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The James Cook Mathematical Notes is published in 3 issues per year, dated January, May and September. The history of JCMN is that the first issue appeared in September 1975, and others at irregular intervals, all the issues up to number 31 being produced and sent out by the Mathematics Department of the James Cook University of North Queensland, of which I was then Professor. In October 1983 this arrangement was beginning to be unsatisfactory, and I changed to publishing the JCMN myself, having three issues per year printed in Singapore and posted from there. I then set a subscription price of 30 Singapore dollars per year. When in 1985 I changed to printing in Australia I kept the same price, for the Singapore dollar is a stable currency.

Since October 1992 it has become clear that the paying of subscriptions by readers is an inefficient operation. Bank charges for changing currency and for international transfers, with postage, together absorb most of the initial input of money. Therefore we have abandoned subscriptions as from the beginning of 1993, issue number 60. To those who want to give something in return for the JCMN, I ask them to make a gift to an animal welfare society in their own country. The animals of the world will be grateful and so will I.

Contributors, please tell me if and how you would like your address printed.

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FUNCTIONAL INEQUALITY (JCMN 60, p.6202)

Don Coppersmith
(Thomas J. Watson Research Center, Yorktown Heights, NY, USA)

Find the maximum of $\int_0^1 f(x)dx$ for $f(x) \geq 0$ satisfying

$$f(x) + \frac{1}{2} f(x/2) + \frac{1}{3} f(x/3) + \ldots + \frac{1}{n} f(x/n) \leq 1.$$

The answer is $\frac{n}{2n-1}$, established as follows.

For any $b > 0$, integrating from b to 2b gives:

$$\int_b^{2b} + \int_{b/2}^{2b/2} + \int_{b/3}^{2b/3} + \ldots + \int_{b/n}^{2b/n} f(x)dx \leq b \quad \text{and so}$$

$$\int_{b/n}^{2b} f(x)\,dx \leq b.$$

Now give b the infinite set of values $\frac{1}{2}, \frac{1}{2}(2n)^{-1}, \frac{1}{2}(2n)^{-2}, \ldots$ in the inequality above and add.

$$\int_0^1 f(x)\,dx \leq \tfrac{1}{2}(1 + (2n)^{-1} + (2n)^{-2} + \ldots\ ) = \frac{n}{2n-1}$$

To show that this bound is the best possible, consider the set S consisting of the union of all the open intervals $(\frac{1}{2}(2n)^{-k}, (2n)^{-k})$ for $k = 0, 1, 2, \ldots$, and let $f(x)$ be the characteristic function of the set, equal to 1 in S and zero elsewhere. We must show that this f satisfies the conditions.

Take any x in the unit interval. Now, for what integer i is x/i in the set S?. In other words, for what i is the number i/x in the union of intervals:- $(1, 2)\cup(2n, 4n)\cup(4n^2, 8n^2)\cup \ldots$? There are certainly some such i, let j be the smallest. We are concerned only with $i \leq n$, and there cannot be more than j of these i, namely $i = j, j+1, j+2, \ldots 2j-1$. Therefore the sum

$$f(x) + \frac{1}{2} f(x/2) + \frac{1}{3} f(x/3) + \ldots + \frac{1}{n} f(x/n)$$
$$\leq \frac{1}{j} + \frac{1}{j+1} + \frac{1}{j+2} + \ldots \frac{1}{2j-1} \leq \frac{j}{j} = 1.$$

SOME PROBLEMS

Paul Erdös
(Mathematical Institute, Hungarian Academy of Sciences)

Here is a question of Sárközy and myself, in fact several questions:-

Let $1 \leq a_1 < a_2 < \ldots < a_{n+1} \leq 2n$ be n+1 integers $\leq 2n$. Trivially $(a_i, a_j) = 1$ for some $i < j$, in fact for some consecutive a_i, a_j. Much less trivial is the question of whether $(a_i, a_j) = 1$ for some $i < j$ with $a_i \leq n$.

Is it true that $a_i < n$ is possible except in the case of the set $(n, n+1, n+2, n+3, \ldots 2n)$?

We have proved (not trivial) that for $(a_i, a_j) = 1$,

$$\max (a_j - a_i) = n - f(n)$$

where f(n) is about log n, i.e. coprime a_i and a_j can be found for which $a_j - a_i$ is nearly n, but you cannot quite get n, it is about n - log n, we do not have an exact value.

Finally, determine or estimate

$$\max(a_i + a_j \,;\, (a_i, a_j) = 1)$$

We have proved that for every sequence the inequality $a_i + a_j > n - o(n)$ is possible with $(a_i, a_j) = 1$, but we know that one can give $a_1 < a_2 < a_3 < \ldots < a_{n+1}$ for which $\max (a_i + a_j \,;\, (a_i, a_j) = 1) = n - f(n)$, where $f(n) \to \infty$ (but fairly slowly).

An old problem of Sárközy and myself states: Let $a_1 < a_2 < \ldots < a_k \leq n$ be such that no a_i divides the sum of two larger a's. Is it then true that $k \leq n/3 + o(1)$, and that the maximum of k is given by the n/3 largest integers? We have no proof.

FACTORS (JCMN 59, p.6173)

For any positive integer n, write the fraction

$$\frac{(2n)!}{n!\ (n + 1000)!}$$

in its lowest terms. Let f(n) be the largest prime dividing the denominator, except that f = 1 if there is no such prime. Find the largest possible value of f(n).

FIRST SOLUTION

Terry Tao (Princeton, U.S.A.)

The answer is 1999, which is the largest prime < 2000.

Lemma 1 If x > 2000 then

$$\left[\frac{n}{x}\right] + \left[\frac{n + 1000}{x}\right] \le \left[\frac{2n}{x}\right]$$

Proof If the non-integer part of n/x is < 1/2, then there is equality, because the two terms on the left are equal. If the non-integer part of n/x is ≥ 1/2, then LHS ≤ 2[n/x] + 1 = RHS.

Lemma 2 If p is a prime, then the largest i for which p^i divides m! is $i = \left[\frac{m}{p}\right] + \left[\frac{m}{p^2}\right] + \ldots$ (This is clear)

Lemma 3 f(n) < 2000 for all n.

Proof Take any prime p > 2000 and calculate the number of times it occurs as a factor firstly in (2n)! and secondly in n!(n+1000)! using Lemma 2. Lemma 1 tells us that the first total ≥ the second total, and therefore there is no factor p of the denominator when the fraction is reduced to its lowest terms.

Now consider n = 999 and p = 1999. Clearly p occurs just once as a factor of n!(n+1000)!, and it is not a factor of (2n)! = 1998! The required result is thus proved.

SECOND SOLUTION

P. H. Diananda (Singapore)

Lemma 1 f(999) = 1999, because $\frac{1998!}{999! \times 1999!} = \frac{1}{999! \times 1999}$.

Lemma 2 For n ≤ 1000, clearly f(n) ≤ 2000, and so f(n) ≤ 1999.

Lemma 3 For n > 1000, the fraction $\frac{(2n)!}{n!(n+1000)!}$

= Integer / (n+1)(n+2) ... (n+1000), and also

= Integer / n(n-1)(n-2) ... (n-999).

Thus f(n) divides both n+a and n-b, for some integers a and b such that 1 ≤ a ≤ 1000 and 0 ≤ b ≤ 999.

Hence f(n) divides (n+a)-(n-b) = a + b, and 1 ≤ a+b ≤ 1999, therefore f(n) ≤ 1999.

From these three results, f(n) has the largest value 1999.

Consider the corresponding problem for $\frac{(2n)!}{n!(n+m)!}$ with integer m ≥ 1, the following can be proved:

If m = 1, f(n) = 1 for all n.

If m > 1, then the largest value of f(n) is the largest prime < 2m, it is attained when n = this prime - m.

