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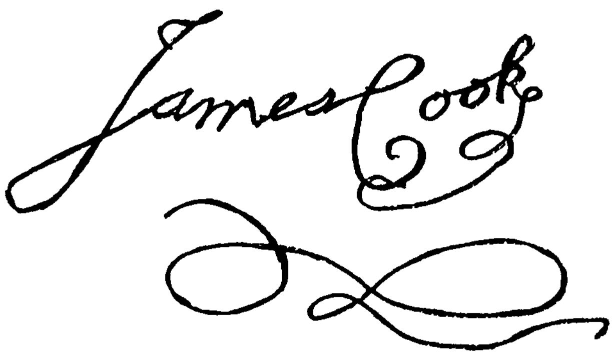

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Editor's note. This issue (number 54) starts volume 6 as we start the new year; the page numbers consequently have a discontinuity. Readers may notice a change - we now print contributors' addresses under their names.

CYCLIC INEQUALITY

Mark Kisin

(112, Summerhill Road, Glen Iris, 3146, Australia)

Let $a(1), a(2), \dots a(n)$ all be positive and have sum = 1. We extend the sequence to be periodic by putting $a(n+i) = a(i)$ for all i. Let k be an integer with $0 < k < n$.

Let $f(x)$ be the sum of a power series $\Sigma c_r x^r$ converging for $|x| \le$ all $a(k+i)/a(i)$, and having all coefficients $c_r \ge 0$.

Then $\Sigma_{i=1}^{n} a(i)f(a(k+i)/a(i)) \ge f(1)$. There is equality only when either $a(1) = a(2) = \dots = a(n) = 1/n$, or $f(x)$ is linear in x.

Proof There is a well-known inequality comparing the weighted power mean of any order $j > 1$ with the weighted arithmetic mean, $\Sigma a(i)x_i^j \ge (\Sigma a(i)x_i)^j$. It may be proved easily from the convexity of the plane set in which $x \ge 0$ and $y \ge x^j$, there is equality only when all the x_i are equal to one another.

Putting $x = a(k+i)/a(i)$, the power mean inequality gives

$$\Sigma a(i)(a(k+i)/a(i))^j \ge (\Sigma a(k+i))^j = 1$$

which is also trivially true when $j = 0$ and when $j = 1$, so that we may multiply by c_j and sum for $j = 0, 1, 2, \dots$ This gives the required result:

$$\Sigma a(i)f(a(k+i)/a(i)) \ge f(1)$$

where there is equality only when either $f(x)$ is linear in x, or the $a(k+i)/a(i)$ are all equal and therefore all equal to 1, so that the $a(i)$ are all equal.

Example 1 Put $f(x) = \exp(x)$. The result is

$\Sigma a(i)\exp(a(i+k)/a(i)) \ge e$, which also follows from the A.M-G.M inequality.

Example 2 Putting $f(x) = x^2$ and $k = 1$ gives

$$a(2)^2/a(1) + a(3)^2/a(2) + \dots + a(1)^2/a(n) \ge a(1) + \dots + a(n)$$

for any positive $a(1) \dots a(n)$, the condition of the sum being 1 may here be dropped because this inequality is homogeneous.

Example 3 Put $f(x) = 1/(m-x)$ where m is a constant such that $ma(i) > a(k+i)$ for all i. The inequality gives

$$\Sigma a(i)^2/(ma(i) - a(k+i)) \ge 1/(m-1).$$

QUOTATION CORNER 34

'The Notice dated 6 November 1990 convening the Annual General Meeting of the Company advised that there was one director retiring by rotation, Mr J.P.Cummings, and he offered himself for re-election. The Company obtained legal advice prior to the issue of this Notice of Meeting that the proper interpretation of the Articles was that only one director was due to retire by rotation. The relevant Article provides for one third of the directors to retire. The Article further provides that "if the number of directors exceeds three and is not a multiple thereof then the number nearest to but not less than one third shall retire from office." The Company has now received legal advice that in the present situation where there are 4 directors, the number who are to retire by rotation should be two.'

– From a notice to the shareholders of a mining company.

STOCHASTIC MATRICES (JCMN 53, p.5287)

Terry Tao

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A stochastic matrix is by definition a real square matrix M with every element non-negative and every column-sum equal to 1. Is it obvious that M has a positive right eigenvector for the eigenvalue one? This algebraic theorem has an interpretation in probability theory, that under certain conditions a Markov chain has a steady state.

