JAMES COOK MATHEMATICAL NOTES

Issue No. 19, Vol. 2

August, 1979.

Our cover picture "Collier brig unloading at Brighton beach, c. 1820" shows the sort of ship used in the coal trade in which James Cook learnt the arts of seamanship. The painting is by Peter Leath of Newport, Isle of Wight, and is reproduced by courtesy of the Shipwrecked Fishermen and Mariners Royal Benevolent Society of 1, North Pallant, Chichester, Sussex, U.K.

GEOMETRY IS ALGEBRA IS GEOMETRY IS . . . .

I am neither an algebraist nor a geometer, only an unrepentant classical analyst. So when I stumbled on a result about the product of three 3×3 zero-diagonal matrices as an algebraic equivalent of Pappus' theorem I did not think it could be new, and I did not rush to publish it. In fact I long failed to find any non-clumsy algebraic proof. I told others about it, asking if they recognized it; others such as Schwerdtfeger (as a matrix expert), Coxeter (as an expert geometer), and Potts (as an old friend, and because I was on sabbatical in Adelaide). Then Potts (as editor of JCMN while Rennie took a sabbatical) posed the result purely as a matrix problem in JCMN 3. In JCMN 4 he gave a solution and saddled the defenceless result with the tag "Guinand's theorem". Parker, in JCMN 5 and 6, detected the conception of the result in Pappus' theorem, and Broyden discussed it in more detail, algebraically, (2). In the meanwhile I had found a less clumsy algebraic proof, and further allied geometric results. I submitted a paper to the Journal of Geometry; it took over a year to get published, after everybody else!

Broyden conjectured that there may be analogous results for matrices of higher orders, and, in a private communication, so did Sahib Ram Mandan. Now there is some weak anology between Möbius pairs of mutually circumscribing tetrahedra in 3-space and Pappus' theorem, when the latter is regarded as a cyclic (or Graves) triad of circumscribing triangles in the plane. This does lead to a rather trivial result about pairs of 4×4 zero-diagonal matrices (5), but I doubt that there can be analogous results for still higher orders.

Nevertheless, this led me to enquire about analogues of Pappus' theorem in more dimensions - with total lack of success. But the other fundamental theorem of projective geometry is Desargues', so what of it? Desargues' theorem says that if the three lines through homologous pairs of vertices of two triangles meet in a point, then the points of intersection of pairs of homologous sides are collinear. So I conjectured the following 3-space analogue:

(I) If the planes through homologous triads of vertices of three tetrahedra all intersect in a point, then the four points of intersection of triads of homologous faces are coplanar. Another way of expression (I) is (I'),

(I') Let A, B and C be three tetrahedra, with A having vertices

$A_r (r = 1, 2, 3, 4)$ and faces a_r opposite A_r, etc. If the tetrahedra are planewise perspective from a point P (that is if for each r the four points A_r, B_r, C_r and P are coplanar) then the tetrahedra are pointwise perspective from some plane ℓ (that is for each r the planes a_r, b_r and c_r meet in a point of ℓ).

Now let us look at some equivalent algebra. Using the Baker and Möbius notation (1) for points, choose the symbols A_r, B_r, C_r $(q, r = 1, 2, 3, 4)$ so that $A_r + B_r + C_r = P$. Then there must exist coefficients a_{qr}, b_{qr}, c_{qr} such that $L_r = \sum a_{qr} A_q = \sum b_{qr} B_q = \sum c_{qr} C_q$, the sums being taken over $q \neq r$. The conclusion of (I) is then that the L_r are linearly dependent. Expressing this in matrix form we get the following proposition:

(II) If (i) Π is the row-vector $\{P, P, P, P\}$, A the row-vector $\{A_1, A_2, A_3, A_4\}$, and similarly for B, C and L;

and (ii) a, b, and c are 4×4 zero-diagonal matrices;

and (iii) $A + B + C = \Pi$ and $Aa = Bb = Cc = L$;

then $\det a = \det b = \det c = 0$, in general*.

(* "in general" means "possibly subject to conditions ruling out degenerate cases").

So now, will somebody please give a simple algebraic proof of (II), and let me get back to analysis?

References:

(1) H.F. Baker, Principles of Geometry, Cambridge U.P. 1929, Ch.I.

