New

JAMES COOK MATHEMATICAL NOTES

FEBRUARY, 1984

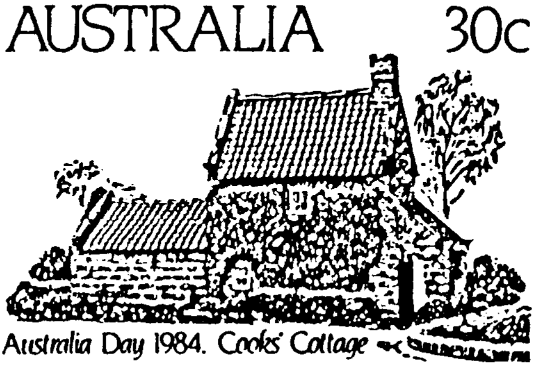
33

The New

JAMES COOK MATHEMATICAL NOTES

Volume 4, Issue number 33

February 1984



This new Australian stamp shows "Captain Cook's Cottage" now rebuilt in the Royal Botanic Gardens in Melbourne. It was originally at Great Ayton in Yorkshire where Cook lived as a boy.

Editor : B. C. Rennie
69 Queens Road,
Hermit Park, N.Q., 4812
Australia.

Three issues per year will be published, in February, June and October.

Subscription rates, in Singapore dollars, per year (covering three issues).

In Singapore (including postage)	$20
Outside Singapore (including air mail postage)	$30

Subscribers in countries with no exchange control (such as Australia, U.K. and U.S.A) may send ordinary cheques in their own currency to the Editor, the amount calculated at the current exchange rate. The rates at the time of writing are approximately

$1(Singapore) = 51 cents (Australian) = 33 pence (U.K.)
= 47 cents (U.S.A.)

Otherwise send whatever kind of currency you can, but do not send a personal cheque unless it can be cleared from Singapore. Back numbers (when available) are $12 (Singapore) each (sent by surface mail from Australia).

Please make cheques payable to B. C. Rennie.

CONTENTS

CONTINUED FRACTIONS AND RANDOM VARIABLES (JCMN 32,p.4022)

C. J. Smyth

The function $f(x) = 1/(x(x + 1) \log 2)$ satisfies

$$x^{-2}f(\tfrac{1}{x}) = \sum_1^{\infty} f(x + n)$$

and so the random variable X with probability density f(x) on $(1, \infty)$ has the property that the reciprocal, 1/X, and the non-integer part, X - [X], both have probability density $g(x) = 1/((x + 1) \log 2)$ on (0, 1).

Gauss in a letter of 30th January, 1812 to Laplace stated that the set (in the unit interval) of numbers for which the n th remainder (in the simple continued fraction expansion) is less than any $x < 1$, has measure $m_n(x)$ converging (as n tends to infinity) to $\log(1 + x)/\log 2$ (which, it may be noted, has derivative $x^{-2}f(1/x) = g(x)$).

The first proof of Gauss's assertion was published by R. Kusmin (Sur un probleme de Gauss, Atti Congr. Itern Bologne 1928, 6, pp. 83-89). From this result it follows easily that for almost all numbers α, the proportion of the occurrences of the integer k in the continued fraction expansion of α is

$$\log\left(1 + \frac{1}{k(k + 2)}\right)/\log 2.$$

E. Wirsing (Acta Arith. 24 (1974) 507-528) estimated the remainder

$$r_n(x) = m_n(x) - \log(1 + x)/\log 2$$

as $r_n(x) = (-\lambda)^n \psi(x) + O(x(1-x)\mu^n)$ $(n \to \infty,\ 0 < x < 1)$
where $\lambda = 0.3036630029\ldots$ and $0 < \mu < \lambda$, and $\psi(x)$ is s real-valued function on [0, 1] with continuous second derivatives, and $\psi(0) = \psi(1) = 0$.

