarrow 0$$

(with $\mathcal{E}_{\tau}^{1,0}$ being the vector bundle of forms of type $(1,0)$ in the complex structure of the fibers of τ).

Using the fact that

$$B^{p+s} \otimes \mathcal{K}_{\mathcal{A}} = \tau^*(\mathcal{E}_X^b) \otimes \mathcal{K}_{\tau}$$

and that the rank of B^1 is $p + s$ we obtain that

$$\begin{aligned} (B^p)^* &= B^s \otimes (B^{p+s})^* \\ &= B^s \otimes (\tau^*(\mathcal{E}_X^b) \otimes \mathcal{K}_{\tau} \otimes (\mathcal{K}_{\mathcal{A}})^*)^* \\ &= B^s \otimes (\tau^*(\mathcal{E}_X^b))^* \otimes (\mathcal{K}_{\tau})^* \otimes \mathcal{K}_{\mathcal{A}} \end{aligned}$$

From the Serre duality and

$$\mathcal{K}_{\mathcal{A}} = \eta^*(\mathcal{K}_{\mathcal{Q}})$$

it follows that

$$\begin{aligned} V_{p,q}(x) &= H^q(\tau^{-1}(x), B^p \otimes \eta^*(V)) \\ &= H^t(\tau^{-1}(x), (B^p)^* \otimes \eta^*(V^*) \otimes \mathcal{K}_{\tau})^* \\ &= H^t(\tau^{-1}(x), B^s \otimes \eta^*(\mathcal{K}_{\mathcal{Q}} \otimes V^*) \otimes (\tau^*(\mathcal{E}_X^b))^*)^* \\ &= H^t(\tau^{-1}(x), B^s \otimes \eta^*(\mathcal{K}_{\mathcal{Q}} \otimes V^*))^* \otimes \mathcal{E}_X^b(x) \\ &= \tilde{V}_{s,t}^*(x) \otimes \mathcal{E}_X^b(x) \end{aligned}$$

and the proof follows. □

Remark. Similar facts hold for the first part of our double fibration, by considering the involutive structure \mathcal{C} instead of the involutive structure \mathcal{E} . The integration over F induces a pairing

$$E_r^{p,q} \times \tilde{E}_r^{s,t} \rightarrow \mathbb{C}$$

between the spectral sequences

$$\begin{aligned} E_2^{p,q} &= H^p(Z, \mathcal{Q}_c(V \otimes \mathcal{H}_c^q)) \implies H^{p+q}(F, \mathcal{A}_c(\eta^*(V))) \\ \tilde{E}_2^{p,q} &= H^p(Z, \mathcal{Q}(\mathcal{K}_{\mathcal{Q}} \otimes V^* \otimes \mathcal{H}^q)) \implies H^{p+q}(F, \mathcal{A}(\eta^*(\mathcal{K}_{\mathcal{Q}} \otimes V^*))) \end{aligned}$$

defined whenever $p + s = \text{rank}(\mathcal{Q}^{0,1})$ and $q + t = d$. When the fibers of η are contractible, $E_2^{p,q}$ is nonzero only when $q = d$. If $q = d$ and $t = 0$ the map

$$E_2^{p,d} \times \tilde{E}_2^{s,0} \rightarrow \mathbb{C}$$

becomes the pairing

$$\int_Z : H^p(Z, \mathcal{Q}_c(V)) \times H^s(Z, \mathcal{Q}(\mathcal{Q}^{g,0} \otimes V^*)) \rightarrow \mathbb{C}.$$

(defined whenever $p + s = \text{rank}(\mathcal{Q}^{0,1})$). Thus in the case when η has contractible fibers the “duality” pairing on Z between the ordinary \mathcal{Q} -cohomology and the compactly supported \mathcal{Q} -cohomology becomes on F the pairing between the \mathcal{A} -ordinary cohomology and \mathcal{A} -compactly supported cohomology. This pairing is induced by integration over F and we have shown above that integration over F becomes the Serre duality on the fibers of τ followed by integration over X . The following theorem follows.

Theorem 3.9. *Consider the picture from Section 3.2, and suppose that the fibers of η are contractible. If \mathcal{Q} is an involutive structure on Z and V is a \mathcal{Q} -compatible vector bundle over Z then the natural bilinear pairing*

$$\int_Z : H^p(Z, \mathcal{Q}_c(V)) \times H^t(Z, \mathcal{Q}(\mathcal{Q}^{g,0} \otimes V^*)) \rightarrow \mathbb{C}$$

on Z becomes the Serre duality on the fibers of τ followed by integration over X .

3.4 The non-oriented case.

We have developed a compactly supported Penrose transform in the case when the manifolds Z , F and X are oriented. We will now explain briefly what happens if we drop the orientability assumption. The only difference is at the pull-back step, where a twist by the orientation bundle $\tilde{\mathcal{E}}_\eta$ of η appears. For this, we first prove the following lemma.

Lemma 3.10. *If the bundle $\eta : F \rightarrow Z$ is not oriented there are isomorphisms*

$$H^k(F, \mathcal{C}_c(\eta^*(V) \otimes \tilde{\mathcal{E}}_\eta)) \cong \Gamma_c(Z, \tilde{\mathcal{H}}_c^k \otimes V),$$

where $\tilde{\mathcal{H}}_c^k$ is the bundle over Z whose fiber over $x \in Z$ is the k -twisted compactly supported de Rham cohomology of $\eta^{-1}(x)$.

Proof. It is enough to consider the case when $F = \mathbb{R}^m \times F_0$ and η is the trivial bundle with typical fiber F_0 . Like in the oriented case, we first calculate $H^k(\mathbb{R}^m \times F_0, \mathcal{C}_{cv}(\tilde{\mathcal{E}}_\eta))$. If F_0 is \mathbb{R}^d the claim follows by applying fiber by fiber the homotopy formula for twisted compactly supported forms on \mathbb{R}^d . If F_0 is not \mathbb{R}^d we choose a good cover $\mathcal{V} := \{U_\alpha\}_\alpha$ of F_0 . Like in Lemma 3.1 (with the same convention of notations) we obtain a double complex

$$\begin{array}{ccccc} & & \vdots & & \vdots \\ & & \uparrow & & \uparrow \\ \rightarrow & \oplus(\mathcal{C}^{0,2} \otimes \tilde{\mathcal{E}}_\eta)_{cv}(\hat{U}_{\alpha\beta}) & \rightarrow & \oplus(\mathcal{C}^{0,2} \otimes \tilde{\mathcal{E}}_\eta)_{cv}(\hat{U}_\alpha) & \rightarrow 0 \\ & & \uparrow & & \uparrow \\ \rightarrow & \oplus(\mathcal{C}^{0,1} \otimes \tilde{\mathcal{E}}_\eta)_{cv}(\hat{U}_{\alpha\beta}) & \rightarrow & \oplus(\mathcal{C}^{0,1} \otimes \tilde{\mathcal{E}}_\eta)_{cv}(\hat{U}_\alpha) & \rightarrow 0 \\ & & \uparrow & & \uparrow \\ \rightarrow & \oplus(\mathcal{C}^{0,0} \otimes \tilde{\mathcal{E}}_\eta)_{cv}(\hat{U}_{\alpha\beta}) & \rightarrow & \oplus(\mathcal{C}^{0,0} \otimes \tilde{\mathcal{E}}_\eta)_{cv}(\hat{U}_\alpha) & \rightarrow 0 \\ \hline & -1 & & 0 & 1 \end{array}$$

but this time applied to sections of $\mathcal{C}^{0,k}$ twisted by $\tilde{\mathcal{E}}_\eta$. Taking the cohomology horizontally and then vertically we see that the cohomology of this double complex is $H^*(\mathbb{R}^m \times F_0, \mathcal{C}_{cv}(\tilde{\mathcal{E}}_\eta))$. On the other hand, taking the cohomology vertically and then horizontally we get that the cohomology of the double complex can also be computed from the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \rightarrow & \oplus_{\alpha<\beta<\gamma} \mathcal{E}(\mathbb{R}^m) & \rightarrow & \oplus_{\alpha<\beta} \mathcal{E}(\mathbb{R}^m) & \rightarrow & \oplus_\alpha \mathcal{E}(\mathbb{R}^m) & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 & & 0 \\ \hline & -2 & & -1 & & 0 & & 1 \end{array}$$

which gives the Cech homology of the cover \mathcal{U} with coefficients in $\mathcal{E}(\mathbb{R}^m)$. Using again the universal-coefficient theorem for homology and the Poincare duality, but this time for non-oriented manifolds, we obtain

$$H_p(\mathcal{U}, \mathcal{E}(\mathbb{R}^m)) = \mathcal{E}(\mathbb{R}^m) \otimes_{\mathbb{C}} H_p(\mathcal{U}, \mathbb{C}) \Gamma(\mathbb{R}^m, \tilde{\mathcal{H}}_c^{d-p}).$$

It follows that $H^k(\mathbb{R}^m \times F_0, \mathcal{C}_{cv}(\tilde{\mathcal{E}}_\eta))$ is isomorphic to $\Gamma(\mathbb{R}^m, \tilde{\mathcal{H}}_c^k)$. The rest of the proof follows by repeating Step 2 and Step 3 from the proof of the Lemma 3.1. \square

The pull-back step of the transform follows now like in the oriented case.

Corollary 3.11. *Let $\eta : F \rightarrow Z$ be a non-oriented fiber bundle with d -dimensional contractible fibers. If \mathcal{Q} is an involutive structure on Z and V is a \mathcal{Q} -compatible vector bundle over Z then there are isomorphisms*

$$H^k(Z, \mathcal{Q}_c(V)) \cong H^{k+d}(F, \mathcal{A}_c(\eta^*V \otimes \tilde{\mathcal{E}}_\eta))$$

for every $k \geq 0$.

There is no difference between the oriented and the non-oriented cases at the intermediate step and at the push down step of the compactly supported Penrose transforms. The compactly supported Penrose transform in the non-oriented case is summarised by the following theorem.

Theorem 3.12. *Consider the manifolds Z , F and X related by the double fibration from Section 3.2 of this chapter, with η having d -dimensional contractible fibers. Suppose none of the manifolds Z , F and X is oriented. If \mathcal{Q} is an involutive structure on Z and V is a \mathcal{Q} -compatible vector bundle over Z , then there is a spectral sequence*

$$E_1^{p,q} = \Gamma_c(X, W_{p,q}) \implies H^{p+q-d}(Z, \mathcal{Q}_c(V))$$

where $W_{p,q} := \tau_*^q(\eta^*(V) \otimes B^p \otimes \tilde{\mathcal{E}}_\eta)$.

3.5 The compactly supported holomorphic Penrose transform.

We consider now the set-up from [7], Chapter 7, consisting of a double fibration like in Chapter 1, Section 1.3, which satisfies the following additional conditions:

1. The manifolds F , Z and X are complex manifolds.
2. The manifold X is Stein of (complex) dimension b .
3. The maps η and τ are holomorphic fiber bundles.
4. The typical fiber of η has real dimension d .

Remarks.

1. The special feature of the compactly supported holomorphic transform which we shall construct in this chapter is that it gives isomorphisms between (compactly supported) Dolbeault cohomology groups on Z and elements of compactly supported top-degree Dolbeault cohomology groups on

X which are annihilated by some differential operators. Differential operators acting between such cohomology groups are not usually studied and we don't know their significance yet. Since in certain situations compactly supported Dolbeault cohomology groups are dual to ordinary Dolbeault cohomology groups (see [27]), one could guess that compactly supported Dolbeault cohomology groups are in a certain sense spaces of distributions and that the operators arising from the compactly supported holomorphic Penrose transform are dual to the operators arising from the usual holomorphic Penrose transform. These statements are just suppositions and they need to be verified.

2. We will see that both the pull-back step and the push-down step of the compactly supported holomorphic Penrose transform contain a shift in dimension. The shift which comes from the pull-back step is by the real dimension of the fibers of η . The shift which comes from the push-down step is by the complex dimension of the complex manifold X .

Involutive structures. We shall use the following involutive structures:

1. The involutive structure \mathcal{Q}_Z on Z is the complex structure of the complex manifold Z . Similarly, \mathcal{Q}_F is the complex structure of the complex manifold F and \mathcal{Q}_X is the complex structure of the complex manifold X .
2. The involutive structure \mathcal{A} on F is the pull back of the complex structure \mathcal{Q}_Z on Z (see also Chapter 1, Section 1.3).
3. The involutive structure \mathcal{E} on F is defined by $\mathcal{E}^{1,0}$ being the annihilator of the τ -vertical vectors which are $(0,1)$ -vectors in the complex structure of the fibers of τ (see also Chapter 1, Section 1.3).

Notation. If M is a complex manifold and \mathcal{E} is its complex structure then for every \mathcal{E} -compatible vector bundle V over M the Dolbeault cohomology group $H^p(M, \mathcal{E}(V))$ will be denoted $H^p(M, V)$, and the compactly supported Dolbeault cohomology group $H^p(M, \mathcal{E}_c(V))$ will be denoted $H_c^p(M, V)$

Lemma 3.13. *The involutive structures \mathcal{A} , \mathcal{Q}_F and \mathcal{E} satisfy the following inclusions*

$$\mathcal{A}^{1,0} \subset \mathcal{Q}_F^{1,0} \subset \mathcal{E}^{1,0}.$$

Moreover,

$$\mathcal{E}^{1,0} / \mathcal{Q}_F^{1,0} \cong \tau^* \mathcal{Q}_X^{0,1}.$$

Proof. The proof follows from the holomorphic Penrose transform. □

Notation. The quotient $(\mathcal{Q}_F^{1,0}/\mathcal{A}^{1,0})^p$ is well-defined and will be denoted Ω_η^p .

Lemma 3.14. *If η has contractible fibers and V is a holomorphic vector bundle over Z , there is a spectral sequence*

$$E_1^{p,q} = H_c^b(X, \tau_*^{q-b}(\Omega_\eta^p \otimes \eta^*V)) \implies H_c^{p+q-d}(Z, V).$$

Proof. The proof has three steps.

1. **Step 1.** Since the typical fiber of η is of real dimension d and is contractible, Corollary 3.3 gives the isomorphism

$$H_c^k(Z, V) \cong H^{k+d}(F, \mathcal{A}_c(\eta^*V)).$$

2. **Step 2.** The involutive structures \mathcal{A} and \mathcal{Q}_F satisfy $\mathcal{A}^{1,0} \subset \mathcal{Q}_F^{1,0}$ and give the spectral sequence

$$E_1^{p,q} = H_c^q(F, \Omega_\eta^p \otimes \eta^*V) \implies H^{p+q}(F, (\mathcal{A})_c(\eta^*V)).$$

3. **Step 3.** We shall calculate the Dolbeault cohomology groups $H_c^q(F, \Omega_\eta^p \otimes \eta^*V)$ in terms of data on X . The involutive structures $\mathcal{Q}_F^{1,0}$ and $\mathcal{E}^{1,0}$ satisfy $\mathcal{Q}_F^{1,0} \subset \mathcal{E}^{1,0}$ and give the spectral sequence

$$\tilde{E}_1^{p,q} = H^q(F, \mathcal{E}_c(\tau^* \mathcal{Q}_X^{0,p} \otimes \Omega_\eta^s \otimes \eta^*V)) \implies H_c^{p+q}(F, \Omega_\eta^s \otimes \eta^*V).$$

Using the identifications

$$H^q(F, \mathcal{E}_c(\tau^* \mathcal{Q}_X^{0,p} \otimes \Omega_\eta^s \otimes \eta^*V)) \cong \Gamma_c(X, \mathcal{Q}_X^{0,p} \otimes \tau_*^q(\Omega_\eta^s \otimes \eta^*V))$$

(provided by the compactly supported smooth Penrose transform) and and passing to the second order the spectral sequence $\tilde{E}^{p,q}$ becomes

$$\tilde{E}_2^{p,q} = H_c^p(X, \tau_*^q(\Omega_\eta^s \otimes \eta^*V)) \implies H_c^{p+q}(F, \Omega_\eta^s \otimes \eta^*V).$$

Since the manifold X is Stein and has complex dimension b the spectral sequence $\tilde{E}^{p,q}$ degenerates to the second order:

$$\tilde{E}_2^{p,q} = 0, \quad \forall p \neq b$$

and it becomes a set of isomorphisms

$$H_c^k(F, \Omega_\eta^s \otimes \eta^*V) \cong H_c^b(X, \tau_*^{k-b}(\Omega_\eta^s \otimes \eta^*V)).$$

Combining the above steps we arrive at the spectral sequence

$$E_1^{p,q} = H_c^b(X, \tau_*^{q-b}(\Omega_\eta^p \otimes \eta^*V)) \implies H_c^{p+q-d}(Z, V).$$

The claim follows. □

Chapter 4

The Penrose transform for currents.

4.1 Introduction.

In this chapter we will extend the Penrose transform to map distribution cohomology groups on Z to kernels (or cokernels) of differential operators on currents on X , where Z and X are related by a double fibration

$$\begin{array}{ccc} & F & \\ \eta \swarrow & & \searrow \tau \\ Z & & X \end{array}$$

which satisfies the conditions from Chapter 1, Section 1.3.

4.2 The Penrose transform for currents.

We consider the set-up of the smooth Penrose transform (see Chapter 1, Section 1.3). We suppose that the dimension of the manifold Z is m , the dimension of the manifold X is b , the dimension of the typical fiber of η is d and that the complex dimension of the typical fiber of τ is n .

Assumption. For simplicity, we shall restrict first to the case when the manifolds F , Z and X are oriented. The orientations of Z and F induce an orientation of the fiber bundle η .

4.2.1 The pull-back step.

Lemma 4.1. *Consider the involutive structure $\mathcal{C}^{1,0} := \eta^*(\mathcal{E}_Z^1)$ on the total space of the fiber-bundle $\eta : F \rightarrow Z$. If V is a vector bundle over Z then the map*

$$\eta^* : \mathcal{J}^0(Z, V) \rightarrow \mathcal{J}^0(F, \eta^*(V))$$

is injective and its image is $H^0(F, \mathcal{C}'(\eta^*V))$.

Proof. Since the property is local in the base direction, we consider η the trivial bundle over Z with typical fiber the d -dimensional manifold F_0 . Let $\pi : Z \times F_0 \rightarrow F_0$ be the other natural projection and $\rho \in \mathcal{D}^d(F_0)$ such that $\int_{F_0} \rho = 1$.

We consider first the case when V is trivial of rank one. We notice that the map η^* is injective, since every form $\alpha \in \mathcal{D}^m(Z)$ can be written as the fiber integral of a form $\gamma \in \mathcal{D}^{m+d}(F)$: it is enough to take γ to be $\pi^*(\rho) \wedge \eta^*(\alpha)$. To show that $\eta^* \mathcal{J}^0(Z) \subset H^0(F, \mathcal{C}')$ consider $\gamma \in \Gamma_c(F, \eta^* \mathcal{E}_Z^m \otimes \pi^* \mathcal{E}_{F_0}^{d-1})$. Then $\eta_*(d\gamma) = 0$ and $(\eta^*(\omega), d\gamma) = 0$ for every $\omega \in \mathcal{J}^0(Z)$. It follows that $\bar{\partial}_{\mathcal{C}}(\eta^*\omega) = 0$ and that the image of η^* is included in $H^0(F, \mathcal{C}')$. In order to prove that the image of η^* defined on $\mathcal{J}^0(Z)$ is the entire $H^0(F, \mathcal{C}')$, let $\underline{\omega} \in \mathcal{J}^0(F)$ such that $\bar{\partial}_{\mathcal{C}}(\underline{\omega}) = 0$. Define $\omega \in \mathcal{J}^0(Z)$ by the formula $(\omega, \alpha)(\underline{\omega}, \eta^*(\alpha) \wedge \pi^*(\rho))$, where $\alpha \in \mathcal{D}^m(Z)$. We will show that $\eta^*(\omega) = \underline{\omega}$. For this let $\beta \in \mathcal{D}^d(F_0)$. Then $(\int_{F_0} \beta) \rho$ and β represent the same cohomology class in $H_c^d(F_0)$ and

$$\eta^*(\alpha) \wedge \pi^* \left(\beta - \left(\int_{F_0} \beta \right) \rho \right) \in d \left(\Gamma_c(F, \mathcal{C}^{m,0} \otimes \mathcal{E}^{d-1}) \right).$$

On the other hand since $\underline{\omega}$ is $\bar{\partial}_{\mathcal{C}}$ -closed, it is zero on $d \left(\Gamma_c(F, \mathcal{C}^{m,0} \otimes \mathcal{E}^{d-1}) \right)$ and it follows that

$$\begin{aligned} (\eta^*(\omega), \eta^*(\alpha) \wedge \pi^*(\beta)) & (-1)^{md} \left(\omega, \left(\int_{F_0} \beta \right) \alpha \right) \\ &= (-1)^{md} \left(\underline{\omega}, \eta^*(\alpha) \wedge \pi^* \left(\left(\int_{F_0} \beta \right) \rho \right) \right) \\ &= (-1)^{md} (\underline{\omega}, \eta^*(\alpha) \wedge \pi^*(\beta)) \end{aligned}$$

The equality

$$\eta^*(\omega) = (-1)^{md} \underline{\omega}$$

follows since the space of finite sums of decomposable compactly supported forms is dense in the space of all compactly supported forms (see [17], page 33).

If V is not trivial of rank one we tensor the injective map

$$\eta^* : \mathcal{J}^0(Z) \rightarrow \mathcal{J}^0(F)$$

by $\eta^*(V)$ to get another injective map

$$\eta^* : \mathcal{J}^0(Z, V) \rightarrow \mathcal{J}^0(F, \eta^*(V))$$

whose image is $H^0(F, \mathcal{C}'(\eta^*V))$.

□

Lemma 4.2. *Consider the involutive structure $\mathcal{C}^{1,0} := \eta^*(\mathcal{E}_Z^1)$ on the total space of the fiber-bundle $\eta : F \rightarrow Z$. If the typical fiber of η is \mathbb{R}^d then $H^d(F, \mathcal{C}'(\eta^*V)) = 0$ for every vector bundle V over Z .*

Proof. Like in the previous proof, we can consider V to be trivial of rank one. By applying fiber by fiber the homotopy formula for compactly supported forms on \mathbb{R}^d (see [11], page 38), there are maps

$$\begin{aligned} K_1 : \Gamma_c(F, \mathcal{C}^{m,1}) &\rightarrow \Gamma_c(F, \mathcal{C}^{m,0}) \\ K_2 : \Gamma_c(F, \mathcal{C}^{m,2}) &\rightarrow \Gamma_c(F, \mathcal{C}^{m,1}) \end{aligned}$$

such that

$$\beta = dK_1(\beta) - K_2d(\beta)$$

for $\beta \in \Gamma_c(F, \mathcal{C}^{m,1})$.

In order to show that $H^d(F, \mathcal{C}') = 0$ we take $\omega \in \mathcal{J}^0(F, \mathcal{C}^{0,d})$ and we define $\underline{\omega} \in \mathcal{J}^0(F, \mathcal{C}^{0,d-1})$ by the formula $(\underline{\omega}, \alpha) = (-1)^d(\omega, K_1(\alpha))$, where $\alpha \in \Gamma_c(F, \mathcal{C}^{m,1})$. Then

$$\begin{aligned} (\bar{\partial}_c(\underline{\omega}), \gamma)(-1)^d(\underline{\omega}, d\gamma) \\ &= (\omega, K_1(d\gamma)) \\ &= (\omega, \gamma) \end{aligned}$$

when $\gamma \in \Gamma_c(F, \mathcal{C}^{m,0})$. For the last equality we have noticed that $K_1d\gamma - \gamma$ is a closed form of zero degree and compactly supported along the fibers of η (which are copies of \mathbb{R}^d), and thus it must be zero. The claim follows. \square

Lemma 4.3. *Let $\eta : F \rightarrow Z$ be a fiber-bundle whose typical fiber is a connected manifold with finite dimensional de Rham cohomology. Then for every V a vector bundle over Z and $\forall k \geq 0$,*

$$H^k(F, \mathcal{C}'(\eta^*V)) \cong \mathcal{J}^0(Z, \mathcal{H}^k \otimes V)$$

where \mathcal{H}^k is the bundle whose fiber over $z \in Z$ is the k -de Rham cohomology group of $\eta^{-1}(z)$.

