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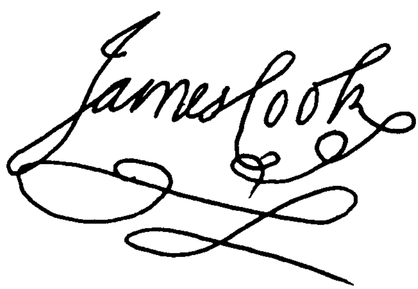

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CONTENTS

VARIATIONS ON PYTHAGORAS

H.S.M. Coxeter, G.F.D. Duff and L. Havercamp

1. Sturm's Inequality Generalized.

In any Euclidean space E^n, consider a finite set of vectors $u_1, \ldots, u_N$. Since

$$\sum_{i<j} (u_i - u_j)^2 = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (u_i - u_j)^2$$

$$= \frac{1}{2} \left(\sum_{i=1}^{N} u_i^2 \sum_{j=1}^{N} 1 - 2 \sum_{i=1}^{N} u_i \sum_{j=1}^{N} u_j + \sum_{i=1}^{N} 1 \sum_{j=1}^{N} u_j^2 \right)$$

$$= N \sum_{i=1}^{N} u_i^2 - \left(\sum_{i=1}^{N} u_i \right)^2 ,$$

The sum of the squares of the $\binom{N}{2}$ *mutual distances of* N *distinct points is less than or equal to* N *times the sum of the squares of the distances of the* N *points from another point* O, *with equality only when* O *is the centre of gravity of the* N *points.*

The case when $N = 3$ is a known property of a triangle, described in a book by R. Sturm [11, p. 71]; see also [1, p. 117 (12.53)].

As a corollary, we see that *The moment of inertia, about a point* O, *of equal masses at fixed points* $P_1, \ldots, P_N$, *is a minimum when* O *is the centre of gravity of the points* P_i.

In particular, *if* N *points are all at the same distance* R *from their centre of gravity, the sum of the squares of their* $\binom{N}{2}$ *mutual distances is equal to* N^2R^2.

J.J. Seidel remarks that such a set of points (with $R = 1$) is a spherical 1-design [3, pp. 259-261].

The present note evolved from the empirical observation that the sum of squares is N^2R^2 whenever the N points are the vertices of a regular or semi-regular polytope.

The particular case of two pairs of opposite points on a circle is readily seen to be equivalent to Pythagoras' theorem.

Though terms such as 'barycentre', 'centroid' or 'centre of mass' might be preferred in certain instances, for consistency we shall use 'centre of gravity' throughout.

2. Mass distributions.

Let μ_x be a measure in E^n. Then

$$\int_{E^n}\int_{E^n} (x - y)^2 d\mu_x d\mu_y$$

$$= \int_{E^n}\int_{E^n} (x^2 - 2x\cdot y + y^2) d\mu_x d\mu_y$$

$$= \int_{E^n} x^2 d\mu_x \int_{E^n} d\mu_y - 2\int_{E^n} x d\mu_x \cdot \int_{E^n} y d\mu_y + \int_{E^n} d\mu_x \int_{E^n} y^2 d\mu_y$$

$$= 2\int_{E^n} x^2 d\mu_x \int_{E^n} d\mu_y - 2\left(\int_{E^n} x d\mu_x \right)^2$$

$$= 2(M_2M_o - M_1^2)$$

where $M_k = \int_{E^n} x^k d\mu_x$, $k = 0,1,2$. We note that M_o (the total mass) and M_2 (the moment of inertia about the origin) are scalars. $\mathbf{M}_1$ is the vector first moment of the mass distribution about the origin so that $\mathbf{M}_1/M_o$ gives the position vector of the centre of gravity. The quotient M_2/M_o is often set equal to R^2, where R is the radius of gyration of the mass distribution about the origin. Hence the above integral is equal to

$$2(M_o^2R^2 - M_1^2) .$$

If the measure consists of a number N of discrete masses m_k at positions u_k, then we have

$$\sum_{j<k}^{N} m_j m_k (u_j - u_k)^2 = \sum_{k=1}^{N} m_k u_k^2 \sum_{k=1}^{N} m_k - \left[\sum_{k=1}^{N} m_k u_k \right]^2 .$$

Note that the sum on the left contains only terms with $j < k$. *The sum of the doubly weighted squares of the mutual distances between the masses is less than or equal to the total mass multiplied by the moment of inertia about the origin, with equality only when the origin is the centre of gravity.*

The earliest derivation of this identity was given by Lagrange in a memoir submitted to the Berlin Academy in 1783 [5], where he gives the following verbal description: "La somme de produits de chaque masse par la carré de sa distance à un point quelconque donné est égal au produit de la somme des masses par la carré de la distance de ce point au centre de gravité de toute ces masses, plus la somme des produits des masses multipliées deux à deux entre elles et par la carré de leur distances respectives, cette dernière étant dividée par la somme même des masses."