Further, ψ can be extended to a holomorphic function on the whole complex plane, with a cut from -1 to $-\infty$. Then $\psi(z)$ satisfies the functional equation

$$\psi(z) - \psi(z + 1) = \frac{1}{\lambda}\psi\left(\frac{1}{z + 1}\right)$$

VARIATION ON EULER-MACLAURIN (JCMN 32, p 4018)

G. Szekeres

$$\int_{-h}^{h} f(x)\,dx = 2h\ f(0) + (h^2/6)\,k_1 - (7h^4/360)\,k_3 + (31h^6/15120)k_5 - \ldots$$

where k_n denotes $f^{(n)}(h) - f^{(n)}(-h)$.

The coefficients 1/6, -7/360, 31/15120, etc. are

$2(2^{k-1} - 1)\, B_k/(2k)!$; they are denoted $-D_{2k}$ by Nörlund in Differenzrechnung (1924) pp. 27, 31. They are generated as coefficients of the Bernoulli polynomials taken at places $x + \frac{1}{2}$. The result may be found by applying the Euler-Maclaurin formula to the intervals (-h, 0) and (0, h) as well as (-h, h).

SERIES EXPANSION

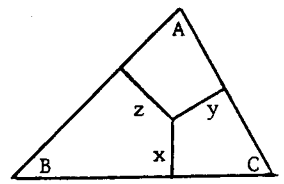
Expand $(2 - 2(1 - x^2)^{\frac{1}{2}})^{\frac{1}{2}}$ in powers of x for positive real x.

(Hall and Knight - Higher Algebra, third ed. 1889)

POINTS IN A TRIANGLE (JCMN 32, p. 4008)

A. P. Guinand

Let x, y and z be the perpendicular distances from a variable point to the sides of a triangle (they are signed variables, all positive for the inside of the triangle).



J. B. Parker's problem was to find a nice geometrical characterization of the point K minimizing $x^2 + y^2 + z^2$. This has been called the symmedian point, the Lemoine point (to French and British) or the Grebe point (to Germans). The point is of interest in navigation because if in the plane three position lines are known, each with the same Gaussian error distribution, then the most probable position is the symmedian point K of the triangle formed by the three position lines. We may use (x, y, z) as coordinates in the plane, noting that $ax + by + cz = 2\Delta$ = twice the area of the triangle; we may also take them as homogeneous coordinates, taking any scalar multiple of (x, y, z) to represent the same point.

In the homogeneous coordinate system K is (a, b, c). To prove this observe that by Schwarz's inequality

$$(x^2 + y^2 + z^2)(a^2 + b^2 + c^2) \geq (ax + by + cz)^2 = 4\Delta^2$$

with equality only if x : y : z = a : b : c.

To develop other properties, first note the coordinates of various important points (the following are in normalised form, with $ax + by + cz = 2\Delta$).

Symmedian point K : $\dfrac{2\Delta}{a^2 + b^2 + c^2}\,(a, b, c)$

Vertex A : $(2\Delta/a)(1, 0, 0)$ (B and C similarly)

Foot D of perpendicular from A to BC : $(2\Delta/a)(0, \cos C, \cos B)$

Mid-point of AD : $(\Delta/a)(1, \cos C, \cos B)$

Mid-point of BC : $(a/2)(0, \sin C, \sin B)$

Centroid, G : $(2/3)\Delta(1/a, 1/b, 1/c)$

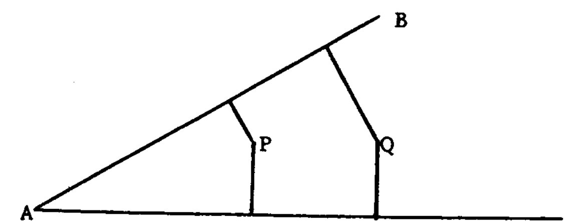
Incentre, I : $\dfrac{2\Delta}{a + b + c}\,(1, 1, 1)$

Excentre, I_a : $\dfrac{2\Delta}{b + c - a}\,(-1, 1, 1)$

Orthocentre, H : $2R \cos A \cos B \cos C\,(\sec A, \sec B, \sec C)$

Circumcentre, O : $\dfrac{2\Delta(\cos A, \cos B, \cos C)}{R(\sin 2A + \sin 2B + \sin 2C)}$

The isogonal conjugate of a point with homogeneous coordinates (x, y, z) may be defined as the point (1/x, 1/y, 1/z). A non-algebraic definition may be obtained as follows.



