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OBITUARY - J.D.E.KONHAUSER (1924-1992)

We regret to report the death at Edina, Minnesota, on 28th February 1992, of Joseph Konhauser, at the age of 67. He had retired from full-time work at Macalester College, Saint Paul, Minnesota in May 1991, and is survived by his wife Aileen, his sister Louise Glaze and his son Daniel Scott and daughter-in-law.

As well as being Chairman of Department at Macalester College, Joseph Konhauser became well known with his work for the Mathematical Olympiad and William Lowell Putnam competitions, and for the Pi Mu Epsilon Journal.

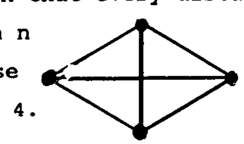
POINTS AND DISTANCES IN THE PLANE (JCMN 57, p.6089)

Paul Erdös
(Hungarian Academy of Sciences, Budapest, P.O.B.127, H-1364)

The theorem that given n distinct points in the plane the maximum distance occurs at most n times is due to Erica Pannwitz, she proved it about 60 years ago.

Harborth in the Elemente der Mathematik determined exactly how often the minimum distance can occur (it was a solution to a problem).

J. Pach and I have the following problem. Given n (> 4) distinct points in the plane, can it happen that every distance except the diameter occurs more often than n times? For n = 4 this is of course possible, but we know of no example for n > 4.



ORTHIC TRIANGLES (JCMN 58, p.6138)

Esther Szekeres & Basil Rennie

If we repeatedly replace a triangle by its orthic triangle (with vertices the feet of the altitudes), do the triangles converge to a point?

Yes. The circumcircle of any triangle has twice the radius of the nine-point circle, which is the circumcircle of the orthic triangle. Therefore consider the circumcircles of the triangles of the sequence; each contains a point inside the previous one, and is half the size; therefore the circles converge to a point. For a formal proof we may treble each radius and so obtain a nested sequence of compact sets.

If the sides of ABC are a, b, c, then the sides of the orthic triangle are $a|\cos A|$, $b|\cos B|$ and $c|\cos C|$. The angles of the orthic triangle are given as follows. If ABC is acute-angled then the angles are $\pi-2A$, $\pi-2B$ and $\pi-2C$, and if ABC is obtuse-angled with the obtuse angle at A the angles are $2A-\pi$, 2B and 2C. The area of the orthic triangle is $2|\cos A \cos B \cos C|$ times the area of ABC, this ratio is $\leq 1/4$ if ABC is acute.

In general the sequence of triples of angles will be infinite, without repetitions, but there is one fixed point of the mapping, when $A = B = C = 60^\circ = \pi/3$. If the sequence comes to a right-angled triangle, the next step gives a degenerate triangle with the three vertices coinciding, having no well-defined orthic triangle. It is possible for the sequence to be periodic, as in the following example, (angles in degrees).

A	B	C
94	60	26
8	120	52
16	60	104
32	120	28
64	60	56
52	60	68
76	60	44
28	60	92
56	120	4
112	60	8
44	120	16
88	60	32
4	60	116
8	120	52

INTEGRAL INEQUALITY (JCMN 57, p.6106 & 58, p.6139)

A. Brown, P. H. Diananda and Terry Tao

Let $B(n)$ be the greatest lower bound of

$$\int_0^1 f(x)^2 dx$$

over all polynomials f of degree n with integer coefficients.

The previous contribution showed that $B(n+1) < B(n)/2$, there are other inequalities like this as follows.

<u>Theorem 1</u> (a) $18\,B(n+2) < B(n)$

(b) $8\,B(n+2) + 2\,B(n+1) < B(n)$

(c) $16\,B(n+2) + B(n+1) < B(n)$

<u>Proof</u> There is a polynomial $f(x)$ of degree n with the integral of its square equal to $B(n)$. Consider the 3 polynomials $f_1(x) = x(1-x)f(x)$,

$f_2(x) = x(1-2x)f(x)$, $\quad f_3(x) = (1-x)(1-2x)f(x)$.

$$\int_0^1 16\, f_1^2 + f_2^2 + f_3^2\, dx = \int_0^1 (24x^4-48x^3+30x^2-6x+1)f^2\, dx$$
$$= \int_0^1 (1 - 6x(1-x)(1-2x)^2) f(x)^2 dx$$
$$< \int_0^1 f(x)^2 dx = B(n).$$

Since the three functions f_1, f_2 and f_3 all have the integrals of their squares $\geq B(n+2)$, (a) follows.

Similarly (b) and (c) follow from the inequalities

$8(x-x^2)^2 + x^2 + (1-x)^2 < 1$ and

$16(x-x^2)^2 + (1-2x)^2 < 1$ (in almost all the interval)

QED

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$.

be a polynomial with integer coefficients, the last coefficient, a_n, being non-zero.

<u>Theorem 2</u> $\int_0^1 (f(x))^2 dx = a_0^2 + a_0a_1 + \ldots + a_{n-1}a_n/n + a_n^2/(2n+1)$

$=$ (integer) + (integer)/3 + ... (integer)/n + (integer)/(2n+1)

$= T_1 + T_2 + T_3 + \ldots$

where $T_1 = (a_0 + a_1/2 + a_2/3 + a_3/4 + a_4/5 + \ldots)^2$

$T_2 = (a_1 + a_2 + 9a_3/10 + 4a_4/5 + 5a_5/7 + 9a_6/14 \ldots)^2/12$

$T_3 = (a_2 + 3a_3/2 + 12a_4/7 + 25a_5/14 + 25a_6/14 \ldots)^2/180$

$T_4 = (a_3 + 2a_4 + 25a_5/9 + 10a_6/3 \ldots)^2/2800$

$T_5 = (a_4 + 5a_5/2 + 45a_6/11 + \ldots)^2/44100$

$T_6 = (a_5 + 3a_6 + \ldots)^2/698544$ etc.

<u>Proof</u> The formulae come from the usual algorithm for expressing a positive definite quadratic form as a sum of squares.

QED

The contribution in JCMN 58 established:- $B(1) = 1/3$, $B(2) = 1/30$ and $B(3) = 1/210$. Now we find $B(4) = 1/630$.

It was shown that $B(4)$ must be (integer)/1260, and (by an example) must be $\leq 1/630$. Suppose that the polynomial $f(x)$ had the integral of its square equal to 1/1260. Then $1/1260 =$ (integer) + (even integer)/2 + (integer)/3 + ... + $(2a_3a_4)/8 + a_4^2/9$, implying that a_3 and a_4 must both be odd, because 1260 has a factor of 4.

Now consider the integral of the square, by Theorem 2 $|T_2| \geq 1/1200 > 1/1260$; this gives the required contradiction, showing that $B(4) = 1/630$.

<u>Theorem 3</u> For any $n > 2$, $B(n) \leq C(n)$ (defined below).

<u>Proof</u> For any $n > 2$, consider any t such that $n - t$ is even and $0 \leq t \leq n$. Let $f(x)$ be the polynomial

$$f(x) = (x - x^2)^{(n-t)/2}(1 - 2x)^t$$

To find the integral of the square, denote the integral by $F(t)$. Then $(n-t+1)F(t) = 2(2t-1)F(t-1)$, and so by induction

$F(t) = F(t-1).(4t-2)/(n-t+1)$

$= ((2t)!/t!)((n-t)!/n!)F(0)$

$= \dfrac{(2t)!\,(n-t)!\,n!}{t!\,(2n+1)!}$

For fixed n this expression has its minimum, denoted by C(n), when t is the number such that n-t is even and $n-4 \le 5t \le n+4$, that is $t = n/5+a$ where $a = 0$ or $\pm 2/5$ or $\pm 4/5$.

n	2	3	4	5	6	7	8	9	10	11
t	0	1	0	1	2	1	2	1	2	3
$\frac{1}{2}n-\frac{1}{2}t$	1	1	2	2	2	3	3	4	4	4

From the contribution in JCMN 58, B(n) is a multiple of D(n) = 2/(the l.c.m. of {1, 2, 3, ... , 2n+1}). Thus $D(n) \le B(n) \le C(n)$.

For n = 7, t = 1 and we may calculate D(7) = C(7) = 1/180180 = 1/(4×5×7×9×11×13), so that this is the value of B(7).

We may calculate B(5) easily as follows. By Theorem 1 above, B(5) > 18 B(7) = 1/10,010, and as B(5) must be a multiple of D(5), the only possible value is 1/6930.

