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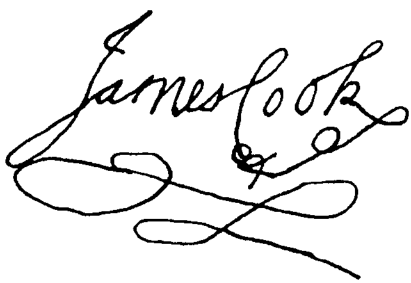

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THE LEMMA OF KÖNIG

George Szekeres

D. König's lemma, referred to in the solution of Blanche Descartes' problem (2), states that a (countably) infinite directed tree in which every vertex is of finite degree contains a path of infinite length. A directed tree T is a connected directed graph with no circuits and with a distinguished vertex 0 called the root of the tree. It is assumed (for the purposes of the lemma) that (i) all branches at 0 are oriented away from 0, and (ii) at any other vertex P exactly one branch is oriented towards P. The degree of a vertex is the number of branches in the path.

A few words about König himself will not be amiss. Dénes König was a noted Hungarian mathematician whose pioneering work in graph theory has been largely responsible for the emergence of a flourishing school of graph theory in Hungary which persists to our day. He wrote the first ever treatise on graph theory at a time when it was still regarded as a sort of poor man's topology. In the book (Theory of Finite and Infinite graphs) he has established the standard terminology of the subject, including the name "graph" (which incidentally he attributes to Sylvester). One of his most often quoted results states that a regular bipartite graph of degree k (one in which every vertex has the same degree k)

always contains a spanning bipartite subgraph of degree 1. This is a beautiful instance of a genuinely graph-theoretical and non-trivial theorem.

König taught at the Technological University of Budapest and gave a course of calculus to first-year chemical engineering students until 1927. I myself enrolled in the faculty the following year and so unfortunately missed out by one year having him as a teacher. In October 1944 he took his own life, to avoid the sufferings and humiliation of his impending deportation to a concentration camp.

Returning to König's lemma, the proof is simple enough. Consider the endpoints P_{11}, $P_{12}, \dots$ of the branches emanating from 0. Since there are altogether infinitely many branches in T, but only finitely many P_{1j}, at least one of them, say $P_1 = P_{11}$, has the property that there are arbitrarily long paths emanating from it. Let P_{21}, P_{22}, ... be the endpoints of branches emanating (away) from P_1. Since there are no closed circuits in T, these are different from 0 and all the P_{1j}. Again there is one, $P_2 = P_{21}$ say, from which arbitrarily long paths emanate, etc. We obtain in this fashion a sequence of distinct points $P_0 = 0$, P_1, P_2, ..., and clearly the path $P_0P_1P_2P_3 \dots$ has infinite length.

The argument is a striking instance of a pure existence proof; it gives no indication whatever how to find, say, P_1. Imagine that both P_{11} and P_{12} are extendible to 1000 steps. What about 10^{10} steps? Even if both survive 10^{10} steps, it is still uncertain what happens afterwards. The only thing that can be said with certainty is that at least one of them will have an infinite extension. The lemma owes its usefulness precisely to the fact that it is non-constructive, that it makes a pure existence statement about infinity. Most of us believe, of course, that if a statement has been proved with the help of König's lemma then it must be true, even if it cannot be verified directly.

TWO PROBLEMS OF BLANCHE DESCARTES

(JCMN 41, p. 4214)

George Szekeres

(1) Is it true that any Gaussian integer $a + bi$ (with a and b integers) can be expressed in one and only one way as a finite sum of distinct integral powers of $(i - 1)/2$?