PRINCETON PROBLEMS

1 Five weightless charged particles are in a frictionless hollow sphere. They have equal electric charges, and so repel one another acccording to the inverse square law. What are the stable positions of equilibrium? In other words, in what positions does the potential energy attain a minimum?

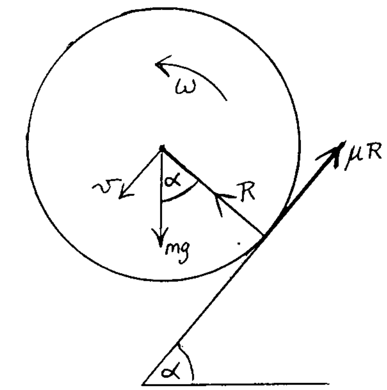
2 A Hausdorff space is called "regular" when any closed set and any point not in it can be contained in disjoint open sets. Prove that if a connected regular space contains at least two points it is uncountable.

3 Consider the following two-person game. Player A chooses two unequal positive numbers and puts them in sealed envelopes. Player B chooses one of the envelopes and opens it to see the number. Then B has to choose either that number or the unknown number in the other envelope. Scoring is that B scores +1 by choosing the larger of the two numbers and -1 by choosing the smaller. Find a strategy for B that gives a strictly positive expectation of score.

There is the obvious answer that B thinks of a number k before opening the envelope, then chooses the number if it is > k. But to be a valid solution this strategy has to be stated carefully; B must choose k so that there is a non-zero probability of k being between A's two numbers, and to do this B must choose from a probability distribution that is non-null on every non-trivial interval, i.e. the probability function must be strictly monotonic in $(0, \infty)$.

NEW PROOF THAT 2 = 1

M. N. Brearley



A uniform sphere is released from rest on a rough inclined plane which is steep enough to cause the sphere to slip initially. Notation is shown on the figure. Let t be the time after the start.

The principles of linear and angular momentum give:

$$m\,dv/dt = mg \sin \alpha - \mu R$$

$$R = mg \cos \alpha$$

$$(2/5)ma\ d\omega/dt = \mu R.$$

Eliminate R, giving $dv/dt = g(\sin \alpha - \mu \cos \alpha),$

$$a\,d\omega/dt = (5/2)\ \mu g \cos \alpha.$$

Integrate, using the fact that v and ω are initially zero,

$$v = g(\sin \alpha - \mu \cos \alpha)t$$

$$a\omega = (5t/2)\ \mu g \cos \alpha$$

At the instant t when slipping ceases, $v - a\omega = 0$.

Therefore $\sin \alpha - \mu \cos \alpha - (5/2)\ \mu \cos \alpha = 0,$

i.e. $(7/2)\ \mu = \tan \alpha.$

In particular, if $\mu = 4/7$ and $\alpha = 45^\circ$, we get

$$2 = 1.$$

Q.E.D

DIRTY STATISTICS IN A DIRTY WORLD

John Parker
(Oak Tree Cottage, Reading Road, RG74QN, U.K.)

The World is rather a dirty place, neither pure white nor jet black, but a sort of middling grey. So it was once when I was asked to comment on two measurements of a certain obscure physical constant not found in the usual reference books. It was measured as 4·4 ± ·1 in one laboratory, and as 8·2 ± ·05 in another laboratory. The boss presented me with this information, saying that he wanted me to give him the best estimate for the constant in two hours time; he hit the roof when I enquired what the ± signs represented (were they standard deviations? probable errors? 95% errors?), of course he didn't know, I wonder what the authors of the figures thought they meant. Probably it doesn't matter anyway.

In both cases the ± terms were probably based on replications. Every good scientist likes to repeat his experiment many times over, just to be on the safe side. After all, if he's got his apparatus nicely set up he might as well go to town with a good long series of measurements.

What had happened here was that the two experimenters were using two quite different techniques, in geographically widely separated laboratories, and of course with different personnel. The quoted errors do not take account of possible systematic biases occurring at one or both stations.

So I just ignored the quoted errors and gave my boss the answer 6·3 ± 1·9, and luckily he did not ask me whether the error term ± 1·9 was a standard deviation, a probable error or a 95% error.

Another example to show the problems facing statisticians in this dirty real world is the following. Suppose that there are 3 independent estimates of a quantity, namely 6·1, 12·4 and 13·0. It is believed that the experimental errors (standard deviations) are of the order of 2·0, but nobody is really

confident about this, a typical state of affairs in a dirty world. What is the best compromise figure to adopt and why?

Finally, a quite different problem, arising from a personal experience. I had written a simple linear regression program, with an option to plot out the data. An irate customer told me the program was up the creek because it gave a ridiculous line. He had not used the plotting option, and when he did the matter was speedily explained, for one of the ordinates (near the end of the range, moreover) had been keyed in wrongly. The real bloomer was mine, for I ought to have issued a health warning in the program write-up. All this was well before the modern era of computer graphics. The lesson is clear, always look at your data before deciding how to analyse them (and key them in carefully).

Mathematical statistics in an ideal world (with the Gaussian law and all that) is marvellous, and moderate departures from a Gaussian hypothesis are often of no serious consequence (one comes across the word "robustness"). But in the real world the occasional blunder crops up from time to time, we hope it happens only rarely, but nobody knows how rarely. Statistical tests for identifying and then cutting out these whoppers exist, though most are based on the Gaussian law and require a reasonably precise knowledge of the internal data errors. Human intervention in the data may or may not be possible (Snooks says that the best way of identifying a blunder is to eyeball the data).

We have a job here —— to reconcile the statistical theory with ordinary Dirty World common sense. Is such a reconciliation possible?

QUOTATION CORNER 40

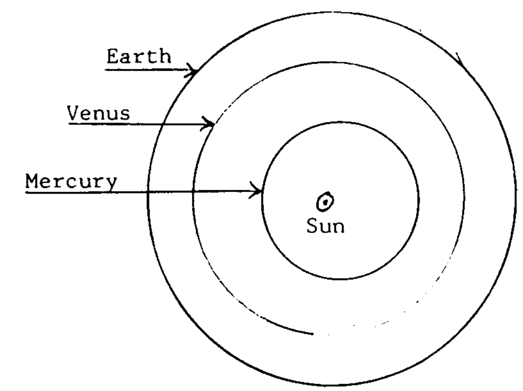
Captain Cook's first Pacific voyage was mainly to observe the 1769 transit of Venus from Tahiti, and for this purpose the astronomer Charles Green sailed on the *Endeavour*. Nevil Maskelyne, the Astronomer Royal, afterwards made some critical comments on the observations that Captain Cook had brought back from Tahiti. —— It must be confessed, that the results of these observations (most of which were made by Mr Green) differ more from one another than they ought to do, or than those made by other observers, with quadrants of the same size, and made by the same artist, the cause of which, if not owing to want of care and address in the observer, I don't know how to assign. ——

Charles Green had died on the return voyage, but Captain Cook wrote (for his Journal of his second voyage, the note was unearthed by J. C. Beaglehole after having been deleted from the 1777 printed edition by the editor) ——

Mr M. might have assigned another reason, he was not unacquainted with the quadrant having been in the Hands of the Natives, pulled to pieces and many of the parts broke, which we had to mend in the best possible manner we could before it could be made use of.

