JAMES COOK MATHEMATICAL NOTES

Volume 5 Issue number 53

October 1990

James Cook

Editor and publisher: B. C. Rennie

Address: 66, Hallett Road,

Burnside, SA 5066

Australia

The James Cook Mathematical Notes is published in 3 issues per year, in February, June and October. The normal subscription rate for one year (3 issues) is 30 Singapore dollars, and for retired or unemployed subscribers it is 15 Singapore dollars, Cheques or other forms of money in any currency are acceptable, the exchange calculated at the rate when the money was sent. The subscription rate includes air mail postage to anywhere outside the country where the issue is posted (now Australia). Readers sometimes ask "Why Singapore dollars?", the answer is that when I first took over publication in 1983 I had it printed in Singapore and posted from there, I fixed the price as 30 Singapore dollars then, and have not changed it since. Some exchange rates at the time of writing (August 1990) are:

One Singapore dollar = 69 cents Australian

= 29 pence U.K.

= 56 cents U.S.A

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EDITOR'S NOTE

A reader has suggested that we include with each contribution the address of the author. In this issue we have not been able to comply with that request, because all the contributions are from the Editor, an unfortunate state of affairs which has not happened before. Will contributors to the next issue please let me know if and how they would like their addresses printed?

ANALYTIC INEQUALITY (JCMN 51, p.5228)

The function f and its derivative f' are positive and continuous in the closed interval [0, c]. Prove that

$$\int_0^c x/f(x)\,dx < \int_0^c 4/f'(x)\,dx.$$

A first look at this problem may be disheartening. If formally we put $f(x) = x^5$ the inequality becomes $J < 4J/5$ where $J = \int_0^c x^{-4}\,dx$, which looks wrong until we remember that the integral diverges. Hopes for a simple proof begin to fade. We give below a proof in two parts. The second part (the reduction to the case of a convex function) which we might expect to be the easier of the two, is obvious in principle but more obscure in detail than the first part.

Lemma The result holds if the derivative f'(x) is non--decreasing.

Proof $f(x) > f(x)-f(x/2) \geq (x/2)f'(x/2)$ by the mean value theorem, and so $x/f(x) < 2/f'(x/2)$. Therefore

$$\int_0^c x/f(x)dx < \int_0^c 2/f'(x/2)dx = \int_0^{c/2} 4/f'(t)dt < \int_0^c 4/f'(x)\,dx.$$

QED

Theorem The result above.

Proof Given the derivative f'(x), let g(x) be the non--decreasing function on [0, c] that is equimeasurable with f'(x) in the interval. Equimeasurability is the property that for any y the subsets on which $g(x) < y$ and on which $f'(x) < y$ have the same measure; the idea is discussed in Hardy, Littlewood and Polya's Inequalities as "rearrangements". We may construct g as the inverse function to $G(y) = \mu\{\text{Set on which } f'(x) > y\}$, provided that we make suitable provision for the possible discontinuities of G which arise from the sub-intervals in which g and f' are constant. Now define h(x) by $h(0) = f(0)$ and $h'(x) = g(x)$. It may be verified that $h(x) \leq f(x)$ in the interval. Finally it follows that

$$\int x/f\,dx \leq \int x/h\,dx < \int 4/h'\,dx = \int 4/f'\,dx \qquad \text{QED}$$

There remains the question of whether the factor 4 in the result could be improved to a smaller value. Looking at values of the ratio $\int x/f\,dx \Big/ \int 1/f'\,dx$ of the integrals, it is hard to find any value greater than $\pi^2/6 = 1{\cdot}644\ldots$.

ANALYTIC INEQUALITY 2

Is the following true?

$$4\int (xf(x))^2dx \int (f'(x))^2dx \geq \left(\int (f(x))^2dx\right)^2$$

where the integrals are from $-\infty$ to ∞. Make any assumptions you like about the function f being smooth and being small at infinity.