The statement is true, provided that we agree that an empty sum represents 0. Indeed $((i - 1)/2)^{-1} = -(1 + i) = g$ and we show first that every Gaussian integer is a sum of distinct positive integral powers of g. Use induction on the Gaussian norm $N(a + ib) = a^2 + b^2$. If $N(a + ib)$ is even then $a + ib$ is divisible by g (since the Gaussian integers form a Euclidean unique factorisation domain, $N(g) = 2$, and the conjugate of g is $i - 1 = -ig$), hence $a + bi = (c + di)g$. Since $N(c + di) = \frac{1}{2}N(a + bi)$, $c + di$ is sum of distinct powers

of g, by the inductive hypothesis, and so is $a + bi$. If $N(a + bi)$ is odd, the division algorithm gives

$$a + bi = (c + di) . g + r + si$$

with $N(r + si) < N(g) = 2$, hence $N(r + si) = 1$. It follows that $r + si$ is one of $1, -1, i, -1$. There are four choices for $c + di$ and one of them gives $r + si = 1$. Induction can now be applied, provided that $N(c + di) < N(a + bi)$. The triangle inequality shows that this will always be so, unless $N(a + bi)$ is 1 or 2. This leaves 8 individual numbers to be represented so as to start off the induction. Here is the list: $1, g, i = 1 + g + g^2, -1 = 1 + g, -1 = 1 + g^2 + g^3 + g^4$, $1 + 1 = (-1) . g = g + g^3 + g^4 + g^5$, $1 - i = i . g = g + g^2 + g^3$,

For uniqueness of the representation it is sufficient to show that 0 cannot be represented as a finite sum or difference of 1 and distinct positive integral powers of g. But clearly $1 \neq \pm g^{k_1} \pm g^{k_2} \pm \ldots$ since the right hand side is divisible by g and the left hand side is not.

(2) Define a finite sequence $p(0), p(1), \ldots p(N)$ of integers to be "phondic" when

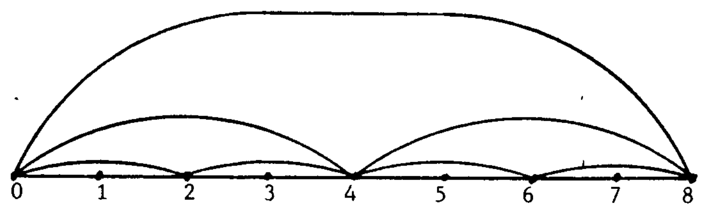
$$p(2^m n) \not\equiv p(2^m(n + 1)) \pmod 4$$

for any non-negative m and n with $2^m(n + 1)$ N. The definition extends in the obvious way to infinite sequences. Let

$$0 \leq s(0) < s(1) < \ldots < s(M) \leq N$$

be a sequence of integers. Prove that there exists a phondic sequence $P(n)$ (for $n = 0, 1, \ldots N$) such that $R(n) = P(s(n))$ is also phondic. A supplementary question is whether the same is true for infinite sequences.

We show that the statement is a consequence of the four-colour theorem; not altogether surprising considering that the problem comes from Blanche D. Consider a graph G whose points are the integers $0, 1, 2, \ldots, N$ and in which $k = 2^m n$, n odd, is joined by edges to the points $k + 2^m$ and $k - 2^m$. For instance, consecutive integers are joined because one of them is odd, and 0 is joined to all points 2^m, $m \geq 0$. Now interpret residues mod 4 as colours, then a phondic sequence is simply a 4-colouring of G in which vertices linked by an edge receive distinct colours. The tail end of the graph looks like this:



The graph from 2^m to 2^{m+1} is just a translation by 2^m of the graph from 0 to 2^m, with an additional edge linking 0 to 2^{m+1}. Therefore G is clearly planar.

Now, given the sequence $0 \leq s(0) < s(1) < \ldots < s(M) \leq N$ construct a similar graph G* with vertices $s(0), s(1), \ldots, s(M)$ such that $s(k)$, $k = 2^m n$ is joined to $s(k + 2^m)$ and $s(k - 2^m)$, but this time we draw the joining arcs <u>below</u> the number line. Clearly the union of G and G* will still be planar (i.e. representable in the plane without crossing edges), hence by the four-colour theorem, 4-colourable. Any 4-colouration

supplies a phondic sequence for G and G*. The result can be extended to infinite sequences because the four-colour theorem does extend to countably infinite planar graphs, by the lemma of König (see article).