Now we can note a few numerical values:

n	D(n)	C(n)	B(n)
2	1/30	1/30	1/30
3	1/210	1/210	1/210
4	1/1260	1/630	1/630
5	1/13,860	1/6930	1/6930
6	1/180,180	1/30,030	
7	1/180,180	1/180,180	1/180,180
8	1/6,126,120	1/1,021,020	
9	1/116,396,280	1/4,157,010	
10	1/116,396,280	1/29,099,070	
11	1/2,677,114,440	1/133,855,722	

Now consider the problem of finding B(6). We saw earlier that it must be q/180180, where q is one of {1, 2, 3, 4, 5, 6}. We eliminate the possibilities that q = 1 or 2 by the observation that B(6) > 2B(7). Now we shall eliminate the possibilities that q = 3 or 4. Let f(x) be any polynomial of degree 6 with integer coefficients.

Let I = integral of the square = q/180,180.

$$I = A + \frac{B}{3} + \frac{C}{4} + \frac{D}{5} + \frac{E}{6} + \frac{F}{7} + \frac{G}{9} + \frac{H}{11} + \frac{J}{13}$$

where A, B, ... J are integers and $J = a_6^2$. It follows:

$$q = 180180\, I = 13(13860A + 4620B + .. + 1260H) + 13860J.$$

Therefore $q \equiv 13860J \equiv 2J \pmod{13}$. The possible cases are:

a_6	$\equiv$ 0	± 1	± 2	± 3	± 4	± 5	± 6
$q \equiv 2a_6^2$	$\equiv$ 0	2	8	5	6	11	7

This has shown that q cannot be 3 or 4. To eliminate the possibility that q = 5, one method is from Theorem 2, but an alternative proof will be given below.

We return to more general considerations. Consider the polynomials

$$p_0(x) = 1$$
$$p_1(x) = 2x - 1$$
$$p_2(x) = 6x^2 - 6x + 1$$
$$p_3(x) = 20x^3 - 30x^2 + 12x - 1$$
$$p_4(x) = 70x^4 - 140x^3 + 90x^2 - 20x + 1, \quad \text{etc.}$$

defined by

$$p_n(x) = \Sigma_{i=0}^{n} (-1)^{n+i}\binom{n+i}{i}\binom{n}{i}x^i = (1/n!)(d/dx)^n(x^2-x)^n = \sum_{r=0}^{n}\binom{n}{r}^2 x^r(x-1)^{n-r}.$$

The polynomials have an orthogonality property:

$$\int_0^1 p_n(x)p_m(x)dx = \delta_{n,m}/(m+n+1)$$

easily proved from the second definition above with repeated integration by parts.

These polynomials are connected with the Legendre polynomials P_n, in fact $p_n(x) = P_n(2x-1)$. Legendre's formula $P_n(\cos\theta) = 2^{-2n}\sum_{r=0}^{n}\binom{2n-2r}{n-r}\binom{2r}{r}\cos(n-2r)\theta$ tells us that $-1 \le P_n(\cos\theta) \le 1$, and so $-1 \le p_n(x) \le 1$ for all x in the unit interval.

We express any polynomial f(x) as $\Sigma c_i p_i(x)$, and then

<u>Theorem 4</u> $\int_0^1 f(x)^2dx = \Sigma c_i^2/(2i+1)$.

Also $p_i(1-x) = (-1)^i p_i(x)$. (Both obvious)

If f(x) is a polynomial with $f(x) = f(1-x)$, then $c_i = 0$ for all odd i; and if $f(x) = -f(1-x)$ then $c_i = 0$ for all even i. It may be noted that the sum of squares above is essentially the same as that in Theorem 2, in fact $(2i+1)T_{i+1} = c_i^2$.

<u>Theorem 5</u> If f is of degree n and $f(x) \neq (-1)^n f(1-x)$, then

$$\int_0^1 f(x)^2 \ge \frac{B(n) + B(n-1)}{4}$$

<u>Proof</u> Let $g(x) = (-1)^n f(1-x)$. Then $f + g$ has degree n and $f - g$ has degree $< n$, but is non-zero.

$$4\int_0^1 f(x)^2dx = \int_0^1 (f+g)^2 + (f-g)^2dx \ge B(n) + B(n-1)$$

<u>Corollary</u> (a) If $3B(n) < B(n-1)$ then all optimal polynomials f of degree n satisfy $f(x) = (-1)^n f(1-x)$.

(b) If $3B(n) = B(n-1)$ then there exists one optimal polynomial f satisfying this condition.

<u>Theorem 6</u> If T is a linear functional on the polynomials, and f has degree n, then

$$\int_0^1 f(x)^2dx \ge \frac{(Tf)^2}{\Sigma_{i=0}^n (2i+1)(Tp_i)^2}$$

<u>Proof</u> With the notation introduced above, $f = \Sigma c_i p_i$, and $Tf = \Sigma c_i Tp_i$. Apply the Cauchy-Schwarz inequality to the sequences $c_i/\sqrt{(2i+1)}$ and $Tp_i\sqrt{(2i+1)}$.

$$(Tf)^2 = (\Sigma c_i Tp_i)^2 \le \Sigma c_i^2/(2i+1)\ \Sigma(2i+1)(Tp_i)^2$$

Further, if f has the property that $f(x) = (-1)^n f(1-x)$, then the inequality may be strengthened by taking the summation over only i for which n-i is even (by Theorem 4).

<u>First Application</u> Let Tf be the coefficient of x^j in f(x). Recalling that $(Tp_m)^2 = \binom{m+j}{j}^2\binom{m}{j}^2$, and that $(Tf)^2$ is a non-negative integer, it follows that either there is no x^j term in f(x) or $\int_0^1 f(x)^2dx \ge 1/\Sigma_{m=j}^n \binom{m+j}{j}^2\binom{m}{j}^2(2m+1)$.

As before the sum may (in certain circumstances) be taken over only the m for which n-m is even.

This gives a criterion for coefficients in optimal polynomials being zero:

If $B(n) < 1/\Sigma_{m=j}^n \binom{m+j}{j}^2\binom{m}{j}^2(2m+1)$ then in every optimal polynomial of degree n the coefficient of x^j is zero.

Specifically, taking $j = 0$, the result becomes:-

If $B(n) < 1/(n+1)^2$ every optimal polynomial has a factor x.

Now take $j = 1$. Note that

$$\Sigma_{m=1}^n (m+1)^2m^2(2m+1) = n^2(n+1)^2(n+2)^2/3.$$

If $n^2(n+1)^2(n+2)^2B(n) < 3$, the formula tells us firstly that every optimal polynomial f(x) has a factor x^2, (because this condition implies the previous $(n+1)^2B(n) < 1$), and secondly that f(x) has a factor $(1-x)^2$ because f(1-x) is also optimal.

In general, for any j, the sum $\sum_{m=1}^{n} \binom{m+j}{j}^2 \binom{m}{j}^2 (2m+1)$ is a polynomial in n, but B(n) decreases geometrically, faster than C(n), which is like 5^{-n}, and so for all sufficiently large n all optimal polynomials have factors $x^j(1-x)^j$.

<u>Second Application</u> Let $Tf = f(\tfrac{1}{2})$. Firstly note that $p_m(\tfrac{1}{2}) = 0$ if m is odd and $= (-1/4)^{m/2} \binom{m}{m/2}$ if m is even.

If 2x-1 is not a factor of f(x), then $|f(\tfrac{1}{2})| \geq 2^{-n}$, and so:

<u>Corollary</u> If $B(n) < 4^{-n}/\sum_{r=0}^{[n/2]} 4^{-2r} \binom{2r}{r}^2 (4r+1)$ then 2x-1 is a factor of every optimal polynomial of degree n. It may be verified that this is so for n = 6, so that the optimal polynomial f(x) has factors $x^2(1-x)^2(2x-1)$. But $3B(6) \leq 3/30030 < B(5)$, and (Theorem 5, Corollary) therefore $f(x) = f(1-x)$ and the remaining factor must be $\pm(2x-1)$. This determines that $B(6) = 1/30030$.

<u>Third application</u> Can we use Theorem 6 to find out about $1 - 5x + 5x^2$ being a factor of optimal polynomials?