Lagrange suggests finding the centre of gravity by finding R^2 for a number of given points, and constructing spheres or circles of radius R about them, so that the centre of gravity is determined by their intersection.

Lagrange also refers to this identity in the Mécanique Analytique [6, Première partie, Section III, §IV, Para 20; Oeuvres, vol. 11, pp. 65-68]. He does not seem to have formulated the identity as a minimum property.

In his Mécanique Celèste, [8, vol. I, Chap. III, pp. 88-89], Laplace describes the identity in a somewhat different way, but without giving an explicit reference to Lagrange. H. Lamb, in his textbook on Statics [7, pp. 165-167] records the result as the second of two theorems of Lagrange.

In the classical n-body problem, this same identity reappears in the study of homographic solutions and of collapse to the centre of gravity [4, Vol. I, Chap. III, pp. 242,258; 8, vol. IV, p. 694]. Laplace showed that under Newtonian gravitation there exist possible motions of n bodies wherein their position vectors remain mutually in a constant cross-ratio while the angular velocities about the centre of gravity are all equal. The equilateral triangle of the Trojan satellite positions is an example of such a homographic solution. Earlier, Euler and Lagrange had found one and two

dimensional homographic solutions. When the number of bodies is at least 4, homographic motion in three space dimensions is possible, each particle moving on a straight line through the centre of gravity in a homothetic configuration. In the event of collapse to the centre of gravity, the above quadratic sums must vanish at that instant [4, Chap. 22; 10, p. 43].

3. A Probabilistic Analogue.

Let X_j, where $j = 1,\ldots,N$ be independent random variables with probability density functions $f_j(x) \geq 0$ where $\int_{-\infty}^{\infty} f_j(x)dx = 1$, [2, p. 9; 12, p. 8]. The joint probability density of $X_1,\ldots,X_N$ is then $f_1(X_1)f_2(X_2) \cdots f_N(X_N)$, [9, p. 159]. The expectation of X_j is $E(X_j) = \int_{-\infty}^{\infty} x_j f_j(x_j)dx_j$. Then

$$\begin{aligned} E\left[\sum_{i<j}(X_i - X_j)^2\right] &= \frac{1}{2}E\left[\sum_{i=1}^{N}\sum_{j=1}^{N}(X_i - X_j)^2\right] \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(x_i - x_j)^2 f_1(x_1)\cdots f_N(x_N)dx_1\cdots dx_N \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_i^2 - 2x_ix_j + x_j^2)f_i(x_i)f_j(x_j)dx_idx_j \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[E(X_i^2)1_j - 2E(X_i)E(X_j) + 1_iE(X_j^2)\right] \\ &= N\sum_{i=1}^{N}E(X_i^2) - \left[\sum_{i=1}^{N}E(X_i)\right]^2 \end{aligned}$$

The expectation of $\sum_{i<j}(X_i - X_j)^2$ *is less than or equal to* N *times the expectation of* $\sum_{j=1}^{N}X_i^2$, *with equality only if the expectation of* $\sum_{j=1}^{N}X_j$ *is zero.*

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BINOMIAL IDENTITY 19

P. A. Moran

If $N = 2M$ then $\sum_{k=0}^{N} \binom{N}{k}^3$

$$2^{-N} \sum_{j=0}^{M} \binom{N}{2j} \left(\frac{(N+2j)!\ (N-2j)!}{(M+j)!\ (M-j)!\ N!} \right)^2$$

$$= 2^{3N-2} \pi^{-2} \int_0^{2\pi} \int_0^{2\pi} \cos^N (x-y) \cos^N x \cos^N y \, dx\, dy$$

The sum with alternating signs, $\sum (-1)^k \binom{N}{k}^3$ is the subject of Binomial Identity 7 in JCMN 21, p.78 and JCMN 32, p.4012.

FROM CAPTAIN COOK'S JOURNAL

Monday, 26th December, 1768 - A Fresh breeze of Wind and Cloudy weather; passed by some Rock Weed. At noon the Observed latitude 26 miles to the Southward of the Log, which I believe is chiefly owing to her being Generally steer'd to the Southward of her Course. Yesterday being Christmas Day the people were none of the Soberest. Wind N.; course S.W.; distance 158 m.; lat. 40° 19' S., long. 54° 30' W.