(2) C.G. Broyden, A note on Guinand's theorem, J. London Math. Soc. (2), 18 (1978), 33-8, and JCMN 12.

(3) L. Gerber, Associated and perspective simplexes, Trans. Am. Math. Soc., 201, (1975), 43-55.

(4) A.P. Guinand, Graves triads in the geometry of the triangle, J. of Geometry, 6, (1975), 131-42.

(5) A.P. Guinand, Graves triads, Möbius pairs, and related matrices, J. of Geometry, 10, (1977), 9-16.

(6) S.R. Mandan, Desargues' theorem in n-space, J. Australian Math.Soc., 1, (1960), 311-8.

A.P. Guinand

NATURAL PHENOMENON

Near Blanefield in January 1979 on a cold morning with bright sun and dead calm, one strand of a somewhat broken-down old barbed wire fence was vibrating in a vertical plane with period about two seconds and amplitude about an inch above and below the mean position. The wire and the trees round about all had snow and frost on them. I noticed this on my way to the village, and again an hour later on my way back. What was the explanation?

ANALYTIC INEQUALITY

The real function $f(x, y)$ is differentiable on the unit square and $f(0, 0) = 0$, $f(0, 1) = f(1, 0) = 2$ and $f(1, 1) = 5$. Find the largest possible M for an inequality of the form:

"Somewhere in the square $(\partial f/\partial x)^2 + (\partial f/\partial y)^2 \geq M$"

ANOTHER BINOMIAL IDENTITY

Show that for any integer $k > 0$, and any real number $0 < p < 1$,

$$\lim_{N\to\infty} \sum_{i=0}^{[N/k]} \binom{N}{ki} p^{ik}(1-p)^{N-ik} = 1/k$$

C.J. Smyth

ALMOST LOWER TRIANGULAR MATRICES

Describe the $n \times n$ matrix A as <u>almost lower triangular</u> if $a_{rs} = 0$ whenever $r + 1 < s$. Denote each element $a_{r-1,r}$ of the superdiagonal by b_r $(r = 2, 3, \ldots, n)$, and let E be the set of r for which $b_r = 0$.

$$A = \begin{pmatrix} a_{11} & b_2 & 0 & 0 & \cdots \\ a_{21} & a_{22} & b_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

<u>Proposition</u> If A is almost lower triangular and k is the number of elements in E, then the nullity of A cannot exceed $k + 1$.

<u>Proof</u> Suppose that A annihilates a $(k+2)$ - dimensional subspace V. The space W of all $\underline{x} = (x_1, \ldots, x_n)^T$ for which $x_1 = 0$ and $x_r = 0$ for all r in E, has dimension $n - k - 1$. Therefore W includes a non-zero vector $\underline{v}$ in V. Let the first non-zero component of $\underline{v}$ be v_s, then $s \geq 2$ (because $\underline{v}$ is in W) and s is not in E (because $v_s \neq 0$ and $\underline{v}$ is in W), therefore $b_s \neq 0$. Now consider component number $s - 1$ of the vector $A\underline{v} = 0$, we see that $b_s v_s = 0$, which is a contradiction.

Corollaries

(i) No eigenspace of A has dimension more than $k + 1$, for if λ is an eigenvalue then $A - \lambda I$ is almost lower triangular

(ii) If the eigenvectors of A span C^n then A has at least $n/(k+1)$ distinct eigenvalues, and therefore exactly n if $k = 0$. Thus any tridiagonal Hermitean $n \times n$ matrix whose superdiagonal elements are all non-zero has n distinct eigenvalues.

H. Kestelman

RECORDS TUMBLE

$2^{44497} - 1$ IS PRIME

In quick succession the largest known prime has grown larger. Last year came the surprising announcement (L.A. Times November 16, 1978) (JCMN 18) that two freshmen students of USC Hayward, Laura Nickel and Curt Noll, had completed a high school computing project by showing that $2^{21701} - 1$ (a number of 6553 digits) is prime. Curt Noll continued this work, and next in February, 1979 found that $2^{23209} - 1$ (6987 digits) is prime. His good fortune is illustrated by the more recent announcement (L.A. Times May 31, 1979) that the next prime in the sequence $2^m - 1$ is $2^{44497} - 1$ (a number of 13,395 digits); this finding is due to Harry Nelson and David Slowinski who apparently had access to considerably faster computing facilities than did Noll; the latter is reported to have remarked that it would have taken him 16 years to duplicate this finding. So now we know of 27 Mersenne primes (primes of the shape $2^m - 1$) and by the same token, of 27 perfect numbers: a number is said to be perfect if it is equal to the sum of its divisors other than itself, and it has long been known (Euclid) that $2^{p-1}(2^p - 1)$ is perfect if $2^p - 1$ is prime. The complete list of known Mersenne primes is $2^m - 1$ with