Following König let me prove then that a countably infinite planar graph is 4-colourable. Let $Q_0, Q_1, Q_2, \ldots$ be the vertices of G. Denote by G_i, $i = 1, 2, 3, \ldots$ the (finite) subgraph of G spanned by the vertices $Q_0, \ldots, Q_i$. Represent each admissible 4-colouring of G_i (with Q_0 receiving a fixed colour) by vertices $P_{i1}, \ldots, P_{ik}$ of a tree T. There exists at least one P_{ij}, because of the four-colour theorem (assumed to be true), and there are of course only a finite number of such colourings, since G_i is finite. Let P_{ij} $P_{i+1,j*}$ be an edge (or branch) of T if and only if P_{ij} has the colouring induced by $P_{i+1,j*}$ in G_i(which is a subgraph of G_{i+1}). We thus get a directed tree to which König can be applied. The result is a 4-colouring of G. Of course it is again impossible to tell which among the 4-colourings of G_i are continuable to a 4-colouring of every G_j, $j > i$; we only know that such a 4-colouring does exist.

It would be interesting to have an effective 4-colouring of Blanche Descartes' graph; perhaps the lady should be consulted about it.

CONGRATULATIONS FROM JCMN

George Szekeres has been elected to the Hungarian Academy of Science.

CAMELOT REVISITED

Marta Sved

King Arthur looked at his knights gathered round the table. - Only n of you are here! -

- The others are out on a quest. -

- Well - said King Arthur - this is just lucky. I had intelligence that we are to have an invasion of n knights from Scotland. -

- One for each of us! - exclaimed Sir Lancelot.

- We can make arrangements who should take on whom. -

- There are too many arrangements - said Sir Bedivere - there is a choice of n knights, and there are n of us, so this makes n^n. -

- Not so fast, Sir Bedivere; what if you and I choose the same Scottish knight, that would mean that one of them is left out. So there is a choice of $n-1$ knights only, making $(n-1)^n$ choices for all of us. So we have to discard that case. That makes only $n^n - (n-1)^n$. — corrected Sir Mordred.

- But, Sir Mordred! - interjected Sir Gawain - you did not take into account that those $n-1$ knights could be selected in $\binom{n}{n-1}$ ways. So we would have only

$$n^n - \binom{n}{n-1}(n-1)^n \text{ possibilities}$$

for the jousts. -

- You are both wrong - cried out Sir Gareth - You forgot that those arrangements you excluded are counted the wrong way. $(n-1)^n$ possibilities for the rounds with $n-1$ knights chosen only means that at most $n-1$ knights are in those fights. What if we counted too many times the arrangement where at least two Scottish knights are left out ... -

At this moment Queen Guinevere raised her voice. She had strolled in casually and overheard the conversation.

- It is so complicated! Perhaps I am simple-minded, but the way I see it is that if Sir Lancelot picks one of the knights he has n choices. That leaves only $n-1$ choices for Sir Gawain and $n-2$ for Sir Mordred, and so on, so that there are

$$n(n-1)(n-2) \;....\; 1 \text{ choices.} -$$

- Listen to simple commonsense - said Merlin, the wise man, who could see things past, present and future so clearly.

- The Queen has certainly the simplest solution. However, if you, knights, would carefully continue your arguments, you would also arrive at the correct answer. You see, there is a simple principle; they will call it the inclusion-exclusion principle in the future.

- Odd name! - said Sir Gawain.

- As odd as the way you wanted to count the number of arrangements for the jousts. -

- ? ? ? -

- It goes like this. Here we have our large, round, black shiny table. Now, in the other rooms of the castle there are N smaller tables, but only N_R of them are round, only N_B are black and only N_S are shiny. So there are T tables which are neither round, nor black, nor shiny. -

- Yes, there are

$N - N_R - N_B - N_S$ tables not round, black or shiny. - said Sir Mordred.

- What of the round black tables, or the black shiny ones? You subtracted them twice! The answer should be

$N - N_R - N_B - N_S + N_{RB} + N_{RS} + N_{BS}$. - said Sir Lancelot.