Proof. We notice that V carries through the argument and we take it to be trivial of rank one. The statement is true in the case when the typical fiber of η is \mathbb{R}^d (when $k = 0$ we apply Lemma 4.1, when $k = d$ we apply Lemma 4.2 and otherwise we apply fiber by fiber the homotopy formula for forms of compact support on \mathbb{R}^d , see [11], page 38, and we dualise it to get a homotopy formula for currents).

When the typical fiber of η is not \mathbb{R}^d we shall use a Cech-de Rham argument to reduce the problem to the case when the typical fiber of η is \mathbb{R}^d . To do this, we consider $F = \mathbb{R}^m \times F_0$ and $\eta : \mathbb{R}^m \times F_0 \rightarrow \mathbb{R}^m$ the trivial bundle with typical fiber F_0 .

Let $\mathcal{U} = \{U_\alpha\}$ be a good cover (see [11], page 42) of the fiber F_0 . Write $U_{\alpha\beta}$ for $U_\alpha \cap U_\beta$, etc, and $\hat{U}_{\alpha\dots\gamma}$ for $\mathbb{R}^m \times U_{\alpha\dots\gamma}$. Also write $\mathcal{J}^{0,k}(\hat{U}_{\alpha\beta})$ for $\mathcal{J}^0(\hat{U}_{\alpha\beta}, \mathcal{C}^{0,k})$, etc. Consider the double complex $K^{p,q}$

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \uparrow & & \uparrow & & \uparrow & \\
\prod_{\alpha_0} \mathcal{J}^{0,2}(\hat{U}_{\alpha_0}) & \rightarrow & \prod_{\alpha_0 < \alpha_1} \mathcal{J}^{0,2}(\hat{U}_{\alpha_0\alpha_1}) & \rightarrow & \prod_{\alpha_0 < \alpha_1 < \alpha_2} \mathcal{J}^{0,2}(\hat{U}_{\alpha_0\alpha_1\alpha_2}) & \rightarrow & \dots \\
& \uparrow & & \uparrow & & \uparrow & \\
\prod_{\alpha_0} \mathcal{J}^{0,1}(\hat{U}_{\alpha_0}) & \rightarrow & \prod_{\alpha_0 < \alpha_1} \mathcal{J}^{0,1}(\hat{U}_{\alpha_0\alpha_1}) & \rightarrow & \prod_{\alpha_0 < \alpha_1 < \alpha_2} \mathcal{J}^{0,1}(\hat{U}_{\alpha_0\alpha_1\alpha_2}) & \rightarrow & \dots \\
& \uparrow & & \uparrow & & \uparrow & \\
\prod_{\alpha_0} \mathcal{J}^0(\hat{U}_{\alpha_0}) & \rightarrow & \prod_{\alpha_0 < \alpha_1} \mathcal{J}^0(\hat{U}_{\alpha_0\alpha_1}) & \rightarrow & \prod_{\alpha_0 < \alpha_1 < \alpha_2} \mathcal{J}^0(\hat{U}_{\alpha_0\alpha_1\alpha_2}) & \rightarrow & \dots
\end{array}$$

where the horizontal maps are the maps of the generalised Mayer-Vietoris sequence (see [11], page 92) and the vertical maps are induced by the operator $\bar{\partial}_{\mathcal{C}}$ at the level of currents. Since the cover \mathcal{U} is good we deduce that the columns are exact except in degree 0. Taking the cohomology vertically and then horizontally we see that the cohomology of the total complex in degree k is isomorphic to the Cech cohomology of F_0 in degree k with coefficients 0-currents on \mathbb{R}^m . This is isomorphic to the space of 0-currents on \mathbb{R}^m with values in the vector bundle \mathcal{H}^k (see [44], page 241 for the universal-coefficient theorem for cohomology). On the other hand, taking the cohomology horizontally and then vertically and using the exactness of the generalised Mayer-Vietoris sequence (see [11], page 94) we get that the cohomology of the double complex in degree k is $H^k(F, \mathcal{C}')$. The conclusion follows. \square

Consider now the set-up from Chapter 1, Section 1.3. If V is a \mathcal{Q} -compatible vector bundle over Z then $\eta^*(V)$ is a \mathcal{A} -compatible vector bundle over F (we recall that $\mathcal{A}^{1,0} := \eta^*(\mathcal{Q}^{1,0})$). The filtration

$$K_p(\Gamma(F, \mathcal{A}^{0,p+q} \otimes \eta^*V)) := \Gamma(F, \mathcal{A}^{0,q} \otimes \eta^*(\mathcal{Q}^{0,p} \otimes V))$$

we had at the smooth Penrose transform induces the filtration

$$K_p(\mathcal{J}^0(F, \mathcal{A}^{0,p+q} \otimes \eta^*V)) := \mathcal{J}^0(F, \mathcal{A}^{0,q} \otimes \eta^*(\mathcal{Q}^{0,p} \otimes V))$$

of the complex

$$\mathcal{J}^0(F, \eta^*V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \mathcal{J}^0(F, \mathcal{A}^{0,1} \otimes \eta^*V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \mathcal{J}^0(F, \mathcal{A}^{0,2} \otimes \eta^*V) \xrightarrow{\bar{\partial}_{\mathcal{A}}} \dots$$

The following lemma follows.

Lemma 4.4. *There is a spectral sequence*

$$E_2^{p,q} = H^p(Z, \mathcal{Q}'(V \otimes \mathcal{H}^q)) \implies H^{p+q}(F, \mathcal{A}'(\eta^*V)).$$

In particular, when the fibers of η are contractible there are isomorphisms

$$H^k(Z, \mathcal{Q}'(V)) \cong H^k(F, \mathcal{A}'(\eta^*V))$$

induced by the pull-back η^ of cohomology representative forms.*

Proof. Applying the spectral sequence of a filtered complex (see [48], page 221) to the complex of currents and the filtration mentioned above we obtain

$$E_1^{p,q} = H^q(F, \mathcal{C}'(\eta^*(V \otimes \mathcal{Q}^{0,p}))) \implies H^{p+q}(F, \mathcal{A}'(\eta^*V)).$$

From Lemma 4.3 we see that $E_1^{p,q} = \mathcal{J}^0(Z, \mathcal{Q}^{0,p} \otimes \mathcal{H}^q \otimes V)$ and passing to the second order of the spectral sequence we get

$$E_2^{p,q} = H^p(Z, \mathcal{Q}'(V \otimes \mathcal{H}^q)).$$

When the fibers of η are contractible the spectral sequence degenerates to isomorphisms (induced by η^*) in every degree

$$H^k(Z, \mathcal{Q}'(V)) \cong H^k(F, \mathcal{A}'(\eta^*V)).$$

□

4.2.2 The intermediate step.

Recall that the involutive structure \mathcal{E} on F is defined by the condition that $\mathcal{E}^{1,0}$ is the annihilator of the τ -vertical vectors which are $(0, 1)$ vectors in the complex structure of the fibers of τ . Then $\mathcal{A}^{1,0} \subset \mathcal{E}^{1,0}$ and we have defined B^1 by the formula $B^1 := \mathcal{E}^{1,0}/\mathcal{A}^{1,0}$ and B^p by the formula $B^p := \wedge^p B^1$. The filtration

$$K_p(\Gamma(F, \mathcal{A}^{0,p+q} \otimes \eta^*V)) := \Gamma(F, B^p \otimes \mathcal{A}^{0,q} \otimes \eta^*V)$$

we had at the smooth Penrose transform induces the filtration

$$K_p(\mathcal{J}^0(F, \mathcal{A}^{0,p+q} \otimes \eta^*V)) := \mathcal{J}^0(F, B^p \otimes \mathcal{A}^{0,q} \otimes \eta^*V)$$

of the complex

$$\mathcal{J}^0(F, \eta^*V) \xrightarrow{\bar{\partial}_A} \mathcal{J}^0(F, \mathcal{A}^{0,1} \otimes \eta^*V) \xrightarrow{\bar{\partial}_A} \mathcal{J}^0(F, \mathcal{A}^{0,2} \otimes \eta^*V) \xrightarrow{\bar{\partial}_A} \dots$$

Applying the spectral sequence of a filtered complex ([48], page 221) to this complex of currents and filtration, we obtain the following lemma.

Lemma 4.5. *There is a spectral sequence*

$$E_1^{p,q} = H^q(F, \mathcal{E}'(B^p \otimes \eta^*V)) \implies H^{p+q}(F, \mathcal{A}'(\eta^*V)).$$

4.2.3 The push-down step.

From the smooth Penrose transform we know that the complex structure of the fibers of τ varies smoothly with respect to the base. We will restrict to the particular case when τ is a fiber-bundle with the typical fiber a compact complex manifold.

We consider W a \mathcal{E} -compatible vector bundle over F such that the dimension of the Dolbeault cohomology groups of W restricted to the fiber of τ over x is independent of $x \in X$. Recall that under this assumption we have defined a vector bundle $\tau_*^q(W) \rightarrow X$ whose fiber over $x \in X$ is the Dolbeault cohomology group $H^q(\tau^{-1}(x), W|_{\tau^{-1}(x)})$.

Lemma 4.6. *There are isomorphisms*

$$H^k(F, \mathcal{E}'(W)) \cong \mathcal{J}^0(X, \tau_*^k(W)).$$

Proof. It is enough to prove the lemma when τ is the trivial bundle over a small domain Δ in \mathbb{R}^b , with typical fiber the compact complex n -dimensional manifold P_0 . We choose Δ small enough such that the bundles $\tau_*^k(W)$ are trivial for every k . We take a hermitian metric on P_0 and a hermitian metric on the vector bundle W . For every $x \in \Delta$ the restriction of W to $\{x\} \times P_0$ will be denoted W_x , and is a hermitian vector bundle over the compact hermitian manifold P_0 . The family $\{W_x\}_{x \in \Delta}$ is a differentiable family of holomorphic vector bundles over P_0 and $\bar{\partial}_{\mathcal{E}}$, as well as the codifferential operator $\bar{\partial}^*$, both defined on the space of forms on $\tau^{-1}(x)$ with values in W_x generate a differentiable family of linear operators on P_0 (see [31], page 324, 325). Since the dimension of the Dolbeault cohomology groups of W_x on P_0 is independent on $x \in \Delta$, the Green operators G_x of W_x generate a differentiable family of linear operators as well (see [31], page 344). It follows that at the level of currents we can define the operators

$$\begin{aligned} \bar{\partial}^* &: \mathcal{J}^0(F, \mathcal{E}^{0,k} \otimes W) \rightarrow \mathcal{J}^0(F, \mathcal{E}^{0,k-1} \otimes W) \\ G &: \mathcal{J}^0(F, \mathcal{E}^{0,k} \otimes W) \rightarrow \mathcal{J}^0(F, \mathcal{E}^{0,k} \otimes W) \end{aligned}$$

by the formulae

$$\begin{aligned} (\bar{\partial}^*(\omega), \gamma_1) &= (\omega, \bar{\partial}^*(\gamma_1)) \\ (G(\omega), \gamma_2) &= (\omega, G(\gamma_2)). \end{aligned}$$

where $\omega \in \mathcal{J}^0(F, \mathcal{E}^{0,k} \otimes W)$, $\gamma_1 \in \Gamma_c(F, \mathcal{E}^{n+b, n-k+1} \otimes W^*)$, $\gamma_2 \in \Gamma_c(F, \mathcal{E}^{n+b, n-k} \otimes W^*)$, with $\bar{\partial}^*$ and G acting along P_0 . We define harmonic, exact and coexact currents in the obvious way.

Let $\{\rho_1, \dots, \rho_p\}$ be a system of smooth local sections of $\mathcal{E}^{0,k} \otimes W$ such that for every $x \in \Delta$ the forms $\rho_i(x)$ (which are of type $(0, k)$ in the complex structure of P_0 and take values in W_x) are harmonic, and $\{[\rho_1(x)], \dots, [\rho_p(x)]\}$ is a basis of the Dolbeault cohomology group $H^{0,k}(P_0, W_x)$. (For every local basis of $\tau_*^k(W)$ we can find harmonic representatives depending smoothly on $x \in \Delta$ by choosing any smooth representatives and taking their harmonic part). We consider the basis $\{[\rho_1^*(x)], \dots, [\rho_p^*(x)]\}$ of $H^{n,n-k}(P_0, W_x^*)$ with $\rho_i^*(x)$ harmonic and smooth in x such that the equality

$$\int_{P_0} \rho_i(x) \wedge \rho_j^*(x) = \delta_{ij}$$

holds for any $x \in \Delta$. (Here δ_{ij} is the Kronecker symbol and \wedge is the usual wedge product on forms followed by the natural contraction between elements of the bundle W and its dual W^*). The proof has three steps.

1. **Step1** First we notice that every class in $H^k(F, \mathcal{E}'(W))$ admits a unique harmonic representative in $\mathcal{J}^0(F, \mathcal{E}^{0,k} \otimes W)$.

For this, let $\omega \in \mathcal{J}^0(F, \mathcal{E}^{0,k} \otimes W)$ represent an arbitrary cohomology class in $H^k(F, \mathcal{E}'(W))$. Using Theorem 2.26, ω is an element of $(\Gamma_c(F, \mathcal{E}^{n+b,n-k} \otimes W^*))'$ which vanishes on $\bar{\partial}_{\mathcal{E}}(\Gamma_c(F, \mathcal{E}^{n+b,n-k-1} \otimes W^*))$. Using a Hodge-de Rham decomposition along the fibers of τ applied to $\beta \in \Gamma_c(F, \mathcal{E}^{n+b,n-k} \otimes W^*)$ we obtain

$$\begin{aligned} (\omega, \beta) &= (\omega, \bar{\partial}_{\mathcal{E}} \bar{\partial}^* G(\beta)) + (-1)^k (\bar{\partial}_{\mathcal{E}} G \bar{\partial}^*(\omega), \beta) \\ &\quad + (\omega, H(\beta)) \end{aligned}$$

where $H(\beta) \in \Gamma_c(F, \mathcal{E}^{n+b,n-k} \otimes W^*)$ is harmonic when restricted to $\{x\} \times P_0$ (for every $x \in \Delta$). The first term on the right hand side vanishes since ω vanishes on $\bar{\partial}_{\mathcal{E}}(\Gamma_c(F, \mathcal{E}^{n+b,n-k-1} \otimes W^*))$. The second term on the right hand side is $\bar{\partial}_{\mathcal{E}}$ -exact and has no contribution at the level of \mathcal{E} -distribution cohomology. Defining the current $H(\omega)$ by the formula $(H(\omega), \beta) := (\omega, H(\beta))$ we see that ω and $H(\omega)$ represent the same cohomology class in $H^k(F, \mathcal{E}'(W))$. Obviously $H(\omega)$ is harmonic and is zero if and only if ω determines the zero class in $H^k(F, \mathcal{E}'(W))$.

2. **Step2** Next we notice that for $[\omega] \in H^k(F, \mathcal{E}'(W))$, there are $\omega_i \in \mathcal{J}^0(\Delta)$ and $s_i \in \Gamma(F, \mathcal{E}^{0,k} \otimes W)$ such that

$$[\omega] = (-1)^{bk} \left[\sum_{i=1}^p \tau^*(\omega_i) \wedge s_i \right]$$

in $H^k(F, \mathcal{E}'(W))$.

To prove this claim, consider an arbitrary class in $H^k(F, \mathcal{E}'(W))$ and $\omega \in \mathcal{J}^0(F, \mathcal{E}^{0,k} \otimes W)$ an harmonic representative of it. Define $\omega_i \in \mathcal{J}^0(\Delta)$ by $(\omega_i, \alpha) = (\omega, \tau^*(\alpha) \wedge \rho_i^*)$ for $\alpha \in \mathcal{D}^b(\Delta)$, and let γ be a form of type $(n, n-k)$ in the complex structure of P_0 , with values in W^* and varying smoothly with respect to $x \in \Delta$. Then $\tau^*(\alpha) \wedge \gamma \in \Gamma_c(F, \mathcal{E}^{b+n, n-k} \otimes W^*)$ and

$$\begin{aligned} \left(\sum_{i=1}^p \tau^*(\omega_i) \wedge \rho_i, \tau^*(\alpha) \wedge \gamma \right) &= (-1)^{bk} \sum_{i=1}^p (\omega_i, \alpha \cdot \tau_*(\rho_i \wedge \gamma)) \\ &= (-1)^{bk} \sum_{i=1}^p (\omega, \tau^*(\alpha) \wedge \tau_*(\rho_i \wedge \gamma) \cdot \rho_i^*) \\ &= (-1)^{bk} (\omega, \tau^*(\alpha) \wedge H(\gamma)) \\ &= (-1)^{bk} (\omega, \tau^*(\alpha) \wedge \gamma) \end{aligned}$$

since ω is harmonic. We obtain the equality

$$\omega = (-1)^{bk} \sum_{i=1}^p \tau^*(\omega_i) \wedge \rho_i$$

and the claim follows.

3. **Step3** To prove the existence of the isomorphism

$$\mathcal{J}^0(\Delta, \tau_*^k(W)) \cong H^k(F, \mathcal{E}'(W))$$

we consider the map

$$T : \mathcal{J}^0(\Delta, \tau_*^k(W)) \rightarrow H^k(F, \mathcal{E}'(W))$$

defined by the formula

$$\sum_{i=1}^p \omega_i \cdot [\rho_i] \rightarrow \left[\sum_{i=1}^p \tau^*(\omega_i) \wedge \rho_i \right].$$

We will check first that T is well defined. It is clear that it is independent of the choice of the representatives ρ_i of the local sections $[\rho_i]$ of $\tau_*^k(W)$. Consider $\{[\tilde{\rho}_1], \dots, [\tilde{\rho}_p]\}$ another local basis of sections of $\tau_*^k(W)$, with $\tilde{\rho}_1(x), \dots, \tilde{\rho}_p(x)$ harmonic on P_0 and with values in W_x . We need to show that if the equality

$$\sum_{i=1}^p \omega_i \cdot [\rho_i] = \sum_{i=1}^p \tilde{\omega}_i \cdot [\tilde{\rho}_i]$$

holds in $\mathcal{J}^0(\Delta, \tau_*^k(W))$ then the equality

$$\left[\sum_{i=1}^p \tau^*(\omega_i) \wedge \rho_i \right] \left[\sum_{i=1}^p \tau^*(\tilde{\omega}_i) \wedge \tilde{\rho}_i \right]$$

holds in $H^k(F, \mathcal{E}'(W))$. For this we notice that if $[\tilde{\rho}_i] = \sum_{j=1}^p a_{ij}[\rho_j]$ for some functions $a_{ij} \in \mathcal{E}(\Delta)$ then $\sum_{j=1}^p a_{ij}\rho_j - \tilde{\rho}_i = \bar{\partial}_{\mathcal{E}}(\beta_i)$ for β_i a (smoothly parametrised over Δ) form of type $(0, k-1)$ along P_0 with values in W . Since $\sum_{i=1}^p a_{ij}\tilde{\omega}_i = \omega_j$ we get

$$\begin{aligned} & \sum_{i=1}^p [\tau^*(\omega_i) \wedge \rho_i] \sum_{i,s=1}^p [\tau^*(\tilde{\omega}_s a_{si}) \wedge \rho_i] \\ &= \sum_{i,s=1}^p [\tau^*(\tilde{\omega}_s) \wedge (a_{si}\rho_i)] \\ &= \sum_{s=1}^p [\tau^*(\tilde{\omega}_s) \wedge \tilde{\rho}_s] + \sum_{s=1}^p [\bar{\partial}_{\mathcal{E}}(\tau^*(\tilde{\omega}_s) \wedge \beta_s)] \\ &= \sum_{s=1}^p [\tau^*(\tilde{\omega}_s) \wedge \tilde{\rho}_s] \end{aligned}$$

It follows that T is well defined.

In order to prove the injectivity of T we notice (from Step 1) that a $\bar{\partial}_{\mathcal{E}}$ -closed current in $\mathcal{J}^0(F, \mathcal{E}^{0,k} \otimes W)$ represents the zero class in $H^k(F, \mathcal{E}'(W))$ if and only if its harmonic part vanishes, that is, if and only if it is zero on the space of forms from $\Gamma_c(F, \mathcal{E}^{n+b, n-k} \otimes W^*)$ harmonic along P_0 . Suppose now that $\sum_{i=1}^p \tau^*(\omega_i) \wedge \rho_i$ represents the zero class in $H^k(F, \mathcal{E}'(W))$ and let $\alpha \in \mathcal{D}^b(\Delta)$. Since ρ_j^* are harmonic along P_0 , we get that $\sum_{i=1}^p (\tau^*(\omega_i) \wedge \rho_i, \tau^*(\alpha) \wedge \rho_j^*) = 0$ or $(\omega_j, \alpha) = 0$. Then $\omega_j = 0$ for every j and the map T is injective. Step 2 shows that T is surjective. □

Consider now instead of W the bundle $\eta^*(V) \otimes B^p$. When V is \mathcal{Q} -compatible this bundle is \mathcal{E} -compatible and we can define the Dolbeault cohomology groups of its restriction to the fibers of τ . We suppose that the dimension of these cohomology groups in every degree is constant (that is, independent of the fibers of τ). In this case we define a bundle $V_{p,q} \rightarrow X$ whose fiber over $x \in X$ is the Dolbeault cohomology group $H^q(\tau^{-1}(x), \eta^*(V) \otimes B^p)$.

Using the identifications

$$H^q(F, \mathcal{E}'(\eta^*V \otimes B^p)) \cong \mathcal{J}^0(X, V_{p,q})$$

provided by Lemma 4.6, the spectral sequence of Lemma 4.5 becomes

$$E_1^{p,q} = \mathcal{J}^0(X, V_{p,q}) \implies H^{p+q}(F, \mathcal{A}'(\eta^*V)).$$

Combining this spectral sequence with Lemma 4.4 we arrive at the Penrose transform for currents:

Theorem 4.7. *If η has contractible fibers and V is a \mathcal{Q} -compatible vector bundle over Z , then there is a spectral sequence*

$$E_1^{p,q} = \mathcal{J}^0(X, V_{p,q}) \implies H^{p+q}(Z, \mathcal{Q}'(V)).$$

4.3 The Penrose transform for compactly supported currents.

In this section we construct a Penrose transform for compactly supported currents, under the same set-up used in the previous section. All the notations and conventions used in Section 4.2 will be used here as well. Recall that we have used the subscript “cv” to denote compact vertical support. (If $\eta : F \rightarrow Z$ is a fiber-bundle then a form or a current on F has compact vertical support if its support is compact when restricted to $\eta^{-1}(K)$, for every compact subset K of Z).

The pull-back step.

Lemma 4.8. *Let $\eta : F \rightarrow Z$ be a fiber-bundle with d -dimensional contractible connected fibers. There is a natural map*

$$T : \mathcal{J}_{\text{cv}}^0(F, \mathcal{C}^{0,d}) \rightarrow \mathcal{J}^0(Z)$$

which induces an isomorphism

$$\hat{T} : H^d(F, \mathcal{C}'_{\text{cv}}) \cong \mathcal{J}^0(Z).$$

Proof. The proof has four steps.

