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EDITORIAL NOTE

Firstly we must record our grateful thanks to Jamie Simpson, who at short notice took over all the work of being Editor and Publisher for JCMN 49. The reason for this arrangement was that my wife and I found ourselves able to go away to visit our son Alastair and his family on the island of St. Helena, where he had recently taken a job. St. Helena is one of the most inaccessible parts of the world. For both passengers and mail the only transport to and from the island is by a ship which sails from England and South Africa in alternate months. The island was first settled by colonists from England about 1673, and was governed by the East India Company until it was transferred to the Crown in 1833. Until the opening of the Suez Canal in 1869 it was a busy place with hundreds of ships per year calling for food and water. When Captain Cook stopped there in May 1771 there were two ships of the Royal Navy and twelve of the East India Company anchored off the island. His Journal makes no mention of going ashore, his only comment on the island was "Pleasant weather." I can confirm about the weather, and add that the hills are lovely for walking.

NON-BINOMIAL IDENTITY

Jamie Simpson

For real positive x, $\sum_{n=1}^{\infty}[2^{-n}x + \tfrac{1}{2}] = [x]$

where [...] denotes integer part.

BINOMIAL IDENTITY 29

(JCMN 49, p.5190)

This identity

$$\sum_{j=0}^{[m/2]}\binom{m}{2j}\binom{2j}{j}2^{m-2j} = \binom{2m}{m}$$

appeared in our last issue with Merlin's prediction that this issue would contain a proof. He was even more right than usual, for we are able to print five proofs below.

A comment on notation — it is often convenient to regard all summations as being over the set of all integers, for it is agreed that the binomial coefficient $\binom{n}{r}$ must be regarded as zero for negative r and for positive integer $n < r$. Also there is usually no need to specify the dummy variable over which the summation is taken, for a little thought will often show that only one is logically possible. These conventions make the typing easier.

BINOMIAL IDENTITY 29(1)

Marta Sved

S is a set consisting of m married couples. The task is to count the number of ways in which it is possible to choose m people from the set S of 2m. For any choice, let there be 2i couples from which we choose either both or neither, of these 2i couples we must choose both partners from exactly i, to get our total right. Then there is just one to be chosen from each of the remaining $m-2i$ couples. Therefore the required number is $\sum\binom{m}{2i}\binom{2i}{i}2^{m-2i}$. But also we know that the number is $\binom{2m}{m}$, and so the identity is established.

BINOMIAL IDENTITY 29(2)

Bob Clarke

By the trinomial theorem

$(\sqrt{x} + 1/\sqrt{x})^{2m} = (x + 2 + 1/x)^m = \sum \frac{m!}{i!\,j!\,k!}\,x^{i-k}\,2^j$ where the summation is over all non-negative i, j and k with sum = m.

Now pick out the constant term on both sides of the equation.

$$\binom{2m}{m} = \sum \frac{m!}{i!\,i!\,(m-2i)!}\, 2^{m-2i} = \sum \binom{m}{2i}\binom{2i}{i} 2^{m-2i}$$

BINOMIAL IDENTITY 29(3)

Mark Kisin

King Arthur was leading a mixed army of Britons and Caledonians. At the end of a day's march he called Sir Lancelot and showed him the m points where sentries would have to be posted round the camp, saying "The watch must be changed in the middle of the night, and so we need 2m men. We shall use m Britons and m Caledonians. You must draw up the orders showing which duties are to be taken by Britons and which by Caledonians. Then each of those two groups will work out among themselves the men to go on watch."

Sir Lancelot began thinking about the number of ways in which he could make the plan. "If at r of the posts Britons take both watches," he said to himself "then it must be at the same number r of posts that Caledonians take both watches. Then at all the other $m-2r$ posts I can choose either BC (a Briton taking the first watch and a Caledonian the second) or CB. Therefore the number of ways is $\sum \binom{m}{r}\binom{m-r}{r} 2^{m-2r}$ if I allow for all possible choices of r." Soon afterwards he realised that the answer was simply $\binom{2m}{m}$, so that he had found the result

$$\sum \binom{m}{r}\binom{m-r}{r} 2^{m-2r} = \binom{2m}{m}$$

which by the multiplication rule $\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$ may also be written $\sum \binom{m}{2r}\binom{2r}{r} 2^{m-2r} = \binom{2m}{m}$.

