JAMES COOK MATHEMATICAL NOTES

Volume 5 Issue number 46

June 1988

Editor and Publisher: B. C. Rennie

Address: 66, Hallett Road,
Burnside, SA 5066,
Australia.

The "James Cook Mathematical Notes" is published in three issues per year, in February, June and October.

The normal subscription rate for one year (3 issues) is 30 Singapore dollars, and for retired or unemployed subscribers it is 15 Singapore dollars. Cheques in any currency are acceptable, the exchange calculated at the current rate when the money is sent. The rate includes air mail postage to anywhere outside the country where the issue is posted (now Australia). Some exchange rates at the time of writing (May 1988) are:

One Singapore dollar = 65 cents Australian
= 27 pence United Kingdom
= 50 cents U.S.A.

We do not normally send out invoices or subscription reminders. Please make cheques payable to B.C.Rennie.

The information about back numbers, which for years we have been printing on the last page, is omitted from this issue, there is no room for it.

CONTENTS

JUBILEE INEQUALITY

Dedicated to JCMN's first five years of independence

Dmitry Mavlo

$$\frac{16}{JCMN + (J+1)(C+1)(M+1)(N+1) + 1 - JM - CN}$$

$$\leq \frac{1}{(J+1)(C+1)M} + \frac{1}{(N+1)(J+1)C} + \frac{1}{(M+1)(N+1)J} + \frac{1}{(C+1)(M+1)N}$$

where J, C, M and N are arbitrary positive numbers. Prove that there is equality if and only if $J=C=M=N=1$.

ADDITION AND MULTIPLICATION PROBLEM

Paul Erdös and R. L. Graham

Consider sequences of integers

$$1 \leq a_1 < a_2 < \dots < a_t \leq n$$

with the following property: for every $1 \leq i < j \leq n$ the sum $a_i + a_j$ is not a factor of the product $a_i a_j$. Determine as accurately as you can $\max t = f(n)$. Is it true that $\lim f(n)/n = \frac{1}{2}$?

Suppose that we strengthen the condition by having the sum not a factor of twice the product, i.e. $a_i + a_j \nmid 2a_i a_j$, then let $\max t = g(n)$. Is it true that $\lim g(n)/n = 0$?

PROBLEM ON CONVEX POLYGONS

Esther Szekeres and Paul Erdös

Let C_n be a convex polygon in the plane. Denote by $S(C_n)$ its circumference, and let $X_1, X_2, \dots X_n$ be the vertices of C_n, and let P be a point either in the interior or on the boundary of C_n. Let $f(C_n)$ be the maximum (for all P) of the sum of the distances from P to the n vertices. Note that this maximum must be attained at a vertex.

Prove that $f(C_3) < S(C_3)$ for every triangle. This is almost trivial. For $n = 4$ there are quadrilaterals for which $f(C_4) < S(C_4)$, and others for which $f(C_4) > S(C_4)$. In particular prove that if two opposite sides of our quadrilateral are equal then $f(C_4) < S(C_4)$.

For $n = 5$ again both cases occur, but prove that for $n \geq 6$ we always have $f(C_n) > S(C_n)$. This is easy for large n but we could not settle it for $n = 6$.

For $n \geq 7$ we expect that

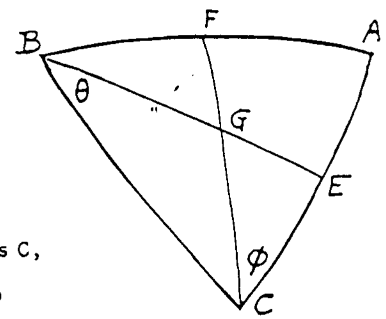
$$\sum_{i=1}^{n} \sum_{j=1}^{n} d(X_i, X_j) > n\,S(C_n) \qquad (1)$$

We could not prove (1), but showed that it fails for $n = 6$ but holds for large n.

EQUAL MEDIANS IN A SPHERICAL TRIANGLE

A. Brown

If a plane triangle has two medians of equal length the triangle is isosceles, but for a spherical triangle it is possible to have two of the medians equal when the triangle is not isosceles. This seems surprising, so let us look at the properties that would be needed for such a triangle.



