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CONTENTS

CAPTAIN COOK

This issue of JCMN has been edited by Jamie Simpson, as the regular Editor, Professor Basil Rennie is holidaying in St Helena. Readers will be aware that Basil is not the only celebrity to have spent time on this island.

"Arriving at St. Helena Cook was extremely annoyed in finding that Hawkesworth's edition of Cook's Journal not only contained inserted material, which insulted the worthy people of St. Helena, but contained an entirely fallacious statement that Cook had approved of Hawkesworth's manuscript. With apparently justified indignation he wrote:

I am well convinced that the island in many particulars has been misrepresented. It is no wonder that the account which is given of it in the narrative of my former Voyage should have given offence to all the principle Inhabitants. It was not less mortifying to me when I first read it, which was not till I arrived now at the Cape of Good Hope, for I never had the perusal of the Manuscript nor did I ever hear the whole of it read in the mode in which it was written, notwithstanding what Dr Hawkesworth has said to the Contrary in the Introduction. In the narrative my Country men at St Helena are charged with exercizing a wanton cruelty over their slaves, they are also charged with want of

ingenuity in not having Wheel carriages, Wheeel Barrows and Porters Knotts to facilitate the task of the labourer. With respect to the first charge, I must say, that perhaps, there is not a Europeean settlement in the world where slaves are better treated than here, out of the many of whom of whom I asked these questions not one had the least shaddow of a complaint. The Second charge, tho' of little consequence is however erronious for I have seen every one of the three Articles that are said not to be used on the island; they have Carts which are drawn by oxen, and Wheel Barrows have been used on the island from the first settlement and some are sent annually out from England in the store Ship."

from "The Explorations of Captain James Cook in the Pacific as told by Selections of his own Journals 1768-1779" by A. Grenfell Price

R. L. GRAHAM PROBLEM

At the last Summer Research Institute in Newcastle Graham began a talk with this problem.

Simplify:

$$(x-a)(x-b)\dots(x-z)$$

k-FOLD REAL FUNCTIONS

H. Burkill and B.C. Rennie

If k is a positive integer or ∞, a k-fold real function on an interval is one that takes each of its values exactly k times. An example given by Marta Sved (JCMN 31, p.3180) of a 2-fold function on $\mathbb{R}$ with infinitely many discontinuities led to the formulation of a general problem ([1] JCMN 39, pp. 4174-4180), namely the evaluation of $\lambda(k)$, the minimum number of discontinuities of a k-fold real function on an open interval, and of $\mu(k)$, $\nu(k)$, the corresponding numbers for half open and compact intervals, respectively. The question was taken up by Jordan Tabov ([2] JCMN 39, pp.4188-4190) who, in particular, proved that $\lambda(2)=\nu(k)=\infty$. The results of [1] and [2] together are summarised as follows:

$$\lambda(2) = \infty, \qquad \mu(2) = 1, \qquad \nu(2) = \infty;$$
$$\lambda(3) = 0, \qquad \mu(3) = 1, \qquad \nu(3) = 1;$$
$$1 \le \mu(4) \le 2, \qquad \nu(4) = 1;$$

for $k = 2r+1$ $(r=1,2,\ldots)$,

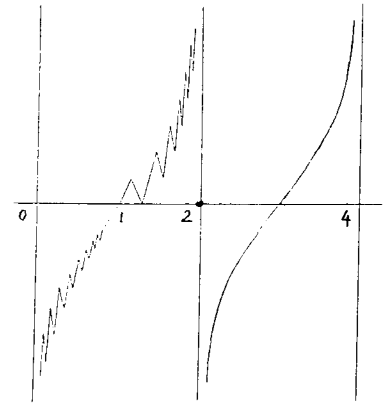
$$\lambda(k) = 0, \qquad 1 \le \mu(k) \le r, \qquad 1 \le \nu(k) \le r;$$

for $k = 2r$ $(r=2,3,\ldots)$,

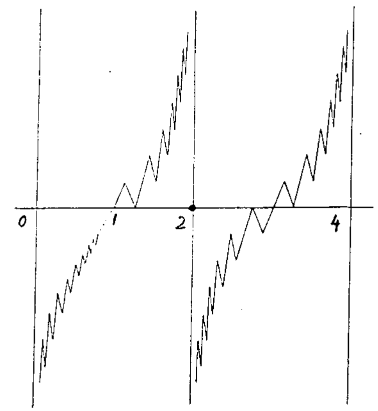
$$1 \le \mu(k) \le r, \qquad 1 \le \nu(k) \le r-1;$$
$$\lambda(\infty) = 0, \qquad \mu(\infty) = 0, \qquad \nu(\infty) = 0.$$

We now show that $\lambda(2r) = 1$, $\mu(2r) = 2$ $(r=2,3,\ldots)$ and obtain small bounds for $\mu(2r+1)$, $\nu(2r)$, $\nu(2r+1)$.

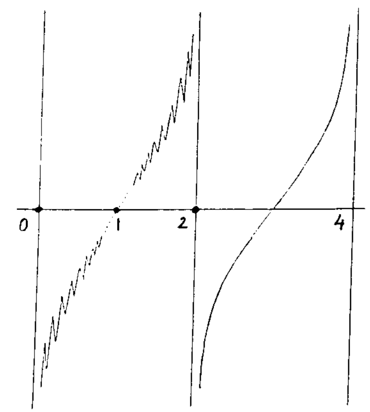
In the first place, examples (A_1), (A_2), etc show that $\lambda(4)\le 1$, $\lambda(6)\le 1$, etc. The left hand portion of each graph (A_1), (A_2) consists of the graph of a continuous 3-fold function on $(0,1)$ with range $(-\infty,0)$, followed by the upper half of the graph of a continuous 3-fold function on $(0,2)$ with range $(-\infty,\infty)$. In (A_1), (A_2), etc., the right hand portions are the graphs of continuous 1-fold, 3-fold, etc. functions on $(2,4)$, all with range $(-\infty,\infty)$.