If P and Q are isogonal conjugates then (by similarity of the two quadrilaterals shown above) the angles BAP and QAC are equal, and similarly for the other two vertices B and C.

Three more properties of the point K are now apparent.

K is the isogonal conjugate of the centroid G.

K is on the line joining the mid-point of AD to the mid-point of BC, because

$$\sin A(1, \cos C, \cos B) + \cos A(0, \sin C, \sin B) = (\sin A, \sin B, \sin C).$$

Similarly K is on the two corresponding lines.

K is the centre of gravity of the system consisting of particles at the vertices A, B and C, with masses a^2, b^2 and c^2 respectively.

Problem. Are K and I always on the same side of the Euler axis? The equation of the Euler axis is

$$x \sin A(\sin 2B - \sin 2C) + y \sin B(\sin 2C - \sin 2A) + z \sin C(\sin 2A - \sin 2B) = 0.$$

GENERALIZED HADAMARD INEQUALITY (JCMN 31, p. 3165 & 32, p.4016)

In the proof of this inequality (Kestelman's last problem) there arose the positive definite $n \times n$ complex matrix

$$W = \begin{pmatrix} I & Z \\ Z^* & I \end{pmatrix}$$

where Z is $p \times q$ and $p + q = n$, and without loss of generality we may suppose that $p \leq q$. The result $\det W \leq 1$ was obtained from the inequality of the arithmetic and geometric means.

Relate the eigenvalues of W to those of the $p \times p$ matrix $I - ZZ^*$, and so obtain an alternative proof that $\det W \leq 1$.

LINEAR EQUATIONS AND LATIN SQUARES

C. J. Smyth

We are seeking specific sets of integer solutions (x_1, x_2, x_3) of an equation $ax_1 + bx_2 + cx_3 = 0$, where a, b, c are given integers. What we are looking for is, for some k, k solutions (x_{1j}, x_{2j}, x_{3j}) $(j = 1,\dots, k)$ such that the three families $S_i = \{|x_{ij}|;\ j = 1, 2, \dots, k\}$ are permutations of one another. For instance, for the equation $5x_1 + 6x_2 + 7x_3 = 0$ we can take the three solutions $(3, 1, -3)$, $(5, -3, -1)$ and $(1, 5, -5)$ and we find $S_1 = \{3, 5, 1\}$, $S_2 = \{1, 3, 5\}$ and $S_3 = \{3, 1, 5\}$, all permutations of one another.

For another example, the equation $3x_1 + 11x_2 + 13x_3 = 0$, take the ten solutions $(1, -5, 4)$, $(1, 8, -7)$, $(2, 3, -3)$, $(3, -2, 1)$, $(4, 6, -6)$, $(5, 1, -2)$, $(6, 9, -9)$, $(7, 4, -5)$, $(8, -1, -1)$, and $(9, 7, -8)$.
Then S_2 and S_3 are both permutations of

$$S_1 = \{1, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Problems.

1. Show that if $|c| > |a| + |b|$, or if there is a prime p such that p divides both a and b but not c, then these solution sets cannot exist.
2. Find a solution set for $6x_1 + 11x_2 + 13x_3 = 0$.
3. Show that if (a, b, c) is a Pythagorean triple then a solution set always exists.

It seems likely that any equation $ax_1 + bx_2 + cx_3 = 0$ has a solution set, except for those equations excluded in problem 1. Any ideas for a proof of this?

