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EDITORIAL

Because of the difficulties with printing in the period round Christmas, we are changing our publication dates, making them all a month earlier, so that JCMN will come out in January, May and September.

Readers will recall the article DEMOCRACY FROM THE POINT OF VIEW OF MATHEMATICS by A. M. Slin'ko in JCMN 55. In it (p. 6040) the author, to show the difficulties that face a democratic system, gave the example of a community of 100 people with monthly salaries of 1, 2, ... 100 roubles, considering a proposal to change all the salaries by subtracting 99 from the largest and adding 1 to all the others; the point being that 99% of the people would have good reason to vote for the change, even though 100 such changes would restore the *status quo*. Her Majesty Queen Elizabeth II (who as far as I know is not one of the readers of JCMN) put forward essentially the same idea in her 1991 Christmas speech, pointing out any democratic system would give bad results if the voters were to use their powers for selfish purposes.

Normally we do not attach much importance to tracing the origins of the mathematical work that we publish, your Editor tries to judge only how clear, interesting, useful and significant it is. However, we like to give credit where credit is due. Looking through some old papers the other day, we found that the idea of OVERARM AND UNDERARM (JCMN 49 and 52) came from the late M. L. Urquhart of the University of Tasmania long ago.

POINTS AND ANGLES IN THE PLANE (JCMN 57, p.6090)

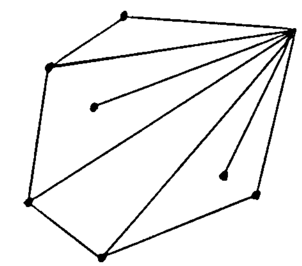
George Szekeres (U of NSW) and Basil Rennie

Given a set of n distinct points in the plane, they form $n(n-1)(n-2)/2$ angles, all in the closed interval between 0 and π. What can we say about these angles? One simple thing is that their mean is $60^\circ = \pi/3$. Perhaps we can say that at least one angle $\le A(n)$ and at least one $\ge B(n)$, where $A(n)$ and $B(n)$ are given by

n	3	4	5	6
A(n)	$60^\circ = \pi/3$	$45^\circ = \pi/4$	$36^\circ = \pi/5$	$30^\circ = \pi/6$
B(n)	$60^\circ = \pi/3$	$90^\circ = \pi/2$	$108^\circ = 3\pi/5$	$120^\circ = 2\pi/3$

How does the table continue?

For $A(n)$ we can find the general result, $A(n) = \pi/n$. To prove this, take the point P, the vertex with smallest angle of the polygon that is the convex hull of the given points; this angle must be $\le (1-2/n)\pi$, and (see drawing) it the sum of n-2 other angles, of which one must therefore be $\le \pi/n$. This bound is the best possible, by the example of the regular n-gon.



The question about $B(n)$ is more difficult. The values up to n=6 are not a good guide to the way the sequence goes on. More values are known, $B(6) = B(7) = B(8) = 2\pi/3$, and more generally $B(2^n) = (1-1/n)\pi$ for all $n \ge 2$.

The question was raised by L.M. Blumenthal in 1939, and there are the following references:

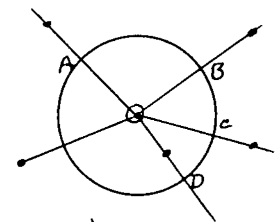
L. M. Blumenthal, American J. Math 61 (1939) 912-922.
G. Szekeres, American J. Math 63 (1941) 208-210.
P. Erdös & G. Szekeres, Annales Univ Sci Budapest 3 14 (1960/61) 53-62.

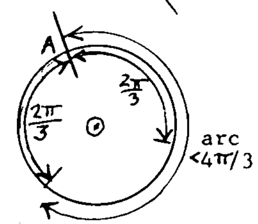
We shall sketch some of the results.

<u>Theorem 1</u> $B(n) \le (1-2/n)\pi$ for all n. This is clear by the example of the regular n-gon. Also, clearly, $B(n+1) \ge B(n)$.

<u>Lemma</u> If the given points are not all on the boundary of their convex hull, then one of the angles $\ge 2\pi/3$.

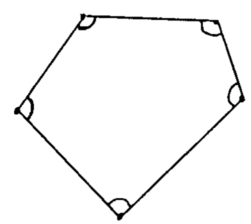
Proof: Let O be the point not on the boundary of the convex hull. Consider the half-lines joining O to the other points, and let A, B, C, ... be where these half-lines meet the unit circle round O. Now consider the $(n-1)(n-2)/2$ angles at O, if they are all $< 2\pi/3$, then all the points A, B, ... are contained in an arc of $< 4\pi/3$, and we may suppose that one of them, say A, is at one end of the arc. But all the points are $< 2\pi/3$ from A, and so all are in an arc of length $< 2\pi/3$. But this contradicts the fact that O is in the convex hull of the points, so the lemma is proved.





<u>Theorem 2</u> $B(n) \ge (1-2/n)\pi$ for $n \le 6$.

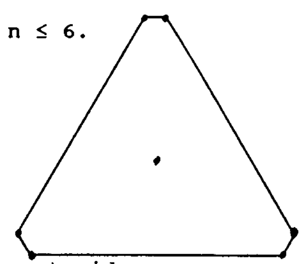
Proof: If the points are all on the polygon that is the boundary of their convex hull, then the sum of the angles of the polygon is $(n-2)\pi$, and so one of them $\ge$ mean $= (1-2/n)\pi$. If not then the lemma above shows that one angle $\ge 2\pi/3 \ge (1-2/n)\pi$.



We now have that $B(n) = (1-2/n)\pi$ for $n \le 6$.

<u>Example 1</u> $B(7) \le 2\pi/3$.