Mr M. should have considered, before he took upon himself to censure these observations, that he had put into his hands the very original book in which they were written in pencil only, the very moment they were taken and I appeal to Mr M. himself, if it is not highly probable that some of them, might from various causes, be so doubtful to the observer, as either to be wholly rejected or to be marked as dubious and which might have been done had Mr Green taken the trouble to enter them in the proper book. Mr M. should also have considered that this was, perhaps, the only original papers of the kind ever put into his hands; does Mr M. publish to the world all the observations he makes good or bad or did he never make a bad observation in his life? ——

MASS OF MERCURY (1)



The mass of the planet Mercury has long been in doubt, for this planet has no moon. One method of estimating the mass is by using the fact that the gravitational attraction of Mercury must perturb the orbit of Venus, and so give an observable effect on the position of Venus in the sky (i.e. the altitude and azimuth or the right ascension and declination), as seen from Earth. The difficulty is that the effect to be measured is comparable with the accuracy of the best astronomical instruments, so that the analysis of the observations involves not only heavy computation, integrating the equations of motion, but also involves tricky questions of mathematical statistics. These statistical questions are hinted at, and (we hope) clarified, by the note MASS OF MERCURY (3) in this issue.

The first attempt at this calculation was by Simon Newcombe in 1895, he came to the conclusion that the mass of Mercury was 1/6,000,000 (with the Sun as unit of mass). Since then other considerations have suggested the value 1/9,000,000, in fact the 1936 edition of W.M.Smart's book *Spherical Astronomy* gave the value 1/9,000,000 in the Appendix B on planetary dimensions, and

H. Jeffreys in 1937 published a calculation giving 1/9,120,000. See the note MASS OF MERCURY (2) below.

The following data on the planetary orbits will give some idea of the difficulties.

Planet	Semi-major axis (astronomical units)	Eccentricity	Mass (☉ = 1)
Earth ⊕	1·000	·017	1/329,000
Venus ♀	0·723	·007	1/403,000
Mercury ☿	0·387	·206	?

The orbits are not quite coplanar, the planes of the orbits of Venus and Mercury are at angles of 3° and 7° respectively to that of Earth, we hope the sketch at the top of this note is not misleading in this respect.

The unknown mass of Mercury not only perturbs Venus and Earth directly by gravitational attraction, but also does so indirectly by perturbing the Sun. These four effects are of the same order of magnitude. The displacement of Venus is of the order of a mile, which subtends from Earth an angle of the order of 1/100 second of arc, but of course more when the planets are close.

STOP PRESS There is a letter from R. A. Lyttleton in the *Journal of the British Astronomical Association*, Volume 103, 1993, Issue 1, pages 8-9. He suggests that Mercury may be a former satellite of Venus. If Venus had ever had a satellite, the effect of the lunar tides in the heavy atmosphere would have been to cause a steady increase in the orbital radius of the satellite, with the final result of the satellite's escape to an orbit round the Sun.

MASS OF MERCURY (2)

For the inner planets other than Mercury, and for the Moon, we have reasonably good estimates for the radius and the mass, and therefore for the mean density, as follows.

	Mean radius miles	Mean radius centimetres	Mean density (water = 1)	1/ Mass (☉ = 1)
Moon ☾	1080	173,800,000	3·34	27,158,000
Mars ♂	2108	339,200,000	3·95	3,093,500
Venus ♀	3788	609,600,000	5·21	403,490
Earth ⊕	3959	637,100,000	5·53	329,390

Problem: for a gravitating sphere of uniform compressible fluid in equilibrium in a uniform gravitational field, find a pressure-density relation for the fluid such that the radius and mean density fit (to within perhaps 2% or 3%) the four values given above. The constant of gravitation is

$$G = 6{\cdot}658 \times 10^{-8} \text{ c.g.s. units.}$$

A satisfactory answer would make plausible the supposition that all these bodies were made of the same material, then if Mercury were also made of the same material we would be able to estimate his mass and mean density from his known radius of 1504 miles or 242,000,000 centimetres. A mass of Sun/9,000,000 would correspond to a mean density of 3·6. The density of rock near the surface of Earth (which may or may not be relevant) is typically between 2·2 for sandstone and 2·77 for slate, and Bullen has taken 3·32 as the density at a depth of 35 kms. It is thought that rock becomes fluid at pressures of the order of 50,000 atmospheres, but evidence is hard to find.

It was R. A. Lyttleton in the early 1960s who drew attention to the mean densities of the inner planets and the Moon, and suggested that they might shed light on the problem of the mass of Mercury; the outer planets, Jupiter, Saturn, Uranus and Neptune, are all much bigger and much less dense.

THE MASS OF MERCURY (3)

(A fable with a moral)

Two research scientists, Dr Able and Dr Baker, were wondering how much mercury there was in a valuable old thermometer which they had just been given. Their colleague Dr Charlie told them how to measure the amount. "If there were no mercury the centre of gravity would always be at the same point of the glass, whatever the temperature," he explained "but as the temperature rises the mercury column will move along the capillary tube, and so will change the centre of gravity." After doing some calculations he added "You see how when the temperature is shown as zero, the mercury just fills the bulb. When the temperature rises to T, the column of mercury in the capillary tube has length proportional to T; the centre of gravity of the thermometer will have been moved a distance in inches equal to M times the square of T, where M is the mass of mercury in the new galactic units that we have recently been ordered to use. And so you will have to measure very accurately."

Dr Able and Dr Baker were confident that they had the best instruments in the country for measuring the position of a centre of gravity, and a week later they were able to show Dr Charlie their results, the position y of the centre of gravity in millionths of an inch as a function of the temperature T.

Temperature	T	0°	6°	12°
Position	y	24	2	40

"One little difficulty" said Dr Charlie "is that if you plot the points (y, T^2) on graph paper you cannot draw a straight line through them. But that is no worry, we have two good statisticians in the Institute, Dr Dog and Dr Easy, just ask them." Both cheerfully took on the problem.

Dr Dog reported "It was a straightforward calculation. The best fit is the formula $y = a + bT^2$, where a and b are chosen to minimize the sum of squares of the deviations, that is to minimize $\Sigma(a + bT^2 - y)^2$. If we do that we find the formula $y = 12 + T^2/6$,

and remembering that y is in millionths of an inch, we have

$$M\,T^2 = 10^{-6}\,T^2/6,$$

so that the mass of Mercury is $M = 1/6{,}000{,}000$."

Dr Easy told them "In my experience the errors in this kind of measurement have a negative exponential distribution, and so by the principle of maximum likelihood the best fit is given by $y = a + bT^2$ where a and b are chosen so as to minimize the function $\Sigma|a + bT^2 - y|$. This gives $y = 24 + T^2/9$, and so the mass of Mercury must be $M = 1/9{,}000{,}000$."

Temperature T	0°	6°	12°
y from data	24	2	40
y by formula D	12	18	36
y by formula E	24	28	40

MASS OF MERCURY (4)

There are fearsome computational problems involved in the traditional methods of estimating the mass of Mercury by using his perturbation of the orbit of Venus. In planning a reliable calculation it is hard to avoid the numerical solution of the six-body problem, for the gravitational fields of Venus, Jupiter and Earth as well as of the Moon all cause perturbations of more than a tenth of a second of arc (rather more than that caused by Mercury) in the position of Venus as seen from Earth. Consequently there might be interest in exploring an alternative method using the theory of almost periodic functions.

