JAMES COOK MATHEMATICAL NOTES

Issue number 8, in February 1977

RECRUITING

James Cook volunteered for the Royal Navy on 17th June 1755 and entered as an able seaman. The manning of naval vessels was mainly by the press gangs as the conscription system was called in those days. In Cook's first ship H.M.S. Eagle his ability was recognized by the captain (Hugh Palliser) who pressed for his promotion. Afterwards Captain Palliser was knighted and became Governor of Newfoundland and Comptroller of the Navy but in later life used to say that his greatest achievement had been to recognise and encourage the genius of James Cook.

Eventually Cook was commissioned and had ships under his command, unlike most other captains he had little difficulty in bringing his crews up to strength, for men volunteered to serve under him. The university now named after him does not find itself in the happy position of having a surplus of would-be students and has been impelled to advertise in newspapers all over Australia as follows:

"What are you going to do in 1977? Have you thought of tertiary study? There are still some places available for new students at James Cook University of North Queensland. The University, at Townsville, a tropical city of about 100,000 people, and a centre of continuing commercial and industrial expansion, is housed in modern air-conditioned buildings on a spacious campus... if you...are interested in a degree in Arts, Commerce, Economics, Education or Engineering (Civil or Electrical), write immediately..."

Strangely the university authorities omitted the Faculty of Science from the list of those with vacancies. James Cook managed without a science degree to be awarded the Copley Medal of the Royal Society (pictured above).

SHIPWRECK DELAYS CAPTAIN COOK

Captain Gordon Cook who set out from Plymouth 200 years after Captain James Cook, in the seventy-foot schooner Wavewalker, was driven by storms on to the rocks of Amsterdam Island (37°S, 77°E) in January. He and his wife Mary and their children Jonathan and Suzanne are being cared for by a party of scientists on the island and are hoping to be able to repair the boat and set sail again for Fremantle.

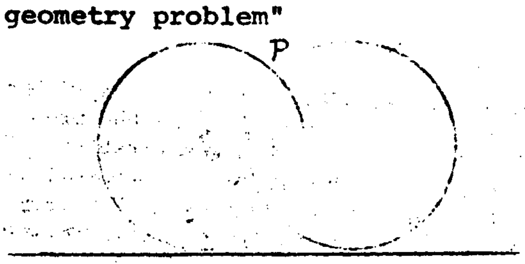
MONOTONY

Another problem from H. Kestelman.

Does every infinite sequence in an ordered set contain an infinite monotonic subsequence? The word "monotonic" is of course meant in the non-strict sense.

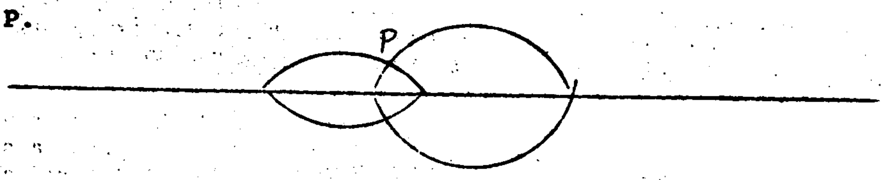
PENNY GEOMETRY

Two readers have responded to this item in JCMN 7. First A.J. Dobson (who is moving to the University of Newcastle at the end of February) wrote "In my school days a penny had the property that its edge could touch a line at a single point. Of course, such an expedient may have been available only in girls' schools, but I used it to solve your geometry problem"



This picture, I think, explains itself. To draw the perpendicular from the given point P to the given line put the penny (in the two possible ways) so that the edge passes through the point and touches the line.

I answered saying that in the strict penny geometry the penny could only be used as Euclid used his straight edge, to join two distinct points. Thereupon she wrote back giving a construction according to the strict rules. Again I think the picture explains itself, the idea is to have any two circles meeting in P, then their mirror images in the given line must meet at the mirror image of P.



Also a letter came from D. Pedoe. He refers to page 124 of his book "A course of Geometry for Colleges and Universities" C.U.P. (1970) (There is no ban on advertising in JCMN). This gives a construction by Paul Pargas for finding the point diametrically opposite to any given point on a circle drawn round the penny. This enables you to find the centre of one of your penny-sized circles, and by a theorem of Poncelet all Euclidean constructions are possible if you have a straight edge and just one circle with its centre.

CONGRUENT QUADRILATERALS

This problem is from G. Szekeres.