BINOMIAL IDENTITY 29(4)

In a trigonometric polynomial (i.e. a sum of finitely many multiples of exp(inx) with integer n) the constant term is rather special, and we often have to pick it out from the others. One importance of the constant term is that it is the integral mean of the function, for which we use the notation IM.

$$\mathrm{IM}\cos^n x = 2^{-n}\,\mathrm{IM}(e^{ix}+e^{-ix})^n$$

is clearly zero if n is odd, and if n is even, say $n = 2j$, it is $2^{-2j}\binom{2j}{j}$. But also

$$\mathrm{IM}\cos^{2m}x = \mathrm{IM}(\tfrac{1}{2}+\tfrac{1}{2}\cos 2x)^m = \mathrm{IM}(\tfrac{1}{2}+\tfrac{1}{2}\cos x)^m = 2^{-m}\sum\binom{m}{r}\,\mathrm{IM}\cos^r x$$

The terms of the sum are zero for odd r, and so we put $r = 2j$, and (using the value above for $\mathrm{IM}\cos^{2j}x$) we have

$$2^{-2m}\binom{2m}{m} = \mathrm{IM}\cos^{2m}x = 2^{-m}\sum\binom{m}{2j}2^{-2j}\binom{2j}{j}$$

which is essentially the result remembered by Merlin.

Binomial Identity 29 (JCMN 49, p. 5190)

C. C. Rousseau

For integer $m > 1$,

$$\sum_{j=0}^{\lfloor m/2\rfloor}\binom{m}{2j}\binom{2j}{j}2^{m-2j} = \binom{2m}{m}.$$

Proof. We use the fact that $2^{-2j}\binom{2j}{j}$ is the average value of $\cos^{2j} x$ over an interval of length $n\pi$ $(n = 1, 2, \ldots)$. Thus

$$\begin{aligned}
\sum_{j=0}^{\lfloor m/2\rfloor}\binom{m}{2j}\binom{2j}{j}2^{m-2j} &= \frac{2^m}{2\pi}\int_0^{2\pi}\left\{\sum_{j=0}^{\lfloor m/2\rfloor}\binom{m}{2j}\cos^{2j}\theta\right\}d\theta \\
&= \frac{2^m}{2\pi}\int_0^{2\pi}\frac{1}{2}\{(1+\cos\theta)^m + (1-\cos\theta)^m\}\,d\theta \\
&= \frac{2^{2m}}{4\pi}\int_0^{2\pi}\left\{\cos^{2m}\frac{\theta}{2} + \sin^{2m}\frac{\theta}{2}\right\}d\theta \\
&= \binom{2m}{m}.
\end{aligned}$$

In the last step, we again use the average value result and that it holds for sine as well as cosine.

SPHERICAL TRIANGLE PROBLEM

Yang Lu (via George Szekeres)

Let ABC be a spherical triangle, and let D, E and F be the mid-points of the sides. If DEF is equilateral, then must ABC be equilateral? If not then determine all non--equilateral ABC for which DEF is equilateral.

SPHERICAL TRIANGLE INEQUALITY

(JCMN 46, p.5104)

This problem from John Parker noted that we had seen in these pages three inequalities for the sides of a spherical triangle,

$$a \leq b + c$$

$$\sin\tfrac{1}{2}a \leq \sin\tfrac{1}{2}b + \sin\tfrac{1}{2}c$$

and $$\sin a \leq \sin b + \sin c,$$

and asked if more generally $\sin ka \leq \sin kb + \sin kc$ for any k in the open interval between 0 and 1.

The answer is YES. To see this, observe that the three numbers ka, kb and kc satisfy the four inequalities (each is less than the sum of the other two, and the sum of all three is less than 2π) that are characteristic of the lengths of the sides of a spherical triangle. From these four the required inequality follows analytically as explained by Archie Brown in JCMN 44, pp.5057-5058.

ORTHOCENTRES

The intersection of the altitudes (the perpendiculars from each vertex to the opposite side) of a plane triangle is called the orthocentre. Why? Of what is it the centre?

