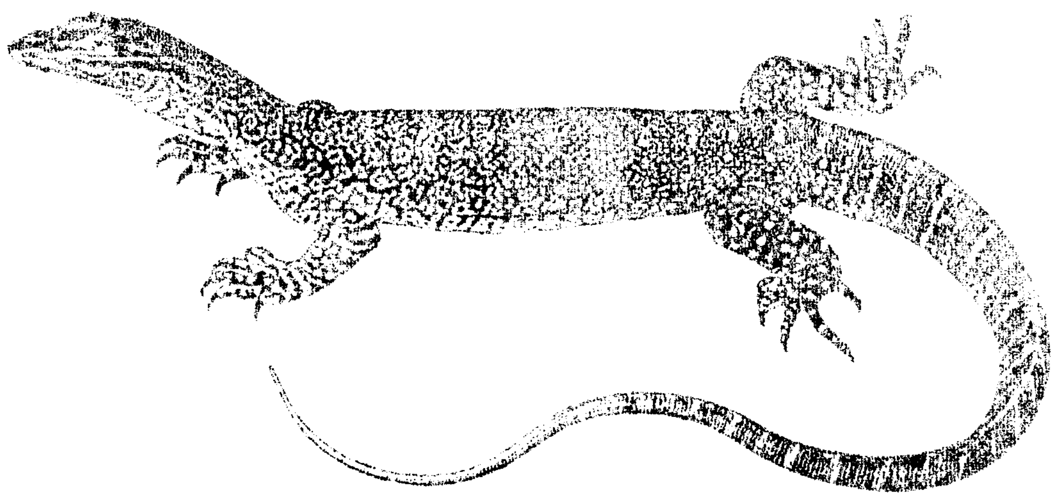
JAMES COOK MATHEMATICAL NOTES

Number 13. November 1977.



In August 1770 Captain Cook and Joseph Banks landed on the island that they named Lizard Island, the former to climb to the summit and look for a passage through the Barrier Reef, and the latter to investigate the wild life. Present day scientists landing there find that large lizards like the one shown above approach to look for scraps of food, as probably Cook and Banks and the crew of the long-boat found in 1770.

A THEOREM OF WIELANDT

Let A be a real nxn matrix of non-negative elements, such that some power of A has all elements strictly positive. Prove that A^k has all elements strictly positive when $k = (n-1)^2 + 1$, and show that this value is the least possible.

H. Fentelman

FROM A BOOKSELLER'S CATALOGUE

Fundamentals of unclear science, with applications in agriculture and biology, by P. N. Tiwari. 1974, $5.20.

CAN YOU SOLVE A QUADRATIC EQUATION? (JCMN 12)

For what real positive a and b does $z^2 + az + b + ia = 0$ have the real parts of both roots negative?

R.B. Potts does this geometrically. Putting $z = x + iy$, the roots are where $x^2 - y^2 + ax + b = 0$ cuts $2xy + ay + a = 0$. Both hyperbolas have centre $(-\frac{1}{2}a, 0)$. For both roots to be in the left half plane the first hyperbola must cut the negative y-axis below where the second does, that is $-\sqrt{b} < -1$ or $b > 1$.

Algebraic proofs came from *H.O. Davies* and *H. Kestelman*:

(i) The roots are given by $2z = -a \pm c\exp(i\theta/2)$ where $c = [(a^2-4b)^2+16a^2]^{1/4}$ and $c^2\cos\theta = a^2-4b$ and $c^2\sin\theta = -4a$ and $2\cos^2\theta/2 = 1+\cos\theta = (c^2+a^2-4b)/c^2$. The conditions for both real parts to be negative is

$a > \pm c\cos\theta/2$ or $2a^2 > 2c^2\cos^2\theta/2 = c^2 + a^2 - 4b$

or $a^2 + 4b > c^2$

or $a^4+8a^2b+16b^2 > c^4 = a^4-8a^2b+16b^2+16a^2$

which leads to $b > 1$.

