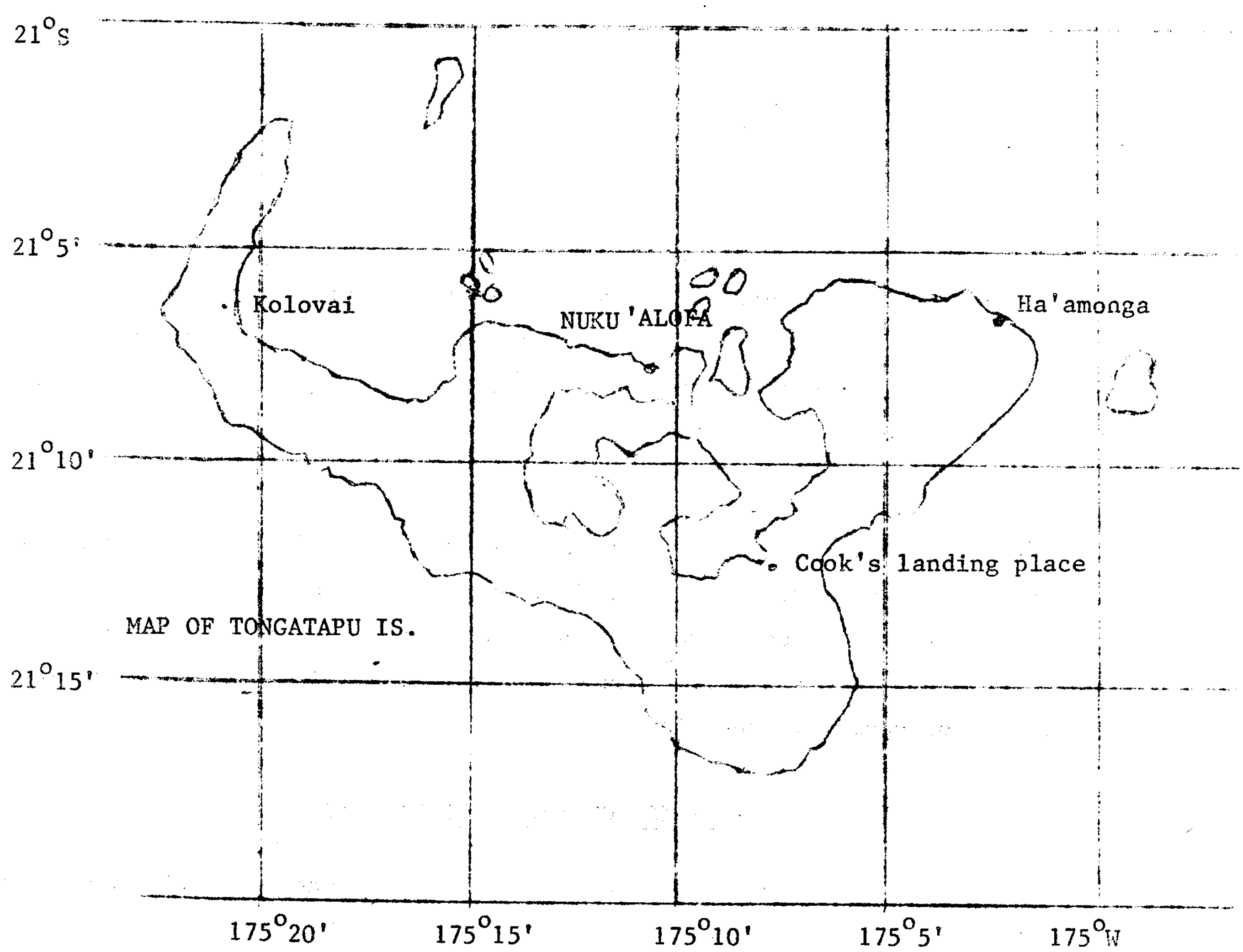
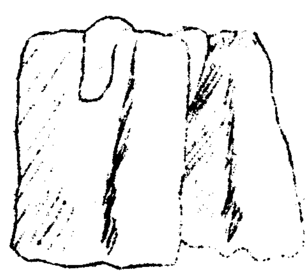
JAMES COOK MATHEMATICAL NOTES

Issue number 9, in May 1977





The Ha'amonga is a trilith of great antiquity, on the eastern coast of Tongatapu, the main island of what Cook called the Friendly Islands, now the Kingdom of Tonga. Marks cut in the upper of the three stones are thought to denote the bearing of the sun at dawn at the equinox and the two solstices.