Let $u = \frac{1+1/\sqrt{5}}{2}$ and $v = \frac{1-1/\sqrt{5}}{2}$, the zeros of the quadratic. Consider the two linear functionals S and T, defined by $Sf = f(u)+f(v)$ and $Tf = \sqrt{5}(f(u)-f(v))$. These functionals operating on the function x^n give:-

n	1	2	3	4	5	6
S	1	3/5	2/5	7/25	1/5	18/125
T	1	1	4/5	3/5	11/25	8/25

Each sequence satisfies the linear recurrence relation

$$t(n) = t(n-1) - t(n-2)/5$$

so their values are of the form $(\text{integer})/5^{[n/2]}$ and $(\text{integer})/5^{[n/2-1/2]}$ respectively. The same must be true

for any polynomial of degree n with integer coefficients. Theorem 6 tells us that $(Sf)^2 \leq (2n+2)^2 \int_0^1 f(x)^2\,dx$, and similarly $(Tf)^2 \leq 5(2n+2)^2 \int_0^1 f(x)^2\,dx$. We have seen that Sf is either zero or $\geq 5^{-[n/2]}$. Thus $Sf = 0$ or $5^{-2[n/2]} \leq (2n+2)^2 \int_0^1 f(x)^2\,dx$. Therefore if $B(n) < 5^{-2[n/2]}(2n+2)^{-2}$ then $Sf = 0$ for all optimal polynomials. Similarly $Tf = 0$ for all optimal polynomials if $B(n) < 5^{-1-2[n/2-1/2]}(2n+2)^{-2}$. We shall see (below) that $B(n) = O(5{\cdot}276^{-n})$ for large n, and so it follows that for all sufficiently large n, for all optimal polynomials f, $Sf = Tf = 0$ and $1-5x+5x^2$ is a factor of f.

What do we know of the asymptotic behaviour of B(n)? Consider the function $g(x) = (x-x^2)^3(1-2x)(1-5x+5x^2)$. It may be calculated that $|g(x)| < M = {\cdot}0005617$ in the unit interval. Put $f = g^m$. Then f(x) is a polynomial of degree $n = 9m$, and $|f| < M^m$, so that $B(9m) \leq \int f^2 dx < M^{2m}$. Therefore $\limsup B(n)^{1/n} \leq M^{2/9} = 1/5{\cdot}276\ldots$ This is asymptotically a little better than $B(n) \leq C(n)$, for $C(n)^{1/n} \to 1/5$. For a bound the other way, we have $B(n) \geq D(n)$, and asymptotically $D(n) \sim \exp(-2n)$. This estimate for D(n) is far from obvious, in fact it is essentially the prime number theorem, see Hardy and Wright, *Theory of Numbers, § 22.7,* p.346. Our asymptotic results are thus:

$$5{\cdot}276 < B(n)^{-1/n} < 7{\cdot}3891.$$

For B(9) we have a better upper bound than C(9), because the polynomial

$$f(x) = x^3(1 - x)^3(1 - 2x)(1 - 5x + 5x^2)$$

gives a value $2/14,549,535 = 16D(9)$ for the integral of the square, which is smaller than $C(9) = 28D(9)$. Similarly for $B(11)$, the polynomial $x^4(1-x)^4(1-2x)(1-5x+5x^2)$ gives a value $1/191,222,460 = 14D(11)$.

The following table gives the known values. The column "Sym" indicates the symmetry, whether every optimal polynomial $f(x)$ is known to have $f(x) = (-1)^n f(1-x)$, with Y for "yes" if $3B(n) < B(n-1)$, see Theorem 5, Corollary (a).

n	B(n)	Sym	Optimal polynomials
1	1/3	N	x or $1-x$ or $2x-1$
2	1/30	Y	$x(1-x)$
3	1/210	Y	$x(1-x)(1-2x)$
4	1/630	N	$x^2(1-x)^2$ or $x^2(1-x)(1-2x)$ or $x(1-x)(1-5x+5x^2)$ or $x(1-x)(1-2x)^2$
5	1/6930	Y	$x^2(1-x)^2(1-2x)$
6	1/30,030	Y	$x^2(1-x)^2(1-2x)^2$
7	1/180,180	Y	$(x-x^2)^3(1-2x)$ or $(x-x^2)^2(1-2x)(1-5x+5x^2)$
8	1/1,021,020	Y	$x^3(1-x)^3(1-2x)^2$

Finally, a table giving the partial information available in the next few cases.

n	lower bound	upper bound	ratio	Sym	known factors
9	1/38,798,760	2/14,549,535	3:16	Y	$(x-x^2)^2(1-2x)$
10	1/116,396,280	1/29,099,070	1:4	?	$(x-x^2)^2(1-2x)$
11	1/2,677,114,440	1/191,222,460	1:14	?	$(x-x^2)^2(1-2x)$

SMALL POLYNOMIALS

Terry Tao

Consider polynomials $f(x)$ of degree n with integer coefficients, as functions on the unit interval.

In the preceding contribution INTEGRAL INEQUALITY (pages 6146 - 6154) we considered the L^2 norms, i.e. we were concerned with the integral $\int_0^1 f(x)^2\,dx$. Now consider the L^∞ norm, that is $\max |f(x)|$. Denote the minimum (over all polynomials f with integer coefficients and of degree n) by $A(n)$.

From Theorem 6 (page 6150 above), if we take the functional Tf to be $f(v)$ (where v is some number in the unit interval), recalling that $|Tp_i| \le 1$ and $\Sigma(2i+1) = (n+1)^2$, we find $(n+1)^2\int_0^1 f(x)^2dx \ge f(v)^2$. This is for all v, and so $A(n)^2 \le (n+1)^2\int_0^1 f(x)^2dx$. This is for all f, and so $A(n)^2 \le (n+1)^2B(n)$. Now consider the f that minimizes $\max |f(x)|$, the integral of its square must be $\le A(n)^2$, and so $B(n) \le A(n)^2$. Thus we have:-

<u>Theorem 1</u> $B(n) \le A(n)^2 \le (n+1)^2B(n)$.

<u>Theorem 2</u> $A(n)^{1/n}$ has a limit as $n \to \infty$.

<u>Proof</u> Let $a = \limsup A(n)^{1/n}$. For any m and n we have $A(n+m) \le A(m)A(n)$, and so in particular the sequence is non--increasing. Therefore $A(m) \le A(n)^{[m/n]}$, and
$a = \limsup A(m)^{1/m} \le \limsup A(n)^{[m/n]/m} = A(n)^{1/n}$.
This is for all n, and so $a \le \liminf A(n)^{1/n}$, proving that $A(n)^{1/n}$ has a limit.

<u>Corollary</u> $B(n)^{1/n}$ has a limit, the square of $\lim A(n)^{1/n}$.

BINOMIAL IDENTITY 36

Firasath Ali and Cecil Rousseau

$$\sum_{p=k}^{[n/2]} (-1)^{p-k}\binom{p}{k}\binom{n-p}{p}2^{n-2p} = \binom{n+1}{2k+1}$$

<u>First Proof</u> Consider the problem of counting the number of 0-1 sequences of length n that have exactly k occurrences of the pattern 01. This was in the 1982 British Mathematical Olympiad. The following calculation by Ian Goulden, given in Ross Honsberger's *Mathematical Gems III*, gives the RHS of the identity.

Enlarge the sequences to length n+2 by adding a 1 on the left and a 0 on the right. Consider the n+1 points between the n+2 members of the sequence, call one a "switch" if it is between unequal digits. There will be k occurrences of 01 if and only if there are k+1 occurrences of 10, and therefore 2k+1 switches altogether. The number of ways is therefore the RHS above.

On the other hand, we shall use the inclusion-exclusion principle to show that the number is the binomial sum on the left above.