(ii) Since the sum of the roots is the real number $-a$, the roots have the form $iy - (\frac{1}{2}+\mu)a$ and $-iy-(\frac{1}{2}-\mu)a$ where $y > 0$ and μ is real. (A)

Since their product is $b+ia$,

$b=y^2 + a^2/4 - a^2\mu^2$ and $2y\mu = 1$ (B)

Since y is positive, μ must be positive and satisfy

$4\mu^2 b = 1 + \mu^2 a^2(1-4\mu^2)$ (1)

and the problem is to find conditions in which the solutions μ of (1) are between 0 and $\frac{1}{2}$. If this is so and (1) holds then $b > 1$. Conversely, suppose that $b > 1$; we have to show that if the numbers given by (A) satisfy (B) then $0 < \mu < \frac{1}{2}$. Set $f(t) = 4t^2a^2 + t(4b-a^2) - 1$. If μ satisfies (1) then $f(\mu^2) = 0$, but $f(0) = -1$ and $f(1/4) = b - 1 > 0$, and so the only positive root of $f(t) = 0$ lies in $(0, 1/4)$, thus $0 < \mu^2 < 1/4$ and so $0 < \mu < 1/2$.

(iii) Increasing both roots by i gives a new quadratic $z^2 + (a-2i)z+b-1 = 0$. The product of the roots is real, equal to $b-1$ and so the real axis bisects the angle between the roots. If $b > 1$ the real parts have the same sign, and because their sum is negative, both are negative. If $b < 1$ the real parts have opposite sign and therefore one of them is positive.

(JCMN 13)

(iv) The schoolboy method would be to say that the critical cases are when one root is pure imaginary and clearly has to be $-i$ so that $b=1$. The answer must be either $b > 1$ or $b < 1$. To decide between these two alternatives just take a simple case like $a = 100$ and $b=2500$ giving $z = -50 \pm 10\sqrt{i}$, which has both real parts negative, so that $b > 1$ is the answer.

QUADRATIC EQUATIONS AGAIN

For what complex b and c does the quadratic $z^2 + bz + c = 0$ have roots of equal modulus? If this is so and if p and q are the roots, in what circumstances does $p^k = q^k$ for some positive integer k?

H. Kestelman

MATRIX OF ZEROS AND ONES

The $n \times n$ matrix A $(n \geq 3)$ has all its elements zero except that $a_{rs} = 1$ when $|r-s| = 1$. If M_j and m_j are the greatest and least elements in A^j, show that $m_j = 0$ and $M_j \to \infty$ as $j \to \infty$.

H. Kestelman

NEW THEOREM: ALMOST EVERY CONVEX POLYHEDRON IS A TETRAHEDRON (JCMN 4 and 7)

Apologies to all the cubes that were forced to doubt their own reality, the new theorem was a hoax.

Consider all the possible probability distributions of any set of things. They may be divided into equivalence classes by the condition of being uniformly continuous with respect to one another. Any assertion beginning "Almost every ...." refers to some class of probability distributions, not just to one distribution. For instance let the set be R^n, there are many classes of distributions but among them one is particularly simple and natural, the class of distributions for which a set has positive probability if and only if it has positive Lebesgue measure. Consequently there is no difficulty about interpreting "almost every point of the plane" or "almost every triangle". However when it comes to convex polyhedra in three dimensions there are two classes of probability distribution with an equal claim to being the natural canonical class: one is derived from the idea of constructing the polyhedron as the convex hull of finitely many points, and the other by regarding the polyhedron as an intersection of half-spaces. Confusion of these two classes leads to the wrong conclusion that almost every convex polyhedron is a tetrahedron.

NUMBER THEORY (JCMN 12)

Problem. Given integers a, b for what positive integers n does $pq \equiv a \mod n$ imply $p+q \equiv b \mod n$.