Capt. Cook came to Tongatapu with HMS Resolution on 10 June 1777

MONOTONY (JCMN 8)

Does every infinite sequence in an ordered set have an infinite monotonic subsequence? G. Szekeres answers that if there is no decreasing subsequence then infinitely many of the terms are "minimal" in the sense of being strictly less than all their successors, and they form a strictly increasing subsequence. Other solutions came from H.O. Davies and J. Hickman.

CENTROIDS

Dan Pedoe writes from Minneapolis.

Your "Centre of Australia" (JCMN 6) arouses some anxious thoughts in my mind. It is an ancient error that lines through the centroid of a closed, simple curve bisect the area, and this is not stressed, or is it?

I was present at a conference of high school teachers on geometry when an "expert" gave a talk in which he showed, correctly, that given a direction, one could find a parallel line which bisected the area, and given another direction, one could find a line parallel to this direction which also bisected the area. "Let G be the intersection of these two bisectors", he said, "Then every line through G bisects the area".

After the lecture, I gently demonstrated the case of an equilateral triangle, for which this is not true. He had said that this was one of the fine teaching ideas in geometry which came from a student's paper. He said: "But I gave her an A for that!" He was not at all abashed, and certainly did not return his fee as a consultant on geometry education.

IS TRIGONOMETRY STILL TAUGHT?

The following is question 9 from the examination paper in trigonometry at Trinity College Cambridge, Wednesday June 7th, 1893, 9-12.

Having given that $\sin x = n \sin(x+\alpha)$, prove that when $n < 1$, $x + k\pi = n \sin \alpha + \frac{1}{2} n^2 \sin 2\alpha + (1/3)\, n^3 \sin 3\alpha + \ldots$ where k is some integer or zero. Explain how the value of k is to be chosen.

Also find a series for y in ascending powers of $\cos\alpha$ where

$$2 \tan y = \sin x \operatorname{cosec} \frac{x+\alpha}{2} \operatorname{cosec} \frac{x-\alpha}{2} .$$

A LITTLE PROBLEM FROM LONDON

Given any real or complex square matrix A, there is a set E of all complex k such that A and kA are similar. In other words $E = \{k;$ There is some nonsingular X such that $XA = kAX\}$. What can you say about E? Show that if the diagonal of A consists of positive real numbers then E contains only the number 1.

The above came from H. Kestelman of University College London, with the title "a little problem for first year students". Does UCL have better first year students than JCUNQ?