Binomial Identity 28 (JCMN 49, p. 5186)

C. C. Rousseau

$$\sum_{r=0}^{n} \binom{2r}{r}\binom{2n-2r}{n-r}(n-2r)^2 = n(n+1)2^{2n-1}$$

Proof. For simplicity of notation, let

$$a_r = 2^{-2r}\binom{2r}{r}, \qquad (r = 0, 1, 2, \ldots).$$

Squaring both sides of the binomial series

$$(1-z)^{-1/2} = \sum_{r=0}^{\infty} a_r\, z^r \qquad (|z| < 1), \tag{1}$$

and comparing cofficients of z^n on each side, we obtain

$$\sum_{r=0}^{n} a_r a_{n-r} = 1. \tag{2}$$

[This is J. B. Parker's elegant solution of Binomial Identity 27 in JCMN 49, p. 5191.] Differentiating (1), we find the series expansion

$$z(1-z)^{-3/2} = 2\sum_{r=0}^{\infty} r a_r\, z^r \qquad (|z| < 1),$$

and by the same technique as before (squaring both sides and comparing coefficients of z^n) obtain

$$\sum_{r=0}^{n} a_r a_{n-r}\, 4r(n-r) = \frac{n(n-1)}{2}. \tag{3}$$

From (2) and (3), it follows that

$$\sum_{r=0}^{n} a_r a_{n-r}\, (n-2r)^2 = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2},$$

which is just another form of the identity

$$\sum_{r=0}^{n} \binom{2r}{r}\binom{2n-2r}{n-r}(n-2r)^2 = n(n+1)2^{2n-1}.$$

PROBABILITY PROBLEM

Jamie Simpson

I have a desk with an infinite number of drawers, labelled D_1, D_2, ... I have left my car-keys in one of these drawers and I cannot remember which one. However I know that the probability that the keys are in drawer D_i is 2^{-i}. If I look in the drawer containing the keys I have a probability $p = 0{\cdot}5$ of finding them. I can look in the drawers in any order, looking in each drawer as many times as I wish.

Question 1. What search strategy minimises the expected number of times that I look in a drawer?

Question 2. With this optimal strategy, what is the expected number of times?

Questions 3 and 4. How do the answers to 1 and 2 change if I must start with D_1? and if after looking in any D_i I must look in one of D_{i-1}, D_i and D_{i+1}?

BINOMIAL IDENTITY 30

C. C. Rousseau

$$\sum_{k=0}^{n} (-4)^k \binom{n+k}{2k} = (-1)^n(2n+1)$$

SUMS OF POWERS OF DISTANCES IN REGULAR POLYGONS AND POLYHEDDRA

Jordan Tabov

Let $A_1A_2 \ldots A_n$ be a regular n-gon. For what positive k does the sum $\sum(A_rM)^k$ remain constant as the point M moves round the circumcircle? Note that we shall consider both non-integer and integer values of k.

This problem is not new; I don't know its origin but the case $n = 4$ is in "Hungarian Mathematical Problems" by L. Csirmaz (Budapest, 1979). An introduction and some results and discussions appeared in the author's article "On an application of Rolle's theorem in geometry" (The Education in Mathematics, 1981, pp.17-20). For the case where k is a positive integer a proof using trigonometric sums was given by O. Mushkarov "Trigonometric polynomials and regular polygons" (Mathematika, Sofia, 1982). I also had asked some students to find extensions to regular polyhedra, and results by K. Yanakiev, A. Petrova and S. Dokov were published in Mathematika in 1981-82.

This problem is closely related to the geometrical identity of Mark Kisin in JCMN 48, p.5168. Indeed its answer reduces the identity to the problem of finding the sum of the k th powers of the diagonals of a regular n-gon.

Our purpose here is to give an exposition of the results mentioned above. First we shall sketch the solution of Mushkarov.

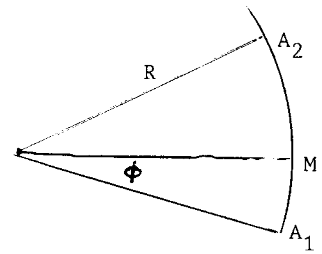
Lemma 1 If k is a positive integer, then

$$\sin^{2k}\phi = 2^{-2k}\binom{2k}{k} + (-1)^k 2^{1-2k}\sum_{j=0}^{k-1}(-1)^j\binom{2k}{j}\cos 2(k-j)\phi,$$

$\sin^{2k+1}\phi = (-4)^{-k}\sum_{j=0}^{k}(-1)^j\binom{2k+1}{j}\sin(2k-2j+1)\phi$.