If a, b, n satisfy these conditions, any a', b' in the residue classes of a, b mod n will also satisfy these conditions. We may assume $0 \leq a, b < n$. Since $1.a \equiv a \equiv (-1)(-a)$

then $1+a \equiv -1-a \equiv b$ so $2(1+a) \equiv 0 \mod n$.

Case 1 $n = 1+a$. Then a is in the multiplicative group of all integers coprime to n, modulo n. Let β be any element in this group, say $\beta^k \equiv 1 \mod n$.

$$a\beta \times \beta^{k-1} \equiv a$$

$$\therefore\ a\beta + \beta^{k-1} \equiv 0 \mod n$$

$$\therefore\ \beta^{k-2} \equiv 1 \mod n \quad (\text{since } a \equiv -1)$$

This contradicts the order of β being k, unless $k \leq 2$. Hence every non-trivial element of the group has order 2. The only n for which this happens is

$$n = 2, 3, 4, 6, 8, 12, 24.$$

This follows from the following result.

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ is the prime decomposition of n, then the multiplicative group of all integers coprime to n mod n, is an abelian group with a direct product decomposition into factors Z_i corresponding to the primes p_i where

$$\text{if } p_i \nmid 2,\ Z_i \text{ is cyclic of order } \phi(p_i^{\alpha_i}) = p_i^{\alpha_i}(1-\frac{1}{p_i})$$

$$\text{if } p_i = 2,\ \text{then } Z_i = \begin{cases} \text{trivial group if } \alpha_i = 1 \\ \text{cyclic of order 2 if } \alpha_i = 2 \\ \text{direct product } C_2 \times C_2 \text{ if } \alpha_i = 3 \\ \text{direct product } C_2 \times C_2^{\alpha_i - 2} \text{ of } \alpha_i > 3 \end{cases}$$

Possible solutions corresponding to case 1 are

a	b	n
1	0	2
2	0	3
3	0	4
5	0	6
7	0	8
11	0	12
23	0	24

For the listed a, b, n we want to prove that $pq \equiv a$ implies $p+q \equiv b$?

if $pq \equiv -1$ then $p+1 \equiv 0$

(JCMN 13)

The residue class p has an inverse $(-q)$ therefore p is in the multiplicative group and so $p^2 \equiv 1$ but $p(-q) \equiv 1$ and so $p \equiv -q$.

Case 2 $n = 2(1+a)$. If a is even, $1 \cdot a = a = 2 \times \frac{a}{2}$ and so $1+a \equiv 2 + a/2$ which implies $a = 2$, and $n = 6$. This does not give a solution.

Suppose a is odd. Then $(n,a) = 1$ so again a lies in the multiplicative group of all integers coprime to n mod n. Let β be any element of this group, say $\beta^k \equiv 1$.

$$\begin{aligned}
a\beta \times \beta^{k-1} &\equiv a \\
\text{so } 2a\beta + 2\beta^{k-1} &\equiv 0 \\
\therefore\ 2a + 2\beta^{k-2} &\equiv 0 \\
\therefore\ 2\beta^{k-2} &\equiv 2 \quad \text{mod } n \\
\therefore\ 2 &\equiv 2\beta^2 \qquad\qquad \text{since } \beta^k \equiv 1 \\
\therefore\ 1 &\equiv \beta^2 \quad \text{mod } a+1
\end{aligned}$$

This implies $\frac{n}{2}$ is one of the n found in case 1. Corresponding solutions for case 2 are

a	b	n
1	2	4
3	4	8
5	6	12
11	12	24

By inspecting it will be seen there are no solutions for the possible values $n = 16, 48$.

B.B. Newman

TWO QUESTIONS ON BINOMIAL COEFFICIENTS (JCMN 12)

Solutions have come in from *H.O. Davies*, *B.B. Newman* and *C.J. Smyth*, no two of them alike.

(a) Find $u(k) = \sum_0^n (-1)^\nu \binom{n}{\nu} \nu^k$ for $k = 0, 1, \ldots n+1$.

Bill Newman points out that 0^0 has to be given the value 1, because the binomial expansion of $(1+0)^0$ is the one term $\binom{0}{0}0^0$.