Form a hexagon by cutting off the corners of an equilateral triangle with small equilateral triangles; for the 7 points take the 6 vertices and a seventh point at the centroid.



Now it is clear that $B(7) = 2\pi/3$, using the example above and the fact that $B(7) \ge B(6) = 2\pi/3$.

<u>Theorem 3</u> $B(2^n) \le (1-1/n)\pi$.

Proof: Let $\beta = e^{i\pi/n}$, so that β^j for $j = 1, 2, \dots n-1$ are the first $n-1$ complex $2n$-th roots of unity, and take a small ϵ and a large R (to be fixed later). Define inductively the sets S_k (of complex numbers) for $k = 1, 2, \dots n$ as follows:

$$S_1 = \{0, 1\}$$

$$S_{k+1} = S_k \cup (R^k\beta^k + S_k), \text{ for } k = 1, 2, \dots n-1.$$

where as usual the notation $a+S$ denotes the set $\{a+b;\ b\in S\}$.

Each S_k has 2^k points, and (by induction) has the property that all the angles in it are $< (1-1/n)\pi+\epsilon$, provided that R is large enough.

From this theorem with $n = 3$, we find $B(8) \le 2\pi/3$, and because $B(8) \ge B(7) = 2\pi/3$, it is proved that $B(8) = 2\pi/3$.

In fact it has been shown (but we shall not attempt proofs here) that $B(2^n-1) = B(2^n) = (1-1/n)\pi < B(2^n+1)$ for all $n \ge 3$. Is it true that $B(2^{n-1} + 1) = B(2^n)$ for all $n \ge 4$?

FOREST ISLAND TREE-COUNTING

C. J. Smyth
University of Edinburgh

The forest ranger wants to count the trees on an island. The country is flat, so that the only obstruction to seeing every tree from every point is that one tree may be hidden behind another. Show how it is possible, after making observations from two points, to find a third point from which every tree will be visible.

A PROBLEM OF SEROV (JCMN 57 p. 6016)

Esther Szekeres

Any 1000 numbers are written in a line. A second line is written down under the first according to the rule: if the number x occurs in the first line k times, then we write down k in the second line under every occurrence of x in the first line. According to the same rule the third line is obtained from the second, and so on. Prove that lines 11 and 12 are the same. Give an example of an initial line that yields eleven distinct lines.

Example (with 13 instead of 1000)

5	6	9	6	4	8	3	3	6	2	4	7	9
1	3	2	3	2	1	2	2	3	1	2	1	2
4	3	6	3	6	4	6	6	3	4	6	4	6
4	3	6	3	6	4	6	6	3	4	6	4	6

First note that the columns of the table may be permuted freely, so that what matters about a line is just the numbers in it, not their positions.

Given the second line, L_2, we can always reconstruct a first line, L_1, with arbitrary numbers, so consider only L_2. This line of the arrangement (and every other below it) represents a partition of the total number, n, of the terms, (1000 in Serov's problem, or 13 in the example above). If a number x figured k times in L_1, then in L_2 we get a group of k occurring k times. I represent this group by (k). We reach the last row, L_r, if we reach a partition of n with all different values.

If L_2 consists of (k_1), (k_2), ... (k_m) then the next line, and each succeeding line contains only multiples of these original brackets. What is the smallest n such that we can have r different rows?

An entry in L_r must be a multiple of some bracket in L_{r-1}, which is a multiple of some bracket in L_{r-2}, ... therefore a multiple of some bracket in L_2. We get the

smallest possible value for n if the initial bracket in L_2 is (1) and each successive line contains a double of the bracket of the previous line. So, to be able to have r different rows we must have in L_2 at least the following:

$L_2 = (1)(1)(2)(4) \ldots (2^{r-3})$, and so

$L_3 = (2)(2)(4) \ldots (2^{r-3})$

......

$L_{r-1} = (2^{r-3})(2^{r-3})$

$L_r = (2^{r-2})$

So the smallest number to achieve r different rows is 2^{r-2}. Now $2^9 < 1000 < 2^{10}$, and therefore a length of 1000 cannot give 12 different lines. The following will yield 11 different lines:

$L_2 = (1)(1)(2)(4) \ldots (2^8)(488)$.

The first line could for instance be:

1, 2, 3, 3, 4, 4, 4, 4, 5 (8 times), 6 (16 times) ... , 10 (256 times), 11 (488 times).

MEASURE THEORY

Terry Tao

This problem originates from W. Rudin. On the closed unit interval [0, 1] there is a non-decreasing bounded $\mu(x)$, and there are continuous non-negative $g_1(x)$, $g_2(x)$, ... such that (a) $\int g_n(x)\, f(x)\, dx \to \int f(x)\, d\mu(x)$ for any continuous function f(x), and (b) $g_n(x) \to 0$ pointwise except on a set of Lebesgue measure zero.

Does it follow that $d\mu(x)$ is a singular measure? i.e. that $\mu(x)$ is almost everywhere continuous with derivative zero.

SET OF EQUATIONS (JCMN 57, p.6099)

The equations
$$x + y^2 + z^6 = 0$$
$$y + z^2 + x^6 = 0$$
$$z + x^2 + y^6 = 0$$
have the obvious solution $x = y = z = 0$. How many other real solutions are there?

FIRST SOLUTION

Mark Kisin

There is just one, $x = y = z =$ the real root of the quintic:
$$x^5 + x + 1 = 0$$
which clearly has only one real root, $x = -0{\cdot}754877\ldots$.

To prove this, first note that x, y and z all ≤ 0, and either all $= 0$ or all < 0. Consider the case of all negative. Change the notation by putting $p = -x$, $q = -y$, and $r = -z$. Our new unknowns are all > 0, and they satisfy
$$\begin{vmatrix} p & q^2 & r^6 \\ q & r^2 & p^6 \\ r & p^2 & q^6 \end{vmatrix} = 0$$
i.e. $$p(r^2q^6-p^8) + q(p^2r^6-q^8) + r(q^2p^6-r^8) = 0$$
or $$p^9+q^9+r^9 = pqr(rq^5+pr^5+qp^5).$$
Consider the R.H.S. of this equation.