Put $\sum_{j=1}^{n}(A_jM)^k = S(n, k)$ and let

R be the radius of the circumcircle.



Theorem 1 Let k be a positive integer.

a) If $1 \leq k \leq n-1$, then

$$S(n, 2k) = n\binom{2k}{k}R^{2k};$$

b) If $k \geq n$, then

$$S(n, 2k) = n\binom{2k}{k}R^{2k} + 2nR^{2k}\sum_{j=1}^{[k/n]}(-1)^{jn}\binom{2k}{k-jn}\cos jn\phi.$$

Theorem 2 Let k be a positive integer, then

$$S(n, 2k+1) = 2R^{2k+1}\sum_{j=0}^{k}(-1)^j\binom{2k+1}{k-j}\cos(j+\tfrac{1}{2})(\phi-\tfrac{\pi}{n})/\sin\tfrac{(j+\frac{1}{2})\pi}{n}$$

Corollary. Let k be a positive integer, and $\sum_{n,k} = \sum_{2}^{n}(A_1A_r)^k$.

a) If $1 \leq k \leq n-1$, then $\sum_{n,2k} = \frac{1}{2}n^2\binom{2k}{k}R^{2k}$.

b) If $k \geq n$ then $\sum_{n,2k} = \frac{1}{2}n^2\binom{2k}{k}R^{2k} + n^2R^{2k}\sum_{j=1}^{[k/n]}(-1)^{jn}\binom{2k}{k-jn}$.

c) For any k, $\sum_{n,2k+1} = nR^{2k+1}\sum_{j=0}^{k}(-1)^j\binom{2k+1}{k-j}\operatorname{cotan}\frac{(2j+1)\pi}{2n}$.

Lemma 2 If the trigonometric polynomial

$$f(\phi) = a_0 + a_1\cos\phi + \dots + a_m\cos m\phi$$

has 2m+1 different roots in the interval $(0, 2\pi)$, then $a_0 = a_1 = \dots = a_m = 0$.

Theorem 3 The only positive values of k for which the sum S(n, k) does not depend on M as M describes the circumcircle, are k = 2, 4, ... 2n-2.

Theorem 4 Let M be any point on the arc $A_{2n+1}A_1$ of the circumcircle of the regular 2n+1-gon $A_1A_2 \dots A_{2n+1}$. Then the non-negative k for which

$$(A_1M)^{2k+1} + (A_3M)^{2k+1} + \dots + (A_{2n+1}M)^{2k+1} = (A_2M)^{2k+1} + (A_4M)^{2k+1} + \dots + (A_{2n}M)^{2k+1}$$

are k = 0, 1, 2, ... , n-1.

Theorem 5 Let M be any point on the circumcircle of the regular 2n-gon $A_1A_2 \dots A_{2n}$. Then the positive k for which

$$(A_1M)^{2k} + (A_3M)^{2k} + (A_5M)^{2k} + \dots + (A_{2n-1}M)^{2k} = (A_2M)^{2k} + (A_4M)^{2k} + \dots + (A_{2n}M)^{2k}$$

are k = 1, 2, 3, ... , n-1.

This completes the exposition of the results from the article of Mushkarov.

The following theorem contains the results established in the four notes by the three students mentioned above.

Theorem 6 In three dimensions let $A_1A_2 \dots A_n$ be a regular polyhedron with n vertices, let the point M describe the circumscribed sphere, and let k be a positive integer. Then the sum $\sum_{j=1}^{n}(A_jM)^k$ is constant (i.e. does not depend on M) if and only if

a) k = 2 or k = 4 for n=4 (tetrahedron)

b) k = 2 or k = 4 or k = 6 for n = 6 (octahedron)

c) k = 2 or k = 4 or k = 6 for n = 8 (cube)

It should be noted that the values of k quoted in the above theorems 3, 4, 5 and 6, are the only ones having the respective properties, not only among the positive integers, but among all positive numbers. This can be proved using Rolle's theorem. For the regular dodecahedron and the regular icosahedron the problem is still open. And what happens in higher dimensions?