If we take E as the mid-point of AC, and F as the mid-point of AB in a spherical triangle ABC, then the cosine formula gives $\cos c =$ $\cos b \cos a + \sin b \sin a \cos C$, and $\cos BE = \cos\frac{b}{2}\cos a + \sin\frac{b}{2}\sin a \cos C$, and eliminating $\sin a \cos C$ leads to

$$2\cos\frac{b}{2}\cos BE = \cos c + \cos a \qquad \dots (1)$$

In the same way,

$$2\cos\frac{c}{2}\cos CF = \cos a + \cos b. \qquad \dots (2)$$

If BE = CF, equations (1) and (2) give

$$\cos\frac{c}{2}(\cos c + \cos a) = \cos\frac{b}{2}(\cos a + \cos b)$$

and this can be written as

$$0 = \left(\cos\frac{b}{2} - \cos\frac{c}{2}\right)\left(1 + \cos a + \cos b + \cos c + 2\cos\frac{b}{2}\cos\frac{c}{2}\right).$$

If $\cos\frac{b}{2} = \cos\frac{c}{2}$, we get $b = c$ and the triangle is isosceles, so we want to look at the case where $b \neq c$ but where we have

$$0 = 1 + \cos a + \cos b + \cos c + 2\cos\frac{b}{2}\cos\frac{c}{2} \qquad \dots (3)$$

Can this condition hold? and what sort of spherical triangle does it give? A simple proof of existence is to give an example. Numerical calculations show that when $b = 150^\circ$ and $c = 110^\circ$ a suitable triangle exists, see details below. There is evidence that such triangles must be fairly large.

If we use equation (3) to replace $\cos c + \cos a$ in (1) we can cancel a positive factor $2\cos\frac{b}{2}$ and obtain

$$\cos BE = -\cos\frac{b}{2} - \cos\frac{c}{2} < 0$$

which shows that $BE > 90^\circ$. Also, we cannot have $\cos\frac{b}{2}$ and $\cos\frac{c}{2}$ both greater than ½, so either b or c must be greater than 120°. It can also be shown that the angles A, B and C must all be greater than 90°. From the triangle AEB

$$\cos\frac{b}{2}\cos c + \sin\frac{b}{2}\sin c \cos A = \cos BE = -\cos\frac{b}{2} - \cos\frac{c}{2}$$

and this leads to

$$\cos A = -\left(1 + 2\cos\frac{b}{2}\cos\frac{c}{2}\right)/\left(2\sin\frac{b}{2}\sin\frac{c}{2}\right) < -\tfrac{1}{2}.$$

In the same way, it can be shown that

$$\cos B = -\left(\cos\frac{b}{2} + (1 + \cos a)\cos\frac{c}{2}\right)/\left(\sin a \sin\frac{c}{2}\right) < 0,$$

$$\cos C = -\left(\cos\frac{c}{2} + (1 + \cos a)\cos\frac{b}{2}\right)/\left(\sin a \sin\frac{b}{2}\right) < 0.$$

For given values of b and c, equation (3) gives a, and the expressions for $\cos A$, $\cos B$ and $\cos C$ then allow A, B and C to be calculated. We have to check that $a + b + c < 360^\circ$ and that $\cos\frac{b}{2} + \cos\frac{c}{2} < 1$. There are some other checks that can be made but for these we need additional relationships between the various arcs and angles in the triangle.

If we use θ and ϕ for the angles EBC and ACF, then from the sine formula

$$\sin\frac{b}{2}\sin BEC = \sin\theta \sin a$$

and

$$\sin\frac{b}{2}\sin BEA = \sin(B - \theta)\sin c$$

and hence

$$\sin(B - \theta)/\sin\theta = \sin a / \sin c \qquad \dots (4)$$

In the same way $\sin(C - \phi)/\sin\phi = \sin b / \sin a \qquad \dots (5)$

When BE = CF, we can also say that

$$\frac{\sin\frac{c}{2}}{\sin\phi} = \frac{\sin CF}{\sin A} = \frac{\sin BE}{\sin A} = \frac{\sin\frac{b}{2}}{\sin(B - \theta)}$$

i.e. $\quad \sin\frac{c}{2}\sin(B-\theta) = \sin\frac{b}{2}\sin\phi.$ ... (6)

It follows from equations (4), (5) and (6) that

$$\sin(C-\phi)\,/\,\sin\theta = \cos\frac{b}{2}\,/\,\cos\frac{c}{2}. \qquad \text{... (7)}$$

From the triangles FGB and EGC, $\sin FG \sin FGB$ $= \sin\frac{c}{2}\sin(B-\theta) = \sin\frac{b}{2}\sin\phi = \sin GE \sin EGC$, and hence $\sin FG = \sin GE$. This means that either FG = GE or FG = 180° - GE. Now if FG = GE then we must have GC = GB, $\theta = C - \phi$ and, from equation (7), $\cos\frac{b}{2} = \cos\frac{c}{2}$. This brings us back to the case b = c, which we ruled out, so we must have

$$FG = 180^\circ - GE \qquad \text{... (8)}$$

and
$$CG + GB = 2(BE - 90^\circ) \qquad \text{... (8a)}$$

Previously we had BE $> 90^\circ$, but now we get a stronger condition, that either FG or GE must be greater than 90°. Once again this is a reminder that we are dealing with very large triangles on the sphere. For the numerical example mentioned earlier ($b = 150^\circ$, $c = 110^\circ$) $a = 94^\circ\!.4$, $A = 145^\circ\!.0$, $C = 147^\circ\!.3$, BE = CF = $146^\circ\!.4$, GE = $119^\circ\!.6$ and GF = $60^\circ\!.4$, $B = 163^\circ\!\cdot 3$.