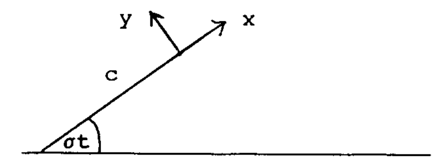
§1 *The gravitational field of a small planet*

Consider a planet of small mass ϵ, rotating in a circular orbit of radius b about a sun of mass $1-\epsilon$. The angular velocity ω and the constant of gravitation G are related by $b^3\omega^2 = G$. Use polar coordinates (r, θ), with the origin at the combined centre of gravity of planet and sun, and with the initial line through the planet. How does the gravitational potential differ from the unperturbed potential G/r due to unit mass at the origin? The perturbation comes from the gravitational attraction of the planet itself and from the fact that the sun is displaced by the planet. We work to first order in the small parameter ϵ. It is not hard to calculate that the perturbing potential, at any point (r, θ) with $r > b$, may be expressed in terms of Legendre polynomials as:-

$$\epsilon G\ \Sigma_{n=2}^{\infty}\ (b^n/r^{n+1})\ P_n(\cos\theta)$$

§2 *Perturbation of a circular orbit*

Consider a test particle of negligible mass in a circular orbit of radius $c > b$, with rate of rotation σ given by $c^3\sigma^2 = G$. How will it be perturbed by the gravitational field of $\Phi = \epsilon G\ \Sigma_{n=2}^{\infty}\ b^n r^{-n-1}\ P_n(\cos(\theta - \omega t))$ given by §1 above? (Now we use a fixed coordinate system)



We shall calculate how the perturbation (x, y) (see the diagram above) from the unperturbed position $(c, \sigma t)$ depends on the corresponding perturbing forces (X, Y). The equations of motion are:-

$$d^2r/dt^2 - r(d\theta/dt)^2 = -G/r^2 + X$$

and $$(d/dt)(r^2 d\theta/dt) = rY$$

Working to first order in x, y, X and Y, the first gives $$d^2x/dt^2 - 2\sigma dy/dt - 3\sigma^2 x = X$$

and, from the second, taking Y* as the indefinite integral (with mean zero) of Y, $$2x\sigma + dy/dt = Y^*$$

The first equation then gives:-

$$d^2x/dt^2 + x\sigma^2 = X + 2\sigma Y^*$$

The periodic terms of the solution, the only ones we want, are now easily found.

Next, what are X and Y?

The Legendre polynomial $P_n(\cos\theta)$ can be expressed as a finite Fourier series,

$$P_2(\cos\theta) = (1/4) + (3/4)\cos 2\theta$$

$$P_3(\cos\theta) = (3/8)\cos\theta + (5/8)\cos 3\theta$$

$$P_4(\cos\theta) = (9/64) + (5/16)\cos 2\theta + (35/64)\cos 4\theta,$$

. . . .

and in general:-

$$P_n(\cos\theta) = 2^{1-2n}\binom{2n}{n}\cos n\theta + 2^{2-2n}\binom{2n-2}{n-1}\cos(n-2)\theta + \dots$$

The perturbing potential Φ is therefore a sum of terms of the form $b^n r^{-n-1}\cos(m\theta - m\omega t)$, where $0 \le m \le n$ and n-m is even, each such term with a factor which is a known multiple of the small parameter ϵ. Such a term contributes to the force $X = \partial\Phi/\partial r$ an amount $-(n+1)b^n c^{-n-2}\cos(m\sigma t - m\omega t)$; similarly the contribution to Y^* is the indefinite time-integral of $Y = (1/c)\partial\Phi/\partial\theta$, which is $Y^* = (\sigma-\omega)^{-1}b^n c^{-n-2}\cos(m\sigma t - m\omega t)$. Thus we come to the DE:-

$$d^2x/dt^2 + x\sigma^2 = (1-n)b^n c^{-n-2}\cos(m\sigma t - m\omega t)$$

with solution $x = \dfrac{(n-1)b^n c^{-n-2}}{m^2(\sigma-\omega)^2 - \sigma^2}\cos(m\sigma t - m\omega t)$

and, for y, $dy/dt = Y^* - 2x\sigma$

$$= \frac{m^2(\sigma-\omega)^2 - \sigma^2 - 2(n-1)\sigma(\sigma-\omega)}{(\sigma-\omega)(m^2(\sigma-\omega)^2 - \sigma^2)}\, b^n c^{-n-2}\cos(m\sigma t - m\omega t)$$

$$y = \frac{m^2(\sigma-\omega)^2 - \sigma^2 - 2(n-1)\sigma(\sigma-\omega)}{m\sigma(\sigma-\omega)^2(m^2(\sigma-\omega)^2 - \sigma^2)}\, b^n c^{-n-2}\sin(m\sigma t - m\omega t)$$

These displacements x and y are what we hope to detect amongst the Fourier components of some observable function. For the perturbation of Venus by Mercury, we note that their rates of rotation are respectively 1·625 and 4·15 rotations per year, (corresponding to σ and ω respectively) and so by looking at the formula above we see that the terms with $m = 1$ are the only ones worth considering. The possible values of n are therefore 3, 5, 7, ... and of these the term with $n = 3$ is the most significant. Therefore let us concentrate our attention on the perturbation potential:

$\epsilon G b^3 r^{-4} P_3(\cos(\theta - \omega t))$ of which the important term is $(3/8)\epsilon\omega^2 b^6 r^{-4}\cos(\theta - \omega t)$. The resulting displacements are

$$x = \frac{3}{4}\,\epsilon\, b^6 c^{-5}\,\frac{\omega}{\omega - 2\sigma}\cos(\omega t - \sigma t)$$

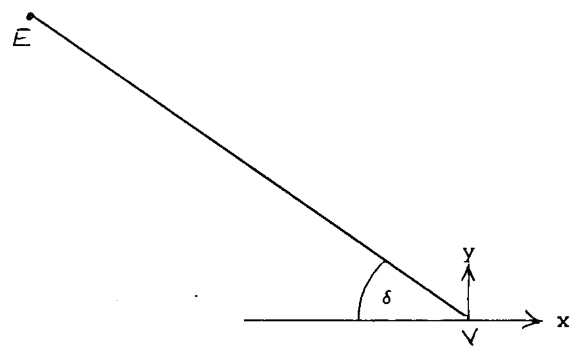
$$y = -\frac{3}{8}\,\epsilon\, b^6 c^{-5}\,\frac{\omega(\omega^2 + 2\sigma\omega - 4\sigma^2)}{(\omega-\sigma)^2(\omega - 2\sigma)}\sin(\omega t - \sigma t)$$

These displacements have a period of 145 days. If we consider the perturbation of Earth by Mercury, similar

considerations apply, and we find smaller displacements, with a period of 116 days.

§3 *The unsolved problems*

If we had complete knowledge of the position of Venus, then we could compare it with the position predicted by Kepler's laws, and the difference would be a small uniformly almost periodic function (caused by the masses of the planets not being negligible) from which we could extract the appropriate Fourier component, thereby giving the mass of Mercury.



One difficulty is that what we can find by observing the celestial longitude of Venus is (see diagram above) the ratio $(x \sin\delta + y\cos\delta)/(\text{length EV})$. The angle δ and the length EV can be calculated for each observation, but where is there an almost periodic function from which we can extract the Fourier components?

Another difficulty is that although the right ascension and declination of Venus have been observed many times over the last 200 years, the times of the observations have not been either uniformly or randomly distributed. They have been when there has been no cloud, and when Venus has not been too close to the Sun or the Moon, etc. Have we any theory of numerical integration to cope with data like this?

Our simplification of the problem by taking all the planetary orbits to be circular and coplanar, and by taking all planets except Mercury to be infinitesimal test particles, will of course introduce errors, but probably not large ones.

Of some relevance, perhaps, is the note HARBOUR MASTER'S DILEMMA, page 3148 of JCMN 30, December 1982. It asked for how long the harbour master had to observe the water level in the harbour before being able to predict the tides. We have not had an answer yet. That question, like this, is one of finding the Fourier components of an almost periodic function from experimental observations.

Your editor is reminded of the story of the present main runway of the airport at Townsville in North Queensland. About 30 years ago the Federal Government allocated money for building a new runway, and the officials in the South sent orders that the direction of the wind should be observed every morning at 9 a.m., and the measurements should be sent South. They presumably analysed the results with care (did they even use a computer?), and finally sent orders that the runway should be aligned in the direction 020° - 200°. Visiting air pilots sometimes ask why the runway was built at right angles to the prevailing wind, which as all the local people know is from the East, though the mornings are usually calm.