1. **Step1** We explain the definition of the map T . From Theorem 2.26 we know that

$$\mathcal{J}^0(F, \mathcal{C}^{0,d}) \cong (\Gamma_c(F, \mathcal{C}^{m,0}))'$$

It follows that the currents from $\mathcal{J}_{\text{cv}}^0(F, \mathcal{C}^{0,d})$ are applied to sections of $\mathcal{C}^{m,0}$ with compact support in the base direction, but not necessarily with compact support in the fiber direction. It is clear that via η^* the space $\mathcal{D}^m(Z)$ is a subspace of the space of sections of $\mathcal{C}^{m,0}$ with compact support in the

base direction, and it makes sense to restrict the currents from $\mathcal{J}_{\text{cv}}^0(F, \mathcal{C}^{0,d})$ to the space $\mathcal{D}^m(Z)$. By definition, the map T is this restriction and is well defined.

In order to prove that T induces a bijective map

$$\hat{T} : H^d(F, \mathcal{C}'_{\text{cv}}) \cong \mathcal{J}^0(Z)$$

we will consider $F = \mathbb{R}^m \times F_0$ (with F_0 contractible, connected and oriented) and η the trivial bundle over \mathbb{R}^m with typical fiber F_0 . Also let $\pi : \mathbb{R}^m \times F_0 \rightarrow F_0$ be the projection on the second factor.

2. **Step2** We show that the map T induces a map \hat{T} defined on $H^d(F, \mathcal{C}'_{\text{cv}})$. For this let $\omega \in \mathcal{J}_{\text{cv}}^0(F, \mathcal{C}^{0,d-1})$ and $\gamma \in \mathcal{D}^m(\mathbb{R}^m)$. Then

$$\begin{aligned} (T(\bar{\partial}_{\mathcal{C}}(\omega)), \gamma) &= (\bar{\partial}_{\mathcal{C}}(\omega), \gamma) \\ &= (-1)^d(\omega, d(\gamma)) \\ &= 0 \end{aligned}$$

since $d(\gamma) = 0$. It follows that $T(\bar{\partial}_{\mathcal{C}}(\omega)) = 0$ and that T descends to a map \hat{T} defined on $H^d(F, \mathcal{C}'_{\text{cv}})$.

3. **Step3** We show that the map \hat{T} is injective.

Since F_0 is contractible there are homotopy operators

$$K_1 : \mathcal{E}^1(F_0) \rightarrow \mathcal{E}(F_0)$$

and

$$K_2 : \mathcal{E}^2(F_0) \rightarrow \mathcal{E}^1(F_0)$$

which satisfy

$$\beta = dK_1(\beta) + K_2d(\beta)$$

for $\beta \in \mathcal{E}^1(F_0)$. We will consider K_i ($i = 1, 2$) defined on the space $\eta^*\mathcal{D}^m(\mathbb{R}^m) \otimes \pi^*\mathcal{E}^i(F_0)$ and acting trivially on the factor $\eta^*\mathcal{D}^m(\mathbb{R}^m)$.

To prove the injectivity of \hat{T} let $[\omega] \in H^d(F, \mathcal{C}'_{\text{cv}})$ such that $\hat{T}([\omega]) = 0$. Thus $\omega \in \mathcal{J}_{\text{cv}}^0(F, \mathcal{C}^{0,d})$, and we define $\underline{\omega}$ on the space $\eta^*\mathcal{D}^m(\mathbb{R}^m) \otimes \pi^*\mathcal{E}^1(F_0)$ by the formula

$$(\underline{\omega}, \underline{\gamma}) = (-1)^d(\omega, K_1(\underline{\gamma}))$$

Since K_1 is continuous, the map $\underline{\omega}$ is continuous as well and belongs to $\mathcal{J}_{\text{cv}}^0(F, \mathcal{C}^{0,d-1})$. We claim that $\bar{\partial}_{\mathcal{C}}(\underline{\omega}) = \omega$. For this let $\gamma \in \eta^*\mathcal{D}^m(\mathbb{R}^m) \otimes$

$\pi^*\mathcal{E}(F_0)$ and notice that

$$\begin{aligned} (\bar{\partial}_{\mathcal{C}}(\underline{\omega}) - \omega, \gamma) &= (-1)^d(\underline{\omega}, \bar{\partial}_{\mathcal{C}}(\gamma)) - (\omega, \gamma) \\ &= (\omega, K_1\bar{\partial}_{\mathcal{C}}\gamma - \gamma) \\ &= 0 \end{aligned}$$

(Here we have used the homotopy property of K_1 and K_2 to obtain that $\bar{\partial}_{\mathcal{C}}(\gamma - K_1\bar{\partial}_{\mathcal{C}}\gamma) = 0$. This implies that $K_1(\bar{\partial}_{\mathcal{C}}\gamma) - \gamma \in \mathcal{D}^m(\mathbb{R}^m)$ and that ω applied to it is 0). The injectivity of the map T follows.

4. **Step4** We show that the map \hat{T} is surjective. For this consider $\omega \in \mathcal{J}^0(\mathbb{R}^m)$ and $\rho \in \mathcal{D}^d(F_0)$ such that

$$\int_{F_0} \rho = 1.$$

Then $\eta^*\omega \wedge \rho \in \mathcal{J}_{\text{cv}}^d(\mathbb{R}^m \times F_0)$ and we will restrict it to the space $\eta^*\mathcal{D}^m(\mathbb{R}^m) \otimes \pi^*\mathcal{E}(F_0)$. This restriction determines an element $\tilde{\omega}$ in $\mathcal{J}_{\text{cv}}^0(F, \mathcal{C}^{0,d})$ whose class in $H^d(F, \mathcal{C}'_{\text{cv}})$ will be denoted $\hat{\omega}$. A simple calculation shows that $T(\tilde{\omega}) = \omega$ and the surjectivity of \hat{T} follows. □

Corollary 4.9. *If η has contractible fibers and V is a vector bundle over Z then there is an isomorphism*

$$T : H^d(F, \mathcal{C}'_{\text{cv}}(\eta^*V)) \cong \mathcal{J}^0(Z, V).$$

Lemma 4.10. *Let $\eta : F \rightarrow Z$ be a fiber-bundle with connected fibers. If \mathcal{Q} is an involutive structure over Z and $V \rightarrow Z$ is \mathcal{Q} -compatible, then for every $k \geq 0$,*

$$H^k(F, \mathcal{Q}'_c(\eta^*V)) \cong \mathcal{J}_c^0(Z, \mathcal{H}_c^k \otimes V)$$

where \mathcal{H}_c^k is the bundle whose fiber over $z \in Z$ is the k -de Rham compactly supported cohomology group of $\eta^{-1}(z)$.

Proof. The proof of this lemma is standard by now.

One first proves that

$$H^k(F, \mathcal{Q}'_{\text{cv}}(\eta^*V)) \cong \mathcal{J}^0(Z, \mathcal{H}_c^k \otimes V).$$

A Cech-de Rham argument reduces this statement to the case of contractible fibers. For the case of contractible fibers, the cohomology in degree less than d is zero (by a homotopy argument), and the cohomology in degree d is given by Corollary 4.9.

One then returns to the compactly supported cohomology using an argument similar to the argument used in Lemma 3.1. □

The pull-back step is summarized by the following lemma.

Lemma 4.11. *Let $\eta : F \rightarrow Z$ be a fiber-bundle with connected fibers and \mathcal{Q} an involutive structure on Z . If V is a \mathcal{Q} -compatible vector bundle over Z then there is a spectral sequence*

$$E_2^{p,q} = H^p(Z, \mathcal{Q}'_c(\mathcal{H}_c^q \otimes V)) \implies H^{p+q}(F, \mathcal{A}'_c(\eta^*V)).$$

The intermediate and the push-down step follow easily.

The intermediate step. There is a spectral sequence

$$E_1^{p,q} = H^q(F, \mathcal{E}'_c(B^p \otimes \eta^*V)) \implies H^{p+q}(F, \mathcal{A}'_c(\eta^*V)).$$

The push down step. Define $V_{p,q} = \tau_*^q(F, B^p \otimes \eta^*V)$. With this notation,

$$H^q(F, \mathcal{E}'_c(B^p \otimes \eta^*V)) \cong \mathcal{J}_c^0(X, V_{p,q}).$$

The Penrose transform for compactly supported currents. The following spectral sequence is the Penrose transform for compactly supported currents:

$$E_1^{p,q} = \mathcal{J}_c^0(X, V_{p,q}) \implies H^{p+q}(F, \mathcal{A}'_c(\eta^*V)).$$

Theorem 4.12. *If the fibers of η are d -dimensional and contractible, there is a spectral sequence*

$$E_1^{p,q} = \mathcal{J}_c^0(X, V_{p,q}) \implies H^{p+q-d}(Z, \mathcal{Q}'_c(V)).$$

4.4 The non-oriented case.

We have constructed Penrose transforms for the non-compactly and compactly supported distribution cohomology under the assumptions that all the manifolds were oriented. The orientations of the manifolds Z and F induced an orientation of the fiber-bundle η . The fiber-bundle τ is always oriented by the complex structure on its fibers, which varies smoothly with respect to the base. In this section we shall briefly describe the transform when none of the manifolds Z , F or X is oriented.

The non-compactly supported transform. The non-compactly supported transform in the non-oriented case works like in the oriented case, and generates the same bundles and operators.

The compactly supported transform. At the compactly supported transform in the non-oriented case a twist by $\tilde{\mathcal{E}}_\eta$ appears at the pull-back step. This is because in the non-oriented case there is an isomorphism

$$H^d(F, \mathcal{C}_{cv}(\tilde{\mathcal{E}}_\eta)) \cong \mathcal{J}^0(Z)$$

having a similar definition as the map \hat{T} from Lemma 4.8. The rest of the transform is identical to the transform in the oriented case. The compactly supported distribution Penrose transform in the non-oriented case gives a spectral sequence

$$E_1^{p,q} = \mathcal{J}_c^0(X, W_{p,q}) \implies H^{p+q-d}(Z, \mathcal{Q}'_c(V))$$

where $W_{p,q} = \tau_*^q(F, B^p \otimes \eta^*V \otimes \tilde{\mathcal{E}}_\eta)$.

Chapter 5

The Penrose transform in practice.

5.1 Introduction.

In this chapter we consider the extension to currents and compactly supported cohomology of some Penrose transforms that appear in the literature. We also show how the natural bilinear pairing between the smooth and the compactly supported cohomology carries through the Penrose transforms.

Notation. If $A : \Gamma(M, V) \rightarrow \Gamma(M, W)$ is a differential operator, we shall use the notation $A_c : \Gamma_c(M, V) \rightarrow \Gamma_c(M, W)$ for the induced operator on the space of compactly supported sections.

5.2 The Euclidean space \mathbb{R}^3 .

A twistor correspondence for \mathbb{R}^3 was constructed by Jones and Tod in [25] and by Hitchin in [21]. The corresponding Penrose transform for the line bundles $\mathcal{O}(n, \lambda)$ was fully considered by Tsai in [46]. Briefly the set-up is as follows.

5.2.1 The set-up.

The twistor space of \mathbb{R}^3 . Consider the double fibration

$$\begin{array}{ccc} & F & \\ \eta \swarrow & & \searrow \tau \\ Z & & \mathbb{R}^3 \end{array}$$

where

1. The space Z is the space of all oriented lines in \mathbb{R}^3 .

2. The space F is the set of pairs (p, l) , where $l \in Z$ and $p \in l$.
3. The maps η and τ are the obvious projections.

A simple check shows that the fibers of η are copies of \mathbb{R} and the fibers of τ are copies of \mathbb{CP}^1 . The space Z is called the twistor space of \mathbb{R}^3 and has several equivalent definitions:

1. Consider the group $\text{ESU}(2)$ which is the set of all matrices

$$\{(A, B) \mid A \in \text{SU}(2), B \in \{2 \times 2 \text{ hermitian trace-free matrices}\}\}$$

with the group operation $(A, B) \circ (A', B') = (AA', AB'A^{-1} + B)$. It acts on

$$\mathbb{R}^3 = \left\{ \left(\begin{array}{cc} z & x - iy \\ x + iy & -z \end{array} \right), x, y, z \in \mathbb{R} \right\}.$$

by associating to each $(A, B) \in \text{ESU}(2)$ the automorphism of \mathbb{R}^3 defined by

$$\mathbb{R}^3 \ni X \rightarrow AXA^{-1} + B \in \mathbb{R}^3.$$

This action of $\text{ESU}(2)$ on \mathbb{R}^3 induces a transitive action on Z with isotropy subgroup

$$L = \left\{ \left(\left(\begin{array}{cc} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{array} \right), \left(\begin{array}{cc} z_0 & 0 \\ 0 & -z_0 \end{array} \right) \right), \theta \in \mathbb{R}, z_0 \in \mathbb{R} \right\}.$$

Thus as a real homogeneous manifold the twistor space of \mathbb{R}^3 is the space $\text{ESU}(2)/L$.

2. The twistor space of \mathbb{R}^3 can be defined as the tangent bundle

$$TS^2 = \{(\underline{n}, \underline{x}) \mid \underline{n}, \underline{x} \in \mathbb{R}^3, \underline{n} \cdot \underline{n} = 1, \underline{n} \cdot \underline{x} = 0\}$$

of the sphere S^2 . The identification between TS^2 and Z is defined by associating to $(\underline{n}, \underline{x})$ the line through the point \underline{x} of direction \underline{n} .

3. The twistor space of \mathbb{R}^3 can be defined as the holomorphic tangent bundle of \mathbb{CP}^1 .

Consider the Lie sub-algebra

$$\mathfrak{q} = \left\{ \left(\left(\begin{array}{cc} t & s \\ 0 & -t \end{array} \right), \left(\begin{array}{cc} z & w \\ 0 & -z \end{array} \right) \right), t, s, z, w \in \mathbb{C} \right\}.$$

of the complexification of the Lie algebra of $\text{ESU}(2)$. The Lie algebra \mathfrak{q} contains the complexification of the Lie algebra of L and determines an homogeneous involutive structure $\mathcal{Q}^{1,0}$ on $Z \cong \text{ESU}(2)/L$, which is the complex structure of the complex manifold $Z \cong T\mathbb{CP}^1$.

Homogeneous line bundles on \mathcal{T} . We will consider the line bundles $\mathcal{O}(n, \lambda)$ on Z which are associated to the (\mathbf{q}, L) -module defined by

$$\rho \left(\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} z_0 & 0 \\ 0 & -z_0 \end{pmatrix} \right) \right) = e^{-in\theta - \lambda z_0},$$

$$\rho' \left(\left(\begin{pmatrix} t & s \\ 0 & -t \end{pmatrix}, \begin{pmatrix} z & w \\ 0 & -z \end{pmatrix} \right) \right) = -nt - \lambda z,$$

where $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$. These line bundles are \mathcal{Q} -compatible and hence holomorphic. A simple calculation shows that the canonical bundle of Z is $\mathcal{O}(-4, 0)$.

The differential operators. Since the action of $\text{ESU}(2)$ on \mathbb{R}^3 is transitive with isotropy subgroup $\text{SU}(2)$, the space \mathbb{R}^3 is the homogeneous manifold $\text{ESU}(2)/\text{SU}(2)$. The spin-bundle \mathcal{E}_A on \mathbb{R}^3 is the homogeneous rank 2 complex vector bundle associated to the defining representation of $\text{SU}(2)$ on \mathbb{C}^2 . The Levi-Civita connection of the Euclidean space \mathbb{R}^3 induces a connection (called the spin connection) on the bundle \mathcal{E}_A and on its dual \mathcal{E}^A . The operators which arise from the smooth Penrose transform for \mathbb{R}^3 applied to the bundles $\mathcal{O}(n, \lambda)$ are presented in the following definition (for a brief account on the Penrose spinor notation, the reader is referred to [46], page 13).

Definition 5.1. 1. *The Laplace operator*

$$\Delta : \mathcal{E}(\mathbb{R}^3) \rightarrow \mathcal{E}(\mathbb{R}^3)$$

is defined by

$$\Delta(f) = \nabla^{AB} \nabla_{AB}(f).$$

2. *The operator*

$$\mathbf{A}_{n,\lambda} : \Gamma(\mathbb{R}^3, \mathcal{E}_A \otimes \odot^n \mathcal{E}^A) \rightarrow \Gamma(\mathbb{R}^3, \odot^{n+1} \mathcal{E}^A)$$

(here “ \odot ” denotes the symmetrised tensor product) is defined by

$$\mathbf{A}_{n,\lambda}(\Psi_A^{BC\dots E}) = \nabla^A{}^H \Psi_A^{BC\dots E} - \lambda \Psi^{(AB\dots E)}.$$

3. *The operator*

$$\mathbf{B}_{n,\lambda} : \Gamma(\mathbb{R}^3, \odot^n \mathcal{E}^A) \rightarrow \Gamma(\mathbb{R}^3, \mathcal{E}_A \otimes \odot^{n+1} \mathcal{E}^A)$$

is defined by

$$\mathbf{B}_{n,\lambda}(\gamma^{C\dots E}) \nabla_A^{(B} \gamma^{C\dots E)} + \lambda \epsilon_A^{(B} \gamma^{C\dots E)}.$$

4. *The operator*

$$\mathbf{E}_{n,\lambda} : \Gamma(\mathbb{R}^3, \odot^n \mathcal{E}_A) \rightarrow \Gamma(\mathbb{R}^3, \mathcal{E}_A \otimes \odot^{n-1} \mathcal{E}_A)$$

is defined by

$$\mathbf{E}_{n,\lambda}(\Theta_{B\dots E}) = \nabla_A^B (\Theta_{B\dots E}) + \lambda \Theta_{AC\dots E}.$$

The smooth Penrose transform. The smooth Penrose transform of the bundles $\mathcal{O}(n, \lambda)$ on Z has been constructed in [46], Chapter 6 and is summarized in the following table:

| n | k | $H^k(Z, \mathcal{O}(n, \lambda))$ | $H^k(Z, \mathcal{O}(-n-4, -\lambda))$ |
|-----------|-----|--|---|
| ≥ -1 | 0 | $\text{Ker}(\mathbf{B}_{n,\lambda})$ | 0 |
| ≥ -1 | 1 | $\text{Ker}(\mathbf{A}_{n+1,\lambda})/\text{Im}(\mathbf{B}_{n,\lambda})$ | $\text{Ker}(\mathbf{E}_{n+2,-\lambda})$ |
| ≥ -1 | 2 | $\text{Coker}(\mathbf{A}_{n+1,\lambda})$ | |
| -2 | 0 | 0 | 0 |
| -2 | 1 | $\text{Ker}(\Delta + 2\lambda^2)$ | $\text{Ker}(\Delta + 2\lambda^2)$ |
| -2 | 2 | $\text{Coker}(\Delta + 2\lambda^2)$ | $\text{Coker}(\Delta + 2\lambda^2)$ |

5.2.2 The compactly supported smooth Penrose transform.

Since the typical fiber of the fiber-bundle η is a copy of \mathbb{R} , a shift in dimension by 1 appears at the compactly supported smooth Penrose transform. Theorem 3.6 together with the smooth Penrose transform for \mathbb{R}^3 imply the following result.

Theorem 5.2. *The compactly supported smooth Penrose transform gives isomorphisms between (smooth) compactly supported Dolbeault cohomology groups of the bundles $\mathcal{O}(n, \lambda)$ and smooth compactly supported fields on \mathbb{R}^3 according to the following table:*

| n | k | $H_c^k(Z, \mathcal{O}(n, \lambda))$ | $H_c^k(Z, \mathcal{O}(-n-4, -\lambda))$ |
|-----------|-----|--|---|
| ≥ -1 | 0 | $\text{Ker}(\mathbf{A}_{n+1,\lambda}_c)/\text{Im}(\mathbf{B}_{n,\lambda}_c)$ | $\text{Ker}(\mathbf{E}_{n+2,-\lambda}_c)$ |
| ≥ -1 | 1 | $\text{Coker}(\mathbf{A}_{n+1,\lambda}_c)$ | |
| -2 | 0 | $\text{Ker}(\Delta + 2\lambda^2)_c$ | $\text{Ker}(\Delta + 2\lambda^2)_c$ |
| -2 | 1 | $\text{Coker}(\Delta + 2\lambda^2)_c$ | $\text{Coker}(\Delta + 2\lambda^2)_c$ |

Remark. Since compactly supported Dolbeault cohomology groups in degree 0 are 0, we deduce the following facts:

1. We rediscover that every compactly supported eigenfunction of the Laplace operator Δ which corresponds to a non-positive eigenvalue is identically 0.
2. If $n \geq -1$ then every compactly supported solution of the operator $\mathbf{E}_{n+2,-\lambda}$ is identically 0.
3. If $n \geq -1$ then

$$\text{Ker}(\mathbf{A}_{n+1,\lambda})_c = \text{Im}(\mathbf{B}_{n,\lambda})_c.$$

5.2.3 The duality pairings.

Since the complex dimension of Z is 2 and the canonical bundle of Z is $\mathcal{O}(-4, 0)$ there is a duality pairing

$$H_c^p(Z, \mathcal{O}(n, \lambda)) \otimes H^q(Z, \mathcal{O}(-n-4, -\lambda)) \rightarrow \mathbb{C}$$

when $p+q=2$. From Theorem 3.9 we obtain the following result.

Theorem 5.3. *Let $n \geq -1$. The Dolbeault duality pairing*

$$H_c^1(Z, \mathcal{O}(n, \lambda)) \otimes H^1(Z, \mathcal{O}(-n-4, -\lambda)) \rightarrow \mathbb{C}$$

becomes the map

$$\text{Coker}(\mathbf{A}_{n+1, \lambda})_c \otimes \text{Ker}(\mathbf{E}_{n+2, -\lambda}) \rightarrow \mathbb{C}$$

induced by contraction of sections followed by integration.

Let $n = -2$. The duality pairing

$$H^1(Z, \mathcal{O}(-2, \lambda)) \otimes H_c^1(Z, \mathcal{O}(-2, -\lambda)) \rightarrow \mathbb{C}$$

becomes the map

$$I : \text{Ker}(\Delta + 2\lambda^2) \otimes \text{Coker}(\Delta + 2\lambda^2)_c \rightarrow \mathbb{C}$$

induced by multiplication of functions followed by the integration.

Remarks.

1. We can show directly that the contraction followed by integration descends to the corresponding quotients, like in the statement of the theorem. Consider for example the case $n = -2$. We will show that the map I is well defined. For this let g and \tilde{g} such that they determine the same class in the quotient $\mathcal{D}(\mathbb{R}^3)/\text{Im}(\Delta + 2\lambda^2)_c$. Then there is $h \in \mathcal{D}(\mathbb{R}^3)$ such that

$$\Delta_c(h) + 2\lambda^2 h = g - \tilde{g}$$

and for every $f \in \text{Ker}(\Delta + 2\lambda^2)$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} f(g - \tilde{g}) &= \int_{\mathbb{R}^3} f(\Delta_c(h) + 2\lambda^2 h) \\ &= \int_{\mathbb{R}^3} \Delta_c(f)h + 2\lambda^2 \int_{\mathbb{R}^3} fh \\ &= -2\lambda^2 \int_{\mathbb{R}^3} fh + 2\lambda^2 \int_{\mathbb{R}^3} fh \\ &= 0 \end{aligned}$$

using an integration by parts. It follows that the induced map

$$I : \text{Ker}(\Delta + 2\lambda^2) \otimes \text{Coker}(\Delta + 2\lambda^2)_c \rightarrow \mathbb{C}$$

is well defined.

2. The nondegeneracy of the pairing

$$H_c^1(Z, \mathcal{O}(-2, 0)) \otimes H^1(Z, \mathcal{O}(-2, 0)) \rightarrow \mathbb{C}$$

(see [27]) provides a nice way of characterising the image of the Laplacian $(\Delta)_c$: a compactly supported function $f \in \mathcal{D}(\mathbb{R}^3)$ is the image through Δ of a compactly supported function if and only if it satisfies

$$\int_{\mathbb{R}^3} fg = 0$$

for every solution $g \in \mathcal{E}(\mathbb{R}^3)$ of the Laplace's equation.

