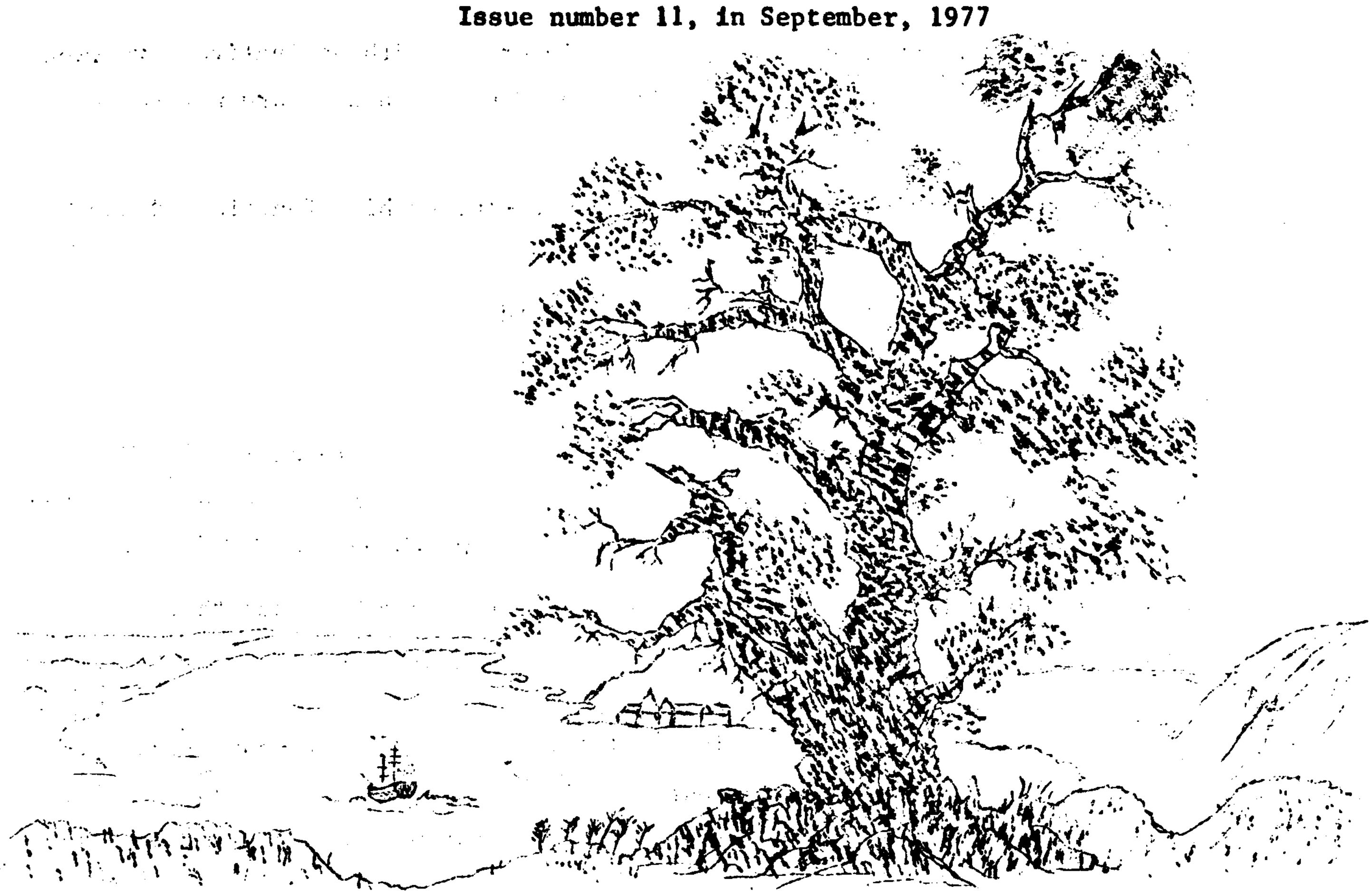
JAMES COOK MATHEMATICAL NOTES

Issue number 11, in September, 1977



Matavai Bay, Tahiti, from One Tree Hill, called by the natives Taharaa, showing the Endeavour at anchor and Fort Venus in 1769. Captain Cook was there again on his last voyage in September 1777.

From a drawing by Sydney Parkinson.

HIGH POWER MATRIX METHODS FOR FIRST YEARS (JCMN 10)

$$\text{Find} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2i & 0 \\ 0 & 0 & i \end{pmatrix}^{100} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$$

Solutions from H.O. Davies, G.M.L. Gladwell, J.B. Parker and G. Szekeres all lie in the convex hull of the following two.

(a) Denoting the matrix by A, it can be shown by induction that

$$A^n (i, 1, 0)^T = \left((2n-1)i^{n-1}, (2n+1)i^n, 0\right)^T$$

(b) The matrix may be written $iI + B$ where $B^2 = 0$, and so by the binomial thereom $A^n = i^n I + n\, i^{n-1} B$.

