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POINTS AND DISTANCES IN THE PLANE

Paul Erdös

(Hungarian Academy of Sciences)

Referring to TWO PROBLEMS (JCMN 56, p. 6060), there are two slightly different questions. Let n_o be the largest n for which we can have n points in the plane in general position (no three on a line and no four on a circle) so that the most frequent distance occurs n-1 times, the next most frequent distance occurs n-2 times, and so on, and finally one distance only once. Ilona Palàsti found 8 such points, and perhaps 9 such points do not exist, but we cannot exclude the possibility that such a set exists for every n. (The Editor's apologies are due for the spelling error "Palànti" for "Palàsti" in the previous issue)

The other problem states: let n_1 be the largest integer n such that there are n points in general position determining $\leq$ n-1 distinct distances. The example in the same issue (p.6060) showed that $n_1 \geq 9$, and clearly $n_1 \geq n_o \geq 8$.

A related question is as follows: let f(n) be the largest integer for which n points in general position determine at least f(n) distinct distances. Probably $f(n)/n \to \infty$, but I do not even see $\lim f(n)/n \geq 1$. It would be of some interest to determine f(n) exactly, at least for small values of n, $f(3) = 1$, $f(4) = 2$, $f(5) =$ either 3 or 4, etc.

Another question Let $x_1, \ldots x_n$ be n points in the plane, all at distances apart $d(x_i, x_j) \geq 1$. Is it true that if the diameter $\max d(x_i, x_j)$ is minimal the points are a subset of a triangular lattice? This proposition has probably been considered by many people, can one prove that it is false for all large n? The same question might be asked about $\Sigma\, d(x_i, x_j)$ or $\Sigma\, (d(x_i, x_j))^2$, the summation in both cases being for i and j satisfying $1 \leq i \leq j \leq n$.

POINTS AND DISTANCES IN THE PLANE

(JCMN 56, p.6062)

Consider a set of n points in the plane; we are interested in the distances between them. Problem (a) of the contribution above is answered by the following:-

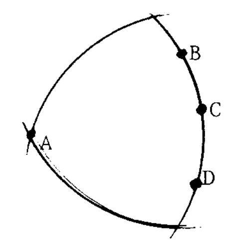
Theorem The maximum distance m, the diameter of the set, is attained no more than n times.

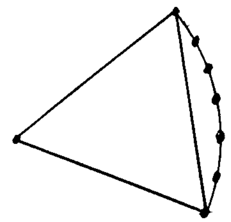
To prove this we need first a bit of graph theory.

Lemma A graph with more edges than nodes must have one node, A, joined to 3 nodes, B, C and D, all of degree 2 or more.

Proof of lemma Consider removing, one by one, all the nodes of degree 0 or 1, and at the same time removing the edges joined to them. At each stage we remove at least as many nodes as edges. Therefore the resulting subgraph has the same property as the original graph, of having more edges than nodes; and so there is one node, A, of degree 3 or more. Let B, C and D be 3 nodes joined to A. The result follows.

Proof of theorem Consider all the line segments that have the length m. They, as edges, and their end-points as nodes, form a graph. We use *reductio ad absurdum*. Suppose that there are more than n such line segments, then the lemma applies. Now consider the four points A, B, C and D given by the lemma. The three points B, C and D are all on the circle of radius m about A, and they are all within an arc of 60° or less, because no two are at distance apart more than m. Label the three points B, C and D as shown, with C in the middle. The set of n points is bounded by the three arcs of radius m from A, B and D (because m is the diameter of the set). There is no point of the set, other than A, at distance m from C; this contradicts the fact that C is a node of degree 2 or more in our graph. Therefore there cannot, as we had supposed, be more than n distances of m between the points. QED





Comment The result of the theorem is best possible, as may be seen from the sketch. Three points are the vertices of an equilateral triangle, and the other n-3 are on the circular arc shown.

Problem (b) asked about the number of times the minimum distance could be attained; is it always < 3n? Yes. Each point can be at the minimum distance from at most 6 others, and this number is attained only when the point and its 6 neighbours are part of a triangular lattice. The inequality is strict because not all points can have 6 others at the minimum distance, in fact any point on the boundary of the convex hull of the set fails to do so.

POINTS AND ANGLES IN THE PLANE

Given a set of n distinct points in the plane, they form $n(n-1)(n-2)/2$ angles, all in the closed interval between 0 and π. What can we say about these angles? One simple thing is that their mean is $60^\circ = \pi/3$. Perhaps we can say that at least one angle $\le A(n)$ and at least one $\ge B(n)$, where $A(n)$ and $B(n)$ are given by

n	3	4	5	6
A(n)	$60^\circ = \pi/3$	$45^\circ = \pi/4$	$36^\circ = \pi/5$	$30^\circ = \pi/6$
B(n)	$60^\circ = \pi/3$	$90^\circ = \pi/2$	$108^\circ = 3\pi/5$	$120^\circ = 2\pi/3$

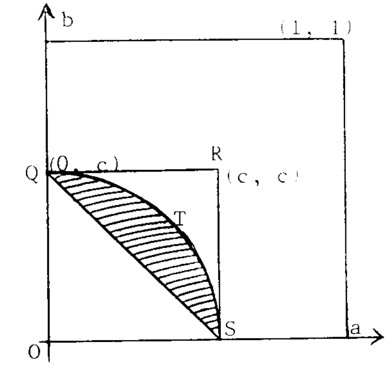
How does the table continue?

