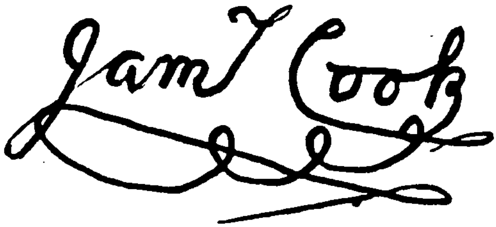
JAMES COOK MATHEMATICAL NOTES

Volume 6   Issue number 60

January 1993



This issue is dedicated to Dr. Frank Smithies,

in celebration of his eightieth birthday.

Editor and publisher: B. C. Rennie

Address: 66, Hallett Road,
Burnside, SA 5066
Australia.

The James Cook Mathematical Notes is now published in 3 issues per year, in January, May and September. The history of JCMN is that the first issue appeared in September 1975, and others at irregular intervals, all the issues up to number 31 being produced and sent out from the Mathematics Department of the James Cook University of North Queensland, of which I was then Professor. In October 1983 this arrangement was beginning to be unsatisfactory, and I changed to publishing the JCMN myself, having three issues per year printed in Singapore and posted from there. I then set a subscription price of 30 Singapore dollars per year. When in 1985 I changed to printing in Australia I kept the same price, for the Singapore dollar is a stable currency.

Now (October 1992) it has become clear that the paying of subscriptions by readers is an inefficient operation. Bank charges for changing currency and for international transfers, with postage, together absorb most of the initial input of money. Therefore we are abandoning subscriptions as from the beginning of 1993, issue number 60. To those who want to give something in return for the JCMN, I ask them to make a gift to an animal welfare society in their own country. The animals of the world will be grateful and so will I.

Contributors, please tell me if and how you would like your address printed.

JCMN 60, January 1993

CONTENTS

ABSOLUTE PRIME NUMBERS

A. T. Kolotov and A. M. Slin'ko

A natural number is said to be an absolute prime if it is prime and remains prime after any permutation of its (decimal) digits. Prove that the decimal representation of an absolute prime number can contain no more than three distinct digits.

— A.T.Kolotov, XVIII-th SMO, 1984.

Find all natural numbers n such that every n-digit number that has n-1 ones and 1 seven in its decimal representation is prime.

—- A.M.Slin'ko, short-listed for the 1990 IMO.

The attention of mathematicians has for a long time been attracted to prime numbers, and specially to those that have some sort of symmetry. The "repunits" $A_n = (10^n - 1)/9$ with decimal representation 111 ... 1 containing only units, form an important class of them. For a repunit A_n to be prime the number n of its digits must be prime, but this condition is far from being sufficient. For instance $A_3 = 111 = 3\times37$ and $A_5 = 11111 = 41\times271$. Some of the repunits are, none the less, prime: A_2, A_{19}, A_{23}, A_{317}, and probably A_{1031} are examples. The question of primeness of the repunits was discussed by M. Gardner [1] and later in [2-4]. It is not known if the number of prime repunits is finite or infinite.

The prime repunits are a subset of the primes that remain prime after any permutation of their digits. The primes with this property were called "permutable primes" by H.-E. Richert who introduced them some 40 years ago [5], and were called "absolute primes" by T.N. Bhargava and P.H. Doyle [6], and by A.W. Johnson [7]. The intent of this note is to give short proofs, not needing number crunching, of all known facts about the absolute primes other than the repunits.

From the table of primes, we find these 21 absolute primes less than 999, other than the repunit 11:

2, 3, 5, 7, 13, 17, 31, 37, 71, 73, 79, 97, 113, 131, 199, 311, 337, 373, 733, 919, 991.

It is easy to see that multidigit absolute primes contain only the four digits 1, 3, 7, 9 in their decimal representation. The digits 0, 2, 4, 5, 6, 8 cannot appear, because by shifting this digit to the units place we obtain a multiple of 2 or 5.

Now we can significantly confine the area of the search, and this will help us.

<u>Lemma 1</u> [6] An absolute prime does not contain in its decimal representation all the four digits 1, 3, 7, 9.

<u>Proof</u> Let N be a number with all these four digits. Shifting them to the right, we can obtain the 7 numbers

$$N_i = 10^4 L + K_i \qquad (i = 0, 1, 2, 3, 4, 5, 6)$$

where $K_0 = 7931 \equiv 0 \pmod 7$, $K_1 = 1793 \equiv 1 \pmod 7$

$K_2 = 9137 \equiv 2 \pmod 7$, $K_3 = 7913 \equiv 3 \pmod 7$

$K_4 = 7193 \equiv 4 \pmod 7$, $K_5 = 1937 \equiv 5 \pmod 7$

and $K_6 = 7139 \equiv 6 \pmod 7$.

The 7 numbers K_i have different residues mod 7, and therefore so do the numbers N_i. Therefore one of the N_i is divisible by 7. Since all these numbers can be obtained from N by permutation of the digits, N is not an absolute prime.

<u>Lemma 2</u> An absolute prime does not contain in its decimal representation any digit a three times with a digit b ≠ a twice.

<u>Proof</u> Suppose that a number N contains digits a, a, a, b, b. By permutation of the digits we can obtain ten numbers N(i, j) of the form

$$N(i, j) = C + (b - a)(10^i + 10^j)$$

for any i and j with $0 \le j < i \le 4$, where C is a number with its last five digits all = a. Since 1001, 1100, 1010, 101, 11, 110 and 10100 give remainders 0, 1, ... 6 on dividing by 7, so do the numbers obtained by multiplying by (b - a), and so do the corresponding N(i, j). Therefore one of these numbers N(i, j) is a multiple of 7.

With Lemma 1 and Lemma 2 and some calculation we find that there cannot be an absolute prime of 4, 5 or 6 digits.

Lemma 3 If N is an absolute prime of n (> 6) digits and it ends with the digits aaaaab (where b ≠ a), so that

$$N = 10^6 K + aA_6 + (b-a)$$

then K is divisible by 7.

Proof By permuting the last 6 digits we can obtain the numbers $N_i = 10^6 K + aA_6 + (b-a)10^i$ for i = 0, 1, 2, 3, 4, 5. Since (b-a) is even and $1 \equiv 1$, $100 \equiv 2$, $10 \equiv 3$, $10000 \equiv 4$, $100000 \equiv 5$ and $1000 \equiv 6 \pmod 7$, the 6 numbers $10^i(b-a)$ have the same property, of having different non-zero remainders on dividing by 7. If the number $10^6 K + aA_6$ had a non-zero remainder on dividing by 7, we could find some i such that $10^i(b-a)$ had the opposite remainder, and N_i would be divisible by 7. Since this is not the case, it follows that $10^6 K + aA_6$ is a multiple of 7. Knowing that 111111 is a multiple of 7, we conclude that $10^6 K$, and hence K, is divisible by 7.

Theorem 1 Every multi-digit absolute prime either is a repunit or can be obtained from a permutation of the digits of the n-digit number $B_n(a, b) = aA_n + b - a = (10^n - 10)a/9 + b$ with decimal representation aaaa....aab, where a and b are two different digits from {1, 3, 7, 9}.

Proof Let n be the number of digits of N. We can suppose that n > 6. By Lemma 1 N does not contain all four of the digits listed, and by Lemma 3 it can contain three of them only if it is a permutation of aaaa....aabc, where a, b and c are three of the numbers. Let's show that this is impossible. Since N is an absolute prime, the n-digit numbers aa...aacaaaaab and aa..aabaaaaac are also absolute primes, and by Lemma 3 the two (n-6)-digit numbers aa...aac and aa...aab are both divisible by 7, therefore their difference c-b is divisible by 7, which is impossible.