First note that $\underline{u}' = (1, 1, \ldots)$ satisfies $\underline{u}'M = \underline{u}'$, so that (row rank being equal to column rank for any matrix over any field) there is a real right eigenvector $\underline{v}$ such that $M\underline{v} = \underline{v}$. From the triangle inequality it follows that $M\underline{w} \geq \underline{w}$, where we have written $\underline{w}$ for the vector with each component the modulus of the corresponding component of $\underline{v}$. But $\underline{u}'(M\underline{w}) = (\underline{u}'M)\underline{w} = \underline{u}'\underline{w}$, so that the vector $M\underline{w} - \underline{w}$ has every component non-negative, with the sum of the components zero, therefore every component is zero, and $\underline{w}$ is our required eigenvector.

An amusing little proof comes from interpreting the vectors as points in Cartesian space. Any vector $\underline{x}$ that has non-negative components with sum = 1 is a point in a simplex. The stochastic matrix M maps the simplex into itself, and so M has a fixed point, where $M\underline{x} = \underline{x}$.

ANALYTIC INEQUALITY

(JCMN 51, p.5228, JCMN 53, p.5276)

Let f be a positive function with positive derivative in the closed interval [0, c]. In the inequality

$$\int_0^c x/f(x)dx < \int_0^c 4/f'(x)dx$$

can the factor 4 be improved? YES.

<u>Lemma</u> $\int_a^b g(x)dx \int_a^b 1/g(x)dx \geq (b-a)^2$ for any function g positive in the interval.

Proof This can be regarded as the arithmetic-harmonic mean inequality, or as the Cauchy-Schwarz inequality for the functions $\sqrt{g}$ and $1/\sqrt{g}$.

<u>Theorem</u> If f and f′ are positive and f′ is continuous for x in the closed interval [0, c], then

$$\int_0^c x/f(x)dx < \int_0^c K/f'(x)dx$$

where $K = (9/4)\log 3 = 2{\cdot}47188\ldots$

Proof $f(x) > f(x) - f(x/3) = \int_{x/3}^x f'(t)dt$, now use the lemma,

$$(2x/3)^2 < f(x) \int_{x/3}^x 1/f'(t)\,dt.$$

Now divide by xf(x):

$$(4/9)x/f(x) < (1/x)\int_{x/3}^x 1/f'(t)dt = \int_{1/3}^1 1/f'(xs)\,ds.$$

Now integrate both sides from 0 to c, and reverse the order of integration on the RHS.

$$(4/9)\int_0^c x/f(x)dx < \int_{s=1/3}^1 \int_{x=0}^c 1/f'(xs)dx\,ds.$$

Change the variable in the inner integral on the RHS.

$$\begin{aligned} \text{RHS} &= \int_{s=1/3}^1 (1/s)\int_{y=0}^{sc} 1/f'(y)dy\,ds \\ &\leq \int_{s=1/3}^1 (1/s)\int_{y=0}^{c} 1/f'(y)dy\,ds \end{aligned}$$

Therefore $(4/9)\int_0^c x/f(x)dx < \log 3 \int_0^c 1/f'(x)dx$ QED

What further improvement is possible? If $f(x) = x^b$ where $1 < b < 2$, and c = 1, then the ratio of the integrals takes the value b, for $\int x/f dx = 1/(2-b)$ and $\int 1/f' dx = (1/b)/(2-b)$.

ANALYTIC INEQUALITY 2 (JCMN 53, p.5277)

$$4\int x^2 f^2 dx \int f'^2 dx \geq \left(\int f^2 dx\right)^2$$

if $f(x)$ is any real function for which the three integrals all exist. All integrals are from $-\infty$ to ∞. This result can be regarded as a special case of the following (Weyl's inequality, see his Gruppentheorie und Quantenmechanik, second edition, 1931). The connection with quantum mechanics will emerge below.

Theorem 1 If $f(x)$ is any real or complex differentiable function for which the three integrals all exist, then

$$4\int x^2 |f(x)|^2 dx \int |f'(x)|^2 dx \geq \left(\int |f(x)|^2 dx\right)^2.$$

The multiplier 4 is the best possible.

Proof The last bit is easy, consider the function $\exp(-x^2/2)$.

Let A and B be the square roots of $\int x^2 |f|^2 dx$ and of $\int |f'|^2 dx$, respectively. By the Cauchy-Schwarz inequality the functions $xf\overline{f'}$ and $xf'\overline{f}$ are both absolutely integrable, and their integrals over the real line are both of modulus $\leq AB$.

