Stochastic Evolution Inclusions.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Boris Bocharov*)

To my family
and
my supervisor, I.Gyongy,
whose generosity, with regards to time and knowledge,
and patience made it possible.

Abstract

This work is concerned with an evolution inclusion of the form

$$u(t) \in u_0 + \int_0^t A(u(s))dU(s) + \int_0^t B(u(s))dM(s),$$

in a triple of spaces "$V \hookrightarrow H \hookrightarrow V^*$", where U is a continuous non-decreasing process, M is a locally square-integrable martingale and the operators A (multi-valued) and B satisfy some monotonicity condition, a coercivity condition and a condition on growth in u. An existence and uniqueness theorem is proved for the solutions, using semi-implicit time-discretization schemes. Examples include evolution equations and inclusions driven by square integrable Lévy martingales.

Contents

Chapter 1

Introduction.

Let $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ be a normal triple of spaces. In many instances, the evolution of the state $u = \{u(t)\}_{t\in[0,T]}$ of a dynamic system, subject to random perturbations, can be described by an equation of the form:

$$u(t) = u_0 + \int_0^t A_s(u(s))ds + \sum_{j=1}^{r} \int_0^t B_s^j(u(s))dW^j(s), \tag{1.0.1}$$

where $(W_t)_{t\geq 0}$ is an r-dimensional Brownian motion and A and B are operators defined on $[0,T]\times\Omega\times V$ with values in V^* and H^r respectively.
Using monotonicity method, the existence and uniqueness of a solution u of (1.0.1) is proved in [16] and [15]. In [7] and [8] this result is extended to stochastic evolution equations driven by martingale measures and semimartingales (which may have jumps). More recently, [10] presents explicit and implicit space/time-discretization schemes and demonstrates how solutions of the aforementioned schemes converge to the solution of (1.0.1).
However, sometimes one may wish to consider a case when the operator A is multi-valued. Then (1.0.1) must be paraphrased as an inclusion problem, in other words,

$$u(t) \in u_0 + \int_0^t A_s(u(s))ds + \sum_{j=1}^{r} \int_0^t B_s^j(u(s))dW^j(s), \tag{1.0.2}$$

Inclusions of this type have been studied in [19] (see also [1], [20], [21], [2]), where solutions have been shown to exist, using the method of Yosida approximations.
In this work we extend the results of [8] to the case

$$u(t) \in u_0 + \int_0^t A(u(s))dU(s) + \int_0^t B(u(s))dM(s),\ 0 \leq t \leq T, \tag{1.0.3}$$

where U is a continuous non-decreasing real-valued process, starting from zero, M is a Hilbert space-valued locally square-integrable martingale, with a continuous bracket process $\langle M\rangle_t$, and the operators A (multi-valued) and B are defined on V.
We proceed in a sequence of stages.
Chapter 2 introduces the notion of the "maximal monotone" operator, central to our discussion, and establishes a few basic results that will be needed later.
In Chapters 3-7, following an approach outlined in [10] and using a variety of techniques from set-valued analysis, we deal with the inclusion (1.0.2), beginning with a (time-independent) deterministic case (i.e., $B \equiv 0$) and then considering a situation when

the operator A is assumed to depend on t and ω explicitly.
Finally, in Chapter 8, we demonstrate how the results of the previous chapters can be generalized in the case (1.0.3).

Chapter 2

Preliminaries.

2.1 Maximal monotone.

We begin by introducing the notion, central to our discussion, of a (maximal) monotone mapping.
Suppose V is a Banach space and let V^* denote its dual. We are given a set-valued mapping $A : V \to 2^{V^*}$. Let $\langle v, v^* \rangle = \langle v^*, v \rangle$ denote the duality product for $v \in V$ and $x \in V^*$. The domain and (effective) range of A are defined as follows:

$$D(A) = \{v \in V : A(v) \neq \emptyset\}, \ R(A) = \cup_{v \in D(A)} A(v).$$

Moreover, the graph G of A is given by

$$G = \{(v, v^*) \in V \times V^* : v \in D(A), v^* \in A(v)\}.$$

Definition 2.1.1. *The operator A is called **monotone**, if*

$$\langle v^* - u^*, v - u \rangle \geq 0, \tag{2.1.1}$$

for all $(u, u^*), (v, v^*)$, such that $u, v \in D(A)$ and $u^* \in A(u), v^* \in A(v)$.

Alternatively, one can say that G is a monotone subset of $V \times V^*$, provided (2.1.1) is satisfied $\forall (u, u^*), (v, v^*) \in G$.

Definition 2.1.2. *The operator A is called **maximal monotone**, if G, viewed as a subset of $V \times V^*$, has no proper monotone extension. In other words, if, and only if,*

$$\langle v^* - u^*, v - u \rangle \geq 0, \ \forall (u, u^*) \in G,$$

implies

$$v \in D(A) \text{ and } v^* \in A(v).$$

Definition 2.1.3. *An operator $B : V \to 2^{V^*}$ is called "monotone" (resp. "maximal monotone") iff $(-B)$ is monotone (maximal monotone).*

2.2 Examples.

Example 2.2.1. *(One-dimensional case) Suppose a function $f : \mathbb{R} \to \mathbb{R}$, is increasing. It follows that, $\forall u \in \mathbb{R}$,*

$$\lim_{x \to u-0} f(x) \text{ and } \lim_{x \to u+0} f(x)$$

exist and are finite. Moreover,

$$\lim_{x\to u-0} f(x) \le f(u) \le \lim_{x\to u+0} f(x).$$

Define $\bar{f} : \mathbb{R} \to \mathbb{R}$ *by*

$$\bar{f}(u) = \{v : \lim_{x\to u-0} f(x) \le v \le \lim_{x\to u+0} f(x)\}.$$

Then, it is easy to show that $\bar{f}$ *is maximal monotone.*
In particular, an increasing (decreasing) continuous function is maximal monotone ("maximal monotone"). Therefore, if $f : \mathbb{R} \to \mathbb{R}$ *is a continuously differentiable convex function, then* f' *is maximal monotone.*

Definition 2.2.1. *A function* $B : V \to V^*$ *is termed* ***hemicontinuous****, if, for all* $x, y, z \in V$,

$$\lim_{\epsilon\to 0}\langle B(x+\epsilon y), z\rangle = \langle B(x), z\rangle,\ \epsilon \in \mathbb{R}.$$

Example 2.2.2. *(Infinite-dimensional case) If* $A : V \to V^*$ *is a monotone hemicontinuous operator on the real reflexive Banach space* V*, then it is maximal monotone (see Proposition 32.7 of [22]).*

Example 2.2.3. *(Subgradients) If* $f : V \to (-\infty, +\infty]$ *is lower semi-continuous, not identically* $+\infty$*, then the subgradient* $\partial f : V \to 2^{V^*}$ *is maximal monotone (Proposition 32.17 of [22]).*

Example 2.2.4. *(Time derivatives) Let* $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ *be an evolution triple. Fix* $T > 0$. *We denote by* $X = L^p([0,T], V)$, $1 \le p < \infty$, *the space of all measurable functions* $u : [0,T] \to V$*, for which*

$$\|u\|_p := (\int_0^T \|u(t)\|_V^p dt)^{\frac{1}{p}} < \infty.$$

It is also known that the dual X^* *(of* X*) can be identified with* $L^q([0,T], V^*)$*, where* $\frac{1}{p} + \frac{1}{q} = 1$.
Then the set of all $u \in X$ *that have generalized derivatives* $u' \in X^*$ *forms a real Banach space, denoted by* $W_p^1([0,T]; V, H)$*, with the norm*

$$\|u\|_{W_p^1} = \|u\|_X + \|u'\|_{X^*}.$$

Define the operator L *by*

$$Lu = u',$$

where

$$D(L) = \{u \in W_p^1([0,T]; V, H) : u(0) = 0\}.$$

Then the operator

$$L : D(L) \subseteq X \to X^*$$

is maximal monotone (see Proposition 32.10 of [22]).

2.3 Some useful results.

Let us establish a few basic results, concerning maximal monotone operators.

Lemma 2.3.1. *Let $A : V \to 2^{V^*}$ be maximal monotone. Then, for any $\lambda > 0$, $v_0^* \in V^*$ and $u_0 \in V$, operators $A_1, A_2, A_3 : V \to 2^{V^*}$, defined by*

(i) $A_1(v) = \lambda A(v)$,

(ii) $A_2(v) = A(v) + v_0^*$,

(iii) $A_3(v) = A(v + u_0)$,

are maximal monotone.

Proof. Monotonicity of all three operators is apparent.
(i) Assume that $\langle v^* - u^*, v - u \rangle \geq 0$, for all $u \in V$ and $u^* \in A_1(u)$. By construction, $u^* \in A_1(u)$ is of the form $u^* = \lambda \tilde{u}^*$, for some $\tilde{u}^* \in A(u)$. Hence the above statement can be rewritten as

$$\langle v^* - \lambda \tilde{u}^*, v - u \rangle \geq 0,$$

or, equivalently,

$$\langle \frac{v^*}{\lambda} - \tilde{u}^*, v - u \rangle \geq 0,$$

for all $u \in V$ and $\tilde{u}^* \in A_1(u)$, since u^* and $\tilde{u}^*$ are in 1-1 correspondence. Using the fact that A is maximal monotone, we conclude that

$$\frac{v^*}{\lambda} \in A(v),$$

or $v^* \in \lambda A(v) = A_1(v)$.
Assertions (ii) and (iii) can be verified in exactly the same manner. □

Chapter 3

Inclusions with time-independent maximal monotone operators. Deterministic case.

3.1 Introduction.

Let $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ be an evolution (equivalently, normal) triple of spaces with dense and continuous embeddings, where V is a reflexive Banach space, H is a Hilbert space and V^* is the dual of V. Fix $T > 0$.
The aim of this chapter is to solve an inclusion

$$u(t) \in u_0 + \int_0^t A(u(s))ds, \ 0 \leq t \leq T, \tag{3.1.1}$$

where $u_0 \in H$ and A is an operator defined on V with values in V^*.

3.2 Assumptions and description of results.

Let $\langle v, x \rangle = \langle x, v \rangle$ denote the duality product for $v \in V$ and $x \in V^*$, $(\cdot, \cdot)$ denote the inner product in H and $\|v\|, |h|, \|v^*\|$ stand for the norms of v, h, v^* in V, H and V^* respectively.

Definition 3.2.1. *We understand the evolution triple $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ to be the following:*

- *(i) V is a real, separable and reflexive Banach space.*

- *(ii) H is a real, separable Hilbert space.*

- *(iii) The embedding $V \hookrightarrow H$ is continuous, i.e.,*

$$|v| \leq const \cdot \|v\|, \ \forall v \in V,$$

and V is dense in H.

Moreover, the inclusion $H \hookrightarrow V^*$ (or, equivalently, $H^* \hookrightarrow V^*$, provided we identify the Hilbert space H with its dual H^* by means of the inner product in H) is the dual of the inclusion $V \hookrightarrow H$.

Remark 3.2.1. *In order to justify that the embedding $H \hookrightarrow V^*$ is continuous and dense, we notice that the mapping $h \to \bar{h}$ from H into V^*, given by*

$$\langle \bar{h}, v \rangle = (h, v), \ v \in V,$$

is linear, injective and continuous. Moreover, since V is reflexive, H is dense in V^* (see Proposition 23.13 of [22] for details).

We proceed by stating the precise assumptions made on the operator A.
There exist constants $\lambda, K_1, K_2 > 0$, such that A satisfies the following conditions:

Assumption 3.2.1. *A is "maximal monotone".*

Assumption 3.2.2. *("Coercivity" condition) For all pairs (v, v^*), such that $v \in V$ and $v^* \in A(v)$,*

$$2\langle v^*, v \rangle + \lambda \|v\|^2 \leq K_1(1 + |v|^2).$$

Assumption 3.2.3. *("Linear growth" condition) For all (v, v^*), with $v \in V$ and $v^* \in A(v)$,*

$$\|v^*\| \leq K_2(1 + \|v\|).$$

Example 3.2.1. *Let f, mapping $\mathbb{R}$ into subsets of $\mathbb{R}$, be a bounded "maximal monotone" function and let $I \subseteq \mathbb{R}$ be a bounded interval. For example, one can have*

$$f(x) = \frac{1}{2}(\frac{x}{|x|} - 1), \ x \neq 0,$$

$$f(0) = \{[-1, 0]\}.$$

Set $V = \overset{\circ}{\mathrm{W}}{}^1_2(I)$ and define an operator $A : V \to 2^{V^*}$ by

$$A(v) = \{\frac{d}{dx}h : \ h(x) \in f(\frac{d}{dx}v(x)) \ for \ dx - a.e. \ x \in I\}.$$

One can show that A is "monotone". Moreover, it has a "maximal monotone" extension, denoted by $\widetilde{A}$. Set $C = \widetilde{A} - J$, where J is the duality mapping from V into V^* (see Appendix C.1 for the definition and useful properties). Then the operator C is "maximal monotone" and "coercive" (Assumptions 3.2.1-3.2.2).

If X is a Banach space, let $L^2([0,T], X)$ denote the Banach space of X-valued Lebesgue-measurable functions $\{z(t) : t \in [0,T]\}$, with the norm

$$\|z\|_{L^2([0,T],X)} = (\int_0^T \|z(t)\|_X^2 dt)^{\frac{1}{2}} < \infty.$$

Definition 3.2.2. *An H-valued continuous function v is a solution of (3.1.1), if $v \in L^2([0,T], V)$ and there exists a function $\alpha \in L^2([0,T], V^*)$, such that, for dt-almost all $t \in [0,T]$,*

$$\alpha(t) \in A(v(t)),$$

and

$$v(t) = u_0 + \int_0^t \alpha(s) ds, \ \forall t \in [0,T].$$

The main result of this chapter is

Theorem 3.2.1. *Under Assumptions 3.2.1-3.2.3, inclusion (3.1.1) has a unique solution.*

Lemma 3.2.1. *Provided inclusion (3.1.1) has a solution, it is unique.*

Proof. Suppose u^1 and u^2 are solutions of (3.1.1). By definition, there exist functions α^1, α^2, such that

$$u^1(t) = u_0 + \int_0^t \alpha^1(s)ds,$$

and

$$u^2(t) = u_0 + \int_0^t \alpha^2(s)ds.$$

Set $h(t) = |u^2(t) - u^1(t)|^2$. Then we have

$$0 \leq h(t) = \int_0^t I(s)ds,$$

where

$$I(s) = 2\langle u^1(s) - u^2(s), \alpha^1(s) - \alpha^2(s)\rangle$$

is non-positive by Assumption 3.2.1. Hence

$$h(t) = 0, \ \forall t \in [0, T],$$

and we conclude that $u^1(t)$ and $u^2(t)$ coincide for every t. □

We proceed by showing that the inclusion (3.1.1) has a solution, using semi-implicit time-discretization schemes.

3.3 Implicit time-discretization scheme.

Take $n \in \mathbb{N}$, divide the interval $[0, T]$ into n equal subintervals of length $\delta_n = \frac{T}{n}$ and set $t_k = k\frac{T}{n}$, $0 \leq k \leq n$.

Definition 3.3.1. *A Lebesgue-measurable function u^n is a solution of a semi-implicit **time-discretization scheme**, if u^n is defined by*

$$u^n(0) := 0,$$

$$u^n(t) := u^n(t_k), \ t \in (t_k, t_{k+1}], \ 0 \leq k \leq n-1,$$

where $u^n(t_k)$ are solutions of a system of inclusions

$$u^n(t_0) = 0,$$

$$u^n(t_1) \in u_0 + \delta_n A(u^n(t_1)),$$

$$u^n(t_{i+1}) \in u^n(t_i) + \delta_n A(u^n(t_{i+1})), \ 0 \leq i \leq n-1. \tag{3.3.1}$$

Theorem 3.3.1. *Under Assumptions 3.2.1-3.2.3, the system (3.3.1), equivalently, **time-discretization scheme**, has a unique solution for all large n.*

Proof. Upon noticing that a typical member of the system (3.3.1),

$$u^n(t_{i+1}) \in u^n(t_i) + \delta_n A(u^n(t_{i+1})),$$

can be re-written as

$$u^n(t_i) \in (I - \delta_n A)(u^n(t_{i+1})),$$

where the identity operator $I : V \to V^*$ is given by

$$I(v) = (\cdot, v),$$

it becomes apparent that it is enough to show that an inclusion

$$u^* \in (I - \delta A)(u) \tag{3.3.2}$$

has a unique solution $u \in V$, for every $u^* \in V^*$ and all (sufficiently) small $\delta > 0$.
We proceed in a sequence of steps.
i) To begin with, I is monotone. Indeed,

$$\langle I(v) - I(u), v - u \rangle = \langle I(v - u), v - u \rangle = |v - u|^2 \geq 0.$$

Moreover, $I : V \to V^*$ is continuous. Therefore, by Proposition 10.0.2 (Appendix B.1), I is maximal monotone.
On the other hand, $-\delta_n A$ is maximal monotone, by part (i) of Lemma 2.3.1.
Since both operators are defined on the whole of V, we are justified in using Theorem 10.0.3 to conclude that the operator $I - \delta_n A : V \to 2^{V^*}$ is maximal monotone ($n = 1, 2, \ldots$).
Let us see that the operator $I - \delta_n A$ is **coercive** (at least, for large n) in the sense of Definition 10.0.7. Suppose $u^* \in (I - \delta_n A)(u)$. This means that $\exists v^* \in A(u)$, such that $u^* = I(u) - \delta_n v^*$. Then, by Assumption 3.2.2,

$$\langle u^*, u \rangle = \langle I(u) - \delta_n v^*, u \rangle = |u|^2 - \delta_n \langle v^*, u \rangle \geq$$

$$|u|^2 + \frac{\delta_n \lambda}{2} \|u\|^2 - \frac{\delta_n K_1}{2}(1 + |u|^2) \geq$$

$$\frac{\delta_n \lambda}{2} \|u\|^2 - \frac{\delta_n K_1}{2},$$

provided $\delta_n < \frac{2}{K_1}$ (equiv., $n > \frac{TK_1}{2}$). Hence

$$\frac{\inf_{u^* \in (I - \delta_n A)(u)} \langle u^*, u \rangle}{\|u\|} \geq$$

$$\frac{\delta_n \lambda}{2} \|u\| - \frac{\delta_n K_1}{2\|u\|} \to +\infty,$$

as $\|u\| \to +\infty$.
The fact that the operator $I - \delta_n A$ is **coercive**, by the above argument, together with Theorem 10.0.4, imply that $R(I - \delta_n A) = V^*$, or, equivalently, that a solution to the implicit scheme exists (at least, for all large n).
ii) Let us check that, for all such n, the inclusion (3.3.2) has a unique solution. Indeed, if this was not the case, then, for some $v^* \in V^*$, there would exist distinct $u_1, u_2 \in V$ and corresponding $v_1^* \in A(u_1)$ and $v_2^* \in A(u_2)$, such that

$$v^* = I(u_1) - \delta_n v_1^* = I(u_2) - \delta_n v_2^*.$$

Then

$$0 = \langle v^* - v^*, u_2 - u_1 \rangle =$$

$$\langle (I(u_2) - \delta_n v_2^*) - (I(u_1) - \delta_n v_1^*), u_2 - u_1 \rangle =$$

$$|u_2 - u_1|^2 - \delta_n \langle v_2^* - v_1^*, u_2 - u_1 \rangle \geq |u_2 - u_1|^2 > 0,$$

by monotonicity of $-\delta_n A$, which is a contradiction.
(iii) Finally, the function u^n is, clearly, Lebesgue-measurable, by construction. □

3.4 Characterization of solutions.

In this section we introduce an approach, which allows one to characterize solutions of the inclusion (3.1.1) as extremal points of some functional.
For notational simplicity, we let $X = L^2([0,T], V)$. Then, one can identify the dual X^* of X with $L^2([0,T], V^*)$.
We make use of the following construction.

Definition 3.4.1. *Let U denote the space of pairs (ξ, a), satisfying the following conditions:*

- $\xi \in H$;
- $a \in X^*$;
- *There exists a function $x \in X$, such that*

$$x(t) = \xi + \int_0^t a(t)ds,$$

for dt-almost every $t \in [0,T]$.

Remark 3.4.1. *Note that, by Theorem 17.0.5 (Appendix E), the function x permits an H-valued continuous modification, denoted x^H, such that*

$$x^H(t) = \xi + \int_0^t a(s)ds, \ \forall t \in [0,T].$$

Let $(\xi, a) \in U$. Take arbitrary $y \in X$ and set

$$F_y(\xi, a) = |u_0 - \xi|^2 + 2 \int_0^T \langle x(t) - y(t), a(t) - y^*(t) \rangle dt,$$

and

$$G(\xi, a) = \sup_y \{F_y(\xi, a) : y \in X\},$$

where $y^* \in W(y)$ and the operator $W : X \to 2^{X^*}$ is defined as follows:

$$W(v) = \{v^* \in X^* : v^*(t) \in A(v(t)), \ dt\text{-a.e. } t \in [0,T]\}.$$

Theorem 3.4.1. *The operator W is defined on the whole of X and is "maximal monotone".*

Proof. See Appendix C.2 for details. □

Due to Assumption 3.2.3,

$$\|v^*\|^2 \le K_2^2(1+\|v\|)^2 \le 2K_2^2 + 2K_2^2\|v\|^2,$$

which means that $y^* \in X^*$, $\forall y^* \in W(y)$. Moreover, since $x-y \in X$ and $a-y^* \in X^*$, an application of Hölder's inequality yields

$$\langle x-y, a-y^*\rangle \in L^1([0,T]).$$

Therefore, F_y is well-defined.
Since one can set $y = x$ (Theorem 3.4.1), when calculating $\sup F_y$, it is evident that $G(\xi, a) \ge 0$, for any choice of $(\xi, a) \in U$.
The main result of this section is

Theorem 3.4.2.

- *(i) Suppose assumptions 3.2.1-3.2.3 hold and let u be a solution to (3.1.1). Then*

$$\inf\{G(\xi,a) : (\xi,a) \in U\} = G(u_0, u^*) = 0,\ \ u^* \in W(u).$$

- *(ii) Assume conditions 3.2.1-3.2.3. If there exists a pair $(\hat{\xi}, \hat{a}) \in U$, such that, $\forall y \in X$,*

$$F_y(\hat{\xi}, \hat{a}) \le 0,$$

then $\hat{\xi} = u_0$, and

$$u^H(t) = u_0 + \int_0^t \hat{a}(s)ds,\ \ t \in [0,T],$$

is a solution of (3.1.1).

Proof. (i) Recall that if u is a solution of (3.1.1), then there is a (square-integrable) V^*-valued function $\alpha = \{\alpha(t) : t \in [0,T]\}$, such that, for dt-a.a $t \in [0,T]$,

$$\alpha(t) \in A(u(t)),$$

and

$$u(t) = u_0 + \int_0^t \alpha(s)ds,\ \ \forall t \in [0,T].$$

Thus

$$F_y(u_0, \alpha) = 2\int_0^T \langle u(t)-y(t), \alpha(t)-y^*(t)\rangle dt \le 0,$$

for every $y \in X$ and $y^* \in W(y)$, due to "monotonicity" condition 3.2.1. It follows that $G(u_0, \alpha) \le 0$.
On the other hand, as we have previously observed, $G(\xi, a) \ge 0$ ($\forall(\xi, a) \in U$). Consequently,

$$G(u_0, \alpha) = 0.$$

(ii) Suppose the pair $(\hat{\xi}, \hat{a})$ satisfies Assumptions 3.2.1-3.2.3. Setting $y = u$ in $F_y(\hat{\xi}, \hat{a})$, we get $|\hat{\xi} - u_0|^2 = 0$, which implies $\hat{\xi} = u_0$.
The condition $F_y(\hat{\xi}, \hat{a}) \le 0$ can now be rewritten as

$$\langle u-y, \hat{a}-y^*\rangle_X =$$

$$\int_0^T \langle u(t)-y(t), \hat{a}(t)-y^*(t)\rangle dt \le 0.$$

Since the operator W is "maximal monotone", by Theorem 3.4.1, we deduce that $\hat{a} \in W(u)$, or

$$\hat{a}(t) \in A(u(t)),$$

for almost every $t \in [0,T]$ (definition of W). □

3.5 Convergence of the implicit scheme.

Recall that for a fixed m, an approximation u^m to u, by the time-discretization scheme, is defined as follows:

$$\begin{aligned} u^m(t_0) &:= 0, \\ u^m(t_1) &\in u_0 + \delta_m A(u^m(t_1)), \\ u^m(t_{i+1}) &\in u^m(t_i) + \delta_m A(u^m(t_{i+1})), \\ u^m(t) &:= u^m(t_i) \text{ for } t \in (t_i, t_{i+1}],\ 0 \le i \le m-1. \end{aligned}$$

We have seen previously that such a scheme has a unique solution for any sufficiently large m. It means that there exist functions $\alpha^m(t_i) \in A(u^m(t_i))$, $1 \le i \le m$, such that the above can be rewritten as a system of equalities:

$$\begin{aligned} u^m(t_0) &= 0, \\ u^m(t_1) &= u_0 + \delta_m \alpha^m(t_1), \\ u^m(t_{i+1}) &= u^m(t_i) + \delta_m \alpha^m(t_i),\ 0 \le i \le m-1. \end{aligned}$$

Using induction on time intervals, it is easy to show that u^m can be cast into integral form as follows[1]:

$$u^m(t) = u_0 \chi_{\{t \ge t_1\}} + \int_0^{\kappa_1(t)} \alpha^m(\kappa_2(s))ds. \tag{3.5.1}$$

Indeed, if $0 \le t < t_1$, then

$$u^m(t) = u^m(t_0) = 0 + \int_0^{\kappa_1(t)=0} \alpha^m(t_1)(t) = 0.$$

Likewise, if $t_1 \le t < t_2$, then

$$u^m(t) = u^m(t_1) = u_0 + \delta_m \alpha^m(t_1) =$$

$$u_0 + \int_0^{\kappa_1(t)=t_1} \alpha^m(\kappa_2(s))ds.$$

Assume further that equality holds for the k's interval, namely, $[\frac{(k-1)T}{m}, \frac{kT}{m}]$. Then, for $t \in [\frac{kT}{m}, \frac{(k+1)T}{m}]$,

$$u^m(t) = u^m(t_k) = u^m(t_{k-1}) + \delta_m \alpha^m(t_k) =$$

[1] The functions $\kappa_1, \kappa_2 : [0,T] \to [0,T]$ are given by $\kappa_1(t) = t_i$ and $\kappa_2(t) = t_{i+1}$, $t \in (t_i, t_{i+1})$, and $\kappa_1(t_i) = \kappa_2(t_i) = t_i$.

$$u_0\chi_{\{t\geq t_1\}} + \int_0^{t_{k-1}} \alpha^m(\kappa_2(s))ds + \int_{t_{k-1}}^{t_k} \alpha^m(\kappa_2(s))ds =$$

$$u_0\chi_{\{t\geq t_1\}} + \int_0^{\kappa_1(t)} \alpha^m(\kappa_2(s))ds.$$

We proceed by establishing some useful estimates on the function u^m.

Lemma 3.5.1. *There exist constants $L_1, L_2, L_3 > 0$, such that, under Assumptions 3.2.1-3.2.3 ,*

$$\sup_{s\in[0,T]} |u^m(s)|^2 \leq L_1,$$

$$\int_0^T \|u^m(\kappa_2(s))\|^2 ds \leq L_2,$$

$$\int_0^T \|\alpha^m(\kappa_2(s))\|^2 ds \leq L_3,$$

for all large m.

Proof. To start with,

$$u^m(t_1) = u_0 + \delta_m\alpha^m(t_1),$$

which implies

$$|u^m(t_1) - \delta_m\alpha^m(t_1)|^2 = |u_0|^2,$$

or, upon expanding and rearranging the terms,

$$|u^m(t_1)|^2 - |u_0|^2 = 2\langle u^m(t_1), \alpha^m(t_1)\rangle\delta_m - |\alpha^m(t_1)|^2\delta_m^2.$$

Similarly,

$$|u^m(t_{i+1})|^2 - |u^m(t_i)|^2 =$$

$$2\langle u^m(t_{i+1}), \alpha^m(t_{i+1})\rangle\delta_m - |\alpha^m(t_{i+1})|^2\delta_m^2,$$

for $1 \leq i \leq m-1$. Adding these equalities up to $i = k-1$, we obtain

$$|u^m(t_k)|^2 \leq |u_0|^2 + \int_0^{t_k} 2\langle u^m(\kappa_2(s)), \alpha^m(\kappa_2(s))\rangle ds.$$

Using "coercivity" condition 3.2.2, we get

$$|u^m(t_k)|^2 \leq |u_0|^2 - \lambda\int_0^{t_k} \|u^m(\kappa_2(s))\|^2 ds + K_1\int_0^{t_k} |u^m(\kappa_2(s))|^2 ds + K_1 T.$$

If we choose m_1 so large that

$$K_1\delta_{m_1} = \frac{K_1 T}{m_1} < \frac{1}{2},$$

then, for all $m \geq m_1$,

$$\frac{1}{2}|u^m(t_k)|^2 + \lambda\int_0^{t_k} \|u^m(\kappa_2(s))\|^2 ds \leq$$

$$(1 - \frac{K_1 T}{m})|u^m(t_k)|^2 + \lambda\int_0^{t_k} \|u^m(\kappa_2(s))\|^2 ds \leq$$

$$C + K_1 \int_0^{t_{k-1}} |u^m(\kappa_2(s))|^2 ds,$$

where $C = K_1 T + |u_0|^2$. So,

$$|u^m(t_k)|^2 \leq 2C + 2K_1 \int_0^{t_{k-1}} |u^m(\kappa_2(s))|^2 ds, \tag{3.5.2}$$

$2 \leq k \leq m$ $(m \geq m_1)$.
Using discrete version of Gronwall's lemma (Appendix D), one can immediately write

$$|u^m(t_k)|^2 \leq 2C(1 + \frac{2K_1 T}{m})^k.$$

Taking into account that $(1 + \frac{1}{x})^x$, $x \geq 1$, is bounded from above by e,

$$|u^m(t_k)|^2 \leq 2C(1 + \frac{2K_1 T}{m})^m =$$

$$2C[(1 + \frac{2K_1 T}{m})^{\frac{m}{2K_1 T}}]^{2K_1 T} \leq 2Ce^{2K_1 T} = L_1 < \infty.$$

Moreover,

$$\lambda \sum_{i=0}^{k-1} \|u^m(t_{i+1})\|^2 \delta_m \leq C + K_1 \sum_{i=0}^{k-1} |u^m(t_{i+1})|^2 \delta_m.$$

Letting $k = m$ and using inequality (3.5.2), with the above estimate, we get

$$\sum_{i=0}^{m-1} \|u^m(t_{i+1})\|^2 \delta_m = \int_0^T \|u^m(\kappa_2(s))\|^2 ds \leq$$

$$\frac{C}{\lambda} + \frac{K_1 L_1 T}{\lambda} = L_2 < \infty.$$

Finally, due to the "linear growth" condition 3.2.3,

$$\int_0^T \|\alpha^m(\kappa_2(s))\|^2 ds = \sum_{i=0}^{m-1} \|\alpha^m(t_{i+1})\|^2 \delta_m \leq$$

$$2K_2^2 \sum_{i=0}^{m-1} (1 + \|u^m(t_{i+1})\|^2) \delta_m =$$

$$2K_2^2 T + 2K_2^2 \int_0^T \|u^m(\kappa_2(s))\|^2 ds \leq$$

$$2K_2^2 T + 2K_2^2 L_2 = L_3 < \infty.$$

□

The above estimates imply that there exists a subsequence, denoted again by u^m, such that

$$u^m(T) \rightharpoonup u^\infty(T) \text{ in } H,$$

$$u^m(\kappa_2(\cdot)) \rightharpoonup v_\infty \text{ in } X,$$

$$\alpha^m(\kappa_2(\cdot)) \rightharpoonup a_\infty \text{ in } X^*,$$

where, as usual, " $\rightharpoonup$ " stands for "converges weakly to".
We are, finally, in the position to demonstrate the validity of the **existence** part of Theorem 3.2.1.

Theorem 3.5.1. *Under Assumptions 3.2.1-3.2.3,*

(i) For dt-a.e. $t \in [0,T]$,

$$v_\infty(t) = u_0 + \int_0^t a_\infty(s)ds. \tag{3.5.3}$$

Moreover,

$$u^\infty(T) = u_0 + \int_0^T a_\infty(s)ds; \tag{3.5.4}$$

(ii) For all $y \in X$,

$$F_y(u_0, a_\infty) \le 0, \tag{3.5.5}$$

and the function v^H_∞ is a solution of (3.1.1);

(iii) The sequence $u^m(T)$ converges strongly to $u^\infty(T)$ in H.

Proof. (i) For a fixed $N \ge 1$, let $\phi : [0,T] \to V$ be a Lebesgue-measurable function that satisfies

$$\|\phi(t)\| \le N, \ t \in [0,T].$$

Recall that

$$u^m(t) = u_0\chi_{\{t \ge t_1\}} + \int_0^{\kappa_1(t)} \alpha^m(\kappa_2(s))ds.$$

Therefore

$$\int_0^T (u^m(t), \phi(t))dt = \int_0^T \chi_{\{t\ge t_1\}}(u_0, \phi(t))dt +$$
$$\int_0^T \langle \int_0^{\kappa_1(t)} \alpha^m(\kappa_2(s))ds, \phi(t)\rangle dt =$$
$$\int_0^T (u_0, \phi(t))dt + J - R_1 - R_2, \tag{3.5.6}$$

where

$$J = \int_0^T \langle \int_0^t \alpha^m(\kappa_2(s))ds, \phi(t)\rangle dt,$$

$$R_1 = \int_0^{t_1} (u_0, \phi(t))dt,$$

$$R_2 = \int_0^T \langle \int_{\kappa_1(t)}^t \alpha^m(\kappa_2(s))ds, \phi(t)\rangle dt.$$

To begin with,

$$|R_1| = |\int_0^{t_1} (u_0, \phi(t))dt| \le \int_0^{t_1} |u_0| \cdot |\phi(t)|dt.$$

Taking into account that $|\phi(t)| \le const \cdot \|\phi(t)\| \le const \cdot N$, we obtain

$$|R_1| \le \frac{const \cdot NT}{m}|u_0| \to 0, \text{ as } m \to \infty.$$

Moreover,

$$|R_2| = |\int_0^T \langle \int_{\kappa_1(t)}^t \alpha^m(\kappa_2(s))ds, \phi(t)\rangle dt| \le$$

$$\int_0^T \| \int_{\kappa_1(t)}^t \alpha^m(\kappa_2(s))ds\| \cdot \|\phi(t)\|dt \le \frac{NT}{m}\int_0^T \|\alpha^m(\kappa_2(t))\|dt \le$$

$$\frac{NT}{m}\int_0^T (1 + \|\alpha^m(\kappa_2(t))\|^2)dt \le \frac{NT}{m}(T + L_3) \to 0.$$

In order to tackle J, the following simple observation proves useful.
Define a (linear) operator $G : X^* \to X^*$ by $G(g)(t) = \int_0^t g(s)ds$, $g \in X^*$. Then

$$\|G(g)\|_{X^*}^2 = \int_0^T \| \int_0^t g(s)ds\|^2 dt \le$$

$$\int_0^T (\int_0^t \|g(s)\|ds)^2 dt \le \int_0^T (\int_0^T 1 \cdot \|g(s)\|ds)^2 dt \le$$

$$\int_0^T T(\int_0^T \|g(s)\|^2 ds)dt \le T^2\|g\|_{X^*}^2.$$

Hence G is a bounded linear operator. This means, in particular, that if $x_i \rightharpoonup x_0$ in X^*, then $G(x_i) \rightharpoonup G(x_0)$ in X^*. Therefore,

$$\lim_{m\to\infty} J = \lim_{m\to\infty}\int_0^T \langle \int_0^t \alpha^m(\kappa_2(s))ds, \phi(t)\rangle dt =$$

$$\int_0^T \langle \lim_{m\to\infty}\int_0^t \alpha^m(\kappa_2(s))ds, \phi(t)\rangle dt =$$

$$\int_0^T \langle \int_0^t \alpha_\infty ds, \phi(t)\rangle dt.$$

Letting $m \to \infty$ in equation (3.5.6), we observe that

$$\int_0^T (v_\infty(t), \phi(t))dt = \int_0^T (u_0, \phi(t))dt + \int_0^T \langle \int_0^t \alpha_\infty(s)ds, \phi(t)\rangle dt,$$

from which it follows (since N in the definition of $\phi(t)$ can be arbitrarily large) that

$$v_\infty(t) = u_0 + \int_0^t \alpha_\infty(s)ds, \text{ dt-a.e. } t \in [0,T],$$

which is (3.5.3).

Repeating the above argument with $\psi \in V$ ($\|\psi\| \leq N$), we obtain (3.5.3).
(ii) We have seen previously that there exists an H-valued continuous modification v_∞^H of v_∞, s.t.

$$v_\infty^H(t) = u_0 + \int_0^t \alpha_\infty(s)ds, \ \forall t \in [0,T].$$

Moreover, by Theorem 17.0.5 (Appendix E),

$$|v_\infty^H(T)|^2 = |u_0|^2 + 2\int_0^T \langle v_\infty(t), \alpha_\infty(t)\rangle dt. \tag{3.5.7}$$

Clearly, $v_\infty^H(T) = u^\infty(T)$.
Take arbitrary $y \in X$, $y^* \in W(y)$ and set

$$F_y^m = 2\int_0^T \langle u^m(\kappa_2(t)) - y(t), \alpha^m(\kappa_2(t)) - y^*(t)\rangle dt =$$

$$2\int_0^T \langle u^m(\kappa_2(t)), \alpha^m(\kappa_2(t))\rangle dt + 2\int_0^T \langle y(t), y^*(t)\rangle dt -$$

$$2\int_0^T \langle u^m(\kappa_2(t)), y^*(t)\rangle dt - 2\int_0^T \langle y(t), \alpha^m(\kappa_2(t))\rangle dt. \tag{3.5.8}$$

Substituting

$$|u^m(T)|^2 - |u_0|^2 \leq 2\int_0^T \langle u^m(\kappa_2(t)), \alpha^m(\kappa_2(t))\rangle dt$$

(Lemma 3.5.1) into (3.5.8) and using the fact that $F_y^m \leq 0$ (by Assumption 3.2.1), we obtain

$$0 \geq F_y^m \geq |u^m(T)|^2 - |u_0|^2 +$$

$$2\int_0^T \langle y(t), y^*(t)\rangle dt - 2B_1 - 2B_2, \tag{3.5.9}$$

where

$$B_1 = \int_0^T \langle u^m(\kappa_2(t)), y^*(t)\rangle dt,$$

$$B_2 = \int_0^T \langle y(t), \alpha^m(\kappa_2(t))\rangle dt.$$

Clearly,

$$B_1 \to \int_0^T \langle v_\infty(t), y^*(t)\rangle dt,$$

and

$$B_2 \to \int_0^T \langle y(t), \alpha_\infty(t)\rangle dt,$$

as $m \to \infty$.