MATHEMATICAL DEMOCRACY (JCMN 55, p. 6036)

This year (1993) democracy is 2499 years old, for it can be said to have started in the city-state of Athens in 507 BC with their adoption of a new method of government. That first democratic constitution lasted for 185 years, and since 322 BC it has never been copied in Athens or anywhere else, but the name "democracy" has lived on, being attached to a great variety of systems.

The (to us) unusual feature of the Athenian system was that there was no election of representatives —— all decisions were taken by a meeting (called the ekklesia or $\epsilon\kappa\kappa\lambda\eta\sigma\iota\alpha$) of all the voters, this body had absolute power. What happened was that only a small proportion of the voters turned up at the meetings, which were every 10 days. On whether the system should be called a success, opinions differ. In a recent article (Cambridge 1992) Peter Jones wrote "The philosopher Plato and the historian Thucydides had little time for democracy. Plato for one regarded ruling as a high skill and could see no reason to make it open-house for any Tom, Dick or Harry (any more than we would give anyone the chance to practice brain surgery)."

The Romans at about the time of the Athenian democracy started their "Senate", showing the beginnings of the modern system in which voters elect representatives who rule the country. All over the world now there are variants of this general idea; most readers of the JCMN will be familiar with at least one such system and with its good and bad features.

In Australia there is a reasonably good voting system for elections to both State and Federal parliaments, that of the "transferable vote". In essence the election system is that each voter puts all the candidates in order of preference; the algorithm for processing the votes is that at each step the candidate with the smallest number of first preferences is eliminated from the lists of all the voters. This step is repeated until only one candidate remains. Some Australians may say that this method does not give good members of Parliament or good government (your editor is trying to be impartial on this question!), but if this is so the fault is probably more with the voters than with the voting system. An interesting attempt to pervert the voting system is now being made —— each party gives to its supporters instructions on how to allocate their preferences, and before an election the parties negotiate with one another about this allocation of preferences, for instance one might say to another "We will give you our second preferences if you give us yours". It is not known just how many voters follow their party voting instructions, but probably most do. If all did, then what would be the effect on the election result? Would it give the same result as a "first past the post" system of counting the votes? A tricky question in the theory of games.

A system of voting to elect the members of a parliament is an algorithm for data processing, its input is the opinions of the voters, and its output is the elected Parliament. Like all such algorithms it obeys the principle of "garbage in —— garbage out".

Mathematical statistics faces the same problem; it applies algorithms to process data, and to give good answers it needs good data. We talk of "robust" algorithms, those that can cope with bad data to a certain extent, but there are limits to what can be done in this direction. A simple example is when we have many measurements of a single unknown quantity. If the likely flaws in the data include a few big blunders then it is a good idea to use the median of the data as the best estimate for the true value of the unknown, but if what is wrong with the data is a lot of small errors in all the values, then the mean is better than the median. Traditionally the median is called more "robust" than the mean, but it would be hard to lay down a general definition of the word "robust". See John Parker's note DIRTY STATISTICS IN A DIRTY WORLD, pages 6234-6235 of this issue.

The lesson for mathematical democrats and statisticians is the same —— to get an appropriate algorithm for any particular problem you should have an idea of the imperfections that you expect in the data. You can change an algorithm to guard against one kind of fault in the data, but only at the expense of making your algorithm more vulnerable to another kind of fault. In order to get a good democratic election system we should look at the nature of the voters, to what extent they are misinformed or stupid or selfish? Not easy!

The average voter in ancient Athens probably did not have the knowledge or wisdom or patience to make good decisions on all the questions that came up —— taxation, road maintenance, foreign alliances, naval ship-building policy, the design of theatres, the major civil and criminal law cases, etc., and so the ekklesia tended to be swayed by the speeches of orators,

often men of ability and patriotism, the name of Pericles comes to mind. In the modern world there are newspapers playing the part that was played in ancient Athens by the orators, trying to influence the voters. What are their motives? Are they as successful at influencing voters as the Athenian orators were? Do they influence the government through influence on the voters? In fact the ekklesia demonstrated what has been called (by R.A.Lyttleton) the "Gold effect", first described by Professor T. Gold (*Lying Truths* Pergamon Press, Oxford, 1980); this phenomenon is that when a lot of people come together to discuss some controversial question on which opinions are initially widely spread, they will change to the condition in which most of the people are agreed on some opinion, often an extreme one.

In the Roman or modern systems of representative democracy the voters do not have to make decisions on all those difficult questions that arise for governments, they have only what might seem an easier task —— to choose honest and competent representatives; but how successful are they? Is it really an easier task? The situation is complicated by the existence of political parties. Many people will vote for a party rather than for a person, so that most members of Parliament are there not because the voters think them competent and honest, but because of having been chosen by their party machine; what sort of candidates will a party organization choose? Presumably the ideal candidate is one likely to be loyal to the party rather than one that wants to ensure that the country is governed well, and probably the ideal candidate is one who will be conscious of having been elected more because of party sponsorship than because of personal worth.

What motivates a political party? In the language of decision theory, what is the party's objective function?

One of the by-products of the party system is that newspapers tend to present elections as contests, with winners and losers; and because the election is treated in the same way as (and reported by the same journalists as) a football match or a horse-race, it gets the same treatment, that the newspapers like to print predictions of the result before the event. Is it that journalists understand the "Gold Effect", and therefore try to use their predictions as a means of influencing the results of the election? There are many interesting problems awaiting those who try to analyse democracy.

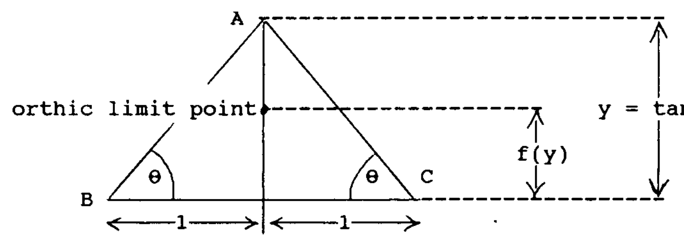
ORTHIC LIMIT POINT OF AN ISOSCELES TRIANGLE

(JCMN 60, p.6209)

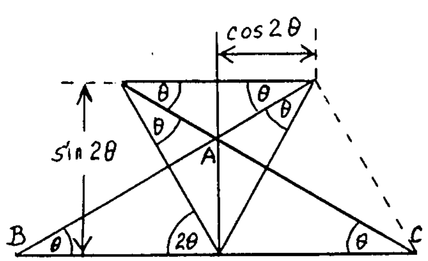
T. C. S. Tao (Princeton)

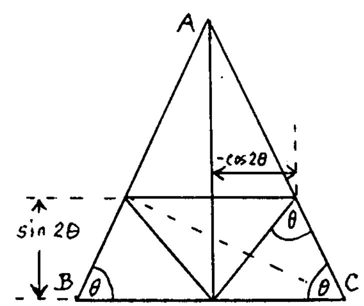
The orthic limit point of a triangle is the limit of the infinite sequence of triangles, starting with the given one, in which each is followed by its orthic triangle. To avoid the ambiguity of degenerate triangles we define the orthic triangle of a degenerate triangle ABC with A = B to be the triangle with three vertices all at A. Thus the orthic limit is defined except when the three vertices are in a line.

Now consider the case of an isosceles triangle ABC.



Let f(y) be the function indicated in the diagram above. An isosceles triangle has base length = 2, with vertices at (±1, 0) and (0, y), with the orthic limit at (0, f(y)). We want to regard this as a definition of the function f for all non-zero values of the variable. It will be sufficient to consider just the two cases, θ between 0 and 45° and between 45° and 90°. With the relation f(-y) = -f(y) these cases cover the whole real variable except zero.