Consider $xf\overline{f'} + xf'\overline{f} + f\overline{f} = (d/dx)(x|f|^2)$, it is absolutely integrable, and (for any real positive u) the integral from $-u$ to u is $u(|f(u)|^2+|f(-u)|^2)$. This expression must therefore tend to some real non-negative limit as u tends to infinity. If the limit were non-zero then $|f(u)|^2+|f(-u)|^2$ would be at least c/u for some positive c for all sufficiently large u, which we know to be impossible because of the square-integrability of the function f. Therefore the limit is zero. Now we have proved that

$$\int f\overline{f} dx = -\int xf\overline{f'} dx - \int xf'\overline{f} dx$$

which is of modulus at most 2AB. Therefore

$$\left(\int |f|^2 dx\right)^2 \leq 4A^2B^2 = 4\int x^2 |f|^2 dx \int |f'|^2 dx. \qquad \text{QED}$$

The result can be extended to:-

Theorem 2 If a and b are real and if the integrals all exist, then $\left(\int |f|^2 dx\right)^2 \leq 4\int (x-a)^2 |f|^2 dx \int |f'-ibf|^2 dx$.

Proof Apply Theorem 1 to $g(x) = f(x+a)\exp(-ibx)$. It gives: $\left(\int |f(x+a)|^2 dx\right)^2 \leq 4\int x^2 |f(x+a)|^2 dx \int |f'(x+a)-ibf(x+a)|^2 dx$. QED

Theorem 1 may be rewritten as a geometrical inequality in the complex Hilbert space $L^2(-\infty, \infty)$. Let Q be the unbounded Hermitean linear operator mapping any element $\varphi(x)$ to $x\varphi(x)$. Let D be the differentiation operator, more precisely if φ is an element corresponding to a differentiable function plus a null function then $D\varphi$ is the element corresponding to the derivative of the first plus any null function. This operator D is an unbounded anti-Hermitean operator. Then we may rewrite Theorem 1 as $\|\varphi\|^2 \leq 2\|Q\varphi\|\|D\varphi\|$. Theorem 2 similarly may be written as $\|\varphi\|^2 \leq 2\|(Q-a)\varphi\|\|(-iD-b)\varphi\|$.

This form of the theorem suggests its relationship to the Uncertainty Principle of Werner Heisenberg in quantum mechanics. See our note below.

HEISENBERG'S UNCERTAINTY PRINCIPLE

Consider the quantum mechanical system consisting of a particle on a straight line. There are two observables, the coordinate Q and the momentum P. To the theoretician each

GEOMETRICAL PROBABILITY (JCMN 53, p.5281)

Suppose that three random points in the unit disc are chosen from the distribution with uniform probability density. Calculate the expectation and variance of the area of the triangle formed by the three points.

We shall show that the expectation is $35/(48\pi) = 0{\cdot}2321$, and the variance is $3/32 - (1225/2304)/\pi^2 = {\cdot}03998$, the standard deviation being $0{\cdot}1997$.

Denoting the three points by A, B and C, we shall first consider only the two points B and C, and the chord joining them. Denote the centre of the circle by O.

<u>Lemma 1</u> The joint probability of B being at radius between r and r+dr and of the length of the perpendicular from O to the chord BC being between p and p+dp, is

$$(4/\pi)(1-2p^2+r^2)(r/\sqrt{(r^2-p^2)})\ drdp.$$

Proof See Figure 1. Regard p and dp and the point B as fixed.

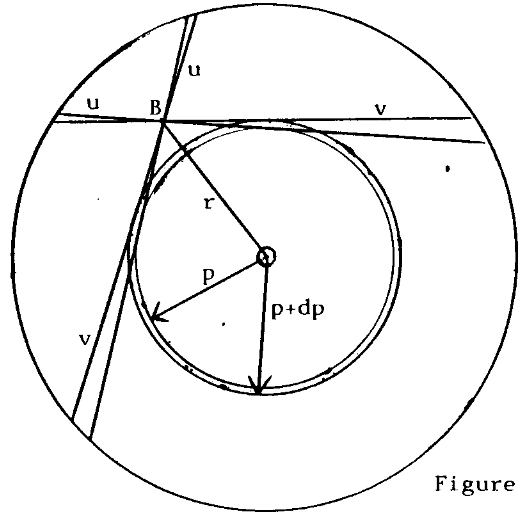

Figure 1

The probability of C being in an appropriate position is $1/\pi$ times the sum of the areas of the 4 thin triangles shown. These triangles (actually not quite triangles, but near enough) all have the angle $dp/\sqrt{(r^2-p^2)}$ at the vertex, and their long sides are either u or v as shown. From the geometry it can be seen that $u+v = 2\sqrt{(1-p^2)}$ and $uv = 1-r^2$. Therefore

$$u^2 + v^2 = (u+v)^2 - 2uv = 2(1 - 2p^2 + r^2)$$

The sum of the areas is $(u^2+v^2)dp/\sqrt{(r^2-p^2)}$. The probability of C being in one of the triangles is therefore