Firstly, take an abstract view of the I-E principle. Suppose that we have a set S of things and a set G of properties, each thing in S may or may not have each of the properties in G. In other words we have a bipartite graph, each edge joining one member of S to one member of G; the existence of the edge meaning that the thing s in S has the property j in G. Each thing of S has a degree k (in the graph theory sense), the number

of properties in G that it has. The function E(k) (for integer $k \geq 0$) is the number of things in S that have degree k. The function W(p) for any integer $p \geq 0$ is defined as follows. Take any p-element subset of G and count the number of s in S that are joined to all members of the subset, i.e. the number of things that have all the p properties; add these numbers for all the p-element subsets of G; the total is W(p). The I-E principle tells us the relation between these two functions. It is

$$E(k) = \sum_p (-1)^{p-k}\binom{p}{k}W(p) \quad \text{and} \quad W(p) = \sum_k \binom{k}{p}E(k)$$

The second equation is clear by counting, and the first may be derived by matrix inversion. (JCMN 45, p.5081 & 47, p.5128)

Considering the present question — let S be the set of all strings such as $s = (s_1, s_2, \dots s_n)$ of symbols 0 and 1. We define G to be the set $\{2, 3, \dots n\}$, but each j in G can also be regarded as a property of an s in S; in fact j is the property that $s_{j-1} = 0$ and $s_j = 1$. Each s in S may or may not have each property j in G. What we want to calculate is (for any k) the number E(k) of strings s in S that have exactly k of the properties in G.

To determine W(p) in our case, first consider the p-element subsets P of G; we may discard those that (as a set of numbers) contain two adjacent numbers, for no string can satisfy those properties. How many subsets P remain? We may call them the consistent subsets. Each is a sequence $1 < j_1 < j_2 < \dots < j_p \leq n$ in which no two of the j's are consecutive. To such a sequence there corresponds the sequence with $i_q = j_q - q$, and $1 \leq i_1 < i_2 < \dots < i_p \leq n-p$.

The number of consistent sets is therefore $\binom{n-p}{p}$. For any such consistent set P, the number of strings s that satisfy the p conditions is 2^{n-2p}, since each condition j in P determines two elements of the string s, the symbol numbered j-1 must be 0 and the symbol numbered j must be 1.

Therefore $W(p) = \binom{n-p}{p} 2^{n-2p}$.

Application of the inclusion-exclusion rule now gives

$$\binom{n+1}{2k+1} = E(k) = \sum_p (-1)^{p-k}\binom{p}{k} W(p) = \sum_p (-1)^{p-k}\binom{p}{k}\binom{n-p}{p} 2^{n-2p}.$$

<u>Second Proof</u> Consider the polynomials

$$f_n(z) = \sum_{p=0}^{[n/2]} \binom{n-p}{p} z^p.$$

Using the identity $\binom{n+1-p}{p} = \binom{n-p}{p} + \binom{n-p}{p-1}$, it follows that

$$f_{n+1}(z) = f_n(z) + zf_{n-1}(z).$$

The general solution of this recurrence is

$$f_n(z) = c_1\lambda_1^n + c_2\lambda_2^n,$$

where λ_1 and λ_2 are the roots of $\lambda^2 = \lambda + z$. The initial conditions are $f_0(z) = f_1(z) = 1$.

Straightforward calculation now yields

$$f_n(z) = (1+4z)^{-\frac{1}{2}} 2^{-n-1}\left((1+\sqrt{1+4z})^{n+1} - (1-\sqrt{1+4z})^{n+1}\right)$$

$$f_n\left(\frac{x-1}{4}\right) = 2^{-n} \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} x^k$$

But from the definition,

$$f_n\left(\frac{x-1}{4}\right) = \sum_{p=0}^{[n/2]} \binom{n-p}{p} 4^{-p}(x-1)^p$$

$$= \sum_{0\le k\le p\le [n/2]} 2^{-2p}(-1)^{p-k}\binom{n-p}{p}\binom{p}{k} x^k$$

Our result now comes from equating the coefficients of x^k in these two expressions.

POINTS AND ANGLES IN THE PLANE (JCMN 58, p.6117)

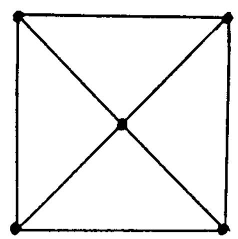
Paul Erdös

(Hungarian Academy of Sciences, Budapest, P.O.B. 127, H-1364)

Let there be given n points in the plane not all on a line. Is it true that they determine at least n-2 distinct angles in the open interval between 0 and π?

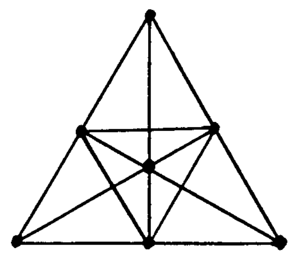
This question is due to Corrádi, Hajnal and myself. It had a curious birth due to a misprint. I sent it to the Math Lapok for high school as an elementary problem but I added "no three on a line". With this condition the problem is very simple. Corrádi and Hajnal asked me – How did you do it? We could not do it. I said it was trivial, but when I looked at the problem I noticed that it was printed as above, "no three on a line" was omitted, and I also could not do it. As far as I know this problem which owes its existence to a misprint is still open.

The special cases of n = 5 and 7 are shown below:-



5 points, 2 angles.

(45° and 90°)



7 points, 4 angles.

(30°, 60°, 90°, 120°)

Are there any other n for which the proposition is untrue?

COMBINATORIAL QUESTION (JCMN 58, p.6137)

Paul Erdös

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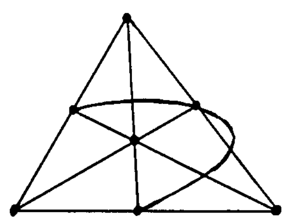
Put the question in another way. For any k let g(k) be the smallest integer m such that from a set of m things one can find m subsets of size k of which any two have at most one element in common.

<u>Lemma</u> $g(k) \geq k(k-1) + 1$.

<u>Proof</u> Take a family of m elements with m subsets having the given property. Take any element, the number of the subsets that contain it must be $\leq (m-1)/(k-1)$. This proves that $mk \leq m(m-1)/(k-1)$, that is $k(k-1) \leq m-1$.

If p is a prime or a power of a prime, then by considering the finite plane projective geometry over the field GF[p], taking the points as our elements and the lines as our subsets, we have the conditions satisfied with sets of size $k = p+1$ and with $m = p(p+1)+1 = k(k-1)+1$ elements. This shows $g(p+1) = p(p+1)+1$. In particular $g(3) = 7$, (the geometry in this case is the Fano plane, which has 3 points on each line and 3 lines through each point). (In drawing a Fano plane the 7 lines cannot all be drawn straight.)



It has been conjectured that if k-1 is not a prime or a prime power then $g(k) > k(k-1)+1$, but this has not been proved for all k. Perhaps $g(k) \leq k(k-1) + 3$ for all k.

It has been shown recently that if 111 11-tuples are chosen from 111 elements then there must be two of them with 2 elements in common, i.e. $g(11) > 111$.

For any m, let $k = k(m)$ be the largest integer such that in a set of m elements it is possible to find m subsets, each of k elements, with no two of the subsets having more than one element in common. It might be thought that $k(m+1) \geq k(m)$, but this has been disproved by Mullin and others from the University of Waterloo, with the example of $k = p+1$ and $m = p(p+1)+1$ for infinitely many p.

In fact let $f(m; k)$ be the largest integer for which in a set of m elements you can find $f(m; k)$ subsets, each of k elements, with no two of the subsets having more than one element in common.

For large m, $f(m+1; k) - f(m; k) > 1$.

In fact $f(m+1; k) - f(m; k) \to \infty$. What is the smallest m for which $f(m+1; k) - f(m; k) > 1$? And what is the smallest M such that this inequality $f(m+1; k) - f(m; k) > 1$ holds for all $m > M$?

In a set of size $p^2 + p + 3$, can one find $p^2 + p + 3$ subsets of size $p + 1$ such that no two of these subsets have more than one element in common?

BINOMIAL IDENTITY 35 (JCMN 58, p.6134)

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It was another meeting of the knights of the Round Table. All m of King Arthur's knights were there, although s of them were about to go on quests and could not stay for long. King Arthur stood up and said:

— There is a great tournament to be held today, as many of you know. Exactly n knights are to attend this tournament.

— That is easily done, my lord, — said Sir Galahad — there are $\binom{m}{n}$ possibilities, surely?

Sir Mordred sneered — You were always naive, Sir Galahad. What about the s of us who are to go on quests today? Surely none of us can attend the tournament.

Sir Lancelot said: — Yes, I have to leave shortly on my quest, and so you must exclude all the arrangements that put me in the tournament.

Sir Gawain noted thoughtfully: — Yes, there are $\binom{m-1}{n-1}$ possibilities that include Sir Lancelot, because after selecting him you must fill the remaining n-1 places from the remaining m-1 knights.

