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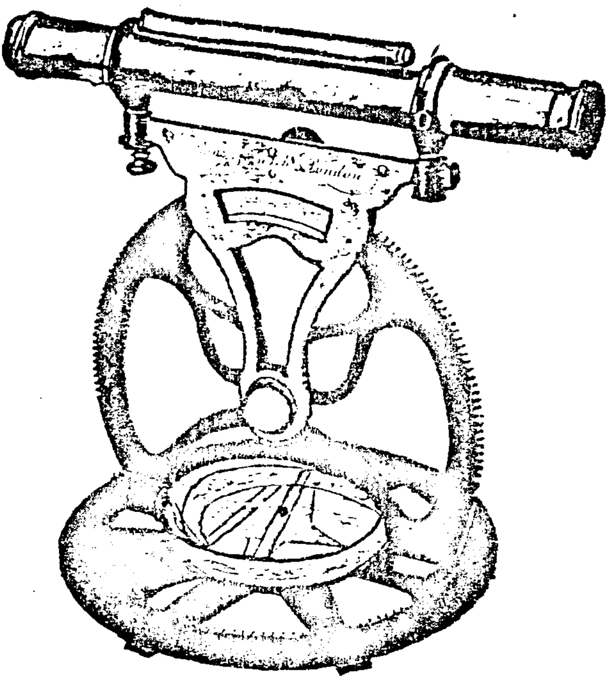


A theodolite of about 1765. This was the principal instrument used in marine surveying. It headed the list of navigational instruments required by Cook and supplied by the Admiralty on his three voyages. The original is housed in the National Maritime Museum, Greenwich.

COMBINATORIAL NUMBER THEORY

C.J. Smyth

Let p, q be relatively prime integers. How large a subset of the first n integers can be chosen so that no element in the subset is p or q times another? We answer the question in the theorem below, which *Marta Sved* asked for p = 2, q = 3 (JCMN 22, Vol. 2, p. 86).

Arrange all integers of the form $p^i q^j$, i, j integers, in ascending order, $1 < p < q \ldots$, and denote them by $\lambda_1 < \lambda_2 < \ldots$. If $\lambda_k = p^i q^j$, we define the parity of λ_k to be the parity of i + j. Define g(x) as follows:

$$g(x) = \max(\#\text{ of } \lambda_k \le x \text{ of even parity}, \#\text{ of } \lambda_k \le x \text{ of odd parit}$$

<u>Theorem</u> Let (p, q) = 1. The largest subset of the first n positive integers that can be chosen so that no element in the subset is p or q times another has cardinality,

$$M = \sum_{k=1}^{\infty}(g(\lambda_k) - g(\lambda_{k-1}))\left([\tfrac{n}{\lambda_k}] - [\tfrac{n}{p\lambda_k}] - [\tfrac{n}{q\lambda_k}] + [\tfrac{n}{pq\lambda_k}]\right)$$

which is asymptotically

$$n(1 - \frac{1}{p})(1 - \frac{1}{q}) \sum_{k=1}^{\infty} \frac{g(\lambda_k) - g(\lambda_{k-1})}{\lambda_k} + O(\log^2 n)$$

as $n \to \infty$. (Here $g(\lambda_0) = 0$).

<u>Example</u> For p = 2, q = 3 we have

λ_k	1<	2<	3<	4<	6<	8<	9<	12<	16<	18<	24<	27<	32<	36<	48<	54<	64<	72<	81<	96<...
parity	0	1	1	0	0	1	0	1	0	1	0	1	1	0	1	0	0	1	0	0
$g(\lambda_k)$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	11

so that $M = n(1 - \frac{1}{2})(1 - \frac{1}{3})(\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{16} + \frac{1}{24} + \frac{1}{32} + \frac{1}{48} + \ldots)$

$\approx 0{\cdot}60\, n$.

<u>Proof of the theorem</u> Every integer can be written uniquely in the form $mp^i q^j$, where $p \nmid m$, $q \nmid m$. Further, one of $mp^i q^j$ and $m'p^{i'}q^{j'}$ is p or q times the other iff $m = m'$ and $|i - i'| + |j - j'| = 1$.