$$(2/\pi)(1-2p^2+r^2)/\sqrt{(r^2-p^2)}dp$$

As the probability of OB being between r and r+dr is 2rdr, the multiplication law of probabilities $P(BC|) = P(C|B)P(B|)$ gives the result that the required probability is

$$(4/\pi)(1-2p^2+r^2)rdrdp/\sqrt{(r^2-p^2)} \qquad \text{QED}$$

<u>Lemma 2</u> The probability of the perpendicular from O to BC being between p and p+dp is

$$(16/3\pi)(1-p^2)^{3/2}dp$$

Proof Integrate the expression given by Lemma 1 with respect to r from p to 1. The evaluation of the integral is elementary, change the variable to $x = \sqrt{(r^2-p^2)}$.

$$\frac{4}{\pi}\int_0^{\sqrt{(1-p^2)}}(1-p^2+x^2)xdx = \frac{4}{\pi}(1-p^2)^{3/2}(4/3) \qquad \text{QED}$$

<u>Lemma 3</u> Given the perpendicular from O to BC to be p, the probability of the radius OB being between r and r+dr is

$$(3/4)(1-2p^2+r^2)(1-p^2)^{-3/2}(r^2-p^2)^{-1/2}rdr$$

Proof This comes from the multiplicative law of probabilities. Using the symbol r to denote the proposition that OB is between r and r+dr, and the symbol p to denote the proposition that the perpendicular is between p and p+dp, Lemma 1 tells us $P(rp|)$, and Lemma 2 gives $P(p|)$. The multiplication law $P(rp|) = P(r|p)P(p|)$ gives us $P(r|p)$, which is what we want.

Lemma 4 Given that the perpendicular from O to BC is p, the expectation of the length BC is $\surd(1-p^2)$.

Proof Referring to Figure 1, where $u+v = 2\surd(1-p^2)$ and $uv = 1-r^2$, note that $u^3+v^3 = (u+v)((u+v)^2-3uv) = 2\surd(1-p^2)(1-4p^2+3r^2)$ and $u^2+v^2 = 2(1-2p^2+r^2)$.

For fixed B, the expectation of BC, which is the expectation of BC for C distributed uniformly in the four thin triangles shown, is

$$\frac{2(u^3+v^3)}{3(u^2+v^2)} = \frac{2\surd(1-p^2)}{3}\,\frac{1-4p^2+3r^2}{1-2p^2+r^2}$$

This expression has to be averaged over all points B, using the distribution of r given by Lemma 3. We get

$$\frac{1}{2(1-p^2)}\int_p^1 \frac{1-4p^2+3r^2}{1-2p^2+r^2}\,\frac{1-2p^2+r^2}{\surd(r^2-p^2)}\,r\,dr$$

To evaluate the integral, change the variable to $x = \surd(r^2-p^2)$, giving:

$$(1/2)\int_0^{\sqrt{1-p^2}} 1 + 3x^2/(1-p^2)\,dx = \surd(1-p^2) \qquad \text{QED}$$

Side-track From Lemma 2 (the distribution of p) and Lemma 4 (the distribution of BC given p) we may find the expectations of the side BC and of the area OBC (= ½p.BC). They are

$$E(BC) = 128/(45\pi) = 0{\cdot}9054$$

$$E(OBC) = 4/(9\pi) = 0{\cdot}1415$$

These are both easily calculated independently, and so we have a check on our results.

Now we go on to consider the other random point, A. The

area of the triangle ABC is h.BC/2 where h is the perpendicular distance from A to the chord BC. If p is fixed then h and BC are independent random variables, so that the expectation of the product is the product of the expectations. In what follows it is convenient to use the angle β given by $p = \cos\beta$ instead of p as our basic variable. See Figure 2.

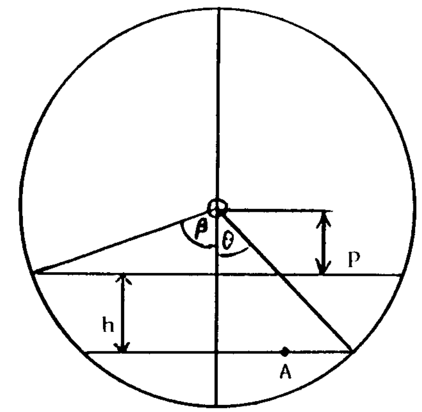

Figure 2

Lemma 5 For fixed β (with $p = \cos\beta$) the expectation of h is

$$(1 - 2\beta/\pi)\cos\beta + \sin\beta\,(5 + \cos 2\beta)/(3\pi).$$

Proof The distribution of the point A is uniform over the disc, so that the probability density of the parameter θ (see Figure 2) is

$$(2/\pi)\sin^2\theta\,d\theta = (1/\pi)(1-\cos 2\theta)d\theta \qquad \text{for } 0 < \theta < \pi.$$