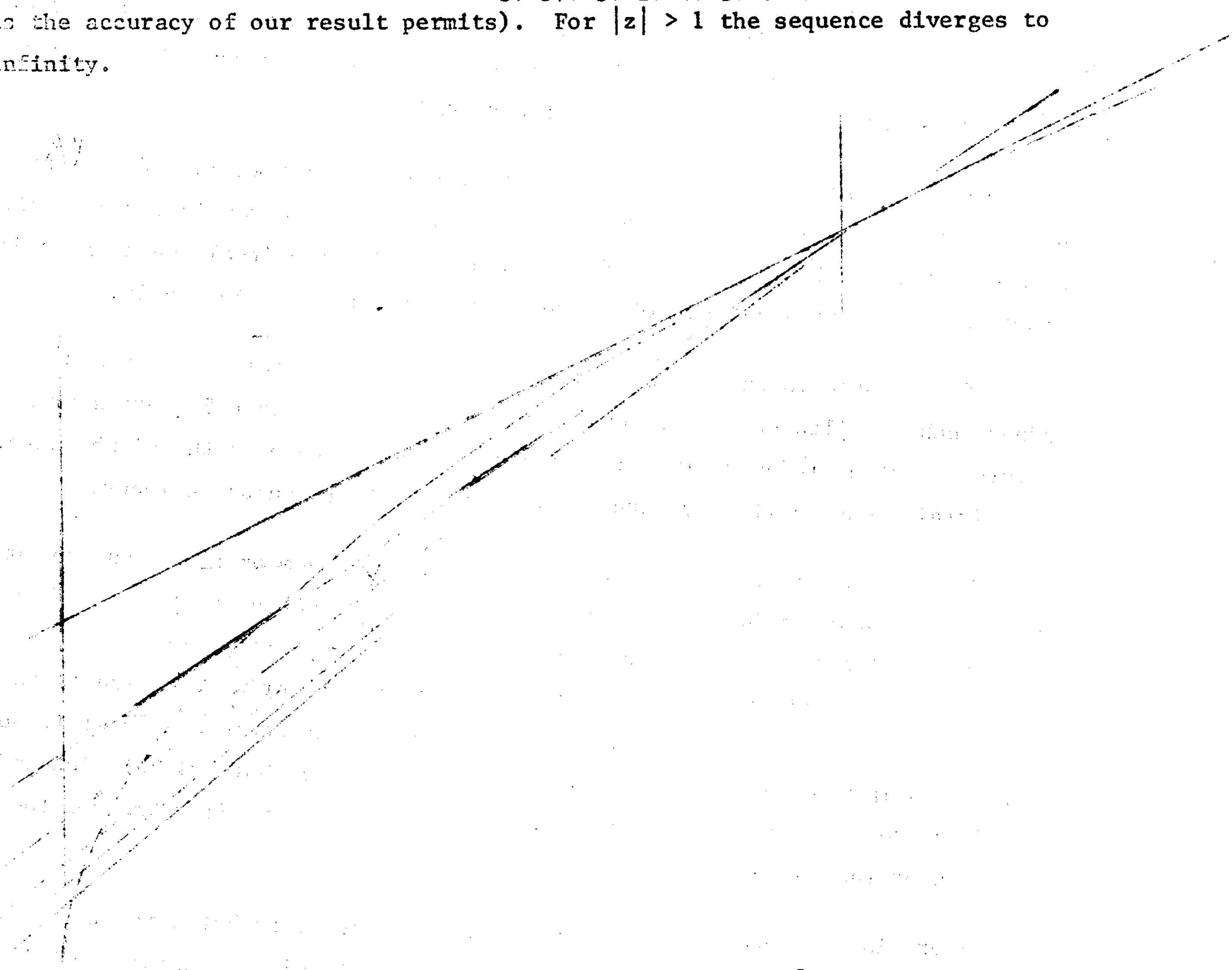
PROBLEM SUITABLE FOR UNDERGRADUATES

From J.M. Hammersley in JCMN 8 -- Investigate

$$\lim(\tfrac{1}{2} + \tfrac{1}{2}(\tfrac{1}{3} + \tfrac{2}{3}(\ldots(\tfrac{1}{n} + \tfrac{n-1}{n} z^2)^2 \ldots)^2)^2)^2$$

The following is compounded from submissions by H.O. Davies, J.B. Parker and G. Szekeres.

For $z \neq \pm 1$ on the unit circle and for $|z| < 1$ the limit is .3314, 0152, 7118, 5670 which in continued fractions is $\frac{1}{3+}\frac{1}{57+}\frac{1}{5+}\frac{1}{2+}\frac{1}{4+}\frac{1}{1+}\frac{1}{8+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{37+}\frac{1}{1+}$ (as far as the accuracy of our result permits). For $|z| > 1$ the sequence diverges to infinity.



Lines $(n+1)y = 1 + nx$ and parabola $y^2 = x$

For real positive z the result is obvious from the diagram, for the rule is to move vertically to a line, horizontally to the parabola, vertically to the next line etc. When the reasoning is formalised it extends naturally to complex variables, for example $f_n(z) = (1 + (n-1)z^2)/n$ is a contraction mapping of the set where $|z| < 1/3$, with contraction factor $< 2/3$, for all $n \geq 4$.

CAPTAIN COOK'S MOUNTAINS

The following story came in from G. Szekeres. -

Coming back from two weeks of camping I found the new issue of JCMN in my mail. It occurred to me that our camping had some links with JCMN. We passed through two of my favourite mountains, Mount Warning and Pigeon House, both named by Captain Cook, who apparently found them excellent for navigational purposes because of their oddly shaped and easily identifiable peaks. The two names are most appropriate and a refreshing change from the unimaginative practice of naming things after some obscure colonial official. Can you dig up anything about the naming of the mountains? - I would be interested.

Incidentally Mt. Warning is now a State Park about 15km out of Murwillumbah, quite close to the Queensland border. The track up to the top is exceptionally well laid out and is a Sunday climber's delight. The top itself on a clear day is to my mind the most beautiful 100 square metres anywhere in Australia.

Pigeon House is on the southern fringes of Morton National Park south of Sydney, behind Milton-Ulladulla on the South Coast. The last 50 metres or so are assisted by steel ladders, ingeniously laid out so as to make the climb slightly non-trivial though perfectly manageable by the ordinary Sunday climber.