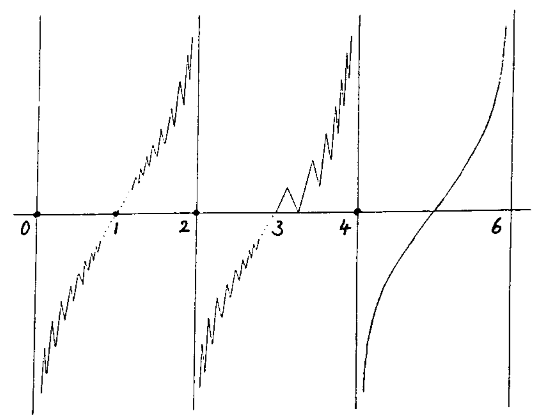
(A_1)



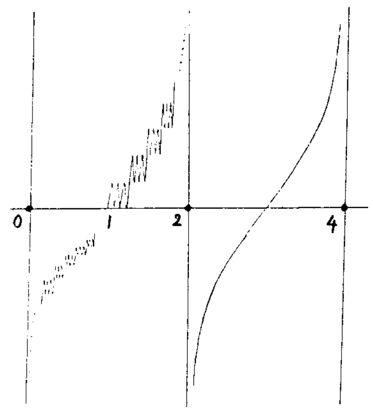
(A_2)



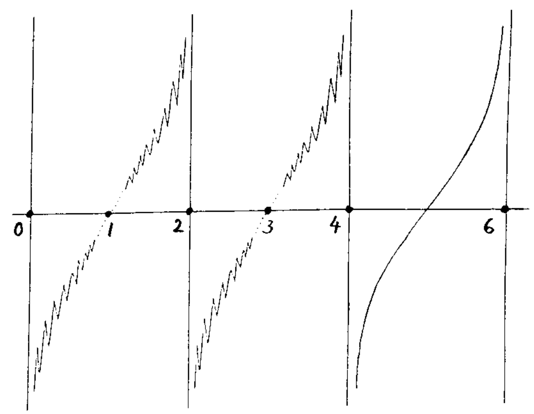
(B_1)



(C_1)



(D_1)



(E_1)

THEOREM 1. For $k = 4,6,8,\ldots$, $\lambda(k) = 1$.

Proof: In view of the examples (A_i) it is only necessary to prove that $\lambda(k)\geq 1$.

Let $k = 2r$, where $r\geq 2$, and suppose that the function $f\colon (0,1)\to \mathbb{R}$ is k-fold and continuous.

We may assume that 0 is a value of f. Then there exist points a_i $(i=1,\ldots,2r)$ such that $0<a_1<a_2<\ldots<a_{2r}<1$ and $f(a_i)=0$ for $i=1,\ldots,2r$.

For each $i\in\{1,2,\ldots,2r-1\}$, at least one of

$$m_i=\inf\{f(x):a_i\leq x\leq a_{i+1}\},\quad M_i=\sup\{f(x):a_i\leq x\leq a_{i+1}\}$$

is not 0. Now $m_i<0$ for at least r values of i or $M_i>0$ for at least r values of i; for, if neither statement is true, there are r values of i for which $m_i=0$ and r values of i for which $M_i=0$, so that there is an i for which $m_i = M_i= 0$, which is impossible. Suppose that $m_i<0$ for s values of i, where $s\geq r$, and denote them by $j_1,\ldots,j_s$, where

$$m_{j_1}\leq \ldots \leq m_{j_s} < 0.$$

Note that

$$m_{j_1} = \inf\{f(x):a_1\leq a_{2r}\}.$$

Also f takes the value m_{j_1} at points b_i $(i=1,\ldots,2r)$, where $b_1<b_2<\ldots<b_{2r}$. There are now two possibilities.

(i) $b_1,\ldots,b_{2r}\in (a_1,a_{2r})$. For $i=1,\ldots 2r-1$, put

$$N_i=\sup\{f(x):b_i\leq x\leq b_{i+1}\}.$$

Then $N_i>m_{j_1}$ $(i=1,\ldots,2r-1)$ since f cannot be constant in $[b_i,b_{i+1}]$. If N is the least of the N_i, then each value in (m_{j_1},N) is taken at least twice in each interval (b_i,b_{i+1}) and so at least $2(2r-1)$ times in (b_1,b_{2r}). As $2(2r-1)>2r$, we have a contradiction.

(ii) At least one of $b_1,\ldots,b_{2r}$ is not in (a_1,a_{2r}). We may suppose that $b_1<a_1$. Then every value in $(m_{j_s},0)$ is taken at least once in $(0,a_1)$ and at least twice in each of the intervals $(a_{j_1},a_{j_1+1}),\ldots,(a_{j_s},a_{j_s+1})$, therefore at least $2s+1$ times altogether. Since $2s+1>2r$, we again arrive at a contradiction.

Examples $(B_1),(C_1)$ prove that $\mu(4)\leq 2$, $\mu(7)\leq 3$. For both (B_1) and (C_1) the graphs on $(0,1)$, $(1,2)$ are those of 3-fold functions with ranges $(-\infty,0)$, $(0,\infty)$, respectively. In (C_1) the graph on $(2,4)$ is the basic one common to the examples (A_i). In (B_1) and (C_1) the right hand portion can be replaced by the graphs of k-fold functions $(k=3,5,\ldots)$ to yield $\mu(6)\leq 2$, $\mu(8)\leq 2$, $\ldots,\mu(9)\leq 3$, $\mu(11)\leq 3,\ldots$. Using also results from [1] we now have

$$1\leq\mu(5)\leq 2,\qquad\qquad 1\leq\mu(k)\leq 3\ (k=7,9,11,\ldots).$$

THEOREM 2. For $k = 4,6,8,\ldots$, $\mu(k)=2$.

Proof: After the examples B_i we need only show that $\mu(k)\geq 2$.

Let $k=2r$, where $r\geq 2$, and suppose that the function $f\colon[0,1) \to \mathbb{R}$ is k-fold. It is known ([1], p.4176) that f cannot be continuous. Suppose that f has exactly one discontinuity. By [1], pp.4176-4177, we may assume that f is bounded; and then from [1], p.4180, the discontinuity is simple and $\lim_{x\to 1^-} f(x)$ exists.