Method (i) $\sum u(k)x^k/k! = \sum_{j=0}^n (-1)^j \binom{n}{j} \sum_{k=0}^\infty j^k x^k/k!$

$= (1-e^x)^n = (-1)^n x^n (1+x/2 + \ldots)^n$

Equating coefficients of powers of x:

$u(k) = 0$ for $k = 0, 1, \ldots n-1$

$u(n) = (-1)^n\ n!$

$u(n+1) = \frac{1}{2}(-1)^n n(n+1)!$

Method (ii) $(-x)^n(1-x)^{-n} = \{1-1/(1-x)\}^n = \sum(-1)^\nu\binom{n}{\nu}(1-x)^{-\nu}$

$$= \sum_0^n(-1)^\nu\binom{n}{\nu}\{1+\nu x + \frac{\nu(\nu+1)}{2}x^2 + \ldots\}$$

Equate coefficients as before

Method (iii) $(1-x)^n = \sum(-1)^\nu\binom{n}{\nu}x^\nu$

Operate k times with $x\,d/dx$ and put $x=1$ to get $u(k)$. For $k<n$ it is clear without doing the calculation that it will give the result zero. For $k=n$ or $n+1$ one is inclined to put $x = \exp y$ so that $xd/dx = d/dy$, coming back to method (i).

(b) Prove: $\sum_0^k(-1)^s\binom{k}{s}\binom{s-k}{r} = (-1)^k\binom{-k}{r-k}$ for k and r non-negative integers.

Method (i) Because $\binom{-j}{r} = (-1)^{r+j-1}\binom{-r-1}{j-1}$ (*)

$$A = \sum_{s=o}^k(-1)^s\binom{k}{s}\binom{s-k}{r} = \sum_{s=o}^k(-1)^{r+k-1}\binom{k}{s}\binom{-r-1}{k-s-1}$$

which is the coefficient of x^{k-1} in the expansion of

$$(-1)^r\,\{\sum_0^k(-1)^s\binom{k}{s}x^s\}\,\{\sum_{j=o}^\infty(-1)^j\binom{-r-1}{j}x^j\}$$

$$= (-1)^r(1-x)^k(1-x)^{-r-1} = (-1)^r(1-x)^{-r-1+k}$$

Hence $A = (-1)^{r+k-1}\binom{-r-1+k}{k-1} = (-1)^k\binom{-k}{r-k}$ using (*) again

where we must take $r\geq k$ for the last expression to have meaning.

Method (ii) Introduce the notation $a(a-h)\ldots(a-nh+h) = a^{n|h}$.

There is a Factorial Binomial Theorem

$$(a+b)^{n|h} = \sum_o^n\binom{n}{r}a^{n-r|h}b^{r|h}$$

and it may be noted that $\binom{n}{r} = n^{r|1}/r!$

$$\binom{s-k}{r} = (-1)^{r+k-s-1}\binom{-r-1}{k-s-1} = (-1)^{r+k-s-1}(-r-1)^{k-s-1|1}\,k^{s|1}(k-s)/k!$$

$$(-1)^s\binom{k}{s}\binom{s-k}{r} = \frac{(-1)^{r+k-1}(-r-1)^{k-s-1|1}\,k^{s|1}}{(k-s-1)!\;s!}$$

$$= (-1)^{r+k-1}\binom{k-1}{s}(-r-1)^{k-s-1|1}\,k^{s|1}/(k-1)!$$

Now apply the factorial binomial theorem

$$\text{Sum} = (-1)^{r+k-1}(-r-1+k)^{k-1|1}/(k-1)!$$

$$= (-1)^r(r+1-k)(r+2-k)\ldots r/(k-1)! = (-1)^r\binom{r}{k-1}$$

Method (iii) $(-z)^k(1+z)^{-k} = (1+z)^{-k}\{1-(1+z)\}^k$

$$= \sum_0^k(-1)^s\binom{k}{s}(1+z)^{s-k}$$

Expand both sides in powers of z and equate coefficients of z^r.