$rq^5+pr^5+qp^5 \le p^6+q^6+r^6$ because $6uv^5 \le u^6 + 5v^6$,

$p^6+q^6+r^6 \le \sqrt[3]{3}\,(p^9+q^9+r^9)^{2/3}$ by the power-mean inequality,

and $pqr \le 3^{-1/3}\,(p^9+q^9+r^9)^{1/3}$ by the A.M.-G.M. inequality.

All these three have the property that the inequality is strict unless the three variables are equal (but for our present argument it would be sufficient if only one had the property). We have R.H.S = L.H.S., and so it follows that the three variables are all equal. Therefore the only non-zero solution has $p = q = r$ and $x = y = z$, as stated above.

SECOND SOLUTION

P. H. Diananda

As before, a non-zero solution must have x, y and z all $<$ 0. If x, y and z are not all equal then by cyclic symmetry there are two cases:

Case A $\quad z \le x < y < 0$

Case B $\quad y < x \le z < 0$

From the second and third equations
$$(y-z) + (z^2-x^2) + (x^6-y^6) = 0 \qquad (*)$$
but in case A $\quad y-z > 0,\quad z^2-x^2 \ge 0\quad$ and $\quad x^6-y^6 > 0$,

and in case B $\quad y-z < 0,\quad z^2- x^2 \le 0\quad$ and $\quad x^6-y^6 < 0$.

In each case the equation (*) is not satisfied. Therefore the only solutions of the set of equations are the ones mentioned above, $x = y = z =$ either 0 or $-0{\cdot}754877666\ldots$.

SYMMETRIC INEQUALITY

Let a, b and c be any real numbers ≥ 0. Let r be the sum, $a+b+c$, let s be the sum of the squares, and let t be the sum of the cubes. Prove that
$$4t \ge rs + 3abc.$$
The inequality above may be expressed in terms of the elementary symmetric functions as
$$9\sigma_3 + 3\sigma_1^3 - 10\sigma_1\sigma_2 \ge 0$$
but your editor does not know if this makes the problem any easier.

Does equality imply that $a = b = c$?

FITTING POLYNOMIALS

Mark Kisin
(112, Summerhill Road, Glen Iris, 3146, Australia)

Consider the following problem: given $n+1$ distinct complex numbers, find a monic polynomial $h(z)$ of degree n to minimize the maximum M of the values of $|h(z)|$ at the given points.

The case $n=1$ is trivial; for given points a and b, the minimum of M is $|a-b|/2$, with the polynomial $h(z) = z - (a+b)/2$ attaining the values $(a-b)/2$ at a and $(b-a)/2$ at b.

The case $n=2$ leads us into triangle geometry. Let a, b and c be the given complex numbers (we may think of them as the vertices of a triangle). Lagrange's interpolation formula tells us that any quadratic $h(z)$ may be expressed in terms of its values at a, b and c as follows:-

$$h(z) = h(a)u(z) + h(b)v(z) + h(c)w(z)$$

where $(a-b)(a-c)u(z) = (z-b)(z-c)$, so that $u(a)=1$ and $u(b)=u(c)=0$; and $v(z)$ and $w(z)$ are defined similarly.

We have $M = \max(|h(a)|, |h(b)|, |h(c)|)$, and we require the quadratic $h(z)$ to be monic, so that

$1 = \text{Coefficient of square term} = \Sigma\, h(a)/((a-b)(a-c))$ where summation is over the three cyclic permutations of (a, b, c). Because $|h(a)| \le M$, etc. it follows that

$$1/M \le \Sigma\, 1/|(a-b)(a-c)| = (|b-c|+|c-a|+|a-b|)/|(b-c)(c-a)(a-b)|$$

where equality will occur if

$$1/|h(a)| = 1/|h(b)| = 1/|h(c)| = \Sigma\, 1/|(a-b)(a-c)|$$

and the phases of $h(a)$, $h(b)$ and $h(c)$ are chosen so that $h(a)/((a-b)(a-c))$ etc. are all real and positive.

Therefore the answer to the problem is that

$$M = |b-c|\,|c-a|\,|a-b|/\Sigma|b-c|$$

and this minimum is attained by the quadratic

$$h(z) = z^2 - z\,\Sigma((b+c)|b-c|)/\Sigma|b-c| + \Sigma\, bc|b-c|/\Sigma|b-c|.$$

Now consider the geometry. Monic polynomials are characterised by their zeros, and so the given triangle has led us to consider a certain pair of points, the zeros of the $h(z)$ given above. What can we say about the mid-point of the line

segment joining them, i.e. the mean of the two zeros? This point is represented by half the sum of the roots of the quadratic above, that is by

$$z = (1/2)\Sigma(b+c)|b-c|/\Sigma|b-c|$$

Expressing this in terms of mechanics, it signifies the centre of gravity of a set of three particles, one at the mid-point of each side of the triangle, of mass equal to the length of the side. In other words it is the the centre of gravity of the triangle made out of uniform heavy wire.

As usual in triangle geometry, we shall now use (instead of their previous meanings) a, b and c as the lengths of the sides. Using (homogeneous) trilinear co-ordinates (x, y, z), where x is the perpendicular distance from the point to the side BC of the triangle, and y and z similarly, the point that we have found has co-ordinates $((b+c)/a, (c+a)/b, (a+b)/c)$. See the note TRIANGLE GEOMETRY (pp. 6127 - 6129) below.