Since $u^m(T)$ converges weakly to $u^\infty(T)$ and using the fact that if $x_n \rightharpoonup x_0$, then

$$\liminf \|x_n\| \geq \|x_0\|,$$

there exists $d \geq 0$, such that

$$\liminf |u^m(T)|^2 = d + |u^\infty(T)|^2 = d + |v_\infty^H(T)|^2.$$

Passing to the limit in (3.5.9) and making use of identity (3.5.7), we obtain

$$0 \geq d + |v_\infty^H(T)|^2 - |u_0|^2 - 2\int_0^T \langle v_\infty(t), y^*(t)\rangle dt -$$

$$2\int_0^T \langle y(t), \alpha_\infty(t)\rangle dt + 2\int_0^T \langle y(t), y^*(t)\rangle dt =$$

$$d + F_y(u_0, \alpha_\infty(t)).$$

Note that

$$F_y(u_0, \alpha_\infty(t)) \leq 0, \ \forall y \in X.$$

Hence, by Theorem 3.4.2, v_∞^H is the solution of (3.1.1).
(iii) Setting $y = v_\infty$, we get $d \leq 0$, and so, in fact, $d = 0$. Thus

$$\liminf |u^m(T)| = |u^\infty(T)|.$$

It follows that

$$\lim |u^m(T)| = |u^\infty(T)|.$$

Indeed, if there is $\epsilon > 0$ and a subsequence $\{m_i\}$, such that

$$||u^{m_i}(T)| - |u^\infty(T)|| \geq \epsilon,$$

then, clearly,

$$|\liminf |u^{m_i}(T)| - |u^\infty(T)|| \geq \epsilon,$$

which is in contradiction with

$$\liminf |u^{m_i}(T)| = |u^\infty(T)|,$$

(just rewrite the above proof with m_i in place of m).
Finally, since

$$u^m(T) \rightharpoonup u^\infty(T) \text{ and } |u^m(T)| \to |u^\infty(T)|,$$

we deduce that

$$u^m(T) \to u^\infty(T) \text{ in } H.$$

□

Chapter 4

Inclusions with time-independent maximal monotone operators. Stochastic case.

4.1 Introduction.

Let $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ be an normal triple of spaces with dense and continuous embeddings. Let $W = (W_t)_{t\geq 0}$ be an r-dimensional Brownian motion carried by a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Fix $T > 0$. In this section we consider the following inclusion problem:

$$u(t) \in u_0 + \int_0^t A(u(s))ds + \sum_{j=1}^r \int_0^t B^j(u(s))dW^j(s),\ 0 \leq t \leq T, \tag{4.1.1}$$

where u_0 is an H-valued F_0-measurable random variable, and A and B are operators defined on V with values in V^* and H^r respectively.
We shall demonstrate that, under certain assumptions on operators A and B, the problem (4.1.1) has a unique solution.

4.2 Assumptions and description of results.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions of right continuity and completeness. Let $\lambda, K_1 > 0$ and $K_2 \geq 0$ be given constants.

Assumption 4.2.1. *("Monotonicity" condition) For any $u, v \in V$ and $u^* \in A(u), v^* \in A(v)$,*

$$2\langle u - v, u^* - v^*\rangle + \sum_{j=1}^r |B^j(u) - B^j(v)|^2 \leq 0.$$

Assumption 4.2.2. *("Coercivity" condition) For all pairs (v, v^*), with $v \in V$ and $v^* \in A(v)$,*

$$2\langle v^*, v\rangle + \sum_{j=1}^r |B^j(v)|^2 + \lambda\|v\|^2 \leq K_1(1 + |v|^2).$$

Assumption 4.2.3. *The operator $A : V \to 2^{V^*}$ is "maximal monotone".*

Assumption 4.2.4. *("Linear growth" condition) For any $v \in V$ and $v^* \in A(v)$,*

$$\|v^*\| \leq K_2(1 + \|v\|).$$

Assumption 4.2.5. *The function $u_0 : \Omega \to H$ is F_0-measurable and such that*

$$E(|u_0|^2) < +\infty.$$

Assumption 4.2.6. *$B^j : (V, \mathcal{B}(V)) \to (H, \mathcal{B}(H))$ is measurable, $0 \leq j \leq r$.*

Remark 4.2.1. *Since, by Assumption 4.2.4,*

$$|2\langle v^*, v\rangle| \leq 2\|v\| \cdot \|v^*\| \leq \|v\|^2 + \|v^*\|^2 \leq$$

$$\|v\|^2 + 2K_2^2\|v\|^2 + 2K_2^2 = (2K_2^2 + 1)\|v\|^2 + 2K_2^2,$$

it follows from Assumption 4.2.2 that

$$\sum_{j=1}^{r} |B^j(v)|^2 \leq K_1|v|^2 + |2\langle v^*, v\rangle| - \lambda\|v\|^2 + K_1 \leq$$

$$K_1|v|^2 + (2K_2^2 + 1)\|v\|^2 + K_3 \leq C(1 + \|v\|^2),$$

for some (positive) constant C (keeping in mind that $|v| \leq const \cdot \|v\|$).

Example 4.2.1. *Let D be a bounded domain of $\mathbb{R}^d$, $p \in [2, \infty)$, $V = \overset{\circ}{\mathrm{W}}{}_p^1(D)$, $H = L^2(D)$ and $V^* = \mathrm{W}_q^{-1}(\mathrm{D})$, $\frac{1}{p} + \frac{1}{q} = 1$.* Define operators A and B by

$$A(u) := \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f_i(x, \nabla u(x)),$$

$$B^k(u) := g^k(x, \nabla u(x)) + h^k(x, u(x)),\ k = 1, 2, \ldots, r,$$

where f_i, g^k and h^k are real-valued functions that meet certain requirements (for details, see [10]).
Then one can show that A and B satisfy Assumptions 4.2.1-4.2.4.

Set $S = ([0,T]\times\Omega, \overline{\mathcal{P}}, dt\times P)$, where $\overline{\mathcal{P}}$ is the completion of $\mathcal{P}$, the predictable σ-algebra, with respect to the measure $dt \times P$.[1] Given a Banach space X, let $L^2(S, X)$ denote the Banach space of X-valued measurable stochastic processes $\{z(t) : t \in [0,T]\}$, with the norm

$$\|z\|_{L^2(S,X)} = (E\int_0^T \|z(t)\|_X^2 dt)^{\frac{1}{2}}.$$

Definition 4.2.1. *An H-valued, $\mathcal{F}_t$-adapted continuous process $v = (v_t)_{t\in[0,T]}$ is a solution of (4.1.1), if $v \in L^2(S, V)$ and there exists a process $\alpha \in L^2(S, V^*)$, such that, for $dt \times P$-a.e. $(t, \omega) \in [0,T] \times \Omega$,*

$$\alpha(t, \omega) \in A(v(t, \omega)),$$

[1] $\mathcal{P}$ is generated by the sets $0 \times A$, $A \in \mathcal{F}_0$, and $(s,t] \times A$, with $0 \leq s < t \leq T$ and $A \in \mathcal{F}_s$.

and, almost surely,

$$u(t) = u_0 + \int_0^t \alpha_s ds + \sum_{j=1}^r \int_0^t B^j(u(s))dW^j(s),$$

for all $t \in [0, T]$.

Theorem 4.2.1. *Under Assumptions 4.2.1-4.2.6, inclusion (4.1.1) has a unique solution.*

As far as the question of **uniqueness** is concerned, then we have the following result.

Lemma 4.2.1. *Inclusion (4.1.1) has, at most, one solution.*

Proof. Suppose u^1 and u^2 are solutions of (4.1.1). Then there exist processes α^1, α^2, such that

$$u_t^1 = u_0 + \int_0^t \alpha_s^1 ds + \sum_{j=1}^r \int_0^t B^j(u_s^1)dW_s^j,$$

and

$$u_t^2 = u_0 + \int_0^t \alpha_s^2 ds + \sum_{j=1}^r \int_0^t B^j(u_s^2)dW_s^j.$$

Set $h(t) = |u_t^2 - u_t^1|^2$. An application of Ito's formula from [7] yields

$$0 \leq h(t) = \int_0^t I(s)ds + N(t),$$

where

$$I(s) = 2\langle u_s^1 - u_s^2, \alpha_s^1 - \alpha_s^2\rangle + \sum_{j=1}^r |B^j(u_s^1) - B^j(u_s^2)|^2$$

is non-positive by Assumption 4.2.1, and

$$N(t) = 2\sum_{j=1}^r \int_0^t (u_s^1 - u_s^2, B^j(u_s^1) - B^j(u_s^2))dW_s^j$$

is a local martingale, starting at zero. It follows that, almost surely,

$$N(t) = 0, \; t \in [0, T],$$

from which we deduce that the processes u^1 and u^2 are modifications of each other. □

We show that the inclusion (4.1.1) has a solution, by considering semi-implicit time-discretization schemes, defined next.

4.3 Time-discretization scheme.

For a given $n \in \mathbb{N}$ and time mesh $\delta_n = \frac{T}{n}$,

Definition 4.3.1. *A (predictable) process* u^n *is a solution of a semi-implicit* ***time-discretization scheme****, if* u^n *is defined by*

$$u^n(0, \omega) := 0,$$

$$u^n(t,\omega) := u^n(t_k,\omega),\ t \in (t_k, t_{k+1}],\ 0 \le k \le n-1,$$

where $u^n(t_k,\omega)$ *are solutions of a system of inclusions*

$$u^n(t_0,\omega) = 0,$$

$$u^n(t_1,\omega) \in u_0(\omega) + \delta_n A(u^n(t_1,\omega)),$$

$$u^n(t_{i+1},\omega) \in u^n(t_i,\omega) + \delta_n A(u^n(t_{i+1},\omega)) + \sum_{j=1}^{r} B^j(u^n(t_i,\omega))\triangle W^j_{t_i}(\omega),$$

where $\triangle W^j_{t_i}(\omega) = W^j_{t_{i+1}}(\omega) - W^j_{t_i}(\omega)$.

Theorem 4.3.1. *Under Assumptions 4.2.1-4.2.6, the* ***time-discretization scheme*** *has a unique solution for all large* n*.*

Proof. Observe that the typical inclusion can be rewritten as

$$(I - \delta_n A)(u^n(t_{i+1},\omega)) \ni u^n(t_i,\omega) + \sum_{j=1}^{r} B^j(u^n(t_i,\omega))\triangle W^j_{t_i}(\omega).$$

Moreover, since

$$2\langle v^*, v\rangle + \lambda\|v\|^2 \le$$

$$2\langle v^*, v\rangle + \sum_{j=1}^{r} |B^j(v)|^2 + \lambda\|v\|^2 \le$$

$$K_1(1 + |v|^2),$$

it follows that, if the pair (A, B) satisfies "coercivity" condition 4.2.2, then the operator A is "coercive" in the sense of Assumption 3.2.2. It follows that the results of Theorem 3.3.1 apply and we are justified in concluding that the **time-discretization scheme** has a unique solution, for every ω and all large n (note that the choice of n only depends on how small δ_n is, but not on ω).
In order to see that u^n is predictable, it is enough, by construction of the process in question, to verify that $u^n(t_i,\cdot)$ is an $\mathcal{F}_{t_i}$-measurable random variable, $0 \le i \le n$. The last assertion follows easily, by mathematical induction, from the fact that the operator $(I - \delta A)^{-1}$ is demicontinuous (at least, for small $\delta > 0$) and hence measurable (see Lemma 10.0.8, Lemma 10.0.9 and Remark 10.0.2, all Appendix B.1). □

4.4 Characterization of solutions.

For notational simplicity, we let $X = L^2(S, V)$. Then, one can identify the dual X^* (of X) with $L^2(S, V^*)$, by means of the duality product

$$\langle x^*, x\rangle_X = E\int_0^T \langle x^*(t), x(t)\rangle dt,$$

where $x \in X$ and $x^* \in X^*$.
We proceed as before.

Definition 4.4.1. *Let* U *denote the space of triplets* (ξ, a, b)*, satisfying the following conditions:*

- *$\xi : \Omega \to H$ is F_0-measurable, with $E|\xi|^2 < \infty$,*
- *$a \in X^*$,*
- *$b^j \in L^2(S, H)$, $1 \le j \le r$,*
- *There exists a process $x \in X$, such that*

$$x_t = \xi + \int_0^t a_s ds + \sum_{j=1}^r \int_0^t b_s^j dW_s^j,$$

for $dt \times P$-almost every $(t, \omega) \in [0, T] \times \Omega$.

Once again, we shall denote by x^H the H-valued continuous modification of x, which exists by Theorem 17.0.5 (Appendix E).
Let $(\xi, a, b) \in U$ and $y \in X$. Set

$$F_y(\xi, a, b) = E|u_0 - \xi|^2 + E \int_0^T [2\langle x(t) - y(t), a_t - y^*(t)\rangle + \sum_{j=1}^r |b_t^j - B^j(y(t))|^2] dt,$$

and

$$G(\xi, a, b) = \sup\{F_y(\xi, a, b) : y \in X\},$$

where $y^* \in W(y)$, and the operator $W : X \to 2^{X^*}$ is defined by

$$W(v) = \{v^* \in X^* : v^*(t, w) \in A(v(t, w)),\ dt \times P\text{-a.e. } (t, \omega) \in [0, T] \times \Omega\}.$$

It turns out that the following assertion is true.

Theorem 4.4.1. *The operator W is "maximal monotone", with $D(W) = X$.*

Proof. See Appendix C.3 for details. □

It follows from Assumption 4.2.4 and Remark 4.2.1 that $F_y(\xi, a, b)$ is well-defined. Moreover, $G(\xi, a, b) \ge 0$.
The main result of this section is

Theorem 4.4.2.
- *(i) Suppose that conditions 4.2.1-4.2.6 hold and let u be a solution of (4.1.1). Then*

$$\inf\{G(\xi, a, b) : (\xi, a, b) \in U\} = G(u_0, u^*, B(u)) = 0,\ \forall u^* \in W(u).$$

- *(ii) Assume conditions 4.2.1-4.2.6. Suppose there exists a triplet $(\hat{\xi}, \hat{a}, \hat{b}) \in U$, such that, $\forall y \in X$,*

$$F_y(\hat{\xi}, \hat{a}, \hat{b}) \le 0.$$

Then $\hat{\xi} = u_0$, and

$$u_t^H = u_0 + \int_0^t \hat{a}_s ds + \sum_{j=1}^r \int_0^t \hat{b}_s^j dW_s^j,\ t \in [0, T],$$

is a solution of (4.1.1).

Proof. (i) Recall that u is a solution of (4.1.1) if there exists a process $\alpha \in X^*$, such that, for $dt \times P$-almost every $(t, \omega) \in [0, T] \times \Omega$,

$$\alpha(t) \in A(u(t)),$$

and, almost surely,

$$u(t) = u_0 + \int_0^t \alpha_s ds + \sum_{j=1}^r \int_0^t B^j(u(s))dW^j(s).$$

It follows that

$$F_y(u_0, \alpha, \beta) = E\int_0^T [2\langle u(t) - y(t), \alpha_t - y^*(t)\rangle + \sum_{j=1}^r |\beta_t^j - B^j(y(t))|^2]dt \leq 0,$$

for any $y \in X$ and $y^* \in W(y)$, due to "monotonicity" condition 4.2.1. Hence, $G(u_0, \alpha, \beta) \leq 0$.
Since, as we have remarked, $G(\xi, a, b) \geq 0$, we conclude that

$$G(u_0, \alpha, \beta) = 0.$$

(ii) Suppose the triplet $(\hat{\xi}, \hat{a}, \hat{b})$ satisfies assumptions of the Theorem. Setting $y = u$ (recall that $D(W) = X$ by Theorem 4.4.1) in $F_y(\hat{\xi}, \hat{a}, \hat{b})$, we note that

$$E|\hat{\xi} - u_0|^2 = 0$$

and

$$E\int_0^T \sum_{j=1}^r |\hat{b}_t^j - B^j(u(t))|^2 dt = 0.$$

Therefore $\hat{\xi} = u_0$, almost surely, and $\hat{b}_t^j = B^j(u(t, \omega))$, for almost all $(t, \omega) \in [0, T] \times \Omega$ $(j = 1, \ldots, r)$.
The condition $F_y(\hat{\xi}, \hat{a}, \hat{b}) \leq 0$ can now be rewritten as follows:

$$\langle u - y, \hat{a} - y^*\rangle_X =$$

$$E\int_0^T \langle u(t) - y(t), \hat{a}(t) - y^*(t)\rangle dt \leq 0.$$

Since $y \in X$ and $y^* \in W(y)$ are arbitrary, and keeping in mind that, by Theorem 4.4.1, W is "maximal monotone", we deduce that $\hat{a} \in W(u)$, or

$$\hat{a}(t, \omega) \in A(u(t, \omega))$$

for almost every $(t, \omega) \in [0, T] \times \Omega$. □

4.5 Convergence of the implicit scheme.

Recall that an approximation u^m to u by an implicit time-discretization scheme is defined by

$$u^m(t_0) := 0,$$

$$u^m(t_1) \in u_0 + \delta_m A(u^m(t_1)),$$

$$u^m(t_{i+1}) \in u^m(t_i) + \delta_m A(u^m(t_{i+1})) + \sum_{j=1}^{r} B^j(u^m(t_i))\triangle W^j_{t_i},$$

$$u^m(t) := u^m(t_i) \text{ for } t \in (t_i, t_{i+1}], 0 \le i \le m-1.$$

We have established that such a scheme has a unique solution for all large m. So, for a (permissible) fixed m, one can find $\alpha^m(t_i) \in A(u^m(t_i))$, $1 \le i \le m$, such that the above system of inclusions can be rewritten as

$$u^m(t_0) = 0,$$

$$u^m(t_1) = u_0 + \delta_m \alpha^m(t_1),$$

$$u^m(t_{i+1}) = u^m(t_i) + \delta_m \alpha^m(t_i) + \sum_{j=1}^{r} B^j(u^m(t_i))\triangle W^j_{t_i}.$$

One can easily check that the process u^m has the form:

$$u^m(t) = u_0\chi_{\{t\ge t_1\}} + \int_0^{\kappa_1(t)} \alpha^m(\kappa_2(s))ds +$$

$$\sum_{j=1}^{r} \int_{t_1}^{\kappa_1(t)} B^j(u^n(\kappa_1(s)))dW^j_s =$$

$$u_0\chi_{\{t\ge t_1\}} + \int_0^{\kappa_1(t)} \alpha^m(\kappa_2(s))ds +$$

$$\sum_{j=1}^{r} \int_0^{\kappa_1(t)} \widetilde{B}^j(u^n(\kappa_1(s)))dW^j_s,$$

where $\widetilde{B}^j(u^n(\kappa_1(s))) = B^j(u^n(\kappa_1(s)))$, $t_1 \le t \le T$, and $\widetilde{B}^j(u^n(\kappa_1(s))) = 0$, $0 \le t < t_1$. We proceed by establishing some useful estimates on the function u^m.

Lemma 4.5.1. *There exist (positive) constants L_1, L_2, L_3, L_4, such that under Assumptions 4.2.1-4.2.6,*

$$\sup_{s\in[0,T]} E|u^m(s)|^2 \le L_1, \tag{4.5.1}$$

$$E\int_0^T \|u^m(\kappa_2(s))\|^2 ds \le L_2, \tag{4.5.2}$$

$$E\int_0^T \|\alpha^m(\kappa_2(s))\|^2 ds \le L_3, \tag{4.5.3}$$

$$\sum_{j=1}^{r} E\int_0^T |\widetilde{B}^j(u^m(\kappa_1(s)))|^2 ds \le L_4 \tag{4.5.4}$$

for all large m.

Proof. Since

$$u^m(t_1) = u_0 + \delta_m \alpha^m(t_1),$$

means

$$|u^m(t_1) - \delta_m \alpha^m(t_1)|^2 = |u_0|^2,$$

it follows that

$$|u^m(t_1)|^2 - |u_0|^2 = 2\langle u^m(t_1), \alpha^m(t_1)\rangle \delta_m - |\alpha^m(t_1)|^2 \delta_m^2.$$

Likewise

$$|u^m(t_{i+1})|^2 - |u^m(t_i)|^2 =$$
$$2\langle u^m(t_{i+1}), \alpha^m(t_{i+1})\rangle \delta_m - |\alpha^m(t_{i+1})|^2 \delta_m^2 +$$
$$2\sum_{j=1}^{r}(u^m(t_i), B^j(u^m(t_i)))\triangle W_{t_i}^j + |\sum_{j=1}^{r} B^j(u^m(t_i))\triangle W_{t_i}^j|^2,$$

for $1 \leq i \leq m-1$.
Adding these equations up to $i = k-1$ and taking expectations, by Ito's isometry of stochastic integrals, we get

$$E|u^m(t_k)|^2 \leq E|u_0|^2 + 2E\int_0^{t_k} \langle u^m(\kappa_2(s)), \alpha^m(\kappa_2(s))\rangle ds +$$
$$\sum_{j=1}^{r} E\int_{t_1}^{t_k} |B^j(u^m(\kappa_1(s)))|^2 ds \leq E|u_0|^2 +$$
$$E\int_0^{t_k} [2\langle u^m(\kappa_2(s)), \alpha^m(\kappa_2(s))\rangle + \sum_{j=1}^{r} |B^j(u^m(\kappa_2(s)))|^2] ds.$$

Using "coercivity" condition 4.2.2, we obtain

$$E|u^m(t_k)|^2 \leq E|u_0|^2 - \lambda E\int_0^{t_k} \|u^m(\kappa_2(s))\|^2 ds +$$
$$K_1 E\int_0^{t_k} |u^m(\kappa_2(s))|^2 ds + K_1 T,$$

or

$$(1 - \frac{K_1 T}{m}) E|u^m(t_k)|^2 + \lambda E\int_0^{t_k} \|u^m(\kappa_2(s))\|^2 ds \leq$$
$$C + K_1 E\int_0^{t_{k-1}} |u^m(\kappa_2(s))|^2 ds,$$

where $C = K_1 T + E|u_0|^2$.
Following precisely the same steps as in the proof of Lemma 3.5.1, we derive estimates (4.5.1), (4.5.2) and (4.5.3).
For (4.5.4), we note that, by Remark 4.2.1 and the above estimates,

$$\sum_{j=1}^{r} E\int_0^{T} |\widetilde{B}^j(u^m(\kappa_1(s)))|^2 ds =$$
$$\sum_{j=1}^{r} E\int_{t_1}^{T} |B^j(u^m(\kappa_1(s)))|^2 ds =$$
$$\sum_{j=1}^{r} E\int_0^{T-\delta_m} |B^j(u^m(\kappa_2(s)))|^2 ds \leq$$

$$C(T + E\int_0^T \|u^m(\kappa_2(s))\|^2 ds) \leq C(T + L_2) = L_4 < \infty.$$

□

The above estimates imply that there exists a subsequence, for simplicity of notation denoted again by u^m, such that

$$u^m(T) \rightharpoonup u^\infty(T) \text{ in } L^2(\Omega, \mathcal{F}_T, H),$$

$$u^m(\kappa_2(\cdot)) \rightharpoonup v_\infty \text{ in } X,$$

$$\alpha^m(\kappa_2(\cdot)) \rightharpoonup a_\infty \text{ in } X^*,$$

$$\widetilde{B}^j(\kappa_1(\cdot)) \rightharpoonup b^j_\infty \text{ in } L^2(S, H),\ j = 1, \ldots, r.$$

We are now in a position to prove the **existence** part of Theorem 4.2.1.

Theorem 4.5.1. *Suppose that conditions 4.2.1-4.2.6 hold. Then*

(i) For $dt \times P$-almost every $(t, \omega) \in [0, T] \times \Omega$,

$$v_\infty(t) = u_0 + \int_0^t a_\infty(s)ds + \sum_{j=1}^r \int_0^t b^j_\infty(s)dW^j_s, \tag{4.5.5}$$

and, almost surely,

$$u^\infty(T) = u_0 + \int_0^T a_\infty(s)ds + \sum_{j=1}^r \int_0^T b^j_\infty(s)dW^j_s; \tag{4.5.6}$$

(ii) For all $y \in X$,

$$F_y(u_0, a_\infty, b_\infty) \leq 0, \tag{4.5.7}$$

and the function v^H_∞ is a solution of (4.1.1);

(iii) The sequence $u^m(T)$ converges strongly to $u^\infty(T)$ in $L^2(\Omega, \mathcal{F}_T, H)$.

Proof. (i) For a fixed $N \geq 1$, let $\phi = \{\phi(t) : t \in [0, T]\}$ be a V-valued adapted stochastic process, such that

$$\|\phi(t, \omega)\| \leq N \text{ for every } t \in [0, T] \text{ and } \omega \in \Omega.$$

Since

$$u^m(t) = u_0\chi_{\{t \geq t_1\}} + \int_0^{\kappa_1(t)} \alpha^m(\kappa_2(s))ds + \sum_{j=1}^r \int_0^{\kappa_1(t)} \widetilde{B}^j(u^m(\kappa_1(s)))dW^j_s,$$

it follows that

$$E\int_0^T (u^m(t), \phi(t))dt = E\int_0^T \chi_{\{t \geq t_1\}}(u_0, \phi(t))dt +$$

$$E\int_0^T \langle \int_0^{\kappa_1(t)} \alpha^m(\kappa_2(s))ds, \phi(t)\rangle dt +$$

$$\sum_{j=1}^{r} E\int_0^T (\int_0^{\kappa_1(t)} \widetilde{B}^j(u^m(\kappa_1(s)))dW_s^j, \phi(t))dt =$$

$$E\int_0^T (u_0, \phi(t))dt + J_1 + J_2 - R_1 - R_2 - R_3, \qquad (4.5.8)$$

where

$$J_1 = E\int_0^T \langle \int_0^t \alpha^m(\kappa_2(s))ds, \phi(t)\rangle dt,$$

$$J_2 = \sum_{j=1}^{r} E\int_0^T (\int_0^t \widetilde{B}^j(u^m(\kappa_1(s)))dW_s^j, \phi(t))dt,$$

$$R_1 = E\int_0^{t_1} (u_0, \phi(t))dt,$$

$$R_2 = E\int_0^T \langle \int_{\kappa_1(t)}^t \alpha^m(\kappa_2(s))ds, \phi(t)\rangle dt,$$

$$R_3 = \sum_{j=1}^{r} E\int_0^T (\int_{\kappa_1(t)}^t \widetilde{B}^j(u^m(\kappa_1(s)))dW_s^j, \phi(t))dt.$$

To begin with,

$$|R_1| = |E\int_0^{t_1} (u_0, \phi(t))dt| \le E\int_0^{t_1} |u_0| \cdot |\phi(t)|dt \le$$

$$\frac{const \cdot NT}{m} E|u_0| \le \frac{const \cdot NT}{m} (E|u_0|^2)^{\frac{1}{2}} \to 0, \text{ as } m \to \infty.$$

Moreover,

$$|R_2| = |E\int_0^T \langle \int_{\kappa_1(t)}^t \alpha^m(\kappa_2(s))ds, \phi(t)\rangle dt| \le$$

$$E\int_0^T \| \int_{\kappa_1(t)}^t \alpha^m(\kappa_2(s))ds\| \cdot \|\phi(t)\|dt \le$$

$$\frac{NT}{m} E\int_0^T \|\alpha^m(\kappa_2(t))\|dt \le$$

$$\frac{NT}{m} E\int_0^T (1 + \|\alpha^m(\kappa_2(t))\|^2)dt \le$$

$$\frac{NT}{m}(T + L_3) \to 0.$$

By the isometry of H-valued stochastic integrals and Hôlder's inequality,

$$|R_3| \le const \cdot NE\int_0^T |\sum_{j=1}^{r} \int_{\kappa_1(t)}^t \widetilde{B}^j(u^m(\kappa_1(s)))dW_s^j|dt \le$$

$$const \cdot N\int_0^T (E|\sum_{j=1}^{r} \int_{\kappa_1(t)}^t \widetilde{B}^j(u^m(\kappa_1(s)))dW_s^j|^2)^{\frac{1}{2}}dt \le$$

$$const \cdot N(\frac{T}{m})^{\frac{1}{2}} \int_0^T (E\sum_{j=1}^r |\widetilde{B}^j(u^m(\kappa_1(t)))|^2)^{\frac{1}{2}} dt \leq$$

$$const \cdot N(\frac{T}{m})^{\frac{1}{2}} (T)^{\frac{1}{2}} (E\int_0^T \sum_{j=1}^r |\widetilde{B}^j(u^m(\kappa_1(t)))|^2 dt)^{\frac{1}{2}} \to 0,$$

by virtue of estimate (4.5.4).
With a view of tackling J_1, let us make the following observation.
Let $G : X^* \to X^*$ be a (linear) operator defined by

$$G(g)_t = \int_0^t g(s)ds, \ g \in X^*.$$

Then, by Holder's inequality,

$$\|G(g)\|_{X^*}^2 = E\int_0^T \| \int_0^t g(s)ds\|^2 dt \leq$$

$$E\int_0^T (\int_0^t \|g(s)\|ds)^2 dt \leq$$

$$E\int_0^T (\int_0^t \|g(s)\|^2 ds)(\int_0^t 1^2 ds)dt \leq$$

$$TE\int_0^T (\int_0^T \|g(s)\|^2 ds)dt = T^2\|g(\cdot)\|_{X^*}^2.$$

The above calculation shows that G is bounded. Hence "$x_i \rightharpoonup x_0$ in X^*" implies "$G(x_i) \rightharpoonup G(x_0)$ in X^*". So,

$$\lim_{m\to\infty} J_1 = \lim_{m\to\infty} E\int_0^T \langle \int_0^t \alpha^m(\kappa_1(s))ds, \phi(t)\rangle dt =$$

$$E\int_0^T \langle \lim_{m\to\infty} \int_0^t \alpha^m(\kappa_1(s))ds, \phi(t)\rangle dt =$$

$$E\int_0^T \langle \int_0^t \alpha_\infty(s)ds, \phi(t)\rangle dt.$$

Likewise, for $j = 1, \ldots, r$ and $g \in L^2(S, H)$, define

$$F_j(g)(t) = \int_0^t g(s)dW_s^j, \ t \in [0, T].$$

An argument similar to the one used above shows that $F_j(g) : L^2(S, H) \to L^2(S, H)$ is continuous, and, consequently,

$$J_2 \to \sum_{j=1}^r E\int_0^T (\int_0^t b_\infty^j(s)dW_s^j, \phi(t))dt.$$

Letting $m \to +\infty$ in equation (4.5.8), we obtain

$$E\int_0^T (v_\infty(t), \phi(t))dt = E\int_0^T (u_0, \phi(t))dt +$$

$$E\int_0^T\langle\int_0^t\alpha_\infty(s)ds,\phi(t)\rangle dt+\sum_{j=1}^r E\int_0^T(\int_0^t b^j_\infty dW^j_s,\phi(t))dt.$$

It follows that

$$v_\infty(t)=u_0+\int_0^t\alpha_\infty(s)ds+\sum_{j=1}^r E\int_0^t b^j_\infty(s)dW^j_s,$$

for $dt\times P$-almost every $(t,\omega)\in[0,T]\times\Omega$, which is (4.5.5).
Repeating the above argument with a function $\psi\in L^2(\Omega,\mathcal{F}_T,V)$ that satisfies $E\|\psi\|\le N$, for a given $N>0$, we get

$$E\langle u^m(T),\psi\rangle=E(u_0,\psi)+$$

$$E\langle\int_0^T\alpha^m(\kappa_2(s))ds,\psi\rangle+$$

$$\sum_{j=1}^r E(\int_0^T\widetilde{B}^j(u^m(\kappa_1(s)))dW^j_s,\psi)=$$

$$E(u_0,\psi)+\tilde{J}_1+\tilde{J}_2,$$

where

$$\tilde{J}_1=E\langle\int_0^T\alpha^m(\kappa_2(s))ds,\psi\rangle\to E\langle\int_0^T\alpha_\infty ds,\psi\rangle,$$

$$\tilde{J}_2=\sum_{j=1}^r E(\int_0^T\widetilde{B}^j(u^m(\kappa_1(s)))dW^j_s,\psi)\to\sum_{j=1}^r E(\int_0^T b^j dW^j_s,\psi).$$

Passing to the limit and taking into consideration that N is arbitrary, we obtain (4.5.6), i.e.,

$$u^\infty(T)=u_0+\int_0^T\alpha_\infty(s)ds+\sum_{j=1}^r E\int_0^T b^j_\infty(s)dW^j_s\ \text{(a.s.)}.$$

(ii) By Theorem 17.0.5 (Appendix E), there exists an H-valued continuous modification v^H_∞ of v_∞, which satisfies

$$E|v^H_\infty(T)|^2=E|u_0|^2+$$

$$E\int_0^T[2\langle v_\infty(t),\alpha_\infty(t)\rangle+\sum_{j=1}^r|b^j_\infty(t)|^2]dt. \tag{4.5.9}$$

By the argument above, $v^H_\infty(T)=u^\infty(T)$ (a.s.).
Now take arbitrary $y\in X$, $y^*\in W(y)$ and set

$$F^m_y=$$

$$E\int_0^T[2\langle u^m(\kappa_2(t))-y(t),\alpha^m(\kappa_2(t))-y^*(t)\rangle+$$

$$\sum_{j=1}^r|B^j(u^m(\kappa_2(t)))-B^j(y(t))|^2]dt=$$

$$2E\int_0^T\langle u^m(\kappa_2(t)),\alpha^m(\kappa_2(t))\rangle dt+2E\int_0^T\langle y(t),y^*(t)\rangle dt-$$

$$2E\int_0^T \langle u^m(\kappa_2(t)), y^*(t)\rangle dt - 2E\int_0^T \langle y(t), \alpha^m(\kappa_2(t))\rangle dt +$$

$$E\int_0^T \sum_{j=1}^r |B^j(u^m(\kappa_2(t)))|^2 dt - 2E\int_0^T \sum_{j=1}^r (B^j(u^m(\kappa_2(t))), B^j(y(t)))dt +$$

$$E\int_0^T \sum_{j=1}^r |B^j(y(t))|^2 dt. \tag{4.5.10}$$

Since (see Lemma 4.5.1)

$$E|u^m(T)|^2 \le E|u_0|^2 +$$

$$2E\int_0^T \langle u^m(\kappa_2(t)), \alpha^m(\kappa_2(t))\rangle dt +$$

$$E\int_0^T \sum_{j=1}^r |B^j(u^m(\kappa_2(t)))|^2 dt,$$

after substituting this expression into (4.5.10) and using the fact that $F_y^m \le 0$ (by "monotonicity" condition 4.2.1), we get

$$0 \ge F_y^m \ge E|u^m(T)|^2 - E|u_0|^2 +$$

$$2E\int_0^T \langle y(t), y^*(t)\rangle dt - 2L_1 - 2L_2 - 2L_3 +$$

$$E\int_0^T \sum_{j=1}^r |B^j(y(t))|^2 dt, \tag{4.5.11}$$

where

$$L_1 = E\int_0^T \langle u^m(\kappa_2(t)), y^*(t)\rangle dt,$$

$$L_2 = E\int_0^T \langle y(t), \alpha^m(\kappa_2(t))\rangle dt,$$

$$L_3 = E\int_0^T \sum_{j=1}^r (B^j(u^m(\kappa_2(t))), B^j(y(t)))dt.$$

Clearly,

$$L_1 \to \int_0^T \langle v_\infty(t), y^*(t)\rangle dt$$

and

$$L_2 \to \int_0^T \langle y(t), a_\infty(t)\rangle dt.$$

Note that

$$E\int_0^T \sum_{j=1}^r (\widetilde{B}^j(u^m(\kappa_1(t))), B^j(y(t)))dt = E\int_0^T \sum_{j=1}^r (\widetilde{B}^j(u^m(\kappa_2(t))), S_m B^j(y(t)))dt =$$

$$E\int_0^T \sum_{j=1}^r (B^j(u^m(\kappa_2(t))), S_m B^j(y(t)))dt = E\int_0^T \sum_{j=1}^r (B^j(u^m(\kappa_2(t))), S_m B^j(y(t)) - B^j(y(t)))dt +$$

$$E\int_0^T \sum_{j=1}^r (B^j(u^m(\kappa_2(t))), B^j(y(t)))dt,$$

where S_m is the averaging operator, defined by

$$(S_m Z)(t) := \delta_m^{-1} \int_{\kappa_1(t)+\delta_m}^{\kappa_1(t)+2\delta_m} Z(s)ds, \text{ if } 0 \le t \le T - \delta_m,$$

and

$$(S_m Z)(t) := 0, \text{ if } T - \delta_m < t \le T,$$

for $Z \in L^2(S, H)$. Since, as one can check,

$$\lim_{m\to\infty} E\int_0^T |(S_m Z)(t) - Z(t)|^2 dt = 0, \ \forall Z \in L^2(S, H),$$

we have

$$L_3 \to E\int_0^T \sum_{j=1}^r (b_\infty^j, B^j(y(t)))dt.$$

Furthermore,

$$\liminf E|u^m(T)|^2 = d + E|u^\infty(T)|^2 = d + E|v_\infty^H(T)|^2, \ d \ge 0.$$

Passing to the limit in equation (4.5.11) and making use of identity (4.5.9), we obtain

$$0 \ge d + E|v_\infty^H(T)|^2 - E|u_0|^2 - 2\int_0^T \langle v_\infty(t), y^*(t)\rangle dt -$$

$$2\int_0^T \langle y(t), a_\infty(t)\rangle dt + 2\int_0^T \langle y(t), y^*(t)\rangle dt +$$

$$E\int_0^T \sum_{j=1}^r |B^j(y(t))|^2 dt - 2E\int_0^T \sum_{j=1}^r (b_\infty^j, B^j(y(t)))dt =$$

$$d + F_y(u_0, a_\infty, b_\infty).$$

Therefore, by Theorem 4.4.2, v_∞^H is a solution of inclusion (4.1.1).
(iii) The last claim can be verified in a manner wholly analogous to that of part (iii) of Theorem 3.5.1. □

Chapter 5

Inclusions with time-independent K-maximal monotone operators.

5.1 Introduction.

This chapter is concerned with the inclusion (4.1.1), i.e.,

$$u(t) \in u_0 + \int_0^t A(u(s))ds + \sum_{j=1}^{r} \int_0^t B^j(u(s))dW^j(s), \tag{5.1.1}$$

but under somewhat different assumptions on the operators A and B, which we now proceed to state.

5.2 Assumptions and description of results.

Let $K, \lambda, K_1 > 0$ and $K_2 \geq 0$ be given constant.

Assumption 5.2.1. *("Monotonicity" condition) For any $u, v \in V$ and $u^* \in A(u), v^* \in A(v)$,*

$$2\langle u - v, u^* - v^* \rangle + \sum_{j=1}^{r} |B^j(u) - B^j(v)|^2 \leq 2K|u - v|^2.$$

Assumption 5.2.2. *("Coercivity" condition) For all pairs (v, v^*), with $v \in V$ and $v^* \in A(v)$,*

$$2\langle v^*, v \rangle + \sum_{j=1}^{r} |B^j(v)|^2 + \lambda \|v\|^2 \leq K_1(1 + |v|^2).$$

Assumption 5.2.3. *The operator A is K-"maximal monotone", that is, the operator $A - KI$ is "maximal monotone".*

Assumption 5.2.4. *("Linear growth" condition) For all pairs (v, v^*), with $v \in V$ and $v^* \in A(v)$,*

$$\|v^*\| \leq K_2(1 + \|v\|).$$

Assumption 5.2.5. *$B^j : (V, \mathcal{B}(V)) \to (H, \mathcal{B}(H))$ is measurable , $0 \leq j \leq r$.*

Assumption 5.2.6. *$u_0 : \Omega \to H$ is F_0-measurable and $E(|u_0|^2) < +\infty$.*

Remark 5.2.1. *Precisely as before, one can check that*

$$\sum_{j=1}^{r} |B^j(v)|^2 \leq C(1 + \|v\|^2),$$

where C is some (positive) constant.