Sir Bedivere, another questing knight, said: – So, after subtracting for each questing knight, we get $\binom{m}{n} - s\binom{m-1}{n-1}$ possibilities.

Sir Galahad, with a slight frown on his face, asked: – But what if two of the questing knights were selected? You would have subtracted the number for that possibility twice!

Merlin, who was attending the meeting, said: — Of course, you have to use the Inclusion-Exclusion Principle. You must add the possibilities for two questing knights, and

then subtract the possibilities for three, and so forth.

Sir Gareth volunteered: – If t questing knights were selected there would be $\binom{m-t}{n-t}$ ways the others could be chosen.

— And also — added Sir Gawain, — the number of ways you can choose t questing knights is $\binom{s}{t}$, so the complete count must be $\binom{m}{n} - \binom{s}{1}\binom{m-1}{n-1} + \binom{s}{2}\binom{m-2}{n-2} - \binom{s}{3}\binom{m-3}{n-3} + \ldots$

By this time all the questing knights had excused themselves hurriedly to begin their quests. King Arthur looked up at the m-s knights who remained, and exclaimed: — Of course ... All I have to do is to select n knights from those that are still here, so there are only $\binom{m-s}{n}$ possibilities.

All the remaining knights then turned expectantly to Merlin, who obligingly peered into the future and said after a while:

– You have just discovered what will be written as Binomial Identity 35 in issue 58 of the James Cook Mathematical Notes in the 1992nd year of our Lord:

$$\binom{m-s}{n} = \sum_{i=0}^{n} (-1)^i \binom{s}{i}\binom{m-i}{n-i}$$

* * * * * * *

It was a few weeks after the tournament, and all the m knights were again at the Round Table. King Arthur stood up and said:

— After our recent successes in quests and the tournament, I have decided on a feast to celebrate. Sir Gawain, I want you to order special seats for our n tourneying knights.

— As you wish, your Majesty. — Sir Gawain replied. — With n special seats to distribute among the m overall, I have $\binom{m}{n}$ possibilities for the seating arrangements.

— Unfortunately, — said King Arthur, — it is not so simple. I have also invited some s knights from Scotland, all great and powerful men, to join us in this feast. Normally, they would take the place of honour on my right, but they would like some of the tourneying knights to sit with them.

Sir Gawain noted distastefully: — I will have to order s special chairs for them. (Readers may recall the story "King Arthur and the m fat knights" in JCMN 31, p.3170 - Editor) Now I have no idea how many possibilities there are.

— I think I can help you. — said Sir Bedivere — If we select t of my fellow tourneying knights to sit among the Scots, then they, together with the visitors, occupy the first s+t seats on the King's right. Thus there are $\binom{s+t}{t}$ seating possibilities for them.

Sir Mordred objected: — That is not quite true, if you say that your t tourneying knights all sit WITHIN the s visitors, then the seat number s+t to the right of the King must always be occupied by a visitor. Hence there are only $\binom{s+t-1}{t}$ possibilities.

Merlin commented: — You also need to account for the remaining seats, which will be shared between the other n - t tourneying knights and the other m - n Camelot knights. The arrangements for these seats number $\binom{m-t}{n-t}$, so the number of seating arrangements for the whole table is $\binom{s+t-1}{t}\binom{m-t}{n-t}$ when there are t tourneying knights sitting with the visiting knights.

Sir Lancelot summarized: — Therefore the total number of seating arrangements is

$$\binom{s-1}{0}\binom{m}{n} + \binom{s}{1}\binom{m-1}{n-1} + \binom{s+1}{2}\binom{m-2}{n-2} + \ldots$$

Sir Galahad remarked: — This formula is just as messy as the one we considered a few weeks ago!

At that moment Queen Guinevere, who had been listening to all that was said, asked King Arthur: — Why don't you seat the tourneying knights first, and then put the Scottish knights in the first available s seats to your right?

King Arthur nodded in agreement. — Yes, there are exactly $\binom{m+s}{n}$ possibilities. So that means

$$\binom{m+s}{n} = \sum_{i=0}^{n} \binom{s+i-1}{i}\binom{m-i}{n-i}$$

Merlin chuckled. — Your Majesty, I note that this new identity you have discovered is related closely to the one we discussed a few weeks ago, before the tournament.

— I see nothing more than superficial similarity. - declared Sir Mordred.

— Well, — said Merlin uncomfortably, — if you consider the first identity, but replace s with -s, then you get today's identity.

Many of the knights laughed on hearing this, and even King Arthur had to smile.

— Have you been dabbling in black magic again? — King Arthur asked. — How could we have a negative number of questing knights? Or a negative number of Scotsmen coming to Camelot? I am afraid, Merlin, that you seem to value these symbolic equations more than the common sense that created them.

Merlin, who had been muttering "Gamma one minus s over gamma one minus s minus t" under his breath replied — I agree, your Majesty, that there is no intuitive way to connect the two, but formally the two identities match. In the future people will use these symbols and derive similar identities without considering their original meaning. There is much to be gained by withholding such restrictions of "common sense".

FAMILY OF POLYNOMIALS (JCMN 57, p.6099 & 58, p.6130)

Terry Tao & A. Brown

The first few members of this family are

$f(0, x) = 1$ $\qquad$ $f(1, x) = 3x - 2$

$f(2, x) = 10x^2 - 12x + 3$

$f(3, x) = 35x^3 - 60x^2 + 30x - 4$

$f(4, x) = 126x^4 - 280x^3 + 210x^2 - 60x + 5$

In the previous contributions three possible definitions were given:-

$$f(n, x) = \sum_{k=0}^{n} (-1)^{n+k}\binom{n+k+1}{n}\binom{n}{k} x^k \tag{1}$$

$$= \sum_{k=0}^{n} \binom{n+k+1}{k}\binom{n}{k}(x-1)^k \tag{2}$$

$$= \sum_{k=0}^{n} \binom{n+1}{n-k}\binom{n}{k}(x-1)^{n-k}x^k \tag{3}$$

The family may also be defined by:-

$$f(n, x) = \frac{1}{n!}\,\frac{1}{x}\,(d/dx)^n\left(x^{n+1}(x-1)^n\right) \tag{4}$$

or

$$= \frac{1}{(n+1)!}\,(d/dx)^{n+1}\left(x^n(x-1)^{n+1}\right) \tag{5}$$

Equation (4) may be shown equivalent to (1) as follows:-

$$x^{n+1}(1-x)^n = \sum_{k=0}^{n} (-1)^k\binom{n}{k} x^{n+k+1}$$

$$n^{th} \text{ derivative } = \sum_{k=0}^{n} (-1)^k n!\binom{n+k+1}{k+1}\binom{n}{k} x^{k+1}$$

which gives the result.

Similarly (5) is equivalent to (2), writing $y = 1-x$,

$$(1-y)^n y^{n+1} = \sum_{k=0}^{n} (-1)^k\binom{n}{k} y^{n+k+1}$$

$$(n+1)^{th} \text{ derivative } = \sum_{k=0}^{n} (-1)^k (n+1)!\binom{n+k+1}{k}\binom{n}{k} y^k$$

so that (5) agrees with (2).

Also, by using the formula of Leibniz for the nth derivative of a product, both (4) and (5) may be shown to be equivalent to (3).

The equivalence of the five definitions is thus established as follows:-

(1) <=> (4) <=> (3) <=> (5) <=> (2).

From (4) and (5) there is a simple derivation of the orthogonality property, as follows:

$$\int_0^1 x f(m, x) f(n, x)\,dx$$

$$= \frac{(-1)^{m+n+1}}{m!(n+1)!}\int_0^1 \left(\frac{d}{dx}\right)^m\left(x^{m+1}(1-x)^m\right)\left(\frac{d}{dx}\right)^{n+1}\left(x^n(1-x)^{n+1}\right)dx$$

Integration by parts m times gives

$$\frac{(-1)^{n+1}}{m!(n+1)!}\int_0^1 x^{m+1}(1-x)^m\left(\frac{d}{dx}\right)^{m+n+1}\left(x^n(1-x)^{n+1}\right)dx$$

which is clearly zero if $m > n$. If $m = n$ it gives

$$\binom{2n+1}{n}\int_0^1 x^{n+1}(1-x)^n dx$$

which is easily evaluated as $1/(2n+2)$, either integrate by parts n times or remember about beta functions.