A solution set may be used to construct a Latin square of $n = 2k$ rows and columns, with determinant zero, and with each row and column containing a, b, c and n-3 zeros. For instance the solution set above for the equation

$5x_1 + 6x_2 + 7x_3 = 0$ gives

$$\begin{pmatrix} 6 & 0 & 5 & 7 & 0 & 0 \\ 0 & 6 & 7 & 5 & 0 & 0 \\ 0 & 7 & 0 & 6 & 5 & 0 \\ 7 & 0 & 6 & 0 & 0 & 5 \\ 5 & 0 & 0 & 0 & 6 & 7 \\ 0 & 5 & 0 & 0 & 7 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \\ -3 \\ 5 \\ -5 \end{pmatrix} = \underline{0}$$

GOAT AND COMPASSES

From John Neilson of Wollongong came a note about what he called the "goat and tether" problem. A flat field is enclosed by a circular fence of unit radius, and a goat is tethered to a point on the fence by a tether of length r such that the goat can graze over an area of $\pi/2$, (half the area of the field).

What is the length r?

Of course it may easily be found numerically, but for arithmeticians the interesting question is whether r is rational, algebraic or transcendental.

A MODEL FOR COMPANY FINANCE

In real life, companies (otherwise known as "corporations" or "limited liability companies") have as their main activity the making and selling of goods and services. However, this article is on a different activity, the manipulation of capital.

A simple model is found which exhibits some of the features observed in real companies. The rules for the model are that anyone can start a company; any company can buy shares in any other company (but not in itself); any company can issue more shares at any time but only in the form of "rights" divided equally among the existing shareholders; each company is controlled by a board of directors, consisting of one or more people, and elected by a majority of shareholders; all share registers and company annual balance sheets are known to the public; all transactions are carried out without any incidental costs such as stamp duties or brokerage.

A consequence of these axioms is that the market value of a share is found by dividing the company's assets by the number of shares issued. Also the market price of a share or of a right can never be negative. If a company's total assets were to become negative then the system would eliminate the company, either by the formal process of liquidation, or by the "bottom of the harbour" method which recently became popular in Australia.

To explain how the model works it would be best to tell a story. Two companies, A and B are started by the financier C (which perhaps is short for Croesus). Each company issues 110 shares for \$6 each, C buys 10 shares in each company (total cost to him is \$120) and then each company buys 100 shares in the other. At this stage C controls two companies, each with assets of \$660. The next event is that A issues another 110 shares at \$4, each shareholder being entitled to one new share for each one held. The market price of the shares drops to \$5 and the rights are worth \$1. C will sell 8 rights for \$8 and use that money to take up the other two shares of his entitlement. Similarly B will sell 80 rights and take up 20 shares. Outsiders (that is some people with \$440 to invest) will buy 88 rights and take up the 88 shares. Then B does the same thing, issuing 110 shares at \$4. The rights become worth \$1 and the new share price is \$5. Both A and C, as before, sell enough rights to take up the rest of their respective entitlements, and outsiders buy the rights and shares.

At this stage each company has \$550 in cash and 120 shares in the other company, worth \$5 each, so that its total assets are \$1100. It may be noted that orthodox accounting gives the right answers, nobody has gained or lost anything. The outside shareholders have 176 shares worth \$880, just what they paid for them, and C has 24 shares of total value \$120, exactly what he put in to the business. The important thing is that C controls both companies, by having 132 of the 220 votes at a meeting of shareholders. In the jargon of the trade the companies are said to be "cross-buttressed".

In the simple model that we have been using nobody ever makes any money; but in real life companies trade, and some grow rich and some grow poor, and company directors can make money either by influencing events or by taking advantage of their inside knowledge of what is happening.

So as to give the story an interesting end, suppose that the two companies, A and B, traded with one another and one gained and the other lost. To be more precise, suppose that (for some reason) A comes to owe B more than \$1150.

Of course C knows about this, and makes appropriate provision by borrowing \$500 from B and using it to buy from A 100 shares in B at the market price of \$5. The share registers of the two companies will be as follows.

	A	B	C	Others	Total
Shares in A	-	120	12	88	220
Shares in B	20	-	112	88	220

Because of the debt, company A is put in the hands of the Receiver, whose job is to liquidate the assets and hand them over to B. All this becomes public knowledge, and the market price of shares in B rises to \$7.50. The Receiver sells the 20 shares for \$150 and gives this, together with the cash holding of \$1100, to B. The total assets of B are then \$1150 cash plus \$500 owing from C (which is why the share price rose to \$1650/220 = \$7.50). At this stage C can sell all his shares for \$840 and pay back his debt of \$500 to the company and be left with \$340, which is more than his initial \$120 investment.