The rest of the proof is divided into several sections. A letter, when used in more than one section, need not retain the same meaning from one section to another.

I: f discontinuous at $a \in (0,1)$.

The essence of the argument is to show that our assumptions lead to the identity $f([0,a))=f((a,1))$.

(i): $f([0,a))\subseteq f((a,1))$.

Suppose that there exists $Y \in f([0,a))$ such that $Y\notin f((a,1))$. Since $f((a,1))$ is an interval, $f(x)<Y$ for all $x\in(a,1)$ or $f(x)>Y$ for all $x\in(a,1)$.

(1): $f(x)<Y$ for $x\in(a,1)$.

Let

$$M=\sup\{f(x):0\le x<a)\}.$$

If f assumes the value M in $[0,a)$, then, since $f(x)<Y\le M$ for $x\in(a,1)$, the $2r$ points in $[0,1)$ at which f takes the value M are all in $[0,a]$. Thus there exist points a_i $(i=1,\dots,2r-1)$ such that $0\le a_1<a_2<\dots<a_{2r-1}<a$ and $f(a_i)=M$ for $i=1,\dots,2r-1$. Moreover $\lim_{x\to a^-}f(x)\le M$. Some values less than M are then assumed at least twice in each interval (a_i,a_{i+1}), $(i=1,\dots,2r-2)$ and at least once in (a_{2r-1},a), therefore at least $4r-3$ times altogether. Since $4r-3>2r$, we have a contradiction.

If f does not assume the value M in $[0,a)$, then $M=\lim_{x\to a^-}f(x)$. Take β so that $\beta\ne f(a)$ and

$$\max(f(0),Y)<\beta<M.$$

There are points b_i $(i=1,\dots,2r)$ such that $0<b_1<b_2<\dots<b_{2r}<a$ and $f(b_i)=\beta$ for $i=1,\dots,2r$. Also, for $i=1,\dots,2r-1$, let

$$m_i=\inf\{f(x):b_i\le x\le b_{i+1}\},M_i=\sup\{f(x):b_i\le x\le b_{i+1}\}.$$

Then, for each i, at least one of m_i, M_i is not β. If r of the m_i are less than β, then some values less than β are taken at least twice in each of the corresponding intervals (b_i,b_{i+1}) and at least once in $(0,b_1)$, therefore at least $2r+1$ times altogether. Since this is impossible, there must be r values of i for which $M_i>\beta$. As $\lim_{x\to a^-}f(x)>\beta$, it follows by an argument similar to that just used that some values greater than β are assumed at least $2r+1$ times. We have therefore again arrived at a contradiction.

(2): $f(x)>Y$ for $x\in(a,1)$.

This is shown to be impossible by similar reasoning involving $\inf\{f(x):0\le x<a\}$. Thus (i) is proved.

(ii): $f((a,1))\subseteq f([0,a))$.

Suppose that there exists $Y\in f((a,1))$ such that $Y\notin f([0,a))$, so that therefore $f(x)<Y$ for $x\in[0,a)$ or $f(x)>Y$ for $x\in[0,a)$.

(1): $f(x)<Y$ for $x\in[0,a)$.

Let

$$M=\sup\{f(x):a<x<1\}.$$

If f assumes the value M in $(a,1)$, then there are points a_i $(i=1,\dots,2r-1)$ such that $a_1<a_2<\dots<a_{2r-1}<1$ and $f(a_i)=M$ for $i=1,\dots,2r-1$. Since $\lim_{x\to a^+}f(x)$, $\lim_{x\to 1^-}f(x)\le M$, f assumes some values less than M at least $4r-2$ times, which is false.

If f does not assume the value M in $(a,1)$, at least one of the inequalities $M\ge\lim_{x\to a^+}f(x)$, $M\ge\lim_{x\to 1^-}f(x)$ is an equality.

in the case of one equality we may assume that $M=\lim_{x\to a^+}f(x)$, and $M>\lim_{x\to 1^-}f(x)$. Take β so that $\beta\ne f(a)$ and

$$\max(\lim_{x\to 1^-}f(x),Y)<\beta<M.$$

There are points b_i $(i=1,\dots,2r)$ such that $a<b_1<b_2<\dots<b_{2r}<1$ and $f(b_i)=\beta$ for $i=1,\dots,2r$. It then follows as in (i), (1) that f takes some values at least $2r+1$ times, and this cannot occur.

Now suppose that $M=\lim_{x\to a^+}f(x)=\lim_{x\to 1^-}f(x)$. Then the value

$$m=\inf\{f(x):a<x<1\}$$

is assumed by f in $(a,1)$. There are $2r$ points in $[0,1)$ at which f takes the value m and so there exist points c_i $(i=1,\dots,2r-1)$ in $[0,1)\setminus\{a\}$ such that $0\le c_1<c_2<\dots<c_{2r-1}<1$ and $f(c_i)=m$ for $i=1,\dots,2r-1$. Note that at least one of the c_i lies in $[0,a)$, then m is also the infimum of f in $[0,a)$. Suppose that exactly n of the c_i are in $[0,a)$, so that $0\le n\le 2r-2$.

If $n=0$, then f assumes some values greater than m at least $2(2r-2)+2=4r-2$ times. If $n\ge1$, then f takes some values greater than m at least $2(n-1)+1$ times in $[0,a)$ and at least $2(2r-n-2)+2$ times in $(a,1)$, therefore $4r-3$ times altogether. So in either case there is a contradiction.

(2): $f(x)>Y$ for $x\in[0,a)$

A contradiction is obtained in a similar way. Hence (ii) holds.

(iii): By (i) and (ii), $f([0,a))=f((a,1))$.