GEOMETRICAL IDENTITY

(JCMN 48, p.5168)

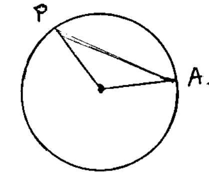
Consider numerical integration of a periodic function. We shall investigate the idea of estimating the integral mean to be the arithmetic mean of the values at n equally spaced points, with spacing equal to $1/n$ times the period. Let $f(x)$ be a function of period 2π (it may be regarded as a function on the unit circle). Let the points be $x_r = 2\pi r/n$ for $r = 1, 2, \ldots n$; this involves no loss of generality because any point can be taken as the starting point of the period of a periodic function. If the function is e^{imx} (with m an integer) then the sum of the values is

$$e^{2\pi im/n} + e^{4\pi im/n} + \ldots + e^{2n\pi im/n}$$

Of course we can calculate this sum as a geometric series, but an easier way is to note that multiplication by $e^{2\pi im/n}$ will permute the terms cyclically and therefore leave the sum unchanged. If m/n is not an integer then the sum is zero, because zero is the only complex number with the property that $kz = z$ for some $k \neq 1$. If m/n is an integer the argument tells us nothing, but that does not matter because each term is 1 and the sum is n.

Now we begin to see something about the accuracy of our numerical integration rule. If the function is e^{imx} then the rule gives the correct answer of zero whenever m/n is not an integer; if $m = 0$ the rule gives the correct answer for the integral mean; but if m/n is a non-zero integer the rule gives the wrong answer 1, it should be 0. Now suppose that the function $f(x)$ is the sum of its Fourier series, as all good functions are, say $f(x) = \sum_{-\infty}^{\infty} c_r e^{irx}$. Then the error in the numerical integration arises only from the terms in which r is a non-zero multiple of n. In fact the error in our estimate of the integral mean is bounded by $\sum_{k=1}^{\infty} |c_{kn}| + |c_{-kn}|$.

Our rule gives the exact answer if $f(x)$ is a trigonometric polynomial of degree less than n, and this gives us a way to approach Mark Kisin's problem in JCMN 48.



The points $A_1, A_2, \ldots A_n$ are uniformly spaced round the unit circle, and P is any point on the circle. What is $\sum (PA_r)^{2m}$?

In polar coordinates let A_r be $(1, 2r\pi/n)$ and let P be $(1, \theta)$. The distance PA_r is $|2\sin(\theta/2 - r\pi/n)|$ and $(PA_r)^2 = 2 - 2\cos(\theta - 2r\pi/n)$. The sum required is $\sum_{r=1}^{n} (2 - 2\cos(\theta - 2r\pi/n))^m$ and may be regarded as the sum of the values of the periodic function $f(x) = (2 - 2\cos x)^m$ at the n equally spaced points $x_r = \theta - 2r\pi/n$. The integral mean of the function is

$$\mathrm{IM}(2 - e^{ix} - e^{-ix})^m = 2^{2m}\mathrm{IM}\sin^{2m}x/2 = 2^{2m}\mathrm{IM}\sin^{2m}x$$
$$= (-1)^m \mathrm{IM}(e^{ix} - e^{-ix})^{2m} = \binom{2m}{m}.$$

If $m < n$ then the sum of the values is n times the integral mean, i.e. $\binom{2m}{m}n$.

Using these techniques a little extension of the previous result is possible, we shall calculate the sum in the case $m = n$.

As before we write the sum $\sum (PA_r)^{2n}$ as $\sum f(x_r)$ where $x_r = \theta - 2\pi r/n$ and $f(x) = (2 - 2\cos x)^n = (2 - e^{ix} - e^{-ix})^n = F(x) + G(x)$, putting $G(x) = 2(-1)^n \cos nx$, so that $F(x)$ is a trigonometric polynomial of degree $n-1$ and the previous result applies to it. The integral mean of $G(x)$ is zero, so that the integral mean of $F(x)$ is the same as that of $f(x)$ which (as we saw above) is $\binom{2n}{n}$.

Since $G(x_r) = 2(-1)^n \cos n\theta$ for all r, and (by the theory above) $\sum F(x_r) = n\,\mathrm{IM}\,F(x) = n\binom{2n}{n}$, we find that the required sum is $\sum f(x_r) = \sum F(x_r) + \sum G(x_r) = n\binom{2n}{n} + 2n(-1)^n \cos n\theta$. which has maxima when P is opposite to a vertex A_r.