Now define f(x) to be the maximum number of elements of $\{(i, j) : i, j \ge 0 , p^i q^j \le x\}$ that can be chosen so that for any two distinct elements (i_1, j_1), (i_2, j_2) of the set, $|i_1 - i_2| + |j_1 - j_2| \ne 1$. Then for fixed m ($\le n$) with $p \nmid m$, $q \nmid m$ the maximum number of elements of the form $mp^i q^j$ ($\le n$) that can be chosen so that none is p or q times another is $f(\frac{n}{m})$. Now f is a step function, increasing by 0 or 1 at every λ_k, as a new lattice point is added to the set $\{(i,j) : p^i q^j \le x\}$. Hence the required number M is

$$M = \sum_{m=1}^{n}{}^{*} f(\tfrac{n}{m}) = \sum_{k=1}^{\infty} \sum^{*}_{\lambda_k \le \frac{n}{m} < \lambda_{k+1}} f(\lambda_k) = \sum_{k=1}^{\infty}(f(\lambda_k) - f(\lambda_{k-1})) \sum^{*}_{m \le \frac{n}{\lambda_k}} 1$$

where $\sum^*$ signifies that the sum is over those m with $p \nmid m$ and $q \nmid m$

$$= \sum_{k=1}^{\infty} (f(\lambda_k) - f(\lambda_{k-1}))\left(\left[\frac{n}{\lambda_k}\right] - \left[\frac{n}{p\lambda_k}\right] - \left[\frac{n}{q\lambda_k}\right] + \left[\frac{n}{pq\lambda_k}\right]\right) .$$

Then, the final expression for M is obtained on remarking that f = g : this follows from the result in "Black and White Cubes" (JCMN 23, Vol. 2, page 125) applied to the set

$$\{(i, j), i, j \ge 0, p^i q^j \le x\}. \quad (\text{i.e. } h_i = 1 + [\log(xp^{-i})/\log q])$$

To calculate the asymptotic value, note that $g(\lambda_k) - g(\lambda_{k-1}) = 0$ or 1, so

$$\left|M - \sum_{k=1}^{\infty}(g(\lambda_k) - g(\lambda_{k-1}))\left(\frac{n}{\lambda_k} - \frac{n}{p\lambda_k} - \frac{n}{q\lambda_k} + \frac{n}{pq\lambda_k}\right)\right| =$$

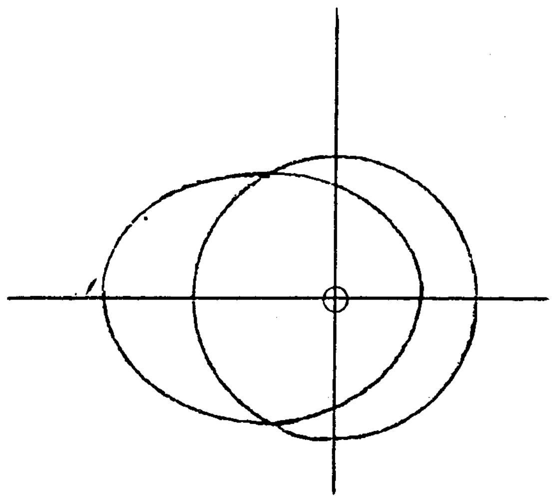
$$= \left|M - n(1 - \frac{1}{p})(1 - \frac{1}{q}) \sum_{k=1}^{\infty} \frac{g(\lambda_k) - g(\lambda_{k-1})}{\lambda_k}\right| \le 4K + n(1 - \frac{1}{p})(1 - \frac{1}{q}) \sum_{k=K+1}^{\infty} \frac{1}{\lambda_k}$$

where K is the least index for which $\lambda_k > n$. It is easily checked that K is $O(\log^2 n)$ and $\sum_{k=K+1}^{\infty} \frac{1}{\lambda_k}$ is $O(\frac{\log n}{n})$, which gives the result.