m = 2 3 5 7 13 17 19 31 61 89 107 127
521 607 1279 2203 2281 3217 4253 4423 9689
9941 11213 19937 21701 23209 44497

A.J. van der Poorten

SOME MATRIX POLYNOMIAL QUESTIONS (JCMN 18)

The following is compounded from contributions by *R.O. Davies* and *G. Szekeres*.

(a) The k^{th} power of the given matrix J has r, s element equal to 1 when $r + k = s$ and zero otherwise. If A commutes with J then equating components of JA and AJ gives $a_{r+1\,s} = a_{r\,s-1}$ (unless $r = n$ or $s = 1$) so that A is of the form

$$\begin{matrix} \alpha & \beta & \gamma & \cdots \\ 0 & \alpha & \beta & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{matrix}$$

which means $A = \alpha I + \beta J + \gamma J^2 + \ldots$.

(b) Because A is normal it is of the form $A = U^* D U$ where U is unitary and D a diagonal matrix of the eigenvalues of A. Then $A^* = U^* D^* U = U^* \overline{D}\, U$ and it follows that A^* is a polynomial in A if and only if $\overline{D}$ is a polynomial in D, that is if there is a polynomial p such that $\overline{\lambda} = p(\lambda)$ for each eigenvalue λ of A. This is so by polynomial interpolation.

(c) Is every matrix commuting with A a polynomial in A? No, because every matrix commutes with the identity I but does not have to be a polynomial in I. However in the 2×2 case if $AB = BA$ there is a linear relation between A and B, in fact $(a_{11}b_{22} - a_{22}b_{11})I + (b_{11} - b_{22})A + (a_{22} - a_{11})B = 0$.

This means that if A is 2×2 and has its diagonal elements unequal then every B commuting with A is a (linear) polynomial in A. But what if A has diagonal elements equal? It can be checked that

$$(a_{21}b_{11} - a_{11}b_{21})I + b_{21}A - a_{21}B = 0$$

and

$$(a_{12}b_{11} - a_{11}b_{12})I + b_{12}A - a_{12}B = 0$$

so that either A is a multiple of I or again B is a (linear) polynomial in A. This establishes for the 2×2 case that if B commutes with a non-degenerate matrix A then B is a polynomial in A. Can this be extended to the $n \times n$ case?

(d) *(From H.O. Davies)*

A problem related to these arose when I was tutoring a postgraduate course at Macquarie University in the '60s. Here is a proof that if A is a normal $n \times n$ matrix with distinct eigenvalues then any matrix B that commutes with A is also normal.

It is known that $A = U^*DU$ for some unitary U and some diagonal D, also $AB = BA$, so that $U^*DUB = BU^*DU$. Now put $UBU^* = C$. Then it follows that $DC = CD$, and by equating r, s components on both sides, $d_r c_{rs} = c_{rs} d_s$. For $r \neq s$, if $d_r \neq d_s$ then $c_{rs} = 0$, that is C is diagonal and therefore $B = U^*CU$ is normal.

BASIC CALCULUS

One way of proving that $\sin x/x \to 1$ as $x \to 0$ is to divide an arc of a circle into n equal bits, then by the theory of rectifiable curves $2n \sin(x/2n) \to x$. But how obvious is it that if f is a function such that (for each x) $n f(x/n) \to x$ as (integer) $n \to \infty$, then $f(x)/x \to 1$ as $x \to 0$?

C.F. Moppert

GEOMETRICAL INEQUALITY

A corridor of unit width has a right-angled corner in it. What is the largest area of a table that can be carried along the corridor and round the corner, while always being kept horizontal?