RANDOM TRIANGLES (JCMN 55, p.6029)

A. Brown

J. B. Parker suggested taking a, b and c from the uniform distribution on the unit interval, if they fail to satisfy the triangle inequalities $a+b > c$, $b+c > a$ and $c+a > b$ the triple is rejected, otherwise they are the sides of a random triangle. What is the probability of the triangle being acute?

A triangle can have only one obtuse angle, and it must be opposite the longest side. By symmetry, the probability of c, say, being the largest side is 1/3. If the opposite angle, C, is obtuse, $c^2 = a^2 + b^2 - 2ab\cos C > a^2 + b^2$, with $c^2 < a^2 + b^2$ if C is acute.



For a given value of c, it will be the largest of the numbers (a, b, c) if the point P with coordinates (a, b) lies within the square OQRS in the diagram (the probability is c^2). If P is in the triangle OQS the triple (a, b, c) is rejected because $c > a+b$ (probability $= c^2/2$). For points within the triangle QRS we have an acceptable triple, and the corresponding triangle is obtuse if $a^2 + b^2 < c^2$. If QTS is a quadrant of the circle $a^2 + b^2 = c^2$, then the shaded area is the location of P for an obtuse triangle, and the area involved is $c^2(\pi/4 - 1/2)$. The remaining portion of the square OQRS gives the location of P for an acute-angled triangle, this area is $c^2(1 - \pi/4)$. These areas can be taken as conditional probabilities (conditional on c having the largest value). So for a triple (a, b, c), with c as the largest value, we have

$$p = \text{probability of rejection} = c^2/2$$
$$q = \text{probability of an obtuse angle} = (\pi/4-1/2)c^2$$
$$r = \text{probability of an acute angle} = (1 - \pi/4)c^2.$$

Since the ratio p : q : r is the same for each value of c, integrating over c does not change this ratio.

We can go through the same argument for the case where a is the largest side or the case where b is the largest side, and the ratio p : q : r will be the same in each case, so we end up with

$$p : q : r = 1/2 : (\pi/4-1/2) : (1-\pi/4)$$
$$= 0{\cdot}5000 : 0{\cdot}2854 : 0{\cdot}2146$$

(integrating over c gives a factor 1/3; allowing for the cases where either a or b or c is the largest number cancels this factor). If we ignore the triples that are rejected, the probability of the triangle being acute is $2 - \pi/2 = 0{\cdot}4292$.

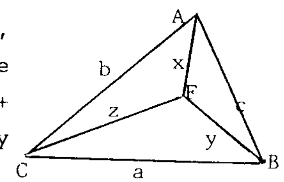
EQUATION TO SOLVE (JCMN 55 p.6054)

J.F.Rigby et al.

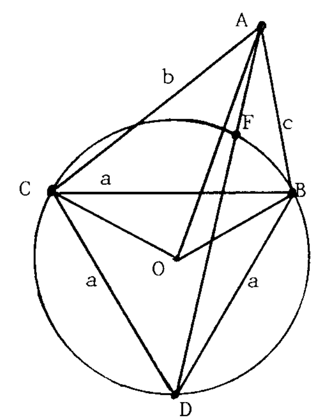
$$y^2 + yz + z^2 = a^2$$
$$z^2 + zx + x^2 = b^2$$
$$x^2 + xy + y^2 = c^2$$

As a first step we shall find one solution when a, b and c are the sides of a triangle with all angles < 120°. For any P, let the lengths PA, PB and PC be x, y and z respectively. The equations will hold if the half-lines PA, PB and PC are all at 120° to one another.

There is such a point P, call it F, sometimes called the Fermat point or the first Fermat point. It is where $x + y + z$ is a minimum (as may be seen by considering the mechanical model of a free particle P attracted by equal forces to A, B and C). There is a simple Euclidean construction for this Fermat point F.



In the diagram, let BCD be equilateral, with centre O. F is where AD meets the circumcircle of BCD.



$$AD^2 = a^2 + c^2 - 2ac\cos(B+60^\circ)$$
$$= a^2 + b^2 - 2ab\cos(C+60^\circ)$$

Adding these equations gives

$$2AD^2 = S + 4\sqrt{3}\Delta = S + 3R$$

where $S = a^2+b^2+c^2$ and Δ is the area of the triangle ABC, and $R = 4\Delta/\sqrt{3} = (1/\sqrt{3})\sqrt{(2b^2c^2+ \dots -a^4- \dots)}$

Similarly

$$AO^2 = a^2/3 + c^2 - (2ac/\sqrt{3})\cos(B+30^\circ)$$

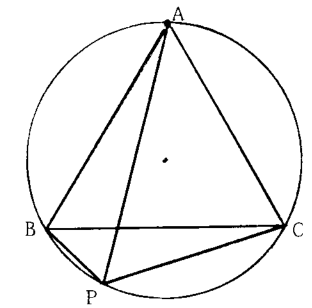
also $= a^2/3 + b^2 - (2ab/\sqrt{3})\cos(C+30^\circ)$,

$2AO^2 = S + 4\Delta/\sqrt{3} - 4a^2/3 = S + R - 4a^2/3$. From circle geometry $x.AD = AO^2 - a^2/3 = S/2 + R/2 - a^2$. Putting $T = 2AD = \sqrt{(2S+6R)}$, we find

$$xT = S + R - 2a^2$$
$$yT = S + R - 2b^2$$
$$zT = S + R - 2c^2.$$

Because this is a solution in radicals it is valid in general, for any real or complex a, b and c, and in general it gives four solutions (x, y, z) for any (a, b, c) because there are two square roots in the formula.