CONVERGENCE OF SERIES

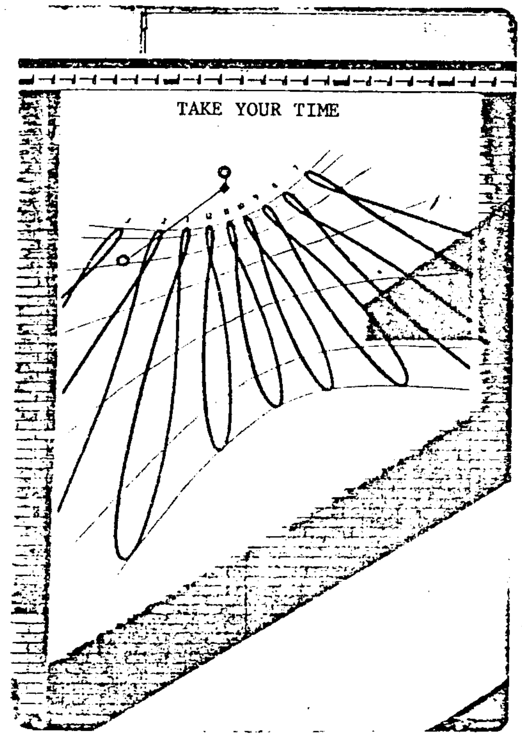
R.L. Agacy

Many text-books explain how rearranging a series may change the sum, but very few point out how rearranging may change the region of convergence. The series $\sum_0^\infty (-z - z^2)^n$ converges for complex z inside an oval on the complex plane, and when rearranged as a power series it becomes $\sum_0^\infty z^{3n} - z^{3n+1}$ which converges inside the unit circle. In eac case the sum is $1/(1 + z + z^2)$ inside the region of convergence.



THE SUNDIAL AT MONASH UNIVERSITY

C.F. Moppert



The shadow is cast by a flat brass annulus, mounted parallel to the wall. The shadow is thus always an annulus and this makes it easy to read. The loops are made of copper strips about 20 mm wide. The month lines are square brass rods of about 5 mm width. The background of the loops is painted red and yellow, the background of the months lines is painted blue.

This sundial is accurate to the minute throughout the year.

The legend is mounted on a painted metal panel. It reads as follows:

The exact time is given when the circular shadow is on the corresponding

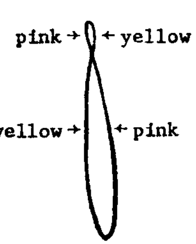
<u>pink line</u>	<u>yellow line</u>
from 22 Dec. through the months of	from 22 June through the months of
Jan	July
Feb	Aug
Mar	Sept
Apr	Oct
May	Nov
to 22 June	to 22 Dec.

The shadow touches the cross-overs on 13 April and 31 August. The <u>blue lines</u> give the path of the circular shadow during the day. They are exact for the following dates:

Top line:	22 June (shortest day)
Second line:	21 May and 24 July
Third line:	21 April and 23 August
Fourth line:	20 March and 23 September
Fifth line:	19 February and 24 October
Sixth line:	21 January and 22 November
Bottom line:	22 December (longest day).

During summer time, add one hour.

As the JCMN is printed only in black and white it must be explained that the pink and yellow lines are those forming the figures of eight, one for each hour. The blue lines are the more or less horizontal curves that indicate the time of year (more precisely they indicate the signs of the Zodiac).



ANALYTIC INEQUALITY (JCMN 19,p.29;20,p.54,Vol 2)

J. Surma (University of Sokoto)

The proposition was a non-linear version of the Hahn-Banach theorem, that any real Lipschitz function (that is f such that $|f(x) - f(y)| \leq k\|x - y\|$) on a subset of a metric space could be extended to the whole space, satisfying the same Lipschitz condition. It appears that this, like the Hahn-Banach theorem, can be proved from the Boolean Prime Ideal Theorem, a statement known to be essentially weaker than the Axiom of Choice in its full generality. This implies the independence of the Axiom of Choice from the Hahn-Banach theorem.

MOVING ROUND CIRCLES

G. and E. Szekeres

Question 3 in the 1979 International Mathematical Olympiad was as follows.

Two circles in a plane intersect, let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal.

Is the clause "in the same sense" in the second sentence necessary?

QUOTATION CORNER (7)

While of course no notice of attendance is required, I would be grateful if I could be told who we might except. —— From a letter announcing a meeting.

GEOMETRIC INEQUALITY

(JCMN 19, p. 32, 21, p. 79 and 22, p. 98, all in Vol. 2)

From *R.K. Guy* comes the following information. This problem, about the largest area of a table that can be carried round a corner in a corridor of unit width, has been about for several years, often under the name "the sofa problem". The value $2/\pi + \pi/2 = 2{\cdot}207416$ has been improved, and the best value so far seems to be $2{\cdot}215649$ obtained by C.E.H. Francis and R.K. Guy. See the American Mathematical Monthly, 84, (1977) p. 811. In the same journal, 83, (1976) p. 188 at the end of the article by N.R. Wagner is the comment "Concerning convex solutions to the sofa problem, the author can only remark that the maximum area is surely larger than the value of $\pi/2$ given by the semicircle".