Let S and $L^2(S, V)$ be as above.

Definition 5.2.1. *An H-valued, $\mathcal{F}_t$-adapted continuous process $v = (v_t)_{t\in[0,T]}$ is a solution of (5.1.1), if $v \in L^2(S, V)$ and there exists a process $\alpha \in L^2(S, V^*)$, such that, for $dt \times P$-a.e. $(t, \omega) \in [0, T] \times \Omega$,*

$$\alpha(t, \omega) \in A(v(t, \omega)),$$

and, almost surely,

$$u(t) = u_0 + \int_0^t \alpha_s ds + \sum_{j=1}^{r} \int_0^t B^j(u(s))dW^j(s),$$

for all $t \in [0, T]$.

Our main result is

Theorem 5.2.1. *Under Assumptions 5.2.1-5.2.6, inclusion (5.1.1) has a unique solution.*

Lemma 5.2.1. *Provided inclusion (5.1.1) has a solution, it is unique.*

Proof. Suppose u^1 and u^2 are solutions of (5.1.1). Then there exist processes α^1 and α^2, such that

$$u_t^1 = u_0 + \int_0^t \alpha_s^1 ds + \sum_{j=1}^{r} \int_0^t B^j(u_s^1)dW_s^j,$$

and

$$u_t^2 = u_0 + \int_0^t \alpha_s^2 ds + \sum_{j=1}^{r} \int_0^t B^j(u_s^2)dW_s^j.$$

Set $h(t) = \exp\{-2Kt\}|u_t^2 - u_t^1|^2$. By Itô's formula from [7],

$$0 \leq h(t) = \int_0^t \exp\{-2Ks\}I(s)ds + N(t),$$

where

$$I(s) = 2\langle u_s^1 - u_s^2, \alpha_s^1 - \alpha_s^2\rangle + \sum_{j=1}^{r} |B^j(u_s^1) - B^j(u_s^2)|^2 - 2K|u_s^1 - u_s^2|^2$$

is non-positive by Assumption 5.2.1, and

$$N(t) = 2\sum_{j=1}^{r} \int_0^t \exp\{-2Ks\}(u_s^1 - u_s^2, B^j(u_s^1) - B^j(u_s^2))dW_s^j$$

is a local martingale, starting from zero. A standard argument shows that, almost surely,

$$N(t) = 0, \ \forall t \in [0, T],$$

and the result follows at once. □

In order to show that the inclusion (5.1.1) has a solution, we consider semi-implicit time-discretization schemes, defined in the next section.

5.3 Implicit time-discretization scheme.

Let $n \in \mathbb{N}$ and $\delta_n := \frac{T}{n}$,

Definition 5.3.1. *A (predictable) process u^n is a solution of a semi-implicit **time-discretization scheme**, if u^n is defined by*

$$u^n(0, \omega) := 0,$$

$$u^n(t, \omega) := u^n(t_k, \omega), \ t \in (t_k, t_{k+1}], \ 0 \le k \le n-1,$$

where $u^n(t_k, \omega)$ are solutions of a system of inclusions

$$u^n(0, \omega) = u^n(t_0, \omega) = 0,$$

$$u^n(t_1, \omega) \in u_0(\omega) + \delta_n A(u^n(t_1, \omega)),$$

$$u^n(t_{i+1}, \omega) \in u^n(t_i, \omega) + \delta_n A(u^n(t_{i+1}, \omega)) + \sum_{j=1}^{r} B^j(u^n(t_i, \omega)) \triangle W^j_{t_i}(\omega), \qquad (5.3.1)$$

where $\triangle W^j_{t_i}(\omega) = W^j_{t_{i+1}}(\omega) - W^j_{t_i}(\omega)$.

Theorem 5.3.1. *Under Assumptions 5.2.1-5.2.6, the time-discretization scheme has a unique solution for all large n.*

Proof. Note that

$$u^n(t_{i+1}) \in u^n(t_i) + \delta_n A(u^n(t_{i+1})) + \sum_{j=1}^{r} B^j(u^n(t_i)) \triangle W^j_{t_i}$$

implies

$$u^n(t_{i+1}) \in u^n(t_i) + \delta_n A(u^n(t_{i+1})) + \sum_{j=1}^{r} B^j(u^n(t_i)) \triangle W^j_{t_i} -$$

$$K\delta_n u^n(t_{i+1}) + K\delta_n u^n(t_{i+1}),$$

from which it follows that

$$((1 - K\delta_n)I - \delta_n(A - KI))(u^n(t_{i+1})) \ni u^n(t_i) + \sum_{j=1}^{r} B^j(u^n(t_i)) \triangle W^j_{t_i},$$

or

$$(I - \frac{\delta_n}{1 - K\delta_n}(A - KI))(u^n(t_{i+1})) \ni \frac{1}{1 - K\delta_n}(u^n(t_i) + \sum_{j=1}^{r} B^j(u^n(t_i)) \triangle W^j_{t_i}),$$

and we are done, provided the operator $I - \frac{\delta_n}{1-K\delta_n}(A - KI)$ can be shown to be maximal monotone and **coercive** (see Theorem 4.3.1).

The first claim is an immediate consequence of Assumption 5.2.3, Lemma 2.3.1 (part (i), with $\delta_n < \frac{1}{K}$) and the fact that the sum of two maximal monotone operators is maximal monotone (Theorem 10.0.3, Appendix B.1). For the second, it suffices to show that $A - KI$ satisfies a condition analogous to 3.2.2. To see this, note that, for a given $v \in V$, $u^* \in (A - KI)(v)$ if, and only if, $u^* = v^* - KI(v)$, for some $v^* \in A(v)$; in other words, elements of $A(v)$ and $(A - KI)(v)$ are in $1-1$ correspondence. Hence, for any $v \in V$ and $u^* \in (A - KI)(v)$, we have

$$2\langle u^*, v\rangle + \lambda\|v\|^2 = 2\langle v^* - KI(v), v\rangle + \lambda\|v\|^2 =$$

$$2\langle v^*, v\rangle + \lambda\|v\|^2 - 2K|v|^2 \le K_1(1 + |v|^2) - 2K|v|^2 \le K_1(1 + |v|^2).$$

The fact that the process u^n is predictable can be verified in a manner identical to that of Theorem 4.3.1. □

5.4 Characterization of solutions.

For simplicity of exposition, we consider a Banach space Y, defined to be $L^2(S, V)$, with the (equivalent) norm

$$\|u\|_Y = (E\int_0^T \exp\{-2Kt\}\|u(t)\|^2 dt)^{\frac{1}{2}} < \infty.$$

Then, using the duality product

$$\langle u^*, u\rangle_Y = E\int_0^T \exp\{-2Kt\}\langle u^*(t), u(t)\rangle dt,$$

one can make an identification of the dual space Y^* and $L^2(S, V^*)$, with the corresponding norm

$$\|u^*\|_{Y^*} = (E\int_0^T \exp\{-2Kt\}\|u^*(t)\|^2 dt)^{\frac{1}{2}} < \infty.$$

Moreover, let $L^2_H(\exp\{-2K\cdot\})$ denote the space $L^2(S, H)$ with the norm

$$\|h\|_{L^2_H(\exp\{-2K\cdot\})} = (E\int_0^T \exp\{-2Kt\}|h(t)|^2 dt)^{\frac{1}{2}} < \infty.$$

Definition 5.4.1. *Let U denote the space of triplets (ξ, a, b), satisfying the following conditions:*

- *$\xi : \Omega \to H$ is F_0-measurable and such that $E|\xi|^2 < \infty$,*
- *$a \in Y^*$,*
- *$b^j \in L^2_H(\exp\{-2K\cdot\})$, $1 \le j \le r$,*
- *There exists a process $x \in Y$, such that*

$$x_t = \xi + \int_0^t a_s ds + \sum_{j=1}^r \int_0^t b^j_s dW^j_s,$$

for $dt \times P$-almost every $(t,\omega) \in [0,T] \times \Omega$.

Precisely as before, we shall denote by x^H the H-valued, continuous modification of the process x.
Let $(\xi, a, b) \in U$ and $y \in Y$. Set

$$F_y(\xi,a,b) = E|u_0 - \xi|^2 + E\int_0^T \exp\{-2Kt\}[2\langle x(t)-y(t), (a_t - KI(x(t))) - y^*(t)\rangle$$

$$+\sum_{j=1}^r |b_t^j - B^j(y(t))|^2]dt,$$

and

$$G(\xi,a,b) = \sup\{F_y(\xi,a,b) : y \in Y\},$$

where $y^* \in W(y)$ and the operator $W : Y \to 2^{Y^*}$ is given by

$$W(v) = \{v^* \in Y^* : v^*(t,w) \in (A - KI)(v(t,w)),\ dt \times P\text{-a.e. } (t,\omega) \in [0,T] \times \Omega\}.$$

Theorem 5.4.1. *The operator W is "maximal monotone" and $D(W) = Y$.*

Proof. Consult Appendix C.4. □

It follows from Assumption 5.2.4 and Remark 5.2.1 that $F_y(\xi,a,b)$ is well-defined. Moreover, $G(\xi,a,b) \geq 0$.
The main result of this section is

Theorem 5.4.2.

- *(i) Suppose that conditions 5.2.1-5.2.6 hold and let u be a solution of (5.1.1). Then*

$$\inf\{G(\xi,a,b) : (\xi,a,b) \in U\} = G(u_0, u^*, B(u)) = 0,\ \forall u^* \in W(u).$$

- *(ii) Assume conditions 5.2.1-5.2.6. If there exists a triplet $(\hat{\xi}, \hat{a}, \hat{b}) \in U$, such that, $\forall y \in Y$,*

$$F_y(\hat{\xi}, \hat{a}, \hat{b}) \leq 0,$$

then $\hat{\xi} = u_0$, and

$$u_t^H = u_0 + \int_0^t \hat{a}_s ds + \sum_{j=1}^r \int_0^t \hat{b}_s^j dW_s^j,\ t \in [0,T],$$

is a solution of (5.1.1).

Proof. (i) Recall that a process u is a solution of (5.1.1), if there exists a process $\alpha \in Y^*$, such that, for $dt \times P$-a.e. $(t,\omega) \in [0,T] \times \Omega$,

$$\alpha(t,\omega) \in A(u(t,\omega)),$$

and, almost surely,

$$u(t) = u_0 + \int_0^t \alpha_s ds + \sum_{j=1}^r \int_0^t B^j(u(s))dW^j(s),$$

for any $t \in [0,T]$. Hence

$$F_y(u_0,\alpha,\beta) = E\int_0^T \exp\{-2Kt\}[2\langle u(t)-y(t),(\alpha_t - KI(u(t))) - y^*(t)\rangle +$$

$$\sum_{j=1}^r |\beta_t^j - B^j(y(t))|^2]dt \le 0,$$

for every $y \in Y$, due to "monotonicity" condition 5.2.1. Therefore, $G(u_0,\alpha,\beta) \le 0$. On the other hand, as we have previously noted, $G(\xi,a,b) \ge 0$. Thus $G(u_0,\alpha,\beta) = 0$.
(ii) Suppose a triple $(\hat{\xi},\hat{a},\hat{b})$ satisfies assumptions of the theorem. Setting $y = u$ in $F_y(\hat{\xi},\hat{a},\hat{b})$, we observe that

$$E|\hat{\xi} - u_0|^2 = 0$$

and

$$E\int_0^T \exp\{-2Kt\}\sum_{j=1}^r |\hat{b}_t^j - B^j(u(t))|^2 dt = 0.$$

Therefore, almost surely, $\hat{\xi} = u_0$ and $\hat{b}_t^j = B^j(u(t))$, for almost all $(t,\omega) \in [0,T]\times\Omega$ $(j = 1,\dots,r)$.
The condition $F_y(\hat{\xi},\hat{a},\hat{b}) \le 0$ now assumes the form

$$\langle u - y, (\hat{a} - KI(u)) - y^*\rangle_Y =$$

$$E\int_0^T \exp\{-2Kt\}\langle u(t)-y(t),(\hat{a}(t) - KI(u(t))) - y^*(t)\rangle dt \le 0,$$

$\forall y \in Y$ and $y^* \in W(y)$. Taking into consideration that, by Theorem 5.4.1, W is "maximal monotone", we infer that $\hat{a} - KI(u) \in W(u)$, or

$$\hat{a}(t,\omega) \in A(u(t,\omega)),$$

for $dt \times P$-almost all $(t,\omega) \in [0,T]\times\Omega$. □

5.5 Convergence of the implicit scheme.

According to Theorem 5.3.1, the time-discretization scheme has a unique solution for all large m. This means that, given any such m, one can find $\alpha^m(t_i) \in A(u^m(t_i))$, $1 \le i \le m$, such that the system of inclusions (5.3.1) can be written as a system of equalities as follows:

$$u^m(t_0) = 0,$$

$$u^m(t_1) = u_0 + \delta_m\alpha^m(t_1),$$

$$u^m(t_{i+1}) = u^m(t_i) + \delta_m\alpha^m(t_i) + \sum_{j=1}^r B^j(u^m(t_i))\triangle W_{t_i}^j.$$

As we have noted in the previous chapter, the process u^m has the form:

$$u^m(t) = u_0\chi_{\{t\ge t_1\}} + \int_0^{\kappa_1(t)} \alpha^m(\kappa_2(s))ds +$$

$$\sum_{j=1}^{r}\int_{0}^{\kappa_1(t)}\widetilde{B}^j(u^n(\kappa_1(s)))dW_s^j.$$

The next result can be verified in a manner wholly analogous to that of Lemma 4.5.1.

Lemma 5.5.1. *There exist (positive) constants L_1, L_2, L_3, L_4, such that, under Assumptions 5.2.1-5.2.6,*

$$\sup_{s\in[0,T]} E|u^m(s)|^2 \le L_1,$$

$$E\int_0^T \|u^m(\kappa_2(s))\|^2 ds \le L_2,$$

$$E\int_0^T \|\alpha^m(\kappa_2(s))\|^2 ds \le L_3,$$

$$\sum_{j=1}^{r} E\int_0^T |\widetilde{B}^j(u^m(\kappa_1(s)))|^2 ds \le L_4, \tag{5.5.1}$$

for all large m.

Set $u_i^m = u^m(t_i)$ and $\alpha_i^m = \alpha^m(t_i)$ for notational simplicity. By construction,

$$u_{i+1}^m = u_i^m + \delta_m \alpha_{i+1}^m + \sum_{j=1}^{r} B^j(u_i^m)\triangle W_i^j,$$

so that

$$u_{i+1}^m = u_i^m + \delta_m \alpha_{i+1}^m - K\delta_m u_{i+1}^m + K\delta_m u_{i+1}^m + \sum_{j=1}^{r} B^j(u_i^m)\triangle W_i^j,$$

or

$$(1-K\delta_m)u_{i+1}^m = u_i^m + \delta_m(\alpha_{i+1}^m - KI(u_{i+1}^m)) + \sum_{j=1}^{r} B^j(u_i^m)\triangle W_i^j.$$

It follows that

$$(1-K\delta_m)^2 E|u_1^m|^2 - E|u_0|^2 \le 2(1-K\delta_n)E\langle u_1^m, \alpha_1^m - KI(u_1^m)\rangle\delta_m$$

$$\vdots$$

$$(1-K\delta_m)^2 E|u_m^m|^2 - E|u_{m-1}^m|^2 \le 2(1-K\delta_m)E\langle u_m^m, \alpha_m^m - KI(u_m^m)\rangle\delta_m + \sum_{j=1}^{r} E|B^j(u_{m-1}^m)|^2\delta_m.$$

Multiplying the i(th) inequality ($i = 1,\ldots,m$) by $(1-K\delta_m)^{2(i-1)}$ and adding them up, we obtain

$$(1-K\delta_m)^{2m}E|u_m^m|^2 - E|u_0|^2 \le$$

$$E\sum_{i=1}^{m} 2(1-K\delta_m)^{2i-1}\langle u_i^m, \alpha_i^m - KI(u_i^m)\rangle\delta_m + E\sum_{i=1}^{m-1} 2(1-K\delta_m)^{2i-2}\sum_{j=1}^{r}|B^j(u_i^m)|^2\delta_m.$$

Recall that

$$t_i = i\frac{T}{m} = i\delta_m,\ i = 1,\ldots,m.$$

Thus

$$E\sum_{i=1}^{m}2(1-K\delta_m)^{2i-1}\langle u_i^m,\alpha_i^m-KI(u_i^m)\rangle\delta_m=$$

$$E\sum_{i=1}^{m}2(1-K\delta_m)^{2\frac{t_i}{\delta_m}-1}\langle u_i^m,\alpha_i^m-KI(u_i^m)\rangle\delta_m=$$

$$2E\int_0^T(1-K\delta_m)^{2\frac{\kappa_2(s)}{\delta_m}-1}\langle u^m(\kappa_2(s)),\alpha^m(\kappa_2(s))-KI(u^n(\kappa_2(s)))\rangle ds=$$

$$2E\int_0^T\phi_m(s)\langle u^m(\kappa_2(s)),\alpha^m(\kappa_2(s))-KI(u^m(\kappa_2(s)))\rangle ds,$$

where

$$\phi_m(s)=(1-K\delta_m)^{2\frac{\kappa_2(s)}{\delta_m}-1}.$$

Likewise

$$E\sum_{i=1}^{m-1}2(1-K\delta_m)^{2i-2}\sum_{j=1}^{r}|B^j(u_i^m)|^2\delta_m=$$

$$\sum_{j=1}^{r}E\int_0^{T-2\delta_m}\psi_m(s)|B^j(u^m(\kappa_2(s)))|^2ds\le$$

$$\sum_{j=1}^{r}E\int_0^{T}\psi_m(s)|B^j(u^m(\kappa_2(s)))|^2ds\le$$

with

$$\psi_m(s)=(1-K\delta_m)^{2\frac{\kappa_2(s)}{\delta_m}-2}.$$

Finally,

$$(1-K\delta_m)^{2m}E|u^m(T)|^2-E|u_0|^2\le$$

$$2E\int_0^T\phi_m(s)\langle u^m(\kappa_2(s)),\alpha^m(\kappa_2(s))-KI(u^m(\kappa_2(s)))\rangle ds+$$

$$\sum_{j=1}^{r}E\int_0^{T}\psi_m(s)|B^j(u^m(\kappa_2(s)))|^2ds. \tag{5.5.2}$$

This inequality will be required shortly.
The above estimates imply that there exists a subsequence, denoted by u^m, such that

$$u^m(T)\rightharpoonup u^\infty(T)\text{ in }L^2(\Omega,\mathcal{F}_T,H),$$

$$u^m(\kappa_2(\cdot))\rightharpoonup v_\infty\text{ in }Y,$$

$$\alpha^m(\kappa_2(\cdot))\rightharpoonup a_\infty\text{ in }Y^*,$$

$$\widetilde{B}^j(\kappa_1(\cdot))\rightharpoonup b_\infty^j\text{ in }L_H^2(\exp\{-2K\cdot\}),\ j=1,\ldots,r.$$

We are now in a position to prove the **existence** part of Theorem 5.2.1.

Theorem 5.5.1. *Suppose that conditions 5.2.1-5.2.6 hold. Then*

(i) *For $dt \times P$-almost every $(t, \omega) \in [0, T] \times \Omega$*

$$v_\infty(t) = u_0 + \int_0^t a_\infty(s)ds + \sum_{j=1}^{r} \int_0^t b_\infty^j(s)dW_s^j, \tag{5.5.3}$$

and, almost surely,

$$u^\infty(T) = u_0 + \int_0^T a_\infty(s)ds + \sum_{j=1}^{r} \int_0^T b_\infty^j(s)dW_s^j; \tag{5.5.4}$$

(ii) *For all $y \in Y$,*

$$F_y(u_0, a_\infty, b_\infty) \le 0, \tag{5.5.5}$$

and the function v_∞^H is the solution of (5.1.1);

(iii) *The sequence $u^m(T)$ converges strongly to $u^\infty(T)$ in $L^2(\Omega, \mathcal{F}_T, H)$.*

Proof. (i) Identities (5.5.3) and (5.5.4) can be verified using the same argument as that put forward in the proof of Theorem 4.5.1.
(ii) As we know, there exists an H-valued continuous modification v_∞^H of v_∞ that satisfies

$$E|v_\infty^H(T)|^2 = E|u_0|^2 +$$
$$E\int_0^T [2\langle v_\infty(t), \alpha_\infty(t)\rangle + \sum_{j=1}^{r} |b_\infty^j(t)|^2]dt.$$

Thus, by Itô's formula from [7],

$$E(\exp\{-2KT\}|v_\infty^H(T)|^2) = E|u_0|^2 +$$
$$E\int_0^T \exp\{-2Kt\}[2\langle v_\infty(t), \alpha_\infty(t) - KI(v_\infty(t))\rangle +$$
$$\sum_{j=1}^{r} |b_\infty^j(t)|^2]dt. \tag{5.5.6}$$

Moreover, $v_\infty^H(T) = u^\infty(T)$ (a.s.).
For $y \in Y$ and $y^* \in W(y)$, set

$$F_y^m =$$
$$E\int_0^T \exp\{-2Kt\}[2\langle u^m(\kappa_2(t)) - y(t), (\alpha^m(\kappa_2(t)) - KI(u^m(\kappa_2(t)))) - y^*(t)\rangle +$$
$$\sum_{j=1}^{r} |B^j(u^m(\kappa_2(t))) - B^j(y(t))|^2]dt =$$
$$2E\int_0^T \exp\{-2Kt\}\langle u^m(\kappa_2(t)), \alpha^m(\kappa_2(t)) - KI(u^m(\kappa_2(t)))\rangle dt +$$
$$2E\int_0^T \exp\{-2Kt\}\langle y(t), y^*(t)\rangle dt -$$
$$2E\int_0^T \exp\{-2Kt\}\langle u^m(\kappa_2(t)), y^*(t)\rangle dt -$$

$$2E\int_0^T \exp\{-2Kt\}\langle y(t), \alpha^m(\kappa_2(t)) - KI(u^m(\kappa_2(t)))\rangle dt +$$

$$E\int_0^T \exp\{-2Kt\}\sum_{j=1}^r |B^j(u^m(\kappa_2(t)))|^2 dt -$$

$$2\sum_{j=1}^r E\int_0^T \exp\{-2Kt\}(B^j(u^m(\kappa_2(t))), B^j(y(t)))dt +$$

$$\sum_{j=1}^r E\int_0^T \exp\{-2Kt\}|B^j(y(t))|^2 dt. \tag{5.5.7}$$

Making use of inequality (5.5.2) and taking into consideration that $F_y^m \leq 0$ (by "monotonicity" condition 5.2.1), we obtain

$$0 \geq F_y^m \geq (1 - K\delta_m)^{2m} E|u^m(T)|^2 - E|u_0|^2 +$$

$$2E\int_0^T [\exp\{-2Kt\} - \phi^m(t)]\langle u^m(\kappa_2(t)), \alpha^m(\kappa_2(t)) - KI(u^m(\kappa_2(t)))\rangle dt +$$

$$\sum_{j=1}^r E\int_0^T [\exp\{-2Kt\} - \psi^m(t)]|B^j(u^m(\kappa_2(t)))|^2 dt +$$

$$2E\int_0^T \exp\{-2Kt\}\langle y(t), y^*(t)\rangle dt -$$

$$2E\int_0^T \exp\{-2Kt\}\langle u^m(\kappa_2(t)), y^*(t)\rangle dt -$$

$$2E\int_0^T \exp\{-2Kt\}\langle y(t), \alpha^m(\kappa_2(t)) - KI(u^m(\kappa_2(t)))\rangle dt +$$

$$2E\int_0^T \exp\{-2Kt\}\sum_{j=1}^r (B^j(u^m(\kappa_2(t))), B^j(y(t)))dt +$$

$$\sum_{j=1}^r E\int_0^T \exp\{-2Kt\}|B^j(y(t))|^2 dt =$$

$$(1 - K\delta_m)^{2m} E|u^m(T)|^2 - E|u_0|^2 + L_1 + L_2 -$$

$$L_3 - L_4 - L_5 + 2E\int_0^T \exp\{-2Kt\}\langle y(t), y^*(t)\rangle dt +$$

$$\sum_{j=1}^r E\int_0^T \exp\{-2Kt\}|B^j(y(t))|^2 dt. \tag{5.5.8}$$

where

$$L_1 = 2E\int_0^T [\exp\{-2Kt\} - \phi^m(t)]\langle u^m(\kappa_2(t)), \alpha^m(\kappa_2(t)) - KI(u^m(\kappa_2(t)))\rangle dt,$$

$$L_2 = \sum_{j=1}^r E\int_0^T [\exp\{-2Kt\} - \psi^m(t)]|B^j(u^m(\kappa_2(t)))|^2 dt,$$

$$L_3 = E\int_0^T \exp\{-2Kt\}\langle u^m(\kappa_2(t)), y^*(t)\rangle dt,$$

$$L_4 = E\int_0^T \exp\{-2Kt\}\langle y(t), \alpha^m(\kappa_2(t)) - KI(u^m(\kappa_2(t)))\rangle dt,$$

$$L_5 = E\int_0^T \exp\{-2Kt\}\sum_{j=1}^{r}(B^j(u^m(\kappa_2(t))), B^j(y(t)))dt.$$

Since, as is easy to check, functions $\phi^m(t), \psi^m(t)$ converge to $\exp\{-2Kt\}$, uniformly in t ($t \in [0,T]$) and keeping in mind estimates (5.5.1), we observe that

$$L_1, L_2 \to 0.$$

Furthermore,

$$L_3 \to \int_0^T \exp\{-2Kt\}\langle v_\infty(t), y^*(t)\rangle dt,$$

$$L_4 \to \int_0^T \exp\{-2Kt\}\langle y(t), a_\infty(t) - KI(v_\infty(t))\rangle dt,$$

$$L_5 \to \sum_{j=1}^{r} E\int_0^T \exp\{-2Kt\}(b_\infty^j, B^j(y(t)))dt.$$

Besides, $\exists d \geq 0$, such that

$$\liminf E((1-K\delta_m)^{2m}|u^m(T)|^2) =$$

$$d + E(\exp\{-2KT\}|v_\infty^H(T)|^2).$$

Passing to the limit in equation (5.5.8) and using identity (5.5.6), we obtain

$$0 \geq d + E(\exp\{-2KT\}|v_\infty^H(T)|^2) - E|u_0|^2 -$$

$$2\int_0^T \exp\{-2Kt\}\langle v_\infty(t), y^*(t)\rangle dt -$$

$$2\int_0^T \exp\{-2Kt\}\langle y(t), a_\infty(t) - KI(v_\infty(t))\rangle dt +$$

$$2\int_0^T \exp\{-2Kt\}\langle y(t), y^*(t)\rangle dt +$$

$$\sum_{j=1}^{r} E\int_0^T \exp\{-2Kt\}|B^j(y(t))|^2 dt -$$

$$\sum_{j=1}^{r} 2E\int_0^T \exp\{-2Kt\}(b_\infty^j, B^j(y(t)))dt =$$

$$d + F_y(u_0, a_\infty, b_\infty).$$

Since, by the above, $F_y(u_0, a_\infty, b_\infty) \leq 0$, $\forall y \in Y$, Theorem 5.4.2 tells us that v_∞^H is the solution of (5.1.1).

(iii) Same as in the relevant section of Theorem 3.5.1. □

5.6 Example with a Lipschitz term.

Suppose we want to solve an inclusion

$$u(t) \in u_0 + \int_0^t (A+F)(u(s))ds+$$

$$\sum_{j=1}^r \int_0^t B^j(u(s))dW^j(s),\ 0 \le t \le T, \tag{5.6.1}$$

under assumptions (on u_0, A and B) of Chapter 4. We further assume that

Assumption 5.6.1. *$F : H \to H$ is Lipschitz-continuous, with a constant $K > 0$.*

Set $\overline{A} = A + F - KI$.

Lemma 5.6.1.
- *The operator $A + F$ is K-"maximal monotone", that is, $\overline{A}$ is "maximal monotone";*
- *Operators $A + F$ and B satisfy "monotonicity", "linear growth" and "coercivity" conditions.*

Proof. We proceed in a sequence of steps.
(i) $\overline{A}$ is "monotone".
Indeed, $F - KI$ is "monotone", since

$$\langle u - v, F(u) - KI(u) - F(v) + KI(v)\rangle =$$

$$(u - v, F(u) - F(v)) - K|u - v|^2 \le$$
$$K|u - v|^2 - K|u - v|^2 = 0.$$

It follows that $\overline{A}$ is "monotone", as the sum of two "monotone" mappings.
(ii) Moreover, since the operator $F - KI : V \to V^*$ is , clearly, continuous and "monotone", by (i), it is, in fact, "maximal monotone" (Theorem 10.0.2). It follows that $\overline{A}$ is "maximal monotone", as the sum of two "maximal monotone" operators (Theorem 10.0.3).
(iii) The pair $(A + F, B)$ satisfies "monotonicity", "linear growth" and "coercivity" conditions.
(a)("Monotonicity" condition) Note that, for any $u, v \in V$ and $u^* \in A(u), v^* \in A(v)$, by Assumption 4.2.1,

$$0 \ge 2\langle u - v, u^* - v^*\rangle + \sum_{j=1}^r |B^j(u) - B^j(v)|^2 \ge$$

$$2\langle u - v, (u^* + F(u)) - (v^* + F(v))\rangle - 2K|u - v|^2 + \sum_{j=1}^r |B^j(u) - B^j(v)|^2,$$

since

$$2\langle u - v, F(u) - F(v)\rangle = 2(u - v, F(u) - F(v)) \le 2K|u - v|^2.$$

(b)("Linear growth" condition) To begin with, fix $u_0 \in V$. Then, for any $u \in V$,

$$|F(u)| \le |F(u) - F(u_0)| + |F(u_0)| \le$$

$$K|u-u_0| + |F(u_0)| \leq |F(u_0)| + K|u_0| + K|u| \leq$$
$$M(1+|u|),$$

where $M = \max\{|F(u_0)| + K|u_0|, K\}$.
Since embeddings $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ are continuous, there exist (non-negative) constants C_1, C_2, such that

$$|u|_H \leq C_1\|u\|_V \text{ and } \|u\|_{V^*} \leq C_2|u|_H.$$

Thus,

$$\|F(u)\|_{V^*} \leq C_2|F(u)|_H \leq C_2M(1+|u|_H) \leq$$
$$C_2M(1+C_1\|u\|_V) \leq C_2MC_3(1+\|u\|_V),$$

with $C_3 = \max\{1, C_1\}$, which implies that F has a "linear growth" property.
Consequently, $A+F$ satisfies "linear growth" condition, since it is the sum of two terms, each of which does.
(c)("Coercivity" condition) For arbitrary $u \in V$ and $u^* \in A(u)$, we have

$$2\langle u, u^* + F(u)\rangle + \sum_{j=1}^{r} |B^j(u)|^2 + \lambda\|u\|^2 \leq$$

$$K_1(1+|u|^2) + 2|(u, F(u))| \leq K_1(1+|u|^2) + |u|^2 + |F(u)|^2 \leq$$
$$K_1(1+|u|^2) + |u|^2 + 2M + 2M|u|^2 \leq P(1+|u|^2),$$

where $P = K_1 + 2M + 1$. □

Chapter 6

Space-time-discretization scheme.

6.1 Introduction.

Let us briefly summarize what we have done so far.
We have studied the inclusion

$$u(t) \in u_0 + \int_0^t A(u(s))ds + \sum_{j=1}^{r} \int_0^t B^j(u(s))dW^j(s),$$

and have demonstrated that, under certain assumptions on the operators A and B, the above problem has a unique solution. Our strategy has been to consider a time-discretized version of the original problem. At the heart of the **existence** argument for the solution of the time-discretization scheme lies a well-known result from set-valued analysis, which states that a weakly coercive, maximal monotone operator is surjective. In general, this is an infinite-dimensional problem and although we are now secure in our knowledge that a solution exists, the above approach tells us nothing about how the latter can be computed.
In order to address this issue, we proceed to discretize not just in time, but in space as well. Here are the details.

6.2 Assumptions and description of results.

Our main result is

Theorem 6.2.1. *Under Assumptions 4.2.1-4.2.6, inclusion (4.1.1) has a unique solution.*

Lemma 6.2.1. *Inclusion (4.1.1) has, at most, one solution.*

Proof. See Lemma 4.2.1. □

The **existence** part is verified, using semi-implicit **time-space-discretization schemes**, defined next.

6.3 Implicit time-space-discretization scheme.

To begin with, there exists a sequence of elements of V, $\{e_i \in V : i = 1, 2, \ldots\}$, which forms a complete orthonormal basis in H. For a fixed $n \in \mathbb{N}$, set $V_n = Span\{e_1, \ldots, e_n\}$

and let V_n^* denote the dual of V_n.
We define an operator $\Pi_n : V^* \to V_n^*$ by

$$\Pi_n(u^*) = \sum_{i=1}^{n} \langle u^*, e_i \rangle e_i^*,$$

where e_i^*, $i = 1, 2, \ldots$, is given by $\langle e_i^*, u \rangle_{V_n} = (e_i, u)_H$, $u \in V_n$.
For $v \in V$, let $\|v\|_{V_n} = \|v\|_V$ denote the restriction of the V-norm to V_n, and let $|v|_{H_n} = |v|_H$ denote the restriction of the H-norm to V_n. We denote by H_n the Hilbert space V_n, endowed with the norm $|\cdot|_{H_n}$. We have $V_n = H_n \equiv H_n^* = V_n^*$, as topological spaces, where H_n is identified with its dual H_n^*, by means of the inner product in H_n.

Lemma 6.3.1. *V_n and Π_n satisfy the following properties:*

- *(i) The sequence $(V_n, n \geq 1)$ is increasing, i.e., $V_n \subseteq V_{n+1}$, and $\bigcup_n V_n$ is dense in V, that is, $\overline{\bigcup_n V_n} = V$.*
- *(ii) For every $u \in V_n$ and $u^* \in V^*$,*

$$(u, \Pi_n(u^*))_{H_n^*} = \langle u, \Pi_n(u^*) \rangle_{V_n} = \langle u, u^* \rangle_V.$$

- *(iii) For every $h \in H$,*

$$|\Pi_n(h)|_{H_n^*} \leq |h|_H,$$

and

$$\lim_{n\infty} |\Pi_n(h)|_{H_n^*} = \lim_{n\infty} |\Pi_n(h)|_{H^*} = |h|_H.$$

Proof. (i) Since $\{e_i\}_{i=1}^{\infty}$ is a complete orthonormal basis in H, for any $v \in V$, one has

$$v = \sum_{i=1}^{\infty} (v, e_i)_H e_i.$$

In other words, $v = \lim_{n\to\infty} f_n$, where

$$f_n = \sum_{i=1}^{n} (v, e_i)_H e_i \text{ and } f_n \in V_n \subseteq \bigcup_n V_n,$$

which justifies the claim.
(ii) We give a sample proof of the fact

$$\langle u, \Pi_n(u^*) \rangle_{V_n} = \langle u, u^* \rangle_V.$$

(the first equality can be dealt with in a similar manner).
Take any $u \in V_n$ and $u^* \in V^*$. Then, by definition of Π_n,

$$LHS := \langle u, \Pi_n(u^*) \rangle_{V_n} = \sum_{i=1}^{n} \langle u^*, e_i \rangle_V (u, e_i)_H.$$

On the other hand, $u = \sum_{i=1}^{n} (u, e_i)_H e_i$. Hence

$$RHS := \langle u, u^* \rangle_V = \langle \sum_{i=1}^{n} (u, e_i)_H e_i, u^* \rangle_V =$$

$$\sum_{i=1}^{n}\langle u^*, e_i\rangle_V (u, e_i)_H = LHS.$$

(iii) Suppose $h \in H$. Then, by construction, $\Pi_n(u) \in H_n^*$. Moreover,

$$|\Pi_n(u)|^2_{H_n^*} = |\Pi_n(I(u))|^2_{H_n^*} = |\sum_{i=1}^{n}\langle I(u), e_i\rangle_V e_i^*|^2_{H_n^*} =$$

$$|\sum_{i=1}^{n}(u, e_i)_H e_i^*|^2_{H_n^*} = \sum_{i=1}^{n}|(u, e_i)_H|^2_H \le \sum_{i=1}^{\infty}|(u, e_i)_H|^2_H = |u|^2_H.$$

The final claim follows from the fact that $\{e_i\}_n$ is a complete orthonormal basis in H. □

Now fix $n, m \in \mathbb{N}$ and set $\delta_n = \frac{T}{n}$.

Definition 6.3.1. *A (predictable) process $u^{n,m}$ is called a solution of a **space-time-discretization scheme**, if it is given by*

$$u^{n,m}(t) := u^{n,m}(t_k),\ t \in (t_k, t_{k+1}],\ 0 \le k \le n-1,$$

$$u^{n,m}(T) := u^{n,m}(t_n),$$

where $u^{n,m}(t_k)$ are solutions of a system of inclusions

$$u^{n,m}(t_0) = 0,$$

$$u^{n,m}(t_1) \in \Pi_m u_0 + \delta_n \Pi_m A(u^{n,m}(t_1)),$$

$$u^{n,m}(t_{i+1}) \in u^{n,m}(t_i) + \delta_n \Pi_m A(u^{n,m}(t_{i+1})) + \sum_{j=1}^{r}\Pi_m B^j(u^{n,m}(t_i))\triangle W^j_{t_i}.$$

Theorem 6.3.1. *Under Assumptions 4.2.1-4.2.6, the **space-time-discretization scheme** has a unique solution for all large n and m.*

Proof. Observe that a typical inclusion problem

$$u^{n,m}(t_{i+1}) \in u^{n,m}(t_i) + \delta_n \Pi_m A(u^{n,m}(t_{i+1})) + \sum_{j=1}^{r}\Pi_m B^j(u^{n,m}(t_i))\triangle W^j_{t_i}$$

can be rewritten as

$$(I - \delta_n \Pi_m A)(u^{n,m}(t_{i+1})) \ni u^{n,m}(t_i) + \sum_{j=1}^{r}\Pi_m B^j(u^{n,m}(t_i))\triangle W^j_{t_i}.$$

Also recall that when we discretize with respect to time alone, the corresponding scheme has a unique solution, due to the fact that the operator $I - \delta_n A$ is surjective (provided A is "maximal monotone" and satisfies a "coercivity" condition).
Let us show that, for each $m \in \mathbb{N}$, $\Pi_m A : V_m \to 2^{V_m^*}$ is a "maximal monotone" operator, satisfying Assumption 4.2.2.
To begin with, the operator $\Pi_m A$ is "monotone". To see this, take arbitrary $u_1, u_2 \in V_m$ and corresponding v_1, v_2, such that $v_1 \in \Pi_m A(u_1)$ and $v_2 \in \Pi_m A(u_2)$. This means that

there exist $u_1^* \in A(u_1)$ and $u_2^* \in A(u_2)$, with

$$v_1 = \Pi_m(u_1^*) \text{ and } v_2 = \Pi_m(u_2^*).$$

Then, by part (ii) of Lemma 6.3.1,

$$\langle u_1 - u_2, v_1 - v_2 \rangle_{V_m} = \langle u_1 - u_2, \Pi_m(u_1^* - u_2^*) \rangle_{V_m} =$$

$$\langle u_1 - u_2, u_1^* - u_2^* \rangle_V \leq 0,$$

since A is "monotone".
In fact, it turns out that $\Pi_m A$ is "maximal monotone" (see Appendix G.2 for details).
Moreover, given $u \in V_m$, $v^* \in \Pi_m A(u)$ and $u^* \in A(u)$, where $v^* = \Pi_m(u^*)$,

$$2\langle u, v^* \rangle_{V_m} + \lambda \|u\|_{V_m}^2 = 2\langle u, u^* \rangle_V + \lambda \|u\|_V^2 \leq$$

$$2\langle u, u^* \rangle + \sum_{j=1}^r |B^j(u)|^2 + \lambda \|v\|^2 \leq K_1(1 + |v|^2).$$

Note that, since the family of operators $\{\Pi_m A\}$ satisfies the same "coercivity" condition, there exists $n_0 \in \mathbb{N}$ (independent of m), such that $\forall n \geq n_0$ and all m, each of the above operators is **coercive** (in a sense of Definition 10.0.7; review the proof of Theorem 3.3.1).
Adaptability can be dealt with in a manner identical to that of Theorem 4.3.1. □

6.4 Characterization of solutions.

Review Section 4.4.