Your editor has no more information about the two mountains, can any other reader help? However here is a little about the way Cook named things. The first landing from H.M.S. Endeavour on Australia was on 28th April 1770 in a large sheltered bay which Cook at first called Stingray Harbour on account of the fish there. However when Banks and Solander collected so many new plants he decided to call it Botany Bay. While the ship was anchored there one of the crew (Forby Sutherland) died and was buried ashore, Cook named the southern headland of the bay after him and to-day there is a town of that name.

There is not much written evidence for my next story but here it is. (Coming from a master mariner who knows the Queensland coast well). Captain Cook kept one of the store-rooms of the ship under his personal control, and one day in June 1770 after supervising the issue of some stores he went up to the quarterdeck and put the key in the binnacle. Later the compass appeared to be behaving erratically and Cook gave the name Magnetical Island to a high island in the offing, thinking that there might be a deposit of iron ore in the hills. Now we know that the magnetic field is quite well-behaved here, but we have kept the name of the island except for the slight change to Magnetic.

CONGRUENT QUADRILATERALS (JCMN 8)

The problem was to find on an ellipse two congruent quadrangles, the congruence not being one of the trivial symmetries of reflections in the principal axes. The only suggestion sent in so far was found to give only the trivial solution. A variation of the problem is as follows. Given ABC not co-normal on an ellipse, find DA'B'C'D' on the ellipse such that ABCD is congruent to A'B'C'D'.

THE AIRPORT WAITING ROOM (JCMN 3, 6 and 8)

A.P. Guinand writes -

Yes, if there are spare copies of earlier JCMN, I would be delighted to see them. Not least the relation between vanishing transforms and airport waiting rooms could begin to prey on my mind were it not for my stern mental self-discipline.

On the subject of vanishing transforms, I have now convinced myself that for any $K > 0$ it is possible to have a pair, $f(x)$ and $g(x) = 2\int_0^\infty f(t)\cos 2\pi xt\,dt$ not identically zero, but both vanishing in the interval $(0,K)$.

If $\sum b_n$ converges then $f(x) = 1/(\pi x)\sum_1^\infty b_n \sin 2\pi nx$ and $g(x) = \sum_{n\geq x} b_n$ are transforms. Now choose $\phi(x) = xf(x)$ vanishing over $(0,1/4)$, and an odd function with unit period, such that the first N Fourier coefficients, $b_1, \ldots b_N$ are all zero. This seems possible, can some other reader supply a simple elegant choice of such a function?

Then you get a pair of transforms $f(x)$ vanishing in $(0,1/4)$ and $g(x)$ constant in $(0,N)$. Taking a difference of two such gives a pair with the same properties except that $g = 0$ in $(0,N)$. Then put $F(x) = f(x/(2\sqrt{N}))$ and $G(x) = 2\sqrt{N}\ g(2x\sqrt{N})$, both vanish for x less than $\frac{1}{2}\sqrt{N}$.

POSITIVE DEFINITE

H. Kestelman writes:

A is a real non-symmetric n×n matrix. If $x^TAx > 0$ for all non zero x in R^n, is it true that all the principal subdeterminants of A are positive? Is the converse true?

AN EASY PROBLEM (JCMN 7)

If $f(x)$ is positive, increasing and concave on $(0,\infty)$ is $g(x) = f(1/x)$ necessarily convex? J.B. Parker sends the counter-example $f(x) = 2x$ for $0 < x < \frac{1}{2}$ and $= x + \frac{1}{2}$ for $x \geqslant \frac{1}{2}$, which is concave but $g(x) = f(1/x)$ is not convex. Another approach is to use second derivatives,

$$x^4 g''(x) = 2xf'(1/x) + f''(1/x)$$

If f'' is negative there is no need for g'' to be positive.

PLEA FOR HELP

When a student asks me for the indefinite integral of $\exp -x^2$, I usually say that it is not expressible in terms of elementary functions. What should I do if asked for a proof?

PROBLEM ON LIMITS

From J.B. Parker in JCMN 8; to show that if $0 < r < 1$ and $f(x) = xe^{rx}/(e^x-1)$ the ratio $\int_{-\infty}^{\infty} x f^n dx / \int_{-\infty}^{\infty} f^n dx$ tends to L where $1/(1-e^{-L})-1/L = r$.