Now that we have studied the case $n = 2$, the general case of $n > 2$ should present no difficulty to the mathematician, but to our word-processor the equations are formidable. So we give just the result: there is such a monic polynomial and it gives the minimum of M to be

$$\frac{1}{\sum_{r=0}^{n} |P_r|}$$

where $P_r = 1/\prod_{j\neq r}(x_r - x_j)$ and where

$x_0, x_1, \dots x_n$ are the $n+1$ given complex numbers.

We give two examples.

<u>Example 1</u> If $x_r = r$ for $r = 0, 1, \dots n$,

then $P_r = 1/(r(r-1)(r-2) \dots 1.(-1)(-2) \dots (r-n))$

$= (-1)^{n-r}/(r!(n-r)!) = (-1)^{n-r}\binom{n}{r}/n!$

and $M = n!/\Sigma\binom{n}{r} = 2^{-n}\, n!$.

Example 2 If $x_r = \beta e^{2\pi ir/(n+1)}$ for $0 \le r \le n$, with $|\beta| = 1$,

put $\omega = e^{2\pi i/(n+1)}$, then we find $x_r = \beta\omega^r$ and

$P_r = \beta^{-n}/(\omega^r-1)(\omega^r-\omega)\ ..\ (\omega^r-\omega^{r-1})(\omega^r-\omega^{r+1})\ ..\ (\omega^r-\omega^n))$,

but $|\omega^r-\omega^k| = |1 - \omega^{k-r}|$, and as k runs from 0 to n

omitting r, k-r will take all values mod n+1 except

zero. Therefore $|P_r| = 1/|(1-\omega)(1-\omega^2)\ ..\ (1-\omega^n)|$

but $(x-\omega)(x-\omega^2)\ ..\ (x-\omega^n) = 1 + x + ... + x^n$, so that

$|P_r| = 1/(n+1)$ and $M = 1$. The polynomial is $h(x) = x^n$.

Now, a little thought will show that the problem we have been considering is equivalent to the following. Given n+1 distinct points $X_0, X_1, \dots X_n$, in the plane, find n points $A_1, A_2, \dots A_n$, so as to minimize $\max(p_0, p_1, \dots p_n)$, where each p_i is the product of the lengths:

$$p_i = (X_iA_1)(X_iA_2) \dots (X_iA_n)$$

For this collection of $A_1, A_2, \dots A_n$, determine

$$\max(p_0, p_1, \dots p_n)$$

(The A_i's correspond to the zeros of the polynomial h(x))

The surprising thing is that if one attempts to solve this related problem directly (i.e. by fiddling with the points $A_1, \dots A_n$), then the problem seems difficult. It is even difficult to obtain a reasonable lower bound for the minimum. Try to show directly that $M \ge 3/2$ (or even ≥ 1) for Example 1 with n = 4.

TRIANGLE GEOMETRY

First, recall a few old facts, some of them to be found in previous issues of JCMN. Given any triangle, the circumcentre O, the centroid G, the nine-point centre N and the orthocentre H are (in that order) on the Euler axis (sometimes called the Euler Line). The ratios of the lengths OG:GN:NH are 2:1:3.

Use (homogeneous) trilinear co-ordinates (x, y, z), where x is the perpendicular distance from the point to the side BC of the triangle of reference, which is the given triangle, and y and z similarly. As usual in triangle geometry we use A, B and C for the angles as well as the vertices of the triangle, and a, b and c as the lengths of the sides BC, CA and AB. The point discovered in FITTING POLYNOMIALS (pages 6124 - 6126 above) has co-ordinates ((b+c)/a, (c+a)/b, (a+b)/c). It is the mean of the zeros of the monic quadratic minimizing the maximum of its modulus on the three vertices, and is the centre of gravity of uniform wire bent to the shape of the triangle. Call it W.

The table below lists some of the important points and lines of a triangle.

	Trilinear coordinates
Vertex A	(1, 0, 0)
Vertex B	(0, 1, 0)
Vertex C	(0, 0, 1)
Line at infinity	ax+by+cz = 0
Circular points	(-1, expiC, exp-iB), and (-1, exp-iC, expiB)
Circumcentre O	(cosA, cosB, cosC)
Centroid G	(bc, ca, ab)
Nine-point centre N	(cos(B-C), cos(C-A), ..)
Centre of Guinand's critical circle	(2cos(B-C)-cosA, ...)
Orthocentre H	(secA, secB, secC)
Euler axis	x sin2A sin(B-C) + .. = 0
Incentre I	(1, 1, 1)
Symmedian point K	(a, b, c)
Our new point W	((b+c)/a, (c+a)/b, ...)
Typical side of orthic triangle	xcosA=ycosB+zcosC
Gergonne point	(1/(1+cosA), ...)

The points IGW are on a line, in that order, and the ratio IG : GW is 2 : 1. To see this, express the coordinates of the points in a normalized form, with $ax+by+cz = a+b+c$; then note

$$3((a+b+c)/(3a),\ (a+b+c)/(3b),\ (a+b+c)/(3c))$$
$$= (1,\ 1,\ 1) + 2((b+c)/(2a),\ (c+a)/(2b),\ (a+b)/(2c)).$$

Now consider some of the circles that we find. Put S for the quadratic form $ayz+bzx+cxy$, and put S′ for $ax^2\cos A+by^2\cos B+cz^2\cos C$. Then we have:-

Circle	Equation	Radius	Centre
Circumcircle	$S = 0$	R	O
where $R = abc/\surd(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)$		$= a/(2\sin A)$	
	$2S = S'$	$(2R/3)\surd(1+p)$	G
	where $p = \cos A\ \cos B\ \cos C$		
Nine-point circle	$S = S'$	$R/2$	N
Guinand's critical	$S = 2S'$	$(R/3)\surd(1-8p)$	
Orthocircle	$S' = 0$	$2R\surd(-p)$	H
Degenerate	$S+S' = 0$	(radical axis & line at infinity)	