- What if a table is round, black and shiny? - said Merlin - It was subtracted three times by Sir Mordred, then added three times by you, Sir Lancelot. -

With the authority so characteristic of him, King Arthur announced:-

- The number of tables T, that are not round, black or shiny is

$$T = N - N_R - N_B - N_S + N_{RB} + N_{RS} + N_{BS} - N_{RBS}. \quad -$$

- Yes, indeed, your Majesty. This is also the way to solve the problem of our guests:

Sir McA, Sir McB, Sir McC, ..., Sir McN.

There are $(n-1)^n$ arrangements which leave Sir McA out, and the same applies when Sir McB or Sir McC or ... Sir McN is left out, hence the sum subtracted from n^n is

$$\binom{n}{1}(n-1)^n.$$

Correct this by adding $\binom{n}{2}(n-2)^n$ arrangements, as the sum of arrangements leaving out Sir McA together with Sir McB or Sir McA together with Sir McC, and so on.

The next correction is achieved by adding the sum of possible arrangements when any group of 3 knights is left without opponents. So we go on to obtain

$$n^n - \binom{n}{1}(n-1)^n + \binom{n}{2}(n-2)^n - \binom{n}{3}(n-3)^n + \dots + (-1)^n . 1^n,$$

or if you prefer, we can instead of $\binom{n}{1}$ write $\binom{n}{n-1}$. instead of $\binom{n}{2}$ put $\binom{n}{n-2}$, for leaving out k knights comes to the same as leaving in the remaining $n-k$.

In that case we get the same sum as above and it reads

$$n^n - \binom{n}{n-1}(n-1)^n + \binom{n}{n-2}(n-2)^n + \dots + (-1)^n . 1^n$$

You can check it, that this gives the same answer as Queen Guinevere's solution. -

Merlin's countenance changed. His eyes stared into the distant future:

- Behold! This is just Binomial Identity 18 which will be sent to JCMN by someone named Jamie Simpson. It will read

$$\sum_{j=1}^{n} \binom{n}{j} (-1)^j j^n = (-1)^n n! \quad -$$

(The factorial sign also represents a happy exclamation mark as Merlin finished his words.)

BOILING THE BILLY

Jim B. Douglas

Consider boiling a saucepan of water over a gas flame. Obviously if the flame is too low it will never boil the water. Equally obviously if the flame is too high much of the heat passes by the sides and is wasted. It seems plausible that there should be an optimum level to which the flame is best adjusted, so as to boil the water with the use of a minimum amount of gas. Formulate a mathematical model, and solve it for the optimum.

BINOMIAL IDENTITY 18

(JCMN 40, p. 4190)

J. B. Parker

Define $W(n, m) = \sum_{j=1}^{n} \binom{n}{j} (-1)^j j^{n-m}$ for $1 \leq n \geq m$.

The formula suggested is $W(n, 0) = (-1)^n n!$

Remark: $W(n, n) = (1-1)^n - 1 = -1$

Lemma 1 $W(n, m) + W(n+1, m)/(n+1) = W(n+1, m+1)$.

Proof

$$\text{LHS} = \sum_{j=1}^{n} (-1)^j j^{n-m} \left\{ \frac{n!}{j!\,(n-j)!} + \frac{n!}{j!\,(n+1-j)!}\, j \right\} + (-1)^{n+1} (n+1)^{n-m}$$

$$= \sum_{j=1}^{n} (-1)^j j^{n-m} \frac{n!\,(n+1-j+j)}{j!\,(n+1-j)!} + (-1)^{n+1} (n+1)^{n-m}$$

$$= \sum_{j=1}^{n+1} (-1)^j \binom{n+1}{j} j^{n-m} = W(n+1, m+1) \qquad \text{QED}$$

Lemma 2 $W(n, m) = 0$ for $0 < m < n$.