Given an ellipse, find two sets of four points, A B C D and A'B'C'D' on it such that the quadrilateral A B C D is congruent to A'B'C'D' but is not obtained from it by one of the trivial symmetries (reflections in the principal axes).

QUESTION VI (from JCMN 4)

The first problem set by Adelaide's first professor of Mathematics can lay some claim to another pair of 'firsts'. It predates by 65 years the first problems in linear programming - F.L. Hitchcock's transportation problem and J. Cornfield's diet problem; and it predates by 19 years Alfred Marshall's recognition of the Giffen paradox.

A senior secondary student of today would solve the problem graphically by considering the problem in the following form: find the quantities, x and y, of claret and ale that are to be bought at the given prices if x is to be maximised subject to the given budget constraint and the 'nutrition' constraint. The effect of an increase in the price of ale on the optimal value of y then becomes apparent.

However, the solution may be exposed more readily if we lapse into a literary approach that would not have been out of place in the nineteenth century.

Clearly, ale is the cheaper source of alcohol, and the man's craving for ale may be satisfied simply by applying part of his purse to the purchase of ale alone. However, since insufficient alcohol can be got by buying claret alone the optimal basket will contain both ale and claret.

We can imagine our bibulous friend filling his basket in the following way. Let him first buy enough of ale alone to satisfy his craving. Then the remainder of his purse can be used to finance the return of some ale in exchange for its alcoholic equivalent in claret.

Now suppose that the price of ale rises (but not so much that his craving cannot be satisfied by ale alone). Then the value of the contents of the basket will increase. Hence, some claret must be 'liquidated' by returning it in exchange for its alcoholic equivalent in ale. We thus have an example of Giffen's paradox - a situation where an increase in the price of a commodity causes more of it to be consumed.

Don Watson

PROBLEM SUITABLE FOR UNDERGRADUATES

J.M. Hammersley sends the following.

Investigate the existence and the value of

$$\lim_{n\to\infty} (\tfrac{1}{2} + \tfrac{1}{2} (\tfrac{1}{3} + \tfrac{2}{3}(\tfrac{1}{4} + \tfrac{3}{4} (\ldots(\tfrac{1}{n} + \tfrac{n-1}{n} z^2)^2 \ldots)^2)^2)^2)^2,$$

where z is a complex number.

The only available undergraduate reader of JCMN, who is about to start his second year studies at the Australian National University, thought the problem rather hard, and so solutions from graduates would be welcome.

THE AIRPORT WAITING ROOM (from JCMN3)

A.P. Guinand sends the following:

"Can a function in L^2 and its Fourier transform both vanish in an interval?" Answer, YES, as the following example shows.

Use cosine transforms for the kernel $2 \cos 2\pi x$, so that a pair of transforms can be denoted by $f(x)$, $g(x)$, with

$$g(x) = 2 \int_0^\infty f(t) \cos 2\pi xt \, dt \;,$$

and its inverse. Then for any real positive k, the functions $f(kx)$, $(1/k)g(x/k)$ are also such transforms.

A particular pair of transforms is

$$f(x) = \frac{[x] - x}{x},$$

$$g(x) = -\sum_{1\le n\le x} \frac{1}{n} + \log x + \gamma,$$

where $[x]$ is the greatest integer function, γ is Euler's constant, and for positive integral x the mid-value $\tfrac{1}{2}\{f(x-0) + f(x+0)\}$ is taken for $f(x)$; similarly for $g(x)$. To show this, consider

$$2\int_0^\infty g(t) \cos 2\pi xt \, dt = -2 \int_0^\infty \{\sum_{1\le n\le t} \frac{1}{n} - \log t - \gamma\} \cos 2\pi xt \, dt \;.$$

Integrating by parts and using Stieltjes integrals, this becomes

$$-2\left[\{\sum_{1\le n\le t} \frac{1}{n} - \log t - \gamma\}\frac{\sin 2\pi xt}{2\pi x}\right]_0^\infty + \frac{1}{\pi x} \int_0^\infty \sin 2\pi xt \, d\,\{\sum_{1\le n\le t} \frac{1}{n} - \log t - \gamma\}$$

$$= \frac{1}{x} \sum_{n=1}^\infty \frac{\sin 2\pi nx}{n} - \frac{1}{x} \int_0^\infty \frac{\sin 2\pi xt}{t} \, dt$$

$$= \frac{1}{x}\{[x] - x + \tfrac{1}{2}\} - \frac{1}{2x} = \frac{[x] - x}{x} = f(x) \;,$$

since the series is the Fourier series of $|x| - x + \frac{1}{2}$, and for $x > 0$,

$$\int_0^\infty \frac{\sin 2\pi xt}{t}\,dt = \tfrac{1}{2}\pi .$$

Note that, for $0 < x < 1$, we have $f(x) = -1$ and $g(x) = \log x + \gamma$. Also $f(x)$ is bounded for all x, and $O(1/x)$ as $x \to \infty$; hence it is in L^2, and so is $g(x)$.