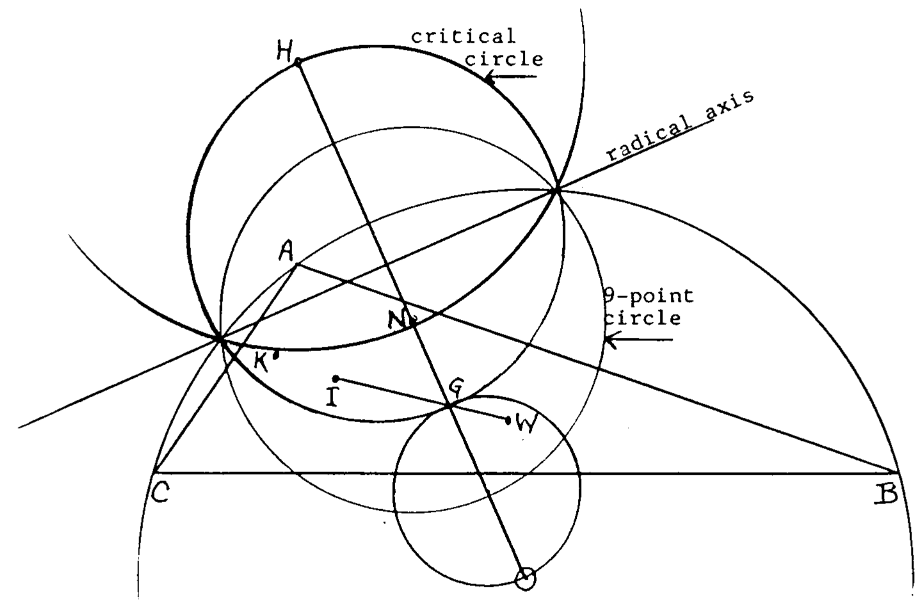
These circles form a coaxal system of which the radical axis is the line $x\cos A+y\cos B+z\cos C = 0$, which can be regarded as a degenerate circle by combining it with the line at infinity to give the locus $0 = (ax+by+cz)(x\cos A+y\cos B+z\cos C) = S + S'$. The radical axis is the polar of the centroid G with respect to the orthocircle. Another property of the radical axis is that it joins the three intersections of the sides of ABC with the corresponding sides of the orthic triangle (the triangle with vertices the feet of the perpendiculars from A to BC, etc); the fact that these three intersections are collinear is given by the theorem of Desargues, because ABC and its orthic triangle are in perspective.

Recall the observation in 1984 by A. P. Guinand (JCMN 33, p.4032 and JCMN 38, p. 4152) about the symmedian point K and the incentre I being inside the critical circle (on GH as diameter) and being on the same side of the Euler axis. Now we can add the fact that they are on the same side of the radical axis, $x\cos A+y\cos B+z\cos C=0$, in fact they are on the side on which the

circumcentre O and centroid G are. To see this, note that for any point (x, y, z), the side is determined by the sign of $(x\cos A+y\cos B+z\cos C)/(ax+by+cz)$; these signs are easily found for these points K, I, O and G from the fact that $\cos A+\cos B+\cos C$ and $a\cos A+b\cos B+c\cos C$ and $bc\cos A+ca\cos B+ab\cos C$ are all positive. For acute-angled triangles this remark is not significant, because the whole of the critical circle is on this side of the radical axis, but for obtuse-angled triangles the radical axis does cut the critical circle.

The fact that the incentre I must be in the critical circle, and the observation above about the points I, G and W, together tell us that W must be in the circle on OG as diameter; also clearly W must be on the opposite side of the Euler axis from I and K.

The drawing below shows these points, circles and lines for a typical triangle ABC, which (as recently pointed out in these pages by John Parker and others) is probably obtuse-angled.



FAMILY OF POLYNOMIALS (1) (JCMN 57, p.6099)

Define the polynomial $f(n, x)$ of degree n as either $\Sigma_{k=0}^{n}(-1)^{n+k}\binom{n+k+1}{n}\binom{n}{k}x^k$ or $\Sigma_{k=0}^{n}(-1)^{k}\binom{n+k+1}{k}\binom{n}{k}(1-x)^k$

<u>Problem (a)</u> Are the two definitions equivalent? YES.

By the first definition f is the coefficient of v^n in

$\Sigma(-1)^k(1-v)^{n+k+1}\binom{n}{k}x^k = (1-v)^{n+1}(1-x+xv)^n$.

Similarly the second definition gives $f(n, x)$ to be the coefficient of u^{n+1} in $(1+u)^{n+1}(x+xu-u)^n$. Now put $v = -1/u$. This means that by the second definition $f(n, x)$ is the coefficient of $(-v)^{-n-1}$ in

$$(1-1/v)^{n+1}(x-x/v+1/v)^n = v^{-2n-1}(v-1)^{n+1}(1-x+xv)^n,$$

which is the coefficient of v^n in $(1-v)^{n+1}(1-x+xv)^n$, agreeing with the first definition.

<u>Problem (b)</u> Orthogonality.

$\int_0^1 x\, f(m, x)\, f(n, x)\, dx$ is the coefficient of $u^m v^n$ in $(1-u)^{m+1}(1-v)^{n+1}\int_0^1 x\,(1-x+xu)^m(1-x+xv)^n\,dx$, which may be written, putting $p = 1-u$ and $q = 1-v$, as

$p^{m+1}q^{n+1}\int_0^1 x\,(1-px)^m(1-qx)^n\,dx$. We shall show the coefficient to be zero when $m \neq n$. We take $m < n$.