BOOK REVIEW

"I AM RIGHT - YOU ARE WRONG" by Edward de Bono; Penguin Books (1990), 203pp. Hardback, £14.99 in U.K. and $29.95 in Australia or Canada.

Mathematicians, more than most people, are aware of the relations between thought and language. When we have mathematical ideas we try to put them into words so as to communicate the ideas to others, but the language does not have the beauty and simplicity of the original idea. To complicate matters we have logicians telling us that mathematics consists of writing rows of symbols and manipulating them according to the axioms and rules of procedure, and that if we follow the rules we get what we may call a theorem. But is this a complete picture of mathematics?.

The thesis of Edward de Bono in this book is that thinking and language are both complicated, and that there is no simple relation between them. Nowadays we can ask how our thinking compares with the activity of a computer, and we have the evidence of neuro-physiology about how the grey matter inside our skulls actually behaves. The author points out that a set of data coming into the brain is not generally stored in its raw state, it is processed, and the processing depends on previous data acquisitions; he describes the brain as a self-organizing system, and emphasises how it differs from a computer. He asks - can we understand the brain or is it too complicated? - and gives an answer that although a simple system may have a complicated behaviour there are sometimes simple things that we can say about the behaviour of very complex systems.

Language has evolved a long way from the first meaningful grunts exchanged between our ancestors of a million years ago. In the hands of poets and scholars it became a beautiful and powerful instrument, but recently in the hands of journalists and politicians it has become a device for manipulating people by obstructing clear thought. The author writes "Language is probably the single most important barrier to progress", and asserts that people can be taught to think more effectively; indeed his methods are now being used in schools in many countries.

The title "I am right - you are wrong" has appended to it a sub-title "from this to the New Renaissance: from rock logic to water logic". This tends to puzzle the reader, perhaps it is meant to. The idea of the title is to sum up how conventional reasoning (as in parliamentary debate or courts of law) is tied to the logical structure that we inherited from ancient Greece and from mediaeval philosophers; this logical structure is essentially concerned with propositions being either right or wrong. This kind of reasoning mixed with conventional language gives a dangerous mixture, because it is easy with words to construct a half-truth and then to get people arguing about whether it be true or false. And such a division of people into two opposing factions is often the objective of politicians,

just as it is of journalists.

The book is divided into 60 short sections. At first it seems disjointed and unplanned, but the central purpose eventually becomes clear. It is all about the way that most people think, how ineffective it is, and what might be done to improve it. This is important to those of us who have spent most of our working lives teaching people mathematics - have we helped them to think accurately and creatively? or have we taught them only how to pass examinations? Can talent be taught? and if we could teach it would it degenerate into just another kind of book-learning?

The author's view is that analytic (or critical) thinking is taught and used a lot, but creativity (or "lateral thinking" or talent) is sadly deficient. Your reviewer would maintain that both are sadly deficient. Even in a university mathematical class it is hard to get critical thought; I sometimes used to set questions of the form: "List the errors, if any, in the following argument: ...", but without much success. Likewise it is hard to encourage, or even detect, any kind of creative thought in an undergraduate class. The average student wants to be told what to memorize for the examination, and what methods to use for each possible type of problem, and in some universities the authorities even encourage this attitude. The objective of a university (in the sense of the body of people with the ultimate power) these days is simply to seek the greatest possible benefit to itself. Its attitude to the students is to give them what they want to keep them happy, easily done because the average student simply wants a degree, not a good education.

This is an example of pursuing the objective of self-preservation, an attitude commonly found in government departments and semi-public institutions, it has been discussed and documented in the books of C. Northcote Parkinson. The author describes this behaviour and gives it the name "ludecy". He makes the point (which is an axiom of classical game theory and decision theory as well as a part of Christian doctrine) that "human nature is selfish, greedy and aggressive, and will always be so", so that the phenomenon of "ludecy" is what we should expect to observe. One might speculate about governments rediscovering these facts about human nature and commissioning game theorists to plan systems that would work well in spite of the faults of the people operating them, a problem with an analogue in electronics, where engineers have had some success in making reliable devices out of less reliable components.