DECK OF CARDS

J. B. Parker

You go through an ordinary pack of 52 cards one by one, if you come to successive cards the same regardless of suit (say two sixes or two kings) you call it a failure. What is the probability of success?

TWO PROBLEMS

Blanche Descartes

(1) Is it true that any Gaussian integer $a + bi$ (a and b integers) can be expressed in one and only one way as a finite sum of distinct integral powers of $(i - 1)/2$?

(2) Define a finite sequence $p(0), p(1), \ldots p(N)$ of integers to be "phondic" when

$$p(2^m n) \not\equiv p(2^m(n+1)) \pmod 4$$

for any non-negative m and n with $2^m(n+1) \leq N$. The definition extends in the obvious way to infinite sequences.

The problem is as follows. Let

$$0 \leq s(0) < s(1) < \ldots < s(M) \leq N$$

be a sequence of integers. Prove that there exists a phondic sequence $P(n)$ (for $n = 0, 1, \ldots N$) such that $R(n) = P(s(n))$ is also phondic. A supplementary question is whether the same is true for infinite sequences.

QUOTATION CORNER 21

The Japan Halley's Comet Association has rented one of the Siding Spring Observatory's telescopes and set up three of its own 20mm reflecting telescopes at vantage points around the town.

The Weekend Australian 28-29 Dec. 1985

HILBERT'S TENTH PROBLEM

Jamie Simpson

A diophantine equation is an equation in which the coefficients and absolute term are integers, and for which we seek integer solutions. A well known example is the equation associated with Fermat's Last Theorem.

Hilbert's 10th problem asks:

"Given a diophantine equation in any number of unknown quantities and with rational integral coefficients: to devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers."

The problem was settled in 1970 when Y. Matijasevic [2] showed that no such process exists. His result uses earlier results by Martin Davis, Hilary Putnam and Julia Robinson. A complete account of the solution is given by Davis in [1].

The proof uses objects called "diophantine sets". A set S of integers is diophantine if there exists a polynomial $f(x, y_1, y_2, \ldots, y_n)$ with integral coefficients such that S contains exactly those x for which

$$f(x, y_1, y_2, \ldots, y_n) = 0 \qquad (1)$$

is solvable in the $y_1, y_2, \ldots, y_n$.

EXAMPLE The set of even integers is diophantine, since these are the values of x for which

$$x - 2y = 0$$

is solvable for y.

Another property that a set of integers may or may not have is that of being "listable". Roughly speaking this means that the set can be listed by a computer. Any diophantine set is listable. This can be seen by noting that we can list all $(n+1)$-tuples $(x, y_1, \ldots, y_n)$ and check to see if they satisfy equation (1).

Matijasevic's great achievement was to show that any listable set is diophantine. Assuming this, the argument then proceeds as follows. The set of diophantine sets is countable. List them as $D_1, D_2, \ldots$ Now let

$$C = \{i \mid i \in D_i\}$$

We claim that the complement of C is not listable. Suppose it is, then it is diophantine and appears somewhere on our list, say as set D_k. We then get a Cantorian-type contradiction when we try to decide whether or not k belongs to the set D_k.

It is easy to show, however, that C is listable. It is therefore diophantine and associated with some equation

$$F(x, y_1, y_2, \ldots, y_n) = 0. \tag{2}$$

Now suppose that the process that Hilbert desired does exist. We could apply it to equation (2) with x taking the values 0, -1, 1, -2, 2, -3, ... Each x would then be assigned either to the set C or to its complement $\overline{C}$. Thus we would be listing $\overline{C}$. This contradiction completes the proof.

References

[1] M. Davis, "Hilbert's tenth problem is unsolvable" American Mathematical Monthly, March, 80 (1973) pages 233-269.

[2] Y. Matijasevic "The Diophantineness of enumerable sets", Doklady Akad. Nauk. SSSR, 191 (1970) 279-282; translation in Soviet Mathematics Doklady 12, (1971), 249-254

Editor's Note We would like to print articles on Hilbert's other problems.

BINOMIAL IDENTITY 18 (JCMN 40, p.4190)

George and Esther Szekeres

$$\sum_{j=1}^{n} \binom{n}{j} (-1)^j j^n = (-1)^n n!$$

This is actually an ancient identity going back to Abel. It is in Netta's old book on combinatorics (the first of its kind), also in Riordan's Combinatorial Identities. Its more general form is

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j (x-j)^n = n!$$

identically in x.