5.2.4 The Penrose transform for compactly supported currents.

The Penrose transform for the compactly supported distribution cohomology is formally the same as the Penrose transform for the compactly supported smooth cohomology, and can be summarised by a table totally similar to the table given in Subsection 5.2.2. Also, compactly supported distribution Dolbeault cohomology groups coincide with compactly supported smooth Dolbeault cohomology groups. Since the compactly supported (smooth and distribution) Penrose transforms for the Euclidean space \mathbb{R}^3 give the isomorphisms

$$H_c^1(Z, \mathcal{O}(-2)) \cong \text{Coker}(\Delta)_c$$

(considered for the smooth and the distribution cohomology respectively) we obtain the interesting result that every compactly supported distribution on \mathbb{R}^3 is the sum of a compactly supported smooth function on \mathbb{R}^3 and of the image through Δ of a compactly supported distribution on \mathbb{R}^3 .

5.3 The Minkowski space \mathbb{R}^4 .

We turn now to the classical twistor correspondence and Penrose transform of Penrose (see [19]).

5.3.1 The set-up.

Consider the space \mathbb{C}^4 with its standard conformal structure and orientation. Every totally null 2-plane in \mathbb{C}^4 has the property that its tangent bivector (determined up to a multiplicative scalar) is either self-dual (in which case the 2-plane is called α -plane) or anti-self-dual (in which case the 2-plane is called β -plane). The set of all α -planes is called the twistor space of \mathbb{C}^4 . As a complex manifold it is the space \mathbb{CP}^3 with a projective line I removed.

Let F be the set of pairs (z, Z) where Z is an α -plane and $z \in Z$. The classical twistor correspondence of Penrose is the natural double fibration

$$\begin{array}{ccc} & \mathcal{F} & \\ p_1 \swarrow & & \searrow p_2 \\ \mathbb{CP}^3 \setminus I & & \mathbb{C}^4 \end{array}$$

where p_1 and p_2 are the obvious projection maps. The fibers of p_1 are copies of \mathbb{C}^2 and the fibers of p_2 are copies of \mathbb{CP}^1 .

In [50] Woodhouse considered the restriction of the classical twistor correspondence to the real Minkowski space \mathbb{R}^4 embedded into \mathbb{C}^4 . He obtained a twistor correspondence for \mathbb{R}^4

$$\begin{array}{ccc} & F & \\ \eta \swarrow & & \searrow \rho \\ \mathcal{P} & & \mathbb{R}^4 \end{array}$$

where the fibers of η are copies of \mathbb{R} , the fibers of ρ are copies of \mathbb{CP}^1 and \mathcal{P} is a real hyper-surface of \mathbb{CP}^3 . He considered the natural bundles $\mathcal{L}(k)$ on \mathcal{P} which are restrictions of the natural bundles $\mathcal{O}(k)$ on \mathbb{CP}^3 , and noticed that the operators which arise from the Penrose transform of these bundles are restrictions of the operators which arise from the classical Penrose transform (see [19]). They are listed in the following definition (for details about spin bundles on \mathbb{R}^4 , Penrose spinor notations, and the spin connection, see [19]).

Definition 5.4. *The wave operator is defined by*

$$\nabla^a \nabla_a : \Gamma(\mathbb{R}^4, \mathcal{O}[-1]) \rightarrow \Gamma(\mathbb{R}^4, \mathcal{O}[-3]).$$

The Dirac-Weyl operator

$$\mathbf{W}_k : \Gamma(\mathbb{R}^4, \underbrace{\mathcal{O}_{(A' \dots D')}[-1]}_{-k-2}) \rightarrow \Gamma(\mathbb{R}^4, \underbrace{\mathcal{O}_{A(B' \dots D')}[-2]}_{-k-3})$$

is defined by

$$\mathbf{W}_k(\pi_{A' \dots D'}) = \nabla^{A'}_A(\pi_{A' \dots D'}).$$

The operator

$$\mathbf{A}_k : \Gamma(\mathbb{R}^4, \underbrace{\mathcal{O}_{A(A' \dots E')}}_{-k-3}[-k-4]) \rightarrow \Gamma(\mathbb{R}^4, \underbrace{\mathcal{O}_{(A' \dots E' F')}}_{-k-2}[-k-5])$$

is defined by the formula

$$\mathbf{A}_k(\pi_{AA' \dots E'}) = \nabla_{A(T' \pi_{A' \dots E'})}^A.$$

The operator

$$\mathbf{B}_k : \Gamma(\mathbb{R}^4, \underbrace{\mathcal{O}_{(A' \dots B')}}_{-k-4}[-k-4]) \rightarrow \Gamma(\mathbb{R}^4, \underbrace{\mathcal{O}_{A(C' A' \dots B')}}_{-k-3}[-k-4])$$

is defined by the formula

$$\mathbf{B}_k(\pi_{A' \dots B'}) = \nabla_{A(C' \pi_{A' \dots B'})}.$$

The operator

$$\mathbf{E} : \Gamma(\mathbb{R}^4, \mathcal{O}^A[-2]) \rightarrow \Gamma(\mathbb{R}^4, \mathcal{O}^{A'}[-3])$$

is defined by the formula

$$\mathbf{E}(\pi^A) = \nabla_A^{A'} \pi^A.$$

The operator

$$\mathbf{F}_k : \Gamma(\mathbb{R}^4, \underbrace{\mathcal{O}_{A(A' \dots F')}}_{-k-3}[-2]) \rightarrow \Gamma(\mathbb{R}^4, \underbrace{\mathcal{O}_{(A' \dots E')}}_{-k-4}[-4])$$

is defined by the formula

$$\mathbf{F}_k(\pi_{AA' \dots F'}) = \nabla^{AA'} \pi_{AA' \dots F'}.$$

Remark. The smooth complex-valued 2-forms on the Minkowski space \mathbb{R}^4 split as a direct sum of self-dual and anti-self-dual:

$$\mathcal{E}^2 = \mathcal{E}_+^2 + \mathcal{E}_-^2$$

1. The operator \mathbf{W}_{-4} is the exterior derivative

$$d : \mathcal{E}_-^2 \rightarrow \mathcal{E}^3.$$

2. The operator \mathbf{A}_{-4} is the exterior derivative (followed by the projection onto the space of self-dual 2-forms)

$$d : \mathcal{E}^1 \rightarrow \mathcal{E}_+^2.$$

3. The operator \mathbf{B}_{-4} is the exterior derivative

$$d : \mathcal{E} \rightarrow \mathcal{E}^1.$$

The smooth Penrose transform. The bundles $\mathcal{L}(k)$ are compatible with the involutive structure \mathcal{Q} which is the CR-structure of \mathcal{P} induced by the complex structure of $\mathbb{C}\mathbb{P}^3$. Following [7] (page 90) and [50] the smooth Penrose transform for the Minkowski space \mathbb{R}^4 applied to the bundles $\mathcal{L}(k)$ is summarized in the following table:

| k | n | $H^n(\mathcal{P}, \mathcal{Q}(\mathcal{L}(k)))$ | $H^n(\mathcal{P}, \mathcal{Q}(\mathcal{L}(-k-4)))$ |
|-----------|-----|--|--|
| ≤ -4 | 0 | 0 | $\text{Ker}(\mathbf{B}_k)$ |
| ≤ -4 | 1 | $\text{Ker}(\mathbf{W}_k)$ | $\text{Ker}(\mathbf{A}_k)/\text{Im}(\mathbf{B}_k)$ |
| ≤ -4 | 2 | $\text{Ker}(\mathbf{F}_k)/\text{Im}(\mathbf{W}_k)$ | $\text{Coker}(\mathbf{A}_k)$ |
| -3 | 0 | 0 | 0 |
| -3 | 1 | $\text{Ker}(\mathbf{W}_{-3})$ | $\text{Ker}(\mathbf{E})$ |
| -3 | 2 | $\text{Coker}(\mathbf{W}_{-3})$ | $\text{Coker}(\mathbf{E})$ |
| -2 | 0 | 0 | 0 |
| -2 | 1 | $\text{Ker}(\nabla^a \nabla_a)$ | $\text{Ker}(\nabla^a \nabla_a)$ |
| -2 | 2 | $\text{Coker}(\nabla^a \nabla_a)$ | $\text{Coker}(\nabla^a \nabla_a)$ |

5.3.2 The compactly supported smooth Penrose transform.

Since the fibers of η are copies of \mathbb{R} , a shift in dimension by 1 will appear at the compactly supported transform. Theorem 3.6 together with the smooth Penrose transform for the Minkowski space \mathbb{R}^4 imply the following result.

Theorem 5.5. *The compactly supported smooth Penrose transform of the bundles $\mathcal{L}(k)$ gives isomorphisms between compactly supported smooth $\bar{\partial}_b$ -cohomology groups of these bundles and compactly supported smooth fields on \mathbb{R}^4 according to the following table*

| k | n | $H_c^n(\mathcal{P}, \mathcal{Q}(\mathcal{L}(k)))$ | $H_c^n(\mathcal{P}, \mathcal{Q}(\mathcal{L}(-k-4)))$ |
|-----------|-----|--|--|
| ≤ -4 | 0 | $\text{Ker}(\mathbf{W}_k)_c$ | $\text{Ker}(\mathbf{A}_k)_c/\text{Im}(\mathbf{B}_k)_c$ |
| ≤ -4 | 1 | $\text{Ker}(\mathbf{F}_k)_c/\text{Im}(\mathbf{W}_k)_c$ | $\text{Coker}(\mathbf{A}_k)_c$ |
| -3 | 0 | $\text{Ker}(\mathbf{W}_{-3})_c$ | $\text{Ker}(\mathbf{E})_c$ |
| -3 | 1 | $\text{Coker}(\mathbf{W}_{-3})_c$ | $\text{Coker}(\mathbf{E})_c$ |
| -2 | 0 | $\text{Ker}(\nabla^a \nabla_a)_c$ | $\text{Ker}(\nabla^a \nabla_a)_c$ |
| -2 | 1 | $\text{Coker}(\nabla^a \nabla_a)_c$ | $\text{Coker}(\nabla^a \nabla_a)_c$ |

Remark. From the compactly supported Penrose transform in degree 0 we deduce following facts.

1. Consider $k \leq -4$. Then

$$\text{Ker}(\mathbf{A}_k)_c \text{Im}(\mathbf{B}_k)_c.$$

In particular, when $k = -4$ it follows that a compactly supported complex-valued 1-form with the self-dual part of its exterior derivative vanishing is necessarily the exterior derivative of a compactly supported function.

Also, every compactly supported solution of the Dirac-Weyl operator \mathbf{W}_k is 0.

2. Consider $k = -3$. It follows that every compactly supported solution of \mathbf{E} and of the Dirac operator \mathbf{W}_{-3} is identically 0.
3. Consider $k = -2$. It follows that every compactly supported solution of the wave operator on \mathbb{R}^4 is identically 0.

5.3.3 The “duality” pairing.

Since the canonical bundle of $\mathbb{C}\mathbb{P}^3$ is $\mathcal{O}(-4)$ and the rank of the bundle $\mathcal{Q}^{0,1}$ is 2, there is a “duality” pairing

$$H^p(\mathcal{P}, \mathcal{Q}_c(\mathcal{L}(k))) \otimes H^q(\mathcal{P}, \mathcal{Q}(\mathcal{L}(-k-4))) \rightarrow \mathbb{C}$$

whenever $p + q = 2$. From Theorem 3.9 we obtain the following result.

Theorem 5.6. *Let $k \leq -4$. The duality maps*

$$H^1(\mathcal{P}, \mathcal{Q}(\mathcal{L}(k))) \otimes H^1(\mathcal{P}, \mathcal{Q}_c(\mathcal{L}(-k-4))) \rightarrow \mathbb{C}$$

and

$$H^1(\mathcal{P}, \mathcal{Q}_c(\mathcal{L}(k))) \otimes H^1(\mathcal{P}, \mathcal{Q}(\mathcal{L}(-k-4))) \rightarrow \mathbb{C}$$

become respectively the maps

$$\text{Ker}(\mathbf{W}_k) \otimes \text{Coker}(\mathbf{A}_k)_c \rightarrow \mathbb{C}$$

and

$$\text{Ker}(\mathbf{F}_k)_c / \text{Im}(\mathbf{W}_k)_c \otimes \text{Ker}(\mathbf{A}_k) / \text{Im}(\mathbf{B}_k) \rightarrow \mathbb{C}$$

induced by contraction of sections followed by integration .

The duality maps

$$H^1(\mathcal{P}, \mathcal{Q}(\mathcal{L}(-1))) \otimes H^1(\mathcal{P}, \mathcal{Q}_c(\mathcal{L}(-3))) \rightarrow \mathbb{C}$$

and

$$H^1(\mathcal{P}, \mathcal{Q}_c(\mathcal{L}(-1))) \otimes H^1(\mathcal{P}, \mathcal{Q}(\mathcal{L}(-3))) \rightarrow \mathbb{C}$$

become respectively the maps

$$\text{Ker}(\mathbf{E}) \otimes \text{Coker}(\mathbf{W}_{-3})_c \rightarrow \mathbb{C}$$

and

$$\text{Coker}(\mathbf{E})_c \otimes \text{Ker}(\mathbf{W}_{-3}) \rightarrow \mathbb{C}$$

induced by contraction of sections followed by integration.

The duality map

$$H^1(\mathcal{P}, \mathcal{Q}(\mathcal{L}(-2))) \otimes H^1(\mathcal{P}, \mathcal{Q}_c(\mathcal{L}(-2))) \rightarrow \mathbb{C}$$

becomes the map

$$\text{Ker}(\nabla^a \nabla_a) \otimes \text{Coker}(\nabla^a \nabla_a)_c \rightarrow \mathbb{C}$$

induced by contraction of sections followed by integration.

5.3.4 The transform for currents.

The Penrose transform for currents is formally the same as the smooth Penrose transform, and can be summarized by a table totally similar to the table given in Subsection 5.3.1. In particular, we obtain an isomorphism between $H^1(\mathcal{P}, \mathcal{Q}'(\mathcal{L}(-2)))$ and distribution solutions of the wave operator on \mathbb{R}^4 . In the case of a CR-manifold distribution cohomology groups are different from smooth cohomology groups. In particular $H^1(\mathcal{P}, \mathcal{Q}'(\mathcal{L}(-2))) \neq H^1(\mathcal{P}, \mathcal{Q}(\mathcal{L}(-2)))$. Also, since the wave operator is hyperbolic, it admits distribution solutions which are not smooth.

5.4 Odd-dimensional hyperbolic space.

Now we apply our results to the Penrose transform of Bailey and Dunne (see [5]) for odd-dimensional hyperbolic space. The set-up is as follows.

5.4.1 The set-up.

The twistor space of H^{2n+1} . On \mathbb{R}^{2n+2} we will consider the symmetric bilinear form \langle, \rangle with matrix $\text{diag}(1, \dots, 1, -1)$. We shall use the same notation \langle, \rangle for the extension of \langle, \rangle to a complex bilinear form on the complexification \mathbb{C}^{2n+2} of \mathbb{R}^{2n+2} .

Consider the double fibration

$$\begin{array}{ccc} & F & \\ \eta \swarrow & & \searrow \nu \\ \mathcal{T} & & H^{2n+1} \end{array}$$

with the following properties:

1. The space H^{2n+1} is the hyperbolic space of dimension $2n + 1$ defined as the set

$$\{x \in \mathbb{R}^{2n+2} \mid \langle x, x \rangle = -1, x_{2n+2} > 0\}.$$

2. The space \mathcal{T} consists of all n -dimensional complex subspaces P of \mathbb{C}^{2n+2} such that the restriction of \langle, \rangle to P is 0 and such that P has trivial intersection with \mathbb{R}^{2n+2} .
3. The space F is the subset of $H^{2n+1} \times \mathcal{T}$ defined by incidence. (We say that $x \in H^{2n+1}$ and $P \in \mathcal{T}$ are incident if $\langle x, y \rangle = 0$ for every $y \in P$).
4. The maps η and ν are the obvious projections.

Since the group $SO_0(2n+1, 1)$ acts on \mathbb{R}^{2n+2} (via its defining representation) preserving the bilinear form \langle, \rangle , it induces an action on the spaces \mathcal{T} , F and H^{2n+1} which is transitive. As a real homogeneous manifold, the complex manifold \mathcal{T} (called the twistor space of H^{2n+1}) is the space $SO_0(2n+1, 1)/L$, with L the subgroup

$$L = \left\{ \begin{pmatrix} U & 0 \\ 0 & A \end{pmatrix}, U \in U(n), A \in SO_0(1, 1) \right\}.$$

of $SO_0(2n+1, 1)$. (Here we are embedding $U(n)$ in $SO(2n, \mathbb{R})$ by identifying \mathbb{C}^n with \mathbb{R}^{2n} in the usual way).

Homogeneous line bundles over \mathcal{T} . Consider the representation of $\text{Lie}(L)$

$$\rho : \text{Lie}(L) \rightarrow \mathbb{C}$$

defined by

$$\rho \begin{pmatrix} u & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix} = -\frac{k}{2}(\text{Tr}u) - \lambda a.$$

where $u \in u(n)$, $a \in \mathbb{R}$ and “Tr” denotes the trace of u considered as a complex $n \times n$ matrix. The representation ρ is the differential of a representation of L which defines an homogeneous rank 1 complex vector line bundle $\mathcal{O}(k, \lambda)$ over \mathcal{T} . One can show that the complex line bundles $\mathcal{O}(k, \lambda)$ are holomorphic. A direct calculation shows that $\mathcal{O}(-2n-2, 0)$ is the canonical bundle of the complex manifold \mathcal{T} .

The differential operators. Let \mathcal{Q} be the complex structure of the complex manifold \mathcal{T} . Since the bundles $\mathcal{O}(k, \lambda)$ are holomorphic, they are \mathcal{Q} -compatible and there is a Penrose transform for them. The following operators arise from this Penrose transform:

1. The hyperbolic Laplacian Δ acting on smooth functions on H^{2n+1} .

2. The Dirac operator \mathbf{D} acting between the space of sections of the spin bundle S on H^{2n+1} defined by the complex 2^n -dimensional spin representation of $SO(2n+1)$. (See [28], page 33 for a concrete description of the spin spaces and their dimension, and Chapter 2 for the definition and properties of spin bundles and Dirac operators).

The smooth Penrose transform. The smooth Penrose transform for the hyperbolic space H^{2n+1} has been constructed in [5] and is summarized in the following table:

| k | q | $H^q(\mathcal{T}, \mathcal{O}(k, \lambda))$ | $H^q(\mathcal{T}, \mathcal{O}(-2n-2-k, -\lambda))$ |
|---------|------------------------|---|--|
| $-2n$ | n | 0 | $\text{Ker}[\Delta - (\lambda^2 - n^2)]$ |
| $-2n$ | $n+1$ | 0 | $\text{Coker}[\Delta - (\lambda^2 - n^2)]$ |
| $-2n$ | $\frac{n(n+1)}{2}$ | $\text{Ker}[\Delta - (\lambda^2 - n^2)]$ | 0 |
| $-2n$ | $\frac{n(n+1)}{2} + 1$ | $\text{Coker}[\Delta - (\lambda^2 - n^2)]$ | 0 |
| $-2n-1$ | n | 0 | $\text{Ker}(\mathbf{D} + i\lambda)$ |
| $-2n-1$ | $n+1$ | 0 | $\text{Coker}(\mathbf{D} + i\lambda)$ |
| $-2n-1$ | $\frac{n(n+1)}{2}$ | $\text{Ker}(\mathbf{D} + i\lambda)$ | 0 |
| $-2n-1$ | $\frac{n(n+1)}{2} + 1$ | $\text{Coker}(\mathbf{D} + i\lambda)$ | 0 |

5.4.2 The compactly supported transform.

Since the dimension of the typical fiber of η is 1, a shift in dimension by 1 appears at the compactly supported transform. Theorem 3.6 together with the smooth Penrose for H^{2n+1} give the following result.

Theorem 5.7. *The compactly supported Penrose transform for $\mathcal{O}(k, \lambda)$ gives isomorphisms between compactly supported (smooth) Dolbeault cohomology groups of these bundles and compactly supported smooth fields on H^{2n+1} according to the following table:*

| k | q | $H_c^q(\mathcal{T}, \mathcal{O}(k, \lambda))$ | $H_c^q(\mathcal{T}, \mathcal{O}(-2n-2-k, -\lambda))$ |
|---------|------------------------|---|--|
| $-2n$ | $n-1$ | 0 | $\text{Ker}[\Delta - (\lambda^2 - n^2)]_c$ |
| $-2n$ | n | 0 | $\text{Coker}[\Delta - (\lambda^2 - n^2)]_c$ |
| $-2n$ | $\frac{n(n+1)}{2} - 1$ | $\text{Ker}[\Delta - (\lambda^2 - n^2)]_c$ | 0 |
| $-2n$ | $\frac{n(n+1)}{2}$ | $\text{Coker}[\Delta - (\lambda^2 - n^2)]_c$ | 0 |
| $-2n-1$ | $n-1$ | 0 | $\text{Ker}(\mathbf{D} + i\lambda)_c$ |
| $-2n-1$ | n | 0 | $\text{Coker}(\mathbf{D} + i\lambda)_c$ |
| $-2n-1$ | $\frac{n(n+1)}{2} - 1$ | $\text{Ker}(\mathbf{D} + i\lambda)_c$ | 0 |
| $-2n-1$ | $\frac{n(n+1)}{2}$ | $\text{Coker}(\mathbf{D} + i\lambda)_c$ | 0 |

The other cohomology groups are zero.

5.4.3 The duality pairings.

Since the canonical bundle of the complex manifold \mathcal{T} is $\mathcal{O}(-2n - 2, 0)$ and the complex dimension of \mathcal{T} is $\frac{n^2+3n}{2}$, there is a duality pairing

$$H^p(\mathcal{T}, \mathcal{O}(k, \lambda)) \otimes H_c^q(\mathcal{T}, \mathcal{O}(-2n - 2 - k, -\lambda)) \rightarrow \mathbb{C}$$

whenever $p + q = \frac{n^2+3n}{2}$. From Theorem 3.9 we obtain the following result.

Theorem 5.8. *The natural Serre-duality maps*

$$H^{\frac{n(n+1)}{2}}(\mathcal{T}, \mathcal{O}(-2n, \lambda)) \otimes H_c^n(\mathcal{T}, \mathcal{O}(-2, -\lambda)) \rightarrow \mathbb{C}$$

and

$$H^{\frac{n(n+1)}{2}+1}(\mathcal{T}, \mathcal{O}(-2n, \lambda)) \otimes H_c^{n-1}(\mathcal{T}, \mathcal{O}(-2, -\lambda)) \rightarrow \mathbb{C}$$

become respectively the maps

$$\text{Ker}[\Delta - (\lambda^2 - n^2)] \otimes \text{Coker}[\Delta - (\lambda^2 - n^2)]_c \rightarrow \mathbb{C}$$

and

$$\text{Coker}[\Delta - (\lambda^2 - n^2)] \otimes \text{Ker}[\Delta - (\lambda^2 - n^2)]_c \rightarrow \mathbb{C}$$

induced by multiplication of functions followed by integration.