M.J.C. Baker and B.J.W. Baker

COOK NEWS

The editor of the book "Captain Cook, Navigator and Scientist" (Australian National University Press, 1970) was knighted by Her Majesty the Queen in June. Congratulations to *Sir Geoffrey Badger*.

PROJECTION MATRICES

Let $P_1, \ldots, P_q$ be $n \times n$ real or complex matrices such that $P_r^2 = P_r$ for all r and $P_r P_s = 0$ for all $r \neq s$. Show that any linear combination of these matrices is a matrix whose eigenvectors span the whole space R^n or C^n.

H. Kestelman.

FRIENDSHIP AND LOVE (JCMN 18)

For hypothesis H, the following facts can be proved.

(i) The number of vertices must be of the form
$$n = u^2(v^2 - 1) \geq 8$$
where u and v are positive integers such that $u < v$, and u and v cannot both be even.

(ii) If J is the matrix whose elements are all 1, then
$$MJ = JM = uvJ$$
$$M^2 = u^2 I + J .$$
Thus each vertex is attached to exactly $u^2 + 1$ double bonds, and $uv - (u^2 + 1)$ single bonds enter and leave each vertex.

(iii) To within isomorphism, the solution in your figure 3 is the unique solution for $n = 8$. I don't know whether solutions exist for any of the possible values $n = 15, 24, \ldots$.

(iv) The eigenvalues of M are uv (with multiplicity 1) and $\pm u$ (with suitable multiplicities to make the trace of M equal to zero). Also M is diagonalizable, and has a basis of eigenvectors.

J.M. Hammersley

A LOVE-HATE THEOREM

The example that I gave (JCMN 18 page 21) for a 7 point solution of the asymmetric version of the friendship problem is really due to Paul Erdős and was described in a joint article by Esther and myself in the Math. Gazette (the English one, not the AMS version) in 1965.

We never thought of an interpretation in terms of love, and for a good reason too: if the arrow indicates love, the opposite direction must surely be interpreted as hate, or at least indifference which is not much better, and it would have been quite inappropriate for us to consider an example in which "a loves b" implies that b doesn't care much for a. (Hammersley's solution of problem H in JCMN18 does not suffer from this embarassing assumption). In the Gazette article we contemplated a problem known among enthusiasts of Erdősian graph theory as Schütte's problem. Assuming that a always loves or dislikes b, and indeed he loves b if and only if b cannot stand him, the problem askes for a love-hate relation among a suitable number of people in which to every set of k people there is at least one other person who hates the whole lot of them. Was the problem perhaps inspired by politicians?

In mathematical terms problem G of JCMN18 is to find a general construction of directed graphs in which every pair of vertices a, b is linked to exactly one c so that both ac and bc are directed towards c. Several years ago I gave an account of this and similar other problems in Mat. Lapok, a journal that only publishes in Hungarian, being the official house journal of the Bolyai Society. Since barbarian readers of JCMN cannot be assumed to possess even the most rudimentary knowledge of the language, I shall give here a few relevant details in a more familiar tongue.

My 7-point example was indeed derived from the simplest instance of a finite projective plane of $v = q^2 + q + 1$ points (q prime power), with a little extra structure added. Readers of Marshall Hall's Combinatorial Theory are of course aware of Singer's ingenious construction, but for the sake of the uninitiated let me sketch the struction, but for the sake of the uninitiated let me sketch the method by the example of $q = 3$, $v = 13$. It is really quite marvellous the way in which mysteries of Galois fields can help to sort out problems in human relations.

The first step is to find an irreducible cubic equation over GF(q) (or, if you like, modulo q if q is prime), whose roots generate the multiplicative group of $GF(q^3)$. The existence of such an equation is one of these miraculous things that happen in Galois fields. If $q = 3$ then $X^3 + 2X + 1 = 0$ is such an equation. Denote by β a root of the equation; technically it is an element of $GF(q^3)$. Now prepare

a table for the first $\frac{q^3-1}{q-1} = q^2 + q + 1 = 13$ powers of β, expressing them as quadratic polynomials of β modulo 3 with the help of $\beta^3 = \beta + 2$. The expressions are shown in the first two columns of the table.