The formula fails in the case where $T = 0$, i.e. $\Sigma(b^2-c^2)^2 = 0$. Then, if a, b and c are real, $a^2 = b^2 = c^2 = (R + S)/2$, and the existence of an infinite set of solutions may be illustrated geometrically. Let P be any point on the circumcircle of the equilateral triangle ABC. If (as in this picture) P is on the arc BC, then $(x, y, z) = (PA, -PB, -PC)$ is a solution of the set of equations in which $a = b = c =$ the side of the triangle.



Another special case of this special case (of T being zero) is when a, b and c are the three cube roots of unity. In this case $R = S = T = 0$ and there is no solution.

BINOMIAL IDENTITY 33

A. Brown

$$\sum_{r=0}^{n}\sum_{s=0}^{n} \frac{(-1)^{r+s}}{(r+2)(s+2)} \binom{n}{r}\binom{n}{s}\binom{r+s}{r}$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n} (-1)^{r+s} \binom{n}{r}\binom{n}{s} \frac{(r+s)!}{(r+2)!(s+2)!}$$

$$= \frac{2n+3}{6(n+1)(n+2)}$$

PROBLEM IN ALGEBRA AND GEOMETRY (JCMN 56, p.6062)

If A, B, C, D and E are five points in the plane, prove

$$\begin{vmatrix} 0 & AB^2 & AC^2 & AD^2 & AE^2 \\ AB^2 & 0 & BC^2 & BD^2 & BE^2 \\ AC^2 & BC^2 & 0 & CD^2 & CE^2 \\ AD^2 & BD^2 & CD^2 & 0 & DE^2 \\ AE^2 & BE^2 & CE^2 & DE^2 & 0 \end{vmatrix} = 0.$$

FIRST SOLUTION

Terry Tao

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Taking E as origin, let $\underline{a}$ be the position vector of A, etc. Then, for example, $AB^2 = (\underline{a}-\underline{b}).(\underline{a}-\underline{b}) = \|\underline{a}\|^2+\|\underline{b}\|^2-2\underline{a}.\underline{b}$ and by subtracting the last row from each of the others and the last column from each of the others, the determinant becomes

$$\begin{vmatrix} -2\underline{a}.\underline{a} & -2\underline{a}.\underline{b} & -2\underline{a}.\underline{c} & -2\underline{a}.\underline{d} & \underline{a}.\underline{a} \\ -2\underline{a}.\underline{b} & -2\underline{b}.\underline{b} & -2\underline{b}.\underline{c} & -2\underline{b}.\underline{d} & \underline{b}.\underline{b} \\ -2\underline{a}.\underline{c} & -2\underline{b}.\underline{c} & -2\underline{c}.\underline{c} & -2\underline{c}.\underline{d} & \underline{c}.\underline{c} \\ -2\underline{a}.\underline{d} & -2\underline{b}.\underline{d} & -2\underline{c}.\underline{d} & -2\underline{d}.\underline{d} & \underline{d}.\underline{d} \\ \underline{a}.\underline{a} & \underline{b}.\underline{b} & \underline{c}.\underline{c} & \underline{d}.\underline{d} & 0 \end{vmatrix}$$

Because $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ are vectors in two dimensions, two must be linearly dependent on the other two. We may take $\underline{a}$ and $\underline{b}$ each to be linearly dependent on $\underline{c}$ and $\underline{d}$. Then by subtracting appropriate multiples of rows 3 and 4 from the first row we can reduce the first row to (0, 0, 0, 0, ?). Similarly the second row. Therefore the determinant is zero.

SECOND SOLUTION

George Szekeres and Anthony Henderson

There is a more general result, that if the five points

are in 3-dimensional space and are on a sphere then the determinant is zero. To prove this it is sufficient to consider any 5 points A, B, C, D and E on the unit sphere with centre at the origin of Cartesian coordinates.

If A = (x, y, z) and B = (u, v, w) then

$AB^2 = 2 - 2xu - 2yv - 2zw$, which, if we now regard A and B as vectors, may be written as 2A.(A-B).

Therefore (omitting the factors 2) we have to show that the following determinant is zero.

$$\begin{vmatrix} A.(A-A) & B.(B-A) & \cdots\cdots & E.(E-A) \\ A.(A-B) & B.(B-B) & \cdots\cdots & E.(E-B) \\ & \cdots\cdots\cdots\cdots & & \\ A.(A-E) & B.(B-E) & \cdots\cdots & E.(E-E) \end{vmatrix}$$

This will be so if the rows are linearly dependent, i.e. if there are non-trivial numbers α, β, ... ϵ such that:

$$\alpha(A-A) + \beta(A-B) + \dots\dots + \epsilon(A-E) = 0$$
$$\alpha(B-A) + \beta(B-B) + \dots\dots + \epsilon(B-E) = 0$$
$$\dots\dots\dots\dots$$
$$\alpha(E-A) + \beta(E-B) + \dots\dots + \epsilon(E-E) = 0$$

These five equations will hold if the following two equations hold: $\alpha A + \beta B + \dots\dots + \epsilon E = 0 = (\alpha + \beta + \dots\dots + \epsilon)E$

These two have a non-trivial solution because the four vectors A-E, B-E, C-E and D-E must be linearly dependent, so that there are scalars α, β, γ, δ (not all zero) such that

$$\alpha(A-E) + \beta(B-E) + \gamma(C-E) + \delta(D-E) = 0.$$

Putting $\epsilon = -\alpha - \beta - \gamma - \delta$ gives the required result.