Other properties of these polynomials may be noted:-

(a) $\int_0^1 f(n, x)\,dx = (-1)^n/(n+1)$

(b) $\int_0^1 f(m, x)f(n, x)\,dx = (-1)^{m-n}\frac{n+1}{m+1}$ if $m \geq n$.

(c) $\int_0^1 x^2 f(m, x)f(n, x)\,dx = \frac{n+1}{(2n+1)(2n+3)}$ if $m = n$,

$= \frac{1}{4(2n+3)}$ if $m = n+1$, and $= 0$ if $m \geq n+2$.

(d) $f(n, x)$ has n simple zeros in the unit interval.

(e) $(2n+1)x(1-x)f'(n, x) = n(n-2nx-x)f(n, x)+n(n+1)f(n-1, x)$

(f) Between two zeros of $f(n, x)$ is one of $f(n-1, x)$.

Also these polynomials are related to those of INTEGRAL INEQUALITY (pp. 6146-54 above) as follows:-

(g) $(n+1)f(n, x) = (x-1)p_n'(x) + (n+1)p_n(x)$

(h) $xf'(n, x) - nf(n, x) = p_n'(x)$

(i) $(x-x^2)f'(n, x) + (1+nx)f(n, x) = (n+1)p_n(x)$

Proofs (a) The function $x^n(x-1)^{n+1}$ has a zero of order n at $x=0$, and a zero of order $n+1$ at $x=1$. The n th derivative is $(-1)^{n+1}n!$ at $x=0$, and zero at $x=1$. It follows that

$\int_0^1 (d/dx)^{n+1} x^n(x-1)^{n+1}\, dx$ is $(-1)^n n!$, and so (a) follows from definition (5) above.

(b) The constant term in $f(n, x)$ is $(-1)^n(n+1)$ by (1) above. Therefore we may put $f(n, x) = x\, g(x) + (-1)^n(n+1)$, where $g(x)$ is some polynomial of degree $n-1$, and so

$$\text{LHS} = \int_0^1 xf(m, x)g(x)\, dx + (-1)^n(n+1)\int_0^1 f(m, x)dx$$

which, because $f(m, x)$ is orthogonal to all polynomials of lower degree, and by (a), equals $(-1)^{m+n}(n+1)/(m+1)$.

(c) Using the definition (4), $m!\, n! \times \text{LHS} =$

$$\int_0^1 \left(\frac{d}{dx}\right)^m\left(x^{m+1}(x-1)^m\right)\left(\frac{d}{dx}\right)^n\left(x^{n+1}(x-1)^n\right)dx$$

Integrating by parts m times, this gives:-

$$(-1)^m \int_0^1 x^{m+1}(x-1)^m\left(\frac{d}{dx}\right)^{m+n}\left(x^{2n+1} - nx^{2n} + \text{lower powers}\right)dx$$

This is zero if $m > n+1$, and for the two cases required is

$\int_0^1 x^{m+1}(1-x)^m\left(\frac{d}{dx}\right)^{m+n}\left(x^{2n+1} - nx^{2n}\right) dx$, easily calculated.

(d) From (4) or (5) using Rolle's Theorem.

(e) Put $C(n, x) = (x-x^2)f'(n, x) + nxf(n, x) = \Sigma c_k x^k$ and $D(n, x) = nf(n, x)+(n+1)f(n-1, x) = \Sigma d_k x^k$, and use (1) to find $d_k/(2n+1) = \frac{(-1)^{n+k}(n+k)!}{(k+1)!(k-1)!(n-k)!} = c_k/n$ for $1 \leq k \leq n$, and $c_0 = d_0 = c_{n+1} = d_{n+1} = 0$. Hence $(2n+1)C(n, x) = nD(n, x)$, which is the result (e).

(f) At successive zeros of $f(n, x)$ the sign of $f'(n, x)$ alternates (the zeros being simple), and therefore, by (e), the sign of $f(n-1, x)$ also alternates.

(g), (h) and (i) are obtained from (4) and (5).

* * * * * *

Binomial Identity 34 (JCMN 57, p.6098) may be written

$$(-1)^{n+k}\binom{n+k+1}{n}\binom{n}{k} = \sum_{i=k}^{n}(-1)^{i+k}\binom{n+i+1}{i}\binom{n}{i}\binom{i}{k}$$

and LHS = Coefficient of x^k in $f(n, x)$ from equation (1),

and RHS = coefficient of x^k in $f(n, x)$ from equation (2).

Also, using the binomial multiplication rule

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{a-b}$$

we may rewrite the equation above (omitting the factors $\binom{n}{k}$ and $(-1)^{n+k}$) as:-

$$\binom{n+k+1}{n} = \sum_{i=k}^{n}(-1)^{n-i}\binom{n+i+1}{i}\binom{n-k}{n-i}$$

$$= \sum_{t=0}^{n-k}(-1)^t\binom{2n+1-t}{n-t}\binom{n-k}{t}$$

which is a case of Binomial Identity 35 (JCMN 58, p.6134) with $2n+1$ for m and $n-k$ for s. Therefore we have two proofs for Binomial Identity 34, one from the work in the previous issue (or above) on these polynomials, and one from the Camelot story in this issue (BINOMIAL IDENTITY 35, pages 6162-5).

ANALYTIC INEQUALITY 3 (JCMN 58, p.6135)

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Let $f(x)$ and $g(x)$ be monotonic non-decreasing in $[0, 1]$ with their integrals equal to p and q respectively. Find the best possible constant K in the inequality

$$\int_0^1 f(x)g(x)\,dx - pq \geq K \int_0^1 |f(x)-p|\,dx \int_0^1 |g(x)-q|\,dx.$$

The answer is $K = 1/4$. To prove this, first take the case where $p = q = 0$. Define a as where $f(x)$ changes sign, that is to say such that $f(x) \leq 0$ for $0 \leq x < a$
and $f(x) \geq 0$ for $a < x \leq 1$.

Define $A = \frac{1}{a} \int_0^a g(x)\,dx$ and $B = \frac{1}{1-a} \int_a^1 g(x)\,dx$.

Then $A \leq 0 \leq B$ and $aA + (1-a)B = 0$.

<u>Lemma 1</u> If in an interval $u(x)$ and $v(x)$ are both monotonic non-decreasing and $v(x)$ has mean zero, then $\int u(x)v(x)\,dx \geq 0$.

<u>Proof</u> There is k such that $u(x)-k$ has the same sign as $v(x)$, and so $\int (u(x)-k)v(x)dx \geq 0$.

<u>Lemma 2</u> $\int_0^1 f(x)g(x)dx \geq \frac{1}{2}(B-A)\int_0^1 |f(x)|dx.$

<u>Proof</u> $\int_0^a f(x)g(x)dx$

$= \int_0^a f(x)(g(x)-A)dx - A\int_0^a |f(x)|dx$

which by Lemma 1, $\geq -A\int_0^a |f(x)|dx$.

Similarly $\int_a^1 f(x)g(x)dx = \int_a^1 f(x)(g(x)-B)dx + B\int_a^1 f(x)dx$

$\geq B\int_a^1 |f(x)|dx$

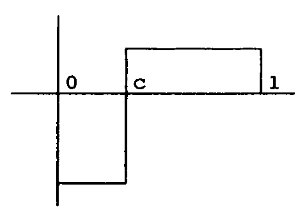
Noting that $\int_0^a = \int_a^1 = \frac{1}{2}\int_0^1 |f(x)|dx$, the result follows.

At this stage we need some more notation. For any c in the open interval $(0, 1)$, define the step-function $h(c, x)$ to be:

$= -1/(2c)$ when $0 \leq x < c$

$= 0$ when $x = c$

$= 1/(2-2c)$ when $c < x \leq 1$



so that $\int_0^1 h(c, x)dx = 0$, and $\int_0^1 |h(c, x)|dx = 1$.

Using this function we may verify that $K = 1/4$ is the best possible constant in the inequality, for if $a < b$ then

$$\int_0^1 h(a, x)h(b, x)dx = \frac{a}{(2a)(2b)} - \frac{b-a}{(2-2a)(2b)} + \frac{1-b}{(2-2a)(2-2b)}$$

$= \frac{1}{4b(1-a)}$ which can be arbitrarily near to $1/4$.

<u>Lemma 3</u> $(B - A)/2 = \int_0^1 h(a, x)g(x)\,dx$

<u>Proof</u> Easily verified.