Proof

Put $m = n$ in Lemma 1, giving $-1 + W(n+1, n)/(n+1) = -1$, so that $W(n+1, n) = 0$ for all n; i.e. $W(n, n-1) = 0$ as well. Thus Lemma 2 is true for $m = n-1$. Now put $m = n-1$ in Lemma 1, giving $W(n, n-1) + W(n+1, n-1)/(n+1) = W(n+1, n)$. Thus (using the $m = n$ result) $W(n+1, n-1) = 0$. Proceeding as before, but setting $m = n-2, n-3, \ldots 1$, we deduce $W(n, m) = 0$ for $0 < m < n$, which proves Lemma 2.

We now write $m = 0$ in Lemma 1, giving

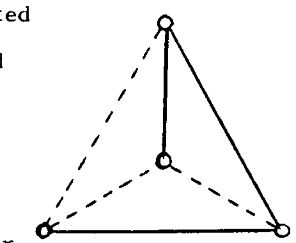
$W(n, 0) + W(n+1, 0)/(n+1) = W(n+1, 1) = 0$ by Lemma 2.

Hence $W(n, 0) = A(-1)^n n!$ where A is some constant, easily seen to be unity by considering the $n = 1$ case.

EDGE-COLOURED GRAPHS

Given any n, for what values of m is the following proposition true? If the complete graph on m nodes has its $m(m-1)/2$ edges coloured with n colours, then the graph contains a monochromatic circuit (i.e. contains a non-trivial simple closed path with all edges the same colour).

For example, when $n = 2$, the proposition holds for $m \geq 5$. The failure for $m = 4$ is illustrated by the graph – red edges are indicated by dotted lines and blue edges by full lines. The truth for $m \geq 5$ and $n = 2$ may be established as follows. Take a complete graph on 5 nodes with edges either red or blue. There are two possibilities. Case 1, one node has three or four edges to it all the same colour, clearly there is a monochromatic triangle. Case 2, every node has two red edges and two blue edges to it. Then through each node there is both a red circuit and a blue circuit. For $m > 5$ the result follows by considering a 5-node subgraph.



AUTOMATIC SPECTRAL ANALYSIS ?

J. H. Loxton

Spectral analysis is a redoubtable weapon in the hands of the chemist and the sun-spot cyclist. Pythagoras, by way of historical precedent, saw how numbers were the basis of music and therefore astronomy (the vibrating string and the music of the spheres) and thereby almost discovered the fourier transform. What follows concerns an attempt to play tunes with some elementary sequences and so to analyse their complexity. Fourier, who complained a lot about mathematicians, would doubtless disapprove.

The famous $r_3(n)$ of Erdös and Turán is the least r such that any sequence $1 \leqslant a_1 < a_2 < \dots < a_r \leqslant n$ of r numbers not exceeding n must contain a three term arithmetic progression. The best bounds so far obtained are

$$n\, e^{-c\sqrt{\log n}} < r_3(n) < c'\, n/\log\log n .$$

The spectacular combinatorics of Szemeredi and the ergodic theory of Furstenberg have shown that a sequence of integers of positive density contains arbitrarily long arithmetic progressions. This is a rich area of mathematics – Szemeredi won \$1000 from Erdös for his theorem – and there are many good problems still to be resolved. For example, the ideas of Szemeredi and Furstenberg have not yet led to quantitative results for $r_k(n)$, defined in the same way as $r_3(n)$ for k term arithmetic progressions.

The obvious way to construct sequences not containing three terms in arithmetic progression is to use the greedy algorithm. The

resulting sequences are not very dense. For example, this process yields the sequence

$$A(1):\ 0,1,3,4,9,10,12,13,27,28,30,31,36,37,39,40,81,82,84,85,\dots .$$

The number of terms of $A(1)$ not exceeding n is approximately $n^{\log 2/\log 3} = n^{0.63\dots}$ which is much smaller than $r_3(n)$. Nevertheless, let $A(m)$ be the sequence obtained by starting with $a_0 = 0$ and $a_1 = m$ and then taking each subsequent a_{n+1} to be the least integer greater than a_n so that $\{a_0, a_1, \dots, a_{n+1}\}$ does not contain three terms in arithmetic progression. The problem raised by Odlyzko and Stanley is to analyse the behaviour of these sequences. (For references to all this and more background, see R. K. Guy, "Unsolved problems in number theory" (Springer, 1981), problem E10.)