As $h = |\cos\theta - \cos\beta|$, the contribution to E(h) from points A below the chord (see Figure 2) is

$$(2/\pi)\int_0^\beta(\cos\theta - \cos\beta)\sin^2\theta\,d\theta$$

$$= (2\sin\beta + \sin\beta\cos^2\beta + 3\beta\cos\beta)/(3\pi).$$

The contribution from points A above the chord is similarly (or by putting $\pi-\beta$ for β in the formula above):

$(2\sin\beta + \sin\beta\cos\beta + 3(\pi-\beta)\cos\beta)/(3\pi)$, and the sum is:

$$(4\sin\beta + 2\sin\beta\cos^2\beta + 3\pi\cos\beta - 6\beta\cos\beta)/(3\pi) \qquad \text{QED}$$

Lemma 6 For fixed β (and $p = \cos\beta$) the expectation of the area of the triangle ABC is:

$$(\tfrac{1}{2}-\beta/\pi)\sin\beta\cos\beta + \sin^2\beta(2+\cos^2\beta)/(3\pi)$$

Proof With β fixed, the random variables h and BC are independent, so that the expectation of the area is

$$E(\tfrac{1}{2}h.BC) = \tfrac{1}{2}E(h)E(BC)$$

From Lemma 4 we know that $E(BC) = \sin\beta$, and $E(h)$ is given by Lemma 5.

Theorem 1 The expectation of the area ABC is $35/(48\pi) = 0{\cdot}2321$.

Proof Take the result of Lemma 6, and average over β using the result of Lemma 2 that the probability density of β is given by the factor $16\sin^4\beta/(3\pi)d\beta$. This leads to the integral:

$$1/(3\pi)\int_0^{\pi/2}(1-2\beta/\pi)\sin 2\beta(1-\cos 2\beta)^2 + (5+\cos 2\beta)(1-\cos 2\beta)^3/(3\pi)d\beta$$

which without much difficulty may be evaluated as $35/(48\pi)$.

Lemma 7 The expectation of the square of the area ABC is 3/32.

Proof With a suitable rotation the 3 random points may be taken as (z, 0), (u, v) and (x, y). The five variables u, v, x, y and z are all uncorrelated (the expectation of a product is zero). The random variable z is statistically independent of the other four; each of u and v is statistically independent of each of x and y. The expectation of a square is 1/4 in the case of u, v, x and y, and is 1/2 in the case of z. The area

of the triangle is $\pm\tfrac{1}{2}((x-z)v-(u-z)y)$. From the square of this expression it is easy to pick out the four terms with non-zero expectation, they are the four squares. The square of xv has expectation 1/16, similarly the square of zv has expectation 1/8, etc. Hence the result.

Theorem 2 The variance of the area ABC is:

$$3/32 - 1225/(48\pi)^2 = 0{\cdot}03988.$$

Proof This comes at once from Theorem 1 and Lemma 7.

GEOMETRICAL PROBABILITY 2

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Suppose that three random points on the unit circle be chosen from the distribution with uniform probability density. Calculate the expectation and the variance of the area of the triangle formed by the three points.

GEOMETRICAL PROBABILITY 3

Two random points in the unit disc (from a uniform probability distribution) give a random line segment (ending at the two points). Find the probability that two such random line segments intersect.

FACTORIZING COMPLETE GRAPHS

(JCMN 42, p.5015, 50, p.5216, 51, pp.5235-6 and 5244)

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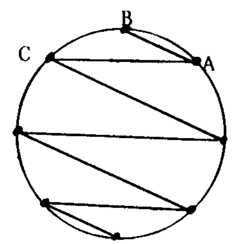
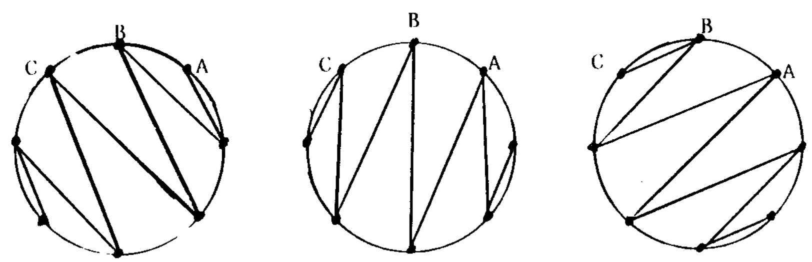
The question was asked in JCMN 51 (Factorizing Complete Graphs) whether the complete graph on 2n nodes was a union of n simple (and therefore Hamiltonian) paths. An equivalent proposition is that the complete graph on 2n+1 nodes is a union of n simple circuits. We shall consider the former question, the answer is YES.