6.5 Convergence of the implicit scheme.

Recall that an approximation $u^{n,m}$ to u by a space-time-discretization scheme is given by

$$u^{n,m}(t_0) := 0,$$

$$u^{n,m}(t_1) \in \Pi_m u_0 + \delta_n \Pi_m A(u^n(t_1)),$$

$$u^{n,m}(t_{i+1}) \in u^{n,m}(t_i) + \delta_n \Pi_m A(u^n(t_{i+1})) + \sum_{j=1}^r \Pi_m B^j(u^n(t_i)) \triangle W_{t_i}^j,$$

$$u^{n,m}(t) := u^{n,m}(t_i) \text{ for } t \in (t_i, t_{i+1}],\ 0 \leq i \leq n-1.$$

Since, as we have observed, such a scheme has a unique solution for any sufficiently large n and arbitrary m, there exist $\alpha^{n,m}(t_i) \in A(u^{n,m}(t_i))$, $1 \leq i \leq n$, such that we have

$$u^{n,m}(t_0) = 0,$$

$$u^{n,m}(t_1) = u_0 + \delta_n \Pi_m \alpha^{n,m}(t_1),$$

$$u^{n,m}(t_{i+1}) = u^m(t_i) + \delta_n \Pi_m \alpha^{n,m}(t_i) + \sum_{j=1}^r \Pi_m B^j(u^{n,m}(t_i)) \triangle W_{t_i}^j.$$

One can easily verify that the process $u^{n,m}$ assumes an integral form:

$$u^{n,m}(t) = \Pi_m u_0 \chi_{\{t \geq t_1\}} + \int_0^{\kappa_1(t)} \Pi_m \alpha^{n,m}(\kappa_2(s))ds +$$

$$\sum_{j=1}^{r} \int_0^{\kappa_1(t)} \Pi_m \widetilde{B}^j(u^{n,m}(\kappa_1(s)))dW_s^j.$$

Lemma 6.5.1. *There exist (non-negative) constants L_1, L_2, L_3, L_4, such that under Assumptions 4.2.5-4.2.6,*

$$\sup_{s \in [0,T]} E|u^{n,m}(s)|^2 \leq L_1,$$

$$E \int_0^T \|u^{n,m}(\kappa_2(s))\|^2 ds \leq L_2,$$

$$E \int_0^T \|\alpha^{n,m}(\kappa_1(s))\|^2 ds \leq L_3,$$

$$\sum_{j=1}^{r} E \int_0^T |\Pi_m \widetilde{B}^j(u^{n,m}(\kappa_1(s)))|^2 ds \leq L_4,$$

for all large n and arbitrary m.

Proof. To begin with,

$$u^{n,m}(t_1) = \Pi_m u_0 + \delta_n \Pi_m \alpha^{n,m}(t_1),$$

which means that

$$|u^{n,m}(t_1) - \delta_n \Pi_m \alpha^{n,m}(t_1)|^2_{H_m^*} = |\Pi_m u_0|^2_{H_m^*},$$

or, upon expanding,

$$|u^{n,m}(t_1)|^2_H - 2(u^{n,m}(t_1), \Pi_m \alpha^{n,m}(t_1))_{H_m^*}\delta_n + |\Pi_m \alpha^{n,m}(t_1)|^2_{H_m^*}\delta_n^2 = |\Pi_m u_0|^2_{H_m^*}.$$

Using parts (ii) and (iii) of Lemma 6.3.1, we get

$$|u^{n,m}(t_1)|^2_H - |u_0|^2_H \leq 2\langle u^{n,m}(t_1), \alpha^{n,m}(t_1)\rangle_V \delta_n.$$

Similarly,

$$|u^{n,m}(t_{i+1})|^2_H - |u^{n,m}(t_i)|^2_H \leq 2\langle u^{n,m}(t_{i+1}), \alpha^{n,m}(t_{i+1})\rangle_V \delta_n +$$

$$2\sum_{j=1}^{r}(u^{n,m}(t_i), B^j(u^{n,m}(t_i)))_H \triangle W^j_{t_i} + |\sum_{j=1}^{r} \Pi_m B^j(u^{n,m}(t_i))\triangle W^j_{t_i}|^2_{H_m^*} \leq$$

$$2\langle u^{n,m}(t_{i+1}), \alpha^{n,m}(t_{i+1})\rangle_V \delta_n + 2\sum_{j=1}^{r}(u^{n,m}(t_i), B^j(u^{n,m}(t_i)))_H \triangle W^j_{t_i} +$$

$$|\sum_{j=1}^{r} B^j(u^{n,m}(t_i))\triangle W^j_{t_i}|^2_H,$$

$1 \le i \le n-1$. Adding these inequalities up to $i = k-1$ and taking expectations, we obtain, by Ito's isometry of stochastic integrals and part (iii) of Lemma 6.3.1,

$$E|u^{n,m}(t_k)|^2 \le E|u_0|^2 +$$

$$E\int_0^{t_k} [2\langle u^{n,m}(\kappa_2(s)), \alpha^{n,m}(\kappa_2(s))\rangle ds + \sum_{j=1}^{r} |B^j(u^m(\kappa_2(s)))|^2 ds],$$

by Ito's isometry of stochastic integrals.
To complete the proof, we proceed as in Lemma 4.5.1. □

The above estimates imply that there exists a subsequence, denoted again by $u^{n,m}$, such that

$$u^{n,m}(T) \rightharpoonup u^\infty(T) \text{ in } L^2(\Omega, \mathcal{F}_T, H),$$
$$u^{n,m}(\kappa_2(\cdot)) \rightharpoonup v_\infty \text{ in } X,$$
$$\alpha^{n,m}(\kappa_2(\cdot)) \rightharpoonup a_\infty \text{ in } X^*,$$
$$\Pi_m \widetilde{B}^j(\kappa_1(\cdot)) \rightharpoonup b^j_\infty \text{ in } L^2(S, H),\ j = 1, \ldots, r.$$

Theorem 6.5.1. *Suppose that conditions 4.2.5-4.2.6 hold. Then*

(i) For $dt \times P$-almost every $(t, \omega) \in [0, T] \times \Omega$

$$v_\infty(t) = u_0 + \int_0^t a_\infty(s) ds + \sum_{j=1}^{r} \int_0^t b^j_\infty(s) dW^j_s.$$

Moreover, almost surely,

$$u^\infty(T) = u_0 + \int_0^T a_\infty(s) ds + \sum_{j=1}^{r} \int_0^T b^j_\infty(s) dW^j_s;$$

(ii) For all $y \in X$,

$$F_y(u_0, a_\infty, b_\infty) \le 0,$$

and the function v^H_∞ is the solution of (4.1.1);

(iii) The sequence $u^m(T)$ converges strongly to $u^\infty(T)$ in $L^2(\Omega, \mathcal{F}_T, H)$.

Proof. (i) Precisely as before, one can show that

$$v_\infty(t) = u_0 + \int_0^t \alpha_\infty(s) ds + \sum_{j=1}^{r} \int_0^t b^j_\infty(s) dW^j_s,$$

for $dt \times P$-almost every $(t, \omega) \in [0, T] \times \Omega$, and

$$u^\infty(T) = u_0 + \int_0^T \alpha_\infty(s) ds + \sum_{j=1}^{r} \int_0^T b^j_\infty(s) dW^j_s,$$

almost surely.
(ii) An H-valued continuous modification v^H_∞ of v_∞ satisfies

$$E|v^H_\infty(T)|^2 = E|u_0|^2 +$$

$$E\int_0^T [2\langle v_\infty(t), \alpha_\infty(t)\rangle + \sum_{j=1}^r |b^j_\infty(t)|^2]dt. \tag{6.5.1}$$

Furthermore, as we have noted previously, almost surely, $v^H_\infty(T) = u^\infty(T)$.
Let $y \in X$ and $y^* \in W(y)$ be arbitrary. Set

$$F^{n,m}_y =$$

$$= E\int_0^T [2\langle u^{n,m}(\kappa_2(t)) - y(t), \alpha^{n,m}(\kappa_2(t)) - y^*(t)\rangle +$$

$$\sum_{j=1}^r |\Pi_m B^j(u^{n,m}(\kappa_2(t))) - \Pi_m B^j(y(t))|^2]dt =$$

$$2E\int_0^T \langle u^{n,m}(\kappa_2(t)), \alpha^{n,m}(\kappa_2(t))\rangle dt + 2E\int_0^T \langle y(t), y^*(t)\rangle dt -$$

$$2E\int_0^T \langle u^{n,m}(\kappa_2(t)), y^*(t)\rangle dt - 2E\int_0^T \langle y(t), \alpha^{n,m}(\kappa_2(t))\rangle dt +$$

$$E\int_0^T \sum_{j=1}^r |\Pi_m B^j(u^{n,m}(\kappa_2(t)))|^2 dt - 2E\int_0^T \sum_{j=1}^r (\Pi_m B^j(u^{n,m}(\kappa_2(t))), B^j(y(t)))dt +$$

$$E\int_0^T \sum_{j=1}^r |\Pi_m B^j(y(t))|^2 dt. \tag{6.5.2}$$

By Lemma 4.5.1,

$$E|u^{n,m}(T)|^2 - E|u_0|^2 \le$$

$$2E\int_0^T \langle u^{n,m}(\kappa_2(t)), \alpha^{n,m}(\kappa_2(t))\rangle dt + E\int_0^T \sum_{j=1}^r |\Pi_m B^j(u^m(\kappa_2(t)))|^2 dt.$$

Substituting this inequality into equation (6.5.2) and using the fact that $F^{n,m}_y \le 0$ (by part (iii) of Lemma 6.3.1 and "monotonicity" condition 4.2.1), we get

$$0 \ge F^{n,m}_y \ge E|u^{n,m}(T)|^2 - E|u_0|^2 +$$

$$2E\int_0^T \langle y(t), y^*(t)\rangle dt - 2L_1 - 2L_2 - 2L_3 +$$

$$E\int_0^T \sum_{j=1}^r |\Pi_m B^j(y(t))|^2 dt, \tag{6.5.3}$$

where

$$L_1 = E\int_0^T \langle u^{n,m}(\kappa_2(t)), y^*(t)\rangle dt,$$

$$L_2 = E\int_0^T \langle y(t), \alpha^{n,m}(\kappa_2(t))\rangle dt,$$

$$L_3 = E\int_0^T \sum_{j=1}^r (\Pi_m B^j(u^{n,m}(\kappa_2(t))), B^j(y(t)))dt.$$

As, $n, m \to \infty$,

$$L_1 \to E\int_0^T \langle v_\infty(t), y^*(t)\rangle dt,$$

$$L_2 \to E\int_0^T \langle y(t), a_\infty(t)\rangle dt,$$

and

$$L_3 \to E\int_0^T \sum_{j=1}^r (b^j_\infty, B^j(y(t)))dt.$$

Moreover, by part (iii) of Lemma 6.3.1,

$$E\sum_{j=1}^r \int_0^T |\Pi_m B^j(y(t))|^2 dt \to E\sum_{j=1}^r \int_0^T |B^j(y(t))|^2 dt.$$

Precisely as before, we have

$$E \liminf |u^{n,m}(T)|^2 = d + E|u^\infty(T)|^2 = d + E|v^H_\infty(T)|^2,\ d \geq 0.$$

Finally, letting $n, m \to \infty$ in equation (6.5.3) and utilizing identity (6.5.1), we obtain

$$0 \geq d + E|v^H_\infty(T)|^2 - E|u_0|^2 - 2E\int_0^T \langle v_\infty(t), y^*(t)\rangle dt -$$

$$2E\int_0^T \langle y(t), a_\infty(t)\rangle dt + 2E\int_0^T \langle y(t), y^*(t)\rangle dt +$$

$$E\int_0^T \sum_{j=1}^r |B^j(y(t))|^2 dt - 2E\int_0^T \sum_{j=1}^r (b^j_\infty, B^j(y(t)))dt =$$

$$d + F_y(u_0, a_\infty, b_\infty).$$

To draw the desired conclusion, we use Theorem 4.4.2.
(iii) Can be dealt with in a manner wholly analogous to that of Theorem 4.5.1.

□

Chapter 7

Inclusions with time-dependent maximal monotone operators.

7.1 Introduction.

In this section we study approximations to an inclusion

$$u(t) \in u_0 + \int_0^t A_s(u(s))ds + \sum_{j=1}^{r} \int_0^t B^j(u(s))dW^j(s), \tag{7.1.1}$$

where A is defined on $[0,T] \times \Omega \times V$, with values in V^*, and B is defined on V, with values in H^r. In other words, in contrast to the cases we have considered thus far, the operator A is assumed to depend on t and ω explicitly.

7.2 Assumptions and description of results.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, satisfying the usual conditions of right continuity and completeness. Fix arbitrary $(t,\omega) \in [0,T] \times \Omega$. Let $\lambda, K_1 > 0$ and $K_2 \geq 0$ be given constants, independent of (t,ω).

Assumption 7.2.1. *("Monotonicity" condition) For any $u, v \in V$ and $u^* \in A_{t,\omega}(u), v^* \in A_{t,\omega}(v)$,*

$$2\langle u - v, u^* - v^* \rangle + \sum_{j=1}^{r} |B^j(u) - B^j(v)|^2 \leq 0.$$

Assumption 7.2.2. *("Coercivity" condition) For all pairs (v, v^*), with $v \in V$ and $v^* \in A_{t,\omega}(v)$,*

$$2\langle v^*, v \rangle + \sum_{j=1}^{r} |B^j(v)|^2 + \lambda \|v\|^2 \leq K_1(1 + |v|^2).$$

Assumption 7.2.3. *The operator $A_{t,\omega} : V \to 2^{V^*}$ is "maximal monotone".*

Assumption 7.2.4. *("Linear growth" condition) For any $v \in V$ and $v^* \in A_{t,\omega}(v)$,*

$$\|v^*\| \leq K_2(1 + \|v\|).$$

Assumption 7.2.5. *The function $u_0 : \Omega \to H$ is F_0-measurable and*

$$E(|u_0|^2) < +\infty.$$

Assumption 7.2.6. *$B^j : (V, \mathcal{B}(V)) \to (H, \mathcal{B}(H))$ is measurable, $0 \le j \le r$.*

Remark 7.2.1. *It follows from Assumptions 7.2.2 and 7.2.4 that there exists a (positive) constant C, such that*

$$\sum_{j=1}^{r} |B^j(v)| \le C(1 + \|v\|^2).$$

Let S and $L^2(S, V)$ be as in Chapter 4.

Definition 7.2.1. *An H-valued, $\mathcal{F}_t$-adapted continuous stochastic process $u \in L^2(S, V)$ is a solution of (7.1.1), if there exists a process $\alpha \in L^2(S, V^*)$, such that, for $dt \times P$ almost all $(t, \omega) \in [0, T] \times \Omega$,*

$$\alpha_t(\omega) \in A(t, \omega, v(t, \omega)),$$

and, almost surely,

$$u(t) = u_0 + \int_0^t \alpha_s ds + \sum_{j=1}^{r} \int_0^t B^j(u(s))dW^j(s),$$

for every $t \in [0, T]$.

Following the same steps as in the proof of Lemma 4.2.1, one can easily demonstrate the validity of the next result.

Lemma 7.2.1. *Inclusion (7.1.1) has, at most, one solution.*

We now consider a semi-implicit time-discretization scheme and show that a (unique) solution of the above scheme converges (in an appropriate sense) to the solution of (7.1.1).

7.3 Implicit time-discretization scheme.

Let $0 \le s < t \le T$. We define the operator $\bar{A}_{(s,t]} : V \times \Omega \to 2^{V^*}$ as follows. Assume that to each $u \in V$, there corresponds $u' \in L^2(S, V^*)$, such that $u'(x, \omega) \in A(x, \omega, u)$, $dx \times P -$ a.e. $(x, \omega) \in (s, t] \times \Omega$. Then, for all those $\omega \in \Omega$, s.t. $\int_0^T \|u'(t, \omega)\|^2 dt < \infty$,

$$\bar{A}_{(s,t]}(u, \omega) = \{u^* \in V^* : u^* = \frac{\int_s^t u'(x, \omega)dx}{t - s}\},$$

and

$$\bar{A}_{(s,t]}(u, \omega) = 0, \text{ otherwise}.$$

For notational simplicity, given a partition

$$0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T,$$

where, as usual, $t_i = i\frac{T}{n}$, $0 \le i \le n$, we shall write

$$\bar{A}_{(t_i, t_{i+1}]} = \bar{A}^n_{i+1}.$$

Let $\omega \in \Omega$. We make the following additional assumptions on $\bar{A}_{(s,t]}$:

Assumption 7.3.1. *The operator $\bar{A}_{(s,t]}$ is defined on the whole of V, i.e., $D(\bar{A}_{(s,t]}) = V$ (a.s.).*

Assumption 7.3.2. *The operator $\bar{A}_{(s,t]}$ is "maximal monotone" (a.s.).*

Remark 7.3.1. *(i) It is not difficult to check that if $A : V \to 2^{V^*}$ is "maximal monotone", then, for any $0 \leq s < t \leq T$, $\bar{A}_{(s,t]} = A$ (a.s.), which shows that Assumptions 7.3.1 and 7.3.2 are justified. Similarly, one can demonstrate that if $A : [0,T] \times \Omega \times V \to V^*$ is hemicontinuous, then the above assumptions are satisfied.*
(ii) On the other hand, suppose $V = H = \mathbb{R}$. We know that there exists C, a subset of $[0,T]$, which is not Lebesgue-measurable. Define $A : \mathbb{R} \to \mathbb{R}$ by

$$A(t,u) = u + \chi_c(t).$$

Then, it is easy to check that $R(\bar{A}_{(s,t]}) = \emptyset$.
This simple observation leads us to conclude that, in general, some kind of "measurability" condition on A is required to ensure that the operator $\bar{A}_{(s,t]}$ does not have a void range and satisfies the above assumptions.

Lemma 7.3.1. *Almost surely, the operator $\bar{A}_{(s,t]}$ satisfies "linear growth" and "coercivity" conditions.*

Proof. (i)("Linear growth") Take $(u, u^*) \in \bar{A}_{(s,t]}$. By definition, $\exists u' \in L^2(S, V^*)$, such that $u^* = \frac{\int_s^t u'(x)dx}{t-s}$, with $u'(x,\omega) \in A(x,\omega,u)$, for $dx \times P -$ a.e. $(x,\omega) \in (s,t] \times \Omega$. Making use of Assumption 7.2.3, we obtain

$$\|u^*\| = \|\frac{\int_s^t u'(x)dx}{t-s}\| \leq \frac{\int_s^t \|u'(x)\|dx}{t-s} \leq$$

$$\leq \frac{\int_s^t K_2(1+\|u\|)dx}{t-s} = K_2(1+\|u\|).$$

(ii)("Coercivity") Upon noticing that

$$\langle u^*, u\rangle = \frac{\int_s^t \langle u'(x), u\rangle dx}{t-s},$$

(this equality holds for step functions; passing to the limit shows that it is true for any u') and using Assumption 7.2.5 inside the integral, we arrive at the desired result. $\square$

With the above notation in mind, we make the following definition.

Definition 7.3.1. *A (predictable) process u^n is called a solution of a* ***time-discretization scheme****, if u^n is defined by*

$$u^n(t) := u^n(t_k),\ t \in (t_k, t_{k+1}],\ 0 \leq k \leq n-1,$$

$$u^n(T) := u^n(t_n),$$

where $u^n(t_k)$ are solutions of a system of inclusions

$$u^n(t_0) = 0,$$

$$u_1^n \in u_0 + \delta_n \bar{A}_1^n(u_1^n)$$

$$u_{i+1}^n \in u_i^n + \delta_n \bar{A}_{i+1}^n(u_{i+1}^n) + \sum_{j=1}^r B^j(u_i^n) \triangle W_{t_i}^j,$$
$$0 \le i \le n-1.$$

Let us recall the line of reasoning of Theorem 4.3.1: the conclusion that the process u^n is predictable rests on the observation that the operator $(I-\delta_n A)^{-1}$ is demicontinuous, and, hence, measurable.
To ensure that the same holds true in the present case, we make the following assumption.

Assumption 7.3.3. *For all large $n \in \mathbb{N}$ and every $0 \le i \le n-1$, the operator $(I - \delta_n \bar{A}_i^n)^{-1}$ is $\mathcal{F}_{t_i} \oplus \mathcal{B}(V)$-measurable.*

Remark 7.3.2. *One can check that if $A_1 : V \to 2^{V^*}$ is "maximal monotone" or $A_2 : [0,T] \times \Omega \times V \to V^*$ is hemicontinuous, then both operators satisfy condition 7.3.3. Furthermore, an argument similar to that advanced in Appendix H.2 can be used to show that if an operator A has the form $A = A_1 + A_2$, then Assumption 7.3.3 is, once more, justified.*

The main result of this section is

Theorem 7.3.1. *Under Assumptions 7.2.1-7.2.6 and 7.3.1-7.3.3, the time-discretization scheme has a unique solution for all large n.*

Proof. Review the proof of Theorem 4.3.1. □

7.4 Characterization of solutions.

Let X denote $L^2(S,V)$. Then, as have seen, the dual X^* can be identified with $L^2(S,V^*)$. We proceed as before.

Definition 7.4.1. *Let U denote the space of triplets (ξ, a, b), satisfying the following conditions:*

- *$\xi : \Omega \to H$ is F_0-measurable and such that $E|\xi|^2 < \infty$,*
- *$a \in X^*$,*
- *$b^j \in L^2(S,H)$, $1 \le j \le r$;*
- *There exists a process $x \in X$, such that*

$$x_t = \xi + \int_0^t a_s ds + \sum_{j=1}^r \int_0^t b_s^j dW_s^j,$$

for $dt \times P$-almost every $(t,\omega) \in [0,T] \times \Omega$.

Let $(\xi, a, b) \in U$ and $y \in X$. Set

$$F_y(\xi,a,b) = E|u_0 - \xi|^2 + E\int_0^T [2\langle x(t) - y(t), a_t - y^*(t)\rangle + \sum_{j=1}^r |b_t^j - B^j(y(t))|^2]dt,$$

and

$$G(\xi, a, b) = \sup\{F_y(\xi, a, b) : y \in X\},$$

where $y^* \in W(y)$ and the operator $W : X \to 2^{X^*}$ is given by

$$W(v) = \{v^* \in X^* : v^*(t, w) \in A(t, \omega, v(t, w)),\ dt \times P\text{-a.e. } (t, \omega) \in [0, T] \times \Omega\}.$$

Assumption 7.4.1. *The operator W is "maximal monotone" and $D(W) = S$ (see Appendix H.1 for justification and examples).*

Following exactly the same steps as in Theorem 4.4.2, one can prove the main result of this section.

Theorem 7.4.1.

- *(i) Suppose that conditions 7.2.1-7.2.6, 7.3.1-7.3.3 and 7.4.1 hold and let u be a solution of (7.1.1). Then*

$$\inf\{G(\xi, a, b) : (\xi, a, b) \in U\} = G(u_0, u^*, B(u)) = 0,\ \forall u^* \in W(u).$$

- *(ii) Assume conditions 7.2.1-7.2.6, 7.3.1-7.3.2 and 7.4.1. Suppose there exists a triplet $(\hat{\xi}, \hat{a}, \hat{b}) \in U$, such that, $\forall y \in X$,*

$$F_y(\hat{\xi}, \hat{a}, \hat{b}) \leq 0.$$

Then $\hat{\xi} = u_0$, and

$$u_t^H = u_0 + \int_0^t \hat{a}_s ds + \sum_{j=1}^r \int_0^t \hat{b}_s^j dW_s^j,\ t \in [0, T],$$

is a solution of (7.1.1).

7.5 Convergence of the implicit scheme.

By Theorem 7.3.1, the time-discretization scheme has a unique solution for all large n, which means that there exist $\alpha_i^n = \alpha^n(t_i) \in \bar{A}_i^n$ and corresponding $\tilde{\alpha}_i^n(\cdot) \in A(\cdot, u_i^n)$, with $\alpha_{i+1}^n = \frac{\int_{t_i}^{t_{i+1}} \tilde{\alpha}_{i+1}^n(x)dx}{\delta_n}$, $1 \leq i \leq n$, such that

$$u^n(t_0) = 0,$$

$$u_1^n = u_0 + \delta_n \alpha_1^n,$$

$$u_{i+1}^n = u_i^n + \delta_n \alpha_{i+1}^n + \sum_{j=1}^r B^j(u_i^n) \triangle W_{t_i}^j,$$

$$0 \leq i \leq n - 1.$$

One can easily derive the following expression for u^n :

$$u^n(t) = u_0 \chi_{\{t \geq t_1\}} + \int_0^{\kappa_1(t)} \alpha^n(\kappa_2(s))ds + \sum_{j=1}^r \int_0^{\kappa_1(t)} \widetilde{B}^j(u^n(\kappa_1(s)))dW_s^j =$$

$$u_0 \chi_{\{t \geq t_1\}} + \int_0^{\kappa_1(t)} \tilde{\alpha}^n_{\frac{\kappa_2(s)}{\delta_n}}(s)ds + \sum_{j=1}^r \int_0^{\kappa_1(t)} \widetilde{B}^j(u^n(\kappa_1(s)))dW_s^j.$$

Lemma 7.5.1. *There exist (positive) constants L_1, L_2, L_3, L_4, such that, under Assumptions 7.2.1-7.2.6 and 7.3.1-7.3.3,*

$$\sup_{s\in[0,T]} E|u^n(s)|^2 \leq L_1,$$

$$E\int_0^T \|u^n(\kappa_2(s))\|^2 ds \leq L_2,$$

$$E\int_0^T \|\tilde{\alpha}^n_{\frac{\kappa_2(s)}{\delta_n}}(s)\|^2 ds \leq L_3,$$

$$\sum_{j=1}^r E\int_0^T |\widetilde{B}^j(u^n(\kappa_1(s)))|^2 ds \leq L_4,$$

for all large n.

Proof. See Lemma 4.5.1. □

Therefore there exists a subsequence, denoted again by u^n, such that

$$u^n(T) \rightharpoonup u^\infty(T) \text{ in } L^2(\Omega, \mathcal{F}_T, H),$$

$$u^n(\kappa_2(\cdot)) \rightharpoonup v_\infty \text{ in } L^2(S, V),$$

$$\tilde{\alpha}^n_{\frac{\kappa_2(\cdot)}{\delta_n}}(\cdot) \rightharpoonup a_\infty \text{ in } L^2(S, V^*),$$

$$\widetilde{B}^j(\kappa_1(\cdot)) \rightharpoonup b^j_\infty \text{ in } L^2(S, H),\ j = 1, \ldots, r.$$

The next result asserts that the inclusion (7.1.1) has a solution.

Theorem 7.5.1. *Suppose that conditions 7.2.1-7.2.6, 7.3.1-7.3.3 and 7.4.1 hold. Then*

(i) For $dt \times P$-almost every $(t, \omega) \in [0, T] \times \Omega$,

$$v_\infty(t) = u_0 + \int_0^t a_\infty(s)ds + \sum_{j=1}^r \int_0^t b^j_\infty(s)dW^j_s,$$

and, almost surely,

$$u^\infty(T) = u_0 + \int_0^T a_\infty(s)ds + \sum_{j=1}^r \int_0^T b^j_\infty(s)dW^j_s;$$

(ii) For every $y \in X$,

$$F_y(u_0, a_\infty, b_\infty) \leq 0,$$

and the function v^H_∞ is a solution of (7.1.1);

(iii) The sequence $u^m(T)$ converges strongly to $u^\infty(T)$ in $L^2(\Omega, \mathcal{F}_T, H)$.

Proof. The above statements can be verified in a manner completely analogous to that of Theorem 4.5.1. □

Chapter 8

Square-integrable martingales.

8.1 Assumptions.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, satisfying the usual conditions of right continuity and completeness.
We are given $M(t)$, a locally square-integrable martingale, taking values in a real separable Hilbert space E, with a continuous bracket process $\langle M\rangle_t$, and $U(t)$, an adapted non-decreasing real-valued continuous process, starting from zero, that satisfies $dU(t) \geq d\langle M\rangle_t$ and $U(T) \leq M$ $(\forall\omega \in \Omega)$, for some positive constant M.
One can show that there exists a predictable process $Q = (Q(t))_{t\geq 0}$, taking values in $L^2(E,E)$, the space of Hilbert-Schmidt operators on E, such that for any fixed orthonormal basis $(e_i)_{i=1}^{\infty}$ in E, almost surely,

$$\langle M^i, M^j\rangle_t = \int_{(0,t]} Q^{ij}(s)dU(s),$$

for all $t \in \mathbb{R}_+$ and integers $i, j \geq 1$, where $M^i = (M, e_i)_E$, $Q^{ij}(t) = (Q(t)e_i, e_j)_E$. Moreover, almost surely, $Q(t)$ is a non-negative self-adjoint nuclear operator for all $t \geq 0$ (see [8] and references therein).
Hence there is a predictable $L^2(E,E)$-valued process $Q^{\frac{1}{2}}$, such that, almost surely, $Q^{\frac{1}{2}}(t)$ is a non-negative definite self-adjoint operator, satisfying

$$(Q^{\frac{1}{2}})^2 = Q, \; t \geq 0.$$

Let us denote by $L^2(E,H)$ the Hilbert space of the Hilbert-Schmidt operators mapping E into H, and by $L_Q(E,H)$ the set of all operators C, mapping $Q^{\frac{1}{2}}E$ into H, such that $CQ^{\frac{1}{2}} \in L^2(E,H)$. We shall denote by $|C|_Q$ the Hilbert-Schmidt norm of $CQ^{\frac{1}{2}}$.
We consider an inclusion problem

$$u_t \in u_0 + \int_0^t A(u_s)dU(s) + \int_0^t B(u_s)dM(s), \; 0 \leq t \leq T, \qquad (8.1.1)$$

where A (generally, multi-valued) and B are operators, defined on V, such that, for every $v \in V$, $A(v)$ is a subset of V^*, $B(v) \in L_{Q(t,\omega)}(E,H)$, $\forall(t,\omega) \in [0,T]\times\Omega$, and the

function $BQ^{\frac{1}{2}} : [0,T] \times \Omega \times V \to L^2(E,H)$ is $\mathcal{O} \times \mathcal{B}(V)$ measurable.[1]
Fix $(t,\omega) \in [0,T] \times \Omega$ and let $K > 0$ be a given constant. Then A and B are assumed to satisfy the following requirements:

Assumption 8.1.1. *("Monotonicity" condition) For any $u, v \in V$ and $u^* \in A(u), v^* \in A(v)$,*

$$2\langle u - v, u^* - v^* \rangle + |B(u) - B(v)|_Q^2 \leq 2K|u - v|^2.$$

Assumption 8.1.2. *The operator $A_K = A - KI$ is "maximal monotone".*

Assumption 8.1.3. *("Coercivity" condition) There exists a constant $\lambda > 0$, such that*

$$2\langle v^*, v \rangle + |B(v)|_Q^2 + \lambda \|v\|^2 \leq K(1 + |v|^2),$$

for all pairs (v, v^*), with $v \in V$ and $v^* \in A(v)$.

Assumption 8.1.4. *("Linear Growth" condition) There is a constant $K_2 \geq 0$, such that, for all pairs (v, v^*), with $v \in V$ and $v^* \in A(v)$, we have*

$$\|v^*\| \leq K_2(1 + \|v\|).$$

Assumption 8.1.5. *u_0 is an H-valued $\mathcal{F}_0$-measurable random variable, with $E|u_0|_H^2 < \infty$.*

Remark 8.1.1. *Using Assumptions 8.1.3 and 8.1.4, one can easily check that*

$$|B(v)|_Q^2 \leq C(1 + \|v\|^2),$$

for some constant $C \geq 0$.

Remark 8.1.2. *Although we have assumed, for simplicity of exposition, that the operator A does not depend on time, our results can be extended to a time-dependent case (see Chapter 7).*

Set $S = ([0,T] \times \Omega, \overline{\mathcal{O}}, dU(t) \times P)$, where $\overline{\mathcal{O}}$ is the completion of $\mathcal{O}$ with respect to the measure $dU(t) \times P$. Given a Banach space X, let $L^2(S,X)$ denote the (Banach) space of X-valued, well-measurable processes $\{z_t : t \in [0,T]\}$, with the norm

$$\|z\|_{L^2(S,X)} = (E \int_0^T \|z_t\|_X dU(t))^{\frac{1}{2}} < \infty.$$

Definition 8.1.1. *An H-valued, $\mathcal{F}_t$-adapted continuous process $v = (v_t)_{t \in [0,T]}$ is a solution of (8.1.1), if $v \in L^2(S,V)$, there exists a process $\alpha \in L^2(S,V^*)$, such that, for $dU(t) \times P$-a.e. $(t,\omega) \in [0,T] \times \Omega$,*

$$\alpha_t(\omega) \in A(v(t,\omega)),$$

and, almost surely,

$$v_t = u_0 + \int_0^t \alpha_s dU(s) + \int_0^t B(v_s) dM(s),$$

for all $t \in [0,T]$.

[1] $\mathcal{O}$ denotes the σ-algebra of the well-measurable sets, i.e., $\mathcal{O}$ is generated by the sets $[s,t) \times A$, where $0 \leq s < t \leq T$, $A \in F_s$.

Our main result is

Theorem 8.1.1. *Under Assumptions 8.1.1-8.1.5, inclusion (8.1.1) has a unique solution u.*

Let us deal with the question of **uniqueness** first.

Lemma 8.1.1. *Inclusion (8.1.1) has, at most, one solution.*

Proof. Suppose u^1 and u^2 are solutions of (8.1.1). Then, by definition, there exist α^1, α^2, such that

$$u_t^1 = u_0 + \int_0^t \alpha_s^1 dU(s) + \int_0^t B(u_s^1) dM(s),$$

and

$$u_t^2 = u_0 + \int_0^t \alpha_s^2 dU(s) + \int_0^t B(u_s^2) dM(s).$$

Set $h(t) = \exp\{-2KU(t)\}|u_t^1 - u_t^2|^2$. Then, by Itô's formula from [7],

$$0 \leq h(t) \leq \int_0^t \exp\{-2KU(s)\} I(s) dU(s) + N(t),$$

where

$$I(s) = 2\langle u_s^1 - u_s^2, \alpha_s^1 - \alpha_s^2 \rangle + |B(u_s^1) - B(u_s^2)|_Q^2 - K|u_s^1 - u_s^2|^2$$

is non-positive (Assumption 8.1.1) and $N(t)$ is a continuous local martingale, starting at zero. It follows that, almost surely, $N(t) = 0$, for every $t \in [0, T]$, and the result now follows at once. □

In order to see that (8.1.1) has a solution, we investigate time-discretized approximations to the original problem, which are discussed in the next section.

8.2 Examples.

(1)(Poisson process) Suppose π_t is a Poisson process with parameter $\lambda > 0$. Recall that

Definition 8.2.1. *An adapted counting process π is a Poisson process if*

- *(i) For any $0 \leq s < t < \infty$, $\pi_t - \pi_s$ is independent of $\mathcal{F}_s$.*
- *(ii) For any $0 \leq s < t < \infty$ and $0 \leq u < v < \infty$, with $t - s = v - u$, the distribution of $\pi_t - \pi_s$ is the same as that of $\pi_v - \pi_u$.*
- *(iii) The random variable π_t has the Poisson distribution with parameter λt, $t \in \mathbb{R}^+$.*

Lemma 8.2.1. *Let π_t be a Poisson process with parameter $\lambda > 0$. Then $\pi_t - \lambda t$ and $(\pi_t - \lambda t)^2 - \lambda t$ are martingales.*

Proof. (i) Since the process $\pi_t - \lambda t$ has zero mean and independent increments, we have

$$E(\pi_t - \lambda t - (\pi_s - \lambda s) | \mathcal{F}_s) = E(\pi_t - \lambda t - (\pi_s - \lambda s)) = 0.$$

(ii) Similarly,

$$E((\pi_t - \lambda t)^2 - \lambda t - ((\pi_s - \lambda s)^2 - \lambda s) | \mathcal{F}_s) =$$

$$E(\pi_s^2 + 2\pi_s(\pi_t - \pi_s) + (\pi_t - \pi_s)^2 - 2\pi_s\lambda t - 2(\pi_t - \pi_s)\lambda t+$$
$$(\lambda t)^2 - \lambda t - (\pi_s^2 - 2\pi_s\lambda s + (\lambda s)^2 - \lambda s)|\mathcal{F}_s) =$$
$$(\lambda(t-s))^2 + \lambda(t-s) + 2\pi_s\lambda(t-s) - 2\pi_s\lambda t - 2\lambda t\lambda(t-s)+$$
$$(\lambda t)^2 - \lambda t + 2\pi_s\lambda s - (\lambda s)^2 + \lambda s = 0.$$

□

In light of this simple observation, one can write

$$\langle \pi - \lambda\cdot\rangle_t = \lambda t.$$

(2)(Wiener process) An argument similar to the above shows that if W_t is a (one-dimensional) Brownian motion, then $W_t^2 - t$ is a martingale and, thus, $\langle W\rangle_t = t$.
(3)(Levy processes with bounded jumps) More generally, if X_t is an arbitrary Levy process, with bounded jumps (i.e., if $\sup_t |\triangle X_t| \leq C < \infty$ a.s., where C is a non-random constant), then there exist constants $\alpha, \beta > 0$, such that $\langle X - \alpha\cdot\rangle_t = \beta t$. For the benefit of the reader we provide a sketch of a proof of this assertion (for details, see [18]).
We proceed in a sequence of steps.