Since it seemed a good idea at the time, I computed some fourier transforms. Figure 1 is an approximation to the transform of the sequence $A(1)$ – the function $\alpha_1(\theta) = \sum_{n<1000} \exp(2\pi i a_n \theta)$, where a_n is the n-th term of the sequence, is plotted against θ.

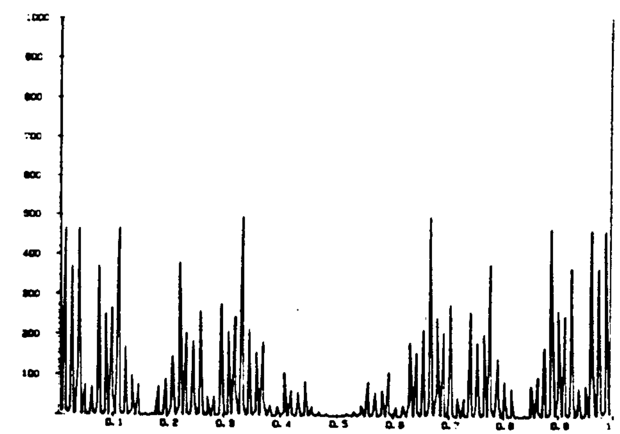


Figure 1. $\alpha_1(\theta)$.

If the terms of the sequence were evenly distributed into the three residue classes mod 3, then $\alpha_1(1/3)$ and $\alpha_1(2/3)$ would be small. The peaks in the transform at 1/3 and 2/3 therefore indicate that the terms a_n are not evenly distributed mod 3 and examination of the sequence suggests a_n is never congruent to 2 mod 3. The smaller peaks at $\theta = 1/9, 2/9$, and so on, indicate bad distribution mod 9 and it seems $a_n \not\equiv 2, 5, 6, 7$, or 8 mod 9. Further peaks at $\theta = p/3^k$ suggest the rule behind the sequence is 3-adic and, in fact, expressing a_n in base 3 makes it clear. The elements of A(1) are just the integers whose base 3 representations do not contain the digit 2. It is not too hard to see that this property yields a sequence which contains no three term arithmetic progressions and that attempting to put in any number with a digit 2 in the original construction by the greedy algorithm would form a three term arithmetic progression.

Now consider the sequence

A(6): 0,6,7,9,10,15,16,19,27,33,34,36,37,42,43,46,81,87,88,90,... .

The transform based on the first 1000 terms, namely $\alpha_6(\theta) = \sum_{n<1000} \exp(2\pi i a_n \theta)$, is shown in figure 2. Again, this is clearly 3-adic and the rule can be discovered by playing around with base 3 representations. (The answer, if needed, can be found at the end

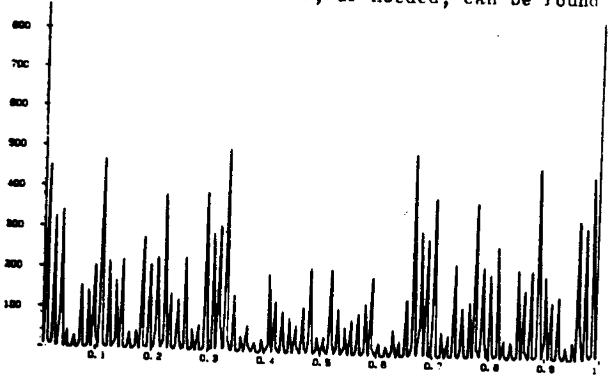


Figure 2. $\alpha_6(\theta)$.

of the article.) In fact, if m is a power of 3, or twice a power of 3, the numbers of the sequence A(m) can be described fairly simply in terms of their base 3 representations. This means that these sequences can be generated by finite automata which are, roughly speaking, computers without memory. In the hierarchy of sequences, the simplest are the periodic sequences and the next are these automatic sequences which are only a little more complicated. (The algebraic view of all this is developed in S. Eilenberg, "Automata, languages and machines" (Academic Press, 1974), vol. A, especially chapter 15.)