Finally, we shall show that the cicumference 2s = a + b + c is between 352.64° and 360°. We can regard b and c as independent, with the remaining side a given by

$$\cos a = -1 - \cos b - \cos c - 2\cos\frac{b}{2}\cos\frac{c}{2}$$

Then $2\,\partial s/\partial b = 1 + \partial a/\partial b$, with $-\sin a\,\partial a/\partial b = \sin b + \sin\frac{b}{2}\cos\frac{c}{2}$, so the condition for $\partial s/\partial b$ to be zero is

$$\sin a = \sin b + \sin\frac{b}{2}\cos\frac{c}{2}$$

In the same way, the condition for $\partial s/\partial c$ to be zero is

$$\sin a = \sin c + \sin\frac{c}{2}\cos\frac{b}{2}$$

For a stationary value of s these conditions must both be satisfied. If we subtract the second from the first we get $\quad \sin\frac{b-c}{2}\left(1 + 2\cos\frac{b+c}{2}\right) = 0$ and this gives either $b+c = 240^\circ$ or b = c. To satisfy the first case, take $b = 120^\circ + 2f$ and $c = 120^\circ - 2f$. Then

$$\cos a = -1 - 2\cos\frac{b+c}{2}\cos\frac{b-c}{2} - \cos\frac{b+c}{2} - \cos\frac{b-c}{2}$$
$$= -1 + \cos 2f + \tfrac{1}{2} - \cos 2f = -\tfrac{1}{2},$$

so $a = 120^\circ$ and $a + b + c = 360^\circ$, which is the upper limit for 2s.

The other case, b = c, is strictly inadmissible, since we started off with the assumption that $b \neq c$. However, we can treat it as a limiting case, with $b \to c$, and see what happens to 2s and to the shape of the triangle in the limiting case. We note that for b = c we have

$$\cos a = -1 - 2\cos b - 2\cos^2\frac{b}{2} = -2 - 3\cos b,$$
$$\sin a = \sin b + \sin\frac{b}{2}\cos\frac{b}{2} = (3/2)\sin b.$$

Elimination of a gives $0 = (1 + \cos b)(7 + 9\cos b)$, and for b between 0° and 180°, we want an isosceles triangle with $\cos b = -7/9$, $\sin b = 4\sqrt{2}/9$, $\cos a = 1/3$, $\sin a = \sqrt{8}/3$. Numerically $b = c = 2a = 141\cdot 06^\circ$ and $2s = 352\cdot 64^\circ$. It is straightforward to work out the second derivatives of s and to check that the conditions for a minimum are satisfied in this case.

For an isosceles triangle, we can expect to have GE = GF and, from equation (8), the limiting case should have GE = GF = 90°. This was verified from the usual spherical triangle relationships. An unexpected result was that for the critical triangle

$$\sin AG = \sin BG = 2/3$$

$$\cos AG = -\cos BG = -\sqrt{5}/3,$$

which means that $AG + BG = 180^\circ$. Th angles are

$$A = \arccos(-11/16) = 133{\cdot}433^\circ$$

$$B = C + \arccos(-7/8) = 151{\cdot}045^\circ,$$

which gives a spherical excess of $1{\cdot}42\pi$.

ISOSCELES SPHERICAL TRIANGLES

Prove that if the centroid, orthocentre and circumcentre of a spherical triangle are on a great circle then the triangle is isosceles.

THREE SQUARES IN ARITHMETIC PROGRESSION

If $a < b < c$ are coprime positive integers and if their squares are in arithmetic progression, that is $a^2 + c^2 = 2b^2$, then what can you say about the possible values of a?

My computer gives the first few values as 1, 7, 17, 23, 31, 41, 47, 49, 71, 73, 79, 89, 97, ...

Show that for each of the possible values of a there are infinitely many ways to choose b and c making $a^2 + c^2 = 2b^2$.

THREE CUBES IN ARITHMETIC PROGRESSION

Can three unequal positive integer cubes be in arithmetic progression?