- (*Step 1*). Fix Λ, a Borel set in $\mathbb{R}$ bounded away from zero (in other words, the closure of Λ does not contain the origin).
  For a given Levy process X, define a sequence $\{T_n\}$ of stopping times recursively as follows
  $$T_\Lambda^1 = \inf\{t > 0 : \triangle X_t \in \Lambda\},$$
  $$T_\Lambda^{n+1} = \inf\{t > T_\Lambda^n : \triangle X_t \in \Lambda\}.$$
  The cadlag paths of X imply that $\lim T_n = \infty$ a.s., since a finite accumulation point would contradict the fact that paths have left limits.

- (*Step 2*). Define N_t^Λ by
  $$N_t^\Lambda = \sum_{0<s\leq t} \chi_\Lambda(\triangle X_s) = \sum_{n=1}^{\infty} \chi_{\{T_n\leq t\}}$$
  and note that N_t^Λ is a counting process without an explosion. In fact, one can show that it is a Levy process with parameter $\nu(\Lambda) = E(N_1^\Lambda)$ $(\nu(\Lambda) < \infty)$.
  Moreover, the following holds true.

  Proposition 8.2.1. *The set function $\Lambda \to N_t^\Lambda$ defines a σ-finite measure on $\mathbb{R}/\{0\}$ for each fixed (t, ω). The set function $\nu(\Lambda) = E(N_1^\Lambda)$ also defines a σ-finite measure on $\mathbb{R}/\{0\}$.*

  Definition 8.2.2. *The measure ν, defined by*
  $$\nu(\Lambda) = E(N_1^\Lambda) = E(\sum_{0<s\leq 1} \chi_\Lambda(\triangle X_s)),$$
  is called the ***Levy measure*** *corresponding to the Levy process X.*

- (*Step 3*). We have

Lemma 8.2.2. *Suppose Λ is as above and let f be Borel and finite on Λ. Then*

$$\int_{\Lambda} f(x)N_t(dx) = \sum_{0<s\leq 1} f(\triangle X_s)\chi_{\Lambda}(\triangle X_s),$$

where for notational simplicity we have suppressed ω.

As a **Corollary** to the above result, we obtain

$$Y_t^{\Lambda} = \int_{\Lambda} f(x)N_t(dx)$$

is a Levy process in its own right.

- (*Step 4*).

Theorem 8.2.1. *Let Λ be a Borel set bounded away from the origin. Let ν be a Levy measure associated to process X and assume that f is Borel and finite on Λ. Then*

$$E(\int_{\Lambda} f(x)N_t(dx)) = t\int_{\Lambda} f(x)\nu(dx)$$

and

$$E\{(\int_{\Lambda} f(x)N_t(dx) - t\int_{\Lambda} f(x)\nu(dx))^2\} = t\int_{\Lambda} f(x)^2\nu(dx).$$

- (*Step 5*). We require two more results.

Proposition 8.2.2.

$$X_t - \int_{\Lambda} xN_t(dx)$$

is a Levy process.

Proposition 8.2.3. *If Λ_1 and Λ_2 are disjoint, then $\int_{\Lambda_1} xN_t(dx)$ and $\int_{\Lambda_2} xN_t(dx)$ are independent Levy processes.*

- (*Step 5*). Set $\Lambda = (-\infty, -1]\bigcup[1, \infty)$. Then, as we have previously remarked,

$$J_t^{\Lambda} = \int_{\Lambda} xN_t(dx) = \int_{|x|\geq 1} xN_t(dx)$$

is a Levy process. It can be thought of as that component of the jump process $\triangle X_t$, which is responsible for "large" jumps. Moreover,

$$Y_t = X_t - J_t^{\Lambda} = X_t - \int_{|x|\geq 1} xN_t(dx)$$

is a Levy process with bounded jumps, since $\sup |\triangle Y_t| \leq 1$.
It turns out that the following result holds.

Theorem 8.2.2. *Let X be a Levy process with jumps bounded by $a > 0$. We know that $E(|X_t|^n) < \infty$, for all $n \in \mathbb{N}$. Set $Z_t = X_t - E(X_t)$. Then Z_t is a martingale and $Z_t = Z_t^c + Z_t^d$, where Z_t^c is a martingale with continuous paths, Z_t^d - a martingale and Z_t^c and Z_t^d are independent Levy processes.*

Proof. (i) To begin with, since Z_t has independent increments and mean zero, it is a martingale (as well as a Levy process).
For a Borel set Λ (bounded away from the origin) define

$$M_t^\Lambda = \int_\Lambda x N_t(dx) - t\int_\Lambda x\nu(dx).$$

By the same argument, M_t^Λ is a martingale.
Moreover, set

$$\Lambda^n = [\frac{1}{n+1}, \frac{1}{n}),$$

$$M_t^n = \int_{\Lambda^n} x N_t(dx) - t\int_{\Lambda^n} x\nu(dx)$$

and

$$W_t^n = M_t^1 + \ldots + M_t^n.$$

Then

$$E(W_t^n) = E(M_t^1) + \ldots + E(M_t^n)$$

and

$$Var(W_t^n) = E((W_t^n)^2) = Var(M_t^1) + \ldots + Var(M_t^n),$$

by Proposition 8.2.3.
One can show that W_t^n and $Z_t - W_t^n$ are independent, which means that, for any n,

$$Var(W_t^n) \leq Var(W_t^n) + Var(Z_t - W_t^n) = Var(Z_t) < \infty.$$

We conclude that W_t^n is L^2-Cauchy and, thus, convergent to a process Z_t^d.
Moreover,

$$Var(Z_t^d) = E((Z_t^d)^2) = \lim E((W_t^n)^2) = \lim Var(W_t^n) =$$

$$\lim(Var(M_t^1) + \ldots + Var(M_t^n)) = \sum_k^\infty Var(M_t^K) =$$

$$\sum_k^\infty t\int_{[\frac{1}{k+1}, \frac{1}{k})} x^2\nu(dx) = t\int_{|x|<1} x^2\nu(dx),$$

where the last equality follows by monotone convergence.
(ii) The process Z_t^d is a martingale. Indeed,
a) Z_t^d is adapted. This follows from the fact that W_t^n (adapted) converges in L^2 and, therefore, in probability, which means that the sequence in question contains an a.e.-convergent subsequence.
b) $E|Z_t^d| < \infty$. Note that the p-norm ($p \geq 1$) is non-decreasing and that Z_t^d is square-integrable for each t by construction.
c) Recall that conditional expectation is L^p-norm preserving ($p \geq 1$), i.e.,

$$X \to Y \Rightarrow E(X|\mathcal{B}) \to E(Y|\mathcal{B}).$$

So, for $0 < s < t$,

$$E(W_t^n|\mathcal{F}_s) \to E(Z_t^d|\mathcal{F}_s).$$

On the other hand,

$$E(W_t^n|\mathcal{F}_s) = W_s^n \to Z_s^d,$$

Due to the fact that W_t^n is martingale.
Therefore, keeping in mind that the L^2 limit is unique,

$$E(Z_t^d|\mathcal{F}_s) = Z_s^d.$$

(iii) Let us show that $\langle Z^d\rangle_t = t\int_{|x|<1} x^2\nu(dx)$. We have

$$E((Z_t^d)^2 - t\int_{|x|<1} x^2\nu(dx)|\mathcal{F}_s) =$$

$$E((Z_s^d)^2+2Z_s^d(Z_t^d-Z_s^d)+(Z_t^d-Z_s^d)^2-s\int_{|x|<1} x^2\nu(dx)-(t-s)\int_{|x|<1} x^2\nu(dx)|\mathcal{F}_s) =$$

$$(Z_s^d)^2 - s\int_{|x|<1} x^2\nu(dx) + E((Z_t^d - Z_s^d)^2) - (t-s)\int_{|x|<1} x^2\nu(dx) =$$

$$(Z_s^d)^2 - s\int_{|x|<1} x^2\nu(dx),$$

due to independent increments and (*Step 4*).
(iv) One can also check that Z_t^c (which is an L^2-limit of $Z_t - W_t^n$) is a martingale with continuous paths and that Z_t^c and Z_t^d are independent (the latter property is a consequence of W_t^n and $Z_t - W_t^n$ being independent).
Z_t^c has continuous bracket process due to continuity of its paths, whereas Z_t^d - by (iii). Furthermore,

$$\langle Z^c + Z^d\rangle_t = \langle Z^c\rangle_t + \langle Z^d\rangle_t.$$

Indeed, let $\{T_n\}$ be an increasing sequence of stopping times, with

$$\lim T_n = \infty \text{ a.s. },$$

such that the stopped process $(Z^c_{t\wedge T_n})^2 - \langle Z^c\rangle_{t\wedge T_n}$ is a martingale for each n. Then

$$E((Z^c_{t\wedge T_n} + Z^d_{t\wedge T_n})^2 - \langle Z^c\rangle_{t\wedge T_n} - \langle Z^d\rangle_{t\wedge T_n}|\mathcal{F}_s) =$$

$$(Z^c_{s\wedge T_n})^2 - \langle Z^c\rangle_{s\wedge T_n} + (Z^d_{s\wedge T_n})^2 -$$

$$\langle Z^d\rangle_{s\wedge T_n} + 2E(Z^c_{t\wedge T_n} Z^d_{t\wedge T_n}|\mathcal{F}_s)$$

and

$$2E(Z^c_{t\wedge T_n} Z^d_{t\wedge T_n}|\mathcal{F}_s) =$$

$$E(Z^c_{s\wedge T_n} Z^d_{s\wedge T_n} + Z^c_{s\wedge T_n}(Z^d_{t\wedge T_n} - Z^d_{s\wedge T_n}) + Z^d_{s\wedge T_n}(Z^c_{t\wedge T_n} - Z^c_{s\wedge T_n}) +$$

$$(Z^c_{t\wedge T_n} - Z^c_{s\wedge T_n})(Z^d_{t\wedge T_n} - Z^d_{s\wedge T_n})|\mathcal{F}_s) = Z^c_{s\wedge T_n} Z^d_{s\wedge T_n},$$

using independence and the fact that Z^c and Z^d are martingales.
So, finally,

$$E((Z^c_{t\wedge T_n} + Z^d_{t\wedge T_n})^2 - \langle Z^c\rangle_{t\wedge T_n} - \langle Z^d\rangle_{t\wedge T_n}|\mathcal{F}_s) =$$

$$(Z^c_{s\wedge T_n} + Z^d_{s\wedge T_n})^2 - \langle Z^c\rangle_{s\wedge T_n} - \langle Z^d\rangle_{s\wedge T_n}.$$

□

(4) Suppose, besides inclusion (8.1.1), we want to consider an inclusion of the form

$$u_t \in u_0 + \int_0^t A(u_s)dU_s + \int_0^t B(u_s)dM_s$$

$$+ \int_0^t \int_Z C(u_s, z)q(ds, dz), \tag{8.2.1}$$

where q is a stochastic martingale measure, which we now proceed to define (consult [6]).
Let $(Z, \sum)$ be a measurable space and let $\{Z_n\}_{n=1}^{\infty}$ be an increasing sequence from $\sum$, such that $Z = \bigcup Z_n$ and let $\sum_n = \{\Gamma \in \sum : \Gamma \subset \sum_n\}$. We assume that $\sum$ is countably generated. For each n and every $\Gamma \in \sum_n$, let $q(\Gamma) = (q(t, \Gamma))_{t \geq 0}$ be a martingale belonging to $\mathcal{M}^2_{loc}(\mathbb{R})$ and let V be an increasing predictable cadlag (real-valued) process starting from zero.

Definition 8.2.3. *We say that $q(dt, dz)$ is a stochastic martingale measure, if $q(t, \Gamma)$ has the following properties:*

- *(i) For every $(t, \omega) \in \mathbb{R}_+ \times \Omega$, there exists a measure on $(Z, \sum)$ - denoted by ρ_t - such that ρ_t is finite on $\sum_n$, the processes $(\rho_t(\Gamma))_{t \geq 0}$ are predictable, for every $\Gamma \in \sum_n$, and*
$$\langle q(\Gamma) \rangle_t = \Pi(t, \Gamma) := \int_0^t \rho_s(\Gamma) dV_s < \infty,$$
for every $(t, \omega, \Gamma) \in \mathbb{R}_+ \times \Omega \times \sum_n$ (for each n).

- *(ii) For every $\Gamma_1, \Gamma_2 \in \sum_n$ (for each n),*
$$\langle q(\Gamma_1), q(\Gamma_2) \rangle_t = \Pi(t, \Gamma_1 \bigcap \Gamma_2).$$

For instance, q can be a Brownian or a Lévy sheet, defined next (the exposition follows closely that of [17]).
(i)(Brownian sheet) Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, P)$ be a filtered probability space and let H be a Hilbert space.

Definition 8.2.4. *An $(\mathcal{F}_t)$-adapted cylindrical Wiener process on H is a linear (in the second variable) mapping $W : [0, \infty) \times H \to L^2(\Omega, \mathcal{F}, P)$, satisfying the following conditions*

- *(i) For all $t \geq 0$ and $x \in H$, $E|W(t, x)|^2 = t|x|_H^2$,*
- *(ii) For each $x \in H$, $(W(t, x), t \geq 0)$ is a real-valued $(\mathcal{F}_t)$-adapted Wiener process.*

Assume that W is a cylindrical Wiener process on $L^2(\mathcal{O})$, where $\mathcal{O} := \{\eta = (\eta_1, \ldots, \eta_d) : \eta_j \in [0, a]\}$. For any $t \geq 0$ and $\eta \in \mathcal{O}$, define

$$\mathcal{W}(t, \eta_1, \ldots, \eta_d) := W(t, \chi_{[0, \eta_1] \times \ldots \times [0, \eta_d]}).$$

Then $\mathcal{W}$ is a Gaussian random field on $[0, \infty) \times [0, a] \ldots \times [0, a]$. Note that it is a real-valued Wiener process with respect to each parameter $t, \eta_1, \ldots, \eta_d$, when the others are fixed. The field $\mathcal{W}$ is usually called a *Brownian sheet.*
(ii)(Lévy sheet) For a possibly unbounded domain $\mathcal{O} \subset \mathbb{R}^d$, consider a Poisson random measure π on $[0, \infty) \times \mathcal{O} \times \mathbb{R}$, with intensity measure $dt\lambda(d\xi)\nu(d\sigma)$, where λ is a non-negative Radon measure on $\mathcal{O}$. Usually λ is the Lebesgue measure l_d. Let π be defined

on a probability space $(\Omega, \mathcal{F}, P)$, with a filtration $(\mathcal{F})_{t\geq 0}$. We assume that π is adapted to $(\mathcal{F}_t)$, that is, $\pi([0,t]\times\Gamma)$ is $(\mathcal{F}_t)$-measurable, for all $t\geq 0$ and $\Gamma\in\mathcal{B}(\mathcal{O}\times\mathbb{R})$. Finally, we assume that $\pi((s,t]\times\Gamma)$ is independent of $\mathcal{F}_s$, for all $0\leq s\leq t$ and $\Gamma\in\mathcal{B}(\mathcal{O}\times\mathbb{R})$. We denote by $\mathcal{P}$ the σ-field of predictable sets in $[0,\infty)\times\Omega$. Generally, ν may be an infinite measure, but we will assume that $\nu(\{0\})=0$ and that

$$\int_{\mathbb{R}} \sigma^2 \nu(d\sigma) < \infty.$$

Let $\widehat{\pi}$ be a compensated Poisson measure. We define, informally (for a formal definition, see [17]), a measure-valued process Z by

$$Z(t,d\xi) := \int_0^t \int_{\mathbb{R}} \sigma\widehat{\pi}(ds,d\xi,d\sigma),\ t\geq 0.$$

Given a compactly supported function ϕ, written $\phi\in C_c(\mathcal{O})$, we define the real-valued process

$$Z(t,\phi) = \int_{\mathcal{O}} \phi(\xi) Z(t,d\xi).$$

Then

Proposition 8.2.4. *(i) For each $\phi\in C_c(\mathcal{O})$, $(Z(t,\phi),t\geq 0)$ is a Lévy process with respect to $(\mathcal{F}_t)$.*
(ii) For all $t\geq 0$ and $\phi\in C_c(\mathcal{O})$, $E(Z(t,\phi))=0$ and

$$E|Z(t,\phi)|^2 = t\int_{\mathbb{R}} \sigma^2\nu(d\sigma)\int_{\mathcal{O}} |\phi(\xi)|^2 \lambda(d\xi).$$

Now let $\lambda = l_d$ be the Lebesgue measure on $\mathcal{O} := [0,a]^d$. For $t\geq 0$ and $\xi\in\mathcal{O}$, define

$$\mathcal{Z}(t,\xi_1,\ldots,\xi_d) := Z(t,\chi_{[0,\xi_1]\times\ldots\times[0,\xi_d]}).$$

One can show that, with respect to each parameter $t,\xi_1,\ldots,\xi_d$, with the others fixed, the process $\mathcal{Z}$ is, up to a multiplicative constant, a Lévy process with the same jump measure ν. This is why $\mathcal{Z}$ is called a *Lévy sheet.*
Following the same steps as in the construction of Itô's integral, one can demonstrate that that the stochastic integral

$$I_t(\varphi) := \int_0^t \int_Z \varphi(s,z) q(ds,dz)$$

is defined for every $\mathcal{P}\times\sum$-measurable function φ, satisfying the condition

$$\int_0^t \int_Z \varphi^2(s,z)\Pi(ds,dz) < \infty \text{ (a.s.) }.$$

Let us now show how $I_t(\varphi)$ may be rewritten as a stochastic integral with respect to an l_2-valued martingale. Let $\{\Gamma_i\}_i$ be a countable family of elements of $\sum$, such that it generates $\sum$ and $q_t(\Gamma_i)<\infty$, for each i and every t. For every fixed (s,ω), we denote by $L^2_{(s,\omega)}$ the Hilbert space of square integrable (with respect to $\rho_s(\omega,dz)$) functions on $(Z,\sum)$. Since, for every fixed (s,ω), the family of functions $\{\chi_{\Gamma_i}\}$ is total in $L^2_{(s,\omega)}$, we can obtain, by the Schmidt orthogonalization procedure (in each $L^2_{(s,\omega)}$), the functions $g^n_{s,\omega}(z)$ having the following properties:

- (i) for each fixed (s,ω), $\{g^n_{s,\omega}\}_n$ is an orthonormal basis in $L^2_{(s,\omega)}$.
- (ii) for every n, the functions $g^n : \mathbb{R}_+ \times \Omega \times Z \to \mathbb{R}$ are $\mathcal{P} \times \sum$-measurable.

For every n, let

$$h^n_t = n^{-1} \int_0^t \int_Z g^n_{s,\omega}(z) q(ds, dz).$$

It is known that $h^n_t \in \mathcal{M}^2_{loc}(\mathbb{R})$ and it is easy to see that $\langle h^n \rangle_t = n^{-2} V_t$. Hence it follows that $H_t = \sum_n h^n e_n \in \mathcal{M}^2_{loc}(l_2)$, where $\{e_n\}$ is an orthonormal basis in l_2. Let

$$\varphi^n_s = \int_Z \varphi(s,z) g^n_{s,\omega}(z) \rho_s(dz),$$

then, for every fixed (s,ω), the series $\sum_n \varphi^n g^n$ converges to φ (with respect to the $L^2_{s,w}$-norm). For every fixed (s,ω), let $\psi_s = \psi_s(\omega)$ be a linear operator in l^2, defined by $\psi_s e_n = n\varphi^n_s$. Then, for each (s,ω), $\psi_s \in L_Q(l^2, \mathbb{R}^d)$, where $Q \in L_1(H,H)$ is a non-negative operator, given by $Q^{ij}_s = \delta_{ij} i^{-2}$, and $L_Q(H, \mathbb{R}^d)$ denotes the set of all linear (not necessarily bounded) operators C, mapping $Q^{\frac{1}{2}}(H)$ into $\mathbb{R}^d$, such that $CQ^{\frac{1}{2}} \in L_2(H, \mathbb{R}^d)$. Now one can see that

$$\int_0^t \int_Z \varphi(s,z) q(ds,dz) = \int_0^t \psi_s dH_s.$$

Finally, let us note that the inclusion (8.2.1) can reduced to the inclusion (8.1.1) if we rewrite the stochastic integral with respect to $q(ds,dz)$ as a stochastic integral with respect to an l^2-valued ,artingale, and if we consider the Hilbert space $H \times l^2$ instead of H and the martingale (M_t, H_t) instead of M_t (see [6]).

8.3 Implicit time-discretization scheme.

Let

$$0 = \tau^n_0 \le \tau^n_1 \le \ldots \le \tau^n_{k_n+1} = T$$

be a random partition of the interval $[0,T]$, where $\{\tau^n_i\}_{i=0}^{k_n+1}$ is a sequence of stopping times, such that

$$\sup_{0 \le i \le k_n} (U(\tau^n_{i+1}) - U(\tau^n_i)) \to 0, \; n \to \infty, \tag{8.3.1}$$

uniformly in ω.

Example 8.3.1. *For instance, one can set*

$$\tau^n_0 := 0,$$

$$\tau^n_1 := \min\{\inf\{t > 0 : U(t) \ge \frac{1}{n}\}, T\},$$

$$\tau^n_{i+1} := \min\{\inf\{t > \tau^n_i : U(t) - U(\tau^n_i) \ge \frac{1}{n}\}, T\}, \; 1 \le i \le k_n,$$

$$\tau^n_{k_n+1} := T,$$

where $k_n = \lfloor Mn \rfloor$ *(see Lemma 23.0.2, Appendix I).*

Let $S \le S'$ be stopping times. By a **stochastic interval** $[[S, S'))$ we mean a subset of $[0, \infty] \times \Omega$, satisfying

$$[[S, S')) = \{(t, \omega) : [S(\omega) \le t < S'(\omega))\}.$$

Definition 8.3.1. *For $n \in \mathbb{N}$, a well-measurable process u^n is called a solution to a semi-implicit **time-discretization scheme**, if u^n is given by*

$$u^n(t, \omega) := u_i^n(\omega),\ (t, \omega) \in [[\tau_i, \tau_{i+1})),\ 0 \le i \le k_n,$$

$$u^n(T, \omega) := u_{k_n+1}^n(\omega),$$

with functions $u_i^n : \Omega \to V$ defined recursively as follows:

$$u_0^n := 0$$

$$u^n(\tau_1^n) \in u_0 + \triangle U_1^n A_1(u^n(\tau_1^n)),$$

$$u_{i+1}^n \in u_i^n + \triangle U_i^n A(u_{i+1}^n) + B(u_i^n) \triangle M_i^n,\ 1 \le 0 \le k_n,$$

where

$$\triangle U_i^n = U(\tau_{i+1}^n) - U(\tau_i^n) \text{ and } \triangle M_i^n = M(\tau_{i+1}^n) - M(\tau_i^n).$$

Theorem 8.3.1. *Under Assumptions 8.1.1-8.1.4, the **time-discretization scheme** has a unique solution for all large n.*

Proof. Note that, for a fixed $1 \le 0 \le k_n$, a typical inclusion problem

$$u_{i+1}^n \in u_i^n + \triangle U_i^n A(u_{i+1}^n) + B(u_i^n) \triangle M_i^n$$

can be rewritten as

$$u_{i+1}^n = u_i^n + B(u_i^n) \triangle M_i^n,\ \text{if } \omega \in \{\triangle U_i^n = 0\},$$

$$u_{i+1}^n \in (I - \triangle U_i^n A)^{-1}(u_i^n + B(u_i^n) \triangle M_i^n),\ \text{if } \omega \in \{\triangle U_i^n > 0\}.$$

(i)(Existence and uniqueness) In view of condition 8.3.1, it suffices to show that the operator $(I - \delta A)^{-1}$ is single-valued, for all small $\delta > 0$, which can be done in the same way as in Theorem 5.3.1.
(ii)(Measurability). Since u^n is cadlag, it is enough to check that the process in question is adapted (see [11]), where the latter assertion will follow at once, by Lemma 23.0.3, provided we can demonstrate that u_i^n is $\mathcal{F}_{\tau_i}$-measurable, $0 \le i \le k_n$. We proceed by induction.
$u_0^n = 0$ is, clearly, $\mathcal{F}_{\tau_0} = \mathcal{F}_0$-measurable. Assume further that u_i^n is $\mathcal{F}_{\tau_i}$-measurable. Recall that

$$u_{i+1}^n = (I - \triangle U_i^n A)^{-1}(u_i^n + B(u_i^n) \triangle M_i^n),\ \omega \in \{\triangle U_i^n > 0\},$$

and

$$u_{i+1}^n = u_i^n + B(u_i^n) \triangle M_i^n),\ \omega \in \{\triangle U_i^n = 0\}.$$

Note that both sets $\{\triangle U_i^n > 0\}$ and $\{\triangle U_i^n = 0\}$ belong to $\mathcal{F}_{\tau_{i+1}}$. Besides, $u_i^n + B(u_i^n(\omega)) \triangle M_i^n$ is $\mathcal{F}_{\tau_{i+1}}$-measurable, by inductive hypothesis. On the other hand, the function

$$(I - \triangle U_i^n A)^{-1}(u_i^n + B(u_i^n)\triangle M_i^n)$$

is $\mathcal{F}_{\tau_{i+1}}$-measurable, as a composition of measurable functions X and f, where $X : \Omega \to (0, \frac{1}{K}] \times V^*$ is given by

$$X(\omega) = (\delta(\omega), v^*(\omega)) = (\triangle U_i^n(\omega), u_i^n(\omega) + B(u_i^n(\omega))\triangle M_i^n(\omega)),$$

and the function $f : (0, \frac{1}{K}] \times V^* \to V$ is defined by

$$f(\delta, v^*) = (I - \delta A)^{-1}(v^*) = (I - \triangle U_i^n A)^{-1}(u_i^n + B(u_i^n)\triangle M_i^n)$$

(See Lemma 23.0.4, Remark 23.0.1 and Lemma 23.0.5, all Appendix I).
Finally, pasting together

$$u_{i+1}^n|_{\{\triangle U_i^n > 0\}} = (I - \triangle U_i^n A)^{-1}(u_i^n + B(u_i^n)\triangle M_i^n)$$

and

$$u_{i+1}^n|_{\{\triangle U_i^n = 0\}} = u_i^n + B(u_i^n)\triangle M_i^n,$$

we obtain a function - precisely, u_{i+1}^n, - which is $\mathcal{F}_{\tau_{i+1}^n}$-measurable, which completes the proof. □

8.4 Characterization of solutions.

For simplicity of exposition, we consider a Banach space Y, defined to be $L^2(S, V)$, with an equivalent norm

$$\|u\|_Y = (E\int_0^T \exp\{-2KU(t)\}\|u_t\|dU(t))^{\frac{1}{2}} < \infty, \ u \in Y.$$

Then, using the duality product

$$\langle u^*, u\rangle_Y = E\int_0^T \exp\{-2KU(t)\}\langle u_t^*, u_t\rangle dU(t),$$

one can identify the dual space Y^* with $L^2(S, V^*)$, with the corresponding norm

$$\|u^*\|_Y = (E\int_0^T \exp\{-2KU(t)\}\|u_t^*\|dU(t))^{\frac{1}{2}} < \infty, \ u^* \in Y^*.$$

Let U denote the space of triplets (ξ, a, b), satisfying the following conditions:

- $\xi : \Omega \to H$ is $\mathcal{F}_0$-measurable, with $E|\xi|^2 < \infty$;
- $a \in L^2(S, V^*)$;
- $bQ^{\frac{1}{2}} \in L^2(S, L^2(E, H))$;
- There exists a V-valued process $x \in Y$, such that

  $$x_t = \xi + \int_0^t a_s dU_s + \int_0^t b_s dM_s,$$

  for $dU(t) \times P$ - almost every $(t, \omega) \in [0, T] \times \Omega$.

Remark 8.4.1. *There is an H-valued continuous modification of x, denoted x^H, such that, almost surely,*

$$x_t^H = \xi + \int_0^t a_s dU(s) + \int_0^t b_s dM(s), \ \forall t \in [0,T].$$

For fixed $(\xi, a, b) \in U$ and $y \in Y$, set

$$I_y(\xi, a, b) = E|u_0 - \xi|^2 +$$

$$E \int_0^T \exp\{-2KU(t)\}\{2\langle x_t - y_t, (a_t - KI(x_t)) - y_t^*\rangle + |b_t - B(y_t)|_Q^2\} dU(t),$$

where $y^* \in W(y)$ and the operator $W : Y \to Y^*$ is defined by

$$W(v) = \{v^* \in Y^* : v^*(t,w) \in (A - KI)(u(t,w)), \ dU(t) \times P\text{-a.e. } (t,\omega) \in [0,T] \times \Omega\}.$$

Theorem 8.4.1. *The operator W is "maximal monotone" and $D(W) = Y$ (see Appendix C.2).*

The next result reveals a connection between solutions of (8.1.1) and the above functional.

Theorem 8.4.2. *Assume conditions 8.1.1-8.1.1. Suppose there exists a triplet $(\xi, a, b) \in U$, such that, $\forall y \in Y$,*

$$I_y(\xi, a, b) \leq 0.$$

Then $\xi = u_0$, and

$$u_t^H = u_0 + \int_0^t a_s dU(s) + \int_0^t b_s dM(s), \ t \in [0,T],$$

is a solution of (8.1.1).

Proof. Suppose the triplet (ξ, a, b) satisfies assumptions of the theorem. Setting $y(\cdot) = u(\cdot)$ (which we are justified in doing, since, by Theorem 8.4.1, W is defined on the whole of Y), we get

$$E|\xi - u_0|^2 = 0$$

and

$$E \int_0^T \exp\{-2KU(t)\}|b_t - B(y_t)|_Q^2 dU(t) = 0,$$

which imply that $\xi = u_0$, almost surely, and $b_t Q_t^{\frac{1}{2}} = B(y_t) Q_t^{\frac{1}{2}}$, for $dU(t) \times P$-a.e. $(t,\omega) \in [0,T] \times \Omega$.
The condition $I_y(\xi, a, b) \leq 0$ can now be rewritten as

$$\langle u - y, (a - KI(u)) - y^* \rangle_Y =$$

$$E \int_0^T \exp\{-2KU(t)\}\langle u_t - y_t, (a_t - KI(u_t)) - y_t^* \rangle dU(t) \leq 0.$$

Keeping in mind the definition of a "maximal monotone" operator, we deduce that $a - KI(u) \in W(u)$, or

$$a_t \in A(u_t),$$

for $dU(t) \times P$-almost every $(t,\omega) \in [0,T] \times \Omega$. □

8.5 Convergence of the implicit scheme.

Recall that at the heart of the time-discretization scheme lies the following system of inclusions:

$$u^n(\tau_0^n) = 0,$$
$$u^n(\tau_1^n) \in u_0 + A_1(u^n(\tau_1^n))\triangle U_1^n,$$
$$u^n(\tau_{i+1}^n) \in u^n(\tau_i^n) + A_{i+1}(u^n(\tau_{i+1}^n))\triangle U_i^n + B(u^n(\tau_i^n))\triangle M_i^n.$$

We have seen previously that such a scheme has a unique solution for all large n, which means that there exist $\alpha^n(\tau_i^n) \in A_i(u^n(\tau_i^n))$, $1 \leq i \leq k_n$, such that the above can be rewritten as a system of equalities:

$$u^n(\tau_0^n) = 0,$$

$$u^n(\tau_1^n) = u_0 + \alpha^n(\tau_1^n)\triangle U_1^n,$$
$$u^n(\tau_{i+1}^n) = u^n(\tau_i^n) + \alpha^n(\tau_{i+1}^n)\triangle U_i^n + B(u^n(\tau_i^n))\triangle M_i^n.$$

Using induction on time intervals, it is easy to show that u_t^n assumes the following integral form[2] :

$$u_t^n = u_0\chi_{\{t\geq t_1\}} + \int_0^{\tau_k^n} \alpha^n(\kappa_2(s))dU(s) + \int_0^{\tau_k^n} \widetilde{B}(u^n(\kappa_1(s)))dM(t),$$

where k is such that $(t,\omega) \in [[\tau_k^n, \tau_{k+1}^n))$, and $\widetilde{B}^j(u^n(\kappa_1(s))) = B^j(u^n(\kappa_1(s)))$, $t_1 \leq t \leq T$, and $\widetilde{B}^j(u^n(\kappa_1(s))) = 0$, $0 \leq t < t_1$.
It turns out that the following result holds.

Lemma 8.5.1. *There exist positive constants L_1, L_2, L_3 and L_4, such that, under assumptions 8.1.1-8.1.5, we have*

$$\sup_{t\in[0,T]} E|u^n(t)|^2 \leq L_1, \tag{8.5.1}$$

$$E\int_0^T \|u^n(\kappa_2(t))\|^2 dU(t) \leq L_2, \tag{8.5.2}$$

$$E\int_0^T \|\alpha^n(\kappa_2(t))\|^2 dU(t) \leq L_3, \tag{8.5.3}$$

$$E\int_0^T |\widetilde{B}(u^n(\kappa_1(t)))|_Q^2 dU(t) \leq L_4, \tag{8.5.4}$$

for all large n.

Proof. To begin with,

$$u_{i+1}^n = u_i^n + \alpha_{i+1}^n \triangle U_i^n + B(u_i^n)\triangle M_i^n,$$

which leads to

$$|u_{i+1}^n|^2 - |u_i^n|^2 = 2\langle u_{i+1}^n, \alpha_{i+1}^n\rangle \triangle U_i^n +$$
$$2(u_i^n, B(u_i^n)\triangle M_i^n) + |B(u_i^n)\triangle M_i^n|^2 - |\alpha_{i+1}^n \triangle U_i^n|^2.$$

[2] $\kappa_1(t) = \tau_i$ and $\kappa_2(t) = \tau_{i+1}$ for $t \in ((\tau_i, \tau_{i+1}))$; $\kappa_1(\tau_i) = \kappa_2(\tau_i) = \tau_i$.

Set $\psi_0 = 1$ and $\psi_i = \Pi_{j=0}^{i-1}(1 - K\triangle U_j^n)$, $i = 1, \ldots, k_n + 1$. Then, clearly, ψ_i(s) are positive (at least, for large n), decreasing in i and $\mathcal{F}_{\tau_i^n}$-measurable.
Using identity

$$\psi_{i+1}C_{i+1} - \psi_i C_i = \psi_i(C_{i+1} - C_i) + C_{i+1}(\psi_{i+1} - \psi_i)$$

one can write

$$\psi_{i+1}|u_{i+1}^n|^2 - \psi_i|u_i^n|^2 \leq \psi_i\{2\langle u_{i+1}^n, \alpha_{i+1}^n\rangle\triangle U_i^n + |B(u_i^n)\triangle M_i^n|^2\}+$$

$$2\psi_i(u_i^n, B(u_i^n)\triangle M_i^n) + |u_{i+1}^n|^2(\psi_{i+1} - \psi_i).$$

Taking expectations and summing up to $i = k - 1$, we obtain

$$E\psi_k|u_k^n|^2 - E|u_0|^2 \leq E\sum_{i=0}^{k-1} 2\psi_i\langle u_{i+1}^n, \alpha_{i+1}^n\rangle\triangle U_i^n +$$

$$E\sum_{i=1}^{k-1}\psi_i|B(u_i^n)|_Q^2\triangle U_i^n + E\sum_{i=0}^{k-1}|u_{i+1}^n|^2(\psi_{i+1} - \psi_i) \leq$$

$$E\sum_{i=0}^{k-1}\psi_i\{2\langle u_{i+1}^n, \alpha_{i+1}^n\rangle + |B(u_{i+1}^n)|_Q^2\}\triangle U_i^n +$$

$$E\sum_{i=0}^{k-1}|u_{i+1}^n|^2(\psi_{i+1} - \psi_i).$$

Making use of "coercivity" condition 8.1.3, one gets

$$E\psi_k|u_k^n|^2 - E|u_0|^2 \leq E\sum_{i=0}^{k-1}\psi_i\{K(1 + |u_{i+1}^n|^2) - \lambda\|u_{i+1}^n\|^2\}\triangle U_i^n +$$

$$E\sum_{i=0}^{k-1}|u_{i+1}^n|^2(\psi_{i+1} - \psi_i),$$

or, after simplification,

$$E\psi_k|u_k^n|^2 + \lambda E\sum_{i=0}^{k-1}\psi_i\|u_{i+1}^n\|^2\triangle U_i^n \leq C_n + E\sum_{i=0}^{k-1}|u_{i+1}^n|^2 d_{i+1},$$

where

$$C_n = E|u_0|^2 + E\sum_{i=0}^{k-1}\psi_i K\triangle U_i^n$$

and

$$d_{i+1} = \psi_{i+1} - \psi_i(1 - K\triangle U_i^n).$$

Taking into account that ψ_i(s) are bounded above by 1, we conclude that $C_n \leq E|u_0|^2 + KE(U(T)) \leq E|u_0|^2 + KM$.
Moreover, as is easy to check, $d_{i+1} = 0$ $(i = 0, \ldots)$.

Therefore we have

$$E\psi_k|u_k^n|^2 + \lambda E\sum_{i=0}^{k-1}\psi_i\|u_{i+1}^n\|^2\triangle U_i^n \le E|u_0|^2 + KM. \tag{8.5.5}$$

Let us take a closer look at $\psi_{k_n+1} = \Pi_{i=1}^{k_n}(1 - K\triangle U_i^n)$, which can be written in exponential form as

$$\psi_{k_n+1} = \exp\{\sum_{i=0}^{k_n}\ln(1 - K\triangle U_i^n)\}.$$

Take $0 < \epsilon < 1$. Since $\sup_{0\le i\le k_n}\triangle U_i^n$ tends to zero, uniformly in ω, we have $K\triangle U_i^n < \epsilon$, for all large n and $0 \le i \le k_n$.
Now, using inequalities

$$\frac{x}{1+x} \le \ln(1+x) \le x,\ \ x > -1,$$

or, equivalently,

$$\exp\{\frac{x}{1+x}\} \le 1 + x \le \exp\{x\},$$

one can write

$$\exp\{\frac{-KM}{1-\epsilon}\} \le \exp\{\frac{-KU(T)}{1-\epsilon}\} \le \psi_{k_n+1} = \Pi_{i=1}^{k_n}(1 - K\triangle U_i^n) \le \exp\{-KU(T)\}.$$

Since ψ_i is non-increasing in i, by construction, the above argument implies

$$0 < C \le \psi_{k_n+1} \le \psi_i,\ \ 0 \le i \le k_n.$$

This simple observation, together with inequality (8.5.5), yield estimates (8.5.1) and (8.5.2).
An application of " linear growth" condition 8.1.4 and Remark 8.1.1 establish estimates (8.5.3) and (8.5.4). □

Recall that

$$u_{i+1}^n = u_i^n + \alpha_{i+1}^n\triangle U_i^n + B(u_i^n)\triangle M_i^n,$$

or

$$u_{i+1}^n - K\triangle U_i^n I(u_{i+1}^n) = u_i^n + (\alpha_{i+1}^n - KI(u_{i+1}^n))\triangle U_i^n + B(u_i^n)\triangle M_i^n.$$

Consequently,

$$(1 - K\triangle U_i^n)^2|u_{i+1}^n|^2 - |u_i^n|^2 \le 2(1 - K\triangle U_i^n)\langle u_{i+1}^n, \alpha_{i+1}^n\rangle\triangle U_i^n +$$

$$2(u_i^n, B(u_i^n)\triangle M_i^n) + |B(u_i^n)\triangle M_i^n|^2.$$

Multiply both sides by ψ_i^2, take expectations and sum up to $i = k_n$ to obtain

$$E\psi_{k_n}^2|u_{k_n}^n|^2 - E|u_0|^2 \le$$

$$E\sum_{i=0}^{k_n}2\psi_i^2(1 - K\triangle U_i^n)\langle u_{i+1}^n, \alpha_{i+1}^n - KI(u_{i+1}^n)\rangle\triangle U_i^n + E\sum_{i=1}^{k_n}\psi_i^2|B(u_i^n)|_Q^2\triangle U_i^n \le$$

$$E\sum_{i=0}^{k_n} 2\psi_i^2(1-K\triangle U_i^n)\langle u_{i+1}^n, \alpha_{i+1}^n - KI(u_{i+1}^n)\rangle \triangle U_i^n +$$

$$E\sum_{i=0}^{k_n} \psi_{i+1}^2 |B(u_{i+1}^n)|_Q^2 \triangle U_i^n. \tag{8.5.6}$$

This inequality will be required shortly.
The above estimates imply that there exists a subsequence, denoted again by u^n, such that

$$u^n(T) \rightharpoonup u^\infty(T) \text{ in } L^2((\Omega, F_T, P), H),$$
$$u^n(\kappa_2(\cdot)) \rightharpoonup v_\infty \text{ in } Y,$$
$$\alpha^n(\kappa_2(\cdot)) \rightharpoonup \alpha_\infty \text{ in } Y^*,$$
$$\widetilde{B}(u^n(\kappa_1(\cdot)))Q^{\frac{1}{2}} \rightharpoonup b_\infty Q^{\frac{1}{2}} \text{ in } L^2(S, L^2(E,H)).$$

The **existence** part of Theorem 8.1.1 is addressed by the next result.