Lemma 2 and Lemma 3 together give

$$\int_0^1 f(x)g(x)dx \geq \int_0^1 g(x)h(a, x)dx \int_0^1 |f(x)|dx$$

and now we apply this result to the first of the two integrals on RHS above, taking b as the point where $g(x)$ changes sign, and assuming (as we may without loss of generality) $a < b$,

$$\text{LHS} \geq \int_0^1 h(b, x)h(a, x)dx \int_0^1 |g(x)|dx \int_0^1 |f(x)|dx$$

where the first of these three integrals (as we saw above) has the value $\frac{1}{4b(1-a)}$, always $> 1/4$.

Now we have proved $\int_0^1 f(x)g(x)dx \geq \frac{1}{4}\int_0^1 |f(x)|dx \int_0^1 |g(x)|dx$ with equality only in the case of null functions.

To extend the result to functions not having mean zero is straightforward, just apply the result above to $f'(x) = f(x)-p$ and $g'(x) = g(x)-q$.

ANALYTIC INEQUALITY 4

The calculation above suggests another (simpler) problem. If $g(x)$ is monotonic non-decreasing with mean zero in the unit interval and if $0 < a < 1$, prove:

$$\frac{2}{1-a}\int_a^1 g(x)dx - \frac{2}{a}\int_0^a g(x)dx \geq \int_0^1 |g(x)|dx.$$

SYMMETRIC INEQUALITY (JCMN 58, p.6123)

Let r be the sum, s be the sum of squares, and t be the sum of cubes, of three numbers a, b and c, all ≥ 0. Prove that

$$4t \geq rs + 3abc.$$

Does equality imply that $a = b = c$?

SOLUTION 1

A. Brown

$$4t - rs - 3abc = 3(a^3+b^3+c^3)-(a^2b+b^2c+c^2a+ab^2+bc^2+ca^2)-3abc$$

$$= \tfrac{1}{2}\,[(a-b)^2(3a+3b+c) + (b-c)^2(3b+3c+a) + (c-a)^2(3c+3a+b)]$$

≥ 0, with equality only if $a = b = c$.

SOLUTION 2

P. H. Diananda

By the power-mean inequality:

$$\left(\frac{t}{3}\right)^{1/3} \geq \left(\frac{s}{3}\right)^{1/2} \geq \frac{r}{3} \geq (abc)^{1/3} \quad \text{with equality iff } a=b=c.$$

Hence $rs + 3abc \leq 3\left(\frac{t}{3}\right)^{1/3}\times 3\left(\frac{t}{3}\right)^{2/3} + 3\left(\frac{t}{3}\right) = 4t.$

Alternative proof: $r \leq \sqrt{3}\sqrt{s}$ and $s \leq \sqrt{r}\sqrt{t}$ by the Cauchy-Schwarz inequality, and so $rs \leq 3t$. By the inequality of arithmetic and geometric means $3abc \leq t$. Add. At each step there is equality if and only if $a = b = c$.

FACTORS

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For any positive integer n, write the fraction

$$\frac{(2n)!}{n!\,(n + 1000)!}$$

in its lowest terms and define $f(n)$ to be the largest prime that divides the denominator of this fraction, (if the denominator is 1, let $f(n) = 1$). What (if any) is the largest value that $f(n)$ can take?

SYMMETRIC SIMULTANEOUS EQUATIONS

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Solve, for (x, y, z) in terms of (a, b, c), the equations

$$x^2 - yz = a$$
$$y^2 - zx = b$$
$$z^2 - xy = c.$$

Hint - there is a solution in radicals.

BINOMIAL IDENTITY 37

$$\sum_{r=m}^{n} \binom{r+m}{m}^2 \binom{r}{m}^2 (2r+1) = \binom{n+m+1}{m}^2 \binom{n}{m}^2 (n+1)^2/(2m+1)$$

VIBRATING RAILS

When your Editor was a boy there was a rumour circulating among schoolboys in England that you could hear a train coming when it was miles away by putting an ear on to the rail, because "the vibrations travel along the rails". I was never able to satisfy myself as to whether the idea was true or false. Long afterwards when I had become a mathematician I tried to work out the theory, not finding the answer, but finding some tantalising questions.

For our model we take the rail to be infinitely long, thin, straight and uniform, vibrating in a vertical plane; at each sleeper the rail is simply supported at an elastic support of rate k, i.e. if y is the upward displacement there is a downward force on the rail of ky. The rail has mass σ per unit length and stiffness EI (i.e. the bending moment is EI times the curvature, E is Young's modulus and I is the appropriate moment of inertia of the cross-section), The spacing between one sleeper and the next is w. Take a coordinate x measured along the line, with t as time.

The equations of motion are easily written down:

Between sleepers: $EI\, \partial^4 y/\partial x^4 + \sigma\, \partial^2 y/\partial t^2 = 0$

At each sleeper $EI\partial^3 y/\partial x^3$ has a discontinuity with jump = $-ky$.

Consider vibrations with a time factor of $e^{i\omega t}$, or (as some prefer to express it) consider this Fourier time--component of the variable y. Now y is a function of x only, for as usual we omit the time factor. The differential equation between sleepers is

$$d^4y/dx^4 = \nu^4 y, \quad \text{where } \nu^4 = \omega^2\sigma/(EI).$$

In the interval $0 < x < w$ between two sleepers the general solution of this DE is

$$y = A \cosh \nu x + B \sinh \nu x + C \cos \nu x + D \sin \nu x.$$

In the next interval, $w < x < 2w$, there will be a similar solution, but with different parameters, A', B', C' and D'.

To find the relation between (A, B, C, D) and (A', B', C', D') we must look at the boundary conditions at $x = w$.

Consider the four functions:-

$$\tfrac{1}{2}y + \tfrac{1}{2}\nu^{-2}d^2y/dx^2 = A \cosh \nu x + B \sinh \nu x$$

$$\tfrac{1}{2}y - \tfrac{1}{2}\nu^{-2}d^2y/dx^2 = C \cos \nu x + D \sin \nu x$$

$$\nu^{-1}dy/dx = A \sinh \nu x + B \cosh \nu x - C \sin \nu x + D \cos \nu x$$

$$\tfrac{1}{2}\nu^{-3}d^3y/dx^3 - \tfrac{1}{2}\nu^{-1}dy/dx = C \sin \nu x - D \cos \nu x$$

Of these four functions the first three are continuous at $x = w$, and the third has a simple discontinuity with the jump equal to $-ky/(2\nu^3 EI) = -k(A' + C')/(2\nu^3 EI)$

Now we simplify the notation a little by putting

$$\beta = k/(4\nu^3 EI) \quad \text{and} \quad \alpha = \nu\omega$$

The boundary conditions at $x = w$ are

$$A \cosh \alpha + B \sinh \alpha = A'$$

$$C \cos \alpha + D \sin \alpha = C'$$

$$A \sinh \alpha + B \cosh \alpha - C \sin \alpha + D \cos \alpha = B' + D'$$

$$C \sin \alpha - D \cos \alpha = -D' + 2\beta(A' + C')$$

These may be written in matrix form

$$(A', B', C', D')^T = M\,(A, B, C, D)^T$$

where M is the 4×4 matrix :-

$$\begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha - 2\beta\cosh\alpha & \cosh\alpha - 2\beta\sinh\alpha & -2\beta\cos\alpha & -2\beta\sin\alpha \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 2\beta\cosh\alpha & 2\beta\sinh\alpha & 2\beta\cos\alpha - \sin\alpha & \cos\alpha + 2\beta\sin\alpha \end{bmatrix}$$

The propagation of vibrations is governed by iteration of this matrix M, and therefore by the eigenvalues of M. An eigenvalue of modulus one indicates the possibility of a progressive wave. From the symmetry of our model (reversing the direction of x) it is clear that if M has an eigenvalue λ, then it must also have an eigenvalue $1/\lambda$. Can we say anything useful about a matrix with this property?