k	β^k	L_k
1	β	(2, 8, 12, 13)
2	β^2	(1, 3, 9, 13)
3	$\beta + 2$	(2, 6, 7, 9)
4	$\beta^2 + 2\beta$	(6, 10, 11, 13)
5	$2\beta^2 + \beta + 2$	(5, 9, 10, 12)
6	$\beta^2 + \beta + 1$	(3, 4, 6, 12)
7	$\beta^2 + 2\beta + 2$	(3, 7, 8, 10)
8	$2\beta^2 + 2$	(1, 7, 11, 12)
9	$\beta + 1$	(2, 3, 5, 11)
10	$\beta^2 + \beta$	(4, 5, 7, 13)
11	$\beta^2 + \beta + 2$	(4, 8, 9, 11)
12	$\beta^2 + 2$	(1, 5, 6, 8)
13	2	(1, 2, 4, 10)

We now declare as "points" of the geometry the equivalence classes of powers of β under the projective equivalence

$$(a_o + a_1\beta + a_2\beta^2) \sim (a_o' + a_1'\beta + a_2'\beta^2)$$

iff there is a non-zero b such that $a_i' = b\, a_i$, $i = 0, 1, 2$. The listed $q^2 + q + 1$ powers of β represent these equivalence classes exactly once.

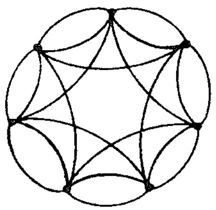
To define the lines of the geometry, let $\phi = a_o + a_1\beta + a_2\beta^2$ be any of the listed expressions. Associate with ϕ the set of points L_ϕ consisting of all $\psi = b_o + b_1\beta + b_2\beta^2$ which satisfy $a_ob_o + a_1b_1 + a_2b_2 \equiv 0 \pmod 3$ (or generally $= 0$ in GF(q)). A simple counting shows that each of the $q^2 + q + 1 = 13$ "lines" L_ϕ contains exactly $q + 1 = 4$ points, and two distinct L_ϕ, $L_{\phi'}$ have exactly one point in common. The reader can easily verify this in the case of $q = 3$ by skimming through the last column of our table. (The numbering of the L_k is self-explanatory; the four numbers in each bracket are the serial numbers from the first column of the points which make up the line). Notice that these lines can also be obtained by taking any one of them, say (1, 2, 4, 10), and forming the sets $S_a = (1 + a, 2 + a, 4 + a, 10 + a)$ modulo 13, $a = 1, 2, \ldots, 13$. Each S_a is a so-called difference set in the sense that every non-zero residue mod 13 can uniquely be obtained as a difference of two members of the set. This is indeed a general feature of Singer's construction; it supplies a difference set modulo any number of the form $q^2 + q + 1$.

The reader may well be puzzled why I picked (1, 2, 4, 10) as my "basis" for the difference set and not, say, (1, 7, 11, 12). The answer is that with the given choice we have the condition

$$a \in S_b \Rightarrow b \notin S_a$$

fulfilled for all a, b. I leave it to the reader as an exercise to show that a difference set always has a basis fulfilling this condition. For example in the next case past 3, namely $q = 4$, $v = 21$, the construction yields (4, 5, 8, 18, 20) for such a basis mod 21.

Now what good is there in my condition? Construct a graph whose vertices are the points of the geometry and the relation $a \lambda b$ (read: *a* loves *b*) holds iff $a \in S_b$. Then by construction, to given b_1, b_2 there is exactly one *a* for which $a \lambda b_1$, $a \lambda b_2$; for it requires $a \in S_{b_1} \cap S_{b_2}$ and there is exactly one such *a* for any pair b_1, b_2. Because of my condition "*a* loves *b*" necessarily implies that *b* doesn't care much for *a*, *a* should not be a member of S_a, (shouldn't love himself) we obtain other equally acceptable solutions which however contain double edges. The figure shows such a solution for $v = 7$.



Perhaps we should call the more restricted problem in which double edges are not allowed the Love-Hate problem. Thus there is a Love-Hate theorem for $v = q^2 + q + 1$ people whenever q is a prime power. Anyone who produces a solution for $v = k^2 + k + 1$ when k is not a prime power deserves a mathematical Nobel prize.

G. Szekeres

HERMITEAN MATRICES

Let A and B be complex square matrices, $m \times m$ and $n \times n$ respectively. A is hermitean ($A^* = A$) and B is skew-hermitean ($B^* = -B$). There is a non-zero $m \times n$ matrix X such that $AX = XB$. Prove that A and B are both singular.