All in all, a puzzling, worrying and important book - try to get your friends to read it.

B.C.R.

BOOK REVIEW

THE EMPEROR'S NEW MIND Subtitle: Concerning Computers, Minds and the Laws of Physics, by Roger Penrose, Oxford University Press, 466 pages, £20 in U.K., $46·95 in Australia.

This book gives you value for money, with explanations of Turing machines, computability (or recursiveness), Gödel-type theorems, Mandelbrot's set, tiling, classical mechanics and electromagnetism, special and general relativity, quantum mechanics and the currently popular theories of cosmology and of the mechanism of the brain. Chapter 6 is a careful examination of quantum mechanics with emphasis on the foundations, and in particular on their weaknesses. Should the state vector of a system have a reality of its own, or should it be subjective, describing only the observer's knowledge of the system? Here is all you need to know about the hidden-variable hypothesis, about Schrödinger's cat and about Einstein's dice.

If you want to tile your kitchen floor the easy way, you can get square or hexagonal tiles from the local tile shop and lay them just by putting each one where it seems to fit; this gives you a periodic pattern; your method is like the way nature makes crystals. But if you want an almost periodic tiling you find it more difficult. Even when you have provided yourself with appropriate tiles the laying is a non-local problem, to lay each tile you need a plan of the distant parts of the pattern for guidance. So how does a quasi-crystal grow? (a quasi-crystal might be described as an almost periodic tiling of space by atoms) The book contains a photograph of a quasi-crystal of an aluminium-lithium-copper alloy. In this connecton the author mentions that there is no algorithmic solution to the tiling problem, of whether copies of a given finite set of polygons can tile the plane. The plan that you need for the tiling of your kitchen floor has to come from a brain, not a computer.

There are minor criticisms that might be made. The index, though comprehensive, is printed in a smaller type-face than the main body of the text. Of all the parts of any book the index is the one part most likely to be read by someone on a ladder in a dark corner of a library, so it should be in the largest type face. The author uses kilometres for describing the size of the universe, and consequently feels constrained to put in a footnote that 10^{20} = 100000000000000000000. The five-page explanation of complex numbers does the job well, but surely anyone unfamiliar with complex numbers would be rash in attempting to read the book at all.

In his "Note to the reader" the author mentions the warning that each equation in a book cuts down the readership by a half, saying that nevertheless he was including a lot of formulae, and suggesting the merits of just letting one's eye slide gently over any intimidating array of symbols. In fact he does a very good job of supplementing or replacing formulae by diagrams. On only one such decision about printing a formula would your reviewer be critical - although there is much about black holes, there is nowhere given the Schwarzschild metric:

$$ds^2 = -dr^2/(1-2m/r)-r^2(\sin^2\theta d\varphi^2+d\theta^2)+(1-2m/r)dt^2$$

which needs no more than second year calculus for a modest degree of understanding, and which sheds a lot of light on black holes. It tells you all that you are ever likely to know about what it would be like in the black hole (where $0 < r < 2m$), and on the other side (where $r < 0$). More deeply hidden in the equation, no doubt, are clues to the boundaries ($r = 0$ and $r = 2m$) where these regions join.

Perhaps this book is a sign that the world of theoretical physics is now in a state like that of 100 years ago, when the current theories, although highly successful in technology, were beginning to show signs of fundamental inconsistencies, at least to the few who had eyes to see the signs. It is interesting to recall that Lord Kelvin, towards the end of a long life spent applying mathematics to technology (and growing wealthy from it) was moved to give a lecture "Nineteenth century clouds over the dynamical theory of heat and light" in April 1900 at the Royal Institution. Eight months later, in December, Max Planck first announced his quantum theory, and five years later Albert Einstein introduced special relativity.

The idea linking all the parts of the book is to enquire into consciousness, into whether our thinking is like the operation of a computer, into how we think and what we think, and indirectly into the question of what mathematics is and what a proof is. Do we create a theorem or discover it? Does a computer feel anything? Does it know of its own existence without being told?