Using this result about points on a sphere, the original question may be answered by considering the limit as the radius of the sphere tends to infinity; the limit process is trivial and not worth setting out here.

The result above extends easily to n+2 points on a hypersphere in n-space or to n+2 points in n-1-dimensional space. The case n = 2, with four points, A, B, C, D on a circle, is Ptolemy's Theorem, the vanishing of the determinant is equivalent to AB.CD ± AC.BD ± AD.BC = 0.

THIRD SOLUTION

J. F. Rigby
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The result can be extended:

Let A, B, C, D and E be five points in 4-dimensional Euclidean space, and denote the determinant

$$\begin{vmatrix} 0 & AB^2 & AC^2 & AD^2 & AE^2 \\ AB^2 & 0 & BC^2 & BD^2 & BE^2 \\ AC^2 & BC^2 & 0 & CD^2 & CE^2 \\ AD^2 & BD^2 & CD^2 & 0 & DE^2 \\ AE^2 & BE^2 & CE^2 & DE^2 & 0 \end{vmatrix}$$

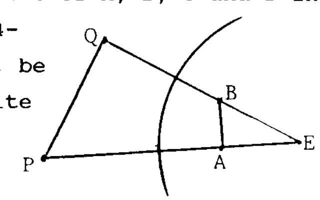
by Δ. Then $\Delta \geq 0$, with equality if and only if A, B, C, D and E lie in a plane or on a sphere. To prove this we need a lemma (perhaps not well known).

Lemma If P, Q, R and S are the vertices of a tetrahedron, then the volume V is given in terms of the edge-lengths by

$$\begin{vmatrix} 0 & PQ^2 & PR^2 & PS^2 & 1 \\ PQ^2 & 0 & QR^2 & QS^2 & 1 \\ PR^2 & QR^2 & 0 & RS^2 & 1 \\ PS^2 & QS^2 & RS^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 288V^2 .$$

Proof of Lemma Left as an exercise, because determinants take up so much space.

Proof of Theorem Let the inverses of A, B, C and D in the 3-sphere (the hypersphere in 4-space) with centre E and radius 1, be P, Q, R and S respectively. Write EA = a, EB = b, EC = c and ED = d. In the figure the triangles EAB and EQP are similar, and EA.EP = EB.EQ = 1, so that AB = ab.PQ and AC = ac.PR, etc. Therefore, by the lemma,



$$\Delta = a^4b^4c^4d^4 \begin{vmatrix} 0 & PQ^2 & PR^2 & PS^2 & 1 \\ PQ^2 & 0 & QR^2 & QS^2 & 1 \\ PR^2 & QR^2 & 0 & RS^2 & 1 \\ PS^2 & QS^2 & RS^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

$$= 288\, a^4b^4c^4d^4V^2 \geq 0.$$

There is equality if and only if P, Q, R and S are coplanar (i.e. in a 2-space), which happens if and only if the five points A, B, C, D and E all lie in a hyperplane (a 3-space), and are either in a plane (a 2-space) or on a sphere (i.e. a 2-sphere in the hyperplane).

FOURTH SOLUTION

Peter W. Donovan
(School of Mathematics, U. of N.S.W., Kensington, 2033, Australia)

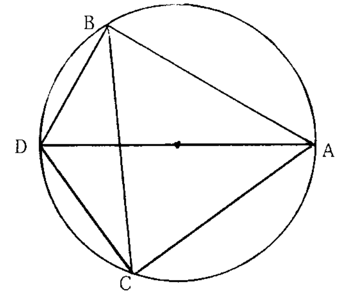
Consider 5 points A, B, C, D and E in Cartesian 3-space, with coordinates $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, etc. The matrix M of the given determinant may be factorized as $M = PRP^T$ where

$$P = \begin{pmatrix} x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ & & \cdots\cdots & & \\ & & \cdots\cdots & & \\ x_5^2+y_5^2+z_5^2 & x_5 & y_5 & z_5 & 1 \end{pmatrix} \text{ and } R = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $\det R = 8$, it follows that $\det M \geq 0$, with equality if and only if the columns of P are linearly dependent, i.e. if and only if the five points are on either a plane or a sphere.

GEOMETRICAL INEQUALITY

Esther Szekeres and Terry Tao



Given that ABDC (in that order) are on a circle with diameter AD, and that the lengths AB, BC and CA are all > 1, prove that the quadrilateral has area > 1/2.

BINOMIAL IDENTITY 34

George Szekeres
(Univ. of NSW, Kensington, NSW, Australia)

$$\sum_{i=k}^{n} (-1)^{n+i} \binom{n}{i} \binom{i}{k} \binom{n+i+1}{n+1} = \binom{n+k+1}{n-k} \binom{2k+1}{k}$$

SUM OF A SERIES

A. Brown

Find the sum of the series $\sum_{n=0}^{\infty} P_n(x)\, y^n/n!$

where $P_n(\ )$ denotes the Legendre polynomial of order n.