H. Kestelman

ANOTHER LIMIT (JCMN 18)

The problem was to show that $\sqrt{\left(1 + \sqrt{\left(1 + 2\sqrt{\left(1 + 3\sqrt{\left(1 + \ldots + (m-1)\sqrt{(1+m)}\ldots\right)}\right)}\right)}\right)}$ $\to 2$.

Let $R_n(x) = \sqrt{(1 + nx)}$ for positive x. If we leave out brackets that are not necessary, the question was about the limit of $f(m) = R_1R_2 \ldots R_{m-1}R_m(1)$. First note that $R_1R_2\ldots R_m(m+2) = 2$ because $R_n(n+2) = n+1$, and $R_1(3) = 2$. Secondly if $0 < k \le x/y < 1$ then $\sqrt{k} < R_n(x)/R_n(y) < 1$ because $k < (1+nx)/(1+ny)$. Putting $k = 1/(m+1)$ and $x = 1$ and $y = m+1$ we have

$$1/\sqrt{(m+1)} < \frac{R_m(1)}{R_m(m+2)} = \frac{R_m(1)}{m+1} < 1$$

$$\text{Therefore} \quad (m+1)^{-1/4} < \frac{R_{m-1}R_m(1)}{R_{m-1}(m+1)} = \frac{R_{m-1}R_m(1)}{m} < 1$$

Repeating the reasoning gives

$$(m+1)^{-1/8} < \frac{R_{m-2}R_{m-1}R_m(1)}{m-1} < 1$$

$$\text{Eventually} \quad (m+1)^{-2^{-m}} < \frac{R_1R_2 \ldots R_m(1)}{2} < 1$$

Since the term on the left hand side tends to 1 it follows that f(m) tends to 2. *J.D.E. Konhauser* writes that the problem was in the 27th Annual Putnam Competition of November 19th 1966, in the form "Justify the statement that $3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{\ldots}}}}}$". *Vichian Laohakosol* writes that the problem was given with solution in the American Mathematical Monthly, volume 63 (1956) pp. 194-195, where it was traced back to S. Ramanujan, Collected Papers (Cambridge 1927) p. 323.

ANOTHER IDENTITY FOR BINOMIAL COEFFICIENTS

(JCMN 17 and 18)

In JCMN 18, pp. 10-12 solutions were given to the problem posed in JCMN 17:

Prove the identity

$$(-1)^n \binom{2n+1}{n} = 2^{2n+1}(2n+1)\binom{\frac{1}{2}}{n+1}$$

Here is yet another short proof using a simple *counting argument*.

Proof By expanding the expression $\binom{\frac{1}{2}}{n+1}$ and simplifying slightly, we obtain

$$\binom{2n}{n} n! = 2^n.1.3.5 \ldots (2n-1)$$

Each of the two sides represents the *number of possible outcomes of the first round of a tournament between 2n participants where each participant plays in exactly one game against one opponent and each game must end with one side winning (i.e. no draws are possible).*

Consider first the right hand side of the identity. There are $(2n-1)(2n-3) \ldots 3.1$ ways in which the first round can be organised: picking any one of the $2n$ players, there are $(2n-1)$ ways of choosing his opponent. Picking any one of the remaining $(2n-2)$ players the opponent can be found in $(2n-3)$ ways, and so on. For each pair the game may end in 2 different ways, hence for the n pairs there are 2^n outcomes, i.e. $2^n(2n-1)(2n-3) \ldots 3.1$ outcomes of the tournament (if the arrangements are not predetermined).

The left hand side of the identity is obtained by noting that the list of the n winners can arise in

$$\binom{2n}{n} \text{ ways,}$$

and there are $n!$ ways in which the remaining n participants could be "allocated" as the losing opponents.

One can carry these counting arguments one step further to extract some other identities.