Hence N either is a repunit or contains only two digits. In the latter case we need Lemma 2 once more to secure that one of the two digits occurs only once.

The prime number 7 played a significant role in the

preceding considerations. Other useful primes also exist and we are going to find some of them. The property of 7 that was most useful for us was the fact that the powers 10^i for i = 0, 1, 2, 3, 4, 5, 6, had all different remainders on dividing by 7. In general, by Fermat's Little Theorem for an arbitrary prime p > 5, we have $10^{p-1} \equiv 1 \pmod p$.

Let h(p) be the least positive integer such that $10^{h(p)} \equiv 1 \pmod p$.

p	7	11	13	17	19	23	29	31
h(p)	6	2	6	16	9	22	28	15

It is obvious that h(p) is a divisor of p-1 and that $10^q \equiv 1 \pmod p$ implies $q \equiv 0 \pmod{h(p)}$. It is also easy to see that the powers 10^i for $0 \le i \le p-1$ have different non-zero remainders on dividing by p if h(p) = p-1. When this is the case, 10 is said to be a primitive root mod p.

Note that the number 10 is a primitive root modulo primes 7, 17, 19, 23 and 29, but 10 is not a primitive root modulo 13, because $10^6 \equiv 1 \pmod{13}$.

Lemma 4 Let $N = A_n$ be a repunit and let p > 3 be a prime. Then $N \equiv 0 \pmod p$ if and only if $n \equiv 0 \pmod{h(p)}$.

Proof As $10^n = 9N + 1$, we have $N \equiv 0 \pmod p$ if and only if

$10^n \equiv 1 \pmod p$, and this is equivalent to $n \equiv 0 \pmod{h(p)}$.

This simple assertion gives us information about divisors of the repunits. In particular, if n is prime and the factorization of A_n into primes is $A_n = p_1 p_2 \ldots p_s$, then $h(p_1) = h(p_2) = \ldots = h(p_s) = n$. For instance, $A_7 = 239 \times 4649$ and $h(239) = h(4649) = 7$.

<u>Lemma 5</u> Let $B_n(a, b)$ be an absolute prime. Suppose that p is a prime, not a factor of a, and that 10 is a primitive root modulo p, and that $n \geq p-1$. Then n is a multiple of $p-1$.

<u>Proof</u> For $i = 0, 1, \ldots p-2$, consider the n-digit numbers

$$B_i = aA_n + (b-a)\times 10^i = 10^{p-1} aA_{n-p+1} + aA_{p-1} + (b-a)\times 10^i,$$

obtained from $B_n(a, b)$ by permutations of the last $p-1$ digits. Since the powers 10^i (for $i = 0, 1, \ldots p-2$) yield all the non-zero remainders on dividing by p, so do the $(b-a)\times 10^i$, and hence all the B_i can be simultaneously prime only in the case when the number $L = 10^{p-1} aA_{n-p+1} + aA_{p-1}$ is divisible by p. But then since A_{p-1} is divisible by p, and 10^{p-1} and a are not, it follows that A_{n-p+1} is divisible by p, and by Lemma 4 n is divisible by $h(p) = p-1$.

<u>Lemma 6</u> If $7 \leq n \leq 16$ then $B_n(a, b)$ is not an absolute prime.

<u>Proof</u> If $a \neq 7$ it follows from Lemma 5 (with $p = 7$) that we need to verify the cases with $n = 12$ only. The case $a = 7$ requires a bit more work. Direct calculations here seem to be unavoidable. These calculations show that the numbers $B_n(7, b)$ by a permutation of digits can be converted to multiples of 3, 17 or 19.

<u>Theorem 2</u> Let N be an absolute prime, not a repunit, with $n > 3$ digits. Then n is a multiple of 11088.

<u>Proof</u> By Lemma 6 we may take $n > 16$. Since 10 is a primitive root modulo 17, Lemma 5 yields that n is a multiple of 16, and hence $n \geq 32$. We can repeat this argument ~~three~~ times, using the primes 19, 23 and 29, to obtain that n is a multiple of 18, 22 and 28, respectively. Hence n divides $LCM(16, 18, 22, 28) = 11088$.

Richert [5] used in addition the primes 47, 59, 61, 97, 167, 179, 263, 383, 503, 863, 887 and 983 to show that the number n of digits of an absolute prime $B_n(a, b)$ must be divisible by 321,653,308,662,329,838,581,993,760. He also mentioned that by using the tables of primes up to 99999 and their primitive roots it is possible to show that $n > 6\times10^{175}$.

Let's discuss now what pairs (a, b) can appear in an absolute prime $B_n(a, b)$ with $n > 3$ (if such a prime exists at all!).

<u>Theorem 3</u> If $n > 3$ and $B_n(a, b)$ is an absolute prime, then $(a, b) \neq (9, 7), (9, 1), (1, 7), (7, 1), (3, 9), (9, 3)$.

<u>Proof</u> Let's write down the following equality

$$9A_n - 2\times10^r = 10^n - 1 - 2\times10^r = (10^n + 1) - 2(10^r + 1).$$

By Theorem 2, n is even but not a power of 2; write $n = ru$, where r is a power of 2 and u is odd. Then $10^n + 1$ is divisible by $10^r + 1$, so that the number above is composite.

But this number can be obtained by a permutation of the digits of $B_n(9, 7)$.

Furthermore $B_n(9, 1) = 10^n - 9 = (10^{n/2} - 3)(10^{n/2} + 3)$ which is composite.

Finally, since n by Theorem 2 is divisible by 3, the sums of the digits of $B_n(1, 7)$ and $B_n(7, 1)$ are also divisible by 3. Hence these numbers as well as $B_n(9, 3)$ and $B_n(3, 9)$ are all divisible by 3.

References

[1] M. Gardner, Mathematical Games, Scientific American, June 1964, p. 118.
[2] H.S. Williams, Some primes with interesting digit patterns, Math. Comp., 1978, vol 32, pp. 1306-1310.
[3] S. Yates, Repunits and repetends, Star Pub. Co., Boynton Beach, Florida, 1982.
[4] P. Ribenboim, Prime number records, a new chapter for the Guiness book.
[5] H.-E. Richert, Om permutable primtall, Norsk Matematiske Tiddskrift, 1951, vol 33, pp. 50-54.
[6] T.N. Bhargava and P.H. Doyle, On the existence of absolute primes, Mathematics Magazine, 1974, vol. 47, p. 233.
[7] A.W. Johnson, Absolute primes, Mathematics Magazine, 1977, vol. 50, pp. 100-103.