Simplifying the equation $\det(M - \lambda I)$ we get the fourth degree equation

$$(1-2\lambda\cosh\alpha+\lambda^2+2\beta\lambda\sinh\alpha)(1-2\lambda\cos\alpha+\lambda^2-2\beta\sin\alpha)+4\beta^2\lambda^2\sinh\alpha\sin\alpha = 0$$

which by the transformation $2\mu = \lambda + 1/\lambda$ is reduced to the quadratic in μ:-

$$(\mu - \cosh\alpha + \beta\sinh\alpha)(\mu - \cos\alpha - \beta\sin\alpha) + \beta^2\sinh\alpha\,\sin\alpha = 0.$$

If this should have a real root between -1 and 1, then we may call the root $\cos\theta$, so that the equation for λ becomes $\lambda + 1/\lambda = 2\cos\theta$, with the solutions $\lambda = \exp(\pm i\theta)$, and this would indicate the possibility of standing or progressive waves. What are the values of the dimensionless parameters α and β for which this happens?

(To save you working this out) The matrix M has determinant = 1, and its inverse is

$$\begin{bmatrix} \cosh\alpha-2\beta\sinh\alpha & -\sinh\alpha & -2\beta\sinh\alpha & 0 \\ 2\beta\cosh\alpha-\sinh\alpha & \cosh\alpha & 2\beta\cosh\alpha & 0 \\ 2\beta\sin\alpha & 0 & 2\beta\sin\alpha+\cos\alpha & -\sin\alpha \\ -2\beta\cos\alpha & 0 & \sin\alpha-2\beta\cos\alpha & \cos\alpha \end{bmatrix}$$

Both M and its inverse are linear in β, does this tell us anything?

It should be noted that an eigenvalue with modulus one for M is a necessary condition for the possibility of a progressive wave, but not sufficient, for the eigenvector might describe a standing wave. To distinguish the two cases we should look at the energy flow.

Lemma: for any complex number λ to be of unit modulus, it is necessary and sufficient that $\lambda + 1/\lambda$ be real and be in the

closed interval [-2, 2].

If $M\underline{u} = \lambda\underline{u}$ then $M^{-1}\underline{u} = \lambda^{-1}\underline{u}$, so that each eigenvalue of N $= (M + M^{-1})/2$ has multiplicity 2, with a 2-dimensional eigenspace. This matrix N is reasonably simple:

$$N = \begin{bmatrix} \cosh\alpha & 0 & 0 & 0 \\ 0 & \cosh\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & \cos\alpha \end{bmatrix}$$

$$+ \beta \begin{bmatrix} -\sinh\alpha & 0 & -\sinh\alpha & 0 \\ 0 & -\sinh\alpha & \cosh\alpha-\cos\alpha & -\sin\alpha \\ \sin\alpha & 0 & \sin\alpha & 0 \\ \cosh\alpha-\cos\alpha & \sinh\alpha & 0 & \sin\alpha \end{bmatrix}$$

Now we want to see if the eigenvalues $\mu = (\lambda+\lambda^{-1})/2$ of N are real and in the closed interval [-1, 1]. They are the zeros of the quadratic:

$$\mu^2-\mu(\cosh\alpha+\cos\alpha-\beta\sinh\alpha+\beta\sin\alpha)+\cosh\alpha\cos\alpha+\beta\cosh\alpha\sin\alpha-\beta\sinh\alpha\cos\alpha$$

The resolvent R of this quadratic is

$$R= \beta^2(\sinh\alpha-\sin\alpha)^2-2\beta(\cosh\alpha-\cos\alpha)(\sinh\alpha+\sin\alpha)+(\cosh\alpha-\cos\alpha)^2$$

and (recalling that the non-dimensional parameters α and β are positive) we may observe that for any β the resolvent will be negative for all sufficiently small α.

Also, by looking at R as a quadratic in β, we may observe that R has a minimum (at a positive value of β) equal to

$$(\cosh\alpha-\cos\alpha)^2((\sinh\alpha-\sin\alpha)^2-(\sinh\alpha+\sin\alpha)^2)(\sinh\alpha-\sin\alpha)^{-2}$$

which is a positive multiple of $-\sinh\alpha\sin\alpha$. Therefore whenever $\sin\alpha$ is positive there is some β for which R is negative; and whenever $\sin\alpha$ is negative then R is positive and N has real eigenvalues for all β.

When R is negative the eigenvalues of N will both be complex, and so the eigenvalues of M can not be of unit modulus, and progressive waves are impossible.

But what more can be said about the set of (α, β) for which the eigenvalues of M have modulus 1? Can we be sure that it is non-empty?

KING ARTHUR'S BLUFF

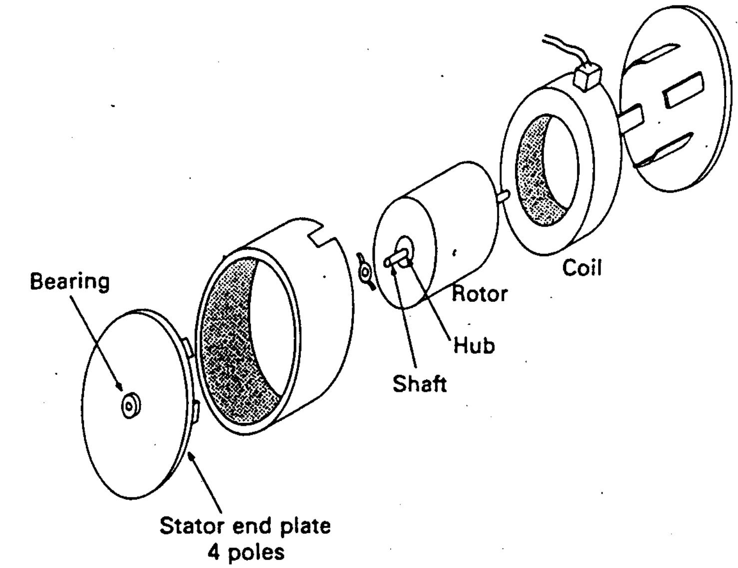
Reports had come to Camelot of the approach of a fleet of Viking longships from which a raiding party was threatening some small fishing villages on the Cornish coast. Some of the knights wanted King Arthur to take his army out to drive off the invaders. But he said "No. I will not leave Camelot undefended, for this may be a trick of theirs. But the villagers must be protected. You must all prepare for action while I make my plans."

Many of the knights had pages with them – boys who had left home in search of adventure, and made themselves useful polishing armour and looking after horses, and who hoped one day to become knights themselves. This fact gave the King an idea, and he summoned Merlin to talk about it. "Suppose" he said "we send a troop of mounted men with banners and trumpeters out along the coast road. The Vikings will see them and will withdraw their landing force rather than risk a fight, for they have learnt that they have no chance against well-trained cavalry. So it will not matter if our force is mostly composed of pages, not of soldiers. Then with luck the Viking chief will think that Camelot is left undefended, and they will make an attack, and we shall have them."

"The difficulty is, your Majesty" said Merlin "that this afternoon there will be a fog rolling in from the sea, and possibly the Viking party will know nothing of our troop advancing on them; and when the two forces meet our untrained boys will be cut to pieces. What I suggest is that you choose r of your n knights to stay to defend Camelot, sending the other n-r to go on this expedition, and make up their numbers to n by choosing r of the available m pages to go with them. Of course I am no soldier and I do not know what will be the best choice for the number r.

"The total number of choices before you is $\binom{m+n}{n}$, but for each r there are $\binom{m}{r}\binom{n}{r}$ choices. So we have found a nice little binomial identity: $\Sigma \binom{m}{r}\binom{n}{r} = \binom{m+n}{n}$, interesting, don't you think?" But the King wasn't listening.

QUOTATION CORNER 38 (JCMN 58, p.6134)



– From Robert Smart we have the comment that the drawing above should be regarded not as an attempt at perspective drawing, but as an attempt at the much easier exercise of isometric projection, which is a limiting case of perspective drawing as the distance from the viewer to the object tends to infinity. Isometric projection may be described, using Cartesian coordinates, as orthogonal projection on to the plane where $x + y + z = 0$; it is very easy to do by hand, for lengths along the three axes are all mapped to the same scale, and parallel lines are drawn as parallel. It would be interesting to learn if the modern CAD (= computer aided design) machines can do isometric projection correctly. Was this drawing produced by hand or by machine?

Below is shown what the isometric projection of a circular cylinder with its axis should look like. To clarify the construction we have put the cylinder in a cubical box with transparent sides. To describe the projection more precisely, the circular face of the cylinder is projected to an ellipse of which the ratio of the major to the minor axes is $\sqrt{3}$, and the minor axis of this ellipse is the projection of the axis of symmetry of the cylinder.