WELL KNOWN FUNCTION

A. van der Poorten

F. Hirzebruch in "Prospects in Mathematics" page 7, Annals of Mathematics Studies, No. 70, Princeton, 1971, writes "... of course the function $f(x)$ with the property that the coefficient of x^n in $(f(x))^{n+1}$ is 1 is well known ..." What is the function?

DRY THE DISHES

C.J. Smyth

"Wash the dishes, dry the dishes, turn the dishes over" goes the children's rhyme. However, if some dishes are "turned over" (i.e. suitably oriented), they may "drain", and not need drying.

The problem is: give necessary and sufficient conditions for a set in 3-space to be one which will "drain". Clearly convexity is sufficient, but not necessary.

ANNIHILATING POLYNOMIALS

H. Kestelman

Let A be a fixed complex n×n matrix. For any complex n×1 vector $\underline{v}$ define the minimal polynomial of $\underline{v}$ with respect to A to be the monic polynomial $\mu(\underline{v}, t)$ in the variable t, of least degree, satisfying $\mu(\underline{v}, A)\underline{v} = 0$. Let f be any polynomial and let $\underline{w}$ be the vector $\underline{w} = f(A)\underline{v} \neq 0$.

Show that $\mu(\underline{w}, t) = \mu(\underline{v}, t)/\phi(t)$ where the monic polynomial $\phi(t)$ is the greatest common divisor of $f(t)$ and $\mu(\underline{v}, t)$.

S.U.M.S. COMPETITION 1980 (JCMN 23, Vol.2, p.124)

R.N. Buttsworth

Introduction

We solve a generalization of problem 3 of the S.U.M.S. competition 1980.

Let A be any set, A* the set of strings of elements of A (of finite length). Strings are called primitive if they are of the form σ σ for some σ in A*. We say τ is derived from σ and στ. A good string is one that can be obtained from primitive strings by applying derivation a finite number of times. Describe all good strings.

The answer is that the good consist of precisely those strings with an even number of occurrence of elements of A.

To show this we need several easy lemmas and some notation.

Notation

A denotes an arbitrary set which remains fixed throughout our discussion.

Λ denotes the empty string.

A* denotes the set of all finite length strings (including formed from elements of A.

$N_a(x)$ for a in A and x in A* is the number of occurrences of a in string x.

P is the set of primitive strings

$$P = \{\sigma\sigma | \sigma \text{ in } A^*\}$$

so we have taken Λ in P.

G is the set of all good strings. For clarity we assume $P \subseteq G$.

Basic Results

Lemma 1 Let x and y be in A*. If y is in G then xyx is in G.

Proof: If either x or y is Λ the result is trivial, since xx is in Otherwise we have yxyx is in P, so if y is in G, xyx is in G by derivation.

Lemma 2 For x and y in A*, xy is in G ⇒ yx is in G.

Proof: We have yy is in P
so xyyx is in G by Lemma 1,
and so if xy is in G
we deduce yx is in G by derivation.

Lemma 3 If $x_1, x_2, \ldots, x_n$ are all in A*, and y is in G, then $x_1x_2 \ldots x_nyx_n \ldots x_2x_1$ is in G.

Proof: This follows immediately by induction on n from Lemma 1.

Lemma 4 If $x_1, x_2, \ldots, x_n$ are all in A* and $x_1x_2 \ldots x_n$ is in G then $x_n \ldots x_2x_1$ is in G.

Proof: Using y = Λ in Lemma 3, we have $x_1x_2 \ldots x_nx_n \ldots x_2x_1$ is in G.
Thus if $x_1x_2 \ldots x_n$ is in G
$x_n \ldots x_2x_1$ is in G by derivation.

We now show that good strings remain good if any two substrings are interchanged.

Theorem If x, y, z, t and w are in A* and xyztw is in G then xtzyw is in G.

Proof: xyztw is in G ⇒ wtzyx is in G by Lemma 3 with n = 5,
and then
w(tzy)x is in G ⇒ xtzyw is in G by Lemma 3 with n = 3.

Corollary If x is in G and x' is a permutation of the elements of A in x, then x' is in G.

Proof: Since x is of finite length x' can be obtained from x by a finite number of transpositions, each of which leaves the string a member of G by the preceding theorem.

Corollary If x is in A* and for every a in A, $N_a(x)$ is even, then x is in G.

Proof: If $N_a(x)$ is even for each a in A, x is a permutation of some string of the form yy.