Proof Represent the 2n nodes as equally spaced points on a circle. The n(2n-1) edges are the chords that join the points. These chords may be classified into 2n equivalence classes of parallel chords. These 2n classes are of two types; each class of the first type consists of n edges, and every node is an end of one of the edges. Each class of the second type consists of n-1 edges, all parallel to a tangent at a node.

Labelling the nodes A, B, C, ... in order as shown, take all the chords parallel to either AB or AC. This gives a simple path starting at B and ending at the diametrically opposite node. The other paths are obtained by rotation.



The idea is best understood from a diagram, the figure above shows the case where n = 4. The path shown may be rotated by 45°, 90° and 135° to give the other three paths. See below.

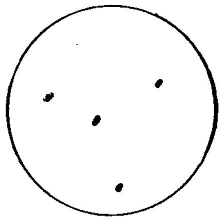


ARROWS IN THE TARGET

King Arthur had been worried by reports of a Saxon raiding party landing near Colchester (in what we now call Essex) while a fleet of longships in the Channel was threatening the castle of Camelot itself. He would have to divide his army into two. The grooms and carters prepared the baggage train for an expedition to the East, while the soldiers had a day of archery practice.

The King and Sir Lancelot set up a round target and measured a distance of thirty paces to a mark from which the archers were to shoot. "From here" commented Sir Lancelot "it is hard even to hit the target." "Of course," answered King Arthur "for my idea is that the weakest archers need the most practice. Each man is to continue shooting until he has put 4 arrows in the target. You will go first."

When Sir Lancelot had reached the required score the King showed him the target

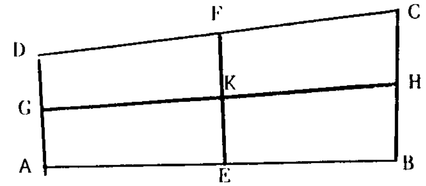


"You see the pattern," he said, "one is inside the triangle formed by the other three. That is a sign of a man in a castle. Therefore you will be in command of the garrison staying to defend Camelot. And I want you to stay here at the butts today, and to look at the pattern of four arrows in the target produced by each soldier. Those that have one arrow in a triangle as you did are to be with you in the Camelot garrison. The others, those whose arrows form a convex quadrangle, are to come with me to hunt down the raiders on the East coast."

How many of the other 335 soldiers did King Arthur expect to come with him?

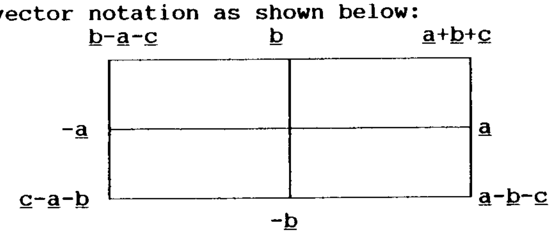
Would his expectation have been more or less if the target had been square instead of round?

HYPERBOLOID AREAS (JCMN 44 p.5058)



It is known that if a plane quadrilateral ABCD is divided into four by the joins of the mid-points of opposite sides, then the sum of the areas of a diagonally opposite pair of the small quadrilaterals is equal to the sum of the other two. In the picture above AEKG+KHCF = GKFD+EBHK. Is the same true of skew quadrilaterals? NO.

Every skew quadrilateral ABCD determines a unique parabolic hyperboloid containing all four sides. We take the origin at the centroid of the four corners, the quadrilateral may be denoted in vector notation as shown below:



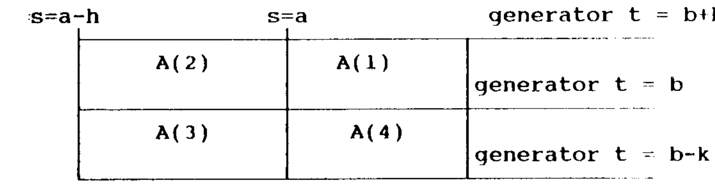
The hyperboloid is given parametrically by $\underline{x} = s\underline{a}+t\underline{b}+st\underline{c}$. The generators are the lines s = constant and t = constant. It is a parabolic hyperboloid because it meets the plane at infinity in two lines. One of these lines (where s is infinite) is where the plane at infinity meets the plane through the origin containing the vectors $\underline{a}$ and $\underline{c}$. The other is given similarly by $\underline{b}$ and $\underline{c}$.