SET OF EQUATIONS

A. Brown

The equations

$$x + y^2 + z^6 = 0$$
$$y + z^2 + x^6 = 0$$
$$z + x^2 + y^6 = 0$$

have the obvious solution $x = y = z = 0$. How many other real solutions are there?

FAMILY OF POLYNOMIALS

For each n = 0, 1, 2, 3, ... we may define a polynomial f(n, x) of degree n as either

$$\sum (-1)^{n+k} \binom{n+k+1}{n} \binom{n}{k} x^k \quad \text{or} \quad \sum (-1)^k \binom{n+k+1}{k} \binom{n}{k} (1-x)^k$$

with both summations over k from 0 to n. The first few are:-

$$f(0, x) = 1$$
$$f(1, x) = 3x - 2$$
$$f(2, x) = 10x^2 - 12x + 3$$
$$f(3, x) = 35x^3 - 60x^2 + 30x - 4$$
$$f(4, x) = 126x^4 - 280x^3 + 210x^2 - 60x + 5$$
$$f(5, x) = 462x^5 - 1260x^4 + 1260x^3 - 560x^2 + 105x - 6$$

Problem (a) Prove that the two definitions above agree.

Problem (b) Prove that $\int_0^1 x\, f(m, x)\, f(n, x)\, dx$ is zero when $m \neq n$, and is equal to $1/(2n+2)$ when $m = n$.

Problem (c) The values f(n, 1/2) for n = 0, 1, 2, .. are

1, -1/2, -1/2, 3/8, 3/8, -5/16, ...

How does this sequence continue?

This family of polynomials is essentially a special case of the Jacobi polynomials, on which comprehensive general information is available in the books by Abramovitz and Stegun or by Erdelyi, Magnus, Oberhettiger and Tricomi; but these polynomials look so simple that we might hope for a simple explanation of their properties.

"NEWSPEAK" AND WAYS OF THINKING

Jordan Tabov
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When discussing the recent changes in Eastern Europe, we usually concentrate on the political and economic changes. But there are other important and interesting changes, and one of them is the change of the way of thinking. We can study this change by looking at language (for indeed it is hard to look at thinking in any other way) and (luckily for most James Cook readers!) the ideas come over just as clearly in the English translation.

To introduce the matter, recall the famous book "1984" by George Orwell, published in 1949. It is a story of a fictitious totalitarian state in which the government introduces step-by-step changes of words and phrases and their meanings in order to create a new language, to be called "Newspeak". The purpose of Newspeak is to impose a special kind of thinking. The slogans

War is peace
Freedom is slavery
Ignorance is strength

represent this new way of thinking.

For similar purposes a kind of "newspeak" (but probably on a smaller scale) was created during the last 45 years in the countries of the former "Eastern Block", and in particular in Bulgaria. I shall try to give some examples of the words and ideas of this "newspeak".

Let us start with the most trivial example, the word "comrade". People were obliged to use it instead of "Mister" or "Sir" or "Madam", and used it equally for friends or enemies.

The next example is more complicated; to do with the words "communism", "socialism", "capitalism", "imperialism" and "fascism". According to Lenin, imperialism is the last stage of capitalism, with a lot of additional disadvantages (let me not go into more detail here). And according to Lenin's disciples, fascism is a form of imperialism which is especially aggressive because it is doomed to failure. In contrast to them, communism is the bright future, something like a paradise to which we can look forward, and socialism is the last step before communism. Of course the description above simplifies the whole picture, but it gives an impression of the ideas and attitudes that were impressed upon the ordinary people (perhaps I should

say almost all people) by the official propaganda using the newspeak. The words "socialism" and "fascism" were treated as having diametrically opposite meanings.

Perhaps I should note that in our "developed socialist society" the "fight against world capitalism and fascism" played an important role. The most famous and honoured people were "the fighters".

Now, after this short (and perhaps superfluous to some readers) explanation, imagine the appearance about ten years ago in Bulgaria of the book "Fascism" by Zh. Zhelev (now President of Bulgaria; the "zh" is pronounced like the "s" in the English word "usual"), containing a description of fascism and its totalitarian structure of the state and the way it works. This description was made on the basis of many investigations, including the works of some famous Communist leaders of the 1930-1939 era. After the book had been on sale for a few days, it disappeared and reading of it was forbidden. Why? Of course the answer is now clear: people saw almost all the attributes of our "socialist life" described as properties of fascism. It began to look as if socialism and fascism were very similar, almost synonymous! This fact tended to destroy the important relation between the words "communism" and "fascism", and so the authorities could not let the idea spread.

Let me also mention the phrase "decadent art" which included jazz, Gershwin, the Beatles, Picasso, Dali, etc. People were arrested in the nineteen fifties for "listening to decadent music". Among other phrases were "Western influence" and "socialist emulation". In the nineteen fifties cybernetics and genetics were "bourgeois pseudo-sciences", the Russian Popov was named as the inventor of radio, and Polzunov as the inventor of the steam engine, etc.

The examples above show only one side of the problems about ways of thinking. One might hope that it would now be sufficient to explain to people the natural former meanings of words, and to get rid of some phrases, so as to make possible clear thinking on political questions. But this would not affect the firm belief of many people that it is the responsibility of "the government" or "the authorities" to look after almost everything, and to provide jobs, homes, food, etc for everybody. This is another (very important) matter, but outside our present discussion.