Theorem 8.5.1. *(i) For $dU(t)\times P$-almost all $(t,\omega)\in[0,T]\times\Omega$,*

$$v_\infty(t) = u_0 + \int_0^t a_\infty(s)dU(s) + \int_0^t b_\infty(s)dM(s). \tag{8.5.7}$$

Moreover, almost surely,

$$u^\infty(T) = u_0 + \int_0^T a_\infty(s)dU(s) + \int_0^T b_\infty(s)dM(s). \tag{8.5.8}$$

(ii) For every $y \in Y$,

$$I_y(u_0, a_\infty, b_\infty) \le 0, \tag{8.5.9}$$

and the function v_∞^H is a solution of (8.1.1).

(iii) The sequence $u^n(T)$ converges strongly to $v_\infty^H(T)$ in $L^2((\Omega, F_T, P), H)$.

Proof. (i) For a fixed $N\ge 1$, let $\phi = \{\phi(t) : t\in[0,T]\}$ be a V-valued adapted stochastic process such that

$$\|\phi(t,\omega)\| \le N \text{ for every } t\in[0,T] \text{ and } \omega\in\Omega.$$

Since

$$u^m(t) = u_0\chi_{\{t\ge\tau_1^m\}} + \int_0^{\tau_k^m} \alpha^m(\kappa_2(s))dU(s) + \int_0^{\tau_k^m} \widetilde{B}(u^m(\kappa_1(s)))dM(s),$$

it follows that

$$E\int_0^T (u^m(t), \phi(t))dU(t) = E\int_0^T \chi_{\{t\ge\tau_1^m\}}(u_0,\phi(t))dU(t) +$$

$$E\int_0^T \langle \int_0^{\tau_k^m} \alpha^m(\kappa_2(s))dU_s, \phi(t)\rangle dU(t) +$$

$$E\int_0^T (\int_0^{\tau_k^n} \widetilde{B}(u^m(\kappa_1(s)))dM(s), \phi(t))dU(t) =$$

$$E\int_0^T (u_0,\phi(t))dU(t)+J_1+J_2-R_1+R_2-R_3, \tag{8.5.10}$$

where

$$J_1=E\int_0^T\langle\int_0^t \alpha^m(\kappa_2(s))dU(s),\phi(t)\rangle dU(t),$$

$$J_2=E\int_0^T(\int_0^t \widetilde{B}(u^m(\kappa_1(s)))dM(s),\phi(t))dU(t),$$

$$R_1=E\int_0^{\tau_1^m}(u_0,\phi(t))dU(t),$$

$$R_2=E\int_0^T\langle\int_{\tau_k^m}^t \alpha^m(\kappa_2(s))dU(s),\phi(t)\rangle dU(t),$$

$$R_3=E\int_0^T(\int_{\tau_k^m}^t \widetilde{B}(u^m(\kappa_2(s)))dM(s),\phi(t))dU(t).$$

One can easily check that

$$|R_1|,|R_2|,|R_3|\to 0,$$

and

$$J_1\to E\int_0^T\langle\int_0^t \alpha_\infty(s)dU(s),\phi(t)\rangle dU(t),$$

$$J_2\to E\int_0^T(\int_0^t b_\infty(s)dM(s),\phi(t))dU(t).$$

Letting $m\to+\infty$ in equation (8.5.10), we obtain

$$E\int_0^T(v_\infty(t),\phi(t))dU(t)=E\int_0^T(u_0,\phi(t))dU(t)+$$

$$E\int_0^T\langle\int_0^t\alpha_\infty(s)dU(s),\phi(t)\rangle dU(t)+E\int_0^T(\int_0^t b_\infty(s)dM(s),\phi(t))dU(t),$$

from which it follows that

$$v_\infty(t)=u_0+\int_0^t\alpha_\infty(s)dU(s)+\int_0^t b_\infty(s)dM(s),$$

for $dU(t)\times P$-almost every $(t,\omega)\in[0,T]\times\Omega$, which is (8.5.7).
Repeating the above argument with $\psi\in L^2(\Omega,V)$, that satisfies $E\|\psi\|\le N$, for a given $N>0$, we obtain (8.5.8).
(ii) Clearly, $v_\infty^H(T)=u^\infty(T)$ (a.s.). Moreover, it can be shown that the process v_∞^H satisfies

$$E\exp\{-2KU(T)\}|v_\infty^H(T)|^2=E|u_0|^2+E\int_0^T\exp\{-2KU(t)\}\{2\langle v_\infty(t),a_\infty(t)\rangle-$$

$$-2K|v_\infty(t)|^2+|b_\infty(t)|_Q^2\}dU(t). \tag{8.5.11}$$

Take arbitrary $y\in Y$, $y^*\in W(y)$ and set

$$I_y^n=$$

$$E\int_0^T \exp\{-2KU(t)\}\{2\langle u^n(\kappa_2(t)) - y(t), (\alpha^n(\kappa_2(t)) - KI(u^n(\kappa_2(t)))) - y^*(t)\rangle +$$

$$|B(u^n(\kappa_2(t))) - B(y(t))|_Q^2\}dU(t) =$$

$$2E\int_0^T \exp\{-2KU(t)\}\langle u^n(\kappa_2(t)), \alpha^n(\kappa_2(t)) - KI(u^n(\kappa_2(t)))\rangle dU(t)+$$

$$2E\int_0^T \exp\{-2KU(t)\}\langle y(t), y^*(t)\rangle dU(t)-$$

$$2E\int_0^T \exp\{-2KU(t)\}\langle u^n(\kappa_2(t)), y^*(t)\rangle dU(t)-$$

$$2E\int_0^T \exp\{-2KU(t)\}\langle y(t), \alpha^n(\kappa_2(t)) - KI(u^n(\kappa_2(t)))\rangle dU(t)+$$

$$E\int_0^T \exp\{-2KU(t)\}|B(u^n(\kappa_2(t)))|_Q^2 dU(t)-$$

$$2E\int_0^T \exp\{-2KU(t)\}(B(u^n(\kappa_2(t)))Q^{\frac{1}{2}}, B(y(t))Q^{\frac{1}{2}})dU(t)+$$

$$E\int_0^T \exp\{-2KU(t)\}|B(y(t))|_Q^2 dU(t), \tag{8.5.12}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(S, L^2(E,H))$.
By (8.5.6),

$$E\psi_{k_n}^2|u_{k_n}^n|^2 - E|u_0|^2 \leq$$

$$E\sum_{i=0}^{k_n} 2\psi_i^2(1-K\triangle U_i^n)\langle u_{i+1}^n, \alpha_{i+1}^n - KI(u_{i+1}^n)\rangle \triangle U_i^n +$$

$$E\sum_{i=0}^{k_n} \psi_{i+1}^2 |B(u_{i+1}^n)|_Q^2 \triangle U_i^n.$$

Substituting the above expression into (8.5.12) and using the fact that $I_y^n \leq 0$ (by "monotonicity" condition 8.1.1), we get

$$0 \geq I_y^n \geq$$

$$E\psi_{k_n}^2|u^n(T)|^2 - E|u_0|^2 + L_1 + L_2 -$$

$$L_3 - L_4 - L_5 + 2E\int_0^T \exp\{-2KU(t)\}\langle y(t), y^*(t)\rangle dU(t)+$$

$$E\int_0^T \exp\{-2KU_t\}|B(y(t))|_Q^2 dU(t). \tag{8.5.13}$$

where

$$L_1 = 2E\int_0^T [\exp\{-2KU(t)\} - \psi_i^2(1-K\triangle U_i^n)]\langle u^n(\kappa_2(t)), \alpha^n(\kappa_2(t)) - KI(u^n(\kappa_2(t)))\rangle dU(t),$$

$$L_2 = E\int_0^T [\exp\{-2KU(t)\} - \psi_{i+1}^2]|B(u^n(\kappa_2(t)))|_Q^2 dU(t),$$

$$L_3 = E\int_0^T \exp\{-2KU(t)\}\langle u^n(\kappa_2(t)), y^*(t)\rangle dU(t),$$

$$L_4 = E\int_0^T \exp\{-2KU(t)\}\langle y(t), \alpha^n(\kappa_2(t)) - KI(u^n(\kappa_2(t)))\rangle dU(t),$$

$$L_5 = E\int_0^T \exp\{-2KU(t)\}(B(u^n(\kappa_2(t)))Q^{\frac{1}{2}}, B(y(t))Q^{\frac{1}{2}})dU(t).$$

Clearly,

$$L_1, L_2 \to 0,$$

$$L_3 \to \int_0^T \exp\{-2KU(t)\}\langle v_\infty(t), y^*(t)\rangle dU(t),$$

$$L_4 \to \int_0^T \exp\{-2KU(t)\}\langle y(t), a_\infty(t) - KI(v_\infty(t))\rangle dU(t),$$

and

$$L_5 \to E\int_0^T \exp\{-2KU(t)\}(b_\infty Q^{\frac{1}{2}}, B(y(t))Q^{\frac{1}{2}})dU(t),$$

as $n \to \infty$.
One can write

$$\liminf \psi_n^2 E(|u^n(T)|^2) = d + E(\exp\{-2KU(T)\}|v_\infty^H(T)|^2), \tag{8.5.14}$$

where $d \geq 0$. Letting $n \to \infty$ in equation (8.5.13) and using (8.5.11), we obtain

$$\begin{aligned} 0 \geq{} & d + E(\exp\{-2KU(T)\}|v_\infty^H(T)|^2) - E|u_0|^2 - \\ & 2\int_0^T \exp\{-2KU(t)\}\langle v_\infty(t), y^*(t)\rangle dU(t) - \\ & 2\int_0^T \exp\{-2KU(t)\}\langle y(t), a_\infty(t) - KI(v_\infty(t))\rangle dU(t) + \\ & 2\int_0^T \exp\{-2KU(t)\}\langle y(t), y^*(t)\rangle dU(t) + \\ & E\int_0^T \exp\{-2KU(t)\}|B(y(t))|_Q^2 dU(t) - \\ & 2E\int_0^T \exp\{-2KU(t)\}(b_\infty Q^{\frac{1}{2}}, B(y(t))Q^{\frac{1}{2}})dU(t) = \\ & d + I_y(u_0, a_\infty, b_\infty). \end{aligned} \tag{8.5.15}$$

Consequently, $I(y) \leq 0$. Taking into consideration that $y \in Y$ is arbitrary, it now suffices to quote Theorem 8.4.2 to arrive at the desired conclusion.
(iii) It follows from (8.5.15) and (8.5.14) that $d = 0$ and

$$\lim E(|u^n(T)|^2) = E(|u^\infty(T)|^2),$$

which, combined with the facts that $u^n(T) \rightharpoonup u^\infty(T)$ and $u^\infty(T) = v_\infty^H(T)$, a.s., imply that the sequence $u^n(T)$ converges strongly to $v_\infty^H(T)$ in $L^2((\Omega, F_T, P), H)$.

□

Chapter 9

Appendix A.

Proposition 9.0.1. ***Convergence principle***. *Suppose $\{u_n\}$ is a sequence in a (Hausdorff) topological space S, which has the following **Property**:*

- *every subsequence $\{u_{n_k}\}$ of the original sequence contains a subsequence, which converges to the (same) limit u_0.*

Then, u_n converges to u_0.

Proof. Recall that $\{u_n\}$ is said to converge to u_0 in S if, for any $U(u_0)$ - a neighborhood of u_0 -, $\exists N$, such that $\forall n > N$,

$$u_n \in U(u_0).$$

Assume the contrary, i.e., $\exists V(u_0)$, s.t. $\forall N, \exists n > N$, for which

$$u_n \not\subseteq V(u_0).$$

Therefore we can choose a subsequence $\{u_{n(k)}\}$, which lies entirely outside $V(u_0)$, and, as a consequence, cannot have a subsequence converging to u_0. A contradiction. □

Taking into account that every subsequence of a convergent sequence u_n in a (Hausdorff) topological space has the same limit, we note that **convergence** and the above **Property** are equivalent.

Definition 9.0.1. *A topological space (S, σ) is called a C_1-space if every element of S has a countable base of neighborhoods ([12]).*

Proposition 9.0.2. *Let $f : (S, \sigma) \to (T, \tau)$ be a mapping between topological spaces. If (S, σ) is a C_1-space, then the following statements are equivalent:*

(i) f is continuous at u_0;

(ii) for any $\{u_n\}$, $u_n \to u_0$,

$$f(u_n) \to f(u_0).$$

Definition 9.0.2. *A **boundary** of a subset A of a topological space (S, σ), denoted by δA, is defined by*

$$\delta A = \{x \in S : \forall U(x), U(x) \cap A \neq \emptyset \text{ and } U(x) \cap A^c \neq \emptyset\},$$

where $U(x)$ denotes a neighborhood of x.

Definition 9.0.3. *A set A, together with its boundary, is called a* ***closure*** *of A, and denoted by $\bar{A}$.*

The following result is a triviality.

Lemma 9.0.2. *Suppose (S, σ) is a C_1-space and let $A \subseteq S$. Then,*

$$x \in \bar{A} \text{ iff } \exists \{x_n\} \in A, \text{ s.t. } x_n \to x.$$

Definition 9.0.4. *Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces. A mapping $f : X \to Y$ is called $\mathcal{F}/\mathcal{G}$-measurable, if*

$$f^{-1}(B) \in \mathcal{F}, \ \forall B \in \mathcal{G}.$$

Let us state without proof two elementary results.

Lemma 9.0.3. *$\mathcal{F}' = \{A \in \mathcal{F} : A = f^{-1}(B), B \in \mathcal{G}\}$ is a σ-algebra.*

Lemma 9.0.4. *$\mathcal{G}' = \{B \in \mathcal{G} : f^{-1}(B) \in \mathcal{F}\}$ is a σ-algebra.*

Using Lemmas 9.0.3-9.0.4, one can easily establish the following result.

Proposition 9.0.3. *A function $f : (X, \mathcal{F}) \to (Y, \sigma(G))$ is $\mathcal{F}/\sigma(G)$-measurable if, and only if,*

$$f^{-1}(B) \in \mathcal{F}, \ \forall B \in G,$$

where, as usual, $\sigma(G)$ denotes the σ-algebra, generated by elements of G.

Definition 9.0.5. *Let (S, d) be a metric space. By Borel σ-algebra on S, denoted by $\mathcal{B}(S)$, we mean a σ-algebra generated by a collection of open subsets of S.*

Bearing in mind that a complement of an open set is closed, and that a σ-algebra is "closed" under comlementation, we conclude that Borel σ-algebra can, alternatively, be viewed as the family of subsets of S generated by closed subsets of S. Moreover,

Proposition 9.0.4. *If S is separable, then*

$$\mathcal{B}(S) = \sigma\{\text{open balls in } S\}.$$

Proof. It suffices to show that an arbitrary open (proper) subset V of S can be written as a countable union of open balls. So take any $x \in V$. Since V is open, $\exists \epsilon > 0$, s.t. an open ball

$$B(x, \epsilon) \subseteq V.$$

Set $\delta(x) = \sup\{\epsilon > 0 : B(x, \epsilon) \subseteq V\}$. Clearly, $\delta(x)$ is finite (due to the fact that V is a proper subset of S). Besides, one can easily verify that

$$B(x, \delta(x)) \subseteq V.$$

Let $C = \{x_1, x_2, \ldots\}$ be a countable dense subset of S, and denote by $C_V = \{x_{n_1}, x_{n_2}, \ldots\}$ its restriction to V.
For an arbitrary $x \in V$, there exists $x_{n_k} \in C_V$, such that $d(x, x_{n_k}) < \frac{\delta(x)}{2}$. This means that $B(x_{n_k}, \frac{\delta(x)}{2}) \subseteq B(x, \delta(x))$. Then we obtain the following sequence of inclusions

$$x \in B(x_{n_k}, \frac{\delta(x)}{2}) \subseteq B(x_{n_k}, \delta(x_{n_k})) \subseteq V,$$

from which it follows that

$$V = \bigcup_k B(x_{n_k}, \delta(x_{n_k})),$$

as required. $\square$

Likewise,

Lemma 9.0.5. *Provided S is separable, $\mathcal{B}(S)$ is generated by closed balls.*

Lemma 9.0.6. *Let X be a separable normed linear space and M its subset. Then M, equipped with a subspace (norm) topology, is also separable.*

Proof. Recall that in a normed linear space (n.l.s.) every point x has a countable base of neighborhoods:

$$\mathcal{B}(x) = \{B(x, \frac{1}{n}), n \in \mathbb{N}\}.$$

It is easy to see that

$$\mathcal{B}'(x) = \{B'(x, \frac{1}{n}), n \in \mathbb{N}\},$$

where

$$B'(x, \frac{1}{n}) = \{y \in M : \|x - y\| < \frac{1}{n}\},$$

is a base of neighborhoods for each $x \in M$.
In order to show that M is separable, we need to produce a countable subset $D = \{y_1, y_2, \ldots\}$ of M, such that each neighborhood of the form $B'(x, \frac{1}{n})$ contains an element of D.
To begin with, suppose $D' = \{x_1, x_2, \ldots\}$ is a dense subset of X. Then

$$C = \{B(x_i, \frac{1}{k}) : i, k \in \mathbb{N}\}$$

is a countable collection of neighborhoods. Although in each particular case, $B(x_i, \frac{1}{k}) \cap M$ may be an empty set, not all such intersections are empty. Therefore, if we choose a single member of M from every non-empty intersection, then we will have obtained at most a countably infinite subset D of M. To show that D dense in M, take an arbitrary $x \in M$ and $n \in \mathbb{N}$. Since D' is dense in X, $\exists x_i \in X$, s.t.

$$x_i \in B(x, \frac{1}{2n}).$$

By the choice of elements of D, $\exists y_j \in M$, s.t.

$$y_j \in B(x_i, \frac{1}{2n}).$$

It follows that

$$\|y_j - x\| \leq \|y_j - x_i\| + \|x_i - x\| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

So,

$$y_j \in B'(x, \frac{1}{n}).$$

$\square$

Proposition 9.0.5. *If V^*, the dual of a real Banach space V, is separable, then V is separable as well ([13]).*

Proof. Let $D = \{f \in V^* : \|f\| = 1\}$. Then, by Lemma 9.0.6, D is separable. Let $D' = \{f_1, f_2, \ldots\}$ be a dense subset of D.
Recall that

$$\|f\| = \sup_{\|x\|=1} |f(x)|.$$

Therefore, there exists a sequence of elements of V, $\{x_n\}$, s.t.

$$|f_n(x_n)| \geq \frac{1}{2} \text{ and } \|x_n\| = 1, \forall n \in \mathbb{N}.$$

Define $M = \{r_1 x_1 + \ldots + r_k x_k : \ r_1, \ldots, r_1 \in \mathbb{Q}, \ k \in \mathbb{N}\}$ and $M' = Span\{x_1, \ldots, x_n, \ldots\}$.
Clearly, $M \subseteq M'$ and is countable.
The claim is that $\bar{M}' = V$.
If this was not the case, then there would exist $x \in (\bar{M}')^c$ and $f \in D$, satisfying

$$f_{|\bar{M}'} = 0 \text{ and } f(x) \neq 0.$$

Then

$$\frac{1}{2} \leq |f_n(x_n)| = |f_n(x_n) - f(x_n)| \leq \|f_n - f\| \|x_n\| = \|f_n - f\|, \forall n \in \mathbb{N},$$

which is a contradiction, as D' is dense.
Finally, it is clear that

$$M \subseteq M' \subseteq \bar{M}.$$

Therefore,

$$\bar{M} = \bar{M}' = V.$$

□

Let us state an easy, but often useful fact about weak convergence.

Lemma 9.0.7. *Suppose V is a real reflexive Banach space and $x_n \rightharpoonup x$ in V. Then*

$$\|x\|_V \leq \liminf \|x_n\|_V.$$

Proof. Take arbitrary $f \in V^*$. From the definition of weak convergence

$$\langle f, x \rangle_V = \lim \langle f, x_n \rangle_V,$$

which implies

$$|\langle f, x \rangle_V| = \lim |\langle f, x_n \rangle_V| \leq \liminf \|f\|_{V^*} \cdot \|x_n\|_V = \|f\|_{V^*} \liminf \|x_n\|_V,$$

and the desired result follows upon recalling that, in a reflexive Banach space V,

$$\|v\|_V = \sup_{\substack{v^* \in V^* \\ v^* \neq 0}} \frac{|\langle v^*, v \rangle_V|}{\|v^*\|_{V^*}}.$$

□

We proceed with a definition.

Definition 9.0.6. *Let U be a metric space and V a Banach space. A function $f : U \to V$ is called* ***demicontinuous****, if*

$$u_n \text{ converges (strongly) to } u \text{ in } U$$

implies

$$f(u_n) \text{ converges (weakly) to } f(u) \text{ in } V,$$

in other words, if

$$u_n \to u \text{ in } U \Rightarrow f(u_n) \rightharpoonup f(u) \text{ in } V.$$

The main result of this section is

Proposition 9.0.6. *Let U be a separable metric space and V - a real separable reflexive Banach space. Moreover, assume that a function $f : U \to V$ is* ***demicontinuous****. Then f is $\mathcal{B}(U)/\mathcal{B}(V)$-measurable.*

Proof. Since V is separable, one can characterize the Borel σ-algebra $\mathcal{B}(V)$ as a family of subsets of V generated by closed balls (Lemma 9.0.5). Then, by Lemma 9.0.3, in order to show that f is measurable, it suffices to check that, for any closed ball $\overline{B(v,r)}$ in V,

$$f^{-1}(\overline{B(v,r)}) \in \mathcal{B}(U).$$

We shall do this by demonstrating that $f^{-1}(\overline{B(v,r)})$ is a closed subset of U.
Assume the contrary: $f^{-1}(\overline{B(v,r)})$ is not closed. In other words, by Lemma 9.0.2, there exist $\{u_n\} \in f^{-1}(\overline{B(v,r)})$ and $u \in (f^{-1}(\overline{B(v,r)}))^c$, such that $u_n \to u$.
Since f is **demicontinuous**, $f(u_n) \rightharpoonup f(u)$, and $f(u_n) - v \rightharpoonup f(u) - v$. Now, by the choice of u_n, $\{f(u_n) - v\} \in \overline{B(0,r)}$. Making use of Lemma 9.0.7, one can write

$$\|f(u) - v\|_V \leq \liminf \|f(u_n) - v\|_V \leq r.$$

It follows that, $f(u) - v \in \overline{B(0,r)}$, and, thus, $f(u) \in \overline{B(v,r)}$. A contradiction. □

Chapter 10

Appendix B.1.

In this section we collect a few basic results, pertaining to monotone operators.

Theorem 10.0.2. *If $A : V \to V^*$ is a monotone hemicontinuous operator on the real, reflexive Banach space V, then it is maximal monotone.*

Proof. Proposition 32.7 of [22]. □

Theorem 10.0.3. *Suppose mappings $A : V \supseteq D(A) \to 2^{V^*}$ and $B : V \supseteq D(B) \to 2^{V^*}$, defined on the real reflexive Banach space V, are maximal monotone and let*

$$D(A) \cap interior D(B) \neq \emptyset.$$

Then, the sum $A + B : V \to 2^{V^*}$ is also maximal monotone.

Proof. Theorem 32.I of [22]. □

Definition 10.0.7. *An operator $A : V \to 2^{V^*}$ is* ***coercive*** *if*

$$\frac{\inf_{u^* \in Au} \langle u^*, u \rangle}{\|u\|} \to +\infty, \text{ as } \|u\| \to +\infty, \ u \in V.$$

Moreover, A is ***weakly coercive*** *if*

$$\inf_{u^* \in Au} \|u^*\| \to +\infty, \text{ as } \|u\| \to +\infty, \ u \in V.$$

Remark 10.0.1. *Note that*

$$\langle u^*, u \rangle \leq \|u^*\| \cdot \|u\|.$$

Therefore,

$$\frac{\inf_{u^* \in Au} \langle u^*, u \rangle}{\|u\|} \leq \inf_{u^* \in Au} \|u^*\|,$$

which means that ***coercive*** *implies* ***weakly coercive****.*

Theorem 10.0.4. *Let $A : V \to 2^{V^*}$ be maximal monotone and weakly coercive mapping on the real reflexive Banach space V. Then,*

$$R(A) = V^*.$$

Proof. Theorem 32.H of [22]. □

Let $\delta > 0$ and set $B_\delta = I - \delta A$. We have seen before (Theorem 3.3.1) that under certain assumptions on the operator A (e.g., "coercivity" condition), the inclusion

$$(I - \delta A)(u) \ni u^*$$

has a unique solution $u \in V$, for every $u^* \in V^*$. To put it differently, the operator B_δ^{-1} is defined on the whole of V^* and is single-valued. Moreover, the following result holds.

Lemma 10.0.8. *The operator $B_\delta^{-1} : V^* \to V$ is demicontinuous for all small δ.*

Proof. Suppose $v_n^* \to v^*$ in V^*. Our aim is to show that $v_n := B_\delta^{-1}(v_n^*) \rightharpoonup v := B_\delta^{-1}(v^*)$. To see this, we check that every subsequence $\{v_{n(i)}\}$ has a weakly convergent subsequence, with the limit v.
Take any subsequence $\{v_{n(i)}^*\}_{i=1}^{\infty}$. It is convergent and, therefore, bounded. Recall (see Theorem 3.3.1) that

$$\delta\lambda\|v\|^2 + |v|^2(2 - \delta K_1) - \delta K_1 \leq$$
$$2\langle v^*, v\rangle \leq 2\|v^*\| \cdot \|v\| \leq$$
$$\frac{\|v^*\|^2}{\epsilon} + \epsilon\|v\|^2, \ \epsilon > 0,$$

for every $v \in V$ and $v^* \in (I - \delta A)(v) = B_\delta(v)$.
Grouping identical terms, we obtain

$$(\delta\lambda - \epsilon)\|v\|^2 + (2 - \delta K_1)|v|^2 \leq \frac{\|v^*\|^2}{\epsilon} + \delta K_1.$$

Choosing ϵ and δ (latter first, then the former) so small, that both $(\delta\lambda - \epsilon)$ and $(2 - \delta K_1)$ are greater than zero, we observe that the sequence $\{v_{n(i)}\}$ is also bounded, which means that it contains a weakly convergent subsequence, for notational simplicity denoted again by $v_{n(i)}$, with a limit $\tilde{v} \in V$.
For any $w \in V$ and $w^* \in B_\delta(w)$, we have

$$\langle v_{n(i)}^* - w^*, v_{n(i)} - w\rangle \geq 0,$$

since B_δ is monotone (as the sum of two monotone operators). So,

$$\langle v_{n(i)}^* - v^*, v_{n(i)} - w\rangle + \langle v^* - w^*, v_{n(i)} - w\rangle \geq 0.$$

As far as the first term is concerned, then

$$|\langle v_{n(i)}^* - v^*, v_{n(i)} - w\rangle| \leq \|v_{n(i)}^* - v^*\| \cdot \|v_{n(i)} - w\| \leq$$
$$M\|v_{n(i)}^* - v^*\| \to 0,$$

as $i \to +\infty$ (where we have used the boundedness of $v_{n(i)}$).
Moreover,

$$\lim_{i \to +\infty} \langle v^* - w^*, v_{n(i)} - w\rangle = \langle v^* - w^*, \tilde{v} - w\rangle.$$

Combining these results, we obtain

$$\lim_{i \to +\infty} \langle v_{n(i)}^* - w^*, v_{n(i)} - w\rangle = \langle v^* - w^*, \tilde{v} - w\rangle \geq 0,$$

$\forall\ w \in V$ and $w^* \in B_\delta(w)$.
Keeping in mind that B_δ is maximal monotone, we deduce that $v \in B_\delta(\tilde{v})$, or $v =$

$\tilde{v} = B_\delta^{-1}(v)$ (due to the fact that B_δ^{-1} is a single-valued operator). Finally, by the Convergence principle (Proposition 9.0.1, Appendix A),

$$v_n \rightharpoonup v,$$

as desired. □

Remark 10.0.2. *We have seen previously (Proposition 9.0.6, Appendix A) that "demi-continuous" implies "measurable". Hence if $f : (\Omega, \mathcal{F}) \to (V^*, \mathcal{B}(V^*))$ is measurable, then, by the above result, so is*

$$B_{\delta_n}^{-1} \circ f : (\Omega, \mathcal{F}) \to (V, \mathcal{B}(V)),$$

as a composition of two measurable mappings.

Lemma 10.0.9. *The process u^n is predictable.*

Proof. Since the process u^n is left-continuous with right limits, by construction, it is enough to check that u_i^n is $\mathcal{F}_{t_i}$-measurable ($i = 0, 1, \dots, n$). We proceed by mathematical induction.
u_0^n is, clearly, $\mathcal{F}_0$-measurable. Assume that u_i^n is $\mathcal{F}_{t_i}$-measurable. Then, since

$$u^n(t_{i+1}) = (I - \delta_n A)^{-1}(u^n(t_i)) = B_{\delta_n}^{-1}(u^n(t_i)),$$

and using the above result, we infer that u_{i+1}^n is $\mathcal{F}_{t_{i+1}}$-measurable. □

Chapter 11

Appendix B.2. Local boundedness of monotone maps.

The aim of this section is to illuminate one fundamental property of monotone maps, namely, local boundedness.
As before, let V denote a real reflexive Banach space and V^* its dual. To begin with,

Definition 11.0.8. *A monotone map $A : V \supseteq D(A) \to 2^{V^*}$ is said to be **locally bounded** at $x \in D(A)$, if there are constants $M > 0$ and $r > 0$, such that*

$$\|y^*\| \leq M, \ \forall y \in D(A) \bigcap \overline{B}(x,r), \ y^* \in A(y),$$

where $\overline{B}(x,r) = \{y \in V : \|y - x\| \leq r\}$.

Moreover,

Definition 11.0.9. *If $C \subseteq V$ is a nonempty set, a point $x \in C$ is an **absorbing point** of C, if the set $C - x$ is absorbing, i.e., $V = \bigcup_{\lambda>0} \lambda(C - x)$.*

Let us note in passing that every interior point of C is, clearly, absorbing.
Our main result is

Proposition 11.0.7. *If $A : V \supseteq D(A) \to 2^{V^*}$ is monotone and $x \in D(A)$ is an absorbing point of $D(A)$, then A is locally bounded at x ([5]).*

Proof. We may assume that $x = 0$ and $0 \in A(0)$ (i.e., $(0,0) \in A$). Otherwise, we can consider a map

$$A_1(y) = A(y + x) - x^*,$$

evidently, monotone, with $D(A_1) = D(A) - x$, which meets the above requirements.
Therefore what remains to do is demonstrate that the operator A is locally bounded at 0.
Define a function $\varphi : V \to [0, +\infty]$ by

$$\varphi(u) = \sup_{y \in D(A), \|y\| \leq 1, y^* \in A(y)} \langle y^*, u - y \rangle.$$

Then φ is convex and lower-semicontinuous.
As far as the first assertion is concerned, given arbitrary $(y, y^*) \in A$, $\|y\| \leq 1$, and $\alpha \in (0,1)$, we have

$$\langle y^*, \alpha u_1 + (1-\alpha)u_2 - y \rangle = \langle y^*, \alpha u_1 + (1-\alpha)u_2 - \alpha y - (1-\alpha)y \rangle =$$

$$\alpha\langle y^*, u_1 - y\rangle + (1-\alpha)\langle y^*, u_2 - y\rangle \leq \alpha\varphi(u_1) + (1-\alpha)\varphi(u_2),$$

and the result follows at once, upon taking the supremum of the LHS.
For the second, assume the contrary. In other words, for some $\epsilon > 0$, $u_0 \in V$ and $u_n \to u_0$

$$\varphi(u_n) \leq \varphi(u_0) - \epsilon, \text{ for every } n \in \mathbb{N}.$$

By the definition of φ,

$$\langle y^*, u_n - y\rangle \leq \varphi(u_0) - \epsilon, \text{ for all } n \in \mathbb{N} \text{ and } (y, y^*) \in A.$$

Since there exists a pair $(y_0, y_0^*) \in A$, with $\|y_0\| \leq 1$, such that

$$\langle y_0^*, u_0 - y_0\rangle \geq \varphi(u_0) - \frac{\epsilon}{2},$$

we conclude that $\forall n \in \mathbb{N}$,

$$\langle y_0^*, u_n - y_0\rangle \leq \langle y_0^*, u_0 - y_0\rangle - \frac{\epsilon}{2},$$

which contradicts the fact that

$$\lim\langle y_0^*, u_n - y_0\rangle = \langle y_0^*, u_0 - y_0\rangle.$$

Set $C = \{u \in V : \varphi(u) \leq 1\}$. Then C is a convex, closed subset of V, containing the origin.
i) Take any $u_1, u_2 \in C$ and $\alpha \in (0, 1)$. Using the fact that φ is convex, we observe that

$$\varphi(\alpha u_1 + (1-\alpha)u_2) \leq \alpha\varphi(u_1) + (1-\alpha)\varphi(u_2) \leq \alpha + (1-\alpha) = 1.$$

ii) Due to the fact that V is a C_1-space, it suffices to show that $u_n \to u_0$ and $u_n \in C$, $n = 1, 2, \ldots$, together imply that $u_0 \in C$, which is, in fact, an immediate consequence of the lower-semicontinuity of φ, since

$$\varphi(u_0) \leq \liminf \varphi(u_n) \leq 1.$$

iii) By monotonicity of A and the assumption that $(0, 0) \in A$,

$$\langle y^*, y\rangle = \langle y^* - 0, y - 0\rangle \geq 0, \text{ for any } (y, y^*) \in A.$$

So

$$\varphi(0) = \sup\langle y^*, 0 - y\rangle \leq 0,$$

and, since we can always set $y = 0$, $y^* = 0$, we deduce that

$$\varphi(0) = 0.$$

We proceed by showing that C is an absorbing set. Take $u \in V$. We have assumed that $D(A)$ is absorbing ($x = 0$ is an absorbing point!), which means that we can find $\lambda > 0$, such that $\lambda u \in D(A)$, that is, the set $A(\lambda u)$ is nonempty. Let $v^* \in A(\lambda u)$. If $(y, y^*) \in A$, by monotonicity of A, we get

$$\langle v^* - y^*, \lambda u - y\rangle \geq 0,$$

or

$$\langle y^*, \lambda u - y\rangle \leq \langle v^*, \lambda u - y\rangle.$$

It follows that

$$\varphi(\lambda u) \leq \sup_{y\in D(A), \|y\|\leq 1} \langle v^*, \lambda u - y\rangle \leq \langle v^*, \lambda u\rangle + \|v^*\| < +\infty.$$

Choose $t \in (0,1)$ in such a way that $t\varphi(\lambda u) < 1$. Due to the fact that φ is convex, we have

$$\varphi(t\lambda u) = \varphi(t\lambda u + (1-t)0) \leq t\varphi(\lambda u) + (1-t)\varphi(0) = t\varphi(\lambda u) < 1.$$

This proves that C is absorbing.

Let $E = (C)\bigcap(-C)$. Then E is clearly closed, convex and symmetric. Furthermore, it is an absorbing set and a neighborhood of the origin.

i) Since C is absorbing, for an arbitrary $u \in V$, there exist $\lambda_1, \lambda_2 > 0$, such that

$$\lambda_1 u \in C \text{ and } \lambda_2(-u) \in C \text{ (or, equivalently, } \lambda_2 u \in -C.)$$

Setting $\lambda = \min\{\lambda_1, \lambda_2\}$ (> 0) and using the fact that φ is convex, we note that

$$\lambda u \in C \text{ and } \lambda u \in -C.$$

Thus E is an absorbing set.

ii) According to the definition, $V = \bigcup_{\lambda>0} \lambda E$. It turns out that we can replace $\bigcup_{\lambda>0}$ with the countable union as follows

$$V = \bigcup_n nE.$$

To see this, take $u \in V$. By above, $\frac{u}{\lambda} \in E$, for some $\lambda > 0$. Choose any $n > \lambda$. Since E contains a line segment, with 0 and $\frac{u}{\lambda}$ as its end points, we deduce that $\frac{u}{n} \in E$. Therefore V can be written as a countable union of closed sets, which, by the Baire Category Theorem, means that for some $u_0 \in V$, $r > 0$ and $n_0 \in \mathbb{N}$,

$$B(u_0, r) \in n_0 E.$$

One can check that

$$B(\frac{u_0}{n_0}, \frac{r}{n_0}) \in E.$$

Since E is symmetric, we also have

$$B(-\frac{u_0}{n_0}, \frac{r}{n_0}) \in E.$$

Finally, keeping in mind that E is convex, one has

$$B(0, \frac{r}{n_0}) \in E.$$

Using the fact that E is a neighborhood of the origin, one can find $\delta > 0$, such that

$$\varphi(u) \leq 1, \ \forall \|u\| \leq 2\delta.$$

This means that

$$\langle y^*, u\rangle \leq 1 + \langle y^*, y\rangle,$$

$\forall y \in D(A),\ \|y\| \leq 1,\ y^* \in A(y),\ \|u\| \leq 2\delta$. Hence, if $y \in D(A) \bigcap \overline{B_\delta}(0)$ and $y^* \in A(y)$, then

$$2\delta\|y^*\| = \sup_{\|u\| \leq 2\delta} \langle y^*, u\rangle \leq 1 + \|y^*\| \leq 1 + \delta\|y^*\|,$$

and so $\|y^*\| \leq \frac{1}{\delta}$. □

Chapter 12

Appendix C.1.

In this section we introduce the notion of the duality mapping, and collect all the relevant (for our discussion) facts about it.
Let V be a real reflexive separable Banach space, and V^* its dual.