Use the fact that the orthic limit of ABC is the orthic

limit of its orthic triangle, then from either diagram we can find the relation

$$f(\tan\theta) + \cos 2\theta\, f(\tan 2\theta) = \sin 2\theta \qquad \ldots\ldots (1)$$

To find a solution for this functional equation, put

$$g(\theta) = f(\tan\theta)\sin 2\theta$$

giving $\quad g(\theta) + \tfrac{1}{2}g(2\theta) = \sin^2 2\theta \qquad \ldots\ldots\ldots\ldots (2)$

with the obvious solution

$$g(\theta) = \sin^2 2\theta - \frac{\sin^2 4\theta}{2} + \frac{\sin^2 8\theta}{4} - \ldots \qquad \ldots\ldots (3)$$

This is not the only solution of (2), but is it the right answer? From (1) we find that f(y) = f(1/y) and so $g(\pi/2-\theta) = g(\theta) = g(-\theta)$. Therefore g has period $\pi/2$. The geometry tells us that f is integrable, and continuous except possibly at zero, therefore g is the Fejér sum of its Fourier series,

of the form $\quad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 4n\theta$

From (2) it is clear that the only possible values of n in this sum are powers of 2, and so (3) is right.

We know the orthic limit point for an equilateral triangle and for a 45°, 45°, 90° triangle. These tell us that $f(\sqrt{3}) = 1/\sqrt{3}$ and f(1) = 1, agreeing with (3).

The series (3) may be written

$$g(\theta) = 1/3 + \sum_{k=1}^{\infty} (-2)^{-k}\cos(2^{k+1}\theta) \qquad \ldots\ldots\ldots\ldots (4)$$

This has a look of non-differentiability, for recall that the first example (due to Weierstrass) of a continuous function nowhere differentiable was

$$\sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \quad \text{where } 0 < a < 1 \text{ and } ab > 1+3\pi/2.$$

The numerical evidence given in JCMN 60, pp. 6209-6211 hints at non-differentiability of f and g, (the function there is xf(1/x) in our present notation). Where is the function g given by (3) or (4) differentiable?

ADDING NUMBERS 2 (JCMN 60, p.6203)

This old problem of Harzheim and Erdös asks: for integer sequences such as

$0 < a_1 < a_2 < \dots \; a_k \le n$ with the sums $\Sigma_{j=u}^{v} a_j$ all distinct, is $k = o(n)$?

Consider the following sequences, the first is that generated by the greedy algorithm, on which subject readers may recall the comments of George Szekeres in JCMN 47, p.5125, SEQUENCES WITHOUT ARITHMETIC PROGRESSIONS. All three sequences satisfy the given condition with n = 40.

1	2	4	5	8	10	14	21	25	26	28	31	36	38	
2	3	4	6	8	11	16	17	20	22	24	30	38		
2	3	4	6	8	11	16	20	22	24	26	30	31	33	38

For each n we want to find the sequence with the biggest k. Of the three sequences above, the first is best for n = 10 (it gives k = 6), the second is best for n = 20 (giving k = 9) and the third for n = 40 (giving k = 15). Therefore we cannot hope for the greedy algorithm, or any other algorithm, to give us a "best" infinite sequence, in which we may see how k depends on n.

Consider the following modification of the problem. Take an integer $T \ge 0$ and consider integer sequences such as

$0 < a_1 < a_2 < \dots < a_k \le n$ for which the sums $\Sigma_{j=u}^{v} a_j$ with $v-u \le T$ are all distinct. For each n there is a greatest possible k, and we define F(T) as lim sup n/(max k). Then $n/k \ge F(T) + o(1)$ for large n. It is trivial that F(0) = 1, and it is easy to show that F(1) = 3/2, the critical sequence consists of the integers not divisible by 3.

Now let us take the case T = 2. Conjecture — is F(2) = 2? In other words we are concerned with sequences of integers $0 < a_1 < a_2 < \dots < a_k \le n$ such that the a_i, the sums of adjacent pairs, such as $a_i + a_{i+1}$, and the sums of adjacent triples such as $a_i + a_{i+1} + a_{i+2}$, are all unequal. For each n what is the largest possible k? Is it close to n/2 for large n?

There is a little evidence bearing on this conjecture that F(2) = 2, as follows.

We know that $F(2) \le 3$. This is easily shown by the the sequence of integers ≡ 1 or 2 or 7 or 8 (mod 12), the sum of an adjacent pair must be ≡ 3 or 9, and of an adjacent triple must be ≡ 4 or 5 or 10 or 11. Could this bound be improved?

The following values of k are attainable (i.e. they are the best I have been able to find), but it is hard to discover if they are maximal.

n	10	20	40	80	160	320	640	1280
k	6	11	22	43	82	160	320	635

For example, k = 22 with n = 40 is given by the integer sequence:-

2, 3, 4, 6, 8, 11, 12, 15, 20, 21, 22, 24,
26, 28, 29, 30, 32, 33, 36, 37, 39, 40.

and k = 43 with n = 80 is given by:-

2, 3, 4, 6, 8, 11, 12, 15, 20, 21, 22, 24,
26, 29, 30, 32, 33, 36, 37, 40, 42, 44, 45, 48,
49, 51, 52, 53, 54, 57, 58, 60, 61, 64, 66, 68,
70, 71, 74, 75, 76, 78, 80.

POLYNOMIAL INEQUALITY 1 (JCMN 60, p.6221)

Cecil Rousseau
(Memphis State University)

Let $f(x)$ be a real polynomial of degree n with $-M \le f(x) \le M$ in the unit interval. It was asked if $|f'(x)| \le 2n^2M$.

This is Markoff's inequality, due to A.A.Markoff, 1889. In Natanson's *Constructive Function Theory,* Volume 1, there is a proof, but it fills four pages, and so we shall not reproduce it here.

The question is given as a problem in the *Aufgaben und Lehrsätz aus der Analysis* of Pólya and Szegö (Problem 83 of Part 6 on page 85 of Volume 2 in the 1970 English translation).

QUOTATION CORNER 41

There are two sides to every question until the truth is known.

—— C. S. Lewis, in *Transportation and other addresses*.

QUOTATION CORNER 42

One of the greatest pains to human nature is the pain of a new idea.

—— Walter Bagehot.

(Both these quotations are from R. A. Lyttleton)

HYPERPLANES PARTITIONING N-SPACE (JCMN 60 p. 6223)

Terry Tao
(Princeton University, U.S.A.)

In the note from Mark Kisin we saw that k hyperplanes in general position divide n-space into

$$f(n, k) = \binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{n}$$

regions. How many are bounded and how many unbounded?

Let $g(n, k)$ be the number of unbounded regions. Draw a large hypersphere about the origin. The hyperplanes divide the sphere into $g(n, k)$ regions, and this number will be unaltered if we move the hyperplanes, each parallel to itself, to go through the origin.

Now we choose one of the hyperplanes, and consider the half-sphere on one side of the chosen hyperplane. This half-sphere is divided into $g(n, k)/2$ regions by the other k-1 hyperplanes. Draw a hyperplane tangent to the sphere parallel to the chosen hyperplane. Project the half-sphere radially on to the tangent hyperplane. We have the tangent hyperplane (a space of n-1 dimensions) divided into $g(n, k)/2$ regions by k-1 hyperplanes in general position. Therefore $g(n, k)/2 = f(n-1, k-1)$.

$$g(n, k) = 2\binom{k-1}{0} + 2\binom{k-1}{1} + \ldots + 2\binom{k-1}{n-1}$$

$$= 1 + \left(\binom{k-1}{0} + \binom{k-1}{1}\right) + \left(\binom{k-1}{1} + \binom{k-1}{2}\right) + \ldots + \binom{k-1}{n-1}$$

$$= 1 + \binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{n-1} + \binom{k-1}{n-1}$$

The number of bounded regions in the original pattern is therefore equal to

$$f(n, k) - g(n, k) = \binom{k}{n} - \binom{k-1}{n-1} = \binom{k-1}{n}.$$

Is there an easier proof?