ENUMERATION OF PAIRS AND KUZNETSOV'S DERIVATIVE

A. M. Slin'ko

Consider the following:

> There are four rows in a rectangular table. Arbitrary natural numbers are written down in the first row. The second row is filled according to the following rule: we look through the numbers of the first row from left to right and write the number k below the number a if the a is found there for the k-th time. In a similar fashion the third row is obtained from the second, and the fourth from the third. Prove that the second and fourth rows coincide.
>
> (A. V. Kuznetsov, XV-th SMO, 1981)

How can the pairs of natural numbers be arranged in a sequence, containing each pair once and only once? Various such methods of enumeration are known, the most known and often used is the Cantor enumeration:

$$(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), \ldots$$

In this sequence all pairs with sums of the two terms equal to 2, 3, 4, ... are written in order, so that pairs with equal sums are in ascending order of the first term. It is easy to compute the position $c(x, y)$ of the pair (x, y) in the sequence. Let's call the sum $x+y$ of a pair (x, y) the "height" of this pair. In the sequence before the pair (x, y) there are $x-1$ pairs of height $x+y$, and $1 + 2 + 3 + \ldots + (x+y-2) = (x+y-2)(x+y-1)/2$ pairs of heights $2, 3, \ldots, (x+y-1)$, respectively. Hence the number of the pair (x, y) in the sequence is:

$$c(x, y) = x + (x+y-2)(x+y-1)/2$$

Given a number n, we can compute the height of the n-th pair of the sequence as the greatest h such that $(h-2)(h-1)/2 < n$. This can be expressed:-

$$h(n) = [\sqrt{(8n+1)}/2 - 1/2]$$

where [...] denotes the integer part.

If the n-th pair is $(\ell(n), r(n))$, then

$$\ell(n) = n - (h(n)-2)(h(n)-1)/2$$

and
$$r(n) = h(n) - \ell(n).$$

Conversely, any functions $c(x, y)$, $\ell(n)$ and $r(n)$ satisfying:

$$c(\ell(n), r(n)) = n, \quad \ell(c(x, y)) = x \quad \text{and} \quad r(c(x, y)) = y$$

define the enumeration:

$$(\ell(1), r(1)), (\ell(2), r(2)), \ldots \quad (\ell(n), r(n)), \ldots$$

The functions $\ell(n)$ and $r(n)$ of any enumeration each has each natural number as a value infinitely many times. Conversely, if a function $f(n)$ has each natural number as its value infinitely many times, we can construct a function $g(n)$ such that the sequence:

$$(f(1), g(1)), (f(2), g(2)), \ldots (f(n), g(n)), \ldots$$

yields an enumeration of the pairs. A. V. Kuznetsov invented a very easy way of doing this, based on the following definition:

Let $f = a_1, a_2, \ldots$ be any (finite or infinite) sequence of integers. Let's define a sequence $f' = b_1, b_2, \ldots$ by (for each i) letting b_i be the number of elements equal to a_i in the sub-sequence $a_1, a_2, \ldots a_i$. We shall call f' the <u>derivative</u> of f.

<u>Theorem 1</u> Let $f = a_1, a_2, \ldots$ be an infinite sequence of natural numbers, and let $f' = b_1, b_2, \ldots$ be the derivative of f. If the sequence f has infinitely many occurrences of every natural number, then the sequence

$$(a_1, b_1), (a_2, b_2), \ldots (a_n, b_n), \ldots$$

yields an enumeration of the pairs.

<u>Proof</u> Take any pair (m, n). It is in the sequence, since m occurs in f infinitely many times, and the n-th occurrence of m will give the pair (m, n). The pair cannot occur twice, for if $(a_i, b_i) = (a_j, b_j)$ then $a_i = a_j = m$ and $b_i = b_j = n$. This is a contradiction since we obtain two n-th occurrences of m in f.

This theorem gives a very useful property of the derivative defined above. But the most intriguing property is the one given by the following theorem.

Theorem 2 $f' = f'''$.

This is a really striking result because one could not expect to find duality here. To prove Theorem 2 we need the following lemma

Lemma Let $f = a_1, a_2, \dots a_n$ be a finite sequence of natural numbers, and let $f' = b_1, b_2, \dots b_n$ be the derivative of f. Define σ_i to be the total number of occurrences of the natural number i in the sequence f′, then $\sigma_1 \geq \sigma_2 \geq \dots$.

Proof of lemma Let $c_1, c_2, \dots c_m$ be the set of distinct numbers occurring in f, and for each i let k_i be the number of occurrences of c_i in f. Renumbering the $c_1, c_2, \dots$ if necessary, we can assume that $k_1 \geq k_2 \geq \dots k_m$.

Note that if $a_i \neq a_{i+1}$ we can transpose a_i with a_{i+1} and simultaneously b_i with b_{i+1} without loss of the property that the second sequence is the derivative of the first. Therefore we can rearrange the sequences as follows

$f = c_1, c_1, \dots c_1, c_2, c_2, \dots c_2, \dots, c_m, c_m, \dots, c_m.$
$f' = 1, 2, \dots k_1, 1, 2, \dots k_2, \dots, 1, 2, \dots k_m.$

Therefore $k_1 \geq k_2 \geq \dots$ implies $\sigma_1 \geq \sigma_2 \geq \dots$

Proof of theorem Clearly it is sufficient to prove the theorem when f is a finite sequence. Let's write down the four sequences as shown below

f	a_1	a_2	a_3	
f′	1			
f″	1			
f‴	1			

We are going to interchange adjacent columns of this table until the statement of the theorem is obvious. Of course,

these interchanges should be done so that each row (except the first) is still the derivative of the row above. The theorem will be proved if these transpositions make $f' = f'''$.