(JCMN 13)

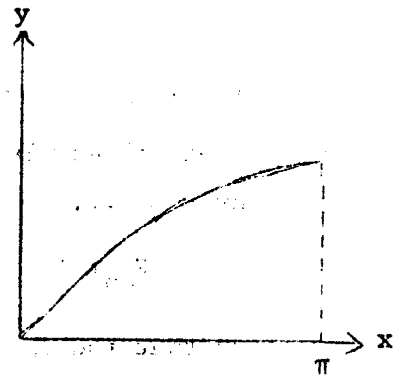
SOME HARD ANALYSIS (JCMN 12)

Let $I_n = \int_0^{\pi} \cot(x/2)\sin(|n|x)\exp(ny)\,dx$

and $J_n = \int_0^{\pi} \cot(x/2)(\exp n(ix+y) - 1)\,dx$

for $n = \pm1, \pm2, \ldots$

where $\sin(x/2) = \sinh(y/2)$ .... (1)



We cannot work with

$$\int_0^{\pi} \cot(x/2)\exp n(ix+y)\,dx$$

because it has infinite real part.

Then $Im\, J_n = \operatorname{sgn}(n)\, I_n$ for $n = \pm1, \pm2, \ldots$

Now (1) gives easily $e^{ix}-1 = w + w^{-1}e^{ix}$ where $w = i\exp\tfrac12(ix+y)$.
Hence $\exp(ix) = (1+w)/(1-1/w)$ and so

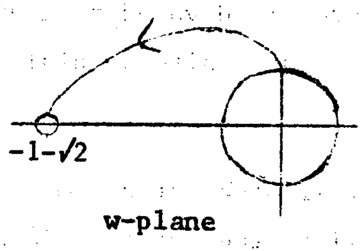
$$\cot x/2\,dx = d(2\log\sin x/2) = d\left(\log\left(\frac{1-\cos x}{2}\right)\right) = d\left(\log\left(2-\frac{1+w}{1-w^{-1}}-\frac{1-w^{-1}}{1+w}\right)\right)$$

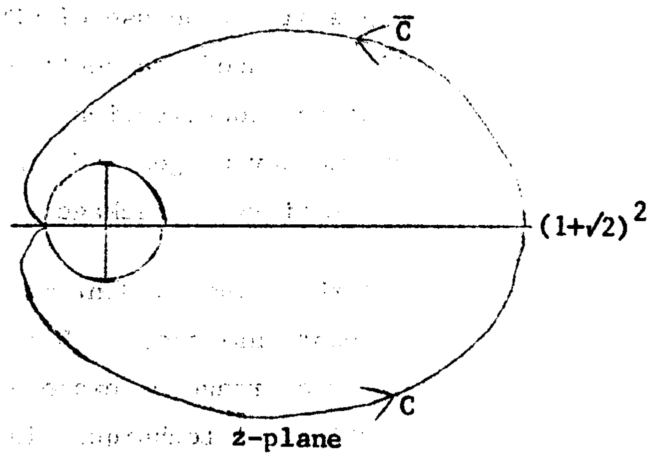
$$= d\left(\log\left(\frac{(w^2+1)^2}{w(w^2-1)}\right)\right) = \left(\frac{4w}{w^2+1} - \frac{1}{w} - \frac{2w}{w^2+1}\right)dw.$$

Put $z = w^2$. Then $e^{ix+y} = -z$ and

$$\cot x/2\,dx = \tfrac12\left(\frac{4}{z+1} - \frac{1}{z} - \frac{2}{z-1}\right)dz.$$

As x goes from 0 to π, $e^{y/2}$ goes from 1 to $(1+\sqrt2)$, w goes from i to $-(1+\sqrt2)$ and so z goes from -1 to $(1+\sqrt2)^2$, on a contour C which passes under the unit circle





Hence $J_n = \int_C F_n(z)\,dz$, where $F_n(z) = \tfrac12\left[(-z)^n - 1\right]\left(\frac{4}{z+1} - \frac{1}{z} - \frac{2}{z-1}\right)$.