ARITHMETIC PROGRESSIONS FROM PASCAL'S TRIANGLE

Stanley Rabinowitz
(P.O. Box 713, Westford, MA 01886, U.S.A.)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Write down any row of Pascal's triangle. Below it, write the next row, omitting the initial "1". Divide corresponding entries of the first row by those of the second. The result is an arithmetic progression.

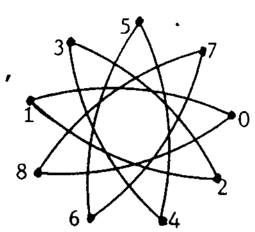
For example, rows 6 and 7 yield the progression.

$$\frac{1}{7},\quad \frac{6}{21},\quad \frac{15}{35},\quad \frac{20}{35},\quad \frac{15}{21},\quad \frac{6}{7},\quad \frac{1}{1}$$

The proof follows from $\binom{n}{k} = \frac{k+1}{n+1}\binom{n+1}{k+1}$.

Adjacent diagonal lines have a similar property, for example $\frac{1}{1}, \frac{5}{4}, \frac{15}{10}, \frac{35}{20}, \frac{70}{35}, \ldots$

PLANE TOPOLOGY



In the plane are n nodes, numbered 0, 1, ... n-1 (mod n). There are n curves C(j), numbered the same way. The curve C(j) joins the nodes numbered j and j+1, and contains no other node. Each curve meets every other curve just once, having a node in common with each neighbour, and crossing each of the other n-3. Note that "crossing" is meant in the strict sense — crossing from one side to the other. Prove that n is odd.

This problem derives from an old question of J. Conway.

EASY QUESTION

Paul Erdös
(Mathematical Institute, Hungarian Academy of Sciences, Budapest)

I thought that the following question was difficult: Denote by $a_1 < a_2 < \ldots$ the integers (1, 2, 3, 4, 6, ...) that are of the form $2^\alpha 3^\beta$. Prove that every integer can be written in the form $n = \Sigma a_i;\ a_i \nmid a_j$ i.e. every n is the sum of a set of these integers, of which no one divides any other. I was wrong, Jantzen and many others found the trivial proof by induction, this is almost too easy for the JCMN. (But not quite —— Editor)

Savin and I proved the related result for integers composed of 2, 5 and 7. Every number n > 31 can be expressed in the form $n = \Sigma a_i$ where $a_i \nmid a_j$ and each $a = 2^\alpha 5^\beta 7^\gamma$.

MEDICAL RESEARCH IN AUSTRALIA

The Australian Broadcasting Commission on 6th February broadcast a talk by a former Director of the Florey Institute for Medical Research in Melbourne. He described with pride how his colleagues had investigated the nature of thirst. They had performed a novel form of surgery on an animal to ensure that any water that the animal drank should not enter its stomach. Then they succeeded in obtaining the striking result that the animal continued to try to relieve its thirst by drinking water.

ORTHOCENTRES IN PHOTOGRAPHY

Geometrically we may regard photography as radial projection on to a plane. This is clear in the case of a pin-hole camera, and for the usual camera with lenses it gives a good approximation to the geometry. For simplicity think of a pin-hole camera, and take Cartesian coordinates with origin at the pin-hole, Q. Suppose that scene being photographed contains a brick, or a rectangular block, showing edges at right angles to one another. Then we choose our Cartesian axes parallel to the edges of the rectangular block.

On the picture, or in the extended plane of the picture, we see three "vanishing points", one for each of the three directions of the edges of the block, the points where the parallel lines meet. These points must be where the Cartesian axes meet the plane of projection, which is the film or plate. Let the vanishing points be A, with coordinates $(1/u, 0, 0)$, B with coordinates $(0, 1/v, 0)$ and C with coordinates $(0, 0, 1/w)$. Then the plane on to which the picture is projected (the film or plate) will have the equation $ux + vy + wz = 1$, because it is through these three points.

It is easily verified that the foot of the perpendicular from A to BC is $(0, v/(v^2+w^2), w/(v^2+w^2))$, and similarly for the other two perpendiculars. The orthocentre H of ABC is therefore the point $(u/(u^2+v^2+w^2), v/(u^2+v^2+w^2), w/(u^2+v^2+w^2))$, which is the foot of the perpendicular from Q on to the plane of projection; it is where the axis of symmetry of the lens system meets the plane of the film, and in most cameras it is in the middle of the negative.

Now start from the other end, suppose that we have only the picture, what can we find out about the camera and the scene that was photographed?

We have the vanishing points A, B and C (if there was a brick in the picture) and therefore we have also the orthocentre H of ABC. The point Q is on the perpendicular to the plane from H, and is at a distance from H equal to the focal length of the lens system. In our notation above $QH = 1/\surd(u^2+v^2+w^2)$, and u, v and w may be related to the parameters of the triangle ABC as follows.

$$a^2 = 1/v^2+1/w^2, \qquad b^2 = 1/u^2+1/w^2, \qquad c^2 = 1/u^2+1/v^2$$

Therefore $2/u^2 = b^2+c^2-a^2 = 2bc \cos A$, and $u^2 = \sec A /(bc)$, etc. and we can express the focal length of the lens system in terms of the triangle parameters by:-

$$\begin{aligned} QH^2 &= 1/(u^2+v^2+w^2) = abc/(a \sec A + b \sec B + c \sec C) \\ &= bc/(\sec A + \sec B \sec C) \\ &= bc \cos A \cos B \cos C /(\sin B \sin C) \\ &= 4R^2 \cos A \cos B \cos C \\ &= -R^2(1 + \cos 2A + \cos 2B + \cos 2C) \end{aligned}$$

where R is the radius of the circumcircle of the triangle ABC.

Now recall the orthocircle or orthic circle, the circle with respect to which the triangle is self-polar, see ORTHOCENTRES, JCMN 51, p.5220 and TRIANGLE GEOMETRY, JCMN 58, p.6127. The orthocircle has centre H, and the square of its radius is $-4R^2\cos A \cos B \cos C$, (it is an imaginary circle when, as in this case, the triangle ABC is acute angled). The point Q can therefore be characterized as the point such that the sphere of zero radius round it meets the plane of ABC in the orthocircle.

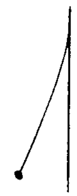
If, as sometimes happens, one of the three perpendicular directions is known to be vertical, say the direction QA, then the angle β between the axis of the camera and the horizontal is easily calculated.

$$\sin \beta = \surd(\cot B \cot C) \quad \text{and} \quad \cos \beta = \surd(\cos A \operatorname{cosec} B \operatorname{cosec} C).$$

Exercise for the student: Given a photograph of a plane rectangle, and knowing the point H (the middle of the negative) can you find out if the rectangle is square?

PENDULUM CLOCKS (JCMN 57 pages 6107-6109)

As explained in JCMN 57, p.6107, the pendulum of a typical clock is not like a rigid body pivoted about a fixed axis. Instead of being on a pivot, it hangs by a short length of flexible steel strip. How does this affect the time-keeping of the clock?



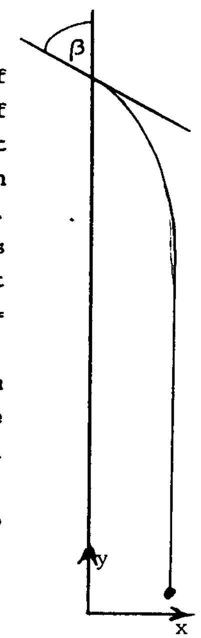
Take a simple model. A light flexible rod is vertical, fixed at the top, and has a heavy particle at the bottom, what is the period of small vibrations? Let M be the mass of the particle, let K be the ratio bending moment/curvature for the rod, and let g = gravity. Then $\sqrt{(K/Mg)}$ has the dimension of a length, call it k. If ℓ is the length of the rod the period of small vibrations is

$$2\pi\sqrt{(\ell - k \tanh(\ell/k))/g}$$

If you are teaching a mechanics class, this might make a nice little question for them, helping to refresh their skills in first year calculus.