Definition 12.0.10. *The duality map J is a function from V, with values in V^*, given by*

$$J(u) = \{u^* \in V^* : \langle u, u^* \rangle = \|u\|^2 = \|u^*\|^2\},\ u \in V.$$

Let us note that, $\forall u \in V$, $J(u)$ is non-empty, which is a consequence of the Hahn-Banach Extension Theorem. Moreover, the operator J has the following useful properties.

Lemma 12.0.10. *Suppose V^* is strictly convex and reflexive. Then J is single-valued, injective and demicontinuous. Moreover,*

$$\langle J(u_n) - J(u), u_n - u \rangle \to 0 \Rightarrow u_n \rightharpoonup u.$$

Proof. (i)(Single-valued) Suppose, for some $u \in V$, there exist u_1^* and u_2^* (in V^*), such that

$$\langle u, u_i^* \rangle = \|u\|^2 = \|u_i^*\|^2,\ i = 1, 2.$$

Then

$$\|u\|^2 = \langle u, \frac{u_1^* + u_2^*}{2} \rangle \leq \|u\| \cdot \|\frac{u_1^* + u_2^*}{2}\| \leq \|u\| \cdot \|u_1^*\| = \|u\| \cdot \|u_2^*\| = \|u\|^2,$$

from which it follows that

$$\|u_1^*\| = \|u_2^*\| = \|\frac{u_1^* + u_2^*}{2}\|,$$

or $u_1^* = u_2^*$ (Lemma 13.2.1).
(ii)(Monotone) By the Cauchy-Schwartz inequality,

$$\langle J(u) - J(v), u - v \rangle = \|u\|^2 + \|v\|^2 - \langle J(u), v \rangle - \langle J(v), u \rangle \geq$$

$$\|u\|^2 + \|v\|^2 - 2\|u\| \cdot \|v\| = (\|u\| - \|v\|)^2 \geq 0.$$

(iii) (Strictly monotone). Indeed,

$$\langle J(u) - J(v), u - v \rangle = 2\langle J(u) - J(\frac{u+v}{2}), \frac{u-v}{2} \rangle + 2\langle J(\frac{u+v}{2}) - J(v), \frac{u-v}{2} \rangle \geq$$

$$2(\|u\| - \|\frac{u+v}{2}\|)^2 + 2(\|v\| - \|\frac{u+v}{2}\|)^2 = 0$$

if, and only if,

$$\|u\| = \|v\| = \|\frac{u+v}{2}\|, \text{ or } u = v.$$

It also follows from the above result that J is injective.
(iv)(Demicontinuous) Suppose $u_n \to u$. Since

$$\langle J(u_n), u_n \rangle = \|u_n\|^2 = \|J(u_n)\|^2,$$

we note that

$$u_n \to u \Rightarrow \|u_n\| \to \|u\| \Rightarrow \|J(u_n)\| \to \|J(u)\|.$$

From boundedness of $\{\|u_n\|\}$, we infer that the sequence $\{\|J(u_n)\|\}$ is also bounded. Keeping in mind that V^* is reflexive, $\exists\{n(i)\}$, such that $J(u_{n(i)}) \rightharpoonup v^*$, for some $v^* \in V^*$. Now, for any $v \in V$,

$$\langle v^*, v \rangle = \lim \langle J(u_{n(i)}), v \rangle \leq \lim \|u_{n(i)}\| \cdot \|v\| = \|u\| \cdot \|v\|.$$

So, $\|v^*\| \leq \|u\|$. On the other hand,

$$\langle v^*, u \rangle = \lim \langle J(u_{n(i)}), u_{n(i)} \rangle = \lim \|u_{n(i)}\|^2 = \|u\|^2.$$

Therefore, $\|v^*\| = \|u\|$. Finally, since

$$\langle v^*, u \rangle = \|u\|^2 = \|v^*\|^2,$$

using the definition of the duality map and the fact that the latter is single-valued (by (i)), we conclude that $v^* = J(u)$, which, by Lemma 9.0.1 (Appendix B.1.), implies that $J(u_n) \rightharpoonup J(u)$, as desired.
(v) Suppose $\langle J(u_n) - J(u), u_n - u \rangle \to 0$. Since, as we have seen in (ii),

$$\langle J(u_n) - J(u), u_n - u \rangle \geq (\|u_n\| - \|u\|)^2 \geq 0,$$

$\|u_n\| \to \|u\|$. Consequently, there exists a subsequence $\{n(i)\}$, such that

$$u_{n(i)} \rightharpoonup u' \text{ and } J(u_{n(i)}) \rightharpoonup u^*.$$

Note that, for any $v^* \in V^*$,

$$\langle v^*, u' \rangle = \lim \langle v^*, u_{n(i)} \rangle \leq \lim \|v^*\| \cdot \|u_{n(i)}\| = \|v^*\| \cdot \|u\|.$$

Hence,

$$\|u'\| \leq \|u\|. \tag{12.0.1}$$

Likewise,

$$\|u^*\| \leq \|u\|. \tag{12.0.2}$$

On the other hand,

$$\langle J(u_n) - J(u), u_n - u \rangle = \|u_n\|^2 + \|u\|^2 - \langle J(u), u_n \rangle - \langle J(u_n), u \rangle \to$$

$$2\|u\|^2 + \langle J(u), u' \rangle + \langle u^*, u \rangle = 0.$$

Thus

$$\langle J(u), u' \rangle = \|u\|^2 \text{ and } \langle u^*, u \rangle = \|u\|^2. \tag{12.0.3}$$

Comparing (12.0.1) and (12.0.2) with (12.0.3), we deduce that

$$\|u'\| = \|u\| \text{ and } \|u^*\| = \|u\|.$$

Now, since

$$\langle J(u), u' \rangle = \|u\|^2 = \|u'\|^2 = \|J(u)\|^2,$$

we get $J(u) = J(u')$, from which, due to the fact that J is injective, it follows that $u = u'$.

Finally, because the above argument can be repeated with any subsequence of the original sequence, we conclude that

$$u_n \rightharpoonup u.$$

□

Let $A : V \to 2^{V^*}$ be a "monotone" map. Then the following holds true.

Theorem 12.0.5. *Suppose V is a real reflexive Banach space, such that V and V^* are strictly convex. Then the "monotone" (monotone) map A is "maximal monotone" if, and only if,*

$$R(-A + J) = V^* \ (R(A + J) = V^*),$$

where J denotes the duality mapping.

Proof. Theorem 32.F of [22]. □

It turns out that the condition "V and V^* are strictly convex" is not too restrictive.

Proposition 12.0.8. *In every reflexive Banach space V, an equivalent norm can be introduced so that V and V^* are locally uniformly convex and thus also strictly convex, with respect to the new norms on V and V^*.*

Proof. Proposition 32.23 of [22]. □

Proposition 12.0.9. *Suppose V is a real reflexive Banach space, and V and V^* are strictly convex. Let the mapping $A : V \to 2^{V^*}$ be "maximal monotone". Then the inverse operator*

$$(-A + J)^{-1} : V^* \to V$$

is single-valued and demicontinuous.

Proof. To begin with, if A is "maximal monotone", then $R(-A + J) = V^*$ (Theorem 12.0.5). This means that the operator $(-A + J)^{-1}$ is defined on the whole of V^*.

(i)(Single-valued) Suppose the operator in question is not single-valued. Then there exist distinct $v_1, v_2 \in V$ and $v_1^* \in A(v_1), v_2^* \in A(v_2)$, such that, for some $v^* \in V^*$,

$$v^* = -v_1^* + J(v_1) = -v_2^* + J(v_2).$$

Since A is "monotone", we have

$$\langle v_2^* - v_1^*, v_2 - v_1 \rangle \leq 0,$$

which implies

$$0 \le (\|v_2\| - \|v_1\|)^2 \le \|v_2\|^2 + \|v_1\|^2 - \langle J(v_1), v_2\rangle - \langle J(v_2), v_1\rangle =$$

$$\langle J(v_2) - J(v_1), v_2 - v_1\rangle \le 0.$$

Therefore,

$$\|v_2\| = \|v_1\| \text{ and } \langle J(v_1), v_2\rangle = \|v_2\| \cdot \|v_1\|.$$

Now

$$\langle J(v_1), v_2\rangle = \|v_2\| \cdot \|v_1\| = \|v_2\|^2 = \|J(v_1)\|^2.$$

Hence, $J(v_1) = J(v_1)$, and, using injectivity of J (part (iii) of Lemma 12.0.10), $v_1 = v_1$. A contradiction.
(ii)(Demicontinuous) Suppose

$$v_n^* = -u_n^* + J(u_n), u_n^* \in A(u_n), \to v^* = -u^* + J(u) \text{ in } V^*.$$

The operator $(-A + J)^{-1}$ is monotone and, hence, locally bounded at v^* (Proposition 11.0.7, Appendix B.2). For us it means that the set $\{\|u\|, \|u_n\|, n = 1, 2, \ldots\}$ is bounded. It follows that

$$\langle v_n^* - v, u_n - u\rangle = \langle(-u_n^* + J(u_n)) - (-u^* + J(u)), u_n - u\rangle =$$

$$\langle -(u_n^* - u^*), u_n - u\rangle + \langle J(u_n) - J(u), u_n - u\rangle \to 0.$$

Since both terms are non-negative, we deduce that

$$\langle J(u_n) - J(u), u_n - u\rangle \to 0,$$

which implies, by part (v) of Lemma 12.0.10, that $u_n \rightharpoonup u$. □

Chapter 13

Appendix C.2.

13.1 Introduction.

We are going to demonstrate the validity of a somewhat more general result.
Suppose $S = (\Lambda, \mathcal{F}, \mu)$ is a complete finite measure space, V is a real, reflexive, separable Banach space and V and V^* are strictly convex. Let $X = L^2(S, V)$ denote a (Banach) space of V-valued $\mathcal{F}/\mathcal{B}(V)$-measurable functions $x = \{x(s) : s \in \Lambda\}$, with the norm

$$\|x\|_X = (\int_\Lambda \|x(s)\|^2 d\mu)^{\frac{1}{2}} < \infty.$$

Then the dual space X^* can be identified with $L^2(S, V^*)$, by means of the duality product

$$\langle x, x^* \rangle_X = \int_\Lambda \langle x(s), x^*(s) \rangle_V d\mu,$$

where $x \in X$ and $x^* \in X^*$.
We define the operator $W : X \to 2^{X^*}$ by

$$W(v) = \{v^* \in X^* : v^*(s) \in A(v(s)), d\mu - \text{a.e. } s \in \Lambda\},$$

and claim that W is "maximal monotone" and $D(W) = X$.
We shall proceed in a sequence of stages. First, we verify that if V and V^* are strictly convex, then X and X^* are also strictly convex. Next, we show that the operator $W + J'$, where J' denotes the duality mapping from X into X^*, is surjective. The latter fact, coupled with a simple observation that W is monotone, will lead us to conclude, by Theorem 12.0.5, Appendix C.1., that the operator W is "maximal monotone". Finally, we check that W is defined on the whole of X.
The details are as follows.

13.2 Strictly convex.

We begin with a definition of what it means for a normed linear space to be strictly convex and derive an alternative characterization of the above property, which lands itself more easily to verification.

Definition 13.2.1. *A real B-space V is said to be strictly convex, if for all (distinct) $x, y \in V$, such that $\|x\| = \|y\| = 1$, and any $\alpha \in (0, 1)$*

$$\|\alpha x + (1 - \alpha)y\| < 1. \tag{13.2.1}$$

It turns out that

Lemma 13.2.1. *A Banach space V is strictly convex iff it has the following* ***Property:***

$$\|\alpha x + (1-\alpha)y\| < \alpha\|x\| + (1-\alpha)\|y\|, \tag{13.2.2}$$

for any $x, y \in V$ (not multiples of each other) and $\forall \alpha \in (0,1)$.

Proof. (i) $\Leftarrow$ Note that if x and y ($x \neq \pm y$) are such that $\|x\| = \|y\| = 1$, then (13.2.2) yields

$$\|\alpha x + (1-\alpha)y\| < \alpha + (1-\alpha) = 1, \forall \alpha \in (0,1).$$

The case $x = -y$ follows immediately from the fact that $|1-2\alpha| < 1$.
So, **Property** implies strictly convex.

(ii) $\Rightarrow$ Suppose V is strictly convex, but (13.2.2) fails for a particular choice of $x_0, y_0 \in V$ (not multiples of each other) and $\alpha_0 \in (0,1)$.
Since, by triangle inequality,

$$\|\alpha x + (1-\alpha)y\| \leq \alpha\|x\| + (1-\alpha)\|y\|,$$

we conclude that

$$\|\alpha_0 x_0 + (1-\alpha_0)y_0\| = \alpha_0\|x_0\| + (1-\alpha_0)\|y_0\|. \tag{13.2.3}$$

Set $z = \frac{\alpha_0 x_0 + (1-\alpha_0)y_0}{\alpha_0\|x_0\| + (1-\alpha_0)\|y_0\|}$ (z is well-defined, since the condition "x and y are not multiples of each other" means that neither is the zero element). Due to (13.2.3), $\|z\| = 1$. Moreover, it is easy to check that

$$\gamma\frac{x_0}{\|x_0\|} + (1-\gamma)\frac{y_0}{\|y_0\|} = z,$$

with $\gamma = \frac{\alpha_0\|x_0\|}{\alpha_0\|x_0\| + (1-\alpha_0)\|y_0\|}$ $(0 < \gamma < 1)$, which is in contradiction with (13.2.1).
So, strictly convex implies the **Property**.

□

Remark 13.2.1. *Later we shall use the* ***Property*** *in the following context: Suppose $x, y \in V$ are non-zero and such that, for some $\gamma \in (0,1)$,*

$$\|\gamma x + (1-\gamma)y\| = \gamma\|x\| + (1-\gamma)\|y\|.$$

Then, by the ***Property****, x and y must be proportional, that is, $x = ky$, $k \in \mathbb{R}$. In fact, one can assume that $k > 0$, since, otherwise,*

$$\|\gamma x + (1-\gamma)y\| = \| - \gamma|k|y + (1-\gamma)y\| =$$

$$|1 - (|k|+1)\gamma| \cdot \|y\| < (1 + (|k|-1)\gamma)\|y\| = \gamma\|x\| + (1-\gamma)\|y\|.$$

The result we are aiming for at this stage is

Proposition 13.2.1. *If V is strictly convex, then $X = L^2(S, V)$ is also strictly convex.*

Proof. In order to be able to use the above **Property**, we take arbitrary $x, y \in X$, not multiples of each other. This means that

$$\{k \in \mathbb{R} : x(s) = ky(s), \text{ for } d\mu\text{-almost all } s \in \Lambda\} = \emptyset.$$

Then, by triangle and Hôlder's inequalities respectively,

$$\|\alpha x + (1-\alpha)y\|_X^2 = \int \|\alpha x(s) + (1-\alpha)y(s)\|^2 d\mu \leq^1$$

$$\int (\alpha\|x(s)\| + (1-\alpha)\|y(s)\|)^2 d\mu = \alpha^2\|x\|_X^2 +$$

$$(1-\alpha)^2\|y\|_X^2 + 2\alpha(1-\alpha)\int \|x(s)\|\cdot\|y(s)\|d\mu \leq^2$$

$$\alpha^2\|x\|_X^2 + (1-\alpha)^2\|y\|_X^2 + 2\alpha(1-\alpha)\|x\|_X\cdot\|y\|_X =$$

$$(\alpha\|x\|_X + (1-\alpha)\|y\|_X)^2.$$

We proceed by showing that either $\leq^1$ or $\leq^2$ (or both) is necessarily strict.
Let us take a closer look at $\leq^2$.
For fixed (non-zero) $u, v \in X$, define a function $f : \mathbb{R} \to [0, +\infty)$ by

$$f(\lambda) = \int (\lambda\|u(s)\| - \|v(s)\|)^2 d\mu =$$

$$\lambda^2\|u\|_X^2 + \|v\|_X^2 - 2\lambda\int \|u(s)\|\cdot\|v(s)\|d\mu.$$

Since, $\forall\lambda \in \mathbb{R}$, $f(\lambda) \geq 0$, it follows that

$$4(\int \|u(s)\|\cdot\|v(s)\|d\mu)^2 - 4\|u\|_X^2\cdot\|v\|_X^2 \leq 0.$$

Moreover,

$$4(\int \|u(s)\|\cdot\|v(s)\|d\mu)^2 = 4\|u\|_X^2\cdot\|v\|_X^2,$$

or

$$\int \|u(s)\|\cdot\|v(s)\|d\mu = \|u\|_X\cdot\|v\|_X$$

if, and only if, $\exists!\lambda_0$, such that $f(\lambda_0) = 0$. Thus

$$\|v(s)\| = \lambda_0\|u(s)\|,\ d\mu - \text{a.a. } s \in \Lambda, \tag{13.2.4}$$

in which case $\lambda_0 > 0$, since we have assumed v not to be (almost everywhere) zero.
In view of this observation, it becomes apparent that we have two possibilities:

1. x and y are as before, plus there does not exist a constant $k > 0$ (independent of s), s.t. (13.2.4) is satisfied. In this case $\leq^2$ is, by the above argument, strict and we are done.

2. x and y are not proportional, but their norms are, i.e.,

$$\|x(s)\| = k\|y(s)\|,\ d\mu - \text{a.e. } s \in \Lambda, \tag{13.2.5}$$

for some $k > 0$.

Let $A = \{s : \|y(s)\| > 0\}$. Since y is not (almost everywhere) zero, $\mu(A) > 0$. Moreover, due to (13.2.5),

$$\|x(s)\| > 0,\ d\mu - \text{a.e. } s \in A.$$

Now,

$$\int \|\alpha x(s) + (1-\alpha)y(s)\|^2 d\mu = \int (\alpha\|x(s)\| + (1-\alpha)\|y(s)\|)^2 d\mu,$$

if, and only if,

$$\|\alpha x(s) + (1-\alpha)y(s)\| = \alpha\|x(s)\| + (1-\alpha)\|y(s)\|,$$

for $d\mu$-a.e. $s \in A$, which means, by the **Property**, that there exists a non-negative real-valued function c (see Remark 13.2.1), such that, for such s,

$$x(s) = c(s)y(s). \tag{13.2.6}$$

On the other hand, if $s \in A^c$, $\|y(s)\| = 0$, by definition of A^c, and hence, taking (13.2.5) into account, $\|x(s)\| = 0$, for $d\mu$-a.e. $s \in A^c$. For such s, one still has an equality similar to (13.2.6).
To sum up, $\leq^1$ is an equality, equivalently,

$$\int \|\alpha x(s) + (1-\alpha)y(s)\|^2 d\mu = \int (\alpha\|x(s)\| + (1-\alpha)\|y(s)\|)^2 d\mu,$$

if, and only if, there exists a function $c : \Lambda \to [0, \infty)$, such that

$$x(s) = c(s)y(s),\ d\mu - \text{a.e. } s \in \Lambda. \tag{13.2.7}$$

Comparing (13.2.5) and (13.2.7), we conclude that $c(s) = k$, for almost all s, and hence

$$x(s) = ky(s),\ d\mu - \text{a.a. } s \in \Lambda,$$

which contradicts our initial assumption that x and y are not proportional.
Therefore, X is strictly convex. □

13.3 Maximal monotone.

To begin with, W is non-empty. To see this, take any $u \in V$ and $u^* \in A(u)$ and set

$$v(s) := u \text{ and } v^*(s) := u^*, \forall s \in \Lambda.$$

A similar argument shows that W is defined for all simple functions. Hence, $D(W)$ is dense in X.
The main result of this section is

Theorem 13.3.1. *W is "maximal monotone", with $D(W) = X$.*

Proof. [1] (Maximal monotone). Define $J' : X \to X^*$ by

$$J'(u(\cdot)) = J \circ u(\cdot).$$

Then $J' : X \to X^*$ is the duality map.
First, note that $J \circ u : \Lambda \to V^*$ is single-valued, since J is. Besides, it is measurable: u is measurable by definition, and J is demicontinuous (Lemma 12.0.10 of Appendix C.1), and hence also measurable (Proposition 9.0.6, Appendix B.1). Recall that in the construction of Bochner integral, we require the given V-valued function to be strongly measurable. However, if V is separable, then "measurability" and "strong

measurability" are equivalent notions (Proposition 18.0.2, Appendix F). Finally,

$$\langle J'(u), u\rangle_X = \int_\Lambda \langle J(u(s)), u(s)\rangle d\mu =$$

$$\int_\Lambda \|u(s)\|^2 d\mu = \|u\|_X^2,$$

and, similarly,

$$\langle J'(u), u\rangle_X = \|J'(u)\|_X^2.$$

So by Definition 12.0.10, J' is the duality map, as claimed.
Moreover, W is "monotone", since

$$\langle v^* - u^*, v - u\rangle_X = \int_\Lambda \langle v^*(s) - u^*(s), v(s) - u(s)\rangle d\mu \leq 0,$$

due to the fact that the integrand is (almost everywhere) non-positive by Assumption 3.2.1.
Crucially, the operator $J' - W$ is surjective. Indeed, take any $v^* \in X^*$ and consider the following inclusion problem:

$$v^* \in -W(u) + J'(u).$$

We claim that, for each $s \in \Lambda$, an inclusion

$$v^*(s) \in -A(u(s)) + J(u(s))$$

has a unique solution. This follows immediately from Proposition 12.0.9 (Appendix C.1).
Thus we can construct a function $u(\cdot)$ pointwise. What remains to be shown is that it is an element of X and is a solution of

$$v^* \in -W(u) + J'(u).$$

Clearly,

$$u := (-A + J)^{-1} \circ v^*,$$

is measurable, as a composition of two measurable functions. Moreover,

$$v^*(s) \in -A(u(s)) + J(u(s)),$$

means $\exists w^*(s) \in A(u(s))$, such that

$$v^*(s) = -w^*(s) + J(u(s)).$$

Making use of condition 3.2.2, we obtain

$$\langle v^*(s), u(s)\rangle = \langle -w^*(s) + J(u(s)), u(s)\rangle =$$

$$-\langle w^*(s), u(s)\rangle + \langle J(u(s)), u(s)\rangle \geq$$

$$\frac{\lambda}{2}\|u(s)\|^2 - \frac{K_1}{2}(1 + |u(s)|^2) + \|u(s)\|^2.$$

On the other hand,

$$\langle u(s), v^*(s)\rangle \leq \|u(s)\| \cdot \|v^*(s)\| \leq \frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|v^*(s)\|^2.$$

Combining these results, we get

$$\frac{1}{2}\|v^*(s)\|^2 + \frac{K_1}{2} \geq (\frac{\lambda}{2} + \frac{1}{2})\|u(s)\|^2.$$

Hence $u \in X$.
Moreover, w^* is in X^*. To see this, recall that

$$v^*(s) = -w^*(s) + J(u(s)),\ s \in \Lambda.$$

Consequently,

$$w^* := J'(u) - v^*$$

is measurable as a linear combination of measurable functions. Furthermore,

$$\|w^*\|^2_{X^*} = \int_\Lambda \|w^*(s)\|^2 d\mu = \int_\Lambda \|J(u(s)) - v^*(s)\|^2 d\mu \leq$$

$$2\int_\Lambda \|(J(u(s))\|^2 d\mu + 2\int_\Lambda \|v^*(s)\|^2 d\mu =$$

$$2\int_\Lambda \|u(s)\|^2 d\mu + 2\int_\Lambda \|v^*(s)\|^2 d\mu =$$

$$2\|u\|^2_X + 2\|v^*\|^2_{X^*} < \infty.$$

So, $w^* \in X^*$.
This completes the proof of the assertion that

$$R(-W + J') = X^*.$$

Therefore, by Theorem 12.0.5, W is "maximal monotone".
[2] ($D(W) = X$). Suppose $u \in X$ is arbitrary. There exists a sequence of simple functions $\{u_n\}$, such that $u_n \to u$. Let $v_n \in W(u_n)$, $n = 1, 2, \ldots$. Then, by Assumption 3.2.3, for each n,

$$\|v_n\|^2_{X^*} = \int_\Lambda \|v_n(s)\|^2 d\mu \leq$$

$$K_2^2 \int_\Lambda (1 + \|u_n(s)\|)^2 ds = 2K_2^2 \mu(\Lambda) + 2K_2^2 \|u_n\|^2_X < \infty.$$

The sequence $\{u_n\}$ is convergent and, therefore, bounded; it follows that $\|v_n\|_{X^*}$ is bounded as well. Hence, it contains a (weakly) convergent subsequence, denoted again by v_n, with the limit $v \in X^*$. Since W is, in particular, "monotone", we have

$$\langle v_n - w^*, u_n - w\rangle_X \leq 0,\ \forall w \in W, w^* \in W(w).$$

Letting $n \to \infty$, we obtain

$$\langle v - w^*, u - w\rangle_X \leq 0,\ \forall w \in W, w^* \in W(w),$$

which means that $u \in D(W)$. □

Chapter 14

Appendix C.3.

To start with, W is non-empty. Indeed, take any $u \in D(A) = V$, $u^* \in A(u)$ and set

$$v(t,\omega) := u \text{ and } v^*(t,\omega) := u^*,\ \forall (t,\omega) \in [0,T] \times \Omega.$$

Furthermore, suppose $v(\cdot)$ can be written in the form

$$v(t,\omega) = \sum_{i=1}^{n} u_i \chi_{B_i},$$

where $u_i \in V$ and $\{B_i\}$ is a partition of $[0,T] \times \Omega$, with every B_i - of the form $B_i = (s_i, t_i] \times A_i$, for some $A_i \in \mathcal{F}_{s_i}$. Then, $v^*(\cdot) \in X^*$, defined by

$$v^*(t,\omega) = \sum_{i=1}^{n} u_i^* \chi_{B_i},$$

with $u_i^* \in A(u_i),\ j = 1, \ldots, n.$, satisfies

$$v^*(\cdot) \in W(v(\cdot)).$$

Finally, let us note that functions of the above kind are dense in X. In other words, $D(W)$ is dense in X.
Now, following the same steps as in the proof of Theorem 13.3.1, one can demonstrate the validity of

Theorem 14.0.2. *Let $A : V \to 2^{V^*}$ be an operator that satisfies Assumptions 4.2.3, 4.2.4 and 4.2.2. Then W is "maximal monotone" and $D(W) = X$.*

Chapter 15

Appendix C.4.

Recall that the operator $W : Y \to 2^{Y^*}$ is defined by

$$W(v) = \{v^* \in Y^* : v^*(t,\omega) \in (A - KI)(v(t,\omega)),\ dt \times P\text{a.e. } (t,\omega) \in [0,T] \times \Omega\}.$$

Theorem 15.0.3. *Let $A : V \to 2^{V^*}$ be an operator that satisfies Assumptions 5.2.2, 5.2.3 and 5.2.4. Then W is "maximal monotone" and $D(W) = Y$.*

Proof. By definition, if A is K-"maximal monotone", then $A_K = A - KI$ is "maximal monotone". Moreover, one can easily check that the operator A_K satisfies "linear growth" and "coercivity" conditions (that is, Assumptions 5.2.2 and 5.2.4, perhaps, with different constants). This means that Theorem 14.0.2 is applicable, and we are justified in concluding that the operator W is "maximal monotone" and $D(W) = Y$. □

Chapter 16

Appendix D.

Gronwall's Lemma 16.0.4. Suppose $\{a_i\}$, $i = 0, \ldots, m$, is a sequence of real numbers that satisfy

$$|a_0| \leq C \text{ and } |a_k| \leq C + K \sum_{i=0}^{k-1} |a_i|,$$

for some positive C, K and $1 \leq k \leq m$.
Then

$$|a_k| \leq C(1+K)^k,\ 1 \leq k \leq m.$$

Proof. Set $b_0 = C$ and $b_k = C + K \sum_{i=0}^{k-1} b_i$. We would like to show that

(i) $b_k = C(1+K)^k$;

(ii) $|a_k| \leq b_k$.

(i) We proceed by induction.
$b_0 = C = C(1+K)^0$. If

$$b_j = C(1+K)^j,\ 0 \leq j \leq k \leq m-1,$$

then

$$b_{k+1} = C + K \sum_{i=0}^{k} b_i = C + K \sum_{i=0}^{k-1} b_i + K b_k =$$

$$(1+K) b_k = C(1+K)^{k+1}.$$

(ii) $|a_0| \leq C = b_0$. Assume further that

$$|a_j| \leq b_j,\ 0 \leq j \leq k \leq m-1.$$

It follows that

$$|a_{k+1}| \leq C + K \sum_{i=0}^{k} |a_i| \leq C + K \sum_{i=0}^{k} b_i = b_{k+1},$$

as required. □

Chapter 17

Appendix E.

Theorem 17.0.5. *Let $x = \{x(t) : t \in [0,T]\}$ be a V^*-valued, $\mathcal{F}_t$-adapted stochastic process of the form*

$$x(t) = x_0 + \int_0^t a(s)ds + \sum_i \int_0^t b^i(s)dW^i(s),$$

where x_0 is an H-valued $\mathcal{F}_0$-measurable random variable, $a = \{a(s)\}_{s\in[0,T]}$ and $b = \{(b^i(s))\}_{s\in[0,T]}$ are $\mathcal{F}_t$-adapted stochastic processes, with values in V^* and in H^r respectively, such that

$$\int_0^T \|a(s)\|ds < \infty, \; \sum_i \int_0^T |b^i(s)|^2 ds < \infty \text{ (a.s.)}.$$

Assume that there exists a V-valued, $\mathcal{F}_t$-adapted stochastic process $v = \{v(t) : t \in [0,T]\}$, such that $v(t,\omega) = x(t,\omega)$, for $dt \times P$ almost every $(t,\omega) \in [0,T] \times \Omega$, and

$$\int_0^T \|v(s)\|ds < \infty, \; \int_0^T \|v(s)\| \cdot \|a(s)\|ds < \infty \text{ (a.s.)}.$$

Then x is, almost surely, an H-valued, continuous, $\mathcal{F}_t$-adapted stochastic process and

$$|x(t)|^2 = |x_0|^2 + \int_0^t [2\langle x(s), a(s)\rangle + \sum_i |b^i(s)|^2]ds +$$

$$2\sum_i \int_0^t (x(s), b^i(s))dW^i(s),$$

for all $t \in [0,T]$.

Proof. See [9]. □

Remark 17.0.1. *We are justified in using the above theorem for the following reasons:*

(i) v_∞ is V-valued, by construction. On the other hand, it is also V^*-valued, since $\int_0^t a_\infty(s)ds$ is.

(ii) Since $\alpha_\infty \in L^2(S, V^*)$, by Hölder's inequality,

$$E\int_0^T \|\alpha_\infty(s)\|ds \leq$$

$$(E\int_0^T \|\alpha_\infty(s)\|^2 ds)^{\frac{1}{2}}(E\int_0^T 1 ds)^{\frac{1}{2}} \leq$$

$$(T)^{\frac{1}{2}} E\int_0^T \|\alpha_\infty(s)\|^2 ds)^{\frac{1}{2}} < \infty.$$

It follows that, almost surely,

$$\int_0^T \|\alpha_\infty(s)\| ds < \infty.$$

(iii) Recall that b^i, $i = 1, \ldots, r$, *are elements of the space* $L^2(S, H)$, *and so*

$$E\sum_i \int_0^T |b^i(s)|^2 ds < \infty.$$

Therefore, a.s.,

$$\sum_i \int_0^T |b^i(s)|^2 ds < \infty.$$

(iv) Finally, keeping in mind that $\alpha_\infty \in L^2(S, V^*)$ *and* $v_\infty(\cdot) \in L^2(S, V)$, *an application of Hölder's inequality yields*

$$\int_0^T \|v_\infty(s)\| ds < \infty, \quad \int_0^T \|v_\infty(s)\| \cdot \|\alpha_\infty(s)\| ds < \infty \text{ (a.s.)}.$$

Chapter 18

Appendix F.

Suppose $(S, \mathcal{F}, \mu)$ is a complete measure space and V a separable Banach space.

Lemma 18.0.1. *Let $f, g : S \to V$ be functions, such that $f = g$, almost everywhere. Then, if f is $\mathcal{F}$-measurable, g is $\mathcal{F}$-measurable as well.*

Proof. Suppose $N \subseteq \mathcal{F}$ is such that $\mu(N) = 0$ and f and g agree on N^c. Since V separable, $\mathcal{B}(V)$ is generated by open balls $\mathbf{B} = B(x, r)$, where $x \in V$ and $r > 0$ (Proposition 9.0.4, Appendix A). By Proposition 9.0.3, in order to see that g is $\mathcal{F}$-measurable, it suffices to check that if f is strongly measurable, then $g^{-1}(\mathbf{B}) \in \mathcal{F}$, which is an immediate consequence of the following simple observation:

$$\{s \in S : g(s) \in \mathbf{B}\} =$$

$$[\{s \in S : f(s) \in \mathbf{B}\} \bigcap N^c] \bigcup [\{s \in S : g(s) \in \mathbf{B}\} \bigcap N^c].$$

□

Definition 18.0.1. *A function $f : S \to V$ is **strongly measurable**, if there exists a sequence $\{f_n\}$ of simple functions, such that, for μ-almost every $s \in S$,*

$$f(s) = \lim_{n \to \infty} f_n(s).$$

*Similarly, f is called **weakly measurable** if, $\forall \varphi \in V^*$, $\varphi(f) = \langle \varphi, f \rangle : (S, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable.*

The following result due to Pettis.

Theorem 18.0.6. *A function $f : S \to V$ is strongly measurable if, and only if, it is weakly measurable.*

Proof. See [4] for details. □

Using this result, one can easily demonstrate the validity of

Proposition 18.0.1. *An almost everywhere limit of a sequence of strongly measurable functions is strongly measurable.*

Proof. Suppose $g(s) = \lim_{n \to \infty} f_n(s)$ (a.e.), where each f_n is strongly measurable. By the Theorem of Pettis, $\langle \varphi, f_n \rangle$ is measurable, $\forall n \in \mathbb{N}$ and $\varphi \in V^*$. Furthermore, for μ-almost every $s \in S$,

$$\langle \varphi, g(s) \rangle = \liminf \langle \varphi, f_n(s) \rangle.$$

Since the RHS is $\mathcal{F}$-measurable, it follows, by Lemma 18.0.1, that $\langle \varphi, g \rangle$ is also $\mathcal{F}$-measurable. Hence, invoking the Theorem of Pettis one last time, g is strongly measurable. $\square$

Moreover,

Proposition 18.0.2. *Under the above conditions, the following statements are equivalent:*

- $f : (S, \mathcal{F}) \to (V, \mathcal{B}(V))$ *is measurable;*
- f *is strongly measurable.*

Proof. $\Leftarrow$ Let $B(x, r) = C$ be an arbitrary open ball. Suppose $\{C_m\}$ is an increasing sequence of open sets, satisfying

$$C_1 \subseteq \overline{C_1} \subseteq C_2 \subseteq \overline{C_2} \subseteq \ldots \subseteq C, \text{ and } \bigcup C_m = C.$$

For instance, one can have $C_n = B(x, r(1 - \frac{1}{n}))$, $n = 1, 2, \ldots$.
(i) Assume that

$$f(s) = \lim_{n\to\infty} f_n(s), \ \forall s \in S,$$

where f_n are simple. Then

$$A = f^{-1}(C) = \bigcup_{m=1} (\bigcup_{n=1} (\bigcap_{k=n} f_n^{-1}(C_m))) = B.$$

To begin with, since simple functions are measurable, by definition, $B \in \mathcal{F}$.
Suppose $s \in A$, or, equivalently, $f(s) \in C$. There exists m, s.t. $f(s) \in C_m$. Due to the fact that C_m is open, $\exists n_0$, such that $\forall n \geq n_0$, $f_n \in C_m$. It follows that, for such n,

$$s \in \bigcap_{n=n_0} f_n^{-1}(C_m).$$

Hence, $s \in B$.
Now, let $s \in B$. Then $\exists m$, such that

$$s \in \bigcup_{n=1} (\bigcap_{k=n} f_n^{-1}(C_m)).$$

This means that there exists n_0, s.t., $\forall n \geq n_0$,

$$s \in f_n^{-1}(C_m), \text{ or } f_n(s) \in C_m.$$

Consequently,

$$f(s) = \lim_{n\to\infty} f_n(s) \in \overline{C_m} \subseteq C.$$

Therefore $s \in f^{-1}(C) = A$.
(ii) If we have μ-almost everywhere convergence, then, using the above notation,

$$A \bigcap N^c = B \bigcap N^c,$$

where the set N is such that

$$\mu(N) = 0 \text{ and } f(s) = \lim_{n\to\infty} f_n(s), \ s \in N^c.$$

It follows that $A \triangle B \subseteq N$, and, keeping in mind that $\mathcal{F}$ is complete, $\mu(A \triangle B) = 0$. Finally, $A \in \mathcal{F}$.
$\Rightarrow$. Let $D = \{x_1, x_2, \ldots\}$ be a dense subset of V. Fix $n \in \mathbb{N}$. Then

$$V = \bigcup_i B(x_i, \frac{1}{n}).$$

Define

$$C_1^n = f^{-1}(B(x_1, \frac{1}{n})),$$

$$C_i^n = f^{-1}(B(x_i, \frac{1}{n}))/(\bigcup_{j=1}^{i-1} f^{-1}(B(x_j, \frac{1}{n}))),\ i \geq 2.$$

Then the sets C_i^n, $i = 1, 2, \ldots$, are measurable (since f is assumed to be measurable), disjoint (though some maybe empty) and satisfy

$$\bigcup_i C_i^n = S.$$

Define

$$f_n(s) = \sum_{i=1}^{\infty} x_i \chi_{C_i^n}$$

and

$$g_m^n = \sum_{i=1}^{m} x_i \chi_{C_i^n}.$$

Each f_n are, at most, countably valued and

$$f(s) = \lim_{n \to \infty} f_n(s),\ \forall s \in S.$$

Moreover, g_m^n are simple and satisfy

$$f_n(s) = \lim_{m \to \infty} g_m^n(s),\ \forall s \in S.$$

Therefore f_n are strongly measurable, by definition, which, in turn, implies (Proposition 18.0.1) that f is also strongly measurable. $\square$

Chapter 19

Appendix G.1. Characterization of maximal monotone operators.

19.1 Introduction.

Recall that an operator $A : V \to 2^{V^*}$ is called maximal monotone if, and only if,

$$\langle u^* - v^*, u - v\rangle \geq 0 \ (\forall v \in V, v^* \in Av) \Rightarrow u^* \in Au.$$

Despite its seeming simplicity, it may be a challenge to decide, on the basis of the above definition, whether a given operator is indeed maximal monotone.
In this section we offer an alternative characterization of maximal monotone maps that is more conducive to testing.
We begin by showing how a monotone operator can be extended.

19.2 Extending monotone maps.

Suppose V is a real reflexive Banach space and $D \subseteq V$ an open, bounded and convex subset. Let an operator $A : D \to 2^{V^*}$ be monotone and bounded, that is,

$$\sup\{\|u^*\| : u^* \in A(u), u \in D\} \leq M < \infty.$$

As before, $\langle x, y\rangle = \langle y, x\rangle$ shall denote the duality product of $x \in V$ and $y \in V^*$.
Set

$$R(A, x, y) =$$
$$\sup\{-\langle x^* - x, y^* - y\rangle + \langle x, y\rangle : x^* \in D, y^* \in A(x^*)\} =$$
$$\sup\{\langle x - x^*, y^*\rangle + \langle x^*, y\rangle : x^* \in D, y^* \in A(x^*)\}.$$

the following result is due to N.Krylov ([14]).