ANOTHER DOUBLE INTEGRAL

George Szekeres and Basil Rennie

$$\int_0^1 \sqrt{\frac{1-y}{y}} \int_0^y \sqrt{\frac{1-x}{x}} \log(y-x)\, dx\, dy = -\frac{\pi^2}{4}(\log 2 + 1/4)$$

First recall a well-known Fourier series:

$$\sum_1^\infty \frac{\cos nx}{n} = -\log|2\sin x/2|$$

or $\quad \log|\sin x| = -\log 2 - \cos 2x - \tfrac{1}{2}\cos 4x - \tfrac{1}{3}\cos 6x - ..$

Transform the given integral (J) as follows. Firstly 2J is the integral over the unit square where $0 < x, y < 1$ if the $\log(y-x)$ is replaced by $\log|y-x|$. Now we have symmetry between x and y. Substitute $x = \sin^2\theta$ and $y = \sin^2\phi$ to get rid of the square roots.

$$2J = \int_0^{\pi/2}\int_0^{\pi/2} (1 + \cos 2\theta)(1 + \cos 2\phi) \log\left|\frac{\cos 2\theta - \cos 2\phi}{2}\right| d\theta d\phi$$

Replace the integral by an integral mean over the whole plane.

$$8J/\pi^2 = \mathrm{IM}\ (1 + \cos 2\theta + \cos 2\phi + \cos 2\theta \cos 2\phi)\log\left|\frac{\cos 2\theta - \cos 2\phi}{2}\right|$$

Recalling that integral means are invariant under linear transformations, put $x = \theta + \phi$ and $y = \theta - \phi$. $8J/\pi^2 =$

$\mathrm{IM}\,(1 + 2\cos x \cos y + \tfrac{1}{2}\cos 2x + \tfrac{1}{2}\cos 2y)(\log|\sin x| + \log|\sin y|)$

which, because cosines have integral mean zero, simplifies to

$\mathrm{IM}\,(\log|\sin x| + \log|\sin y| + \tfrac{1}{2}\cos 2x \log|\sin x| + \tfrac{1}{2}\cos 2y \log|\sin y|)$

$= -2\log 2 - \tfrac{1}{2}$

because as we see from the Fourier series above

$\mathrm{IM}\log|\sin x| = -\log 2$ and $\mathrm{IM}\log|\sin x|\cos 2x = -\tfrac{1}{2}$.

THUS SPOKE MERLIN

Marta Sved

King Arthur surveyed his 2n knights, all returned from their quests, on which they had gone in pairs.

— I have good news for you. There is a great tournament to be held in Scotland, and I have received an invitation for p of you to go.—

— Choosing p out of 2n is not a difficult task — said Sir Bedivere. — We all know that you have to consider $\binom{2n}{p}$ possibilities. —

Sir Lancelot interjected: — We have just come back in pairs from our quests. I do not think that after all the hardships endured together it would be fair to break up the pairs just now. Each pair should either go or stay here together. —

— Sir Lancelot, you are very thoughtful; — answered the King — we should indeed exclude all the arrangements where pairs are separated. —

— Since there are n pairs of us — volunteered Sir Gawain — why not look at each of the pairs and strike out all those arrangements where only one knight is chosen out of the pair? This would mean discarding $2\binom{2n-2}{p-1}$ arrangements for each pair, since there are two ways of choosing one of the pair, and the rest to be sent to the tournament are chosen out of the remaining 2m-2 knights. Hence subtract $2n\binom{2n-2}{p-1}$. —

Merlin, who had listened to all this exchange, could not keep quiet any more.

— All of us, here in Camelot, have encountered that famous INCLUSION-EXCLUSION principle several times, so without too many further arguments I have to point out that you must add the $2^2\binom{n}{2}\binom{2n-4}{p-2}$ arrangements that break up two pairs (as there are $\binom{n}{2}$ ways of choosing 2 pairs out of the n pairs), and then subtract $2^3\binom{n}{3}\binom{2n-6}{p-3}$ for the cases where three pairs are separated, and so on. Thus the correct number is

$$\binom{2n}{p} - 2\binom{n}{1}\binom{2n-2}{p-1} + 2^2\binom{n}{2}\binom{2n-4}{p-2} - 2^3\binom{n}{3}\binom{2n-6}{p-3} + \dots$$

Queen Guinivere stopped arranging her flowers around the Round Table, and looked up.