In actual clocks the ratio k/ℓ is small, of the order of 1/50, in fact only the top part of the rod is made flexible. In our next calculation, studying a flexible pendulum with non-small displacement, we assume ℓ to be large. We take the pendulum to be a long weightless flexible wire with the bob, a particle of weight W = Mg, at the bottom, the parameters K and k = $\sqrt{(K/W)}$ being as above.

Consider the static problem of the pendulum at rest with the top fixed at an angle β to the vertical. The wire hangs in a curve with a vertical asymptote (thinking of ℓ as infinite). What is the sideways displacement, x, of the bob (or of the asymptote)? And what is the vertical displacement y of the bob caused by turning the top through the angle β?



The curve has radius of curvature K/(Wx) at the top. Consider increasing β to $\beta+d\beta$. The curve remains the same shape except for an extra arc of length $(K/Wx)d\beta$ added at the top and an equal length removed from the bottom. Therefore $dx = (K/Wx)\sin\beta\, d\beta$. This gives us a differential equation for the relation between x and β. The solution is $x = 2k \sin(\beta/2)$. By similar reasoning $dy = (K/Wx)(1 - \cos\beta)d\beta$, and by using the value found for x, $dy = k \sin \beta/2 \; d\beta$. Solving this differential equation we find that $y = 2k(1 - \cos(\beta/2))$.

The elastic potential energy in the rod is $4\sqrt{KW} \sin^2\beta/4$, equal to the gravitational potential energy.

Now we are in a position to consider the pendulum swinging, and more particularly to find out how the motion differs from that of a pivoted pendulum. In particular we want to find out how the "circular error" factor differs from the $(1 + \alpha^2/16)$ that is known for the pivoted pendulum (where α is the swing from the vertical). We are investigating the question asked in the first contribution (page 6109) — does this suspension decrease or increase the circular error of the pendulum?

Make the quasi-static assumption, that the elastic distortion of the pendulum rod is always what it would be if at rest with the force from the bob equal to what would be predicted by simple pendulum theory.

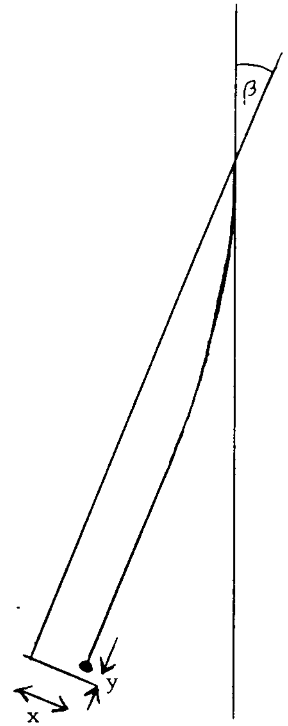
The simple theory tells us that if the bob has weight W and the pendulum swings through an angle α each side of the vertical, then the tension at the bottom of the rod when it is at an angle β to the vertical is $W(3\cos\beta - 2\cos\alpha)$.

Now consider our model. We take units so that the acceleration of gravity, the mass of the bob and the length of the rod are all = 1. Some readers may be worried because on the previous two pages we have been assuming that the length is infinite. But actually 1 is a good approximation to infinity in the relevant sense here, of being large compared with K/W.

For a typical grandfather clock, the parameter $k = \surd(K/W)$ is about 1/100 of the pendulum length.

Our first calculation showed that the main effect of having a flexible suspension instead of a pivot is to reduce the effective length of the pendulum from ℓ to ℓ-k, but we are more interested in how the period depends on the amplitude.

The parameter k, the angle ß and the angle α (the maximum of ß) are all small, and from now on we shall discard the higher powers of these quantities as seems appropriate.



From simple pendulum theory, the force on the end of the pendulum rod is

$W = 1 + \alpha^2 - 3\beta^2/2$, therefore (see diagram)

$x = 2\surd(K/W)\sin\beta/2 = k\beta(1 - \alpha^2/2 + 17\beta^2/24)$

$y = 2\surd(K/W)(1 - \cos\beta/2) = (k\beta^2/4)(1 - \alpha^2/2 + 35\beta^2/48)$

Elastic potential energy = $(k\beta^2/4)(1 + \alpha^2/2 - 37\beta^2/48)$

Height of bob = $(\beta^2/2)(1 - \beta^2/12 - 3k/2 + 3k\alpha^2/4 - 31k\beta^2/32)$

The coordinates of the bob relative to fixed axes are

$(\sin\beta - x\cos\beta - y\sin\beta,\ 1 - \cos\beta - x\sin\beta + y\cos\beta)$.

Differentiating, squaring and adding, we find the square of the speed of the bob to be

$$((1-y)d\beta/dt - dx/dt)^2 + (x\,d\beta/dt - dy/dt)^2$$

The second term can be discarded, every term in it being too small for our level of approximation. Thus we have the kinetic energy, which added to the gravitational and elastic potential energies must give a constant, the value at the end of the swing; this gives us the equation:-

$$\tfrac{1}{2}(d\beta/dt)^2(1 - k)^2(1 + k\alpha^2 - 19k\beta^2/4) = \text{kinetic energy} =$$
$$\tfrac{1}{2}(\alpha^2 - \beta^2)(1 - k)(1 - \alpha^2/12 - \beta^2/12 - k(7\alpha^2 + 23\beta^2)/16).$$

To find the period, change the variable by putting $\beta = \alpha\sin\varphi$, then (with the integrals being over ß from 0 to α or over φ from 0 to π/2)

$$\text{Quarter-period} = \int \frac{d\beta}{d\beta/dt} = \int \frac{\surd(\alpha^2 - \beta^2)d\varphi}{d\beta/dt}$$
$$= \sqrt{1 - k}\ \frac{\pi}{2} \int 1 + \alpha^2(1/24 + 23k/32) + \beta^2(1/24 - 53k/32)\ d\varphi$$
$$= \sqrt{1 - k}\ \frac{\pi}{2}\ (1 + (\alpha^2/16)(1 - 7k/4))$$

This shows that the use of the spring suspension (instead of a pivot) decreases the circular error, but, in most of the clocks that your editor has come across, makes little difference to it. In fact a careful designer of a pendulum clock would hesitate to use a large value of k because it would tend to make the timekeeping more dependent on temperature; in the factor $\surd(\ell - k)$ the length ℓ will increase and the k will decrease with rising temperature except when certain rather special alloy steels are used.

PLATONIC SOLIDS

Consider the following three of the Platonic solids:

	faces F	edges E	vertices V
Tetrahedron	4	6	4
Octohedron	8	12	6
Icosahedron	20	30	12

They all have the property that $3V = E + 6$, which means that if you make a model, using lengths of plastic drinking straw for the edges, and lengths of thin elastic threaded through the straws to hold them together, then each becomes a rigid body. The other two Platonic solids, the cube and the dodecahedron, do not have this property.

Those of our readers who (like the Editor) have grand-children living far away, may note how these models, sent in the form of their parts, make convenient birthday presents. They may be accompanied by little lessons in mathematics.

As the makers of plastic drinking straws often produce them in several different colours, the idea of edge-colouring of a graph may be introduced. For the three solids listed above, the number of edges meeting at each vertex is 3, 4 and 5 respectively; and this is the chromatic number, in the sense that with this number of colours it can be arranged that no two edges of the same colour meet at a vertex.

Edge-colouring leads naturally to the idea of a Hamiltonian circuit, that is a closed path visiting every vertex just once. In the case of the tetrahedron ond in the case of the octohedron, the paths that use only two of the edge colours are Hamiltonian. What is the corresponding property of the icosahedron?