The first equation may be proved as follows. Consider words of length n written with an alphabet of n letters. The number of words that use the whole of the alphabet is n!. The number that can be written using part or all of the alphabet is $\binom{n}{n} n^n = n^n$. A term $\binom{n}{j} j^n$ counts the number of words that may be written by first choosing a subset of j of the alphabet, and then writing all possible words with this subset. The equation

$$\sum_{j=1}^{n} \binom{n}{j} (-1)^{n-j} j^n = n!$$

follows from the inclusion-exclusion principle.

A similar argument can be applied to show:

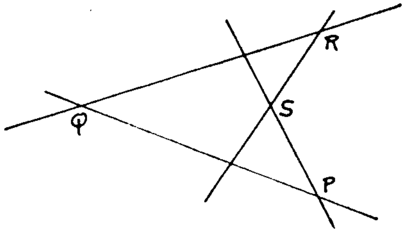
$$\sum_{j=1}^{n} \binom{n}{j} (-1)^j j^k = 0 \quad \text{where } 0 < k < n.$$

COCKED HATS

J. B. Parker

Readers who have followed the recent contributions on the symmedian point of a triangle will recall the position lines that are drawn by a navigator on a (plane) chart to represent the information from observations of position. The triangle formed in this way from three observations is sometimes called a "cocked hat". There is a theorem in the Admiralty Manual of Navigation that the probability of the true position being inside the cocked hat is 1/4. This result does not depend on the errors being Gaussian or of any other particular type of distribution, but only on the true position having probability ½ of being on each side of the position line.

The analogous problem for four position lines might be suitable for JCMN. Given four position lines in the plane, what is the probability that the true position is inside the (triangular) convex hull of the six points of intersection of the four lines?



Or perhaps instead of the convex hull PQR above it might be better to consider the region PQRS, which consists of all points that are inside two of the four cocked hats that may be obtained by omitting one of the position lines.

There is an analogous one-dimensional problem. Given n estimates $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ for an unknown, what is the probability of the true value being between x_1 and x_n?

THE HONEYMOON PROBLEM

A lucky bride and bridegroom had two wedding presents, one was a round-the-world air ticket for each of them with stops at N places, and the other was a voucher for a fortnight's hotel accommodation at any one of the N places at which they would call. They knew nothing in advance about the N places at which they might stay. How should they choose where to stop? This is a one-person game (note - "made one flesh" - from the marriage service) and in its abstract form is as follows.

There are N objects, arranged in an order of merit unknown to the player. They are offered to the player in turn and at stage r (r = 1, 2, ... N) the player is told the ordering of the r objects already offered, and must decide whether to accept the currently offered object. The game is over when an object is chosen. The player must choose one of the objects, and therefore must take the last object if none of the first N-1 has been chosen. The player tries to choose an object as near as possible to the top of the list. The score is calculated by giving N-1 points if the best object is chosen, N-2 if the second best, etc.

A strategy for the player is a sequence (s(1), s(2), ... s(N)) with the meaning that at stage r the object will be accepted if it is among the top s(r) of the r already offered. The last element s(N) of the strategy must be N. For example if N=3 the best strategy is (0,1,3) which means to reject the first object offered, then to accept the second if it is better than the first, and otherwise to reject the second and take the third. The expected score for this strategy is 4/3. For N=4 the two strategies (0,1,1,4) and (0,1,2,4) are equally good, they each give an expected score of 17/8.

Different settings for this problem have been appearing in various places for years. One of the latest is in the "Puzzle Corner" of the Bulletin of the IMA (Volume

22, 1986, page 30) in which there is a deplorably sexist story of a bachelor trying to choose the best of N girls for a wife. Other versions describe it as "The Secretary Problem" and tell a story of how someone interviews in succession N candidates for a job as secretary, and must decide at the end of each interview whether to offer the job to the candidate. Many of these problems ask for the best of the "restricted strategies" defined as follows.