<u>Lemma</u> If $0 < s \le n \le t$ and $r \le n$ then the coefficient of v^n in $F(s) = q^{r+1}\int_0^1 x^s\,(1-qx)^t\,dx$ is zero.

Proof: Integration by parts gives

$$(t+1)F(s) = -q^r(1-q)^{t+1} + q^r s\int_0^1 x^{s-1}\,(1-qx)^{t+1}\,dx \qquad (*)$$

if $s = 1$, the RHS becomes $-(1-v)^r v^{t+1} + (1-v)^{r-1}(1-v^{t+2})/(t+2)$ in which the coefficient of v^n is clearly zero. Having established our result for $s = 1$, we complete the proof by

induction on s. Consider the formula (*) above, in the first term the coefficient of v^n is clearly zero; now take the second term, if $s > 1$ then $0 < s-1 \le n \le t+1$ and $r-1 \le n$, so that the induction hypothesis tells us that the coefficient of v^n is zero. This proves the lemma.

The orthogonality follows without difficulty, for the integral $q^{n+1}\int_0^1 x\,(1-x(1-u))^m\,(1-qx)^n\,dx$ is a sum of terms of the form $q^{r+1}\int x^s\,(1-qx)^t\,dx$ where $0 < s \le m+1 \le n = t$ and $r = n$. (We ignore the powers of u) The lemma tells us that the coefficient of v^n is zero in every term.

But what can be done about the case $m = n$?

<u>Problem (c)</u> Values at $x = \frac{1}{2}$.

From the first definition, $f(n, \frac{1}{2})$ is the coefficient of v^n in $2^{-n}(1-v)^{n+1}(1+v)^n = 2^{-n}(1-v)(1-v^2)^n$. If n is odd, put $n = 2m-1$, $f(2m-1, \frac{1}{2})$ is the coefficient of v^{2m-1} in $2^{1-2m}(-v)(1-v^2)^{2m-1}$, i.e. $f(2m-1, \frac{1}{2}) = -2^{1-2m}(-1)^{m-1}\binom{2m-1}{m-1} = (-1)^m 2^{-2m}\binom{2m}{m}$. If $n = 2m$ then $f(2m, \frac{1}{2})$ is the coefficient of v^{2m} in $2^{-2m}(1-v^2)^{2m}$ which is $(-1)^m 2^{-2m}\binom{2m}{m}$, the same as $f(2m-1, \frac{1}{2})$.

FAMILY OF POLYNOMIALS (2) (JCMN 57 p 6099)

A. Brown

If $f_n(x) = \Sigma_{k=0}^{n}(-1)^{n+k}\binom{n+k+1}{k}\binom{n}{k}x^k$, it is easy to check that $f_n(x)$ can be written as

$$f_n(x) = (-1)^n (n+1) F(-n,\ n+2;\ 2;\ x), \tag{1}$$

where F(a, b; c; x) is the usual hypergeometric series.

This means that $f_n(x)$ satisfies the differential equation

$$x(1-x)f_n'' + (2-3x)f_n' + n(n+2)f_n = 0 \tag{2}$$

or

$$(d/dx)(x^2(1-x)f_n') + n(n+2)xf_n = 0 \tag{2a}$$

From standard Sturm-Liouville theory

$$\int_0^1 x\ f_n(x)\ f_m(x)\ dx = 0 \qquad \text{for } m \neq n.$$

In the same way, if $y = 1-x$, then

$$g_n(y) = \sum_{k=0}^{n} (-1)^k \binom{n+k+1}{k} \binom{n}{k} y^k = F(-n,\ n+2;\ 1;\ y)$$

and $g_n(y)$ satisfies the equation obtained from (2) by changing from x to y. The polynomial solutions must be proportional to each other and it is fairly obvious that $f_n(x) = g_n(y)$. One argument that could be used is that $f_n(x)$ satisfies a recurrence relation

$$0 = (n+2)(2n+1)f_{n+1} + \{4(n+1)^2 - (2n+1)(4n+6)x\}f_n + n(2n+3)f_{n-1} \tag{3}$$

and $g_n(y)$ satisfies the corresponding equation (on setting $x = 1-y$). If f_0 and f_1 are specified, the rest of the sequence $\{f_n(x)\}$ follows from equation (3). So if $f_0 = g_0$ and $f_1 = g_1$, this should ensure that $f_n(x) = g_n(y)$ for n = 2, 3, 4, ...

If we put $x = \frac{1}{2}$ in equation (3), we get

$$(n+2)(2n+1)f_{n+1}(\tfrac{1}{2}) = -f_n(\tfrac{1}{2}) - n(2n+3)f_{n-1}(\tfrac{1}{2}) \tag{3a}$$

This can be used to give $f_6(\frac{1}{2}) = -5/16$, $f_7(\frac{1}{2}) = f_8(\frac{1}{2}) = 35/128$. Thus

$$f_{2n-1}(\tfrac{1}{2}) = f_{2n}(\tfrac{1}{2}) = (-1)^n 2^{-2n} \binom{2n}{n} \tag{4}$$

for n = 1, 2, 3, 4. We can then proceed by induction to show if equation (4) holds for n = N, the recurrence relation ensures that $f_{2N+1}(\frac{1}{2})$ and $f_{2N+2}(\frac{1}{2})$ have the same form.