Consider f′, the second row. Step by step we shall move all the 1's in this row to the left, without jumping one over another. Then, similarly, we shall gather the 2's, then the 3's, and so on. Let σ_i be the number of i's in the sequence f′ (in the second row). By the lemma, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m$, where m is the largest number in the second row. If we really can carry out all the transpositions considered, we shall obtain the following table

f											
f′	1	1	.	.	1	2	2			m	m
f″	1	2	.	.	σ_1	1	2				σ_m
f‴	1	1	.	.	1	2	2			m	m

as required.

But we must be sure that all the transpositions conceived are permitted. To prove this, consider two adjacent columns as shown below

a_i	a_{i+1}
b_i	b_{i+1}
c_i	c_{i+1}
d_i	d_{i+1}

If $a_i \neq a_{i+1}$, $b_i \neq b_{i+1}$ and $c_i \neq c_{i+1}$, the interchange of these two columns will be permitted for sure. As we know, only the interchanges with $b_i > b_{i+1}$ were needed, and in this case obviously $a_i \neq a_{i+1}$ (for otherwise $b_i = b_{i+1} - 1 < b_{i+1}$). Moreover, applying the lemma to the sub-sequence $b_1, \dots b_i$, we conclude that the number of elements equal to b_{i+1} in it is not less than the number equal to b_i. Therefore in the subsequence

$$b_1, \dots b_i, b_{i+1},$$

the number of b_{i+1}'s is strictly greater than the number of b_i's. By the definition of the derivative, this means that $c_i < c_{i+1}$. The interchange of these two columns is therefore permitted, and the proof of the theorem is complete.

Another problem:

> Any 1000 numbers are written in a line. A second line is written down under the first according to the rule: if the number a occurs in the first line k times, then we write down k in the second line under every occurrence of a in the first line. According to the same rule the third line is obtained from the second, and so on.
>
> Prove that lines 11 and 12 are the same. Give an example of an initial line that yields eleven distinct lines.
>
> (M. I. Serov, X-th SMO, 1976)

INTEGRAL INEQUALITY

Consider polynomials of degree n (= 1, 2, ...) with integer coefficients. Let B(n) be the greatest lower bound of

$$\int_0^1 (f(x))^2 dx$$

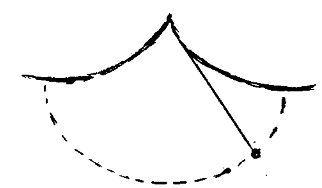
over all such polynomials. The first value is easily seen to be B(1) = 1/3. What are B(2) and B(3)? What is the asymptotic behaviour of B(n) for large n?

PENDULUM CLOCKS

Legend has it that some time about the year 1582 Galileo observed a swinging chandelier with about the same frequency as his own heart-beat, and he had the idea of using a pendulum as the essential time-keeping element of a clock.

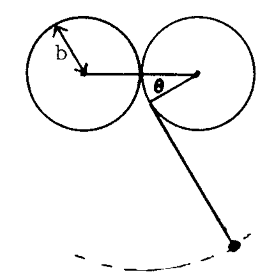
But it was Christiaan Huygens (1629-1695) who made the first pendulum clock, which he presented to the Estates-General of Holland in 1657. His book Horologium Oscillatorium (1673) gives the relevant theory: (a) that the period of a simple pendulum is $2\pi\sqrt{}(\text{length}/\text{gravity})$ for small amplitudes, (b) that the period increases with the amplitude, he probably knew the factor $1 + a^2/16$, called the "circular error", the factor showing the dependence of period on amplitude, see the calculation below. (c) that a particle free to move on a cycloid has period independent of amplitude, and (d) that the evolute (= locus of centre of curvature) of a cycloid is an identical cycloid.

A consequence of (d) is that one can (at least in the slightly unreal world of theoretical mechanics) constrain a particle to move without friction on a cycloid by suspending the particle from a light flexible string between cycloidal barriers (see diagram).



This elegant bit of theory, however, did not lead to the use of cycloidal pendulums in clocks; for of course no suspension, however flexible, can wrap itself round a curve of which the curvature is unbounded.

As a footnote to this work of Huygens we shall sketch the theory of what may be called the "involute pendulum", consisting of a particle on a flexible string constrained between two circular barriers, so that the particle moves in an involute. (see diagram). To simplify the equations we take units so that the acceleration of gravity, the length of the string and the mass of the bob are all unity. Let



θ (> 0) be the inclination of the straight part of the string to the vertical, let a be the maximum of θ, and let b be the radius of the two circular barriers. We call a the "amplitude" or "swing", though some people would call it the "semi-amplitude".