Considering the possible outcomes of the tournament if we allow draws as well as winning or losing, we obtain by similar counting procedures the identity

$$\sum_{k=o}^{n-1} \binom{2n}{2k} \binom{2k}{k} k! \prod_{j-1}^{n-k} (2j-1) + \binom{2n}{n} n! = 3^n.1.3.5. \ldots (2n-1)$$

This can be written in terms of binomial coefficients of type $\binom{\frac{1}{2}}{r}$ as

$$\sum_{j=o}^{n} \frac{(j+1)!}{(2j)!} \frac{1}{(n-j)!} (-2)^j \binom{\frac{1}{2}}{j+1} = \frac{(n+1)!}{(2n)!} (-6)^n \binom{\frac{1}{2}}{n+1} .$$

Any alternative proofs of the above identity?

Marta Sved.

MATRIX NUMBER THEORY

When setting homework questions on quadratic forms for my Mathematical Methods class, I want to find a real symmetric matrix whose components are integers, preferably between 0 and 9, with eigenvalues rational and therefore integers. The obvious way is to use a rational orthogonal transformation on a diagonal matrix of integers.

To construct a rational orthogonal $n \times n$ matrix it is necessary to have a square expressible as a sum of n or fewer squares, because if the top row is a/m, b/m, c/m, ... then $m^2 = a^2 + b^2 + \ldots$. The first few non-trivial cases with $n = 3$ are $3^2 = 2^2 + 2^2 + 1$, $7^2 = 6^2 + 3^2 + 2^2$ and $9^2 = 8^2 + 4^2 + 1^2 = 7^2 + 4^2 + 4^2$ and these lead without difficulty to rational orthogonal matrices, and so to integer symmetric matrices with integer eigenvalues. For instance

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4p + 4q + r & 4p - 2q - 2r & 2p - 4q + 2r \\ 4p - 2q - 2r & 4p + q + 4r & 2p + 2q - 4r \\ 2p - 4q + 2r & 2p + 2q - 4r & p + 4q + 4r \end{pmatrix}$$

must have eigenvalues $9p$, $9q$ and $9r$.

This leads to a matrix with biggish components, but there is a possible simplification. Choosing p, q and r so that their sum is divisible by 3, the matrix components will all divide by 3. Then taking advantage of the fact that we may add any integer multiple of the unit matrix, we come to the conclusion that $\begin{pmatrix} u + v & 2v + 2w & 2w \\ 2v + 2w & u + w & 2v \\ 2w & 2v & u - v - w \end{pmatrix}$

has eigenvalues $u + 3v + 3w$, $u - 3v$ and $u - 3w$. This solves our original problem of finding symmetric 3×3 matrices with integer components chosen from 0, ... 9 and integer eigenvalues.

Now have we uncovered any interesting questions in matrix number theory? A natural speculation is that if we can find $m^2 = a^2 + b^2 + c^2$ (all integers) then there is an integer matrix M with top row (a, b, c) such that $MM^T = m^2 I$, but this turns out to be wrong. In fact $17^2 = 12^2 + 12^2 + 1^2$ does not lead to such a matrix. Is there a rational unit vector orthogonal to (12, 12, 1)?

SOME CHARACTERISTIC POLYNOMIAL QUESTIONS

(i) M is an $n \times n$ matrix whose characteristic polynomial $\det(tI - M)$ has the form $t^n - (c_o + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1})$ where the c_r are all ≥ 0, their sum is 1, and there is at least one consecutive pair c_k, c_{k+1} both of which are positive. Show that the sequence $\{M^j\}$ converges to a matrix of rank 1.

(ii) A is a matrix with characteristic polynomial

$$t^{km} + c_{m-1} t^{k(m-1)} + \ldots + c_r t^{kr} + \ldots + c_o \quad ;$$

if $k > 1$ and $\{A^j\}$ converges then its limit is 0.

H. Kestelman

INTEGRAL ROOTS (JCMN 17 and 18)

Here is a proof that *J.B. Parker's* contribution did not omit any of the solutions.

If both roots of both equations

$$x^2 + ax + b = 0, \quad y^2 + by + a = 0 \qquad (1)$$

are integers, then a and b are integers. There are two families of solutions:-

<u>Family 1</u> Both equations have 1 as a root if and only if $a + b + 1 = 0$, the other roots being $x = b$, $y = a$.

<u>Family 2</u> At least one equation has 0 as a root if and only if $ab = 0$, and then $(a,b) = (0, -x^2)$ or $(-y^2, 0)$ where x and y are integers.