Now take the integral over the closed curve $C \cup \bar C$ formed by C and its reflection in the real axis, anticlockwise. Then since

$$F_n(\bar z) = \overline{F_n(z)},\ \text{we have}\ \int_{\bar C} F_n = -\overline{\int_C F_n},\ \text{and so}$$

$$2\,i\,Im\,J_n = \int_{C\cup\overline{C}} F_n(z)\,dz$$

and hence for $n = \pm 1, \pm 2, \ldots$

$$\operatorname{sgn}(n)\,I_n = \pi[\text{sum of residues of } F_n \text{ at } 0 \text{ and } 1] \qquad (2)$$

(N.B. F_n has no pole at -1.) At $z = 1$, residue of F_n = residue of $\frac{(1-(-z)^n)}{z-1}$, which is easily seen to be $1-(-1)^n$. At $z=0$, residue of F_n is clearly ½ when $n=1, 2, \ldots$ For $-n=-1, -2, \ldots$, we have at $z=0$

$$F_{-n}(z) = \tfrac{1}{2}\left[(-z)^{-n} - 1\right] \left[4\left(1-z+z+\ldots+(-1)^{n-1}z^{n-1}+\ldots\right)-z^{-1}+2\left(1+z+\ldots+z^{n-1}+\ldots\right)\right]$$

so that the residue is $\tfrac{1}{2}\left[(-1)^n 4(-1)^{n-1}+1+2\,(-1)^n\right] = -2+\tfrac{1}{2}+(-1)^n$.

Hence for $n=1, 2, \ldots$, $I_n = \left(1-(-1)^n+\tfrac{1}{2}\right)\pi = \left[\frac{3}{2} - (-1)^n\right]\pi$ and

for $-n = -1, -2, \ldots$ $I_{-n} = -\left(1-(-1)^n - 2 + \tfrac{1}{2} + (-1)^n\right)\pi = \frac{\pi}{2}$.

C.J. Smyth

"STATISTICS IN ACTION" by Peter Sprent.
Published by Pelican as a paperback.
Retail Price $2.95. ISBN 0 14 02.1955 2

This introduction to statistics is written for readers with no previous mathematical training beyond simple arithmetic. Because of this, the author concentrates more on statistical concepts and ideas than on computation. He covers a range of topics including hypothesis testing, analysis of variance, regression, the use of CUSUM charts, queueing theory and Bayesian statistics. Also included are sections on more unusual ideas, such as the differences in sentence construction of two authors. One chapter is devoted to a discussion on the advantages and disadvantages of computer usage, with special reference to statistical packages.

A mathematics student may find this book rather light-hearted compared with conventional texts. But for the non-mathematician, the author's informal approach and wide range of examples would turn the task of understanding and applying some statistical techniques into a rather interesting experience.

M. Kahn

ANYONE FOR A SPOT OF RIGOUR?

The partial differential equation

$$\partial/\partial x(r^2\partial y/\partial x) = r^2\partial^2 y/\partial t^2 \quad \text{where } r = r(x) > 0$$

arises naturally from considering sound in a voice-pipe with circular cross-section of variable radius $r(x)$, or (after some manipulation) from a vibrating string of variable density and tension, or from long waves in a non-uniform canal. It is well known that if r is constant there is a solution $y=y(x-t)$ which can be interpreted as a signal moving only in the positive direction, but a step in the function $r(x)$ (that is $r=a$ for $x<b$ and $r=c$ for $x>b$) causes partial reflection of any signal reaching the point $x=b$. Partial reflections are of widespread technical importance and so one wants to clarify how they arise in this partial differential equation. What is wrong with the following discussion?