MONTE CARLO INTEGRATION

(JCMN 46, p.5104 and 47, p.5129)

Recall from our earlier contributions under this title that if we put $g = \frac{1}{2}\sqrt{5} - \frac{1}{2}$ and write x_r for the non-integer part of rg, and take any continuous function f(x) on the unit interval, then there are interesting things to observe about

$$S(N) = \sum_{r=1}^{N} \left(f(x_r) - \int_0^1 f(x)\,dx\right).$$

Our previous contributions considered particularly the values of S(N) when N goes through the Fibonacci numbers, 1, 2, 3, 5, 8, 13, ... Now let us return to the case of general N.

Gerry Myerson's comment that S(N) was unbounded in the case f(x) = x is the motivation for considering S*(N), defined like S(N) except that the summation is from -N to N, with the proviso that in the term for r=0 we interpret $f(x_0)$ to mean $\frac{1}{2}f(0) + \frac{1}{2}f(1)$.

Investigation of S*(N) is essentially investigation of the accuracy of taking

$$\sum_{r=-N}^{N} f(x_r) \,/\, (2N+1)$$

as a numerical estimate for the integral $\int_0^1 f(x)\,dx$. For the function f(x) = x it is clear that S*(N) = 0 and the numerical estimate is exact, whatever the value of N.

Let L and U be the lower and upper bounds of S(N) for N = 1, 2, 3, 4, ..., and similarly let L* and U* be the bounds of S*(N). Now let us look at what seems to emerge from the output of my computer. Firstly (perhaps not surprising but not obvious) the bounds L* and U* are also the lower and upper limits of S*(N) as N tends to infinity. Secondly L* = -U*,

I did not expect this. Thirdly there are the numerical coincidences (to five decimal places) that may be observed in the table of results below.

f(x)	$\int_0^1 f(x)dx$	U*=-L*	U	L
$6x(1-x)$	1	1	1	0
$30x^2(1-x)^2$	1	1·02819	1·01409	-·01409
$140x^3(1-x)^3$	1	1·20275	1·10137	-·10137
$630x^4(1-x)^4$	1	1·46470	1·23235	-·23235
$4x(1-x^2)$	1	1	1	0
$4x(1-x)(2-x)$	1	1	1	0
$12x^2(1-x)$	1	1	1·04658	-·00053
$12x(1-x)^2$	1	1	1·00053	-·04658
$12x(1-x)(3x-1)$	1	1	1·63198	-·34860
$12x(1-x)(2-3x)$	1	1	1·34860	-·63199
$4/(1+x^2)+2x-4$	$\pi-3$	·14159	·14255	-·01227

Some of the bounds are attained, for example in the case of the function $12x(1-x)^2$ the sum S(N) attains its lower bound of $-15g^{12} = 2160g-1335$ at N=3.

CONVERGENCE QUESTION

Let $g = \frac{1}{2}\sqrt{5} - \frac{1}{2}$. Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2|\sin n\pi g|}$$

Does it converge? If so then one might ask if this property of g is shared by other quadratic irrationals.

EDGE-COLOURED GRAPHS
(JCMN 42, p.5015)

Given any n, for what m is it true that a complete graph on m nodes with n edge-colours must contain a monochromatic circuit? The previous contribution under this title showed that for $n = 2$ the answer is all $m \geqslant 5$. Now it can be shown that for $n = 3$ the answer is all m from 7 upwards.

Lemma A graph in which the number of edges is greater than or equal to the number of nodes must contain a circuit. This is obvious.

Theorem If the complete graph on $m \geqslant 7$ nodes has its edges coloured in 3 colours then it contains a monochromatic circuit.

Proof First take $m = 7$. There are 21 edges, therefore there are at least 7 edges in one of the colours. The graph made up of these edges and the nodes to which they join must (by the lemma) contain a circuit. This is a monochromatic circuit in the original graph. The case of $m > 7$ presents no difficulty, you just consider any 7-node subgraph.

Example The complete graph sketched below has 6 nodes and 15 edges, with 5 edges in each of 3 colours, but it has no monochromatic circuit. The three colours are represented on the drawing by curved, straight and dotted lines.