— All that you have to do is to select the pairs that you want to send! —

— Of course, how simple, — said King Arthur — there should be just $\binom{n}{p/2}$ arrangements to consider, if we want to find the Camelot team of p knights.—

This was the moment for Sir Mordred:

— The task is impossible! With a side-glance at the letter of invitation I noticed that p is odd. That nice little formula of Merlin comes to 0, because there are no arrangements containing p knights with no questing pairs separated! —

— Your sight failed you, Sir Mordred. The number happens to be even and so we have

$$\binom{2n}{p} - 2\binom{n}{1}\binom{2n-2}{p-1} + 2^2\binom{n}{2}\binom{2n-4}{p-2} - \dots + 2^p\binom{n}{p}\binom{2n-2p}{0} = \binom{n}{p/2}$$

However, I must admit that if p had been odd, you would have been right, because in that case

$$\binom{2n}{p} - 2\binom{n}{1}\binom{2n-2}{p-1} + 2^2\binom{n}{2}\binom{2n-4}{p-2} - \dots - 2^p\binom{n}{p}\binom{2n-2p}{0} = 0$$

Merlin's eyes glazed. It was his look into the future:

— I can spot something very similar in James Cook Mathematical Notes, Issue number 45, in the 1988th year of our Lord. It will be called Binomial Identity 21, and it will state that

$$\sum_{k=m}^{n} (-\tfrac{1}{2})^k \binom{n}{k}\binom{2k}{k-m} = (-\tfrac{1}{2})^n \binom{n}{\frac{1}{2}n+\frac{1}{2}m} \quad \text{if } n+m \text{ is even}$$

$$= 0 \text{ if } n+m \text{ is odd.} \text{ —}$$

— You call this similar? — countered Sir Mordred.

— I admit that it needs some slight modifications, such as multiplying both sides by $(-2)^n$; then setting $l = n - k$, with $p = n - m$, it reads:

$$\sum_{l=p}^{0} (-2)^l \binom{n}{n-l}\binom{2n-2l}{p-l} = \binom{n}{n-p/2},$$

where $\binom{n}{n-l} = \binom{n}{l}$ and $\binom{n}{n-p/2} = \binom{n}{p/2}$, —

Sir Gareth, who had kept quiet throughout the whole discussion, said:

— We could have different rules for selecting the p knights. —

— Indeed we could have done so. — answered Merlin. He looked thoughtful, then added:

— Do I see a Binomial Identity in JCMN 46? It reads:

$$\sum_{q=0}^{[p/2]} (-1)^q \binom{n}{q}\binom{2n-2q}{p-2q} = 2^p \binom{n}{p}$$

BINOMIAL IDENTITY 23

Marta Sved

$$\sum_{q=0}^{[p/2]} (-1)^q \binom{n}{q}\binom{2n-2q}{p-2q} = 2^p \binom{n}{p}$$

DIFFERENTIAL INEQUALITY

Jordan Tabov

The function $F(x, y)$ has continuous second derivatives everywhere in the plane, i.e. $F \in C^2(\mathbb{R}^2)$. Also

$$y^2F_{xx} - 8xyF_{xy} + 16x^2F_{yy} - 4xF_x - 4yF_y \geqslant 0$$

where the subscripts x and y denote partial differentiation. Prove that $F_x(0, 0) = 0$.

POLYNOMIAL PROBLEM

J. M. Hammersley

In the polynomial $Q(x)$ the term of highest degree is $(-1)^n x^{2n}$. All the other terms are of odd degree. Also $Q(1-x) = Q(x)$. Show that these conditions determine $Q(x)$ uniquely. Show also that $Q(x) = P(x-x^2)$ where P is a polynomial of degree n, and that in P all the coefficients except that of the constant term (which is zero) are positive integers.

COMBINATORIAL INEQUALITY

Ross Talent

Given n unequal real numbers, show that there are at least $(n-1)!$ ways of ordering them so that for each $k = 1, 2, \ldots n$, the mean of the first k numbers is not less than the mean of all the numbers.

SPHERICAL TRIANGLE INEQUALITY

J. B. Parker

Following Esther Szekeres, we may regard a spherical triangle ABC also as a plane triangle. Because the sides of the plane triangle are $2\sin\frac{1}{2}a$, etc., we have another spherical triangle inequality

$$\sin\tfrac{1}{2}a \leq \sin\tfrac{1}{2}b + \sin\tfrac{1}{2}c$$

in addition to the two already noted in these pages:

$$a \leq b + c \qquad \text{and} \qquad \sin a \leq \sin b + \sin c.$$

Can we generalize these to $\sin ka \leq \sin kb + \sin kc$ if $0 < k < 1$?

MONTE CARLO INTEGRATION

The basic idea is that if x_1, x_2, ... are random variables (uniformly distributed) on the unit interval, then $(1/N)\sum_{n=1}^{N} f(x_n)$ is an approximation to $\int_0^1 f(x)\,dx$. In fact it is a bad approximation, with an error of order $1/\sqrt{N}$ for large N; it is only in higher dimensions that the Monte Carlo method has practical value. Nevertheless, the simple case is interesting.