The parameters $\quad u=r(y_t + y_x)$ and $v = r(y_t - y_x)$

satisfy

$$u_x - u_t = r'v/r$$

and

$$v_x + v_t = r'u/r$$

and the energy conservation equation

$$\partial/\partial x\ (v^2-u^2) + \partial/\partial t\ (v^2+u^2) = 0$$

The mean velocity of energy flow is $(v^2-u^2)/(v^2+u^2)$ from left to right, and this is always between plus and minus one. Generally we may suppose that locally the disturbance may be separated into one part which is a progressive wave in the positive direction (that is from left to right) and another part representing a progressive wave the other way. With each of these two parts there will be associated an energy flow equal to the appropriate energy density multiplied by the group velocity. The energy conservation equation above forces us to conclude that v is the parameter measuring the progressive wave in the positive direction. The equation $v_x+v_t = r'u/r$ tells us that v is propagated unchanged at unit velocity in the positive direction except that where $r'\neq0$ the quantity v is added to by a term that can be interpreted as a partial reflection of the wave that is travelling in the negative direction.

A few years ago a cable railway was being built in the Atherton Tablelands and when the suspension cable was being set up they wanted to check its tension by hitting it with an iron bar and timing the return of the pulse. The tension was appreciably different at the two ends and the simple wave equation did not apply. By reasoning like that above I came to the conclusion that when the partial differential equation was put in the form considered here (replacing length s along the cable by $x = \int(\rho/T(s))^{\frac{1}{2}}ds)$ the partial reflection phenomenon would not seriously impair the transmission of the signal. Consequently I told the engineer involved in the construction that he would get the correct relation between the tension and the time of return of the pulse by numerical integration using the elementary value for the local wave speed.

The cable railway is now in service successfully, but would it be right to say that in our original partial differential equation we may associate partial reflections with non-constancy of the function r(x)?

THIRD TIME LUCKY

If I press the $\frac{1}{x}$ key twice on my calculator, I do not necessarily obtain ~~by~~ my original number. However, if I press it three times I get exactly the same result as pressing it once. Can you prove that this is always so?

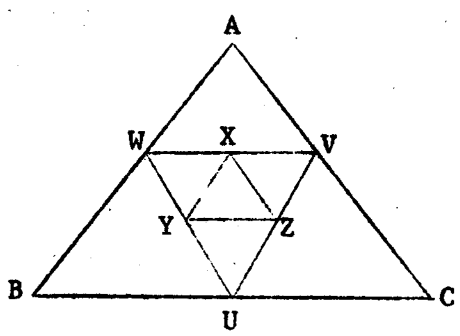
Since this is a discrete problem, it is perhaps best formulated precisely in terms of the integers. As the exponent of 10 of the original number plays no essential part in the problem we can ignore it, and suppose that we have an integer n in the range $10^9 \le n < 10^{10}$. Define

$$f(n) = \text{nearest integer to } \frac{10^{19}}{n},$$

(½ odd integers rounded upwards). Then $10^9 \le f(n) \le 10^{10}$. The problem is to show that for n in this range, $f(f(f(n))) = f(n)$. We can also ask what values $f(f(n))-n$ can take and what is the smallest n with $f(f(n)) \neq n$.

C.J. Smyth

A NEST OF TRIANGLES



There are three triangles, ABC, UVW and XYZ, as above, the second is inscribed in the first and the third in the second.

Prove that if two of the three pairs of triangles are in perspective then so is the third pair.

A.P. Guinand

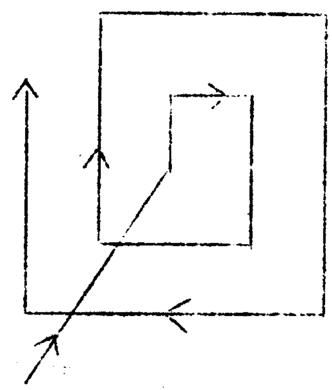
ALGORITHM WANTED

Given n points in k-dimensional Cartesian space how do you calculate the volume of their convex hull?