What about the sequence

A(4): 0,4,5,7,11,12,16,23,26,31,33,37,38,44,49,56,73,78,80,95,... ?

Its transform, $\alpha_4(\theta) = \sum_{n<1000} \exp(2\pi i a_n \theta)$, shown in figure 3, is much more erratic and the nice 3-adic behaviour seems to be missing. Can A(4) be generated by a finite automaton, or is it more complicated? If so, why? Note that the construction by the greedy algorithm is not automatic because it requires memory of all the earlier terms of the sequence. My analytic machinery is not yet sharp enough to make any firm deductions from figure 3. Incidentally, it seems that A(4) grows at the same rate as A(1), but I cannot even prove this.

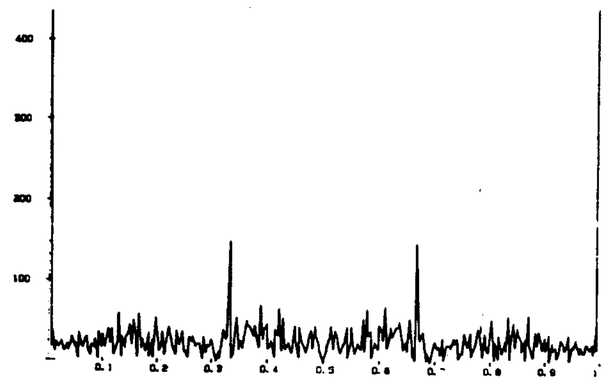


Figure 3. $\alpha_4(\theta)$.

Footnote. The numbers in the sequence A(6) are again those whose representations in base 3 have no twos, except possibly in the three least significant digits. These three digits can be

000, 020, 021, 100, 101, 120, 121 or 201.

POWER MEAN INEQUALITY

Dmitry P. Mavlo

Let $\underline{x}$ be a vector of non-negative components $(x_1, x_2, \ldots, x_n)$, and let $\underline{x}^k$ denote the vector $(x_1^k, \ldots, x_n^k)$. Write A and Γ for the arithmetic and geometric means respectively. Prove that

$$A^k(\underline{x}) - \Gamma^k(\underline{x}) \geq n^{1-k}(A(\underline{x}^k) - \Gamma(\underline{x}^k))$$

for positive integer k. Find the cases of equality.

QUAINT IDENTITY

$$\left\{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-1}\right\}^{\frac{1}{2}} - \left\{\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}-1}\right\}^{\frac{1}{2}} = \sqrt{2}$$

BINOMIAL IDENTITY 19

(JCMN 41, p. 4213)

J. B. Parker

The identity was that if N = 2M then $\sum_{k=0}^{N} \binom{N}{k}^3$

$$= 2^{-N} \sum_{j=0}^{M} \binom{N}{2j} \left\{\frac{(N+2j)!\,(N-2j)!}{(M+j)!\,(M-j)!\,N!}\right\}^2$$

$$= 2^{3N-2}\,\pi^{-2} \int_0^{2\pi}\int_0^{2\pi} \cos^N(x-y)\cos^N x \cos^N y \, dx\, dy.$$

First note that $(1+x)^N(1+y)^N(1+\frac{1}{xy})^N$

$$= (1+x)^N(1+y)^N(1+xy)^N(xy)^{-N} = T^{-N}(1+T)^{2N}(1+S/(1+T))^N$$

where S = x + y and T = xy.