Let \langle, \rangle be a $\text{Spin}(2n + 1)$ -invariant complex bilinear form on the spin bundle S . Then the natural Serre-duality maps

$$H_c^n(\mathcal{T}, \mathcal{O}(-1, \lambda)) \otimes H^{\frac{n(n+1)}{2}}(\mathcal{T}, \mathcal{O}(-2n - 1, -\lambda)) \rightarrow \mathbb{C}$$

and

$$H_c^{n-1}(\mathcal{T}, \mathcal{O}(-1, \lambda)) \otimes H^{\frac{n(n+1)}{2}+1}(\mathcal{T}, \mathcal{O}(-2n - 1, -\lambda)) \rightarrow \mathbb{C}$$

become respectively the maps

$$\text{Coker}(\mathbf{D} - i\lambda)_c \otimes \text{Ker}(\mathbf{D} - i\lambda) \rightarrow \mathbb{C}$$

and

$$\text{Ker}(\mathbf{D} - i\lambda)_c \otimes \text{Coker}(\mathbf{D} - i\lambda) \rightarrow \mathbb{C}$$

induced, modulo a factor, by the map

$$\Gamma_c(H^{2n+1}, S) \times \Gamma(H^{2n+1}, S) \ni (\sigma, \nu) \rightarrow \int_{H^{2n+1}} \langle \sigma, \nu \rangle.$$

Proof. We only need to check the statement involving the Dirac operator (the statement involving the hyperbolic Laplacian follows from Theorem 3.9). Since \langle, \rangle and the Serre-duality map along the fibers of τ are complex bilinear and $\text{Spin}(2n + 1)$ -invariant, they differ by a multiplicative constant factor (see [36], page 170, together with the fact that $\text{Spin}(2n+1)$ generates all the endomorphisms of the representation space of the 2^n -dimensional complex spin representation of $SO(2n + 1)$). The conclusion follows. \square

Chapter 6

Relative involutive cohomology and applications.

6.1 Introduction.

In this chapter we introduce the relative involutive cohomology for an arbitrary involutive structure and we prove that it has the excision property. As an application of the methods presented in the previous chapters we construct a Penrose transform for $\mathbb{R}^3 \setminus \{0\}$, which involves a relative Dolbeault cohomology group. Here we need to calculate the involutive cohomology of the total space of a rank one fiber-bundle with a local section removed, with respect to the pull-back of involutive structures from the base. The de Rham theory for manifolds which vary smoothly with respect to a parameter has been studied in [43] using abstract methods, but in this particular case we give a very explicit formulation of the fiber-wise de Rham cohomology. We hope that the theory of relative involutive cohomology will be successfully applied to other Penrose type transforms. We believe that a first step in this direction is to calculate the involutive cohomology of a fiber-bundle with a sub-bundle removed, with respect to the involutive structures induced from the base. We treat this problem in Section 6.4.

6.2 Relative involutive cohomology.

The involutive cohomology can be studied using the usual tools of algebraic topology. In particular there is a Mayer-Vietoris sequence for the involutive cohomology as stated in the following theorem.

Theorem 6.1. *Let $\{D_1, D_2\}$ be an open cover of the involutive manifold (M, \mathcal{A}) . Then the sequence of complexes ($k \geq 0$)*

$$0 \rightarrow \Gamma(M, \mathcal{A}^{0,k}) \rightarrow \Gamma(D_1, \mathcal{A}^{0,k}) \oplus \Gamma(D_2, \mathcal{A}^{0,k}) \rightarrow \Gamma(D_1 \cap D_2, \mathcal{A}^{0,k}) \rightarrow 0$$

is exact and there is an associated long exact sequence in cohomology

$$\rightarrow H^k(M, \mathcal{A}) \rightarrow H^k(D_1, \mathcal{A}) \oplus H^k(D_2, \mathcal{A}) \rightarrow H^k(D_1 \cap D_2, \mathcal{A}) \rightarrow H^{k+1}(M, \mathcal{A}) \rightarrow$$

Similar considerations hold for the \mathcal{A} -involutive cohomology of an \mathcal{A} -compatible vector bundle $V \rightarrow M$.

In this section we define and study the relative involutive cohomology for an arbitrary involutive structure.

Consider a smooth map $f : (S, \mathcal{E}) \rightarrow (M, \mathcal{A})$ between the involutive manifolds (S, \mathcal{E}) and (M, \mathcal{A}) which satisfies $f^*(\mathcal{A}^{1,0}) \subset \mathcal{E}^{1,0}$, and let V be a \mathcal{A} -compatible vector bundle over M . Then $f^*(V)$ is \mathcal{E} -compatible and we define the complex

$$\Omega^*(f) := \bigoplus_{q \geq 0} \Omega^q(f)$$

by

$$\begin{aligned} \Omega^q(f) &:= \Gamma(M, \mathcal{A}^{0,q} \otimes V) \oplus \Gamma(S, \mathcal{E}^{0,q-1} \otimes f^*(V)) \\ \bar{\partial}(\gamma, \theta) &:= (\bar{\partial}_{\mathcal{A}}(\gamma), f^*(\gamma) - \bar{\partial}_{\mathcal{E}}(\theta)) \end{aligned}$$

(see also [11], page 78). Like in [11] a simple calculation shows that $\bar{\partial}^2 = 0$. Also, by definition we have the exact sequence

$$0 \rightarrow \Gamma(S, \mathcal{E}^{0,q-1} \otimes f^*(V)) \xrightarrow{\alpha} \Omega^q(f) \xrightarrow{\beta} \Gamma(M, \mathcal{A}^{0,q} \otimes V) \rightarrow 0$$

with the obvious maps α and β : $\alpha(\theta) = (0, \theta)$ and $\beta(\gamma, \theta) = \gamma$.

Definition 6.2. Let S be a closed subset of the involutive manifold (M, \mathcal{A}) and V a \mathcal{A} -compatible vector bundle over M . The relative involutive cohomology $H_S^q(M, \mathcal{A}(V))$ is defined to be $H^q(\Omega^*(i))$ where $i : (M \setminus S, \mathcal{A}) \rightarrow (M, \mathcal{A})$ is the inclusion.

Theorem 6.3. The relative involutive cohomology $H_S^*(M, \mathcal{A})$ of the involutive manifold (M, \mathcal{A}) fits into a long exact sequence

$$\rightarrow H^{q-1}(M, \mathcal{A}(V)) \rightarrow H^{q-1}(M \setminus S, \mathcal{A}(V)) \rightarrow H_S^q(M, \mathcal{A}(V)) \rightarrow H^q(M, \mathcal{A}(V)) \rightarrow$$

Proof. This is the long exact sequence in cohomology associated to a short exact sequence (see also Definition 6.2). \square

Theorem 6.4. The relative involutive cohomology has the excision property. More precisely, consider (M, \mathcal{A}) an involutive manifold and V a \mathcal{A} -compatible vector bundle over M . If S is a closed subset of M and D is an open neighborhood of S then for every $q \geq 0$,

$$H_S^q(M, \mathcal{A}) \cong H_S^q(D, \mathcal{A}).$$

Proof. We first show that any cohomology class in $H_S^q(M, \mathcal{A}(V))$ admits a representative with support in N , where N is an open subset of M containing S . For this, let f be a smooth function which is 0 in a small neighborhood of S (included in N) and 1 on a neighborhood of $M \setminus N$. Also, let $(\gamma, \theta) \in \Gamma(M, \mathcal{A}^{0,q} \otimes V) \oplus \Gamma(M \setminus S, \mathcal{A}^{0,q-1} \otimes V)$ represent a cohomology class in $H_S^q(M, \mathcal{A})$: $\gamma|_{M \setminus S} = \bar{\partial}_{\mathcal{A}}(\theta)$ and $\bar{\partial}_{\mathcal{A}}(\gamma) = 0$. Then $(\gamma - \bar{\partial}_{\mathcal{A}}(f\theta), (1-f)\theta)$ represents the same cohomology class as (γ, θ) because it is equal to $(\gamma, \theta) - \bar{\partial}(f\theta, 0)$ and has support in N because $f = 1$ on a neighborhood of $M \setminus N$.

In order to prove the excision property we need to show that the map $I : H_S^q(M, \mathcal{A}(V)) \rightarrow H_S^q(D, \mathcal{A}(V))$ induced by restriction is an isomorphism. Since every cohomology class in $H_S^q(D, \mathcal{A}(V))$ admits a representative with support close to S , the surjectivity of I is obvious. To prove the injectivity of I we take (γ, θ) with support included in D representing a cohomology class in $H_S^q(M, \mathcal{A}(V))$ such that $[(\gamma, \theta)] = 0$ in $H_S^q(D, \mathcal{A}(V))$. Then there is $(\sigma, \beta) \in \Gamma(D, \mathcal{A}^{0,q-1} \otimes V) \oplus \Gamma(D \setminus S, \mathcal{A}^{0,q-2} \otimes V)$ such that $\bar{\partial}(\sigma, \beta) = (\gamma, \theta)$. We claim that (σ, β) can be chosen with support included in D . (Then the equality $\bar{\partial}(\sigma, \beta) = (\gamma, \theta)$ would hold on M as well and the injectivity of I would follow). For this it is enough to replace (σ, β) with $(\sigma, \beta) - \bar{\partial}(f\beta, 0)$. We claim that $(\sigma, \beta) - \bar{\partial}(f\beta, 0)$, which is equal to $(\sigma - \bar{\partial}_{\mathcal{A}}(f\beta), (1-f)\beta)$ has support included in D : the statement is clear for $(1-f)\beta$, since $f = 1$ on a neighborhood of $M \setminus D$, and for the same reason on a neighborhood of $M \setminus D$ (intersected with D), $\sigma - \bar{\partial}_{\mathcal{A}}(f\beta)$ is equal to $\sigma - \bar{\partial}_{\mathcal{A}}(\beta)$ which is equal to θ and has support included in D . \square

Remark. Consider S a closed subset of the involutive manifold (M, \mathcal{A}) which is also a submanifold of M , and V a \mathcal{A} -compatible vector bundle over M . Then for every q the sequence

$$0 \rightarrow \Gamma(M, \mathcal{A}^{0,q} \otimes V) \rightarrow \Gamma(M \setminus S, \mathcal{A}^{0,q} \otimes V) \rightarrow \frac{\Gamma(M \setminus S, \mathcal{A}^{0,q} \otimes V)}{\Gamma(M, \mathcal{A}^q \otimes V)} \rightarrow 0$$

is exact, since $M \setminus S$ is dense in M . Considering the long exact sequence in cohomology and using a five-lemma argument (see [11], page 44) it follows that

$$H_S^q(M, \mathcal{A}(V)) \cong H^{q-1} \left(\frac{\Gamma(M \setminus S, \mathcal{A}^* \otimes V)}{\Gamma(M, \mathcal{A}^* \otimes V)} \right).$$

6.3 A Penrose transform for $\mathbb{R}^3 \setminus \{0\}$.

In Chapter 5, Section 5.2 we have considered a Penrose transform for the Euclidean space \mathbb{R}^3 . The set-up was the double fibration

$$\begin{array}{ccc} & F & \\ \eta \swarrow & & \searrow \tau \\ Z & & \mathbb{R}^3 \end{array}$$

where the space Z was the space of all oriented lines in \mathbb{R}^3 , the space F was the set of pairs (p, l) , where $l \in Z$ and $p \in l$, and the maps η and τ were the obvious projections.

We now study the induced transform on \mathbb{R}^3 with the origin 0 removed. Let $F_0 := \tau^{-1}(\mathbb{R}^3 \setminus \{0\})$. We obtain a double map

$$\begin{array}{ccc} & F_0 & \\ \eta_0 \swarrow & & \searrow \tau_0 \\ Z & & \mathbb{R}^3 \setminus \{0\} \end{array}$$

where τ_0 is a fiber-bundle with typical fiber $\mathbb{C}\mathbb{P}^1$ but η_0 has the typical fiber \mathbb{R} on the (oriented) lines which do not intersect the origin, and the typical fiber $\mathbb{R} \setminus \{0\}$ on the lines which intersect the origin. In other words, the map η_0 is the fiber-bundle η with the sub-bundle Σ removed, where $\Sigma \rightarrow \mathbb{C}\mathbb{P}^1$ is a fiber-bundle with the typical fiber just a point.

Lemma 6.5. *Let \mathcal{C} be the involutive structure on F_0 defined by $\mathcal{C}^{1,0} := \eta_0^*(\mathcal{E}_Z^1)$. If V is a vector bundle over Z then*

$$H^0(F_0, \mathcal{C}(\eta^*V)) = \Gamma(Z, V)$$

and

$$H_\Sigma^2(F, \mathcal{C})H^1(F_0, \mathcal{C}(\eta^*V)) = \frac{\Gamma(Z \setminus \mathbb{C}\mathbb{P}^1, V)}{\Gamma(Z, V)}.$$

Proof. The vector bundle V carries through the argument and we take it to be trivial of rank one. The statement for the cohomology in degree zero follows easily. To prove the other statements we notice that since the typical fiber of η is contractible, the \mathcal{C} -involutive cohomology groups on F are zero in degree one and two. This implies that $H^1(F_0, \mathcal{C})$ is isomorphic to $H_\Sigma^2(F, \mathcal{C})$ and the second statement follows. Since $H^1(F_0, \mathcal{C})$ is isomorphic to $H_\Sigma^2(F, \mathcal{C})$ it has the excision property (see Theorem 6.4). Restricting F if necessary, we can extend the fiber-bundle Σ to a fiber-bundle $\tilde{\Sigma} \rightarrow Z$ which is a sub-bundle of the bundle $F \rightarrow Z$.

The long exact sequence associated to the Mayer-Vietoris sequence of the cover $\{\eta^{-1}(Z \setminus \mathbb{C}\mathbb{P}^1), F \setminus \tilde{\Sigma}\}$ of F_0 gives the exact sequence

$$0 \rightarrow \Gamma(Z, \mathcal{E}) \rightarrow \begin{array}{c} \Gamma(Z \setminus \mathbb{C}\mathbb{P}^1, \mathcal{E}) \\ \oplus \\ \Gamma(Z, \mathcal{E} \oplus \mathcal{E}) \end{array} \rightarrow \Gamma(Z \setminus \mathbb{C}\mathbb{P}^1, \mathcal{E} \oplus \mathcal{E}) \rightarrow H^1(F_0, \mathcal{C}) \rightarrow 0$$

where the first map is

$$f \rightarrow \begin{array}{c} f \\ \oplus \\ (f, f) \end{array}$$

and the second map is

$$\begin{array}{c} h \\ \oplus \\ (g_+, g_-) \end{array} \rightarrow (g_+ - h, g_- - h).$$

This exact sequence contains the exact subsequence

$$0 \rightarrow \Gamma(Z, \mathcal{E}) \rightarrow \Gamma(Z, \mathcal{E}) \rightarrow 0$$

and the quotient sequence

$$0 \rightarrow \begin{array}{c} \Gamma(Z \setminus \mathbb{C}\mathbb{P}^1, \mathcal{E}) \\ \oplus \\ \Gamma(Z, \mathcal{E}) \end{array} \rightarrow \Gamma(Z \setminus \mathbb{C}\mathbb{P}^1, \mathcal{E} \oplus \mathcal{E}) \rightarrow H^1(F_0, \mathcal{C}) \rightarrow 0$$

has the first map

$$\begin{pmatrix} h \\ g \end{pmatrix} \rightarrow (g - h, -h)$$

and is exact. Again, this exact sequence contains the exact subsequence

$$0 \rightarrow \Gamma(Z \setminus \mathbb{C}\mathbb{P}^1, \mathcal{E}) \rightarrow \Gamma(Z \setminus \mathbb{C}\mathbb{P}^1, \mathcal{E}) \rightarrow 0$$

and the quotient

$$0 \rightarrow \Gamma(Z, \mathcal{E}) \rightarrow \Gamma(Z \setminus \mathbb{C}\mathbb{P}^1, \mathcal{E}) \rightarrow H^1(F_0, \mathcal{C}) \rightarrow 0$$

is exact. The last statement follows. □

Theorem 6.6. *Let $\mathcal{O}(-2)$ be the holomorphic line bundle over Z defined in Chapter 5, Section 5.2. If $\Delta_{\mathbb{R}^3}$ is the Laplace operator on $\mathcal{E}(\mathbb{R}^3)$ and $\Delta_{\mathbb{R}^3 \setminus \{0\}}$ is the Laplace operator on $\mathcal{E}(\mathbb{R}^3 \setminus \{0\})$ then there is a short exact sequence*

$$0 \rightarrow \text{Ker}(\Delta_{\mathbb{R}^3}) \rightarrow \text{Ker}(\Delta_{\mathbb{R}^3 \setminus \{0\}}) \rightarrow H_{\mathbb{C}\mathbb{P}^1}^1(Z, \mathcal{O}(-2)) \rightarrow 0.$$

Proof. Let \mathcal{Q} be the complex structure of the complex manifold Z . We consider the spectral sequence

$$E_1^{p,q} = H^q(F_0, \mathcal{C}(\eta^*(\mathcal{Q}^{0,p} \otimes \mathcal{O}(-2)))) \implies H^{p+q}(F_0, (\eta^*\mathcal{Q})(\eta^*(\mathcal{O}(-2))))$$

associated to the involutive structures $\eta^*(\mathcal{Q})$ and \mathcal{C} on F_0 . From Lemma 6.5 the second order level of this spectral sequence is

$$E_2^{p,q} : \begin{array}{c} \left| \begin{array}{ccc} 0 & 0 & 0 \\ H_{\mathbb{C}\mathbb{P}^1}^1(Z, \mathcal{O}(-2)) & H_{\mathbb{C}\mathbb{P}^1}^2(Z, \mathcal{O}(-2)) & H_{\mathbb{C}\mathbb{P}^1}^3(Z, \mathcal{O}(-2)) \\ H^0(Z, \mathcal{O}(-2)) & H^1(Z, \mathcal{O}(-2)) & H^2(Z, \mathcal{O}(-2)) \end{array} \right. \end{array}$$

The smooth Penrose transform for \mathbb{R}^3 identifies $H^2(Z, \mathcal{O}(-2))$ with the cokernel of the Laplacian Δ on $\mathcal{E}(\mathbb{R}^3)$. Since the cokernel of Δ on $\mathcal{E}(\mathbb{R}^3)$ is zero, the cohomology group $H^2(Z, \mathcal{O}(-2))$ is zero as well. The spectral sequence degenerates to give the short exact sequence

$$0 \rightarrow H^1(Z, \mathcal{O}(-2)) \rightarrow H^1(F_0, \eta^*(\mathcal{Q})(\eta^*(\mathcal{O}(-2)))) \rightarrow H_{\mathbb{C}\mathbb{P}^1}^1(Z, \mathcal{O}(-2)) \rightarrow 0.$$

The smooth Penrose transform for \mathbb{R}^3 also identifies $H^1(Z, \mathcal{O}(-2))$ with the kernel of Δ on $\mathcal{E}(\mathbb{R}^3)$. Similarly we see that $H^1(F_0, \eta^*(\mathcal{Q})(\eta^*(\mathcal{O}(-2))))$ can be identified with the kernel of Δ on $\mathcal{E}(\mathbb{R}^3 \setminus \{0\})$. The claim follows. \square

Theorem 6.7. *Let \mathcal{H} be the sheaf of harmonic functions of \mathbb{R}^3 . Then there is an isomorphism of relative cohomology groups*

$$H_{\mathbb{C}\mathbb{P}^1}^1(Z, \mathcal{O}(-2)) \cong H_{\{0\}}^1(\mathbb{R}^3, \mathcal{H}).$$

where Z is the twistor space of \mathbb{R}^3 and $\mathcal{O}(-2)$ is the holomorphic line bundle over Z defined as in Chapter 5, Section 5.2.

Proof. Consider the exact sequence of sheaves

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{E} \xrightarrow{\Delta} \mathcal{E} \rightarrow 0$$

on \mathbb{R}^3 . Using the fact that the sheaf \mathcal{E} is fine and that Δ is onto on $\mathcal{E}(\mathbb{R}^3)$ it follows that

$$H^1(\mathbb{R}^3, \mathcal{H}) = \text{Coker}(\Delta_{\mathbb{R}^3}) = 0$$

From the long exact sequence involving the relative cohomology groups we obtain

$$0 \rightarrow H^0(\mathbb{R}^3, \mathcal{H}) \rightarrow H^0(\mathbb{R}^3 \setminus \{0\}, \mathcal{H}) \rightarrow H_{\{0\}}^1(\mathbb{R}^3, \mathcal{H}) \rightarrow 0$$

The claim now follows from Theorem 6.6 (because $H^0(\mathbb{R}^3, \mathcal{H})$ is the space of harmonic functions on \mathbb{R}^3 and $H^0(\mathbb{R}^3 \setminus \{0\}, \mathcal{H})$ is the space of harmonic functions on $\mathbb{R}^3 \setminus \{0\}$). \square

Remark. Since relative cohomology groups have the excision property there is also an isomorphism

$$H_{\mathbb{C}P^1}^1(Z, \mathcal{O}(-2)) \cong H_{\{0\}}^1(U, \mathcal{H}).$$

for any open subset U of \mathbb{R}^3 which contains the origin.

6.4 General theory.

Under mild conditions the following more general result holds but we shall not give a proof for it.

Let

$$\begin{array}{ccc} \Sigma & \hookrightarrow & F \\ \downarrow & & \downarrow \eta \\ S & \hookrightarrow & Z \end{array}$$

where $\eta : F \rightarrow Z$ is a fiber-bundle and $\Sigma \rightarrow S$ is a sub-bundle with S a proper submanifold of Z which is also a closed subset of Z . Extend $\Sigma \rightarrow Z$ to a fiber-bundle $\tilde{\Sigma} \rightarrow N$ (with N open in Z) which is also a sub-bundle of $F \rightarrow Z$ to obtain the picture

$$\begin{array}{ccccc} \Sigma & \hookrightarrow & \tilde{\Sigma} & \hookrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \eta \\ S & \hookrightarrow & N & \hookrightarrow & Z \end{array}$$

Define the bundle $\mathcal{R}^q \rightarrow N$ to be the bundle whose fiber over $x \in N$ is the relative de Rham cohomology group $H_{\Sigma_x}^q(F_x)$. Suppose Z has an involutive structure \mathcal{Q} and that V is a \mathcal{Q} -compatible vector bundle over Z . Then there is a long exact sequence

$$\rightarrow H^p(Z, \mathcal{Q}(\mathcal{H}^q \otimes V)) \rightarrow E_2^{p,q} \rightarrow H_S^{p+1}(N, \mathcal{Q}(\mathcal{R}^q \otimes V)) \rightarrow H^{p+1}(Z, \mathcal{Q}(\mathcal{H}^q \otimes V)) \rightarrow$$

where $E_2^{p,q}$ are the second order terms of the spectral sequence

$$E_1^{p,q} = H^q(F \setminus \Sigma, \mathcal{C}(\eta^*(\mathcal{Q}^{0,p} \otimes V))) \implies H^{p+q}(F \setminus \Sigma, (\eta^* \mathcal{Q})(\eta^*(V))).$$

We haven't investigated the applications of this general result, but we believe it is useful for the pull-back step of other Penrose type transforms.

Chapter 7

The Atiyah-Ward Ansätze on 3-dimensional Einstein-Weyl spaces.

7.1 Introduction.

Consider \mathbb{M} a complex self-dual conformal 4-manifold, Z its twistor space and \widehat{W} a holomorphic line bundle over Z trivial on the twistor lines. The idea of the A_k Atiyah-Ward Ansatz (see [2] or [48]) is to write down self-dual Yang-Mills solutions which are the Penrose-Ward transform of an extension E

$$0 \rightarrow \mathcal{O}(-k) \otimes \widehat{W} \rightarrow E \rightarrow \mathcal{O}(k) \otimes \widehat{W}^* \rightarrow 0$$

over the twistor space Z of \mathbb{M} . Since the data which defines E is the line bundle \widehat{W} and the extension class $H^1(Z, \mathcal{O}(-2k) \otimes \widehat{W}^2)$, the A_k Atiyah-Ward Ansatz provides a method of generating a $SL(2, \mathbb{C})$ self-dual Yang-Mills field from a self-dual Maxwell field (which corresponds to \widehat{W}) and a solution of an auxiliary equation (which corresponds to $H^1(Z, \mathcal{O}(-2k) \otimes \widehat{W}^2)$). There are similar Ansätze also on Riemannian self-dual conformal 4-manifolds, obtained by restricting the Ansätze from complex self-dual 4-manifolds to the real Euclidean slices.

In this chapter we will reduce the A_k Atiyah-Ward Ansatz (for $k = 0, 1, 2$) from 4-dimensional Riemannian self-dual manifolds to 3-dimensional real Einstein-Weyl manifolds. For $k = 0$ the reduced Ansatz will provide an interpretation of the affine monopole equations. For $k = 1, 2$ the reduced Ansätze will generate a solution of the Einstein-Weyl Bogomolony equations from a solution of the abelian monopole equation and a solution of an auxiliary equation.