Now put $F(x) = 32f(4x) - 44f(2x) + 17f(\frac{1}{2}x) - 5f(\frac{1}{4}x)$.

Then its transform is

$$G(x) = 8g(\tfrac{1}{4}x) - 22g(\tfrac{1}{2}x) + 34g(2x) - 20g(4x) .$$

For $0 < x < \frac{1}{4}$ we have

$$F(x) = -32 + 44 - 17 + 5 = 0 ,$$

and
$$G(x) = 8(\log \tfrac{1}{4}x + \gamma) - 22(\log \tfrac{1}{2}x + \gamma) + 34(\log 2x + \gamma) - 20(\log 4x + \gamma)$$
$$= (8 - 22 + 34 - 20)(\log x + \gamma) + \log 2(-16 + 22 + 34 - 40)$$
$$= 0.$$

That is, both $F(x)$ and $G(x)$ vanish for $0 < x < \frac{1}{4}$.

More generally, for any k in $0 < k < 1$, the functions

$$(k^2+2k)f(k^2x) - (k^2+2k+3)f(kx) + (k^{-2}+2k^{-1}+3)f(k^{-1}x) - (k^{-2}+2k^{-1})f(k^{-2}x),$$

and

$$(1+2k^{-1})g(k^{-2}x) - (k+2+3k^{-1})g(k^{-1}x) + (k^{-1}+2+3k)g(kx) - (1+2k)g(k^2x)$$

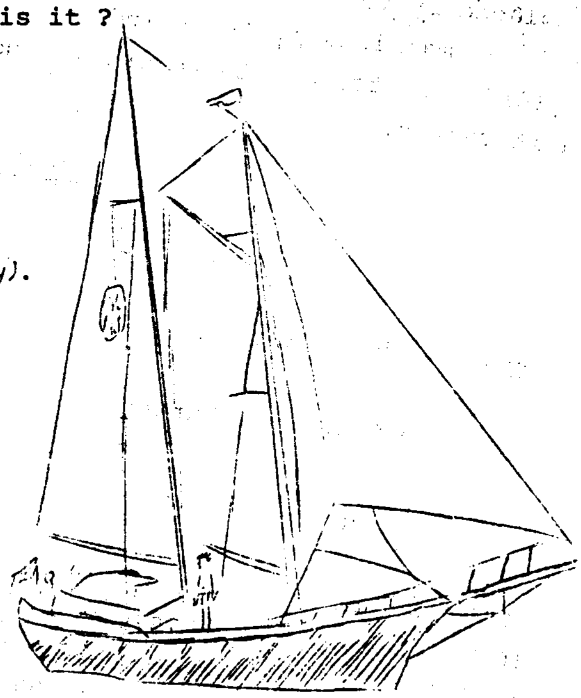
are cosine transforms, and both vanish for $0 < x < k^2$.

That is, a pair of transforms in L^2 with respect to the cosine kernel $2 \cos 2\pi x$ can both vanish in any interval $0 < x < K < 1$.

QUESTION. Is the above possible for any $K \geq 1$? Is there an upper bound for such K ? If so, what is it ?

STOP PRESS

The schooner Wavewalker was expected at Fremantle about the time this issue was being produced (19th February).



SPANNING BY COLUMNS (from JCMN 7)

H. Kestelman's problem was to show that if A is real $n \times n$ and if $p_1 \ldots p_q$ are real polynomials with no common divisor the columns of $p_1(A), \ldots p_q(A)$ together span R^n.

The solution by G. Szekeres is as follows. By the theory of polynomials there exist $P_1 \ldots P_q$ such that $p_1P_1 + \ldots + p_qP_q = 1$; and this polynomial equation is satisfied if the variable x is replaced by the matrix A. The space spanned by all the columns of all the matrices $p_i(A)$ includes the n columns of the product $p_1(A)P_1(A)$ etc, and therefore includes all the columns of the unit matrix.