It remains only to show that every good string x has the property.

Theorem If x is a good string, $N_a(x)$ is even for each a in A.

Proof: The number of occurrences of an element a in a primitive string is even, the process of derivation (deriving τ from σ and στ) can therefore never lead to a good string with an odd number of occurrences of the element a.

We note for completeness that when A = {a, b} we may say ab is not a good string.

QUOTATION CORNER (8)

PUSH ON	PUSH OFF

— Labels on the overhead projector in a lecture room.

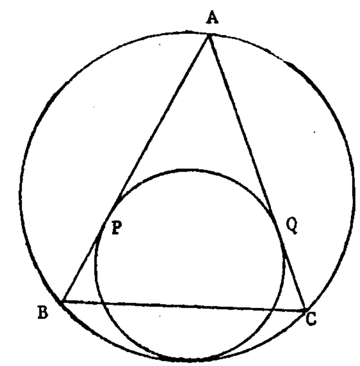
CAN WE BELIEVE ANYTHING?

From *L.M. Marsh* comes the following paragraph, a footnote culled from J.R. Ravetz, Science as Craftsman's Work, Scientific Knowledge and its Social Problems, Clarendon Press, Oxford, 1971.

In fields where the data are found, the problem of coping with "outlying" data is particularly severe, for one can only rarely repeat the process under observation. One might try to use statistical tests to estimate the significance of such data, deciding whether they are indications of a regular cause, or merely freaks; but all such tests depend on hypotheses about the universe from which the data are drawn, and so are less likely to be genuinely applicable in this extreme case. The extent to which such outlying data occur in experimental and observational work is likely to be underestimated by those unfamiliar with such work, since it is frequently suppressed. A most striking example of such suppression (in the teaching literature) is the classic study on deaths by horse-kicks in German army corps, which is the paradigm case for the Poisson distribution, describing randomly occurring events. The original study of horse-kick deaths by von Bortkewitsch extended over fourteen army corps, but the data from four of these was rejected as being anomalously high, and the data from the remaining ten was shown to fit very well the theoretical numbers. A note of this suppression remains in some of the literature; see J.S. Coleman, Introduction to Mathematical Sociology (Collier-MacMillan: The Free Press, Glencoe and London, 1964), p. 291. But this is the exception; there is no hint that the horse-kick deaths data is other than 'raw' in R.A. Fisher, Statistical Methods for Research Workers, 13th ed. (Oliver & Boyd, Edinburgh and London, 1955), p. 55.

A CIRCLE AND A TRIANGLE

C.S. Davis



A circle touches the sides AB and AC of a triangle ABC in points P and Q, and also touches the circumcircle internally. Show that the mid-point of PQ is the in-centre of the triangle.

QUOTATION CORNER (9)

But in Applied Mathematics all was different. One did not say: "Let us assume that the turning moment is the product of the force times the perpendicular distance", one said: "It is so". This seemed to me dogmatic. I used to rework the examples out with the force times the <u>square</u> of the distance, and then the <u>cube</u> of the distance, and would compare the results and make up little dream worlds in which these conditions applied. —— From "The Hot-House Plant" an autobiography by Y. Stevenson, recounting how the author gave up mathematics and became a Marxist and a psychologist.

BINOMIAL IDENTITY NUMBER ELEVEN (JCMN 23,Vol.2,p.

With a small change of notation the identity was:

$$4^n = \sum_{s=0}^{n} \binom{2s}{s} \binom{2n-2s}{n-s} .$$

From *J.B. Parker*: Firstly, using beta and gamma functions,

$$\int_0^{\frac{\pi}{2}} \sin^{2n-2s}\theta \cos^{2s}\theta \, d\theta = 2B(n-s+\tfrac{1}{2}, s+\tfrac{1}{2})$$

$$= 2\Gamma(n-s+\tfrac{1}{2})\,\Gamma(s+\tfrac{1}{2})/\Gamma(n+1)$$

which by using the duplication formula

$$= 2^{-2n} \frac{(2n-2s)!}{n!(n-s)!} \frac{(2s)!}{s!} \frac{\pi}{2} .$$

Therefore

$$1 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\cos^2\theta + \sin^2\theta)^n \, d\theta$$

$$= \sum_0^n \frac{2}{\pi} \binom{n}{s} \int_0^{\frac{\pi}{2}} \sin^{2n-2s}\theta \cos^{2s}\theta \, d\theta$$

$$= \sum 2^{-2n} \binom{n}{s} \frac{(2n-2s)!\ (2s)!}{(n-s)!\ n!\ s!}$$

$$= 4^{-n} \sum_{s=0}^{n} \binom{2n-2s}{n-s} \binom{2s}{s} .$$

From *C.S. Davis*: Firstly $\sum_{r=0}^{\infty} \binom{2r}{r} x^r = (1-4x)^{-\frac{1}{2}}$. (This identity is equivalent to $\binom{2r}{r} = (-4)^r \binom{-1/2}{r}$ and there was a proof in JCMN 18, Volume 2, page 11). It follows that