In the interval $[0,a)$, f must attain at least one of

$$m = \inf\{f(x):0\le x<a\} = \inf\{f(x):a<x<1\},$$
$$M = \sup\{f(x):0\le x<a\} = \sup\{f(x):a<x<1\}.$$

Suppose that f takes the value m in $[0,a)$ and therefore also in $(a,1)$. Then there are points c_i $(i=1,\dots,2r-1)$ in $[0,1)\backslash\{a\}$ such that $0\le c_1<c_2<\dots<c_{2r-1}<1$, $c_1\in[0,a)$, $c_{2r-1}\in(a,1)$ and $f(c_i)=m$ for $i=1,\dots,2r-1$. A contradiction is now obtained as in (ii), (1).

II: f is discontinuous at 0.

Take any two points p,q such that $0<p<q<1$ and $f(p)=f(q)$. The function $g:[p,1+p)\to\mathbb{R}$ defined by

$$g(x) = \begin{cases} f(x) & \text{for } p\le x<q, \\ f(x-q) & \text{for } q\le x<p+q \\ f(x-p) & \text{for } p+q\le x<1+p \end{cases}$$

is then k-fold and continuous except at the point $q\in(p,p+1)$ since $g(q) = f(0) \ne \lim_{x\to 0^+} f(x) = \lim_{x\to q^+} g(x)$. By I, this is an impossible situation.

Finally we consider $\nu(k)$. It follows from examples (D_1), (E_1) that $\nu(8)\le 3$, $\nu(7)\le 4$. In (D_1) the graph on $(0,2)$ is of a similar type to that in (A_1) except that 7-fold functions are used in the construction rather than 3-fold ones. For E_1, the graphs on $(0,2)$, $(2,4)$ are copies of the $(0,2)$ portion in (B_1). In both (D_1) and (E_1) replacement of the right hand continuous portion by the graphs of k-fold functions $(k=3,5,\dots)$ gives $\nu(10)\le 3$, $\nu(12)\le 3,\dots,\nu(9)\le 4,\dots$. By use also of [1] and [2] we therefore have

$$1\le\nu(6)\le 2, \qquad 1\le\nu(k)\le 3 \qquad (k=8,10,\dots),$$
$$1\le\nu(5)\le 2, \qquad 1\le\nu(k)\le 4 \qquad (k=7,9,\dots).$$

BINOMIAL IDENTITY 24 (JCMN 47, p.5120)

J.B.Barker

$$\sum_{j=1}^{m} \binom{n+j-1}{r} = \sum_{j=1}^{m} \binom{n}{r+1-j}\binom{m}{j}$$

Proof: First note that

$$\binom{n+j-1}{r} = \binom{n}{r} + \sum_{j=1}^{m} \binom{j-1}{s}\binom{n}{r-s}$$

which follows by repeated application of the $\binom{N+1}{r} = \binom{N}{r} + \binom{N+1}{r-1}$ formula. So,

$$\begin{aligned}\sum_{j=1}^{m}\binom{n+j-1}{r} &= m\binom{n}{r} + \sum_{j=2}^{m}\sum_{s=1}^{j-1}\binom{j-1}{s}\binom{n}{r-s} \\ &= m\binom{n}{r} + \sum_{s=1}^{m-1}\sum_{j=s+1}^{m}\binom{j-1}{s}\binom{n}{r-s} \\ &= m\binom{n}{r} + \sum_{s=1}^{m-1}\binom{n}{r-s}\sum_{j=0}^{m-s-1}\binom{j+s}{j} \\ &= m\binom{n}{r} + \sum_{j=2}^{m}\binom{n}{r+1-j}\sum_{s=0}^{m-j}\binom{s+j-1}{s}\end{aligned}$$

So the identity is proved if we can show that

$$= \sum_{s=0}^{m-j}\binom{s+j-1}{s} = \binom{m}{j}.$$

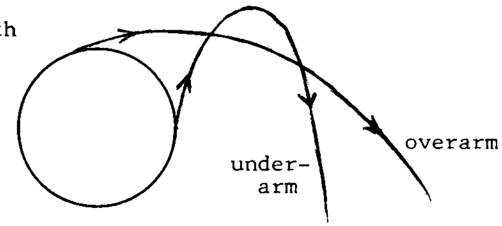
Proof of this is by induction on m. Set

$$f(m) = \sum_{s=0}^{m-j}\binom{s+j-1}{s} - \binom{m}{j},$$

and note that $f(j) = 0$. Then $f(m+1) = f(m) + \binom{m}{j} + \binom{m}{j-1} - \binom{m-1}{j} = f(m)$ using the above formula, and hence $f(m) = 0$ for all $m\ge j$ and the result is proven.

OVERARM AND UNDERARM THROWING

Imagine a machine for throwing balls, consisting of a holder at the end of an arm (of length = a) rotating at speed ω (in either direction) about a fixed horizontal axis. The ball can be released with speed $V = a\omega$ at any position of the arm. (see the picture)



Does the "over-arm" or the "under-arm" throw give greater range? Use the simple theory of projectiles, regarding the ball as a particle in a uniform gravitational field g.

In the limit when the radius a becomes zero, there is no distinction between overarm and underarm, the range is determined by the "enveloping parabola" (familiar to every schoolboy, one hopes). The trajectory (you recall) is given in terms of the time of flight t and the angle α of elevation by

$$x = Vt\cos\alpha \qquad \text{and} \qquad y = Vt\sin\alpha - \tfrac{1}{2}gt^2$$

so that the accessible points (x, y) satisfy

$$g^2x^2 + 2gV^2y \leq V^4$$

with equality when $V = gt\sin\alpha$ and $y/x = -\cot 2\alpha$.

QUOTATION CORNER 27

Dr Victor Morris ... said: "The molecule looked hexagonal or a little like blancmange. We were astonished at the results."

— Description of "tunneling microscope" in the "Australian" (newspaper). Tuesday, October 4th 1988, page 29.