We look for solutions which do not belong to either family. If one equation, say the first, has both its roots -1, the other equation, $y^2 + y + 2 = 0$, does not have integer roots. Hence we may assume that each equation has at least one root not equal to 0 or ± 1. We write x and y for these two roots, and write $\xi = |x|$, $\eta = |y|$. We may therefore assume

$$2 \leq \xi \leq \eta , \qquad (2)$$

since we can, if necessary, interchange a and b. Thus $xy \neq 1$, and we can solve (1) for a and b in terms of x and y, obtaining

$$a + x = \frac{y^2 - x}{xy - 1}, \quad b + y = \frac{x^2 - y}{xy - 1}, \qquad (3)$$

$$(a + b + 1) + (x + y + 1) = (x + y + 1)(x + y - 2)/(xy - 1), \qquad (4)$$

$$(a - b) + (x - y) = -(x - y)(x + y + 2)/(xy - 1) \qquad (5)$$

Thus we have

$$x + y \neq -1 , \qquad (6)$$

for otherwise (4) implies $a + b + 1 = 0$ and gives Family 1. Also $y^2 - x \neq 0$; for otherwise $a + x = 0$, and thence $b = 0$ by (1), so that Family 2 occurs. Similarly $x^2 - y \neq 0$. Hence from (3)

$$xy - 1 \text{ divides both } y^2 - x \text{ and } x^2 - y. \qquad (7)$$

Hence $\xi\eta - 1 \leq |xy - 1| \leq |x^2 - y| \leq \xi^2 + \eta$; and thence $\eta(\xi - 1) \leq \xi^2 + 1 = (\xi - 1)(\xi + 1) + 2$. Therefore $\eta \leq \xi + 1 + \frac{2}{\xi - 1}$. Since ξ and η are integers we conclude that

$$\text{either} \quad 2 \leq \xi \leq \eta \leq \xi + 1 \qquad (8)$$

$$\text{or} \quad (\xi, \eta) = (2, 4) \text{ or } (2, 5) \text{ or } (3, 5). \qquad (9)$$

Since $y \neq x^2$, there are 10 cases to consider under (9), namely

$$(x, y) = (\pm 2, -4), (\pm 2, \pm 5), (\pm 3, \pm 5).$$

Only 2 of these 10 cases satisfy (7), namely $(x, y) = (-2, -5)$ and $(-3, -5)$; and, from (3), the corresponding values of a and b are

$$(a, b) = (5, 6) \text{ and } (6, 5). \qquad (10)$$

This leaves (8), which in view of (6) can be written as

$$\text{either} \quad x + y = 0, 1, \qquad (11)$$

$$\text{or} \quad x - y = \pm 1, \qquad (12)$$

$$\text{or} \quad x - y = 0. \qquad (13)$$

If (11) holds, the right hand side of (4) is $-2/(xy - 1)$ and is an integer; so $\xi\eta \leq 3$, contradicting (8). If (12) holds, the right hand side of (5) is the integer $\pm(x + y + 1)/(xy - 1)$ with a non-zero numerator. So $\xi\eta - 1 \le |xy - 1| \le |x + y + 1| \le \xi + \eta + 1$. Hence $\xi\eta - \xi + \eta + 2 \le 2\xi + 3$; $\eta \le 2 + 3/\xi$; whence $\eta \le 3$. So $(x, y) = (2, 3)$ or $(-2, -3)$, neither of which satisfies (7). So (13) is the only possibility. By (5) it gives $a = b$; and from (1) we see that $(a - 2)^2 - 4 = a^2 - 4a = (2x + a)^2$. The only two squares that differ by 4 are 0 and 4, so that $a = 0$ or 4, and $a = 0$ is excluded by Family 2. Therefore

$$(a, b) = (4, 4) \tag{14}$$

and (10) and (14) are the only solutions not belonging to Families 1 and 2.

J.M. Hammersley

QUOTATION CORNER (1)

He who can, does. He who cannot, teaches.

– G.B. Shaw, Maxims for Revolutionists, an appendix to the play Man and Superman (1903).

If Shaw were alive to-day he might have commented on the next development in our educational revolution; those unable either to do or to teach anything give advice on how to teach.

BOUND VOLUME

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Your editor would like to hear from you anything connected with mathematics or with James Cook.

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