Another line of attack comes from using $X = 2x-1$ and writing $f_n(x) = h_n(X)$. The differential equation becomes

$$(1-X^2)d^2h_n/dX^2 + (1-3X)dh_n/dX + n(n+2)h_n = 0, \tag{2b}$$

and this equation is satisfied by the Jacobi polynomial

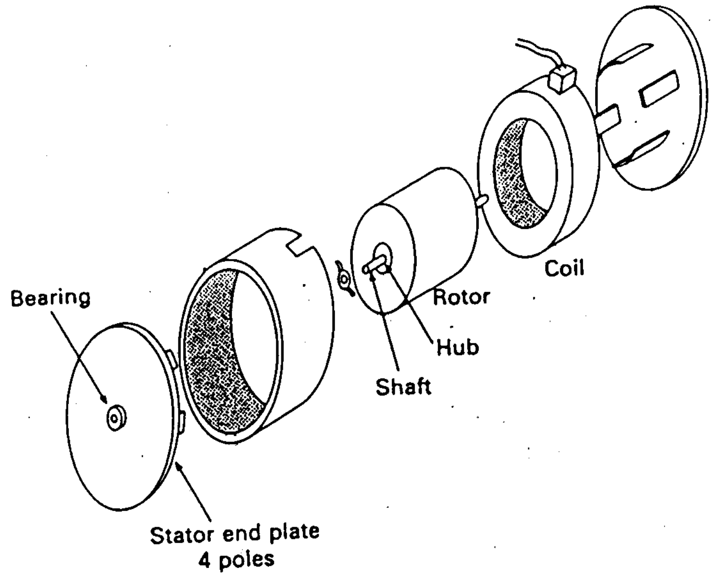
$$P_n^{(0,\ 1)}(X) = 2^{-n} \sum_{m=0}^{n} \binom{n}{m}\binom{n+1}{n-m}(X-1)^{n-m}(X+1)^m.$$

If we take $h_n(X) = P_n^{(0,\ 1)}(X)$, it is easy to check that $h_n(X) = f_n(x)$ for n = 0, 1, 2, and that $h_n(X)$ satisfies the recurrence relation that corresponds to equation (3) (on setting $x = (1+X)/2$). As before, this ensures that $f_n(x) = h_n(X)$ for n = 3, 4, 5, ...

This gives us a third formula for defining our functions:

$$f_n(x) \text{ (or } f(n,\ x) \text{ in the original notation) } = \sum_{m=0}^{n} \binom{n}{m}\binom{n+1}{n-m}(x-1)^{n-m}x^m.$$

QUOTATION CORNER 38



— From an article in Bull. I. M. A. vol 27, p. 232, in which this drawing is described as being from a thesis. It is a sad commentary on modern education that graduates in engineering and mathematics seem to be baffled by the problem of how to make a perspective drawing of a circular cylinder.

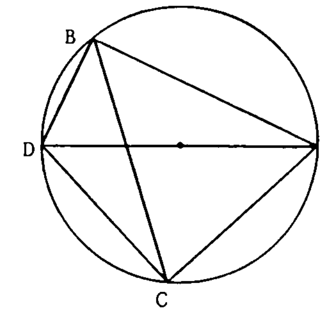
BINOMIAL IDENTITY 35

$$\binom{m-s}{n} = \sum_{t=0}^{n} (-1)^t \binom{s}{t} \binom{m-t}{n-t}$$

GEOMETRICAL INEQUALITY (JCMN 57, p.6098)

Mark Kisin

(112, Summerhill Road, Glen Iris, 3146, Australia)



It is given that AB, BC and AC are all > 1, and that AD is a diameter of the circle.

Area of ABDC = triangle ABD + triangle ADC

= (AB.BD + AC.CD)/2 > (BD + CD)/2 > BC/2 > 1/2.

This bound of 1/2 is the best possible, (B = D!)

ANALYTIC INEQUALITY 3

Let $f(x)$ and $g(x)$ be non-decreasing in the unit interval. Put $\int f(x)dx = p$ and $\int g(x)dx = q$. Find the best constant K in the inequality

$$\int f(x)g(x)dx - pq \geq K\int |f(x)-p|dx \int |g(x)-q|dx$$

where all integrals are over the unit interval.

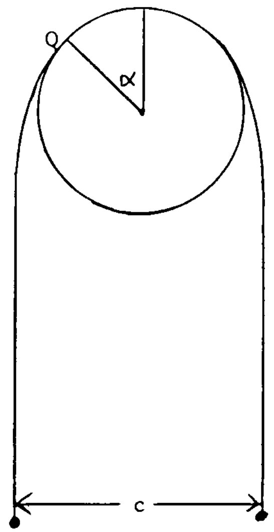
JCMN readers with long memories may recall the superficially similar (but really very different) formula:

$$\left|\int f(x)g(x)dx - pq\right| \leq \pi^{-2}\left(\int f'(x)^2dx \int g'(x)^2dx\right)^{\frac{1}{2}}$$

in KNOWING THE ANSWER, JCMN 26, p.3027, September 1981.

NICE LITTLE ELASTICITY PROBLEM (JCMN 57, p.6109)

This question was suggested by the more difficult (and important) problem of understanding the usual suspension of a clock pendulum.



Two particles of unit weight are fixed to the ends of a long thin weightless wire, inextensible but with the property that the bending moment at any point is K times the curvature. The wire hangs symmetrically in equilibrium over a frictionless horizontal circular cylinder of radius r.

The answer is that the angle α is given by

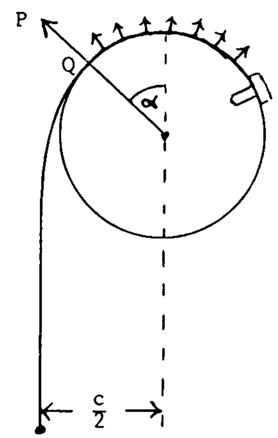
$$\sin \alpha = 1 - K/(2r^2)$$

and the distance c is given by

$$c = 2r + K/r,$$

and (happy surprise) these values may be found without solving a differential equation.

Now look at a different model, see the drawing below, the left-hand half of the drawing is the same as before, but the right-hand bit of the wire is cut off and fastened to the cylinder.