BINOMIAL IDENTITY 25 (JCMN 47, P.5123)

J.B.Parker

$$\sum_{r=0}^{2m+j} (-1)^r \binom{4m+p}{2r} / (2r+1) = (-4)^m k/(4m+p+1)$$

where for any p = 0,1,2 or 3 the numbers j and k are as follows.

p	0	1	2	3
j	0	0	1	1
k	1	2	2	0

Proof: First define, for i= 1 to 4,

$$K_i = (4m+p+1) \sum_{r=0}^{2m+j} (-1)^r \binom{4m+p}{2} / (2r+1).$$

These K_i correspond to the possibilities of the table. K_4 is easy for this is

$$\sum_{r=0}^{2m+1} (-1)^r \frac{(4m+3)!}{(2r)!(4m-2r+3)!} \, \frac{4m+4}{2r+1}$$

$$= \sum_{r=0}^{2m+1} (-1)^r \binom{4m+4}{2r+1},$$

and this is zero since the r terms cancel with the (2m+1-r) terms. By considering separately the summands for r = 0 to m-1 and r = m+1 to 2m and using the well-known identity $\binom{n+1}{s+1} = \binom{n}{s+1} - \binom{n}{s}$ we get

$$K_1 = \sum_{r=0}^{2m} (-1)^r \binom{4m+1}{2r+1}$$

$$= \sum_{r=0}^{m-1} (-1)^r \binom{4m+2}{2r+1} + (-1)^m \binom{4m+1}{2m+1}.$$

Similarly $\quad K_2 = \sum_{r=0}^{2m} (-1)^r \binom{4m+2}{2r+1}$

$$= 2\sum_{r=0}^{m-1} (-1)^r \binom{4m+2}{2r+1} + (-1)^m \binom{4m+2}{2m+1},$$

and since $\binom{4m+2}{2m+1} = 2\binom{4m+1}{2m+1}$ we have $K_2 = 2K_1$.

$$
\begin{aligned}
K_3 &= \textstyle\sum_{r=0}^{2m+1}(-1)^r\binom{4m+3}{2r+1} \\
&= \textstyle\sum_{r=0}^{m}(-1)^r\binom{4m+3}{2r+1} - \sum_{r=0}^{m}(-1)^r\binom{4m+3}{2r} \\
&= \textstyle\sum_{r=0}^{m}(-1)^r\{\binom{4m+2}{2r+1} + \binom{4m+2}{2r}\} - \sum_{r=1}^{m}(-1)^r\{\binom{4m+2}{2r-1} + \binom{4m+2}{2r}\} -1 \\
&= \textstyle\sum_{r=0}^{m}(-1)^r\binom{4m+2}{2r+1} + 1 + \sum_{s=0}^{m-1}(-1)^s\binom{4m+2}{2s+1} - 1 \\
&= 2\textstyle\sum_{r=0}^{m-1}(-1)^r\binom{4m+2}{2r+1} + (-1)^m\{\binom{4m+2}{2m+1} \\
&= K_2.
\end{aligned}
$$

So if we can prove the identity for K_1 we are done. Now

$$K_1 = \binom{4m+1}{1} - \binom{4m+1}{3} + \ldots - \binom{4m+1}{4m-1} + \binom{4m+1}{4m+1},$$

which is $\dfrac{(1+i)^{4m+1} - (1-i)^{4m+1}}{2i}$,

but $(1+i)^2 = 2i$ and $(1-i)^2 = -2i$ so $(1+i)^{4m} = (1-i)^{4m} = (-4)^m$.

So $K_1 = (-4)^m\{\dfrac{1+i-(1-i)}{2i}\} = (-4)^m$, the required result.

QUOTATION CORNER 28

In all 758 people were surveyed.... The balance of members between city and country was evenly divided with 47.5% being metropolitan Adelaide members and 52.5% from country South Australia. The remaining members are from interstate.

- From "National Trust News", the official publication of the National Trust of South Australia.

BINOMIAL IDENTITY 26 (JCMN 48, p.5145)

In the summer of 516 A.D. King Arthur and his knights were called to South-eastern England to repel a landing by the Saxons. Queen Guinevere and her ladies-in-waiting travelled with the knights to set up a base camp. The spot chosen was at Broadhalfpenny Down, ten miles from the Channel coast, and just within the region where King Arthur's influence was strong. After a short campaign the returning warriors were welcomed back to the camp, and although they had no Round Table they were able to sit down to a memorable feast. Afterwards the Queen said to King Arthur "We must stay here for a few days, for some of the wounded are not yet fit to travel." "That would be wise," answered the King, "but some of the young men will be restless with nothing to do. Merlin, what do you suggest?" "This Broadhalfpenny Down is a lovely stretch of turf, your Majesty," answered Merlin, "and in about a thousand years the people here will invent a game to play on it. They will call it cricket." "That's great news," said the King, "and there's no need to wait a thousand years, we'll have the first game tomorrow. Tell us what we have to do." Merlin looked into the distant future, a worried frown gathering on his face. "The laws of cricket, sire, will be very complicated, and will be changing a lot over the years." "I don't want to know about the difficulties, Merlin, I am tired and I'm going to bed. Tomorrow morning when I get up you will have made all the arrangements. Get some of the knights to help you. Call yourselves the Merlin Cricket Committee, or M.C.C. for short." "It shall be done, your Majesty."

In a few minutes Merlin was standing with Sir Gawain and Sir Lancelot looking over the smooth green turf of the Downs in the evening sunshine. "There must be r men in each of two teams," announced Merlin, "and we shall call one team English and the other Australian." "But there are 2n of us," put in Sir Gawain, "so that 2n - 2r will have to be spectators." "Perhaps it would make it more interesting," said Sir Lancelot, "if we also divided the spectators into English and Australian. We shall first choose the n English and from them choose the r players in the team. Then we choose a team of r from the n

Australians. That gives $\binom{2n}{n}\binom{n}{r}^2$ ways of arranging the game. But before going any further we must obtain Queen Guinevere's approval." When this had been explained to her, the Queen thought for a while and then commented "There is a lot to be said for first choosing the 2r players, and then choosing which of the players are to be called the English, and which of the 2n - 2r spectators. And so it seems that the number of ways of arranging the game is $\binom{2n}{2r}\binom{2n-2r}{n-r}\binom{2r}{r}$." "Most interesting," said Merlin, "what you have pointed out shows that $\binom{2n}{n}\binom{n}{r}^2/\binom{2n}{2r} = \binom{2n-2r}{n-r}\binom{2r}{r}$, which is an even number, and so you have answered the question that will be asked on page 5145 of JCMN 48. Now it is all coming back to me. There will be a village called Hambledon on the far side of this valley. The villagers will start a cricket club, and they will beat the Rest of England at Lords in 1793, without knowing how they have helped to clarify the theory of Legendre polynomials!"