SYMMETRIC SIMULTANEOUS EQUATIONS (JCMN 59, p. 6173)

Harry Alexiev, A. Brown and J. B. Parker

Problem: Solve for (x, y, z) in terms of (a, b, c) the equations

$x^2 - yz = a, \qquad y^2 - zx = b, \qquad z^2 - xy = c.$

If we write $A = a+b+c$, $B = a^2 + b^2 + c^2 - bc - ca - ab$,

$C = AB = a^3 + b^3 + c^3 - 3abc$,

$X = x + y + z, \qquad Y = x^2 + y^2 + z^2 - yz - zx - xy,$

then the equations for a, b, c give

$A = Y$, $a-b = (x-y)X$, $b-c = (y-z)X$, $c-a = (z-x)X$,

$AX^2 = X^2Y = \frac{1}{2}\{(x-y)^2 + (y-z)^2 + (z-x)^2\}X^2 = B$

and $AX = \pm\surd(AB) = \pm\surd C$

Note that if (x, y, z) is a solution then (-x, -y, -z) is also a solution. With this in mind we can take (if $C \neq 0$)

$x+y+z = (1/A)\surd C$, $x-y = A(a-b)/\surd C$, $y-z = A(b-c)/\surd C$

and solve these three linearly independent equations for x, y and z. This gives the solution:

$(x, y, z) = \pm(a^2-bc,\ b^2-ca,\ c^2-ab)/\surd C.$

Two special cases occur when (i) $X = 0$ or (ii) $a+b+c = 0$.

(i) (From above) $a = b = c$; let $D = 2\surd(a/3)$. Then

$x = D\cos\varphi$, $y = D\cos(\varphi-2\pi/3)$, $z = D\cos(\varphi-4\pi/3)$

provides a solution for every value of φ, because

$2(x^2-yz)D^{-2} = 1 + \cos 2\varphi - \cos 2\varphi - \cos 2\pi/3 = 3/2$, etc.

(ii) If $A = a+b+c = 0$, then (see above) $B = AX^2 = 0$ (assuming there is a solution), and $bc+ca+ab = 0$. Then $a^2-bc = a(a+b+c) - (bc+ca+ab) = 0$, etc. Therefore we have

$a^3 = b^3 = c^3 = abc$, so that a, b and c are either equal (= 0) or of the form k, $k\omega$, $k\omega^2$, where k is a complex number and ω is a complex cube root of unity. If these conditions are not satisfied there is no solution. But what if they are?

For the general case, with $X \neq 0$ and $A \neq 0$, the form of solution suggests that it might have been better to work out a^2-bc, b^2-ca and c^2-ab in the first place. We obtain

$a^2 - bc = x(x^3 + y^3 + z^3 - 3xyz) = xXY$

$b^2 - ca = yXY, \qquad c^2 - ab = zXY.$

and the previous working gives $XY = XA = \pm\surd C$.

Another way of approaching this problem is to note that from the original equations we find:

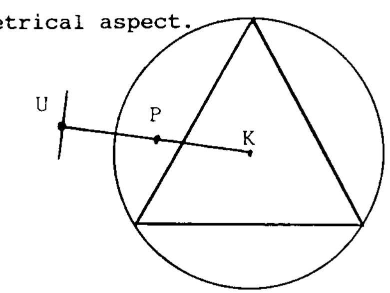
$bx+cy+az = 0, \qquad cx+ay+bz = 0, \qquad ax+by+cz = XY.$

Now use Cartesian coordinates. For points H (a, b, c), J (c, a, b) and M (x, y, z), OM is perpendicular to OJ and to OL, so the vector $\underline{OM}$ is in the direction of the vector product $\underline{OJ} \times \underline{OL}$. Hence (x, y, z) is proportional to

$(a^2 - bc,\ b^2 - ca,\ c^2 - ab)$

and the scalar product $\underline{OM}.\underline{OH} = ax+by+cz = XY = \surd(AB)$ gives the appropriate length for $\underline{OM}$.

The problem has another geometrical aspect. Take an equilateral triangle, and consider inversion in the circumcircle. Using trilinear coordinates, with the triangle as triangle of reference, the



circumcircle has equation $yz+zx+xy = 0$, and its centre, K, is the point (1, 1, 1). If P has coordinates (a, b, c) then its polar is: $x(b+c) + y(a+c) + z(a+b) = 0$, and the line PK is $x(b-c) + y(c-a) + z(a-b) = 0$. The inverse U of P in the circle is the intersection of these two lines, and so its coordinates (u, v, w) are given by

$$\frac{u}{a^2 - bc} = \frac{v}{b^2 - ca} = \frac{w}{c^2 - ab}$$

Since inversion is a self-reciprocal operation, the same form of equation will give (a, b, c) in terms of (u, v, w). Thus the geometry has given us an algebraic result, that if

$$\frac{a}{x^2 - yz} = \frac{b}{y^2 - zx} = \frac{c}{z^2 - xy}$$

then $$\frac{x}{a^2 - bc} = \frac{y}{b^2 - ca} = \frac{z}{c^2 - ab} \quad (= M, \text{ say})$$

In order to solve the given equations it remains only to determine M. From the first of the given equations:

$$a = x^2-yz = M^2\{(a^2-bc)^2 - (b^2-ca)(c^2-ab)\}$$
$$= M^2a(a^3 - 2abc + b^3 + c^3 - abc)$$

It follows that $M = \pm(a^3 + b^3 + c^3 - 3abc)^{-1/2}$, and we find again the general solution $x = M(a^2 - bc)$, etc.

The special cases mentioned above of $a=b=c$ and $a+b+c=0$ have a geometrical interpretation. They correspond to the point (a, b, c) being the centre K of the circle or a point on the line at infinity, respectively. Generally the inverse of any point on the line at infinity is the centre. However, the points $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ are the two circular points on the line at infinity, and their inverses are a little uncertain. On the one hand, we like to think that points on the circle invert into themselves, but on the other hand we like to think that all points on the line at infinity invert into the centre.

BINOMIAL COEFFICIENTS

Terry Tao

We are all familiar with the binomial coefficients as set out in Pascal's triangle

```
                1
             1     1
          1     2     1
       1     3     3     1
    1     4     6     4     1
  ...........................
```

and we represent them by the symbol $\binom{n}{r}$ where n and r are non-negative integers. But can we extend the definition to other values of n and r? We want to do so in such a way as to preserve what we can of the algebraic structure of the original family. The obvious first attempt is to extend the range of values of r to all integers, by assigning the value zero to the new binomial coefficients. This changes Pascal's triangle to

```
......   0    0    0    1    0    0    0   ....
.....  0    0    0    1    1    0    0    0   ....
 ......   0    0    1    2    1    0    0   .....
.....  0    0    1    3    3    1    0    0   ....
 ......   0    1    4    6    4    1    0   ......
       ..............................
```

and preserves the two simplest properties of the original: that when you add adjacent values on one line you get the value below, and that the pattern is symmetrical about a vertical line. This is a useful convention, for it often (but not quite always) means that in a summation we may sum over all integer values of the free variable, instead of having to worry about the range of summation. But can we go further? Can we have negative integer values for the n?

Using the binomial expansion of $(1 + x)^n$ for negative n, we naturally choose to extend Pascal's triangle by adding the corresponding rows above that for n = 0, this gives the pattern shown below

....

...	0	0	0	1	-2	3	-4	5	...
...	0	0	0	1	-1	1	-1	1	.....
.....	0	0	1	0	0	0	0	0	...
...	0	0	1	1	0	0	0	0	....
....	0	1	2	1	0	0	0	0	...
...	0	1	3	3	1	0	0	0	.....

....

We have kept the addition property but lost the symmetry. We no longer have $\binom{n}{n-r} = \binom{n}{r}$. This is a warning of dangers ahead!