The answer is the transpose of $(-199i, 201, 0)$.

A CAMBRIDGE PROBLEM (JCMN 7)

Solution by A.P. Robertson:

There exist real functions f and g on an interval, with derivatives everywhere and such that $f'^2 + g'^2$ is Riemann-integrable but f' and g' are not Reimann-integrable.

The construction below provides two (Lebesgue-integrable) functions ϕ and ψ such that, if

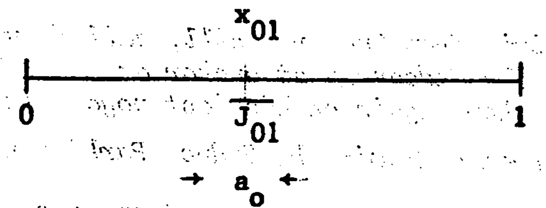
$$f(x) = \int_0^x \phi(t)\,dt \text{ and } g(x) = \int_0^x \psi(t)\,dt$$

then $f'(x) = \phi(x)$ and $g'(x) = \psi(x)$ for <u>all</u> x in $[0,1]$;

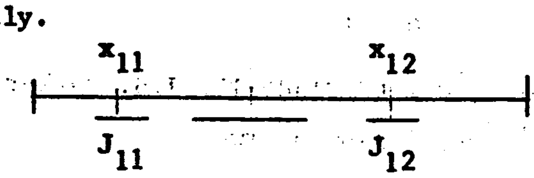
but though $\phi^2+\psi^2 = 1$, neither ϕ nor ψ is Riemann-integrable on $[0,1]$. We concentrate on the construction of ϕ; arranging it so that $1 - \sqrt{(1-\phi^2)}$ has the same properties as ϕ; then we take this as $1-\psi$, which ensures $\phi^2+\psi^2 = 1$.

Take any decreasing sequence (a_n) of positive real numbers such that $\sum_{n=0}^{\infty} 2^n a_n = \alpha < 1$. For example, to fix ideas, we could take $a_n = 4^{-n-1}$

We perform an operation on $I = [0,1]$ similar to that used in constructing Cantor's ternary set. This is best described in stages. In stage 0 we let x_{01} be the mid-point of I and "remove" an open interval J_{01} of length a_o and centre x_{01}.



In stage 1 we let x_{11} and x_{12} be the mid-points of the two "remaining" pieces of I and "remove" open intervals J_{11}, J_{12}, each of length a_1, and centred at x_{11}, x_{12} respectively.



and so on; in stage n we "remove" 2^n open intervals J_{nr} $(1 \le r \le 2^n)$ of length a_n each, centred at the points x_{nr}.

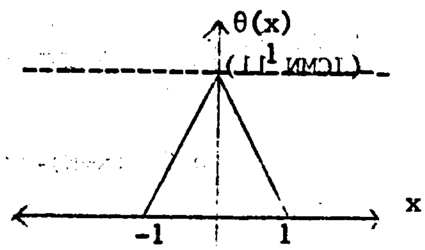
The total set "removed" from I is the open set

$$G = \bigcup_{n=0}^{\infty} \bigcup_{r=1}^{2^n} J_{nr},$$

a set of Lebesgue measure $\sum_{n=0}^{\infty} \sum_{r=1}^{2^n} \mu(J_{nr}) = \sum_{n=0}^{\infty} 2^n a_n = \alpha$

The complementary closed set $F = I \setminus G$ has positive measure $1-\alpha(>0)$. To construct ϕ, we take any function θ like the one shown, $\theta(x) = 1-|x|$. Specifically we require

$0<\theta(x)\leq 1$ for $-1\leq x\leq 1$,
$\theta(-1)=\theta(1) = 0$,
$\theta(0) = 1$,
θ is continuous on $|-1,1|$.

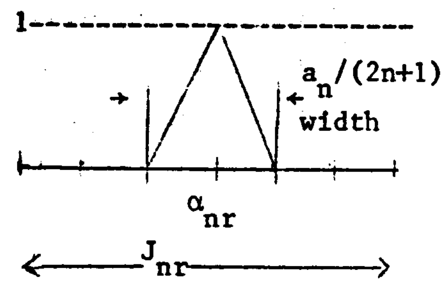


We now define ϕ on G as follows: for each J_{nr}, divide the interval into $2n+1$ equal pieces and erect a scaled version of ϕ on the middle piece, putting $\phi(x) = 0$ on the rest of J_{nr}. Thus, on J_{nr},

$$\phi(x) = \theta\left(\frac{4n+2}{a_n}(x-x_{nr})\right)$$

$$\text{for } |x-x_{nr}| < \frac{a_n}{4n+2}$$

and $\phi(x) = 0$ otherwise.



Thus ϕ has a spike in the middle of each interval J_{nr} and is zero on the rest of G. We also put $\phi(x) = 0$ on F. Then clearly ϕ is continuous on G. But every point x of F is a point of discontinuity of ϕ, at which $\limsup_{t\to x} \phi(t) = 1$ & $\liminf_{t\to x} \phi(t) = 0$, because every neighbourhood of x contains an interval J_{nr}, on which ϕ takes all values between 0 and 1. Since $\mu(F)>0$, ϕ cannot be Riemann-integrable.