Looking back from the twentieth century, we can add a footnote. The tide of Saxon invasion never reached Hambledon, which is six miles West of the boundary of Sussex (the land of the South Saxons). The site of King Arthur's legendary victory over the Saxons at Mount Badon is now unknown.

BINOMIAL IDENTITY 28

$$\sum_{r=0}^{n}\binom{2r}{r}\binom{2n-2r}{n-r}(n-2r)^2 = n(n+1)\,2^{2n-1}$$

BINOMIAL IDENTITY 26 (JCMN 48, p.5145)

J.B. Parker

Is $\binom{2n}{n}\binom{n}{r}^2/\binom{2n}{2r}$ always an even integer, for $0\le r\le n$ and $n>0$?

Proof: It is easily shown that

$$\binom{2n}{n}\binom{n}{r}^2/\binom{2n}{2r} = \binom{2n-2r}{n-r}\binom{2r}{r}$$

and that if $m>0$ then

$$\binom{2m}{m} = 2\binom{2m-1}{m}$$

is even. Since at least one of the terms on the right hand side of the first equation has this form the result follows.

If $0<r<n$ then both terms are even and so the original expression is divisible by 4.

GEOMETRICAL IDENTITY (JCMN 48, p.5168)

Mark Kisin

Let $A_1, A_2, \dots, A_n$ be uniformly spaced around the circumference of a circle, and let P be any point on the circle. Then if $m<n$

$$\sum_{r=1}^{\infty}|PA_n|^{2m} = \binom{2m}{m}n.$$

To prove this we used two lemmas.

LEMMA 1. Let $\theta = 2\pi/n$ where n is a positive integer. Let j be a positive integer satisfying $2j-1<n$, and let β be any real number. Then,

$$\sum_{i=0}^{n-1}\cos^{2j-1}(\theta i+\beta) = 0$$

Proof: Using a known identity (see Gradshteyn and Ryslick, "Tables of Integrals, Series and Products) we have,

$$\Sigma_{i=0}^{n-1}\cos^{2j-1}(\theta i+\beta)$$

$$= \Sigma_{i=0}^{n-1} \ \frac{1}{2^{2j-2}} \ \Sigma_{k=0}^{j-1}\binom{2j-1}{k} \cos((2j-2k-1)(\theta i+\beta))$$

$$= \frac{1}{2^{2j-2}} \ \Sigma_{k=0}^{j-1}\binom{2j-1}{k} \ \ \Sigma_{i=0}^{n-1} \cos((2j-2k-1)(\theta i+\beta)).$$

Now let $2j-2k-1 = N$ and let $(2j-2k-1)\beta = \alpha$. The inner sum becomes

$$\Sigma_{i=0}^{n-1} \cos(N\theta i+\alpha).$$

Now let A_0, $A_1,\ldots,A_n$ be points on the unit circle with centre O such that $\angle A_0OA_1 = \alpha$ and $\angle A_iOA_{i+1} = \theta$ for $i=1,\ldots,n-1$. Let w_i be the vector $\overline{OA}_i$ for $i=0$ to n.

$$\Sigma_{i=0}^{n-1} \cos(N\theta i+\alpha) = \Sigma_{i=1}^{n} \ w_0 \cdot w_i$$

$$= w_0 \cdot \Sigma_{i=1}^{n} \ w_i.$$

The sum here equals zero since the points $A_1,\ldots,A_n$ are evenly spaced around the circle and cannot all coincide since this would imply that $n|N$, i.e. $n|2j-2k-1$ which is clearly impossible. □

LEMMA 2. With θ, j, n and β as in lemma 1 and with $2j<n$,

$$\Sigma_{i=0}^{n-1}\cos^{2j}(\theta i+\beta) = n\binom{2j}{j}/2^{2j}.$$

Proof: Using another identity from Gradshteyn and Ryslick,

$$\Sigma_{i=0}^{n-1}\cos^{2j}(\theta i+\beta)$$

$$=\Sigma_{i=0}^{n-1}\frac{1}{2^j}\{\Sigma_{k=0}^{j-1} \ 2\binom{2j}{k} \cos(2(j-k)(\theta i+\beta)) + \binom{2j}{j}\}$$

$$=\frac{1}{2^j}\{\{\Sigma_{k=0}^{j-1} \ 2\binom{2j}{k} \ \Sigma_{i=0}^{n-1} \cos(2(j-k)(\theta i+\beta))\} + n\binom{2j}{j}\}$$

The inner sum is 0 by the same argument as in lemma 1, which gives the required result. □

We can now prove our

THEOREM. Let a regular n-gon, with vertices $A_1, A_2,\ldots,A_n$ be inscribed in a unit circle with centre O. Let P be any point on the circumference, and let $m<n$ be a positive integer. Then

$$|\overline{PA}_1|^{2m} + |\overline{PA}_2|^{2m} + \ldots + |\overline{PA}_n|^{2m} = \binom{2m}{m}n.$$