VECTORS WITH UNEQUAL COMPONENTS

C.J. Smyth

Given any positive integer n find the smallest k such that every k-dimensional subspace of R^n contains a vector with one component differing from all the other components.

SOLVING ALGEBRAIC EQUATIONS (JCMN 29, p.3115)

How can you find the real part of a complex root of a real algebraic equation, using only a simple programmable calculator?

Given the real polynomial

$$f(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$$

we can find the real zeros and divide by the corresponding linear factors of f. Therefore we may suppose without loss of generality that the zeros of f are all complex. Let $p \pm iq$ be a conjugate complex pair of roots of $f = 0$, then p is a zero of the polynomial

$$g(z) = z^{n(n-1)/2} + \tfrac{1}{2}(n-1)a_{n-1}z^{(n-2)(n+1)/2} + \dots$$

defined to have as its zeros all the $n(n-1)/2$ half-sums of pairs of roots of $f = 0$. Each coefficient in the polynomial g is a symmetric polynomial in the n zeros of f, and is therefore a polynomial in the coefficients of f.

For example, taking $n = 4$, let

$$f(z) = z^4 + az^3 + bz^2 + cz + d,$$

then g may be calculated to be

$$g(z) = z^6 + 3az^5/2 + (3a^2 + 2b)z^4/4 + (a^3 + 4ab)z^3/8$$
$$+ (2a^2b + ac + b^2 - 4d)z^2/16 + a(b^2 + ac - 4d)z/32$$
$$+ (abc - c^2 - a^2d)/64.$$

The n/2 real parts of the complex zeros of f will be among the real zeros of g. With luck they will be the only real zeros of g, but if two zeros of f have the same imaginary part then some care will be needed to disentangle the wanted values from the other real zeros of g. For example, if the zeros of f are $3 \pm i$ and $5 \pm i$ then g will have four

real zeros, 3, 5 and 4 repeated, and two complex zeros, $4 \pm i$.

Sometimes (in questions of stability, for example) we do not want to know the real parts of the zeros of f, but only to know if they are all negative. The calculation is then much easier, because the following result can be used.

__Theorem.__ Let f be any real polynomial and let g be defined as above. The real parts of the zeros of f are all negative if and only if the coefficients in g are all positive.

__Proof.__ The "if" is trivial, because $g = 0$ cannot have any positive or zero roots. To prove "only if" let f have complex zeros $-2p_m \pm 2iq_m$, and real zeros $-2r_m$ where all r_m and p_m are positive. Then $g(z)$ will be a product of real linear or quadratic factors of the form

$$z + 2p_m \quad \text{or} \quad (z + p_m + p_s)^2 + (q_m \pm q_s)^2$$

$$\text{or} \quad z + r_m + r_s \quad \text{or} \quad (z + r_m + p_s)^2 + q_s^2,$$

and therefore all the coefficients in g will be positive.

The algebra above illustrates the point, first noted by Ferrari in the 16th century, that in order to solve a biquadratic equation it is sufficient to be able to solve cubics and take square roots. We work in the complex number field, and take (as we may without loss of generality)

$$f(z) = z^4 + bz^2 + cz + d$$

in the equation $f(z) = 0$. In the notation above

$$g(z) = z^6 + bz^4/2 + (b - 4d)z^2/16 - c^2/64,$$

a cubic in z^2.

Let f have zeros p, q, r and s, and let
$p + q = -r - s = 2u$ and $p + r = -q - s = 2v$
and $p + s = -q - r = 2w$.

Then the six roots of $g = 0$ are $\pm u$, $\pm v$ and $\pm w$, and the eight values of $\pm u \pm v \pm w$ are a permutation of the eight values $\pm p$, $\pm q$, $\pm r$ and $\pm s$. Therefore if we solve $g = 0$ to find the six roots we have the eight values from which we can pick out the four zeros of f.