Now look at some examples — From BINOMIAL IDENTITY 35 in JCMN 59, pp 6162-6164, we see that the identity

$$\binom{m-s}{n} = \sum_{i=0}^{n} (-1)^i \binom{s}{i}\binom{m-i}{n-i}$$

may be transformed into $\binom{m+s}{n} = \sum_{i=0}^{n} \binom{s+i-1}{i}\binom{m-i}{n-i}$ by changing the sign of s, provided that we use the rule that

$$\binom{-n}{r} = (-1)^r\binom{n+r-1}{r} \quad \text{when n and r both} \ge 0,$$

as suggested by the binomial theorem.

That was the good news, now the bad. In THE DERANGED KNIGHTS OF CAMELOT, JCMN 52, pp 5254-5257, we find the proof of Binomial Identity 31:

$$\sum_{m=0}^{n} m^2 \binom{n}{m} \sum_{r=0}^{m} (-1)^r m!/r! = (n^2 - 2n + 2)n!$$

$$\text{or} \quad \sum_{m=0}^{n} \frac{m^2}{(n-m)!} \sum_{r=0}^{m} (-1)^r/r! = n^2 - 2n + 2$$

(it was proved where n = the number of knights > 1). Now let's check this last formula for a few values of n.

n	0	1	2	3	4	5	6
LHS	0	0	2	5	10	17	26
RHS	2	1	2	5	10	17	26

Is there any principle like analytic extension applying to binomial coefficients?

Confine ourselves to the case where r is a non-negative integer, 0, 1, 2, ... etc, but n can be any complex number. Define the function C(n, r) to be

$$C(n, r) = \frac{n(n-1)(n-2)\ \dots\ (n-r+1)}{r!}$$

Of course if n is a positive integer, $C(n, r) = \binom{n}{r}$, not only when $n \ge r$, but also when $n < r$ and the function = 0. If n is not a positive integer, how does the function behave as r tends to infinity, with fixed n? It is easy to show that $C(n, r) \sim (-1)^r r^{-n-1}/\Gamma(-n)$.

Now consider the story of the deranged knights. Put

$$d(m) = \sum_{s=0}^{m} (-1)^s/s! \quad \text{and} \quad f(n) = \sum_{m=0}^{\infty} C(n, m)m^2 d(m).$$

The series for f(n) will converge only if $C(n, m)m^2 \to 0$ as $m \to \infty$, that is only if either n has real part > 2 or n is a non-negative integer. Thus f(n) is analytic only in the half-plane where n has its real part > 2, though the series exists at other isolated points. We should not be upset by the failure of the binomial identity at the two points n = 0 and 1.

If we have a binomial identity proved for integer $n \ge 0$, is there a theorem by which we can show the identity to be valid for other n?

Consider the following three rules:-

(a) Make sure the n never appears in factorials, and is always on top in any binomial coefficient. For example

$\binom{n}{r} = \binom{n}{n-r}$ is NOT suitable, but $\sum_{r=0}^{n}\binom{n}{r} = 2^n$ is.

(b) Replace any summation over a range dependent on n by summation over all integers.

(c) Replace $\binom{x}{y}$ by $C(x, y)$.

Conjecture With these changes, if the binomial identity holds for positive integer n, it holds for all n for which the summations are absolutely convergent.

QUOTATION CORNER 39

I cannot give any scientist of any age better advice than this: the intensity of the conviction that a hypothesis is true has no bearing on whether it is true or not.

— P. B. Medawar, *Advice to a young scientist,* 1979.

(Contributed to JCMN by R. A. Lyttleton)

BINOMIAL IDENTITY 37 (JCMN 59, p.6173)

John Parker

(Oak Tree Cottage, Reading Road, RG74QN, U.K.)

$$\sum_{r=m}^{n}\binom{r+m}{m}^2\binom{r}{m}^2(2r+1) = \binom{n+m+1}{m}^2\binom{n}{m}^2(n+1)^2/(2m+1)$$

Put f(n) = RHS, then f(n+1) - f(n) is easily seen to be $\binom{n+m+1}{m}^2\binom{n+1}{m}^2(2m+3)$, and when n = m the two sides are the same. The result follows by induction on n.

ADDING NUMBERS

Ròbert Freud

(Eötvös University, Budapest, Múzeum krt. 6-8, 1088, Hungary)

Paul Erdös has asked the following question. Let $1 \le a_1 < a_2 < \dots < a_k \le n$ be integers such that no a_i is the sum of (any 2 or more) consecutive a_j-s. Is it possible for k to be significantly larger than n/2?

Pomerance showed that k = 12 with n = 20 can be attained by the example {4, 5, 6, 7, 8, 10, 12, 14, 16, 17, 19, 20}; and more generally if n = 4m with m odd, then $\{m-1, m, m+1, \frac{3m-1}{2}, \frac{3m+1}{2}, 2m, 2m+2, 2m+3, \dots, 4m\}\setminus\{\frac{5m+1}{2}, 3m, \frac{7m+1}{2}\}$ is a set having $\frac{n}{2}+2$ elements, which is one more than the trivial set {2m, 2m+1, 2m+2, ... 4m}

The following construction shows that k = 19n/36 + O(1) may be attained.

Let x and y be positive integers, and consider the following sets of numbers.

(A) 2x-2y, 2x-2y+1, ... 2x, 2x+1, ... 2x+2y

(B) 3x-3y+1, 3x-3y+2, 3x-3y+4, ... 3x-1, 3x+1, ... 3x+3y-1

(C) 4x-4y+2, 4x-4y+4, 4x-4y+6, ... 4x-2, 4x, ... 4x+4y-2

(D) 4x+4y+2, 4x+4y+3, 4x+4y+4, ... 8x+8y+4

Explaining these, we took for (A) the 4y+1 adjacent integers round 2x, for (B) the 4y integers around 3x not divisible by 3, for (C) the 4y-1 even integers around 4x, and for (D) all the numbers from 4x+4y+2 to 8x+8y+4. We shall attain our desired sequence by the suitable choice of x and y, and by deleting some numbers from (D).

The elements in (A), ... (D) are distinct and are in increasing order if

$2x+2y \le 3x-3y$ and $3x+3y \le 4x-4y+2$ .... (i)

The sum of two adjacent elements of (A) is never equal to an element in (C) or (D), and such a sum cannot be an element of (B) if

$4x-4y+1 > 3x+3y-1$ .... (ii)

The sum of three or more consecutive elements of (B), (C) or (D) is different from all elements of (A), ... (D) if

$9x-9y+7 > 8x+8y+4$ .... (iii)

The sum of five or more consecutive elements (of (A)) is different from all elements in (A), ... (D) if

$10x-10y+10 > 8x+8y+4$ .... (iv)

Now we turn to the final stage of our construction. There are altogether 4x+16y+3 numbers in (A), ... (D). We delete those elements that are equal to a sum of some consecutive elements. It is clear that only elements of (D) have to be deleted, namely those that are:

I. sums of three consecutive elements of (A)

II. sums of two consecutive elements of (B)

III. sums of four consecutive elements of (A)

IV. sums of two consecutive elements of (C)

V. sums of two or three consecutive elements at the border of (A)-(B), (B)-(C) or (C)-(D).

Our construction guarantees that sets I and II coincide, and that III and IV coincide, hence we have to delete altogether $(4y-1)+(4y-2)+5 = 8y+2$ elements. The conditions (i), ... (iv), in fact basically (iii), require $x \ge 17y-2$. Choosing here equality, our sequence contains 76y-7 elements up to $n = 8x+8y+4 = 144y-12$, which yields the proportion 19/36 as claimed.