Proof: As in lemma 1 let w_i be the vector $\overline{OA}_i$ for $i=1$ to n, let $w_0 = \overline{OP}$ and let $\theta = 2\pi/n$, then

$$\Sigma_{i=1}^{n} \ |\overline{PA}_i|^{2m}$$

$$= \Sigma_{i=1}^{n}|w_0-w_i|^{2m}$$

$$= \Sigma_{i=1}^{n}(w_0 \cdot w_0 - 2w_0 \cdot w_i + w_i \cdot w_i)^m$$

$$= \Sigma_{i=1}^{n}(1 - 2w_0 \cdot w_i + 1)^m$$

$$= \Sigma_{i=1}^{n} \ 2^m(1 - w_0 \cdot w_i)^m$$

$$= \Sigma_{i=1}^{n} \ 2^m(1 - \cos(i\theta + \beta))^m$$

$$= 2^m \ \Sigma_{i=0}^{n-1} \ \Sigma_{j=0}^{m} \ \binom{m}{j}(- \cos(i\theta + \beta))^j$$

$$= 2^m \ \Sigma_{i=0}^{n-1} \ \{\Sigma_{j=0}^{[m/2]}\binom{m}{2j}(\cos(i\theta + \beta))^{2j} - \Sigma_{j=1}^{m^*}\binom{m}{2j-1}(\cos(i\theta + \beta))^{2j-1}\},$$

where $m^* = [m/2]$ or $[m/2]+1$ depending on whether m is, respectively, even or odd.

$$= 2^m\{\sum_{j=0}^{[m/2]} \binom{m}{2j} \sum_{i=0}^{n-1} \cos^{2j}(i\theta + \beta) -$$

$$\sum_{j=0}^{m^*} \binom{m}{2j-1} \sum_{i=0}^{n-1} \cos^{2j-1}(i\theta + \beta)\}.$$

By lemma 1 the second inner sum = 0, and applying lemma 2 to the first inner sum gives

$$= 2^m \sum_{j=0}^{[m/2]} \binom{m}{2j} n \binom{2j}{j}/2^{2j}$$

$$= 2^m n \sum_{j=0}^{[m/2]} \binom{m}{2j}\binom{2j}{j}/2^{2j}.$$

The proof is completed by an appeal to the supernatural. The sum here appears in Binomial identity 29 below. We are reliably informed that Merlin remembers that a proof will appear in the next issue of JCMN. This gives $n\binom{2m}{m}$, as required. □

BINOMIAL IDENTITY 29

Merlin

For integral m > 1,

$$\sum_{j=0}^{[m/2]} \binom{m}{2j}\binom{2j}{j} 2^{m-2j} = \binom{2m}{m}.$$

QUOTATION CORNER 29

...Output is now measured in tens of hundreds instead of thousands,...

-Country Life, December 1, 1988, page 226.

BINOMIAL IDENTITY 27 (JCMN 48, p.5146)

$$\sum_{r=0}^{n} \binom{2r}{r}\binom{2n-2r}{n-r} = 2^{2n}$$

This identity has an interesting history. Marta Sved writes that it has appeared before in JCMN in about 1982. The acting editor has not been able to find it in his incomplete set of back copies. Marta set it as an open problem in her article "Counting and Recounting" in the Mathematical Intelligencer, (5(1983), 21). It generated a number of replies all using path-counting techniques which she published in "Counting and Recounting : the Aftermath", (Mathematical Intelligencer 6(1984), 44-45). Among the solvers was Charles Pearce who proved it verbally immediately after reading the article. The following very neat proof is due to John B. Parker.

For $|v| < 1$ we have

$$\sum_{r=0}^{\infty} \frac{1}{v^r} = (1-v)^{-1}$$

$$= [(1-v)^{-1/2}]^2$$

$$= \sum_{r=0}^{\infty} \binom{2r}{r} v^r 2^{-2r} \sum_{s=0}^{\infty} \binom{2s}{s} v^s 2^{-2s}$$

The result is then obtained by picking out the coefficient of v^n on each side.

BICYCLES AND SHERLOCK HOLMES

In JCMN 48, page 5166, R. A. Lyttleton asked how Sherlock Holmes in "The Adventure of the Priory School" was able to tell the direction of motion of a bicycle by examining its tracks.

On a modern bicycle (figure 1), if the axis about which the handlebars turn is extended it meets the ground close to the point at which the front wheel touches the ground. This point will not depend on the angle at which the handlebars are turned relative to