DECIMALS

Notes from M.L. Cartwright

To-day we have in England	23·45,
U.S.A.	23.45,
Continent of Europe	23,45,
or	23_{45}, or $23{\scriptstyle 45}$
Cheques in U.S.A.	$\$23\frac{45}{100}$.

1631 Oughtred $0\lfloor\underline{56}$ for 0·56

1657 van Schooten 17579625... (3) for 17579·625

1571-1635 Adriaen van Metius 47852°:8'0"4'''

or 47852/8'0"4''' for 47852·804

and he spoke of "$\frac{481481}{1000000}$. often 4'8"1'''4""8"""1"""" ."

1616 Johann Hartman Beyer wrote similarly to Kepler

314,1'5"9'''2""6"""5""""

1626 Stevin wrote 1993, (0) 2, (1) 7, (2) 3 for 1993·273.

See D.E. Smith, History of Mathematics Vol 2, Ginn U.S.A. (1925) pp. 244-246.

Australian usage is rather constrained by the way U.S.A. makers have cornered the market in mathematical typewriters - Editor.

MORE ANECDOTAGE (from JCMN 7)

G. Szekeres writes as follows

(i) The product $x(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})\ldots = \sin x$ is due to Euler, but he was quite aware (I think) that he had no rigorous proof (he later supplied several).

He did it through the series

$$\pi \cot \pi x = 1/x + \sum_1^\infty (1/(x+n) + 1/(x-n))$$

Knopp's Infinite Series gives details in Chapter 6, too lengthy to reproduce here.

(ii) (On the question 4 from page 203 of Todhunter, asking for the solution of a certain quintic equation)

$$x^5 - 10x^3 + 20x - 8 = (x+2)(x^4 - 2x^3 + 12x - 4)$$

$$= (x+2)(x^2 - x - 1 - \sqrt{5}(x-1))(x^2 - x - 1 + \sqrt{5}(x-1))$$

(very mild use of Galois theory)

The comment about making "very mild" use of Galois theory reminded your editor of a story recounted by the Warden of a College.

During the war there was an Army camp near a town in northern N.S.W., and one Saturday morning a soldier and his girl friend went to the Anglican church and asked the Rector if he could marry them that afternoon. The Rector explained that he could not because of the State law which required that due notice be given before a marriage could legally take place. The soldier thought for a bit and then asked "Well, in that case, I wonder if you would say a few words, just enough to see us over the week-end."

PROBLEM ON LIMITS

Show that if $0 < r < 1$ the ratio

$$\int_{-\infty}^{\infty} \frac{x^{n+1} e^{nrx}}{(e^x-1)^n} dx \Big/ \int_{-\infty}^{\infty} \frac{x^n e^{nrx}}{(e^x-1)^n} dx$$

tends to a limit as $n \to \infty$, and that this limit L satisfies the equation

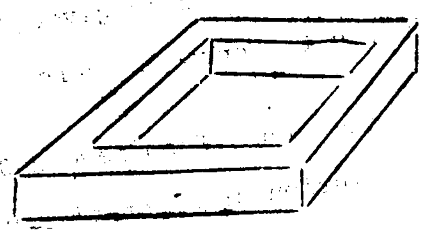
$$\frac{1}{1-e^{-L}} - \frac{1}{L} = r.$$

J.B. Parker

THE WINDOW FRAME

The following anecdote is from C.F. Moppert.

In my examination for elementary maths teacher (up to eighth school year) I was asked the Euler polyhedron formula. I said it was $v + f = e + 2$, the deplorable form in which I had learnt it in high school. The examiner asked me then whether the formula was valid for polyhedra which were not regular. I said yes, it was also valid e.g. for a window frame (my high school teacher had shown us that this is indeed the case)



The examiner said this was not true, he didn't even consent to check it. Who was right?

RATIONAL ROOTS (JCMN 7)

J.B. Parker's solution for the equation $1 + 2x + 3x^2 + \ldots + nx^{n-1} = n^2$ is to rewrite it

$$nx^{n+1} - (n+1)x^n + 1 = n^2(x-1)^2$$

and by writing $x = 1 + a$ to get

$$(1 + a)^n(n + na - n - 1) = n^2a^2 - 1$$

This is true for $a = 1/n$ so that $(n+1)/n$ is a rational root. Another solution came in from G. Szekeres.

Your editor would like to hear from you anything connected with mathematics or with James Cook.

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