As for an upper bound, it is easy to prove that the proportion cannot exceed 2/3, moreover this holds if we exclude only $a_i = a_j + a_{j+1}$ (and for this case it is the best possible)

Later I learned from D. Coppersmith and Steven Phillips (Thomas J. Watson Research Center, Yorktown Heights, NY, USA) that they had rediscovered my result above and improved it; they have a construction giving $13n/24 + O(1)$. They also improved the upper bound to $2/3 - 1/3584$.

Consider an infinite sequence. Let A(n) denote the number of elements $\le n$. We can achieve $\limsup A(n)/n = 19/36$ using the previous construction. We simply repeat the construction with very rapidly growing values of y and omit a few problematic terms.

More precisely, assume that we have repeated the process several times, and the sum of all our elements is T. Choose now (say) $y = T^2$, and form the next segment of the sequence, using the (finite) construction described above. Now we delete the sets of elements z for which the following inequalities hold:-

$$2x-2y+1 \le z \le 2x-2y+T$$
$$4x-4y+T \le z \le 4x-4y+2T+1$$
$$6x-6y+2T+3 \le z \le 6x-6y+3T+3$$
$$8x-8y+3T+1 \le z \le 8x-8y+4T+6$$

For the remaining elements no one is the sum of consecutive others and the "loss" of about $4T = 4\sqrt{y}$ elements is negligible compared with $n = 144y-12$, hence the proportion 19/36 is not violated.

FUNCTIONAL INEQUALITY

Given any positive integer n, for all non-negative real functions f(x) on the unit interval with the property that

$$f(x) + \frac{1}{2} f(x/2) + \frac{1}{3} f(x/3) + \ \dots + \frac{1}{n} f(x/n) \le 1,$$

find the maximum of $\int_0^1 f(x)\, dx$.

ADDING NUMBERS 2

Paul Erdös

An old problem of Harzheim and myself asks:-

Let $a_1 < a_2 < \dots < a_k \le n$ be such that all the sums $\sum_{j=u}^{v} a_j$ are distinct. Is it true that $k = o(n)$? (It is easy to prove that for an infinite sequence the lower density must be zero) Can one find a large k (such as for example n/power of log n) for which the sums $\sum_{j=u}^{v} a_j$ are distinct?

ORTHIC TRIANGLES (JCMN 58, p.6138, 59, p.6145)

Terry Tao

Given a plane triangle, consider the operation of forming the orthic triangle, with vertices at the feet of the altitudes. Repetition of the operation gives a sequence of triangles converging to a point. The original question — where is this limit point? — has not yet been answered in our pages.

Nevertheless, there are interesting observations to be made about the shapes of the triangles of the sequence. The shape of any triangle may be represented by the angles (A, B, C), all ≥ 0, and with sum $= \pi$. The operation of forming the orthic triangle gives a new family of angles (A′, B′, C′) as follows:

If the triangle is acute-angled then

$A' = \pi - 2A$, $B' = \pi - 2B$ and $C' = \pi - 2C$.

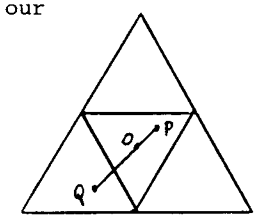
If the triangle has an obtuse angle at A, then

$A' = 2A - \pi$, $B' = 2B$ and $C' = 2C$.

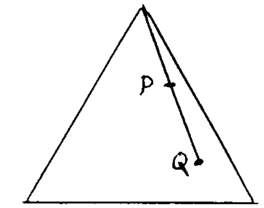
Similarly for other obtuse-angled triangles.

Now we shall form a geometrical picture of the algebraic operation described above for any (A, B, C).

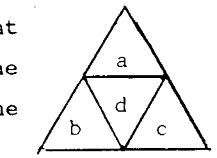
Take (A, B, C) to be trilinear coordinates of a point in an equilateral triangle. Joining the mid-points of the sides, we obtain an inner triangle containing all the points that represent acute-angled triangles; the boundary lines, $2A = B + C$, etc. represent right-angled triangles. Let our original triangle be represented by the point P, and let Q represent the image, that is the orthic triangle. If P is in the inner triangle, then Q is as



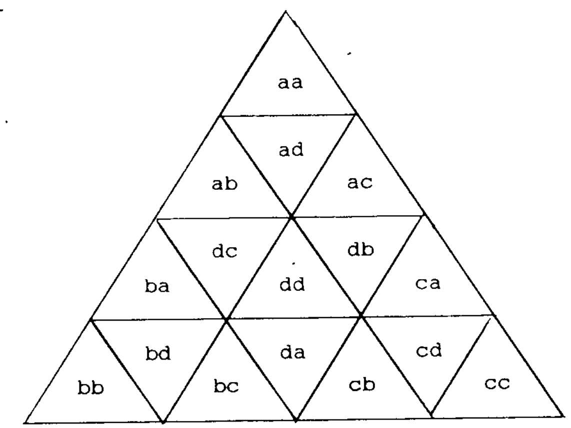
shown, with $2PO = OQ$, and POQ a straight line, where O is the centroid. If P represents an obtuse-angled triangle, then Q is as shown in the second picture.



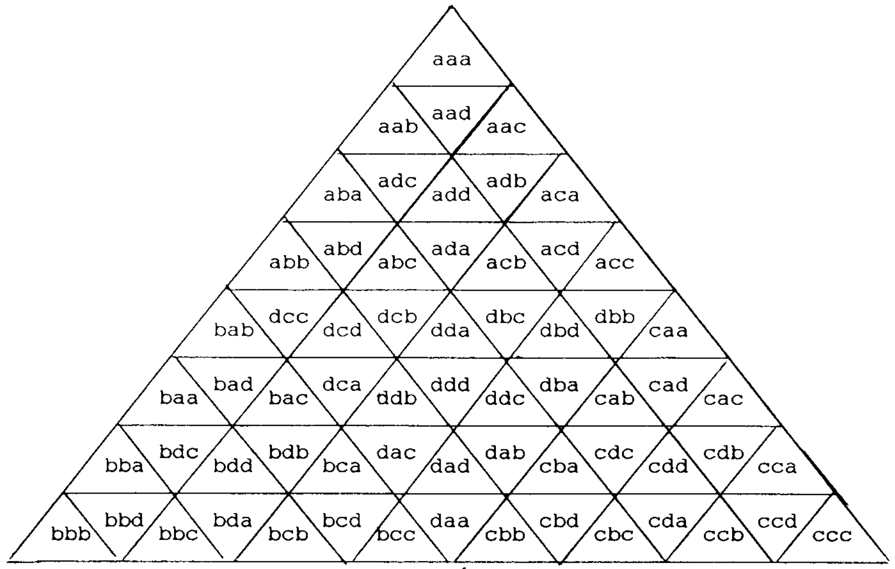
Let's label our space of triangles by the letters a, b, c and d in each of the four triangular subsets as shown. To describe the position of any point P in the space we use as first approximation the letter a, b, c or d of the region in which the point is. Note that if P is in d, then its image Q can be anywhere in the space, therefore we may subdivide d into da, db, dc and dd, according to where the image Q is. Similarly a, b and c all map into the whole space, and therefore each of them may be subdivided in the same way.



Thus we obtain this picture, with 16 subdivisions of our space:-



The subdivision may be repeated in the obvious way. The region adc, for example, is the part of ad that has its image in dc.