First note that f has a strict maximum at L because f' changes sign only at this point, let the maximum value be M. For sufficiently small $\varepsilon > 0$ the function $g(x) = f(x)/(M-\varepsilon)$ is > 1 on an arbitrarily small interval $A < x < B$ containing L, and $0 < g < 1$ in the open complementary intervals, Suppose for the moment that r is greater than a half, that is L is positive, and we may take A and B to be positive.

$$\int_{-\infty}^{A} + \int_{A}^{B} + \int_{B}^{\infty} (x-A)g^n \, dx \text{ tends to infinity}$$

because the contributions from the first and last intervals tend to zero. Therefore this expression is ultimately positive and so $\int x\, g^n dx / \int g^n dx$ is ultimately greater than A. A similar argument shows that the ratio is ultimately less than B. This is so for all positive ε, and so the ratio tends to L.

The case of r less than a half may be dealt with similarly.

BOOK REVIEWS

GEOMETRY AND THE LIBERAL ARTS by DAN PEDOE, Penguin Books, 1976, ISBN 014055.1190. The prices printed on the cover are United Kingdom £2.50 and New Zealand $7.50.

This is a book for dipping into at random (or is that just one of my bad habits?)

The reader will expect much about art and will not be disappointed, finding Vitruvius, Dürer, Leonardo da Vinci and many more. However, the author interprets "liberal arts" in a very liberal way and most readers browsing through the index will find irresistible the temptation to look up something. How many modern mathematicians have ever followed one of Euclid's own proofs? I used to think that the principle of least time dated from Fermat, and I had no idea why the pediments of Greek buildings were held on the heads of stone Caryatides. To sum up, a book in which you will find nice little bits of entertainment and instruction.

B.C.R.

ELEMENTARY MATHEMATICAL ANALYSIS by I.T. ADAMSON Longman Mathematical Texts, 1975, ISBN 0-582-44285-0, 216 pages, paperback, $8.50.

What this book is and does can be put quite simply in one sentence. It writes elementary calculus of matriculation and first year level in the language of set theory, doing it carefully and well. That having been said, most of our readers will have their minds firmly made up as to whether the book is worth having. There is no need to say more. However it is hard to resist commenting on the author's insistence that $D_1 f(x,y)$ and $D_2 f(x,y)$ are the only reasonable notations for partial derivatives, saying about others that "the inconvenience... is too high a price to pay for the curious pleasure most of us derive from writing the symbol ∂." Your reviewer's opinion is that, pleasure or not, the ∂ is almost a necessity for those who teach thermodynamics, and even for doing a simple pure mathematical calculation like $\partial x/\partial y . \partial y/\partial z . \partial z/\partial x = -1$.

B.C.R.

Dan Pedoe writes: (on his book reviewed above)

Did I tell you that a review appeared in the "Journal of the Institute of Health Education", Jan. 1977, which ended "An interesting book for those engaged in design at all levels but valuable too to the general reader who will gain understanding of the need for care in the creation of man's spatial environment."!!!

EULER'S PRODUCT FOR SINE

Take any positive x and expand the function $\cos\theta x$ for $-\pi < \theta < \pi$ as a Fourier cosine series of period 2π. The coefficients are given by

$$a_n = (2/\pi) \int_0^{\pi} \cos\theta x \cos n\theta \, d\theta$$
$$= (1/\pi)\,(-1)^n \sin\pi x \left(\frac{1}{x+n} + \frac{1}{x-n}\right)$$

The Fourier series converges to the function and so

$$\cos\theta x = (1/\pi)\sin\pi x \left\{\frac{1}{x} + \sum(-1)^n \left(\frac{1}{x+n} + \frac{1}{x-n}\right)\cos n\theta\right\}$$

at the point $\theta = \pi$, and therefore

$$\pi\cot\pi x = 1/x + \sum\left(\frac{1}{x+n} + \frac{1}{x-n}\right)$$

For $0 < x < 1$ the series converges uniformly (the M test of Weierstrass) and so we may integrate term by term from zero to x.

$$\left[\log\sin\pi x - \log x\right]_0^x = \sum\log(1 - x^2/n^2)$$
$$(\sin\pi x)/(\pi x) = \Pi(1 - x^2/n^2)$$

It is not hard to show that $x\,\Pi(1 - x^2/n^2)$ is odd and has period = 2, and so the product formula is established for all real x.

THE INTERNATIONAL SCENE

Readers may be interested to know about "Eureka" from which we occasionally lift items for JCMN (and vice versa). Eureka is published at Algonquin College, Ottawa, it consists of 10 issues per year, and the subscription rate (outside USA and Canada) is 7 Canadian dollars per year. Enquiries about subscriptions or back numbers to F.G.B. Maskell, Algonquin College, 200 Lees Avenue, Ottawa, Ont. KIS CC5, Canada.

Your editor would like to hear from you anything connected with mathematics or with James Cook.

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