In all, an excellent book.

B.C.R

GEOMETRICAL PROBABILITY

Suppose that three random points in the unit disc are chosen from the distribution with uniform probability density. Calculate the expectation amd the variance of the area of the triangle formed by the three points.

QUOTATION CORNER 33

" ... It is anticipated that the person appointed will have a successful record in research and in attracting research funding."

- From an advertisement for a chair in LMS Newsletter 171.

Perhaps the work of selection committees would be eased if the mathematical world were to follow the example of the professsional tennis circuit, keeping computerised records of the leading players, with their "season's winnings" and "total career winnings".

BINOMIAL IDENTITY 21

(JCMN 45, p.5072)

$$\sum_{k=m}^{n} (-1/2)^k \binom{n}{k}\binom{2k}{k-m} = \begin{cases} 0 & \text{if } n+m \text{ is odd} \\ (-1/2)^n \binom{n}{n/2+m/2} & \text{otherwise.}\end{cases}$$

Proof Firstly note that $\sum_{k=0}^{n} (-1/2)^k \binom{n}{k}(1+x)^{2k} x^{-k}$

$$= (1 - (1+x)^2/(2x))^n = (-\tfrac{1}{2x} - \tfrac{x}{2})^n$$

$$= (-1/2)^n x^{-n} (1+x^2)^n$$

In this identity we shall equate the coefficients of x^m in the two sides. The coefficient in the LHS is

$$\sum_0^n (-1/2)^k \binom{n}{k} (\text{coefficient of } x^{m+k} \text{ in } (1+x)^{2k})$$

$$= \sum (-1/2)^k \binom{n}{k}\binom{2k}{m+k} = \sum (-1/2)^k \binom{n}{k}\binom{2k}{k-m}$$

where the summation may be taken over the k for which $m \leq k \leq n$, for the other terms are zero.

The coefficient of x^m in the RHS of the equation above is

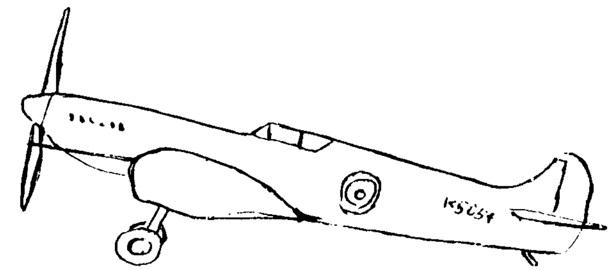
$$(-1/2)^n (\text{coefficient of } x^{m+n} \text{ in } (1+x^2)^n)$$

which is zero if m+n is odd, and otherwise is $(-\frac{1}{2})^n \binom{n}{n/2+m/2}$.

STOCHASTIC MATRICES

Suppose that M is a real square matrix with every element non-negative, and with every column-sum = 1. Is it obvious that there must exist a non-zero column vector $\underline{x}$ such that $M\underline{x} = \underline{x}$? with $\underline{x} \geq 0$

EXAMPLE IN MECHANICS



Prototype Spitfire, first flown in March 1936.

Why would an aircraft be designed with a tail skid instead of a tail wheel? The main reason was the difficulty of making a good castoring wheel, i.e. a wheel of which the axle is freely pivoted about a vertical axis. The difficulty will be familiar to many readers from the trolleys used in supermarkets and in airports, they show a kind of instability called "shimmy". A simple explanation is given below.

Firstly we need a little pure mathematical result.

<u>Theorem</u> Consider the cubic equation $z^3 + Bz^2 + Cz + D = 0$, with real coefficients. The three roots all have negative real parts if and only if $0 < D < BC$ and $0 < B$.

<u>Proof</u> Let the roots be r_1, r_2 and r_3. Then

$$r_1 + r_2 + r_3 = -B$$
$$r_1r_2 + r_2r_3 + r_3r_1 = C$$
$$r_1r_2r_3 = -D$$

Either all three roots are real or one is real and two complex, in the latter case let r_1 be the real root.