Proposition 19.2.1.

- *(1) $R(A, x, y)$ is a function on $V \times V^*$.*
- *(2) $R(A, x, y) \geq \langle x, y\rangle$ on $D \times V^*$, $R(A, x, y) = \langle x, y\rangle, x \in D, y \in A(x)$.*
- *(3) The set $Z(A) = \{(x, y) : x \in D, R(A, x, y) = \langle x, y\rangle\}$ is closed and monotone.*
- *(4) For any $x \in D$ the set $Z(A, x) = \{y : (x, y) \in Z(A)\}$ is the closure of the convex hull of the set of partial (weak) limits of $\{y_n\}$, where $y_n \in A(x_n)$, as $x_n \to x$.*

NB: In the finite-dimensional case this statement can be strengthened as follows: $Z(A,x)$ is the convex hull of the set of partial limits.

- *(5) Given $n \geq 1, \alpha_i \geq 0, (x_i, y_i) \in Z(A), i = 1, \ldots, n, \sum \alpha_i = 1$, then*

$$\langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{n} \alpha_i y_i \rangle \leq \sum_{i=1}^{n} \alpha_i \langle x_i, y_i \rangle.$$

Proof. (1) Finiteness of R follows from the boundedness of A (on D) and D itself. So R is well-defined.
Moreover, it is convex. Indeed, take $(x_1, y_1), (x_2, y_2) \in V \times V^*$ and arbitrary $\alpha \in (0,1)$. Then, for any $x^* \in D$ and $y^* \in Ax^*$,

$$LHS = \langle \alpha x_1 + (1-\alpha)x_2 - x^*, y^* \rangle + \langle x^*, \alpha y_1 + (1-\alpha) y_2 \rangle =$$

$$\langle \alpha x_1 + (1-\alpha)x_2 - \alpha x^* - (1-\alpha)x^*, y^* \rangle + \langle x^*, \alpha y_1 + (1-\alpha) y_2 \rangle =$$

$$\alpha(\langle x_1 - x^*, y^* \rangle + \langle x^*, y_1 \rangle) + (1-\alpha)(\langle x_2 - x^*, y^* \rangle + \langle x^*, y_2 \rangle) \leq$$

$$\alpha R(A, x_1, y_1) + (1-\alpha) R(A, x_2, y_2),$$

and the result follows by taking the sup of the LHS.
Note that the product space $V \times V^*$ can be made into a normed linear space, if we define the algebraic operations coordinate-wise and define the norm, for instance, by $\|(x,y)\|_{V \times V^*} = \|x\|_V + \|y\|_{V^*}$. Moreover, it is not difficult to see that topology induced by the above-mentioned norm coincides with a product (norm$\times$norm) topology. it is known that if R is a proper, convex and bounded function, defined on a topological vector space, then it is continuous.
Since $V \times V^*$, by the above construction, is a C_1-space, the continuity properties of R can be stated in terms of sequences, rather than neighborhoods, the fact that we shall often use.
(2) For any $(x,y) \in D \times V^*$, one can set $x^* = x$ (Definition of R), which means that $R(A,x,y) \geq \langle x, y \rangle$.
The second claim is a direct consequence of the monotonicity of A.
(5) Keeping in mind that D is convex, we get, by (2),

$$R(A, \sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{n} \alpha_i y_i) \geq \langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{n} \alpha_i y_i \rangle.$$

On the other hand, using the fact that R is convex, we deduce that

$$R(A, \sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{n} \alpha_i y_i) \leq \sum_{i=1}^{n} \alpha_i R(A, x_i, y_i) = \sum_{i=1}^{n} \alpha_i \langle x_i, y_i \rangle,$$

where the last equality follows from the fact that $(x_i, y_i) \in Z(A)$, i.e., $R(A, x_i, y_i) = \langle x_i, y_i \rangle$.
(3) Take $\{(x_n, y_n)\} \in Z(A)$, with $x_n \to x$ and $y_n \to y$ (in respective norm topologies). Then, by (2) and continuity of R,

$$R(A,x,y) = \lim R(A, x_n, y_n) = \lim \langle x_n, y_n \rangle = \langle x, y \rangle.$$

So $(x,y) \in Z(A)$ and, hence, $Z(A)$ is closed.
The assertion that $Z(A)$ is monotone follows from (5), with $n = 2$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$.

(4) Denote the closure of the convex hull in question by $P(A, x)$.
To begin with, $Z(A, x)$ is closed. Indeed,

$$y_n \to y \text{ and } R(A, x, y_n) = \langle x, y_n \rangle$$

imply

$$R(A, x, y) = \langle x, y \rangle.$$

Furthermore, $Z(A, x)$ is convex. Take $y_1, y_2 \in Z(A, x)$ and $\alpha \in (0, 1)$. Then, by (2), (1) and the definition of $Z(A, x)$,

$$\langle x, \alpha y_1 + (1 - \alpha) y_2 \rangle \leq R(A, x, \alpha y_1 + (1 - \alpha) y_2) \leq$$

$$\alpha R(A, x, y_1) + (1 - \alpha) R(A, x, y_2) = \alpha \langle x, y_1 \rangle + (1 - \alpha) \langle x, y_2 \rangle =$$

$$\langle x, \alpha y_1 + (1 - \alpha) y_2 \rangle,$$

from which it follows that

$$R(A, x, \alpha y_1 + (1 - \alpha) y_2) = \langle x, \alpha y_1 + (1 - \alpha) y_2 \rangle.$$

Thus, $\alpha y_1 + (1 - \alpha) y_2 \in Z(A, x)$.
Suppose $x, \{x_n\}_{n=1}^{\infty} \in D$ and $\{y_n\}_{n=1}^{\infty}$ are such that $x_n \to x$ and $y_n \in A(x_n)$. Let y be a partial (weak) limit of the sequence $\{y_n\}$. To put it differently, there exists a subsequence $\{n(i)\}_{i=1}^{\infty}$, such that $x_{n(i)} \to x$ and $y_{n(i)} \rightharpoonup y$. Then, by monotonicity of A, for any $x^* \in D$, $y^* \in A(x^*)$,

$$-\langle x^* - x_{n(i)}, y^* - y_{n(i)} \rangle + \langle x_{n(i)}, y_{n(i)} \rangle \leq \langle x_{n(i)}, y_{n(i)} \rangle,$$

or, passing to the limit,

$$-\langle x^* - x, y^* - y \rangle + \langle x, y \rangle \leq \langle x, y \rangle,$$

which implies, due to arbitrariness of x^* and y^*,

$$R(A, x, y) \leq \langle x, y \rangle,$$

and, finally, by (2),

$$R(A, x, y) = \langle x, y \rangle.$$

Therefore the set of partial (weak) limits is contained in $Z(A, x)$. Furthermore, taking into consideration that $Z(A, x)$ is both closed and convex, we infer

$$P(A, x) \subseteq Z(A, x).$$

NB: If the set $P'(x)$ is defined by $P'(x) = \{$ partial limits of $\{y_n\} : y_n \in A(x_n), x_n \to x\}$, then in the finite-dimensional setting, $P'(x)$ is closed. To see this, suppose $z_n \to z$, $z_n \in P'(x)$. Then $\exists$ $\{x_n\}$ and $\{y_n\}$, such that

$$x_n \to x,$$

and

$$\|z_n - y_n\| \leq \frac{1}{n},\ y_n \in A(x_n),$$

which makes it apparent that z is a partial limit in its own right (since $y_n \to z$), and,

consequently, an element of $P'(x)$.
Moreover, $P'(x)$ is bounded, due to the fact that A is a bounded operator. Therefore, $P'(x)$ is compact. Recalling that a convex hull of a compact set is compact, we conclude that $P(A,x)$ is closed.
We have seen above that $P(A,x) \subseteq Z(A,x)$. Suppose there exists $y \in Z(A,x)/P(A,x)$. By the strict separation theorem for Banach spaces (which we are justified in using since $P(A,x)$ is closed and convex), one can find $x_0 \in V$, such that

$$\sup_{z \in P(A,x)} \langle x_0, z \rangle = a < b = \langle x_0, y \rangle. \tag{19.2.1}$$

We proceed to show that

$$\varphi = \limsup \langle x_0, y_n \rangle \leq a, \;\; y_n \in A(x_n), x_n \to x. \tag{19.2.2}$$

From the definition of lim sup and the fact that A is bounded on D, it follows that there are $\{x_{n(i)}\}$ and $\{y_{n(i)}\}$, satisfying

$$x_{n(i)} \to x, \;\; x_{n(i)} \in D,$$

$$y_{n(i)} \rightharpoonup y', \;\; y_{n(i)} \in A(x_{n(i)}).$$

Since y' is a partial weak limit, we have $y' \in P(A,x)$. Consequently,

$$\varphi = \limsup \langle x_0, y_n \rangle = \lim \langle x_0, y_{n(i)} \rangle = \langle x_0, y' \rangle \leq a.$$

Now, due to the fact that $y \in Z(A,x)$, $R(A,x,y) = \langle x, y \rangle$. Moreover, for any $x^* \in D$ and $y^* \in A(x^*)$

$$\langle x^* - x, y^* - y \rangle \geq 0,$$

otherwise, there would exist $x' \in D$ and $y' \in A(x')$, such that

$$\langle x' - x, y' - y \rangle < 0,$$

which would, in turn, imply

$$R(A,x,y) \geq -\langle x' - x, y' - y \rangle + \langle x, y \rangle > \langle x, y \rangle = R(A,x,y).$$

Let us choose $\{x_n\}$ in such a way that $x_n \to x$ and $x_n - x$ is in the same direction as x_0. Then, by the above observation,

$$\langle x_0, y_n - y \rangle \geq 0, \;\; y_n \in A(x_n).$$

It follows from (19.2.1) and (19.2.2) that

$$0 \leq \limsup \langle x_0, y_n - y \rangle = \limsup \langle x_0, y_n \rangle - \langle x_0, y \rangle \leq a - b < 0,$$

a contradiction. Hence

$$P(A,x) = Z(A,x).$$

□

19.3 Maximal monotone extension.

We have just demonstrated that if $A : D \to 2^{V^*}$ is monotone and bounded, then $Z(A)$ is a monotone subset of $V \times V^*$, containing A. On the other hand, by Zorn's lemma, A permits a maximal (monotone) extension $\overline{A}$. A natural question is how the two are related, which is addressed by the following

Lemma 19.3.1. *Let $A : V \to 2^{V^*}$ be a monotone map. Define $Z'(A) : V \to 2^{V^*}$ point-wise by "$Z'(A, x)$ is the closure of the convex hull of the set of partial (weak) limits of $\{y_n\}$ $(y_n \in A(x_n))$, as $x_n \to x$". Then*

$$Z'(A) = \overline{A}.$$

Proof. Clearly, $Z'(A) \subseteq \overline{A}$ (viewed as subsets of $V \times V^*$). In particular, for any $x \in V$,

$$Z(A, x) \subseteq \overline{A}(x).$$

To verify the reverse inclusion, take arbitrary $x \in V$. Recall that, by Theorem 11.0.7 (Appendix B.2), if x is an absorbing point of $D(A)$ (for instance, an interior point), then A is locally bounded at x, that is, $\exists r > 0$ and non-negative constant M, such that

$$\sup\{\|u^*\| : u^* \in A(u), u \in B(x, r)\} \leq M. \tag{19.3.1}$$

Since, in our case, $D(A) = V$, x is absorbing and, therefore, (19.3.1) holds. If we set $D = B(x, r)$, then results of the previous section apply, and we deduce that

$$Z'(A, x) = Z(A, x).$$

By definition, $Z(A, x)$ consists of all those y's, for which $R(A, x, y) = \langle x, y\rangle$. However, this condition is obviously satisfied when $y \in \overline{A}(x)$:

$$\langle x, y\rangle \leq R(A, x, y) =$$

$$\sup\{-\langle x^* - x, y^* - y\rangle + \langle x, y\rangle : x^* \in D, y^* \in A(x^*)\} \leq \langle x, y\rangle.$$

So,

$$\overline{A}(x) \subseteq Z(A, x) = Z'(A, x).$$

□

Chapter 20

Appendix G.2.

Suppose the operator $A : V \to 2^{V^*}$ is "maximal monotone", satisfying Assumptions 4.2.1-4.2.6.
For $n \in \mathbb{N}$, define $\Pi_n : V^* \to V_n^*$ by

$$\Pi_n(u^*) = \sum_{i=1}^{n} \langle u^*, e_i \rangle_V e_i^*,$$

where e_i^*, $i = 1, 2, \ldots$, are given by $\langle e_i^*, u \rangle_{V_n} = (e_i, u)_H$, $u \in V_n$.

Proposition 20.0.1. *For any $n \in \mathbb{N}$, $\Pi_n A : V_n \to 2^{V_n^*}$ is "maximal monotone".*

Proof. We have previously seen that the operator $\Pi_n A$ is "monotone".
In order to show that it is, in fact, "maximal", we have to check, in view of Proposition 19.2.1 (Appendix G.1), that for any $u \in V_n$, a set $\{\Pi_n A(u)\}$ is a convex hull of (partial) limit points.
To begin with, we know that $\forall u \in V$, $\{A(u)\}$ is, in particular, convex. Since Π_n is linear, $\{\Pi_n A\}$ is convex as well. Consequently, it suffices to show that the above set contains all limit points. To this end, suppose $\{u_m\} \in V_n$ and $\{v_m^*\} \in V_n^*$ are sequences, satisfying

$$u_m \to u \in V_n$$

and

$$v_m^* \to v^* \in V_n^*, \; v_m^* \in \Pi_n A(u_m).$$

Then $\exists u_m^* \in A(u_m)$, $m = 1, 2, \ldots$, such that $v_m^* = \Pi_n(u_m^*)$.
Since A satisfies the "linear growth" condition and $\{u_m\}$ is bounded (due to the fact that it is convergent), we conclude that the sequence $\{u_m^*\}$ is also bounded. Thus it contains a weakly convergent subsequence $\{u_{m(i)}^*\}$, i.e.,

$$u_{m(i)}^* \rightharpoonup u^*, \; u^* \in V^*.$$

Recall that A is "maximal monotone" iff

$$\langle u - v, u^* - v^* \rangle \leq 0 \; \forall v, v^* \in A(v) \Rightarrow u^* \in A(u).$$

Suppose $\{u_m\} \in V$ and $\{u_m^*\} \in V^*$ are sequences, such that

$$u_m \to u,$$

and

$$u_m^* \rightharpoonup u^*, \ u_m^* \in A(u_m).$$

Then for any (v, v^*), with $v^* \in A(v)$,

$$\langle u_m - v, u_m^* - v^* \rangle \leq 0,$$

by "monotonicity" of A, and, passing to the limit,

$$\langle u - v, u^* - v^* \rangle \leq 0.$$

Since (v, v^*) is arbitrary, it means that $u^* \in A(u)$.
Finally,

$$v_{m(i)}^* = \Pi_n(u_{m(i)}^*) = \sum_{j=1}^{n} \langle u_{m(i)}^*, e_j \rangle e_j^* \to \sum_{j=1}^{n} \langle u^*, e_j \rangle e_j^* = \Pi_n(u^*).$$

Due to uniqueness of the limit, we note that

$$v^* = \Pi_n(u^*) \in \Pi_n(A(u)).$$

□

Chapter 21

Appendix H.1.

21.1 Maximal Monotone.

Proposition 21.1.1. W is K-maximal monotone.

Proof. Since we have used a similar argument before, a sketch of the proof will suffice. Recall that A is called K-maximal monotone if and only if $A - KI$ is maximal monotone. We will show that W', defined by:

$$W'(v) = \{v^* \in Y^* : v^*(t,w) \in (A_{t,\omega} - KI)(v(t,w)),\ dt \times dP \text{ -a.e.},\ v \in Y\}.$$

W' is non-empty as well (at least, for step functions). Moreover, one can easily see that it is also monotone. In order to show that it is, in fact, maximal monotone, we use a well-known result, which states that a monotone operator $A : V \to 2^{V^*}$ is maximal monotone iff $R(-A + \delta J) = V^*$ for all (equivalently, at least, one) $\delta > 0$. It means that demonstrating that W' is maximal monotone is equivalent to establishing that an inclusion

$$v^*(\cdot) \in -W'(u(\cdot)) + \delta J'(u(\cdot))$$

has a solution for any $v^*(\cdot)$ in X.
Notice that the above inclusion can be solved point-wise for every (s, ω), since $A_{s,\omega} - KI$ is maximal monotone (definition).
Moreover, u is measurable (see Appendix H.2). $\square$

Chapter 22

Appendix H.2.

22.1 Introduction.

Suppose the operator $A : [0, T] \times \Omega \times V \to V^*$ is such that, for every $(t, \omega) \in [0, T] \times \Omega$, $A(t, \omega, u)$ is "maximal monotone". Let $X = L^2(S, V)$ be as before. Then, $X^* = L^2(S, V^*)$.
In order to see that the operator $W : X \to 2^{X^*}$ is maximal monotone in its own right, it suffices to check that, for $\forall u^* \in X^*$, the function u, defined point-wise by

$$u(t, \omega) = (A_t + \lambda J)^{-1} u^*(t, \omega), \ \lambda > 0,$$

is measurable.
As we have observed before, if $A_1 : V \to 2^{V^*}$ is (multi-valued) maximal monotone or $A_2 : [0, T] \times \Omega \times V \to V^*$ is a (single-valued) hemicontinuous operators, then the above condition is satisfied. Our aim is to show that if the operator B is of the form $B = A_1 + A_2$ the above statement holds true.
To this end, take arbitrary $u^* \in X^*$ and $\lambda > 0$. Set $A_1' = A_1 + \lambda J$ and $A_2' = A_2 + \lambda J$. Then the function u, given by

$$u = (A_1 + A_2 + 2\lambda J)^{-1} u^* = (A_1' + A_2')^{-1} u^*$$

is a (unique) solution of the inclusion problem

$$u^* \in A_1'(u) + A_2'(u).$$

Using the fact that A_1' is maximal monotone (as the sum of two maximal monotone operators), the above inclusion can, equivalently, be re-written as a variational inequality:

$$\langle u - v, u^* - A_2'(u) - v^* \rangle \geq 0, \ (v, v^*) \in A_1'. \qquad (22.1.1)$$

Using Galerkin method, we intend to demonstrate that, given a sequence of finite-dimensional subspaces of V, corresponding variational inequalities all have unique, measurable solutions u_n and that this sequence of solutions converges weakly to u, the solution of (22.1.1), i.e.,

$$u_n \rightharpoonup u,$$

(See Theorem 32.A of [22]). This means that, for any $u^* \in V^*$,

$$\langle u^*, u_n(t, \omega) \rangle_V \to \langle u^*, u(t, \omega) \rangle_V,$$

and, consequently, that $\langle u^*, u(t,\omega)\rangle_V$ is measurable.
Finally, quoting Theorem 18.0.6 (Appendix F) , we conclude that $u(t,\omega)$ is indeed measurable.

22.2 Galerkin approximations.

Let V_n and Π_n be as in Chapter 6. We are looking for $u \in V_n$, which solves

$$\langle u - v, u^* - A_2'(u) - v^* \rangle \geq 0,$$

where $(v, v^*) \in A_1'$ and $v \in V_n$.
Recalling the properties of Π_n, one can see that the above inequality is equivalent to the following one:

$$\langle u - v, \Pi_n(u^*) - \Pi_n A_2'(u) - \Pi_n(v^*) \rangle \geq 0. \tag{22.2.1}$$

Let us show that the operator $A_2' = A_2 + \lambda J$ is pseudomonotone. Indeed, since J is demicontinuous, it is hemicontinuous. According to Proposition 27.6 ([22]), "hemicontinuous + monotone" implies pseudomonotone. By the same argument, A_2 is also pseudomonotone. Finally, a sum of two pseudomonotone operators is again pseudomonotone (Propotition 27.6 of [22]). Finally, by Theorem 32.A ([22]), inequality (22.2.1) has a solution.
Moreover, as we have observed before (Proposition 20.0.1, Appendix G.2), the operator $\Pi_n A_1$ is maximal monotone. Set

$$\overline{\Pi_n A_1} = \Pi_n A_1 + \lambda \Pi_n J.$$

Let us check that $J_n = \Pi_n J : V_n \to 2^{V_n^*}$ is the duality mapping.

Lemma 22.2.1. *J_n is the duality mapping.*

Proof. Recall that $J : X \to X^*$, where X is a real, reflexive, strictly convex Banach space, is the duality map if, and only if,

$$\langle u, J(u)\rangle_X = \|u\|_X^2 = \|J(u)\|_{X^*}^2, \ \forall u \in X.$$

Since J is positive homogeneous and Π_n linear, it suffices to consider those $u \in V_n$ that satisfy $\|u\|_{V_n} = \|u\|_V = 1$. So, what we have to demonstrate is that, for such u,

$$\langle u, J_n u\rangle_{V_n} = \|J_n u\|_{V_n^*}^2 = 1.$$

To begin with,

$$\langle u, J_n(u)\rangle_{V_n} = \langle u, \Pi_n J(u)\rangle_{V_n} = \langle u, J(u)\rangle_V = \|u\|_V^2 = 1.$$

Furthermore,

$$1 = \langle u, \Pi_n J(u)\rangle_{V_n} \leq \|\Pi_n J(u)\|_{V_n^*} =$$
$$\sup_{\|v\|_{V_n} \leq 1, v \in V_n} \langle v, \Pi_n J(u)\rangle_{V_n} = \sup_{\|v\|_{V_n} \leq 1, v \in V_n} \langle v, J(u)\rangle_V \leq$$
$$\sup_{\|v\|_V \leq 1, v \in V} \langle v, J(u)\rangle_V \leq \|J(u)\|_{V^*} = 1.$$

Therefore
$$\|J_n(u)\|^2_{V_n^*} = \|\Pi_n J(u)\|^2_{V_n^*} = 1.$$
□

Since $\Pi_n J$ is the duality mapping, it is maximal monotone (Proposition 32.21, [20]). It follows that the operator $\Pi_n A_1 + \lambda \Pi_n J$ is also maximal monotone, for it is the sum of two maximal monotone operators (Theorem 10.0.3, Appendix B.1). Keeping in mind the definition of a maximal monotone operator, we observe that u_n is a solution of (22.2.1) if, and only if, it is a solution of an inclusion problem

$$(\Pi_n A_1 + \lambda \Pi_n J)(u_n) \in \Pi_n(u^*) - \Pi_n A_2'(u_n).$$

That u_n is a unique solution becomes apparent if we rewrite the above problem as follows:
$$((\Pi_n A_1 + \Pi_n A_2') + \lambda \Pi_n J)(u_n) \in \Pi_n(u^*).$$

Since $\Pi_n A_1 + \Pi_n A_2'$ is monotone and $\Pi_n(V^*) = V_n^*$, the existence of a solution implies, by Theorem 12.0.5 (Appendix C.1), that $\Pi_n A_1 + \Pi_n A_2'$ is, in fact, maximal monotone. Moreover, Proposition 12.0.9 (Appendix C.1) tells us that the operator $((\Pi_n A_1 + \Pi_n A_2') + \lambda \Pi_n J)^{-1}$ is single-valued. Finally, one can write

$$u_n = (\Pi_n A_1 + \lambda \Pi_n J)^{-1}(\Pi_n(u^*) - \Pi_n A_2'(u_n)). \tag{22.2.2}$$

Let us summarize what we have shown so far. We have established that, for $n \in \mathbb{N}$, variational inequality (22.2.1), corresponding to V_n, has a unique solution $u_n \in V_n$, which has the above form. Moreover, the following holds true.

Proposition 22.2.1. *The function $u_n : [0,T] \times \Omega \to V_n$ is measurable.*

Proof. Note that identity (22.2.2) can be written as

$$u_n = g \circ h((t,\omega), u_n),$$

with
$$g = (\Pi_n A_1 + \lambda \Pi_n J)^{-1}$$

and
$$h((t,\omega), u_n) = \Pi_n(u^*(t,\omega)) - \Pi_n A_2'(u_n).$$

As we know, g is demicontinous. However, since V_n is finite-dimensional, it is, in fact, continuous. Similarly, h is continuous in u_n (hemicontinuity is equivalent to continuity in a finite-dimensional setting) and measurable in (t,ω). So

$$u_n = G((t,\omega), u_n),$$

where $G := g \circ h$ is measurable in (t,ω), continuous in u_n and has a unique solution for each pair (t,ω).
Taking into consideration $\mathcal{B}(V_n^*)$ is generated by closed balls, it suffices to show that, given an arbitrary closed ball $\mathbf{B}$,

$$C_1 = \{(t,\omega) : u - G((t,\omega), u) = 0, u \in \mathbf{B}\}$$

is a measurable set.

Set

$$C_2 = \{(t,\omega) : \inf_{u \in \mathbf{B}} \|u - G((t,\omega),u)\|_V = 0\}.$$

Clearly, $C_1 \subseteq C_2$.
On the other hand,

$$\inf_{u \in \mathbf{B}} \|u - G((t,\omega),u)\|_V = 0,$$

means that $\exists \{u_m\} \in \mathbf{B}$, such that

$$\|u_m - G((t,\omega),u_m)\|_V \to 0.$$

Hence, there is a convergent subsequence $\{u_{m_i}\}$, satisfying

$$u_{m_i} \to u_0 \in \mathbf{B}.$$

Using the fact that G is continuous in u, we, finally, obtain

$$u_0 - G((t,\omega),u_0) = 0.$$

To conclude, $C_1 = C_2$. Suppose $\{v_m\}$ is a dense subset of $\mathbf{B}$.
Set

$$C_3 = \{(t,\omega) : \forall n \in \mathbb{N}, \exists i, \text{ s.t. } \|v_i - G((t,\omega),v_i)\|_V \le \frac{1}{n}\}$$

and

$$C_4 = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \{(t,\omega) : \|v_i - G((t,\omega),v_i)\|_V \le \frac{1}{n}\}.$$

Then, one can show that $C_2 = C_3 = C_4$.
Note that since C_4 is a measurable set, C_1 is also measurable. □

Chapter 23

Appendix I.

Lemma 23.0.2. *Let functions $\tau_i^n : \Omega \to [0,T]$, $0 \le i \le k_n + 1$ ($k_n = \lfloor Mn \rfloor$), be defined as follows:*

$$\tau_0^n := 0,$$

$$\tau_1^n := \min\{\inf\{t > 0 : U(t) \ge \frac{1}{n}\}, T\},$$

$$\tau_{i+1}^n := \min\{\inf\{t > \tau_i^n : U(t) - U(\tau_i^n) \ge \frac{1}{n}\}, T\},\ 1 \le i \le k_n,$$

$$\tau_{k_n+1}^n := T.$$

Then $\{\tau_i^n\}_{i=0}^{k_n+1}$ is a sequence of stopping times, satisfying (8.3.1).

Proof. Recall that $\sigma : \Omega \to [0,\infty]$ is a stopping time, provided $\{\sigma \le t\} \in \mathcal{F}_t$, $\forall t \ge 0$.
(i) Clearly, τ_0^n and $\tau_{k_n+1}^n$ are stopping times.
(ii) As far as τ_1^n is concerned, we consider three cases:

- $t = 0$. Then $\{\tau_1^n \le 0\} = \emptyset \in \mathcal{F}_0$, due to continuity of $U(t)$.
- $t = T$. In this case, $\{\tau_1^n \le T\} = \Omega \in \mathcal{F}_T$, since $\tau_1^n \le T$.
- $0 < t < T$. Due to the fact that $U(t)$ is adapted,

$$\{\tau_1^n \le t\} = \{\omega : U(t) \ge \frac{1}{n}\} \in \mathcal{F}_t.$$

(iii) In general, ignoring two trivial cases, if $0 < t < T$, then

$$\{\tau_i^n \le t\} = \{\omega : U(t) \ge \frac{i}{n}\} \in \mathcal{F}_t.$$

Moreover,

$$\sup_{0 \le i \le k_n} \triangle U_i^n \le \frac{1}{n} \to 0,$$

uniformly in ω, as desired. □

Lemma 23.0.3. *If $u_i^n : \Omega \to V$, $0 \le i \le k_n + 1$, is $\mathcal{F}_{\tau_i^n}$-measurable, then the process u^n is adapted.*

Proof. To start with, the space V is separable, which implies that its Borel σ-algebra $\mathcal{B}(V)$ is generated by open balls (Theorem 9.0.4, Appendix A). So, it is enough to check measurability on such sets.

(a) If $t = T$, then $u_T^n = u_{k_n+1}^n$, which is $\mathcal{F}_{\tau_{k_n+1}^n} = \mathcal{F}_T^n$, by assumption and the fact that $\tau_{k_n+1}^n = T$.
(b) Fix $t < T$. Then, for an arbitrary open ball $B \subseteq V$, we have

$$\{\omega : u_t^n \in B\} = \bigcup_{i=0}^{k_n} (\{\omega : u_i^n \in B\} \bigcap \{\omega : \tau_i^n \le t < \tau_{i+1}^n\}).$$

Let us check that each of the sets on the right belongs to $\mathcal{F}_t$. Now

$$\{\omega : u_i^n(\omega) \in B\} \bigcap \{\omega : \tau_i^n(\omega) \le t < \tau_{i+1}^n(\omega)\} =$$

$$\{\omega : u_i^n(\omega) \in B\} \bigcap (\{\omega : \tau_i^n(\omega) \le t\} / \{\omega : \tau_{i+1}^n(\omega) \le t\}).$$

Using the identity

$$A \bigcap (B/C) = (A \bigcap B)/(A \bigcap C),$$

one can write

$$\{\omega : u_i^n \in B\} \bigcap \{\omega : \tau_i^n \le t < \tau_{i+1}^n\} =$$

$$(\{\omega : u_i^n \in B\} \bigcap \{\omega : \tau_i^n \le t\}) / (\{\omega : u_i^n \in B\} \bigcap \{\omega : \tau_{i+1}^n \le t\}).$$

Recall that if τ is a stopping time, then the σ-algebra $\mathcal{F}_\tau$ is defined by

$$\mathcal{F}_\tau = \{\Lambda \in \mathcal{F} : \ \Lambda \bigcap \{\tau \le t\} \in \mathcal{F}_t, \ \forall t\}.$$

A useful observation is that, if $S \le T$ are stopping times, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.
Since u_i^n is $\mathcal{F}_{\tau_i}$-measurable and by definition $\mathcal{F}_{\tau_i}$, we have

$$\{\omega : u_i^n \in B\} \bigcap \{\omega : \tau_i^n \le t\} \in \mathcal{F}_t.$$

Likewise, due to the fact that $\tau_i \le \tau_{i+1}$, $\{\omega : u_i^n(\omega) \in B\} \in \mathcal{F}_{\tau_{i+1}}$ and, by the same argument,

$$\{\omega : u_i^n(\omega) \in B\} \bigcap \{\omega : \tau_{i+1}^n(\omega) \le t\} \in \mathcal{F}_t.$$

□

We require the following construction. Let (S_1, d_1) and (S_2, d_2) be separable metric spaces. The set $S = S_1 \times S_1 = \{(s_1, s_2) : s_1 \in S_1 \text{ and } s_2 \in S_2\}$ can be made into a metric space in a variety of ways. We shall define a metric d on S by

$$d((u_1, v_1), (u_2, v_2)) = \max\{d_1(u_1, u_2), d_2(v_1, v_2)\}.$$

It is not difficult to check that

- (1) d is, indeed, a metric and, thus, (S, d) is a metric space.
- (2) (S, d) is separable.
- (3) A sequence $(u_n, v_n) \to (u, v)$ in S if, and only if, $u_n \to u$ in S_1 and $v_n \to v$ in S_2.

An immediate consequence of (2) is that a Borel σ-algebra $\mathcal{B}(S)$ is generated by a collection of open balls $B(x, r) = \{y \in S : d(y, x) < r\}$, where $x \in S$ and $r > 0$.

Moreover, note that

$$B((s,t),r) = \{(u,v) \in S : d((u,v),(s,t)) < r\} =$$

$$\{(u,v) \in S : \max\{d_1(u,s), d_2(v,t)\} < r\} =$$

$$\{(u,v) \in S : d_1(u,s) < r \text{ and } d_2(v,t) < r\} =$$

$$\{u \in S_1 : d_1(u,s) < r\} \times \{v \in S_2 : d_2(v,t) < r\} =$$

$$B_1(s,r) \times B_2(t,r),$$

where B_1 and B_2 denote open balls in S_1 and S_1 respectively.
Suppose $(\Omega, \mathcal{F})$ is a measurable space and let $X_1 \to S_1$ and $X_2 \to S_2$ be functions. Define $X \to S$ by $X = (X_1, X_2)$. Then

Lemma 23.0.4. *X is $\mathcal{F}/\mathcal{B}(S)$-measurable if, and only if, X_1 is $\mathcal{F}/\mathcal{B}(S_1)$-measurable and X_2 is $\mathcal{F}/\mathcal{B}(S_2)$-measurable.*

Proof. (i)$\Rightarrow$ Define projection operators $\pi_i : S \to S_i$, $i = 1, 2$, by $\pi_i(x_1, x_2) = x_i$. Since, as is easy to verify, π_i is continuous (hence, $\mathcal{B}(S)/\mathcal{B}(S_i)$-measurable), $X_i = \pi_i \circ X$ is measurable, as a composition of measurable functions.
(ii)$\Leftarrow$ On the other hand, as we have remarked above, $\mathcal{B}(S)$ is generated by open balls, so, in order to see that X is $\mathcal{F}/\mathcal{B}(S)$-measurable, it is enough to check measurability on sets of this kind.
Take an open ball $B \subseteq S$. We know that $B = B_1 \times B_2$, where $B_1 \subseteq S_1$ and $B_2 \subseteq S_2$ are open balls. Then

$$\{\omega : X(\omega) \in B\} = \{\omega : X_1(\omega) \in B_1\} \bigcap \{\omega : X_2(\omega) \in B_2\} \in \mathcal{F}.$$

□

Remark 23.0.1. *Set $\Omega_{i+1} = \{U^n(\tau^n_{i+1}) > U^n(\tau^n_i)\} \in \mathcal{F}_{\tau^n_{i+1}}$. Then $(\Omega_{i+1}, \mathcal{F}_{\tau^n_{i+1}} \bigcap \Omega_{i+1})$ is a measurable space.*
Define $X_1 : \Omega_{i+1} \to (0, \delta_0]$ and $X_2 : \Omega_{i+1} \to V^*$ by

$$X_1 = U(\tau^n_{i+1}) - U(\tau^n_i) = \triangle U^n_i$$

and

$$X_2 = u^n_i + B(u^n_i)(M(\tau^n_{i+1}) - M(\tau^n_i)).$$

Note that X_1 is $\mathcal{F}_{\tau^n_{i+1}}/\mathcal{B}((0,\delta_0])$-measurable, and X_2 is $\mathcal{F}_{\tau^n_{i+1}}/\mathcal{B}(V^*)$-measurable. Therefore, by the above lemma, a function $X : \Omega_{i+1} \to (0, \delta_0] \times V^*$, defined by $X = (X_1, X_2)$, is $\mathcal{F}_{\tau^n_{i+1}}/\mathcal{B}((0,\delta_0] \times V^*)$-measurable.

Recall that a function $f : S \to V$, where S is a metric space and V is a separable reflexive Banach space, is called **demicontinuous** if

$$u_n \to u \text{ in } S \Rightarrow f(u_n) \rightharpoonup f(u) \text{ in } V.$$

Let us demonstrate that

Lemma 23.0.5. *A function $f : (0, \frac{1}{K}] \times V^* \to V$, given by $f(\delta, v^*) = (I - \delta A)^{-1} v^*$, is demicontinuous.*

Proof. Suppose $(\delta_n, v^*_n) \to (\delta, v^*)$, or, equivalently (see (3) above), $\delta_n \to \delta$ and $v^*_n \to v^*$. It follows that

- $\exists \mu > 0$, such that $\delta_n \geq \mu$, $\forall n \in \mathbb{R}$;
- The sequence $\{\|v_n^*\|\}$ is bounded.

Set $u_n = (I - \delta_n A)^{-1} v_n^*$ and $u = (I - \delta A)^{-1} v^*$ (recall that, by Theorem 3.3.1, the operator $I - \delta A$ is single-valued). This means that $\exists \{u_n^*\} \in V^*$, with $u_n^* \in Au_n$, and $u^* \in V^*$, $u^* \in Au$, such that

$$v_n^* = Iu_n - \delta_n u_n^* \text{ and } v^* = Iu - \delta u^*.$$

Now, by "coercivity" condition 8.1.3,

$$2\langle v_n^*, u_n\rangle = 2|u_n|^2 - 2\delta_n\langle v_n^*, u_n\rangle \geq$$

$$2|u_n|^2 + \delta_n \lambda \|u_n\|^2 - K\delta_n - K\delta_n |u_n|^2 \geq \mu\lambda\|u_n\|^2 - 1.$$

On the other hand,

$$2\langle v_n^*, u_n\rangle \leq \frac{\|v_n^*\|^2}{\epsilon} + \epsilon\|u_n\|^2, \ \forall \epsilon > 0.$$

Choosing ϵ in such a way that $\mu\lambda - \epsilon > 0$, and combining the above estimates, we obtain

$$(\mu\lambda - \epsilon)\|u_n\|^2 \leq \frac{\|v_n^*\|^2}{\epsilon} + 1 \leq C,$$

due to the boundedness of $\{\|v_n^*\|\}$.
It follows that $\{u_n\}$ is also bounded and thus contains a weakly convergent subsequence $\{u_{n(i)}\}$, such that $u_{n(i)} \rightharpoonup \tilde{u} \in V$.
Take arbitrary $w \in V$ and $w' \in (I - \delta A)w$. One can find $w^* \in Aw$, such that $w' = Iw - \delta w^*$. Moreover, keeping in mind that A is, in particular, "monotone", we have

$$\langle v_{n(i)}^* - w', u_{n(i)} - w\rangle =$$

$$\langle (Iu_{n(i)} - \delta_{n(i)} u_{n(i)}^*) - (Iw - \delta w^*), u_{n(i)} - w\rangle =$$

$$|u_{n(i)} - w|^2 - \langle \delta_{n(i)} u_{n(i)}^* - \delta w^*, u_{n(i)} - w\rangle =$$

$$|u_{n(i)} - w|^2 - \delta_{n(i)}\langle u_{n(i)}^* - w^*, u_{n(i)} - w\rangle - (\delta_{n(i)} - \delta)\langle w^*, u_{n(i)} - w\rangle \geq$$

$$-(\delta_{n(i)} - \delta)\langle w^*, u_{n(i)} - w\rangle.$$

Letting $i \to \infty$, we obtain

$$\langle v^* - w', \tilde{u} - w\rangle \geq 0.$$

Since (w, w') is arbitrary and the operator $I - \delta A$ is a maximal monotone (Theorem 3.3.1), we deduce that

$$v^* \in (I - \delta A)\tilde{u} \text{ or } \tilde{u} = (I - \delta A)^{-1} v^*,$$

from which it follows, due to the fact that $(I - \delta A)^{-1}$ is single-valued, that $\tilde{u} = u$, as required. □

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