J.B. Parker

SEARCH PATTERNS

Consider how best to find a stationary object at sea, given an inaccurate estimate for its position. The common method is the "square search". If your aircraft or ship is capable of searching a strip of width w, that is $w/2$ on each side, then you first go to the estimated position, then steer a distance w North, then w East, $2w$ South, $2w$ West, $3w$ North and so on (adding w to the length of the leg at alternate turns). Is there a more efficient way? Two possible disadvantages of the square search are that there is a bit missed on the outside of each corner, and that some bits of sea are searched twice on account of the journey in from base to the centre where the square search starts.



J.B. Parker

INEQUALITY WANTED (JCMN 10 and 12)

Let $0 < a < b$ and let m be a probability measure on $[a, b]$. Put $\mu = \int t\,dm$, $\sigma^2 = \int (t-\mu)^2 dm$ and $G = \exp \int \log t\ dm$. The inequality $\sigma^2/(2b) \le \mu - G \le \sigma^2/(2a)$ (for the case where m consists of finitely many points of equal weight) was suggested by K.S. Williams in Eureka (Volume 3, 1977) and proved by G. Szekeres in JCMN 12. This special case can be shown equivalent to the general case by a theorem in functional analysis. The following is a summary of a paper accepted for Proc. Amer. Math.Soc. entitled "A Refinement of the Arithmetic Mean-Geometric Mean Inequality" by D.I. Cartwright and M.J. Field.

Lemma If $0 \le q \le 1$ and $t \ge 0$ then

$$1 + qt + \tfrac{1}{2}q(1 - 1)t^2 \le (1 + t)^q \le 1 + qt + \tfrac{1}{2}q(q - 1)t^2/(1 + t)$$

This lemma leads to a proof of the theorem when m is a two-point distribution. Now we try to establish the theorem when m is any n-point distribution, points x_r with weights p_r $(r = 1, 2, \ldots n)$. We use induction on n. Let the x_r be fixed, we may assume them distinct. Consider

$$f(p) = f(p_1, \dots p_n) = \mu - G - \sum p_k(x_k-\mu)^2/(2b)$$

as a function of p in the set S where all $p_k \geq 0$. Note that in this function we do not assume $\sum p_k = 1$ and also that μ and G depend on p as before, that is $\mu = \sum p_k x_k$ and $G = \exp \sum p_k \log x_k$. There is a point p^o when f is minimized subject to $\sum p_k = 1$. If p^o is an interior point of S then there is a Lagrange multiplier λ such that $\partial f/\partial p_j = \lambda(\partial/\partial p_j)(-1 + \sum p_k) = \lambda$

$$x_j - G \log x_j - (x_j - \mu)^2/(2b) = \lambda$$

for each j at the point p^o. Thus each x_j is a solution of the equation $\xi - G \log \xi - (\xi-\mu)^2/(2b) = \lambda$. Between any two roots there is by Rolle's theorem a root of

$$1 - G/\xi - (\xi-\mu)/b = 0 \quad \text{or } \xi^2 - (b+\mu)\,\xi + b\,G = 0$$

which has at most one solution between a and b. It follows that there cannot be more than two distinct values of x_j and so $n \leq 2$. In this case the theorem is proved. If on the other hand p^o is not an interior point of the set S then one $p_j = 0$ and the theorem follows by induction. The other inequality is proved similarly.

D.I. Cartwright and M.J. Field

RANK ONE MATRICES

A square matrix has rank one. Show that it is similar to a diagonal matrix if and only if its square is non-zero.

H. Kestelman

NON-NEGATIVE MATRICES

P is a real n×n matrix whose elements are all ≥ 0. For every choice of i,j, $1 \leq i,j \leq n$, there is a k such that the (i,j)th element of P^k is positive. If P has a positive element on its diagonal, prove that all elements of P^q are positive if q is large enough.

H. Kestelman

Your editor would like to hear from you anything connected with mathematics or with James Cook.

Prof. B.C. Rennie, Mathematics Department,
James Cook University of North Queensland,
Post Office James Cook University, Q.4811,
Townsville, Australia.