If we take any irrational g, and let x_n be the non-integer part of ng, i.e. $x_n = \{ng\}$, then we seem to get better results than when the x_n are random variables. Let f(x) be any well-behaved function on (0, 1), and let

$$S(N) = \sum_{r=1}^{N} \left(f(x_r) - \int_0^1 f(x)\,dx\right)$$

This S(N) is N times the error in the estimate for the integral. My "Peach" computer indicates that S(N) is bounded as N tends to infinity, but of course cannot prove it, even for one

particular function. I can find no theoretical basis for the result.

Even stranger things happen when we invoke the magic powers of the Golden Ratio. Let $g = \frac{1}{2}\sqrt{5} - \frac{1}{2}$ $= 0\cdot61803398874989 49...$ Recall the Fibonacci numbers, $F(1) = 1$, $F(2) = 1$, $F(3) = 2$, ... and the Lucas numbers, $G(1) = 1$, $G(2) = 3$, $G(3) = 4$, ... , both sequences with the property that each number is the sum of its two predecessors. My computer suggests that

$$S(F(n)) = (-1)^n \frac{2-g}{5}(f(1) - f(0)) + O(1/F(n))$$

for large n. Similarly, stopping the summation at Lucas numbers instead of at Fibonacci numbers:

$$S(G(n)) = (-1)^n (g-1)(f(1) - f(0)) + O(1/G(n))$$

Again, I can suggest no theoretical basis for the phenomenon. In the table on the next page are given some numerical results, the sum S(N) where N is a Fibonacci number, $N = F(n)$, for three different functions, $f(x) = x$, $f(x) = 6x(1-x)$ and $f(x) = 2/(1+x^2)$.

It further seems that this property of Fibonacci numbers and Lucas numbers is shared by any other sequence with the same recurrence relation.

Hypothesis If N(1), N(2), ... are positive integers with $N(n+2) = N(n+1) + N(n)$ then $(-1)^n S(N(n))$ tends to a constant multiple of $f(1) - f(0)$.

Sum S(F(n)) for three different functions

n	F(n)	x	6x(1-x)	$2/(1+x^2)$
2	1	·1180340	·4164079	-·1235827
3	2	-·1458980	·4984472	·2000481
4	3	·2082039	·2461180	-·2143380
5	5	-·2294902	·2336955	·2633794
6	8	·2492236	·1081775	-·2521632
7	13	-·2589070	·0952358	·2739323
8	21	·2658514	·0434690	-·2670627
9	34	-·2697767	·0372289	·2758232
10	55	·2723427	·0169218	-·2728184
11	89	-·2738751	·0143434	·2762296
12	144	·2748425	·0065102	-·2750261
13	233	-·2754327	·0054966	·2763386
14	377	·2758004	·0024935	-·2758708
15	610	-·2760265	·0021021	·2763735
16	987	·2761667	·0009534	-·2761936
17	1597	-·2762532	·0008033	·2763858
18	2584	·2763067	·0003643	-·2763170
19	4181	-·2763397	·0003069	·2763904
20	6765	·2763601	·0001392	-·2763641
21	10946	-·2763728	·0001172	·2763921
22	17711	·2763806	·0000532	-·2763821
23	28657	-·2763854	·0000448	·2763928
24	46368	·2763884	·0000203	-·2763890

EXTRAPOLATION

Consider increasing sequences of positive integers such as $A = \{a_1 < a_2 < a_3 \quad \dots\}$. It is a famous conjecture of P. Erdös that if the set of lengths of arithmetic progressions in A is bounded then the sum of reciprocals, $\sum 1/a_n$, converges.

A special case is

<u>First conjecture</u>: If A does not contain three numbers in arithmetic progression then $\sum 1/a_n$ converges.

or <u>Second conjecture</u>: There is a constant C(3) such that if A does not contain 3 numbers in arithmetic progression then $\sum 1/a_n \leq C(3)$.

The equivalence of the first and second conjectures can be proved as follows. Trivially the second implies the first. We shall show that if the second proposition is untrue then the first is untrue.

<u>Theorem</u> Consider all finite or infinite sequences such as $A = \{a_1 < a_2 < \dots\}$ of positive integers such that $a_i + a_k \neq 2a_j$ for all $i < j < k$. For any such sequence A let $S(A) = \sum 1/a_n$. If the set of all S(A) is unbounded then there is one A for which S(A) is infinite.

<u>Proof</u> We are given that for finite sequences A the sums S(A) take arbitrarily large values. We shall construct an infinite sequence B for which $S(B) = \infty$. The essence of the proof is in the following lemma.