We begin with a brief review of conformal geometry, in order to fix our notations and conventions. This material is well known, and references for it will be given. Next we present the Atiyah-Ward Ansätze in the natural framework of conformal geometry and Weyl derivatives. The Atiyah-Ward Ansätze have been

well understood in the literature (see [2] and [48]) but using our treatment they become very suitable for the reduction process. Since self-dual Maxwell fields in dimension 4 correspond to solutions of the abelian monopole equation in dimension 3 the main step in the process of reduction is to reduce the auxiliary equations determined by the extension class from 4 to 3 dimensions. This will be done in detail in Section 7.4. In Section 7.5 we obtain the explicit form of the reduced Ansätze.

7.2 Conformal geometry.

The material from this section can be found in [15] and [16].

Conventions. We shall be interested in real conformal geometry, so unless otherwise specified the manifolds will always be real. Unlike the previous chapters, in this chapter we shall use the notation TM and \mathcal{E}_M^k for the *real* tangent bundle of a manifold M and for the bundle of *real* k -forms on M respectively. The complexified tangent bundle of M will be denoted $T_{\mathbb{C}}M$, and more generally, the complexification of an arbitrary real vector bundle V will be denoted $V_{\mathbb{C}}$. If V is a real vector bundle and W is a complex vector bundle then the tensor product $V \otimes_{\mathbb{R}} W$ will be denoted simply $V \otimes W$.

Density line bundles and Weyl derivatives. If V is a real n -dimensional vector space and w any real number, the one dimensional linear space $L^w = L^w(V)$ carrying the representation $A \rightarrow |\det A|^{w/n}$ of $GL(V)$ is called the space of densities of weight w . For M an arbitrary manifold, we will use the density line bundle $L^w = L^w(TM)$ of M which is defined to be the bundle over M whose fiber at $x \in M$ is $L^w(T_x M)$. The density line bundles are oriented, hence trivialisable. However, they are not canonically trivial. We will say that the bundle L^w (and any element or section of it) has weight w . More generally the tensor bundle $L^w \otimes (TM)^j \otimes (\mathcal{E}_M^1)^k$ (and any sub-bundle, quotient bundle, element or section) will be said to have weight $w + j - k$.

Conformal structures. A conformal structure on a manifold M is a positive definite bilinear form c on $L^{-1}TM$ (when tensoring with a density line bundle we generally omit the tensor product sign). It allows us to identify vector fields (of a certain weight) with covector fields (of the same weight). We shall make use of this identification without further comment. Also, a conformal structure c on M defines a whole class of metrics (which we call compatible) $\rho^2 c$, for every

$\rho \in L^{-1}$. We notice that the local sections ρ and ν of L^{-1} determine the same compatible metric if and only if $\rho = \nu$ or $\rho = -\nu$. If the conformal manifold (M, c) is also oriented and has dimension n , there is a Hodge star operator defined by the orientation and the conformal structure. In our conventions it satisfies the properties

$$\begin{cases} i_X * (\alpha) &= *(X \wedge \alpha) \\ *(1) &= \text{or}_M. \end{cases}$$

(Here $\text{or}_M \in \Gamma(M, L^n \otimes \mathcal{E}_M^n)$ is the unit section given by the orientation). A direct calculation shows that $*^2 = (-1)^{\frac{1}{2}n(n-1)}$ and that

$$\alpha \wedge * \beta = (-1)^{\frac{1}{2}k(k-1)} \langle \alpha, \beta \rangle \text{or}_M$$

when α and β are k -forms on M . Using the $*$ operator we will identify the (weightless) top forms on M with functions on M . In our conventions, the (weightless) top form α is identified with the function $(-1)^{\frac{1}{2}n(n-1)} * \alpha$.

Suppose now that M is 4-dimensional. Then every 2-form γ on M decomposes as the sum of its self-dual part γ_{SD} and anti-self-dual part γ_{ASD} , where $\gamma_{SD} := \frac{1}{2}(\gamma + *\gamma)$ and $\gamma_{ASD} := \frac{1}{2}(\gamma - *\gamma)$. The space of self-dual 2-forms on M will be denoted \mathcal{E}_+^2 , and the space of anti-self-dual 2-forms on M will be denoted \mathcal{E}_-^2 . Since an (anti)-self-dual 2-form is zero if its contraction with a non-vanishing vector field is 0, the formula $\gamma = 2|X|^{-2} (X \wedge i_X(\gamma))_{ASD}$ holds when γ is anti-self-dual and X is an arbitrary non-vanishing vector field on (an open subset of) M . We shall use this formula in our calculations.

Weyl connections. A Weyl connection on a conformal manifold is a connection on L^1 . The fundamental theorem of conformal geometry (see [49]) states that on a conformal manifold (M, c) there is a bijective correspondence between Weyl connections and torsion free connections on TM which preserve the conformal structure c . The corresponding linear map sends a 1-form γ to the $co(TM)$ -valued 1-form Γ defined by $\Gamma_X = \gamma(X)\text{Id} + \gamma \wedge X$, where “Id” denotes the identity endomorphism. (Here and elsewhere we use the convention $(X \wedge Y)(Z) : \langle X, Z \rangle Y - \langle Y, Z \rangle X$ for the wedge product $X \wedge Y$).

Curvatures on conformal manifolds. Consider (M, c) a conformal n -dimensional manifold and D a Weyl derivative on it. The curvature F^D of D on L^1 is a 2-form on M called the Faraday curvature. If $F^D = 0$ D is said to be closed and there are local scales $\mu \in L^1$ with $D(\mu) = 0$. Equivalently, D is locally the Levi-Civita connection the compatible metric $\mu^{-2}c$. If such a length scale is global, D is said to be exact and is the Levi-Civita connection of the (global) compatible metric $\mu^{-2}c$.

One important property we shall use in our calculations is the decomposition of the curvature $R^{D,w}$ of D on $L^{w-1}TM$:

$$R_{X,Y}^{D,w} = W_{X,Y} + wF^D(X,Y)\text{Id} - r^D(X) \wedge Y + r^D(Y) \wedge X$$

Here $W_{X,Y}$ is the Weyl curvature (which depends only on the conformal structure c), F^D is the Faraday curvature (which depends only on D) and r^D is a section of $L^{-2}\text{End}(TM)$ called the normalized Ricci tensor. Also, the normalized Ricci tensor of D decomposes under the orthogonal group as

$$r^D = r_0^D + \frac{1}{2n(n-1)}\text{scal}^D \cdot \text{Id} - \frac{1}{2}F^D$$

where r_0^D is symmetric trace free and scal^D is the scalar curvature of D (which is a section of L^{-2}).

Two conformally invariant operators. Let M be a conformal oriented n -dimensional manifold and D a Weyl connection on M . We shall consider the conformally invariant Laplacian $\Delta : L^{\frac{-n+2}{2}} \rightarrow L^{\frac{-n-2}{2}}$ defined by the formula $\Delta(\rho) = \text{tr}D^2(\rho) - \frac{n-2}{4(n-1)}\text{scal}^D \rho$. There is also a coupled version of the conformally invariant Laplacian: if W is a vector bundle on M with a connection ∇ , the coupled conformally invariant Laplacian Δ is defined on $L^{\frac{-n+2}{2}} \otimes W$ by exactly the same formula, except that in the term $\text{tr}D^2(\rho)$ the Weyl connection D is coupled with ∇ . In our calculations we shall write down explicitly the expression $\text{tr}D^2(\rho)$ using the formula $\text{tr}D(\alpha) = (-1)^{\frac{1}{2}n(n-1)}(*d^D*)(\alpha)$ which holds for any 1-form α on the n -dimensional manifold M (see [8], page 34).

We will also consider the Hessian $\mathcal{H} : L^1 \rightarrow L^{-1}\text{End}(TM)$ defined by the formula

$$\mathcal{H}(\mu) = \text{sym}_0(D^2(\mu) + r^D\mu)$$

where “ sym_0 ” denotes the symmetric trace free part of a tensor. There is also a coupled version of the Hessian: if W is a vector bundle on M with a connection ∇ , the coupled Hessian \mathcal{H} is defined on $L^1 \otimes W$ by the same expression, except that in the term $\text{sym}_0D^2(\mu)$ the Weyl connection D is coupled with ∇ . Both the Laplacian and the Hessian (coupled or not) are conformally invariant (that is, they are independent of the choice of the Weyl connection D).

Conformal submersions. Let $\pi : M \rightarrow B$ be a smooth surjective map between conformal manifolds and let the horizontal bundle \mathcal{H} be the orthogonal complement of the vertical bundle ν of π in TM . Following [14], π is called a conformal submersion if for all $x \in M$ $d\pi_x|_{\mathcal{H}_x}$ is a nonzero conformal linear map.

The simplest class of conformal submersions are the conformal submersions generated by conformal vector fields: if K is a non-vanishing conformal vector field on (an open set of) a conformal manifold M , then M/K is a manifold carrying a conformal structure induced by the conformal structure of M , and the natural projection $\pi : M \rightarrow M/K$ is a conformal submersion. Let $\xi := K/|K|$. If M is oriented, then there is an induced orientation on B such that the formula

$$*_M(\xi \wedge \alpha) = (-1)^{k+1} *_B(\alpha)$$

(with α a k -form on B) holds.

The Jones and Tod construction. A conformal manifold (M, c) together with a Weyl connection D (called the Einstein-Weyl connection) is said to be Einstein-Weyl if $r_0^D = 0$. The importance of the Einstein-Weyl 3-dimensional spaces is that they are natural reductions of self-dual 4-dimensional spaces. This result has been proved by Jones and Tod who showed (see [25]) that if M is a 4-dimensional self-dual oriented conformal manifold and K is a non-vanishing conformal vector field then $B := M/K$ inherits an orientation, a conformal structure c_B and a Weyl derivative D^B which respect to which it is Einstein-Weyl. Moreover, the Einstein-Weyl space B comes with a solution (w, A) (where $A \in \mathcal{E}^1(B)$ and $w \in L^{-1}$) of the abelian monopole equation $*_B D^B(w) = dA$ from which M can be recovered: the real line bundle M is locally isomorphic to $U \times \mathbb{R}$; the conformal structure on M is given locally by the formula $\pi^*(c_B) + w^{-2}(dt + A)^2$ and the conformal vector field K is $\frac{\partial}{\partial t}$. In other words there is a correspondence between self-dual 4-spaces with symmetry and Einstein-Weyl 3-spaces with monopoles.

Consider now M and B related by the Jones and Tod correspondence, like above. Then there is also a correspondence between solutions (w_1, ∇_1) (where ∇_1 is a connection on a vector bundle V over B and $w_1 \in L^{-1} \otimes \text{End}(V)$) of the Einstein-Weyl Bogomolny equations $*_B(D^B \otimes \nabla_1)(w_1) = F^{\nabla_1}$ on B and self-dual Yang-Mills fields on M . The correspondence is defined in the following way: the self-dual Yang-Mills field ∇ on M which corresponds to (w_1, ∇_1) on B acts on $\pi^*(V)$ and is defined by the formula $\nabla := \pi^*\nabla_1 - w_1\xi$ (where $\xi := w^{-1}(dt + A)$). In particular, there is a correspondence between solutions of the abelian monopole equation on B and self-dual Maxwell fields on M (in the case when the rank of V is 1).

Some distinguished Weyl derivatives. Consider (B, c_B) a 3-dimensional oriented Einstein-Weyl space with Einstein-Weyl connection D^B . Let (w, A) be a solution of the abelian monopole equation on B and $\pi : M \rightarrow B$ the conformal submersion it generates (via the Jones and Tod construction) with (M, c_M)

oriented conformal 4-dimensional and self-dual. Following [15], we will consider on M the Weyl connection D^0 defined by the condition $D^0(|K|) = 0$, where K is the conformal vector field on M which defines π . The Weyl connection D^0 is the Levi-Civita connection of the compatible metric $|K|^{-2}c_M$, and a direct calculation shows that it descends on B .

We will also consider the Weyl derivative D^{sd} on B defined by the formula $D^{sd} := D^B + \frac{1}{2}w^{-1}D^B(w)$. We remark that via the isomorphism $\pi^*(L_B^{-1}TB) \cong L_M^{-2}\Lambda_-^2(TM)$ given by $\chi \rightarrow \xi \wedge \chi - *_M(\xi \wedge \chi)$ the Weyl connection D^{sd} on $L_M^{-2}\Lambda_-^2(TM)$ is the pull-back of the Weyl connection D^B on $L_B^{-1}TB$ (see [14]).

7.3 The Ansätze for 4-dimensional conformal manifolds.

The Atiyah-Ward Ansätze for 4-dimensional conformal manifolds can be found for example in [2] or [48]. We present it here in the natural framework of conformal geometry. This treatment of the Atiyah-Ward Ansätze does not appear in the literature.

7.3.1 The case $k = 0$.

Lemma 7.1. *Consider ∇ a connection on a complex vector bundle W over the 4-dimensional conformal oriented manifold M .*

If $\mathbf{A} \in \Gamma(M, \mathcal{E}^1 \otimes \text{End}(W^, W))$ then the connection*

$$\begin{pmatrix} \nabla & 0 \\ 0 & \nabla \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{A} \\ 0 & 0 \end{pmatrix}$$

on $W \oplus W^$ is self-dual if and only if ∇ is a self-dual Yang-Mills field and \mathbf{A} satisfies the equation $(d^\nabla \mathbf{A})_{ASD} = 0$.*

Proof. The connection on $W \oplus W^*$ has the curvature

$$\begin{pmatrix} F^\nabla & 0 \\ 0 & -(F^\nabla)^* \end{pmatrix} + \begin{pmatrix} 0 & d^\nabla(\mathbf{A}) \\ 0 & 0 \end{pmatrix}$$

where $(F^\nabla)^*$ is a two form with values in $\text{End}(W^*)$ such that for every $X, Y \in TM$ the endomorphism $(F^\nabla)_{X,Y}^*$ is the adjoint of $F_{X,Y}^\nabla$. It follows that the connection on $W \oplus W^*$ is self-dual if and only if ∇ is a self-dual Yang-Mills field and $d^\nabla(\mathbf{A})$ is self-dual. \square

7.3.2 The case $k = 1$.

Notation. On a 4-dimensional conformal oriented (spin) manifold M we shall use the notation S_- for the weight $1/2$ spin bundle. It is a complex rank 2 vector bundle over M such that $\odot S_- = \Lambda_-^2(L^{-1/2}TM)_\mathbb{C}$ and $\Lambda^2(S_-) = L_\mathbb{C}^1$.

Lemma 7.2. *Let M be a 4-dimensional oriented conformal manifold and ρ a section of L^{-1} . Then on S_- the Levi-Civita connection of the compatible metric ρ^2c has the curvature*

$$R_{X,Y} = W_{X,Y}^- - \rho \mathcal{H}_{X,Y}(\rho^{-1}) + \frac{1}{2} \rho^{-1} \Delta(\rho) \cdot (X \wedge Y)_{ASD}$$

In particular, if M is a self-dual conformal 4-manifold and ρ is a solution of the conformally invariant Laplacian, then the Levi-Civita connection on S_- has self-dual curvature, isomorphic to $-\rho \mathcal{H}(\rho^{-1})$.

Proof. The proof has three steps.

1. **Step1** We determine the image of $\rho \mathcal{H}(\rho^{-1})$ under the isomorphism (see [8], page 50)

$$S_0(TM) \cong \mathcal{E}_+^2 \otimes \Lambda_-^2(L^{-1}TM) \oplus \mathcal{E}_-^2 \otimes \Lambda_+^2(L^{-1}TM)$$

defined by

$$s(X, Y) = X \wedge s(Y) - Y \wedge s(X).$$

(Here $S_0(TM)$ is the space of symmetric traceless real-valued bilinear forms on TM , $s \in S_0(TM)$ and $X, Y \in TM$).

For this we first calculate $\text{tr}[\rho D^2(\rho^{-1})]$ in terms of the Laplacian $\Delta(\rho)$.

$$\begin{aligned} \text{tr}[\rho D^2(\rho^{-1})] &= \text{tr}[D(\rho D \rho^{-1}) - D(\rho) \otimes D(\rho^{-1})] \\ &= \text{tr}[D(\rho D \rho^{-1})] + \rho^{-2} \text{tr}[D(\rho) \otimes D(\rho)] \\ &= -\text{tr}[D(\rho^{-1} D \rho)] + |\rho^{-1} D(\rho)|^2 \end{aligned}$$

On the other hand,

$$\begin{aligned} \rho^{-1} \Delta(\rho) &= \rho^{-1} \text{tr}(D^2 \rho) - \frac{1}{6} \text{scal}^D \\ &= \text{tr}[D(\rho^{-1} D \rho) - D(\rho^{-1}) \otimes D \rho] - \frac{1}{6} \text{scal}^D \\ &= \text{tr}[D(\rho^{-1} D \rho)] + |\rho^{-1} D(\rho)|^2 - \frac{1}{6} \text{scal}^D \end{aligned}$$

and this implies that

$$\begin{aligned} \text{tr}[\rho D^2(\rho^{-1})] &= -\text{tr}[D(\rho^{-1} D \rho)] + |\rho^{-1} D(\rho)|^2 \\ &= -\frac{1}{6} \text{scal}^D + 2|\rho^{-1} D(\rho)|^2 - \rho^{-1} \Delta(\rho) \end{aligned}$$

Since

$$\text{sym}_0[\rho D^2(\rho^{-1})] = \text{sym}[\rho D^2(\rho^{-1})] - \frac{1}{4}\text{tr}[\rho D^2(\rho^{-1})]$$

it follows that $\rho\mathcal{H}(\rho^{-1})$ in $\mathcal{E}^2 \otimes \Lambda^2(L^{-1}TM)$ is the tensor

$$\begin{aligned} E_{X,Y} &= X \wedge r_0^D(Y) - Y \wedge r_0^D(X) \\ &\quad + X \wedge \text{sym}[\rho D^2(\rho^{-1})](Y) - Y \wedge \text{sym}[\rho D^2(\rho^{-1})](X) \\ &\quad + \left(\frac{1}{12}\text{scal}^D - |\rho^{-1}D(\rho)|^2 + \frac{1}{2}\rho^{-1}\Delta(\rho) \right) \cdot X \wedge Y \end{aligned}$$

2. **Step2** We determine the curvature on S_- of an arbitrary Weyl connection D . Because S_- has weight 1/2 the curvature of the connection D on S_- has the expression

$$R_{X,Y}^D = W_{X,Y}^- + \frac{1}{2}F^D(X,Y) \cdot \text{Id} - (r^D(X) \wedge Y - r^D(Y) \wedge X)_{ASD},$$

where

$$r^D = r_0^D + \frac{1}{24}\text{scal}^D \cdot \text{Id} - \frac{1}{2}F^D$$

Using the relation $F^D = 2 \cdot \text{skew}[\rho D^2(\rho^{-1})]$ we get

$$r^D(Y) \wedge X = r_0^D(Y) \wedge X + \frac{1}{24}\text{scal}^D \cdot Y \wedge X - \text{skew}[\rho D^2(\rho^{-1})](Y) \wedge X$$

The curvature $R_{X,Y}^D$ on S_- becomes

$$\begin{aligned} R_{X,Y}^D &= W_{X,Y}^- + \frac{1}{2}F^D(X,Y) \cdot \text{Id} + \{r_0^D(Y) \wedge X - r_0^D(X) \wedge Y \\ &\quad + \frac{1}{12}\text{scal}^D \cdot Y \wedge X - \text{skew}[\rho D^2(\rho^{-1})](Y) \wedge X \\ &\quad + \text{skew}[\rho D^2(\rho^{-1})](X) \wedge Y\}_{ASD}. \end{aligned}$$

3. **Step3** We determine the curvature on S_- of the Levi-Civita connection of the metric ρ^2c . The Levi Civita connection of this metric is the Weyl connection $D + \gamma$ where $\gamma := \rho^{-1}D(\rho)$. On TM it is the connection $D + \Gamma$, were $\Gamma \in \mathcal{E}_M^1 \otimes \text{End}(TM)$ is defined by

$$\Gamma_X = \gamma(X) \cdot \text{Id} + \gamma \wedge X$$

for $X \in TM$. A direct calculation shows that

$$\begin{aligned} (d^D\Gamma)_{X,Y} &= \{D_X(\gamma(Y)) - D_Y(\gamma(X)) - \gamma[X,Y]\} \cdot \text{Id} \\ &\quad + D_X(\gamma) \wedge Y - D_Y(\gamma) \wedge X \\ &= \{D_X(\gamma)(Y) - D_Y(\gamma)(X)\} \cdot \text{Id} \\ &\quad + D_X(\gamma) \wedge Y - D_Y(\gamma) \wedge X \\ &= d(\gamma)(X,Y) \cdot \text{Id} + D_X(\gamma) \wedge Y - D_Y(\gamma) \wedge X \\ &= D_X(\gamma) \wedge Y - D_Y(\gamma) \wedge X - F^D(X,Y) \cdot \text{Id} \end{aligned}$$

using the fact that $F^D + d(\gamma) = 0$ because $D + \gamma$ is exact. Also from a direct calculation we see that

$$\begin{aligned} \Gamma_Y(\Gamma_X(s)) - \Gamma_X(\Gamma_Y(s)) &= |\gamma|^2(Y \wedge X)(s) + \gamma(Y)(X \wedge \gamma)(s) \\ &\quad - \gamma(X)(Y \wedge \gamma)(s) \end{aligned}$$

The difference between the curvatures $R^{D+\gamma}$ and R^D on the spin bundle S_- (of weight 1/2) is the expression

$$\begin{aligned} R^{D+\gamma} - R^D &= \{D_X(\gamma) \wedge Y - D_Y(\gamma) \wedge X \\ &\quad + \gamma(X)Y \wedge \gamma - \gamma(Y)X \wedge \gamma \\ &\quad - |\gamma|^2 Y \wedge X\}_{ASD} - \frac{1}{2}F^D(X, Y) \cdot \text{Id} \end{aligned}$$

Using the previous step we obtain the curvature of $D + \gamma$ on S_- :

$$\begin{aligned} R_{X,Y}^{D+\gamma} &= W_{X,Y}^- + \{r_0^D(Y) \wedge X - r_0^D(X) \wedge Y + \frac{1}{12}\text{scal}^D \cdot Y \wedge X \\ &\quad - \text{skew}[\rho D^2(\rho^{-1})](Y) \wedge X + \text{skew}[\rho D^2(\rho^{-1})](X) \wedge Y \\ &\quad + D_X(\gamma) \wedge Y - D_Y(\gamma) \wedge X + \gamma(X)Y \wedge \gamma - \gamma(Y)X \wedge \gamma \\ &\quad - |\gamma|^2 Y \wedge X\}_{ASD} \\ &= W_{X,Y}^- - E_{X,Y} + \left\{ \frac{1}{2}\rho^{-1}\Delta(\rho) \cdot X \wedge Y + X \wedge \text{sym}[\rho D^2(\rho^{-1})](Y) \right. \\ &\quad \left. - Y \wedge \text{sym}[\rho D^2(\rho^{-1})](X) - \text{skew}[\rho D^2(\rho^{-1})](Y) \wedge X \right. \\ &\quad \left. + \text{skew}[\rho D^2(\rho^{-1})](X) \wedge Y + D_X(\gamma) \wedge Y - D_Y(\gamma) \wedge X \right. \\ &\quad \left. + \gamma(X)Y \wedge \gamma - \gamma(Y)X \wedge \gamma \right\}_{ASD} \\ &= W_{X,Y}^- - E_{X,Y} + \left\{ -\rho D^2(\rho^{-1})(Y) \wedge X + \rho D^2(\rho^{-1})(X) \wedge Y \right. \\ &\quad \left. + \frac{1}{2}\rho^{-1}\Delta(\rho) \cdot X \wedge Y + D_Y(\rho D(\rho^{-1})) \wedge X - D_X(\rho D(\rho^{-1})) \wedge Y \right. \\ &\quad \left. + \rho D_X(\rho^{-1})Y \wedge \rho D(\rho^{-1}) - \rho D_Y(\rho^{-1})X \wedge \rho D(\rho^{-1}) \right\}_{ASD} \end{aligned}$$

On the other hand

$$\begin{aligned} D_X(\rho D(\rho^{-1})) &= D_X(\rho)D(\rho^{-1}) + \rho D_X(D\rho^{-1}) \\ &= -\rho^2 D_X(\rho^{-1})D(\rho^{-1}) + \rho D_X(D\rho^{-1}) \\ &= -\rho D_X(\rho^{-1})\rho D(\rho^{-1}) + \rho D_X(D\rho^{-1}) \end{aligned}$$

and consequently

$$D_X(\rho D(\rho^{-1})) \wedge Y = \rho D_X(D\rho^{-1}) \wedge Y - \rho D_X(\rho^{-1})\rho D(\rho^{-1}) \wedge Y$$

The curvature $R^{D+\gamma}$ on S_- becomes

$$R_{X,Y}^{D+\gamma} = W_{X,Y}^- - E_{X,Y} + \frac{1}{2}\rho^{-1}\Delta(\rho) \cdot (X \wedge Y)_{ASD}$$

Via the identification of the tensor E with $\rho\mathcal{H}(\rho^{-1})$ given by Step 1, we obtain the expression for the curvature

$$R_{X,Y}^{D+\gamma} = W_{X,Y}^- - \rho\mathcal{H}_{X,Y}(\rho^{-1}) + \frac{1}{2}\rho^{-1}\Delta(\rho) \cdot (X \wedge Y)_{ASD}$$

The conclusion follows. □

Theorem 7.3. *Let D be a Weyl connection on a 4-dimensional oriented conformal manifold M and ∇ a connection acting on a complex line bundle W over M . Let ρ be a section of $L^{-1}W$ and suppose that W has a square root. Then the curvature of the connection $[D + \rho^{-1}D(\rho)] \otimes \nabla$ on $S_- \otimes W^{-1/2}$ has the form*

$$R_{X,Y} = W_{X,Y}^- - \rho\mathcal{H}_{X,Y}(\rho^{-1}) + \frac{1}{2}\{\rho^{-1}\Delta(\rho) \cdot X \wedge Y + F^\nabla(X) \wedge Y - F^\nabla(Y) \wedge X\}_{ASD}$$

In particular if M is a self-dual oriented conformal 4-manifold, ρ satisfies the background coupled wave equation and the connection ∇ is a self-dual Maxwell field, then the connection $[D + \rho^{-1}D(\rho)] \otimes \nabla$ on $S_- \otimes W^{-1/2}$ has self-dual curvature.