We imagine the cylinder pivoted about its axis, and we ask what is the couple on it. It is obviously c/2, but let's find it by the method of virtual work. If the cylinder turns through an angle of β anticlockwise, then there is a decrease of $r\beta$ in gravitational potential energy, and a decrease of $K\beta/(2r)$ in elastic potential energy (because the part touching the

cylinder has energy $K/(2r^2)$ per unit length). Therefore we have the equation: $c\beta/2 = r\beta + K\beta/(2r)$

or $c = 2r + K/r$.

Now to find the angle α, consider (in either model) the part of the wire to the left of the point Q, the left-hand end of the part in contact with the cylinder. The bending moment is K/r, and by taking moments about Q we find

$$K/r = c/2 - r \sin \alpha$$

or (using the equation above for c) $\sin \alpha = 1 - K/(2r^2)$.

The validity of our calculation depends on α being non-zero, so that $2r^2 > K$. If the radius r had the critical value $\sqrt{(K/2)}$ or less then the wire would touch the cylinder only at the top, and the distance apart of the particles would be $c = 2\sqrt{(2K)}$.

It is instructive to look at more details of the system. The part of the wire in contact with the cylinder is under a uniform tension T, and exerts a uniform force of T/r per unit length on the cylinder. By considering the equilibrium of a small bit of the wire at Q we find that $T = \sin \alpha$, and that there is a concentrated normal force of $P = \cos \alpha$ on the wire at Q.

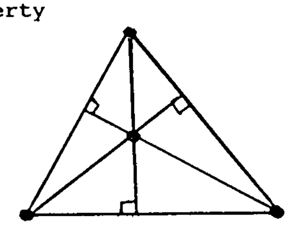
COMBINATORIAL QUESTION

Given m things, we want to choose the same number m of k-element subsets so that no two of these subsets have more than one element in common. For each m, what is the largest possible k? The first few seem to be as follows:

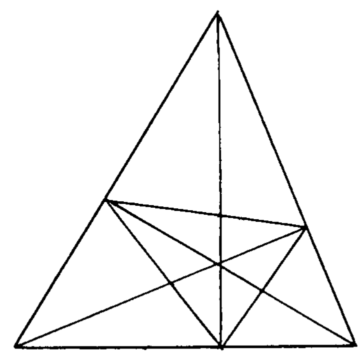
m	1	2	3	4	5	6	7
k	1	1	2	2	2	2	2

TRIANGLES - ARE THEY UNFAIR TO THEIR ORTHOCENTRES?

Consider three points, A, B and C in the Euclidean plane. One of the first things that they lead us to consider is their orthocentre H. But the defining property of H is that of the four points A, B, C and H, the line joining any two is perpendicular to the line joining the other two. In other words each of the four points is the orthocentre of the triangle formed by the other three.



This relation is symmetric among the four points, and so we are left with no reason for treating H any differently from A, B and C. Now, what can we say about our figure (above) of four points and the 6 lines joining them? There are 3 more points demanding attention — the intersections of the pairs of perpendicular lines. And so we are back where we started, with a triangle! This triangle is the orthic triangle of each of the 4 triangles that can be formed from the four points. The family of 4 points that was the basis of our figure is just the set of the four tritangent centres of the new triangle.



Perhaps the answer to our question is yes, the three points are unfair to their orthocentre; they have to be, for otherwise they would be for ever generating three new points and then having to adopt the fourth point that goes with the three. An infinite sequence of points, even if convergent, is not really in the spirit of Euclidean geometry.

If we repeatedly replace a triangle by its orthic triangle, do these triangles converge to a point? And, if they do, where is the point?

INTEGRAL INEQUALITY (JCMN 57, p.6106)

Let B(n) be the greatest lower bound of

$$\int_0^1 f(x)^2dx$$

over all polynomials f of degree n with integer coefficients. Clearly B(1) = 1/3. What are the other values?

We can find B(2) = 1/30 and B(3) = 1/210 as follows. If f is a quadratic with integer coefficients, the integral of its square is of the form: (integer)/5 + (even integer)/4 + (integer)/3 + (even integer)/2 + (integer), which is an integer divided by 30. The function $f(x) = x(x-1)$ leads to the value 1/30 for the integral, and so no other function can do better. A similar argument with cubics shows that the lowest value for the integral is 1/210, which is attained by the polynomial $x(2x-1)(x-1) = 2x^3 - 3x^2 + x$.

This reasoning applied to fourth degree polynomials is less successful. It shows that the integral must be (integer)/1260, but is there any f(x) that gives a value of 1/1260? The polynomials

$$x^2(x-1)^2 = x^4 - 2x^3 + x^2$$
$$x^2(2x-1)(x-1) = 2x^4 - 3x^3 + x^2$$
$$x(2x-1)^2(x-1) = 4x^4 - 8x^3 + 5x^2 - x$$

and $$x(x-1)(5x^2-5x+1) = 5x^4 - 10x^3 + 6x^2 - x$$

and, of course, those obtained from these by putting $-f(x)$ or $f(1-x)$ for $f(x)$, all give a value of 1/630 for the integral.

About the asymptotic behaviour of B(n), we may make the simple observation that $B(n+1) < \frac{1}{2}B(n)$. To prove this, take any f of degree n, and consider the two polynomials

$$f_1(x) = xf(x) \qquad \text{and} \qquad f_2(x) = (1-x)f(x)$$

Let A_1 and A_2 be the integrals of the squares of f_1 and f_2 respectively, then $A_1+A_2 = \int(1-2x+2x^2)f(x)^2dx < \int f(x)^2dx$, so that either A_1 or A_2 must be $< \frac{1}{2}\int f(x)^2dx$.

SOLID GEOMETRY

Given a triangle ABC in 3-dimensional Euclidean space, what can be said about the locus of points P such that the 3 angles APB, BPC and CPA are equal?