<u>Lemma</u> Assuming the set of all S(A) to be unbounded, then given any finite sequence A there is another finite sequence B which is an extension of A (i.e. $b_n = a_n$ for all $n \leqslant$ the length of A), and which has $S(B) > S(A) + 1$.

<u>Proof of lemma</u> Take some irrational G between $2K$ and $2K + 1$, where K is the largest member of A. Classify the integers $> G$ into three classes, each consisting of countably many intervals:

$$P = (G,\ rG) \cup (r^3G,\ r^4G) \cup (r^6G,\ r^7G) \cup \ \ldots$$
$$Q = (rG,\ r^2G) \cup (r^4G,\ r^5G) \cup (r^7G,\ r^8G) \cup \ \ldots$$
$$R = (r^2G,\ r^3G) \cup (r^5G,\ r^6G) \cup (r^8G,\ r^9G) \cup \ \ldots$$

where $r=3/2$. Clearly P, Q and R cover (G, ∞), we are concerned only with integers, so that the end-points of the intervals above do not matter.

Having started with the finite sequence A, we choose a finite sequence C such that $S(C) > 1+\frac{1}{2}+1/3+..+1/(2K)+3$. Let C' be the sequence obtained from C by omitting all the numbers $\leq 2K$. Then $S(C') > 3$.

Let B_1, B_2 and B_3 be the intersections of C' with P, Q and R respectively. Then for some $m = 1$, 2 or 3, we must have $S(B_m) > 1$. Define B to be the union of this B_m with A, then $S(B) = S(A) + S(B_m) > S(A) + 1$. It remains only to verify that B has the property of not containing any three numbers in arithmetic progression. As A and B_m both have the property it will be sufficient to show the impossibility of there being such a sequence $b_i < b_j < b_k$ in B with b_i in A and b_k in B_m. Firstly $b_j = \frac{1}{2}b_i + \frac{1}{2}b_k > G/2$, and is therefore not in A. Now to show that b_j is not in B_m.

Observe that $r^2 > 2$ and $r^3+1 = 4{\cdot}375 < 4{\cdot}5 = 2r^2$. Therefore $2Gr^n < Gr^{n+2}$ and $Gr^{n+3} + G < 2Gr^{n+2}$. Plunging now into the notation of set theory:

$$2\,(Gr^n,\ Gr^{n+2}) \supseteq (Gr^{n+2},\ Gr^{n+3}) + (0,\ G).$$

This means that (for any $m = 1$, 2 or 3) if we take any number

in B_m and any number in A then half the sum will not be in B_m. If b_i is in A and b_k is in B_m then $b_j=\frac{1}{2}b_i+\frac{1}{2}b_k$ will not be in B_m, which shows that B has the required property. This completes the proof of the lemma; the theorem follows easily.

The rest of this article is concerned with $C(3)$. We shall show that it is at least as great as the constant

$$C = \int_0^1 \prod_{r=0}^{\infty}(1 + x^{(3^r)})\,dx = \int_0^1 (1+x)(1+x^3)(1+x^9)\ \ldots\ dx$$
$$= 1 + \tfrac{1}{2} + 1/4 + 1/5 + 1/10 + 1/11 + 1/13 + 1/14 + 1/28 + \ldots$$
$$= 3{\cdot}007938999898937\ldots$$

Consider the following sequence of finite rows of the symbols X and O.

XX
XXOXX
XXOXXOOOOXXOXX
XXOXXOOOOXXOXXOOOOOOOOOOOOOXXOXXOOOOXXOXX
...

Row number n has $\frac{1}{2}(3^n+1)$ symbols of which 2^n are X and the rest are O. The rows are defined inductively, row $n+1$ being obtained by copying row n, then adding $\frac{1}{2}(3^n-1)$ symbols all O, and then adding another copy of row n. (Readers will remember the Thue sequences, JCMN 20 and JCMN 28) The first row does not have any X midway between two others, and by induction the other rows have the same property. There is an infinite sequence of which each row above is an initial segment. Let a_1, a_2, ... be the positions of the symbol X in the infinite sequence defined above, and let $A = \{a_1, a_2, ..\}$ Then $\sum 1/a_n = S(A)$ is the constant C mentioned above, and so $C \leq C(3)$. But is $C = C(3)$?