Proof. The proof of this lemma is similar to the previous proof.

1. **Step1** Using the coupled conformally invariant Laplacian one can prove that the image of $\rho^{-1}\mathcal{H}(\rho)$ under the same isomorphism

$$S_0(TM) \cong \mathcal{E}_+^2 \otimes \Lambda_-^2(L^{-1}TM) \oplus \mathcal{E}_-^2 \otimes \Lambda_+^2(L^{-1}TM)$$

is the tensor E defined by

$$\begin{aligned} E_{X,Y} = & X \wedge r_0^D(Y) - Y \wedge r_0^D(X) \\ & + X \wedge \text{sym}[\rho D^2(\rho^{-1})](Y) - Y \wedge \text{sym}[\rho D^2(\rho^{-1})](X) \\ & + \left(\frac{1}{12}\text{scal}^D - |\rho^{-1}D(\rho)|^2 + \frac{1}{2}\rho^{-1}\Delta(\rho) \right) \cdot X \wedge Y \end{aligned}$$

2. **Step2** We calculate the curvature R^D of the Weyl connection D on the bundle S_- . The expression of the curvature R^D is formally almost the same as in the uncoupled case, the only difference being the term

$$\frac{1}{2} (F^\nabla(X) \wedge Y - F^\nabla(Y) \wedge X)_{ASD}.$$

This term appears since $2 \cdot \text{skew}[\rho D^2(\rho^{-1})]_{X,Y}$ calculates the curvature $[F^D - F^\nabla](X, Y)$, and not the curvature F^D as before. Thus on S_- the curvature of the Weyl connection D is

$$\begin{aligned} R_{X,Y}^D = & W_{X,Y}^- + \frac{1}{2} F^D(X, Y) \cdot \text{Id} + \{r_0^D(Y) \wedge X - r_0^D(X) \wedge Y \\ & + \frac{1}{12} \text{scal}^D \cdot Y \wedge X - \text{skew}[\rho D^2(\rho^{-1})](Y) \wedge X \\ & + \text{skew}[\rho D^2(\rho^{-1})](X) \wedge Y + \frac{1}{2} F^\nabla(X) \wedge Y - \frac{1}{2} F^\nabla(Y) \wedge X\}_{ASD} \end{aligned}$$

3. **Step3** We calculate the curvature $[D + \rho^{-1}D(\rho)] \otimes \nabla$ on $S_- \otimes W^{-1/2}$. Let $\gamma := \rho^{-1}D(\rho)$. The Weyl connections $D + \gamma$ and D differ on TM by the 1-form Γ with values in $\text{End}(TM)$, which has the same formal expression as in the uncoupled case. A direct calculation using the equality $D_X(\gamma)(Y) - D_Y(\gamma)(X) = F^\nabla - F^D$ gives that

$$(d^D \Gamma)_{X,Y} = D_X(\gamma) \wedge Y - D_Y(\gamma) \wedge X + (F^\nabla(X, Y) - F^D(X, Y)) \cdot \text{Id}$$

The expression for $\Gamma_X \Gamma_Y - \Gamma_Y \Gamma_X$ is the formally same as in the uncoupled case. It follows that on S_- the difference between the curvatures $R^{D+\gamma}$ and R^D is

$$\begin{aligned} R^{D+\gamma} - R^D = & \{D_X(\gamma) \wedge Y - D_Y(\gamma) \wedge X \\ & + \gamma(X)Y \wedge \gamma - \gamma(Y)X \wedge \gamma \\ & - |\gamma|^2 Y \wedge X\}_{ASD} + \frac{1}{2} (F^\nabla(X, Y) - F^D(X, Y)) \cdot \text{Id} \end{aligned}$$

Using Step 2 the curvature of the Weyl connection $D + \rho^{-1}D(\rho)$ on S_- becomes

$$\begin{aligned} R_{X,Y}^{D+\gamma} = & W_{X,Y}^- - E_{X,Y} + \frac{1}{2} \rho^{-1} \Delta(\rho) \cdot (X \wedge Y)_{ASD} \\ & + \frac{1}{2} (F^\nabla(X) \wedge Y - F^\nabla(Y) \wedge X)_{ASD} + \frac{1}{2} F^\nabla(X, Y) \cdot \text{Id} \end{aligned}$$

It follows that the curvature $R^{(D+\gamma) \otimes \nabla}$ on $S_- \otimes W^{-1/2}$ is

$$\begin{aligned} R_{X,Y}^{(D+\gamma) \otimes \nabla} = & W_{X,Y} - E_{X,Y} + \frac{1}{2} \{ \rho^{-1} \Delta(\rho) \cdot X \wedge Y + F^\nabla(X) \wedge Y \\ & - F^\nabla(Y) \wedge X \}_{ASD} \end{aligned}$$

and the conclusion follows. \square

Remark. The bundle $S_- \otimes W^{-1/2}$ is a $SL(2, \mathbb{C})$ bundle: $\Lambda^2(S_- \otimes W^{-1/2}) = W^{-1} \otimes L^1$, and ρ defines an isomorphism between W and $L^1_{\mathbb{C}}$.

7.3.3 The case $k = 2$.

Remark. We make a remark about Weyl connections which will be used in the proof of the following lemma. Recall that the space of Weyl connections on a conformal manifold M is an affine space over $\mathcal{E}^1(M)$. Thus if D is a Weyl connection on M any other Weyl connection \hat{D} is of the form $\hat{D} := D + \gamma$ with $\gamma \in \mathcal{E}^1(M)$. Then on $L^{-1/2}TM$, $\hat{D}_X - D_X = \frac{1}{2}\gamma(X) \cdot \text{Id} + \gamma \wedge X$ and on $L^{-1}\Lambda^2(TM)$ it extends as a derivation. We obtain that $\hat{D} - D = \Gamma$ on $L^{-1}\Lambda^2(TM)$ where the 1-form Γ with values in $\text{End}[L^{-1}\Lambda^2(TM)]$ is defined by the formula

$$\begin{aligned} \Gamma(X)(v_1 \wedge v_2) = & \gamma(X)v_1 \wedge v_2 + \gamma(v_1)X \wedge v_2 - \gamma(v_2)X \wedge v_1 \\ & - \langle X, v_1 \rangle \gamma \wedge v_2 + \langle X, v_2 \rangle \gamma \wedge v_1 \end{aligned}$$

on decomposable bivectors. Also, if X is any vector field Γ_X preserves the bundle of anti-self-dual bivectors (of weight 1).

Lemma 7.4. *Let D be a Weyl connection on a conformal 4-dimensional oriented manifold M and $\rho \in \mathcal{E}_-^2(M)$ non-degenerate. Define $F \in \mathcal{E}_M^1 \otimes \text{End}[L^{-1}\Lambda_-^2(TM)]$ by the formula $F_X(\alpha) = [\eta(\alpha) \wedge X]_{ASD}$ where $X \in TM$, $\alpha \in L^{-1}\Lambda_-^2(TM)$ and $\eta(\alpha)$ is a vector field determined by the condition $i_{\eta(\alpha)}\rho = \langle D(\rho), \alpha \rangle$. Then the connection $D + F$ on $L^{-1}\Lambda_-^2(TM)$ induces a connection on the annihilator of ρ in $L^{-1}\Lambda_-^2(TM)$ which is independent of the choice of the Weyl connection D .*

Proof. First we show that $D + F$ induces a connection on the annihilator of ρ in $L^{-1}\Lambda_-^2(TM)$. For this, consider $\alpha \in L^{-1}\Lambda_-^2(TM)$ orthogonal to ρ . It is enough to notice that

$$\begin{aligned} \langle D_X(\alpha) + F_X(\alpha), \rho \rangle &= -\langle \alpha, D_X(\rho) \rangle + \rho(\eta(\alpha), X) \\ &= -\langle \alpha, D_X(\rho) \rangle + \langle D_X(\rho), \alpha \rangle \\ &= 0 \end{aligned}$$

Next we show that the connection induced by $D + F$ on the annihilator of ρ in $L^{-1}\Lambda_-^2(TM)$ is independent of the choice of the Weyl connection D . For this we consider $\hat{D} := D + \gamma$ another Weyl connection. Then \hat{D} on $L^{-1}\Lambda_-^2(TM)$ is the connection $D + \Gamma$ with Γ a 1-form with values in $\text{End}[\Lambda_-^2(TM)]$ (see the above remark). A direct calculation shows that

$$i_X(\Gamma(X)(\alpha)) \wedge X = i_\gamma(\alpha) \wedge X \cdot |X|^2$$

where $X \in TM$ and $\alpha \in L^{-1}\Lambda_-^2(TM)$.

Denote by $\hat{\eta}(\alpha)$ the vector field determined by α and defined using \hat{D} , and by the \hat{F} the corresponding 1-form with values in $\text{End}[L^{-1}\Lambda_-^2(TM)]$. We need to

show that for $\alpha \in L^{-1}\Lambda_-^2(TM)$ and $X \in TM$ the equality

$$\hat{D}_X(\alpha) + \hat{F}_X(\alpha) = D_X(\alpha) + F_X(\alpha)$$

holds, or equivalently that the equality

$$[(\eta(\alpha) - \hat{\eta}(\alpha)) \wedge X]_{ASD} = \Gamma_X(\alpha)$$

holds. For this we notice that $\Gamma_X(\alpha)$ being anti-self-dual, it is equal to $2|X|^{-2}[X \wedge i_X(\Gamma_X(\alpha))]_{ASD}$ or equivalently to $2(X \wedge i_\gamma(\alpha))_{ASD}$. Thus it is enough to prove that $\hat{\eta}(\alpha) - \eta(\alpha) = 2i_\gamma(\alpha)$, and this follows from the non-degeneracy of ρ together with the following calculation: for every $X \in TM$,

$$\begin{aligned} \rho(\eta(\alpha) - \hat{\eta}(\alpha), X) &= -\langle \Gamma_X(\rho), \alpha \rangle \\ &= \langle \rho, \Gamma_X(\alpha) \rangle \\ &= 2\langle \rho, (X \wedge i_\gamma(\alpha))_{ASD} \rangle \\ &= 2\rho(X, i_\gamma(\alpha)) \end{aligned}$$

(for the first equality we have used the definition of $\eta(\alpha)$ and $\hat{\eta}(\alpha)$, and for the fourth equality we have used the fact that self-dual 2-forms are orthogonal to anti-self-dual 2-forms).

□

Remark. In the proof we have used the antisymmetry

$$\langle \Gamma_X(\rho), \alpha \rangle = -\langle \rho, \Gamma_X(\alpha) \rangle$$

This follows since ρ is orthogonal to α . Thus the connection $D + F$ is independent of the choice of the Weyl connection D only on the annihilator of ρ in $L^{-1}\Lambda_-^2(TM)$.

Theorem 7.5. *Let ∇ be a self-dual Maxwell field acting on a line bundle W over the 4-dimensional oriented conformal self-dual manifold M . Let $\rho \in \Gamma(M, \mathcal{E}_-^2 \otimes W)$ be non-degenerate such that $d^\nabla(\rho) = 0$, and D be a Weyl connection on M . Define $F \in \mathcal{E}_M^1 \otimes \text{End}[L^{-1}\Lambda_-^2(TM) \otimes W^{-1/2}]$ by the formula $F_X(\alpha) = [\eta(\alpha) \wedge X]_{ASD}$ where $\alpha \in L^{-1}\Lambda_-^2(TM) \otimes W^{-1/2}$, $X \in TM$ and $\eta(\alpha)$ determined by the condition $i_{\eta(\alpha)}(\rho) = \langle (D \otimes \nabla)(\rho), \alpha \rangle$. Then on the annihilator of ρ in $L^{-1}\Lambda_-^2(TM) \otimes W^{-1/2}$ the connection $D \otimes \nabla + F$ has self-dual curvature.*

Proof. See [2].

□

Remark. We can show directly that the annihilator of ρ in

$$L^{-1}\Lambda_-^2(TM) \otimes W^{-1/2}$$

has trivial determinant. For this we first consider the orthogonal decomposition

$$L^{-1}\Lambda_-^2(TM) \otimes W^{-1/2}L^3W^{-3/2}\rho \oplus \langle \rho \rangle^\perp$$

of $L^{-1}\Lambda_-^2(TM) \otimes W^{-1/2}$. Then we notice that since $\Lambda_-^2(TM)$ is 3-dimensional, the determinant of $L^{-1}\Lambda_-^2(TM) \otimes W^{-1/2}$ is equal to $L^{-3}W^{-3/2} \otimes \det[\Lambda_-^2(TM)]$, which is equal to $L^3W^{-3/2}$. The claim follows.

7.4 The reduced equations.

Assumptions. In this section we consider (B, D^B) a 3-dimensional oriented Einstein-Weyl space, (w, A) a solution of the abelian monopole equation on B and $\pi : M \rightarrow B$ the conformal submersion it generates with M an oriented 4-dimensional self-dual manifold. On B we fix a vector bundle V with a connection ∇_1 acting on it and a section $w_1 \in L^{-1} \otimes \text{End}(V)$. We define a connection ∇ on $\pi^*(V)$ by the formula $\pi^*(\nabla_1) - w_1\xi$, where $\xi := w^{-1}(dt + A)$ with t the fiber coordinate of π . In other words, the horizontal part of ∇ is ∇_1 and the vertical part of ∇ is w_1 .

Notations. In order to simplify the notations, the tensor product connection $D^B \otimes \nabla_1$ on B applied to a section of $L^k \otimes V$ (or to a section of $L^kTB \otimes V$) will be denoted simply ∇_1 . Similarly, the tensor product connection $D^B \otimes \nabla$ on M applied to a section of $L^k \otimes \pi^*(V)$ (or to a section of $L^kTM \otimes \pi^*(V)$) will be denoted simply ∇ .

7.4.1 The case $k = 0$.

Lemma 7.6. *Let $A_2 \in \Gamma(B, \mathcal{E}_B^1 \otimes \text{Hom}(V^*, V))$ and $w_2 \in L^{-1} \otimes \text{Hom}(V^*, V)$ such that they satisfy the relation*

$$d^{\nabla_1}(A_2) = *_B(\nabla_1(w_2) - w_1A_2)$$

*Then the 1-form $\mathbf{A} \in \Gamma(M, \mathcal{E}_M^1 \otimes \text{Hom}(\pi^*V^*, \pi^*V))$ defined by the formula $\mathbf{A} = \pi^*(A_2) - w_2\xi$ satisfies the equation $d^\nabla(\mathbf{A})_{ASD} = 0$.*

Proof. A direct calculation shows that

$$\begin{aligned}
d^\nabla(\mathbf{A}) &= d^{\pi^*(\nabla_1) - w_1\xi}(\mathbf{A}) \\
&= d^{\pi^*(\nabla_1)}(\mathbf{A}) - w_1\xi \wedge \mathbf{A} \\
&= d^{\pi^*(\nabla_1)}(\pi^*(A_2) - w_2\xi) - w_1\xi \wedge A_2 \\
&= \pi^*(d^{\nabla_1}(A_2)) - \nabla_1(w_2) \wedge \xi - w_2d^{D^B}(\xi) - w_1\xi \wedge A_2 \\
&= \pi^*(d^{\nabla_1}(A_2)) + \xi \wedge (\nabla_1(w_2) - w_1A_2) - w_2d^{D^B}(\xi)
\end{aligned}$$

First we notice that, since (w, A) satisfies the abelian monopole equation, the form $d^{D^B}(\xi)$ is self-dual, being equal to $w^{-1}(\xi \wedge D^B(w) + *_M(\xi \wedge D^B(w)))$. Also, since $*_M(\alpha) = -\xi \wedge *_B(\alpha)$ for $\alpha \in \mathcal{E}^2(B)$ we obtain that the anti-self dual part of $d^\nabla(\mathbf{A})$ is 0 if and only if $d^{\nabla_1}(A_2) = *_B(\nabla_1(w_2) - w_1A_2)$. The conclusion follows. \square

7.4.2 The case $k = 1$.

In order to find the reduced equations in the case $k = 1$ we need the following lemma.

Lemma 7.7. *The scalar curvatures $\text{scal}_M^{D^B}$ and $\text{scal}_B^{D^B}$ are related by the formula*

$$\text{scal}_M^{D^B} = \text{scal}_B^{D^B} - \frac{9}{2}|w^{-1}D^B(w)|^2.$$

Proof. If scal_M^0 denotes the scalar curvature of the Weyl connection D^0 on M and scal_B^0 denotes the scalar curvature of D^0 on B (recall that D^0 is basic) then the formula

$$\text{scal}_M^0 = \text{scal}_B^0 - \frac{1}{2}|\gamma_0|^2$$

holds, with $\gamma_0 := w^{-1}D^B(w)$ (see [16]). On the other hand $D^0 = D^B + \gamma_0$ and applying the formula (see [16])

$$\text{scal}^{D+\gamma} = \text{scal}^D - 2(n-1)\text{tr}D(\gamma) - (n-1)(n-2)|\gamma|^2$$

which holds for any Weyl connection D and 1-form γ on a conformal n -dimensional manifold, we obtain that

$$\begin{aligned}
\text{scal}_M^0 &= \text{scal}_M^{D^B} - 6\text{tr}_M D^B(\gamma_0) - 6|\gamma_0|^2 \\
\text{scal}_B^0 &= \text{scal}_B^{D^B} - 4\text{tr}_B D^B(\gamma_0) - 2|\gamma_0|^2
\end{aligned}$$

Next we calculate the traces tr_M and tr_B . Using the formula $\text{tr}_M D^B(\gamma_0) = (*_M d^{D^B} *_M)(\gamma_0)$ and the formula $*_M(\alpha) = (-1)^{k+1} \xi \wedge *_B(\alpha)$ (where α is a k -form on B) we obtain

$$\begin{aligned}
& \text{tr}_M D^B(\gamma_0) \text{tr}_M D^B(w^{-1} D^B w) \\
&= *_M \circ d^{D^B} (*_M w^{-1} D^B w) \\
&= *_M \circ d^{D^B} (\xi \wedge *_B(w^{-1} D^B w)) \\
&= *_M \circ d^{D^B} (w^{-2} (dt + A) \wedge dA) \\
&= *_M \circ [-2w^{-3} D^B(w) \wedge (dt + A) \wedge dA] \\
&= -2 *_M [w^{-3} D^B(w) \wedge (dt + A) \wedge *_B D^B w] \\
&= 2w^{-2} *_M [\xi \wedge D^B(w) \wedge *_B D^B w] \\
&= 2w^{-2} |D^B w|^2 *_M (\xi \wedge \text{or}_B) \\
&= 2w^{-2} |D^B w|^2 *_B (\text{or}_B) \\
&= -2|\gamma_0|^2
\end{aligned}$$

Using the formula $\text{tr}_B D^B(\gamma_0) = -(*_B d^{D^B} *_B)(\gamma_0)$ we obtain

$$\begin{aligned}
& \text{tr}_B D^B(\gamma_0) - *_B \circ d^{D^B} (*_B w^{-1} D^B w) \\
&= -*_B \circ d^{D^B} (w^{-1} dA) \\
&= *_B [w^{-2} D^B(w) \wedge dA] \\
&= w^{-2} *_B [D^B(w) \wedge *_B D^B(w)] \\
&= w^{-2} |D^B w|^2 *_B (\text{or}_B) \\
&= -|\gamma_0|^2
\end{aligned}$$

We obtain

$$\begin{aligned}
\text{scal}_M^0 &= \text{scal}_M^{D^B} + 6|\gamma_0|^2 \\
\text{scal}_B^0 &= \text{scal}_B^{D^B} + 2|\gamma_0|^2.
\end{aligned}$$

Returning to the relation between scal_M^0 and scal_B^0 mentioned at the beginning of the proof the conclusion follows. \square

Lemma 7.8. *If $s \in L_B^{-1/2} \otimes V$ is a solution of the equation*

$$\text{tr}_B \nabla_1^2(s) - \frac{1}{6} \text{scal}^{D^B} s + w_1^2 s = 0$$

then $w^{1/2} \pi^(s) \in L_M^{-1} \otimes \pi^*(V)$ is a solution of the conformally invariant Laplacian coupled with the connection ∇ .*

Proof. Since $\nabla = \pi^*(\nabla_1) - w_1\xi$, we get that $\nabla(w^{1/2}s) = \nabla_1(w^{1/2}s) - w_1\xi \cdot w^{1/2}s$ and

$$\begin{aligned} *_M\nabla(w^{1/2}s) &= \xi \wedge *_B\nabla_1(w^{1/2}s) + w_1w^{1/2}s \cdot \text{or}_B \\ &= w^{-1}(dt + A) \wedge *_B\nabla_1(w^{1/2}s) + w_1w^{1/2}s \cdot \text{or}_B \end{aligned}$$