$\sum_0^n \binom{2n-2s}{n-s} \binom{2s}{s}$ is the coefficient of x^n in $(1-4x)^{-1}$, and this coefficient is 4^n.

COMMUTING MATRICES

H. Kestelman

Suppose that E is a set of square matrices all commuting with one another.

(a) Do the matrices of E have a common eigenvector?

(b) If in addition each matrix M in E is diagonalizable, that is, for some X, $X^{-1}MX$ is diagonal, does it follow that the matrices of E are simultaneously diagonalizable, that is, for some X, all $X^{-1}MX$ are diagonal?

SIMILAR MATRICES

H. Kestelman

Let A be a complex $n \times n$ matrix, not a multiple of the unit matrix, and let $\underline{u}$ be a column $(n \times 1)$ vector, not a multiple of $(1, 0, 0, \ldots)^T$. Show that M exists such that $M^{-1}AM$ has first column $\underline{u}$.

ORTHOGONAL MATRICES (JCMN 23,Vol.2, p.126)

Both *H. Kestelman* and *R.N. Buttsworth* have pointed out that the answer is no to this question whether when a real square matrix has sum of squares in each row and column equal to one it is possible by changes of sign to make the matrix orthogonal. For instance take all elements $1/\sqrt{3}$ in a 3×3 matrix.

WHITHER SYSTEMS?

The "call for papers" of a forthcoming congress of cybernetics and systems has the rule that papers should

(a) start with a clear formulation of the problem

(b) include convincing empirical evidence for the author's propositions

(c) have a firm theoretical anchorage in current developments

(d) make use of recent developments in explaining the relevar phenomena.

If the world of theoretical physics had followed these rules sixty years ago they would have excluded Einstein's general relativity and the quantum theories of Bohr, Schrodinger and Heisenberg, because none of these great advances made use of any recent theory, in fact they were all based on mathematics that had been known for fifty years

NAMES

For many years Mr. Justice Hanger has administered the laws of Queensland, and in the James Cook University Dr. Monypenny lectures in the Commerce Department. Are there such happy coincidence in the world of mathematics? Did Klein ever write on infinitesimals? or Poisson on the motion of a body through a fluid? My only example is that W.L. Edge wrote a book, "Ruled Surfaces" published by Cambridge University Press about 1930.

ALMOST LOWER TRIANGULAR MATRICES (JCMN 19, Vol.2, p.29)

H. Kestelman

Describing $A = \begin{pmatrix} a_{11} & b_2 & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & b_3 & 0 & \dots & 0 \\ & & & & & \\ a_{n1} & a_{n2} & a_{n3} & & & a_{nn} \end{pmatrix}$

as "almost lower triangular" the theorem was that the nullity of A cannot exceed k + 1 where k is the number of zeros in $\{b_2, b_3, \dots b_n\}$. A simpler proof is to consider the $(n+1) \times (n+1)$ matrix B,

$$B = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ a_{11} & b_2 & 0 & \dots & 0 & 0 \\ & \dots\dots\dots\dots & & & & \\ a_{n-1,1} & & \dots & & b_n & 0 \\ a_{n1} & a_{n2} & \dots & & a_{nn} & 0 \end{pmatrix}$$

This matrix B is lower triangular and has n − k non-zero elements on its diagonal, so that its rank is at least n − k. The row-rank of B is either the same as that of A or one more, and so A has rank at least n − k − 1, and therefore nullity at most k + 1, attaining that value only if (1, 0, 0, ...) is linearly independent of the rows of A.

MATRIX EQUATIONS

H. Kestelman

The $n \times n$ matrix M has n distinct eigenvalues and f is a polynomial of degree k. Show that in the complex field the equation $f(X) = M$ has at least one and at most k^n solutions.

Your editor would like to hear from you anything connected with mathematics or with James Cook.

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