DIVERGENT MACLAURIN SERIES (JCMN 32, p. 4018)

R. Vyborny

Here are two examples of a function with divergent Maclaurin series,

$$F(x) = \int_0^\infty e^{-t}/(1 + x^2 t^2)dt$$

exists for all x and is differentiable any number of times and has series:

$$1 - 2!\, x^2 + 4!\, x^4 - 6!\, x^6 + \ldots$$

For the other example let G be defined by

$$G(x) = \int_0^\infty e^{-t}/(1 + xt)dt \quad \text{for } x \geqq 0$$

and $$G(x) = \int_0^{-1/(2x)} e^{-t}/(1 + xt)dt \quad \text{for } x < 0.$$

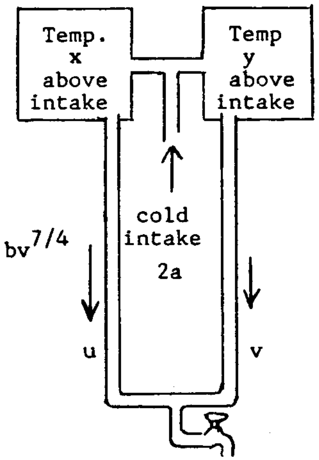
This function G has Maclaurin series

$$1 - x + 2!\, x^2 - 3!\, x^3 + 4!\, x^4 - \ldots$$

SOLAR HOT WATER (JCMN 32, p. 4023)

In the new Federal Government building on Dalrymple Road in Townsville (it is rumoured) the solar water heaters on the roof fail to share the load equally; consequently the water does not come out of the taps hot enough. A possible explanation is as follows.

Firstly any arrangement of the pipe-work would probably lead to unequal flows through the various heaters, even without taking into account the effect of the heat. Secondly the temperature of a heater (consisting of a solar panel and hot water tank on the roof of the building) decreases with the amount of hot water supplied from it, and because the supply is to taps below the level of the heaters there is a convection effect which will aggravate the inequalities. To give an idea of the pressures involved, the pressure gradient due to a flow of 2 gallons per minute in a 3/4 inch pipe is the same as that due to a 35°C temperature difference in a vertical pipe, about .0075 p.s.i. per foot.



Consider the simple model indicated in the diagram. The equations are

Water balance : $u + v = 2a$

Pressure balance : $x + bu^{7/4} = y + bv^{7/4}$

Heat balance : $dx/dt = c - x - xu$

$dy/dt = c - y - yv$

There is of course a symmetrical solution with $u = v = a$ and $x = y$.

A good undergraduate question would be to investigate the stability of the symmetrical solution.

BESSEL FUNCTIONS AND ELLIPTIC INTEGRALS (JCMN 32, p. 4008)

E.R. Love

$$\int_0^\infty J_0(x)\ J_0(Rx)\,dx \;=\; \frac{1}{\pi}\int_0^\pi (R^2 - \sin^2\theta)^{-\frac{1}{2}}\,d\theta \qquad \text{for all } R > 1.$$

According to Erdelyi et al., Tables of Integral Transforms (Bateman Manuscript Project), both sides of the equation may be expressed in terms of hypergeometric functions as

$(1/R)\ F\ (\frac{1}{2}, \frac{1}{2}; 1; R^{-2})$. See volume 1, p. 387 and volume 2, p. 48.

EDITORIAL

Reprints of issues 1- 31 (published by the University) are still available from

Mathematics Department,
James Cook University of North Queensland,
Post Office James Cook,
Townsville, N.Q. 4811, Australia,

bound as paperback volumes;

Volume 1 (Issues 1-17)
Volume 2 (Issues 18-24)
Volume 3 (Issues 25-31).

The price is $10 (Australian) per volume (including sea mail postage), and cheques for these volumes should be made payable to James Cook University.

Since Issue 32 (October 1983) the JCMN has been published by me (the Editor). Subscriptions and contributions will be welcomed. My address is either at the University (see above) or at home (see page 4026).

Basil Rennie