For a simple example take the hyperboloid xy = z, given parametrically by $\underline{x} = (x, y, z) = (s, t, st)$. An infinitesimal vector in the surface is $\partial\underline{x}/\partial s ds + \partial\underline{x}/\partial t dt = (ds, dt, tds+sdt)$. The infinitesimal vector element of area is the vector product

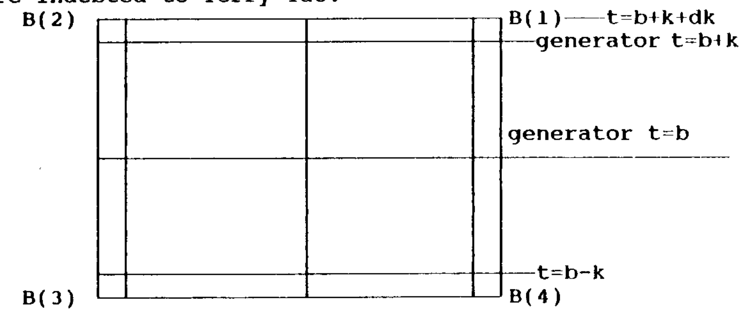
$$(ds, 0, tds) \times (0, dt, sdt) = (t, -s, 1)dsdt$$

The area of any set is therefore $\iint\surd(1+s^2+t^2)dsdt$, an integral over the appropriate region of the s-t plane.

A typical quadrilateral is formed by the generators $s = a\pm h$ and $t = b\pm k$ (these parameters a and b are not related to the vectors $\underline{a}$ and $\underline{b}$ mentioned above). Let D(a, b, h, k) be the difference A(1) - A(2) + A(3) - A(4), where A(1), ... A(4) are the areas of the four quarters of the quadrilateral as shown below.



We shall determine the second derivative $\partial^2D/\partial h\partial k$; for this idea we are indebted to Terry Tao.



D(a, b, h+dh, k+dk) - D(a, b, h+dh, k) - D(a, b, h, k+dk) + D(a, b, h, k) = B(1) - B(2) + B(3) - B(4), where B(1), ... B(4) are the areas of the small skew quadrilaterals in the corners as shown. For instance B(1) is the area bounded by the generators s = a+h, s = a+h+dh, t = b+k and t = b+k+dk. Now divide by dhdk and let dh and dk tend to zero. This gives:-

$$\partial^2D/\partial h\partial k = \surd(1+(a+h)^2+(b+k)^2) - \surd(1+(a-h)^2+(b+k)^2) + \surd(1+(a-h)^2+(b-k)^2) - \surd(1+(a+h)^2+(b-k)^2).$$

Consider the RHS of this equation as a function of h and k. Expanding it in a power series, the leading term is

$$4hk(\partial^2/\partial a\partial b)\surd(1+a^2+b^2).$$

Now we can see that if D is expanded in a power series in h and k, the leading term is $4h^2k^2(\partial^2/\partial a\partial b)\surd(1+a^2+b^2)$.

Knowing D for all infinitesimal skew quadrilaterals, does it help us to find D for any one? Yes. It is a matter of convolution products.

Consider the algebra of convolution of functions on the infinite rectangular lattice. Firstly, in one dimension

$$(1, -1)*(1, 2, 3, 4, 3, 2, 1) = (1, 1, 1, 1, -1, -1, -1, -1),$$

and similarly for longer strings, in fact (1, -1) is a convolution factor of the function consisting of a string of n values 1 followed by an equal number of values -1. The idea extends to two dimensions, for example

$$\left\|\begin{matrix} -1 & 1 \\ 1 & -1 \end{matrix}\right\| * \left\|\begin{matrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 6 & 4 & 2 \\ 1 & 2 & 3 & 2 & 1 \end{matrix}\right\| = \left\|\begin{matrix} -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{matrix}\right\|$$

Coming back to the question of areas of skew quadrilaterals, a simple example is as follows. Suppose that we bisect the sides to give four smaller quadrilaterals and then do the same again. We get 16 little areas as shown below.

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

The convolution equation:-

$$\left\|\begin{matrix} -1 & 1 \\ 1 & -1 \end{matrix}\right\| * \left\|\begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix}\right\| = \left\|\begin{matrix} -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{matrix}\right\|$$

tells us that (C+D+G+H)-(A+B+E+F)-(K+L+O+P)+(I+J+M+N) (where A, B, ...P are areas as in the diagram above) is expressible as the sum of the following 9 differences

$$\begin{aligned} &(B+E-A-F) + 2(C+F-B-G) + (D+G-C-H) \\ &+ 2(F+I-E-J) + 4(G+J-F-K) + 2(H+K-G-L) \\ &+ (J+M-I-N) + 2(K+N-J-O) + (L+O-K-P). \end{aligned}$$

Similarly any skew quadrilateral may be divided into a large number (2m)(2n) of smaller ones by partitioning the parameters s and t into equal sub-intervals.