Now it is clear that the subdivision may be continued without end, so that each element in the space (forgetting for the moment the points on the boundaries of the little triangles) may be denoted by an infinite sequence of the digits a, b, c, and d. For example the element

·adadadadadadad............

is the isosceles triangle with angles A = 108°, B = C = 36°. Its orthic triangle is denoted by ·dadadadadada.... and has angles A = 36°, B = C = 72°.

This representation of elements of our space by sequences of digits is like the representation of the unit interval of the real variable by sequences of decimal digits, 0, 1, ... 8, 9. There is the same difficulty that some elements do not have unique representation. In the real variable case,

for instance, the notations ·23999999... and ·2400000.... both represent the same number. In the case of our space of triangles, the triangle with angles A = 90°, B = 30° and C = 60° may be represented by ·dbcbcbc... or by ·acbcbcb...

In both cases we try to console ourselves by saying that almost all the elements of our space are well-behaved, with unique representation; this requires us to set up a measure theory on the space (this is not quite true, the assumptions we need are weaker, see below) but no difficulties arise, we use the obvious Lebesgue measure in both cases. The exceptional cases are those triangles for which the sequence of orthic triangles comes eventually to a right-angled triangle, and at the next stage to a degenerate triangle with two vertices coincident, and the angles therefore indeterminate.

The important property of our representation of triangles is that deletion of the first digit of the representation gives the orthic triangle.

Now, digressing for the moment to the real variable, we define a real number to be "normal" when any finite sequence of digits occurs with the "correct" frequency. More precisely, let $d_1 d_2 \ldots d_k$ be any finite sequence of digits, and let x be in the unit interval, for any n count the number of occurrences of the sequence in the first n digits of x, this number divided by n-k+1 should converge to 10^{-k} as n tends to infinity; if so then x is "normal". It can be shown that almost all real numbers are normal.

Guided by these ideas, let us turn back to our space of triangles. The earnest student of measure theory should be able to prove that almost all triangles are normal; therefore if we start at almost any element, and repeatedly form the orthic triangle, we get a trajectory that is dense in the set. Such a trajectory generates a measure on the set, in fact the measure of any subset E is the limit (as n tends to infinity) of (number of points in E from the first n points of the trajectory)/n. This measure turns out to be the same as the Lebesgue measure of the equilateral triangle, which was the measure that we introduced in order to define "almost all". In any space, there is an equivalence relation among the possible measures, two measures being equivalent if every subset null in one is null in the other. Therefore if instead of using Lebesgue measure in our definition of "almost all", we had used any other equivalent (in the sense above) measure, we would have reached the same result — that the trajectory starting at almost any point defines on the space a measure equal to the Lebesgue measure on the equilateral triangle.

Recall the questions raised in JCMN 55 (RANDOM TRIANGLES, p.6028) about what is a random triangle and what is the probability that it be acute-angled. Perhaps we here have an answer — that the infinite sequence of orthic triangles from almost any starting triangle has asymptotically one in four of the triangles acute-angled.

ORTHIC TRIANGLES 2 (JCMN 58, p.6138 & 59, p.6145)

Where is the limit point of the sequence of orthic triangles starting from an arbitrary triangle? Let us call it the "orthic limit point". If the starting triangle is isosceles, then all the triangles of the sequence are isosceles, with the same line of symmetry. This seems to be a simple special case worth investigating.

As the starting triangle take the isosceles triangle with vertices at (0, 1), (b, 0) and (-b, 0). The orthic limit will be a point (0, c). It is not hard with a computer to find c as a function of b. There are points where c is not defined, for instance b=1, but the computer tells us that this is a removable discontinuity, for some neighbouring points are:-

b	c
·99999970	·99999983
·99999980	·99999976
·99999990	·99999992
·99999995	·99999994
·99999999	·99999999
1·00000001	1·00000001
1·00000005	1·00000004
1·00000010	1·00000012
1·00000020	1·00000016
1·00000030	1·00000043
1·00000040	1·00000048

The following figures give more idea of the function:

b	c	0·5	0·09816
0·1	-0·01706	0·6	0·44102
0·2	0·07554	0·7	0·67189
0·3	0·06063	0·8	0·73707
0·4	-0·00950	0·9	0·93637

Having doubts about the differentiability of the function, let's look at a little bit of it in more detail. Magnify the scale by 100, so that b goes up in steps of ·001. First differences are given in the last column.

b	c	diff
·553	·230741	
		+2962
·554	·233703	
		+1506
·555	·235209	
		- 233
·556	·234976	
		+1171
·557	·236147	
		+2281
·558	·238428	
		+2803
·559	·241231	
		+2870
·560	·244101	
		+4881
·561	·248982	
		+5514
·562	·254496	
		+4389
·563	·258885	
		+2864
·564	·261749	
		+3783
·565	·265532	
		+4841
·566	·270373	

Now magnify the scale by another factor of 100, so that b goes up in steps of ·00001.

b	c	diff
·55618	·23519440	
		+ 803
·55619	·23520243	
		+1917
·55620	·23522160	
		- 253
·55621	·23521907	
		- 395
·55622	·23521512	
		-1168
·55623	·23520344	
		-3499
·55624	·23516845	
		-2222

b	c	diff
·55625	·23514623	
		- 123
·55626	·23514500	
		- 738
·55627	·23513762	
		- 278
·55628	·23513484	
		+ 160
·55629	·23513644	
		+ 352
·55630	·23513996	
		+ 201
·55631	·23514197	
		+2212
·55632	·23516409	
		+3461
·55633	·23519870	

Now magnify the scale by 100 again.

b	c	diff
·5562999	·2351401123	
		-1494
·5563000	·2351399629	
		+ 559
·5563001	·2351400188	
		-1101
·5563002	·2351399087	
		-3459
·5563003	·2351395628	
		-2383
·5563004	·2351393245	
		-3723
·5563005	·2351389522	
		-2726
·5563006	·2351386796	
		-4889
·5563007	·2351381907	
		-6220
·5563008	·2351375687	
		-3990
·5563009	·2351371697	
		-3531
·5663010	·2351368166	
		-1227
·5563011	·2351366889	
		-2426
·5563012	·2351364463	

These figures suggest that c is a continuous but non-differentiable function of b (apart from having a dense set of removable discontinuities).

COMBINATORIAL QUESTION (JCMN 58, p.6137 & 59, p.6160)

James Geelen and Jamie Simpson
(Curtin University of Technology, Perth, W.A.)

Given m things, we want to choose the same number m of k-element subsets so that no two of these subsets have more than one element in commom. For each m, what is the largest possible k? We will call this k(m).

The problem was discussed by Paul Erdös in JCMN 59, p.6160, giving results equivalent to theorems 1 and 2 below. The problem can be viewed as a problem in graph theory: in K_m (the complete graph on m vertices) what is the largest k such that K_m contains m edge-disjoint copies of K_k? The edge-disjointness ensures that any two of the copies of K_k have at most one vertex in common. Now K_m contains $\binom{m}{2}$ edges, and each K_k contains $\binom{k}{2}$ edges, so we must have $\binom{m}{2} \geq m\binom{k}{2}$, or $m \geq k(k-1) + 1$, giving the following upper bound for k(m):

<u>Theorem 1</u> $k(m) \leq \dfrac{1 + \sqrt{(4m-3)}}{2}$

The most interesting case ocurs when we have equality in this theorem. In this case m and k have the forms

$$m = n^2 + n + 1$$
$$k = n + 1$$

and the graph and its subgraphs are equivalent to a projective plane of order n. Recall that in such a plane there are $n^2 + n + 1$ lines (one for each of the K_k graphs) and n^2+n+1 points (the vertices of the K_m). Each line contains n+1 points and there are n+1 lines through every point. The

question of equality in theorem 1 is then equivalent to asking for which orders n do projective planes exist. Unfortunately this is not known. The following information appears in *A First course in Combinatorial Mathematics* by I. Anderson, Clarendon Press, (1979), and in "The non-existence of projective planes of order 10" by C.W.H. Lam, L. Thiel and S. Swiercz in Can. J. Math. 41 (1989) pp. 1117-1123.