The value $C = S(A)$ that we want to calculate is

$1 + \frac{1}{2} + \frac{1}{4} + 1/5 + 1/10 + 1/11 + 1/13 + 1/14 + 1/28 + 1/29 + 1/31 + \ldots$

Grouping these terms in subsets of 2^n terms, we want the sum of the series $1 + u_1 + u_2 + u_3 + \ldots$, where

$$u_1 = 1/2$$
$$u_2 = 1/4 + 1/5$$
$$u_3 = 1/10 + 1/11 + 1/13 + 1/14$$
$$\ldots\ldots$$

Each u_n is the sum of the reciprocals of 2^{n-1} numbers, of which the smallest is $3^{n-1} + 1$ and the largest is $\frac{1}{2}(3^n + 1)$. My computer gives the following values for u_n, for $S_n = 1 + u_1 + u_2 + \ldots + u_n$, and for the ratios $r_n = u_n/u_{n-1}$.

TABLE 1

n	Term u_n	Sum S_n	Ratio $r_n = u_n/u_{n-1}$
1	·5	1·5	
2	·45	1·95	·9
3	·3392607392607393	2·2892607392607393	·7539127539127539
4	·2364381692542547	2·5256989085149940	·6969216944155151
5	·1600415891549443	2·6857404976699383	·6768855877193119
6	·1072419419658678	2·7929824396358061	·6700879598367492
7	·0716171089722268	2·8645995486080329	·6678087664154811
8	·0477720172298228	2·9123715658378557	·6670475521198269
9	·0318540776937432	2·9442256435315989	·6667936491042119
10	·0212374001716254	2·9654630437032243	·6667089964371140
11	·0141585664445818	2·9796216101478062	·6666807768447401
12	·0094391108900863	2·9890607210378925	·6666713700876463

Convergence of the series is slow. The extrapolation methods of Aitken or Lubkin may be used to give a better

estimate for the sum, they both give an accuracy of 5 decimal places.

This table seems to show that the ratio r_n converges to the limit 2/3. In order to investigate this phenomenon numerically we calculate the differences $r_n - r_{n-1}$ and then the ratios of these differences, i.e. $(r_n - r_{n-1})/(r_{n-1} - r_{n-2})$. That is the second column in Table 2 below. These values seem to converge to 1/3. Intrigued by this, we repeat the process, calculating the ratios of differences from the previous column, these values are given in the third column. Again we find convergence to 1/3. Repeating the process again, the same thing happens, though by now there is not much accuracy left in our figures, because of the subtractions.

TABLE 2

4	·390117			
5	·351566			
6	·339269	·318979		
7	·335292	·323369		
8	·333984	·329038	1·291142	
9	·333550	·331770	·482014	
10	·333406	·332797	·375703	·131389
11	·333357	·333153	·346789	·271981
12	·333341	·333273	·338711	·279364

Now we must try to interpret these figures. If the ratios r_n were all exactly 2/3 then the terms would be $u_n = (2/3)^{n-1}a$ for some constant a ($= u_1$); in consequence the sum $S = \lim S_n$ would be $S_n + 2u_n$. Therefore we take $S_n + 2u_n$ as the first asymptotic approximation to the sum S. It gives an accuracy of 6 decimal places when $n = 12$.

Now suppose that the ratios of differences in the second column of Table 2 had all been 1/3. Then the ratios r_n would have had to be of the form $2/3 + 3^{-n}c$ for some constant c. This would have implied that $u_n = (2/3)^{n-1}a + (2/9)^{n-1}b + O((2/27)^n)$ as $n \to \infty$. If u_n were exactly $(2/3)^{n-1}a + (2/9)^{n-1}b$ then S would be exactly $S_n + 4(5u_n - u_{n-1})/7$. Therefore this is taken as the second asymptotic approximation. Similarly the convergence of the third column of Table 2 to 1/3 indicates that $u_n \sim (2/3)^{n-1}a + (2/9)^{n-1}b + (2/27)^{n-1}c$ and gives the next asymptotic approximation:

$$S_n + (554u_n - 148u_{n-1} + 8u_{n-2})/175$$

With n = 12 this gives the sum of the series as 3·007938999898937, probably correct to the 14th decimal place.

The terms u_n may similarly be analysed for asymptotic properties by looking at ratios of differences of successive values. It appears that

$$u_n \sim a(2/3)^{n-1} + b(2/9)^{n-1} + c(2/27)^{n-1} + d(2/81)^{n-1}$$

where $a = \cdot 8164629274136638$

$b = -\cdot 510207329$

$c = \cdot 30707$

$d = -\cdot 176$

There are heuristic reasons for expecting the first term in the asymptotic expression above to be a multiple of $(2/3)^n$, for u_n is a sum of 2^n terms each of order 3^{-n}. But could a more detailed argument predict the coefficient? And could the form of the other terms be established? The moral of this exercise seems to be that if you want the sum of a series it may be a good idea to calculate a few terms very accurately, and study them carefully.