It follows that

$$\begin{aligned} d^\nabla(*_M\nabla(w^{1/2}s)) &= -w^{-2}D^B(w) \wedge (dt + A) \wedge *_B\nabla_1(w^{1/2}s) \\ &\quad + w^{-1}dA \wedge *_B\nabla_1(w^{1/2}s) - \xi \wedge d^{\nabla_1}(*_B\nabla_1(w^{1/2}s)) \\ &\quad + d^\nabla(w_1w^{1/2}s \cdot \text{or}_B) \\ &= \xi \wedge w^{-1}D^B(w) \wedge *_B\nabla_1(w^{1/2}s) - \xi \wedge d^{\nabla_1}(*_B\nabla_1(w^{1/2}s)) \\ &\quad - w_1^2w^{1/2}s \cdot \xi \wedge \text{or}_B \\ &= \xi \wedge [w^{-1}D^B(w) \wedge *_B\nabla_1(w^{1/2}s) - d^{\nabla_1}(*_B\nabla_1(w^{1/2}s))] \\ &\quad - w_1^2w^{1/2}s \cdot \text{or}_B \end{aligned}$$

A simple calculation shows that the terms which contain first derivatives in s vanish and we obtain

$$\begin{aligned} d^\nabla(*_M\nabla(w^{1/2}s)) &= \xi \wedge [\text{or}_B \langle w^{-1}D^B(w), D^B(w^{1/2}) \rangle s \\ &\quad - d^{D^B}(*_B D^B(w^{1/2}))s - w^{1/2}d^{\nabla_1}(*_B\nabla_1(s)) - w_1^2w^{1/2}s \cdot \text{or}_B] \\ &= -w^{1/2}\xi \wedge [d^{\nabla_1}(*_B\nabla_1(s)) + w^{-1/2}d^{D^B}(*_B D^B(w^{1/2}))s \\ &\quad - \langle w^{-1}D^B(w), w^{-1/2}D^B(w^{1/2}) \rangle s \cdot \text{or}_B + w_1^2s \cdot \text{or}_B] \\ &= -w^{1/2}\xi \wedge [d^{\nabla_1}(*_B\nabla_1(s)) - \frac{3}{4}|\gamma_0|^2s \cdot \text{or}_B + w_1^2s \cdot \text{or}_B] \end{aligned}$$

because

$$w^{-1/2}d^{D^B}(*_B D^B(w^{1/2})) = -\frac{1}{4}|\gamma_0|^2\text{or}_B$$

and

$$\langle w^{-1}D^B(w), w^{-1/2}D^B(w^{1/2}) \rangle \frac{1}{2}|\gamma_0|^2.$$

It follows that

$$\begin{aligned} \text{tr}_M\nabla^2(w^{1/2}s) &= *_M[-w^{1/2}\xi \wedge d^{\nabla_1}(*_B\nabla_1(s)) + \frac{3}{4}|\gamma_0|^2w^{1/2}s \cdot \xi \wedge \text{or}_B \\ &\quad - w_1^2w^{1/2}s \cdot \xi \wedge \text{or}_B] \\ &= -w^{1/2}*_B d^{\nabla_1}(*_B\nabla_1(s)) - \frac{3}{4}|\gamma_0|^2w^{1/2}s + w_1^2w^{1/2}s \\ &= w^{1/2}\text{tr}_B\nabla_1^2(s) - \frac{3}{4}|\gamma_0|^2w^{1/2}s + w_1^2w^{1/2}s \end{aligned}$$

Using Lemma 7.7 we can calculate now Δ (the conformally invariant Laplacian on M coupled with the connection ∇) applied to $w^{1/2}s$:

$$\begin{aligned}
\Delta(w^{1/2}s) &= \text{tr}_M \nabla^2(w^{1/2}s) - \frac{1}{6} \text{scal}_M^{D^B} w^{1/2}s \\
&= w^{1/2} [\text{tr}_B \nabla_1^2(s) - \frac{3}{4} |\gamma_0|^2 s + w_1^2 s] \\
&\quad - \frac{1}{6} (\text{scal}_B^{D^B} - \frac{9}{2} |\gamma_0|^2) w^{1/2}s \\
&= w^{1/2} [\text{tr}_B \nabla_1^2(s) - \frac{1}{6} \text{scal}_B^{D^B} s + w_1^2 s] \\
&= 0
\end{aligned}$$

The conclusion follows. \square

Remark. It is worth pointing out that the reduced equation we have obtained in Lemma 7.8 does not involve the conformally invariant Laplacian coupled with the connection ∇_1 : the coefficient of the scalar curvature in the reduced equation is $-\frac{1}{6}$, and not $-\frac{1}{8}$. It follows that in the expression of the reduced equation the Weyl derivative D^B is essential, and cannot be replaced by another Weyl derivative (like in the case of the conformally invariant Laplacian).

7.4.3 The case $k = 2$.

Lemma 7.9. *If $\alpha \in \Gamma(B, \mathcal{E}_B^1 \otimes V)$ is a solution of the system*

$$\begin{cases} d^{\nabla_1}(\alpha) = w_1 *_B(\alpha) \\ d^{\nabla_1}(*_B \alpha) = 0 \end{cases}$$

then $\pi^(w\alpha) \in \Gamma(M, \mathcal{E}_-^2 \otimes \pi^*(V))$ satisfies*

$$d^\nabla \pi^*(w\alpha) = 0.$$

Proof. First we recall that the isomorphism $\pi^*(L^{-1}\mathcal{E}_B^1) \cong \mathcal{E}_-^2$ used implicitly in the statement of the lemma associates to $\beta \in L^{-1}\mathcal{E}_B^1$ the anti-self-dual 2-form $\xi \wedge \beta - *_M(\xi \wedge \beta)$. Then we notice that

$$\begin{aligned}
d^\nabla [\xi \wedge (w\alpha) - *_M(\xi \wedge (w\alpha))] &= d^{\pi^*(\nabla_1)}((dt + A) \wedge \alpha) \\
&\quad - d^\nabla(*_B \alpha)w - *_B(\alpha) \wedge D^B(w) \\
&= *_B D^B(w) \wedge \alpha - (dt + A) \wedge \pi^*(d^{\nabla_1}(\alpha)) \\
&\quad - d^\nabla(*_B \alpha)w - *_B(\alpha) \wedge D^B(w) \\
&= - (dt + A) \wedge \pi^*(d_1^\nabla(\alpha)) - d^{\nabla_1}(*_B \alpha)w \\
&\quad + w_1 w \xi \wedge *_B \alpha
\end{aligned}$$

By identifying the horizontal and vertical parts it follows that

$$d^\nabla[\xi \wedge (w\alpha) - *_M(\xi \wedge (w\alpha))] = 0$$

if and only if

$$\begin{cases} d^{\nabla_1}(\alpha) = w_1 *_B(\alpha) \\ d^{\nabla_1}(*_B\alpha) = 0 \end{cases}$$

The conclusion follows. □

7.5 The Ansätze for 3-dimensional Einstein-Weyl spaces.

7.5.1 The case $k = 0$

The reduced Ansatz for $k = 0$ gives an interpretation of the affine monopole equations:

$$\begin{cases} *_B\nabla_1(w_1) = F^{\nabla_1} \\ d^{\nabla_1}(A_2) = *_B(\nabla_1(w_2) - w_1 A_2) \end{cases}$$

where $A_2 \in \Gamma(B, \mathcal{E}_B^1 \otimes \text{Hom}(V^*, V))$ and $w_2 \in L^{-1} \otimes \text{Hom}(V^*, V)$. The first of these equations is the abelian monopole equation. The second of these equations is obtained by reducing the auxiliary equation of the A_0 Ansatz from 4 to 3 dimensions (see Lemma 7.6). The following lemma holds.

Lemma 7.10. *The affine monopole equations on 3-dimensional Einstein-Weyl manifolds are natural reductions of the A_0 Atiyah-Ward Ansatz on self-dual conformal 4-manifolds.*

7.5.2 The case $k = 1$.

Notation. For B an oriented conformal (spin) 3-manifold, S will denote the weight 1/2-spin bundle. It is a rank two complex vector bundle on B such that $\odot^2 S = T_{\mathbb{C}}B$.

Theorem 7.11. *Let (∇_1, w_1) be a solution of the abelian monopole equation defined on the line bundle V over the 3-dimensional oriented Einstein-Weyl space B . Suppose that V has a square root. If $s \in L^{-1/2}V$ is a solution of the equation*

$$\text{tr}\nabla_1^2(s) - \frac{1}{6}\text{scal}^{D^B}s + w_1^2s = 0$$

then the Higgs field

$$-\frac{1}{2}s^{-1}\nabla_1(s) \in L_B^{-1} \otimes \text{End}(L_B^{-1/4}S \otimes V^{-1/2})$$

and the connection $(D^B + \Gamma) \otimes \nabla_1$ on $L_B^{-1/4}S \otimes V^{-1/2}$ with

$$\Gamma_X = \frac{1}{2}s^{-1}(\nabla_1)_X(s) \cdot \text{Id} - \frac{1}{2}w_1X + \frac{1}{2} *_B (s^{-1}\nabla_1(s) \wedge X)$$

(where $X \in TB$) satisfy the Einstein-Weyl Bogomolny equations on B .

Proof. Consider (w, A) a solution of the abelian monopole equation on B and $\pi : M \rightarrow B$ the conformal submersion it generates with M a 4-dimensional self-dual oriented conformal manifold. From Theorem 7.3 and Lemma 7.8 we obtain a self-dual Yang-Mills field on $S_- \otimes \pi^*(V^{-1/2})$ defined by the formula $(D^B + \rho^{-1}\nabla(\rho)) \otimes \nabla$, where S_- is the spin bundle over M such that $\odot^2 S_- = L^{-1}\Lambda_-^2(TM)$, $\rho := w^{1/2}s \in L^{-1} \otimes \pi^*(V)$ and the connection ∇ on $\pi^*(V^{-1/2})$ is induced by the connection $\nabla := \pi^*(\nabla_1) - w_1\xi$ on $\pi^*(V)$. The proof has three steps.

1. **Step1** We show that on S_- the connection $D^B + \rho^{-1}\nabla(\rho)$ is the connection

$$\pi^*(D^B) + \frac{1}{2} \left(\frac{1}{2}w^{-1}D^B(w) + s^{-1}\nabla_1(s) - w_1\xi \right) \cdot \text{Id} + F$$

where $F \in T^*M \otimes \text{End}(S_- \otimes \pi^*(V^{-1/2}))$ is

$$F_X = [(s^{-1}\nabla_1(s) - w_1\xi) \wedge X]_{ASD}$$

For this we first notice that

$$\nabla(\rho) = \frac{1}{2}w^{-1/2}D^B(w)s + w^{1/2}(\nabla_1(s) - w_1\xi \cdot s)$$

and

$$\rho^{-1}\nabla(\rho) = \frac{1}{2}w^{-1}D^B(w) + s^{-1}\nabla_1(s) - w_1\xi$$

It follows that on L^1 the connection $D^B + \rho^{-1}\nabla(\rho)$ is equal to

$$D^B + \frac{1}{2}w^{-1}D^B(w) + s^{-1}\nabla_1(s) - w_1\xi.$$

Now recall that on L^1 we have defined the Weyl connection D^{sd} to be $D^B + \frac{1}{2}w^{-1}D^B(w)$ (see Section 7.2), and that it has the property that on $L^{-1/2}S_-$ it is the pull-back of D^B on $L^{-1/2}S$. Since on L^1 the connection $D^B + \rho^{-1}\nabla(\rho)$ is equal to $D^{sd} + s^{-1}\nabla_1(s) - w_1\xi$, it follows that on $L^{-1/2}S_-$ (which is of weight 0) it is equal to $\pi^*(D^B) + F$, with $F \in \Gamma(M, \mathcal{E}_M^1 \otimes \text{End}[L^{-1/2}S_-])$ defined by

$$F_X = [(s^{-1}\nabla_1(s) - w_1\xi) \wedge X]_{ASD}$$

(for every $X \in TM$). Then on $L^{1/2} \otimes L^{-1/2}S_-$ the connection $D^B + \rho^{-1}\nabla(\rho)$ is the tensor product $[D^{sd} + \frac{1}{2}(s^{-1}\nabla_1(s) - w_1\xi) \cdot \text{Id}] \otimes [\pi^*(D^B) + F]$, which is equal to $[D^B + \frac{1}{2}(\frac{D^B(w)}{w} + s^{-1}\nabla_1(s) - w_1\xi) \cdot \text{Id}] \otimes [\pi^*(D^B) + F]$, because $D^{sd} = D^B + \frac{1}{2}w^{-1}D^B(w)$ on L^1 . The claim follows.

2. **Step2** We change the weight of the bundle $S_- \otimes \pi^*(V^{-1/2})$ in order to get a self-dual Yang-Mills field with horizontal part independent of w .

The change of weight is realized by the diffeomorphism

$$w^{1/4} : S_- \otimes \pi^*(V^{-1/2}) \rightarrow L_M^{-1/4} S_- \otimes \pi^*(V^{-1/2})$$

The self-dual Yang-Mills field $[D^B + \rho^{-1} \nabla(\rho)] \otimes \nabla$ on $S_- \otimes \pi^*(V^{-1/2})$ induces a self-dual Yang Mills field

$$[\pi^*(D^B) + \frac{1}{2}(s^{-1} \nabla_1(s) - w_1 \xi) \cdot \text{Id} + F] \otimes \nabla$$

on $L_M^{-1/4} S_- \otimes \pi^*(V^{-1/2})$, with F having the same formal expression as in Step 1.

3. **Step3** We determine the horizontal and the vertical parts of the self-dual Yang-Mills field on $L_M^{-1/4} S_- \otimes \pi^*(V^{-1/2})$. They will provide the connection and the Higgs field which form a solution of the Einstein-Weyl Bogomolny equations on B .

For this, we first notice that on $\pi^*(V^{-1/2})$ the connection ∇ is $\pi^*(\nabla_1) + \frac{1}{2} w_1 \xi$, and that the Yang-Mills field on $L_M^{-1/4} S_- \otimes \pi^*(V^{-1/2})$ becomes $\pi^*(D^B \otimes \nabla_1) + \frac{1}{2} s^{-1} \nabla_1(s) \cdot \text{Id} + F$, with F a 1-form with values in $\text{End} \left(L_M^{-1/4} S_- \otimes \pi^*(V^{-1/2}) \right)$ defined by

$$F_X = [(s^{-1} \nabla_1(s) - w_1 \xi) \wedge X]_{ASD}$$

($X \in TM$). It follows that the vertical part of the self-dual Yang-Mills field is $F_\xi = -\frac{1}{2} s^{-1} \nabla_1(s)$ (since $(D^B \otimes \nabla_1)_\xi(s)$ is 0, the connection ∇_1 being a pull-back connection on $L^{-1/2} \otimes \pi^*(V)$, ξ being vertical and s being the pull-back of a section on the base). The horizontal part of the self-dual Yang-Mills field is $D^B \otimes \nabla_1 + \Gamma$, where

$$\Gamma_X = \frac{1}{2} s^{-1} (\nabla_1)_X(s) \cdot \text{Id} + i_\xi(F_X)$$

($X \in TB$). A simple calculation shows that

$$i_\xi(F_X) = -\frac{1}{2} w_1 X + \frac{1}{2} *_B (s^{-1} \nabla_1(s) \wedge X)$$

and the conclusion follows. □

7.5.3 The case $k = 2$.

Theorem 7.12. *Let (w_1, ∇_1) be a solution of the abelian monopole equation defined on the line bundle V over the 3-dimensional oriented Einstein-Weyl space B . Suppose that V has a square root. If $\alpha \in \Gamma(B, \mathcal{E}_B^1 \otimes V)$ satisfies*

$$\begin{cases} d^{\nabla_1}(\alpha) = w_1 *_B(\alpha) \\ d^{\nabla_1}(*_B\alpha) = 0 \end{cases}$$

then the Higgs field $H \in L^{-1} \otimes \text{End}(L^{-1/2}TB \otimes V^{-1/2})$ defined by

$$H(\chi) = |\alpha|^{-2} *_B(\alpha \wedge \langle \nabla_1(\alpha), \chi \rangle) + \frac{1}{2} w_1 \chi$$

and the connection $D^B \otimes \nabla_1 + G$ with $G \in T^*B \otimes \text{End}(L^{-1/2}TB \otimes V^{-1/2})$ given by

$$G(X)(\chi) = |\alpha|^{-2} [\langle (\nabla_1)_\alpha(\alpha), \chi \rangle X + \langle (\nabla_1)_X(\alpha), \chi \rangle \alpha - \langle \nabla_1(\alpha), \chi \rangle \alpha(X)]$$

(where $X \in TB$) induces a solution the Einstein-Weyl Bogomolny equations on B defined on the orthogonal complement of α in the bundle $L^{-1/2}TB \otimes V^{-1/2}$.

Proof. Consider (w, A) a solution of the abelian monopole equation on B and $\pi : M \rightarrow B$ the conformal submersion it generates with M a 4-dimensional self-dual oriented conformal manifold. Let $\rho := w(\xi \wedge \alpha - *_B\alpha)$ and ∇ the connection on $\pi^*(V)$ defined by $\nabla := \pi^*(\nabla_1) - w_1\xi$. Preserving the notations from Theorem 7.5, we obtain the self-dual Yang-Mills field $D \otimes \nabla + F$ acting on the annihilator of ρ in $L^{-1}\Lambda_-^2(TM) \otimes \pi^*(V^{-1/2})$. Since $L^{-1}\Lambda_-^2(TM) \cong \pi^*(TB)$, the self-dual Yang-Mills field can be considered to act on the annihilator of $\pi^*(\alpha)$ in $\pi^*(TB \otimes V^{-1/2})$. Also, the self-dual Yang-Mills field is independent of the choice of the Weyl connection D (see Lemma 7.4). We shall choose the Weyl connection D to be D^{sd} (see Section 7.2).

The proof has three steps.

1. **Step1** We show that the self-dual Yang-Mills field acting on $\pi^*(TB \otimes V^{-1/2})$ is

$$\pi^*(D^B \otimes \nabla_1) + \frac{1}{2} (w^{-1}D^B(w) + w_1\xi) \cdot \text{Id} + F$$

where F is a 1-form with values in $\text{End}(\pi^*(TB \otimes V^{-1/2}))$ such that when X is basic

$$\begin{aligned} F_X(\chi) &= -|\alpha|^{-2} \langle (\nabla_1)_\alpha(\chi), \alpha \rangle X - |\alpha|^{-2} \alpha(X) \langle (\nabla_1)(\alpha), \chi \rangle \\ &\quad + |\alpha|^{-2} \langle (\nabla_1)_X(\alpha), \chi \rangle \alpha \end{aligned}$$

and

$$F_\xi(\chi) = |\alpha|^{-2} *_B(\alpha \wedge \langle \nabla_1(\alpha), \chi \rangle)$$

for $\chi \in TB \otimes V^{-1/2}$.

For this, we first note that $D^{sd} = \pi^*(D^B) + \frac{1}{2}w^{-1}D^B(w) \cdot \text{Id}$ on $\pi^*(TB)$, and that the product connection $D^{sd} \otimes \nabla$ on $\pi^*(TB \otimes V^{-1/2})$ is the connection $\pi^*(D^B \otimes \nabla_1) + \frac{1}{2}(w^{-1}D^B(w) + w_1\xi) \cdot \text{Id}$. Next we determine F as a 1-form with values in $\text{End}(\pi^*(TB \otimes V^{-1/2}))$. For this let $\chi \in TB \otimes V^{-1/2}$ orthogonal to α and $\beta := \xi \wedge \chi - *_M(\xi \wedge \chi) \in L^{-1}\Lambda_-^2(TM) \otimes \pi^*(V^{-1/2})$. Then (see also the definition of F from Lemma 7.4) the 1-form F with values in $\text{End}(\pi^*(TB \otimes V^{-1/2}))$ is defined by

$$F_X(\chi) = i_\xi([\eta(\beta) \wedge X]_{ASD})$$

where $X \in TM$ and $\eta(\beta) \in L^{-1}TM \otimes \pi^*(V^{-1/2})$ is determined by the equality

$$\rho(\eta(\beta), Y) = -\langle \rho, (D^{sd} \otimes \nabla)_Y(\beta) \rangle$$

which holds for every $Y \in TM$. Our next aim is to determine $\eta(\beta)$ explicitly. Let $\eta(\beta)h(\beta) + v(\beta)$ with $h(\beta)$ horizontal and $v(\beta) = \lambda(\beta) \cdot \xi$ is vertical. The connections $D^{sd} \otimes \nabla$ and $\pi^*(D^B \otimes \nabla_1)$ differ by a multiple of the identity on $\pi^*(TB \otimes V^{-1/2})$. This multiple will be ignored since applied to β it is always going to be killed by the inner product with ρ . Also, recall that in our convention of notations the tensor product connection $D^B \otimes \nabla_1$ (or its pull-back) applied to a section of $TB \otimes V^{-1/2}$ (or to a section of its pull-back $\pi^*(TB \otimes V^{-1/2})$) will be simply denoted ∇_1 and $\pi^*(\nabla_1)$ respectively.

Let $Y = \xi$ in the relation $\rho(\eta(\beta), Y) = -\langle \rho, (D^{sd} \otimes \nabla)_Y(\beta) \rangle$. Using the fact that $\pi^*(\nabla_1)_\xi(\beta) = 0$ we get $\rho(\eta(\beta), \xi) = 0$ or equivalently $\langle \alpha, h(\beta) \rangle = 0$. Now let Y be basic. A simple calculation shows that

$$\rho(\eta(\beta), Y) = w\lambda(\beta)\alpha(Y) + w*_B(h(\beta) \wedge Y \wedge \alpha).$$

On the other hand we have

$$\begin{aligned} & \langle \rho, (D^{sd} \otimes \nabla)_Y(\beta) \rangle \langle \rho, (\pi^*\nabla_1)_Y(\beta) \rangle \\ &= \langle \rho, \xi \wedge (\nabla_1)_Y(\chi) - *_M(\xi \wedge (\nabla_1)_Y(\chi)) \rangle \\ &= 2w\langle (\nabla_1)_Y(\chi), \alpha \rangle \end{aligned}$$

and we obtain

$$\lambda(\beta)\alpha(Y) + *_B(h(\beta) \wedge Y \wedge \alpha) - 2\langle (\nabla_1)_Y(\chi), \alpha \rangle.$$

Now this relation determines the vertical as well as the horizontal part of $\eta(\beta)$: to determine the vertical part $v(\beta)$ we take $Y := \alpha$ to get

$$\lambda(\beta) = -2|\alpha|^{-2}\langle(\nabla_1)_\alpha(\chi), \alpha\rangle.$$

To determine the horizontal part $h(\beta)$ we take Y orthogonal to α to get

$$*_B(h(\beta) \wedge Y \wedge \alpha) = -2\langle(\nabla_1)_Y(\chi), \alpha\rangle$$

Since the horizontal part of $\eta(\beta)$ is orthogonal to α we can write it down explicitly from the above relation:

$$h(\beta) = -2|\alpha|^{-2}*_B(\alpha \wedge \langle\nabla_1(\alpha), \chi\rangle).$$

The claim now follows.

2. **Step2** We change the weight of the bundle $\pi^*(TB \otimes V^{-1/2})$ in order to get a self-dual Yang-Mills field with horizontal part independent of w . The change of weight is realized by the diffeomorphism $w^{1/2} : \pi^*(TB \otimes V^{-1/2}) \rightarrow \pi^*(L^{-1/2}TB \otimes V^{-1/2})$. On the annihilator of α in $\pi^*(L^{-1/2}TB \otimes V^{-1/2})$ we obtain the self-dual Yang-Mills field

$$\pi^*(D^B \otimes \nabla_1) + \frac{1}{2}w_1\xi \cdot \text{Id} + \tilde{F}$$

the 1-form \tilde{F} with values in $\text{End}(\pi^*(L^{-1/2}TB \otimes V^{-1/2}))$ having the same formal expression as F .

3. **Step3** We identify the horizontal and vertical parts of the self-dual Yang-Mills field on $\pi^*(L^{-1/2}TB \otimes V^{-1/2})$ in order to get the connection and the Higgs field which form a solution of the Einstein-Weyl Bogomolny equations on B . This follows from a simple calculation, and we obtain the statement of the theorem.

□

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