(a) A projective plane is usually defined to have 4 non-collinear points, which prohibits the existence of a plane of order 1. If we do allow such a plane then the next theorem gives k(3) = 2, a result which can otherwise be obtained using theorem 1 and a construction.

(b) A plane of order n exists if $n \geq 2$ is a prime or a power of a prime.

(c) No plane of any other order is known to exist.

(d) There is no plane of order n where n is congruent to 1 or 2 modulo 4, and is not the sum of two squares.

(e) No plane of order 10 exists. The proof of this result required a massive amount of computation.

With these observations Theorem 1 leads to the following result.

<u>Theorem 2</u> If n is the order of a projective plane then

$$k(n^2+n+1) = n+1$$

and if not then $k(n^2+n+1) \leq n$.

We now return to our graph-theoretic version of the problem. If a graph exists with m vertices and with p edge-disjoint copies of K_q, and such that each vertex belongs to exactly r of these copies, we say that (m, p, q, r) is good. The projective planes correspond to good quadruples

$$(n^2 + n + 1,\ n^2 + n + 1,\ n + 1,\ n + 1).$$

The following lemma is immediate from the definition.

Lemma 1: (m, m, q, r) is good for some r if and only if

$$k(m) \geq q.$$

Two other lemmas can be proved with a little more effort.

Lemma 2: (m, p, q, r) is good if and only if (p, m, r, q) is good.

Proof: The fact that (m, p, r, q) is good means that the complete graph G with vertices $g_1, g_2, \ldots g_m$ has p edge-disjoint complete subgraphs of order q. Call these $G_1, \ldots G_p$. Now construct a complete graph H with vertices $h_1, \ldots, h_p$. We form complete subgraphs $H_1, \ldots, H_m$ in H according to the rule,

h_i is a vertex of H_j if and only if g_j is a vertex of G_i.

Each H_j has order r. If the edge (h_1, h_2), say, belonged to both H_i and H_j we would have the edge (g_i, g_j) belonging to both G_1 and G_2, which is impossible. Thus $H_1, \ldots, H_m$ are edge-disjoint. Let $q = |G_i|$ = the number of elements in G_i. Finally each vertex h_i clearly belongs to q of the subgraphs $H_1, \ldots, H_m$.

Lemma 3: If n is the order of a projective plane then

$$(n^2 + n,\ n^2,\ n + 1,\ n) \quad \text{is good.}$$

Proof : Consider the complete graph and copies of K_{n+1} associated with a projective plane of order n. Remove one vertex and the incident edges. The resulting graph has $n(n+1)$ vertices; n+1 of the original copies of K_{n+1} have been destroyed, so it contains n^2 edge-disjoint copies of K_{n+1}. Each vertex originally belonging to a copy of K_{n+1} which also contained the removed vertex. Since this copy of K_{n+1} has been destroyed, the vertex now belongs to n copies of K_{n+1}. The structure we have produced here is an affine plane.

Using these lemmas we prove our main result.

Theorem 3 : If n is the order of a projective plane, then $k(m) = n$ for all $m = n^2-1, \ldots, n^2+n$.

Proof : From lemmas 2 and 3, $(n^2, n^2+n, n, n+1)$ is good. So we have n^2+n edge-disjoint copies of K_n in a complete graph K_{n^2} on n^2 vertices. It follows that we have at least n^2+n edge-disjoint copies of K_n in each K_m for all m that satisfy $n^2 \leq m \leq n^2 + n$. This means that $k(m) \leq n$ for these values of m, and Theorem 1 shows that $k(m) \leq n$. This proves the theorem except in the case $m = n^2-1$.

Now we deal with this case. As before we have K_{n^2} with n^2+n edge-disjoint copies of K_n; remove one vertex. In doing so we will decrease the number of copies of K_n by $n + 1$, leaving $n^2 - 1$ copies. We thus have $k(n^2-1) \geq n$. Theorem 1 gives $k(n^2-1) \leq n$, which completes the proof.

Theorems 2 and 3 give us quite a few values of k(n) for low values of n. The first value not given is k(14).

<u>Theorem 4</u> : $k(14) = 4$.

<u>Proof</u> : Theorem 1 shows that $k(14) \le 4$. The following collection of 14 subsets of the set {1, 2, ... 14} shows that $k(14) \ge 4$.

{1, 5, 9, 13}	{1, 8, 10, 12}	{1, 6, 7, 11}
{2, 6, 10, 14}	{2, 7, 9, 12}	{2, 5, 8, 11}
{3, 5, 7, 14}	{3, 6, 8, 13}	{3, 9, 10, 11}
{4, 8, 9, 14}	{4, 7, 10, 13}	{4, 5, 6, 12}
{1, 2, 3, 4}	{11, 12, 13, 14}.	

The next table gives the values of k(m), for $m \le 30$, that follow from our theorems and from the contribution below.

m	k(m)
1, 2	1
3, 4, 5, 6	2
7, 8, 9, 10, 11, 12	3
13, 14, 15, 16, 17, 18, 19, 20, 22	4
21, 24, 25, 26, 27, 28, 29, 30	5

The first unknown value is k(23). Theorem 1 shows that $k(23) \le 5$, but we have not been able to construct an example to show that this bound is attainable. The evaluation of $k(22) = 4$ in the contribution below shows that k(m) is not monotonic, (a fact already noted by Mullin and others).

COMBINATORIAL QUESTION
(JCMN 58, p.6137, 59, p.6160 and above)

This note is to prove that $k(22) = 4$; for the definition of k(m) see the previous article. Firstly, (the hard part) we shall show that $k(22) < 5$.

We suppose that we have 22 elements, and that there are 22 sets of 5 elements each, with no two sets having more than one element in common. The word "set" from now on means one of these 22 sets. The following theorems are almost obvious.

<u>Theorem 1</u> No element can be in 6 or more sets.

<u>Theorem 2</u> Every element is in exactly 5 sets.
<u>Corollary</u> A set may be identified by any two of its elements.

<u>Theorem 3</u> To each element x corresponds precisely one element (to be denoted by x^*) such that no set contains both x and x^*. Clearly $x^{**} = x$.

<u>Theorem 4</u> To any set $S = \{a, b, c, d, e\}$ corresponds another set $S^* = \{a^*, b^*, c^*, d^*, e^*\}$, disjoint from it.
<u>Proof</u> There is precisely one set not meeting S (by a counting argument as in Theorems 1 and 2), call it S^*. Take any x in S^*. This x is in 4 other sets, say J, K, L, M, all $\neq S$. Each of J, K, L, M has one element in common with S; therefore there is one of a, b, c, d, e not in any set containing x, this element must be x^*. Therefore x is one of a^*, b^*, c^*, d^*, e^*.

<u>Theorem 5</u> Each set S shares one element with every one except S^* of the other 21 sets.