The distance of the bob below the centre of the circle is $\cos\theta + b(\sin\theta - \theta\cos\theta)$, so that the available energy is

$$\cos\theta - \cos a + b(\sin\theta - \theta\cos\theta) - b(\sin a - a\cos a)$$

but this is equal to the kinetic energy $((1 - b\theta)d\theta/dt)^2/2$. Therefore the time for a quarter-period is $\int_0^a d\theta/\dot\theta =$

$$\int_0^a \frac{(1 - b\theta)\,d\theta}{\sqrt{R}}$$

where $R = 2\cos\theta - 2\cos a + 2b(\sin\theta - \theta\cos\theta) - 2b(\sin a - a\cos a)$.

To a sufficient approximation $1 - \cos x = x^2/2 - x^4/24$ and $\sin x - x\cos x = x^3/3$. Therefore

$$R = a^2 - \theta^2 - (a^4 - \theta^4)/12 + (2b/3)(\theta^3 - a^3)$$
$$= (a^2 - \theta^2)\{1 - (a^2+\theta^2)/12 - (2b/3)(a^2+a\theta+\theta^2)/(a+\theta)\}$$

and $(1-b\theta)/\sqrt{R}$ is, to the same approximation,

$$\{1 + (a^2+\theta^2)/24 + (b/3)(a^2-2a\theta-2\theta^2)/(a+\theta)\}/\sqrt{(a^2-\theta^2)}$$

In the integral we change the variable by putting $\theta = a\sin\varphi$, and the quarter-period may be evaluated (to this degree of approximation) as

$$\int_0^{\pi/2} 1 + a^2(1 + \sin^2\varphi)/24 + (ab/3)(1 - 2\sin\varphi - 2\sin^2\varphi)/(1 + \sin\varphi)\,d\varphi$$
$$= \pi/2 + \pi a^2/32 - ab/3$$
$$= (\pi/2)(1 + a^2/16 - (2/3\pi)ab)$$

This expression (as a function of the amplitude a) has a stationary value when $a/b = 16/(3\pi) = 1{\cdot}70$.

In interpreting these formulae recall how we chose units at the start, making the pendulum length = 1, so that a is either the angular displacement of the pendulum or (to first order) the horizontal displacement of the bob from the mid-point.

Putting b = 0 we get the case of the simple pendulum, the factor $1 + a^2/16$ measures what is called the "circular error" of a simple pendulum. Its magnitude is about 30 seconds per day when a is 4° or 0·07 radians, which is typical for a "grandfather" or "long-case" clock. The significance of the circular error for accurate timekeeping is because variations in the amplitude a are inevitable. For instance, changes in air density, or in the frictional losses in the driving train, or (in the case of spring-driven clocks) in the torque given by the mainspring, will change a, and therefore change the rate of the clock. For instance a drop of 3% in the barometric pressure will increase a by 1%, and so make your grandfather clock lose

about half a second per day. In a typical spring-driven 8-day mantel clock, the mainspring torque might drop 25% from what it is in the fully-wound state, decreasing the swing from 6° to five and a half, and causing a loss of 10 seconds per day on the first day after winding.

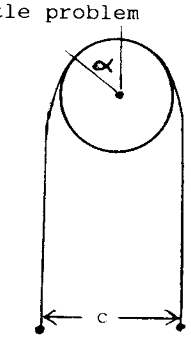
A possible benefit of using an involute pendulum now emerges. If we choose the radius b so that the amplitude a (the displacement of the pendulum bob from the central position) is close to 1·7b, then we would expect better timekeeping than with a simple pendulum.

The other major source of inaccuracy, also well-known to the early clock-makers, is the expansion with heat of the pendulum rod. With a steel rod, the clock will lose 5 seconds per day for every rise of 10°C or 18°F in temperature. The clock-makers of the 18th century knew of two remedies; the gridiron pendulum (using two metals, steel and brass, with different coefficients of expansion), and the mercury pendulum, which uses the fact that mercury expands five times as much as steel.

The pendulum of a typical grandfather (long-case) clock is not pivoted to rotate as a solid body about a horizontal axis, it is suspended by a short moderately flexible steel strip, which typically would be 0·012" (0·3 mm.) thick and 3/16 inch (5 mm.) wide. An interesting problem is the investigation of how the motion of such a pendulum differs from that of one with an ideal frictionless pivot; does this suspension aggravate or mitigate the circular error of the simple pendulum?

For those hesitant about taking that plunge into the murky depths of elasticity theory, here is a nice little problem suggested by the pendulum question.

Two particles of unit weight are fixed to the ends of a long wire, inextensible, but with the property that the bending moment at any point is K times the curvature. The wire hangs symmetrically in equilibrium over a fixed frictionless horizontal circular cylinder of radius r, see the diagram. Find the horizontal distance c between the weights, and find the angle 2α subtended by the part of the wire that is in contact with the cylinder.



ANALYSIS PROBLEM (JCMN 56 p.5084)

If a continuous function of the real variable has derivative zero at the rationals, then must it be constant?

NO. Enumerate the non-negative rationals as r_1, r_2, ... Construct an open set G from an open interval of length 1/2 containing r_1, an open interval of length 1/4 containing r_2, etc. Then G is of measure ≤ 1. Let F be the complement of G. Define the function f(x) on the positive real variable as the measure of the intersection $F \cap (0, x)$. It is continuous and is not constant, and it has derivative zero at every rational point.

Here is a variant of the problem:-

If a differentiable function of the real variable has derivative zero at the rationals, then must it be constant?

QUOTATION CORNER 37

This year we are trialling the literary assessment.

— From an Australian school magazine, 1991.