<u>Only if</u> If $B \leq 0$ the result follows. If $D \leq 0$ then $r_1r_2r_3 \geq 0$; in the case of all roots real, one is ≥ 0, and in the case of only one real root $r_1|r_2|^2 \geq 0$.

If $BC \leq D$ then (using $f(z)$ to denote the cubic) $f(-B) = D - BC \geq 0$. Because $f(-\infty) = -\infty$, there must be a real root $r_1 \leq -B = r_1 + r_2 + r_3$, and therefore $r_2 + r_3 \geq 0$. It follows that one of these two roots must have real part ≥ 0.

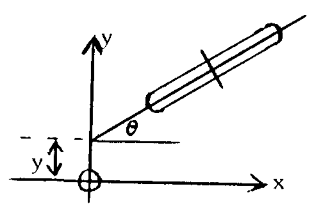
<u>If</u> Suppose that $0 < D < BC$ and $0 < B$. Then $f(-B) = D - BC < 0$, and $f(0) > 0$. There must be a real root r_1 between $-B$ and 0. Therefore $r_2 + r_3 = -B - r_1 < 0$. Now there are two cases to consider.

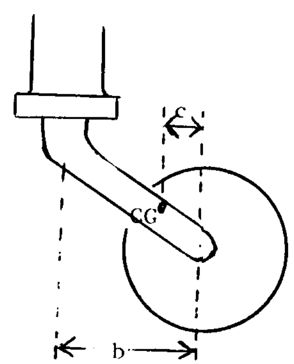
Case 1, all roots are real. Since r_1 and $r_1 r_2 r_3 = -D$ are both negative, it follows that $r_2 r_3 > 0$, and so r_2 and r_3 have the same sign, but their sum is negative and so both are.

Case 2, r_2 & r_3 are complex, they have the same real part, which must be $(r_2 + r_3)/2 < 0$.

Now all cases are dealt with, and the theorem is proved.

Getting down to our mechanical problem, suppose that a vehicle is constrained to move at constant speed V in the negative x-direction, and that the castor wheel on a slightly flexible strut rolls without slipping on the plane $z = 0$. We look into the stability of the steady state.





If there is oscillation then the vertical strut holding the castor wheel will be subject to forces in the y-direction.

The equation of motion for the wheel, axle and forks, moving as a rigid body in the horizontal plane is

$$Pb = J\ddot{\theta} - mc(\ddot{y} + b\ddot{\theta})$$

where m is the mass, P is the force in the y-direction exerted by the pivot on the strut, J is the moment of inertia about the vertical axis through the point of contact, and c is the distance from this axis to the centre of gravity (see drawing).

The equation of motion for the bottom of the strut may be assumed to be of the form $P = M\ddot{y} + Ry$. Also there is a constraint (for no slipping) $\dot{y} + b\dot{\theta} + V\theta = 0$. In this way we get a linear differential equation with constant coefficients for y and θ. There is a solution with time factor exp(pt) if

$$(Mbp^2 + mcp^2 + Rb)y - (J - mbc)p^2\theta = 0$$

$$py + (bp + V)\theta = 0$$

The parameter p must be a root of the cubic

$$Ap^3 + Bp^2 + Cp + D = 0$$

where $A = J + Mb^2$ $\quad B = V(Mb + mc)$

$C = Rb^2$ $\quad D = RbV$

The system will be stable if $AD < BC$, by our theorem above. This condition is $J < mbc$. A more elaborate calculation is possible, taking into account damping effects and the gyroscopic effect of the rotating wheel, but it still comes down to our theorem on the roots of a cubic.

On the same topic of aircraft tail wheels, it may be noted that an aircraft with a good stable frictionless castoring tail wheel is in some danger of the undignified manoeuvre of "ground-looping", a result of directional instability when taxying. The theory is a good exercise for young mathematicians.

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