Take any set, call it T = {a, b, c, d, e}, and consider the class Σ of the 20 sets other than T and T*. Let W be the class of all elements other than a, a*, b, b*, c, c*, d, d*, e, e*, there are 12 of them. We shall use the symbols 1, 2, 3, 4, ... 11, 12, for the elements of W.

Each set S in Σ contains one element of T and one element of T*. Therefore the 20 sets in Σ may be set out in a table as follows, the 20 rectangles other than those on the principal diagonal. Each set can be identified by the elements that it has in common with T and T*, identified by the rows and columns respectively. The rectangles on the principal diagonal are blanked out, because a and a* cannot be in the same set, etc. We denote the 20 sets of Σ by symbols such as (a, b*), meaning the set in row a and column b*, i.e. the set that contains the element a of T and the element b* of T*. Note that each element of W must occur just once in each row and just once in each column of the table.

Now we try to fill in the table. Firstly, let 1, 2 and 3 be the elements of W in the set (a, b*), i.e. the five elements {a, b*, 1, 2, 3} form one of our sets. There is one element 1 in row b, and by permuting the symbols c, d and e we can ensure that this 1 is in (b, c*). Next, by interchanging if necessary the symbols d and e, we can ensure that the 1 that is in row c will be in column d*. This gives us the table as below.

	a*	b*	c*	d*	e*
a		1, 2, 3			
b			1		
c				1	
d					
e					

There must be one 2 and one 3 in column c*, neither can

be in row a (because they already occur in this row) nor in row b (because 1 and 2 are together in the set (a, b*) and so they cannot occur together in any other set, similarly for 3). By interchanging if necessary the symbols 2 and 3, we can have the 2 in (d, c*) and the 3 in (e, c*). This gives the table below.

	a*	b*	c*	d*	e*
a		1, 2, 3			
b			1		
c				1	
d			2		
e			3		

Next consider the 3 that must be in row c. It cannot be in column b* (there is already one in the column) or in column d* (it cannot be in a set with 1 again) or in column e* (because 3 is in (e, c*) therefore 3* is in (c, e*), and 3 cannot be in a set with 3*). Therefore this 3 must be in (c, a*). Consider the 1 that must be in row e. It must be in column a* because all the other columns already have a 1. Filling in these two we get the table below.

	a*	b*	c*	d*	e*
a		1, 2, 3			
b			1		
c	3			1	
d			2		
e	1		3		

Now we come to the contradiction. Consider the 1 and the 3 that must be in row d. Neither can be in column a* or column b* or column c* (because both are already in each of these columns), but they cannot both be in (d, e*).

This contradiction has proved that $k(22) < 5$. The fact that $k(22) \geq 4$ is shown by the following 22 sets of 4 elements.

{1, 2, 3, 4}	{19, 20, 21, 22}	
{1, 6, 7, 19}	{2, 6, 8, 18}	{3, 9, 10, 18}
{1, 8, 9, 20}	{2, 7, 11, 20}	{3, 8, 11, 19}
{1, 10, 11, 21}	{2, 9, 12, 21}	{3, 7, 13, 21}
{1, 12, 13, 22}	{2, 10, 15, 22}	{3, 14, 16, 22}
{4, 11, 12, 18}	{5, 13, 14, 18}	
{4, 9, 13, 19}	{5, 10, 12, 19}	
{4, 6, 10, 20}	{5, 15, 16, 20}	
{4, 7, 8, 22}	{5, 6, 17, 21}	

Thus we have proved that $k(22) = 4$, as shown in the table on p.6216 above.

NUMBER PUZZLE

<u>Question</u>

n	1	2	3	4	5	6	7	8	9
f(n)	1	3	7	19	51	141	393	1107	3139

How does the sequence go on?

<u>Answer</u>

For positive integer n, if ω is a primitive cube root of unity, $f(n) = (-\omega)^n \sum_{r=0}^{n} \binom{n}{r}^2 \omega^r$.

<u>Further question</u> What is the asymptotic behaviour of $f(n)$ for large n?

POLYNOMIAL INEQUALITY 1

Terry Tao

Let $f(x)$ be a real polynomial of degree n such that $-M \leq f(x) \leq M$ in the unit interval. Prove that

$$|f'(x)| < n(n+1)(n+2)M/\sqrt{3}$$

and

$$|f''(x)| < (n-1)n(n+1)(n+2)(n+3)M/\sqrt{10}$$

in the unit interval. Can these inequalities be made stronger?

<u>Conjecture</u> Is $|f'(x)| \leq 2n^2 M$?

ANALYTIC INEQUALITY 5

Prove (or disprove) for any positive integer n and and real coefficients $a_1, a_2, \ldots a_n$:

$$\max_x \sum_{r=1}^{n} \left|a_r \sum_{t=1}^{r} \cos(r+1-2t)x\right| \leq n \max_x \left|\sum_{r=1}^{n} a_r \cos rx\right|$$

The case $n = 1$ is trivial, and the case $n = 2$ is easy, in fact it is $\max_x |a|+|2b \cos x| \leq 2 \max_x |a \cos x + b \cos 2x|$, which simplifies to $|a| + 2|b| \leq 2|a| + 2|b|$.

With $n = 3$ it becomes a little more difficult, (and so on!).

POLYNOMIAL INEQUALITY 2

Let $f(x)$ be a polynomial of degree n, with real coefficients, regarded as a function on the unit interval.

Let $\|f\|_p$ denote the norm of order p, defined as follows:

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx \right)^{1/p}, \quad \text{with } \|f\|_\infty \text{ denoting } \max |f(x)|.$$

What are the inequalities relating $\|f\|_1$, $\|f\|_2$ and $\|f\|_\infty$?

In particular, is it true that $\|f\|_\infty/\|f\|_1 < 4{\cdot}3903126890496$ when $n = 2$?

HYPERPLANES PARTITIONING N-SPACE

Mark Kisin
(Monash University, Vic, Australia)

Into how many regions do k hyperplanes in general position divide n-dimensional Eulidean space? Call it $f(n, k)$.

The case $n = 1$ is trivial, $f(1, k) = k+1$. In general, if we have k hyperplanes $h_1, h_2, \ldots h_k$ in $\mathbf{R}^n$ and add another, h_{k+1}, then the new hyperplane will be divided into $f(n-1, k)$ regions by the other k; and therefore it will increase the number of regions in the $\mathbf{R}^n$ by $f(n-1, k)$. Thus we have

$$f(n, k+1) = f(n, k) + f(n-1, k) \qquad \text{............ (1)}$$

This relation together with

$$f(1, k) = k + 1 \qquad \text{................ (2)}$$

and

$$f(n, 0) = 1 \qquad \text{................ (3)}$$

are sufficient to determine f completely.

Now we shall derive the general formula

$$f(n, k) = \binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{n} \qquad \text{............ (4)}$$

in two different ways.

From (1), (2) and (3) by double induction it is clear that $f(n, k)$ is a polynomial of degree n in k. Also $f(n, k) = 2^k$ for $0 \le k \le n$, so that (4) holds when $k \le n$. But a polynomial of degree n is uniquely determined by $n + 1$ values, and so (4) holds for all k.

The second proof is simply noting that (4) clearly satisfies (2) and (3), and it satisfies the recurrence (1) because $\binom{k+1}{n} = \binom{k}{n} + \binom{k}{n-1}$.

Is there a combinatorial proof of (4)?

How many of the regions are bounded?