A "restricted strategy" is a strategy (s(1), s(2), ...) in which s(N)=N and s(r) for all r = 1,2,...z all =0, and all other s(r)=1. The expected score for a restricted strategy may be calculated as follows. The probability that no choice has been made before stage r (where $z < r < N$) is reached can be seen by induction to be z/(r-1), the probability of the choice being made at stage r being (1/r)z/(r-1) = z/(r-1)-z/r. The expected score if the choice is made at stage r is the expectation of the largest of r numbers chosen at random from the set $\{0,1,2,\ldots,N-1\}$, which is (Nr-1)/(r+1). The probability of the last object being chosen is z/(N-1) and in that case the expected score is (N-1)/2. The expectation of score at the end of the game is therefore:

$$\sum_{r=z+1}^{N-1} \frac{z}{r(r-1)} \frac{Nr-1}{r+1} + \frac{z}{2}$$

$$= zN \sum \frac{1}{(r-1)(r+1)} - z \sum \frac{1}{r(r-1)(r+1)} + z/2$$

$$= N - \frac{N+1}{2}\left(\frac{z}{N} + \frac{1}{z+1}\right)$$

The integer z to maximize this is easily calculated numerically. For large N the first approximation is $z = \sqrt{N} - 1$, and this gives an expected score of $N - \sqrt{N} + \frac{1}{2} + O(1/\sqrt{N})$.

It is easy to see that the best strategy for N greater than 4 is not one of the restricted strategies, for an investigation of s(N-1) for the best strategy shows that it should be (N-1)/2 if N is odd and either N/2-1 or N/2 if N is even.

A computer search finds the best strategies for N up to 9 to be as shown in the table below.

N	Best strategy	Expected score	Score from the best restricted strategy
3	(0, 1, 3)	1·33333	1·33333
4	(0, 1, 1 or 2, 4)	2·125	2·125
5	(0, 1, 1, 2, 5)	2·95	2·9
6	(0,0,1,1 or 2,2 or 3,6)	3·78333	3·66667
7	(0,0,1,1,2,3,7)	4·72381	4·52381
8	(0,0,1,1,1,2,3 or 4,8)	5·6	5·375
9	(0,0,0,1,1,2,3,4,9)	6·50397	6·22222

For larger N what can we say about the expected score of the best strategy? It is greater than the expected score of the best restricted strategy, therefore

$$> \text{max (over all integer z) } N - \frac{N+1}{2}\left(\frac{z}{N} + \frac{1}{z+1}\right)$$

$$\geq N - (1+1/N)((4N+1)^{\frac{1}{2}} - 1)/2$$

This last inequality would make a nice little exercise for an elementary calculus class.

BESSEL FUNCTIONS (JCMN 38, p.4157)

J. B. Parker

How can we find $\int_0^\infty J_0(x)\,dx$? Call it W.

Hankel's transform of order zero tells us that

$$f(x) = \int_0^\infty t\,F(t)\,J_0(xt)\,dt \quad \text{when} \quad F(t) = \int_0^\infty x\,f(x)\,J_0(xt)\,dx.$$

Putting $F(t) = 1/t$ we find that

$$f(x) = \int_0^\infty J_0(xt)\,dt = W/x$$

and therefore $F(t) = W\int_0^\infty J_0(xt)\,dx = W^2/t$ and so

$W^2 = 1$ and $W = \pm 1$. The ambiguity of sign was to be expected because changing the sign of J_0 will not affect the validity of the Hankel Transform formula.

In order to decide between the two signs we use the differential equation $J_0'' + J_0'/x + J_0 = 0$, the first terms of the power series $J_0(x) = 1 - x^2/4 + \ldots$ and Bessel's integral formula $J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin y)\,dy$ which bounds the function between -1 and 1.

$$\int_0^x J_0(t)\,dt = J_0'(0) - J_0'(x) - \int_0^x J_0'(t)/t\,dt$$

(now integrating by parts)

$$= -J_0'(x) + \left[(1-J_0(t))/t\right]_0^x + \int_0^x (1-J_0(t))/t^2\,dt.$$

The third term is positive and the first two terms tend to zero as x tends to infinity, and so we must give W the plus sign.

As a bonus we have found another formula

$$\int_0^\infty (1-J_0(t))/t^2\,dt = 1.$$

QUOTATION CORNER 22

Those who write books without indexes will be the librarians' assistants in Hell.

Quoted by Lt. Col. R. W. Oldfield (10th. October, 1930) as being a favourite saying of a distinguished contemporary librarian.

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BACK NUMBERS

The JCMN for its first eight years (1975-1983) was published by the Mathematics Department of the James Cook University. Issues 1-31 of this period have been reprinted in three paperback volumes, on sale for 10 Australian dollars each, postage included, from

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Since 1983 the JCMN has had no connection with the James Cook University. Of issues from 32 onwards, back numbers are available from me.

This issue is being prepared (in September 1986) at my old home, 69, Queen's Road, Hermit Park, NQ 4812, but I am intending to sell this house. My more permanent address is

66, Hallett Road,
Burnside, SA 5066,
Australia.

Basil Rennie