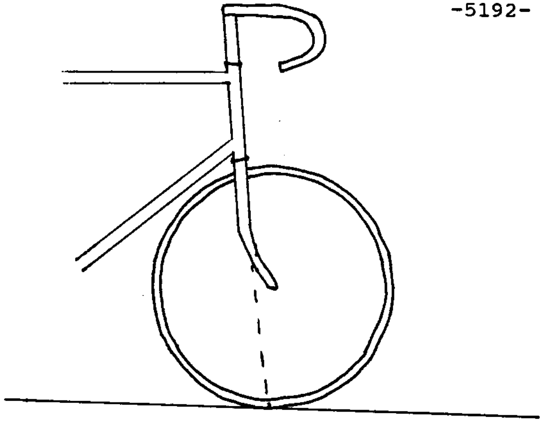


Figure 1

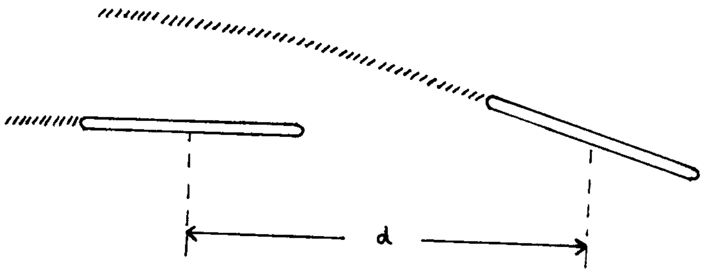


Figure 2

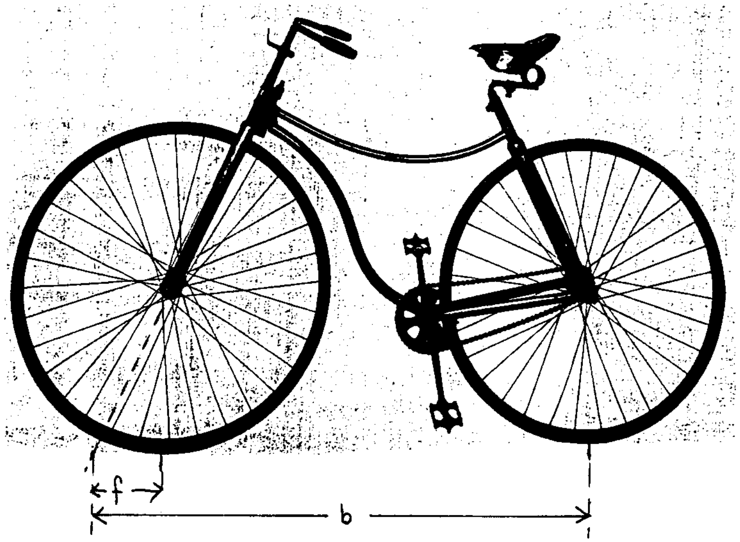


Figure 3

the bicycle's frame, and hence the distance between the front and back wheels' points of contact is fixed. Say this distance is d. Furthermore the point will lie in the same plane as the bicycle's frame. The rear wheel also lies in this plane, so if we take a tangent to its path, extend it a distance d, we will meet the path of the front wheel. See figure 2. By drawing tangents to each track in each direction at a few points the direction of motion of a modern bicycle may be determined.

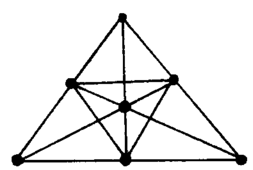
Sherlock Holmes had a more difficult problem. The bicycle in the (1904) "Adventure of the Priory School" was probably similar to the Rover Safety Bicycle of figure 3, which was introduced in 1895. In this case the axis about which the handlebars turn meets the ground at a point in front of the front wheel's point of contact. A tangent to the front wheel extended a distance f, and to the rear track extended a distance b will meet at this point. Knowing this Holmes was able to determine which track was which and the direction in which the bicycle was travelling.

SYLVESTER CONFIGURATIONS

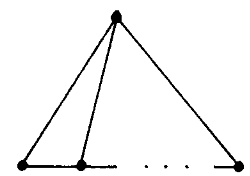
W.F.Smyth

Modern interest in Sylvester configurations begins, as does so much else, with Erdös in 1933; articles by Crowe & McKee (Math. Mag. 41 (1968) 30-34) and Kelly & Moser (Canad. J. Math. 10 (1958) 210-219) give historical references. More recently, related work has been done by Erdös, Mullin, Sós & Stinson (Disc. Math. 47 (1983) 49-62), Lin (Amer. Math. Monthly 95 (1988) 932-933), and Erdös & Purdy (Congressus Numerantium, to appear).

Imagine that in the Euclidean plane R^2 all parallel lines in a direction ϕ meet in a single point p_ϕ, and let τ denote the collection of all points p_ϕ, $0 \le \phi < \pi$. Then take τ to be the "line at infinity", so that $R^2 \cup \tau$ is a model of the projective plane, here called simply the plane. A Sylvester configuration, or simply configuration, is a finite set of $n \geq 2$ points in the plane together with all possible straight lines through them. A configuration is trivial if all its points are collinear. A line is ordinary if it contains exactly two of the n points; otherwise, special. If every line of a configuration C is ordinary, then C is said to be maximal. A point is ordinary if it lies on at least one ordinary line; otherwise, special. The notation $C_{i,j}$ is used to denote a configuration on i ordinary and j special points. Configurations of particular interest are the trivial configuration $C_{0,n}$, the failed Fano plane $C_{3,4}$, the near-pencil $C_{n,0}$, and the maximal configuration $C_{n,0}$.



Failed Fano Plane



Near-Pencil

The Sylvester graph of a non-trivial configuration C is the undirected graph whose vertices are its ordinary points and whose edges are its ordinary lines. Then the Sylvester graph of C is complete if and only if every pair of ordinary points defines an ordinary line. A non-trivial configuration will be called complete if its Sylvester graph is complete. Imagine now that a positive integer weight is assigned to every point of a Sylvester configuration, and that the weight of a line is defined to be the sum of the weights of the points on it. Then a configuration is said to be magic if its points can be given weights in such a way that the weights of all its lines are equal. With these definitions several interesting conjectures can be stated.

(A) If $j \geq 6$, then the special points of every configuration $C_{i,j}$ are collinear.

(B) Every configuration $C_{i,j}$ for which $i < j$ is either the failed Fano plane or trivial.

(C) (Due to Sylvester.) A configuration C is complete if and only if C is either the failed Fano plane or maximal.

(D) (Due to Murty, Amer. Math. Monthly 78-9 (1971) 1000-1002.) A configuration C is magic if and only if C is one of: the failed Fano plane, a near-pencil, trivial, or maximal.

It is easy to see that if (A) is true, then (B) holds for all $j \geq 6$. A more difficult chain of reasoning shows that (C) implies (D), and a lengthy but elementary argument then establishes the fact that (C) is in fact true whenever $j < 6$ or $i \geq j$. Indeed,

$$(A) \vee (B) \Rightarrow (C) \Rightarrow (D)$$

In order to establish (A) or (B), it appears that the proof of yet another conjecture is required. Given a configuration C, let S denote some non-empty subset of the special points, and let P denote the largest non-empty set of points with the property that for every $p \in P$ and every $s \in S$, the line ps contains no other point of $P \cup S$. Then $\{P,S\}$ is called an S-partition of C, and the new conjecture is as follows:

(E) Suppose $\{P,S\}$ is an S-partition of a configuration C. Then if either $|P| = 1$ or $|S| = 1$, there exist at least $|P| + |S| - 1$ points of C not in $P \cup S$; otherwise, at least $|P| + |S| - 2$.

This conjecture is equivalent to a conjecture of Erdös & Purdy. It appears to underlie not only conjectures (A)-(D), but also the following conjecture, due to Sylvester, perhaps the most famous one of all in this area:

(F) Every configuration on n points contains at least $\lceil (n-1)/2 \rceil$ ordinary lines.

Kelly & Moser showed that there were at least 3n/7 ordinary lines. It is claimed that a doctoral dissertation written about 10 years ago contained a very lengthy and complex proof of (F), but the result is unpublished.

R. L. GRAHAM PROBLEM

The solution to the problem on page 5173 is 0. Consider the 24th term.

PUZZLE

Find the deliberate mistake in this issue of JCMN.