The coefficient of unity on the LHS is the required $\sum_{k=0}^{N}\binom{N}{k}^3$. Now on the RHS we may omit the odd powers of S, which are irrelevant, and obtain

$$T^{-N}(1+T)^{2N} \sum_{r=0}^{M} \binom{N}{2r} S^{2r}(1+T)^{-2r}$$

Put $S^{2r} = \binom{2r}{r}T^r$ + irrelevant terms which we leave out. We therefore have $\sum_{r=0}^{M}\binom{N}{2r}T^{r-N}\binom{2r}{r} \sum_{u=0}^{2N-2r}\binom{2N-2r}{u}T^u$

The coefficient of unity is $\sum_{r=0}^{M}\binom{N}{2r}\binom{2r}{r}\binom{2N-2r}{N-r}$

and we therefore have the new Binomial Identity 19*

$$\sum_{k=0}^{N} \binom{N}{k}^3 = \sum_{r=0}^{M} \binom{N}{2r} \binom{2r}{r} \binom{2N-2r}{N-r}$$

It holds also for odd N if the summation is taken for r from 0 to the integral part of N/2.

To prove the last lap of B.I. 19 is straightforward, viz.:-

$$2^{3N-2}\,\pi^{-2} \int_0^{2\pi}\int_0^{2\pi} \cos^N(x-y)\cos^N x \cos^N y \,dx\,dy$$

$$= 2^{3N-2}\,\pi^{-2} \int_0^{2\pi}\int_0^{2\pi} \cos^N x \cos^N y \sum_{j=0}^{M} \binom{N}{2j} \cos^{2j}x \cos^{2j}y \sin^{N-2j}x \sin^{N-2j}y \,dx\,dy$$

(odd powers don't contribute)

$$= 2^{3N-2}\,\pi^{-2} \sum_{j=0}^{M} \binom{N}{2j} \left\{\int_0^{2\pi} \cos^{2j} t \sin^{N-2j} t \,dt\right\}^2$$

$$= 2^{3N} \sum_{j=0}^{M} \binom{N}{2j} \left\{\frac{(N+2j)!\,(N-2j)!}{(M+j)!\,(M-j)!\,N!}\, 2^{-2N}\right\}^2 \qquad \text{QED}$$

In order to establish the first part of B.I.19, express the double integral as an integral mean

$$2^{3N-2}\,\pi^{-2} \int_0^{2\pi}\int_0^{2\pi} \cos^N(x-y)\cos^N x \cos^N y \,dx\,dy$$

$$= 2^{3N} \underset{(x,\,y)}{\text{I.M.}} \cos^N(x-y)\cos^N x \cos^N y$$

(Now changing the variables by $u = x + y$ and $v = x - y$)

$$= 2^{2N} \underset{(u,\,v)}{\text{I.M.}} \cos^N v\,(\cos u + \cos v)^N$$

$$= 2^{2N} \underset{(u,\,v)}{\text{I.M.}} \sum_{r=0}^{N} \binom{N}{r} \cos^r u \cos^{2N-r} v$$

$$= 2^{2N} \sum_{r=0}^{N} \binom{N}{r} (\underset{u}{\text{I.M.}} \cos^r u)\,(\underset{v}{\text{I.M.}} \cos^{2N-v} v)$$

(recalling that $\underset{x}{\text{I.M.}} \cos^r x$ is zero for odd r and $2^{-2j}\binom{2j}{j}$ where $r = 2j$)

$$= \sum_{j=0}^{N} \binom{N}{2j} 2^{2N-2j-2N+2j} \binom{2j}{j} \binom{2N-2j}{N-j}$$

and by invoking B.I.19* the first part of B.I.19 follows.

INTEGRAL INEQUALITY

George Szekeres

In a Ph.D. thesis on hidden variables in quantum mechanics by D.E. Liddy, the following inequality appeared.

$$\int_{\phi=0}^{\pi} \sin^2\phi \,d\phi \int_{\psi=0}^{\pi} \left\{1 - (\sin\theta \sin\phi \cos\psi + \cos\theta \cos\phi)^2\right\}^{\frac{1}{2}} d\psi \geqslant \frac{4\pi}{3}\cos\theta$$

for $0 \leqslant \theta \leqslant \pi/2$.

It was verified by computer; is there an analytic proof?

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