Now we can express D(a, b, h, k) as a double integral over a rectangle in the s-t plane. The integral will be the limit of the Riemann sums given by the convolution method described above, in fact the sum:-

$$\Sigma D(x, y, h/m, k/n)(m-m|x-a|/h)(n-n|y-b|/k)$$

Letting m and n tend to infinity gives the integral,

$$\iint 4(h-|x-a|)(k-|y-b|)xy(1+x^2+y^2)^{-3/2}dxdy.$$

It is clear that this in general will be non-zero.

ACUTENESS OF RANDOM TRIANGLES

John Parker and Terry Tao

Taking three points A, B and C in the plane at random, let p be the probability that the triangle ABC has an obtuse angle at A. Then the probability of the triangle being acute is 1-3p. Before asking what p is, we must first make the question precise by defining the distribution from which our three random points are drawn. For example:-

(a) Uniform distribution on a circle.
(b) Uniform distribution in a disc.
(c) Uniform distribution in a square.
(d) Gaussian distribution over the whole plane.

For case (a) the calculation is not hard, it gives p = 1/4. For the other cases we can offer no more than some Monte Carlo estimates. They indicate values p = 0·250 for (b), p = 0·276 for (c) and p = 0·249 for (d).

EXPONENTIAL MEANS

in the book "Inequalities" by Hardy, Littlewood and Pólya, (Chapter 3) is mentioned the following "generalized mean". Take any monotonic continuous function φ. Then for any values x, y, ... having weights p, q, ... the mean m is defined by

$$(p + q + \dots)\varphi(m) = p\varphi(x) + q\varphi(y) + \dots$$

The case $\varphi(x) = x$ gives the arithmetic mean

The case $\varphi(x) = x^p$ gives the power mean of order p.

The case $\varphi(x) = 1/x$ gives the harmonic mean.

The case $\varphi(x) = \log x$ gives the geometric mean.

The essential feature of this generalized mean is worth noticing. The feature ensures, for example, that (using the notation M(...) for a mean with equal weights) we always have

$$M(M(a, b), M(c, d)) = M(a, b, c, d), \text{ etc.}$$

To explain this better, an algebraic notation is needed. A typical element of the algebra is a pair (x, p), (think of x as a value and p as its weight). In the algebra a binary operation $\perp\!\!\perp$ is defined by $(x, p) \perp\!\!\perp (y, q) = (m, p+q)$ where m is the mean given by $(p+q)\varphi(m) = p\varphi(x) + q\varphi(y)$. This operation $\perp\!\!\perp$ is easily seen to be commutative and associative, i.e. $u \perp\!\!\perp v = v \perp\!\!\perp u$ and $u \perp\!\!\perp (v \perp\!\!\perp w) = (u \perp\!\!\perp v) \perp\!\!\perp w$, so that without ambiguity we may write $u \perp\!\!\perp v \perp\!\!\perp w \perp\!\!\perp \ldots$ etc.

The idea is not purely artificial, it reflects the way that scientists treat data. As different measurements of an observable come in, the current estimate for the true value (which is a kind of mean of the available measurements) is made by taking a weighted mean of the new measurement and the previously calculated mean, for this gives the same answer as recalculating the mean from all the measurements. The estimate does not depend on the order in which the data have been received, or on how they are grouped.

An example of a generalized mean is what we might call the exponential mean, obtained by taking $\varphi(x)$ to be $\exp x$.

<u>Theorem</u> The exponential mean of any set of convex functions is convex.

Proof Firstly note that (because of the associativity mentioned above) it is sufficient to prove the result for a set of two functions. Secondly , it is sufficient to prove the result only for the case of equal weights, because $p \exp(f(x)) + q \exp(g(x)) = (1/2)\exp(f(x)+\log(2p)) + (1/2)\exp(g(x)+\log(2q))$. Thirdly, it is sufficient to consider only twice-differentiable functions, because they can approximate uniformly any convex function. The proof is now reduced to a simple exercise in differential calculus.

$$\log(\exp(f(x)) + \exp(g(x)))$$

has second derivative equal to

$$(f''\exp f + g''\exp g)/(\exp f + \exp g) + (f'-g')^2/(2+2\cosh(f-g))$$

which is non-negative if f'' and g'' are.

Another way of stating this theorem is that if two functions f and g both have the property that the graph looks convex when drawn on log-log graph paper, then the sum $f+g$ also has that property.

The theorem above holds if the exponential mean is replaced by the power mean of order $p \geq 1$, but not with the geometric or harmonic means or the power mean of order $p < 1$.