Let $f(x) = \int_0^x \phi(t)\,dt$ (Lebesgue integral)

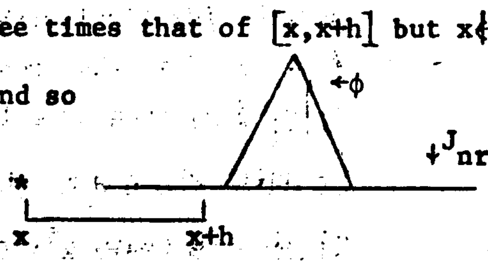
Then if x is in G, ϕ is continuous at x and so $f'(x)$ exists and equals $\phi(x)$. Now consider the case of x in F; we need to show that in that case $f'(x)$ exists and is zero. So take any $\varepsilon > 0$. There exists an m with $1/m < \varepsilon$ and we take $\delta=a_m/3$. Now suppose that $0<h<\delta$.

$$\int_x^{x+h} \phi(t)\,dt = \sum_{n=0}^{\infty} \sum_{r=1}^{2^n} \int_{H_{nr}} \phi \text{ , where}$$

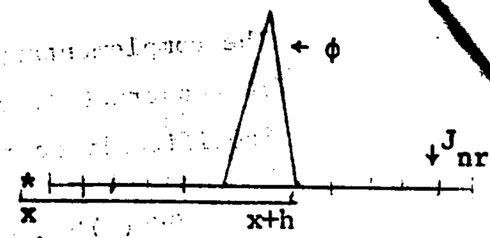
$$H_{nr} = J_{nr} \cap [x,x+h].$$

If $n\leq m$, the length of J_{nr} is at least three times that of $[x,x+h]$ but $x \notin J_{nr}$. Hence (see the picture) $\phi(t) = 0$ on H_{nr} and so

$$\int_{H_{nr}} \phi = 0 \text{ for } n\leq m.$$



If $n>m$, a similar argument shows that the part of H_{nr} on which $\phi(t)\neq 0$ is at most a fraction $\frac{1}{n+1}$ of the length of H_{nr}. Thus



$$\int_{H_{nr}} \phi \leq \frac{1}{n+1}\, \mu(H_{nr}) \leq \frac{1}{m+1}\, \mu(H_{nr}) \leq \varepsilon\, \mu\, (H_{nr})$$

$$\text{So } 0<f(x+h)-f(x) = \int_x^{x+h} \phi \leq \sum_{n=m+1}^{\infty} \sum_{r=1}^{2^n} \varepsilon\, \mu(H_{nr}) \leq \varepsilon h$$

It follows that f is differentiable on the right at x, with derivative zero. Similarly the left hand derivative is also zero, and so $f'(x) = 0$, as required.

To construct ψ, let $\omega(x) = 1-\surd\{1-(\theta(x))^2\}$; then ω has all the properties required of θ. The above construction with ω in place of θ leads to a function χ with all the properties of ϕ proved above. To complete the proof, we put $\psi = 1 - \chi$. It is easy to check that, for all x in I, $\chi(x)=1-\surd\{1-(\phi(x))^2\}$ and so $\phi^2+\psi^2 = 1$.

A LITTLE PROBLEM FROM LONDON (JCMN 9 AND 10)

H. Kestelman asked what is the set E of all complex k for which a given $n \times n$ matrix A is equivalent to kA. The following complete answer is from G. Szekeres.

(a) If $A = 0$ then $E = C$, the whole complex plane, this is trivial.

(b) If A is nilpotent, $A^m = 0$ for some m but $A \neq 0$, then E is the set of all non-zero complex numbers. This can be seen by considering the Jordan normal form to which A is similar, it consists of boxes like:
$$\begin{pmatrix} 0 & 1 & .. & 0 \\ & \cdot & \cdot & \cdot \\ 0 & 0 & & 0 \end{pmatrix}$$
with ones above the principal diagonal.

(c) If A is not nilpotent, it has a minimal equation
$$A^m + c_1 A^{m-1} + \ldots + c_m = 0$$
with not every c_j zero. Suppose that A is similar to kA, $k \neq 0$. Then the minimal equation of kA, namely $A^m + (c_1/k)A^{m-1} + \ldots + c_m k^{-m} = 0$ must be identical with that of A, hence $k^i = 1$ for all i for which $c_i \neq 0$. If d is the greatest common denominator of the non-empty set $\{i|c_i \neq 0\}$ then $k^d = 1$ and so at any rate E consists only of d'th roots of unity where d may be one. But if A is similar to kA then it is similar to every $k^i A$ for $i = 1, 2, \ldots d$, and so E consists of all the n^{th} roots of unity for some divisor n of d. There are no other possibilities.

RATIONAL APPROXIMATIONS

The problem is from G.M. Gladwell, of Waterloo University.

If $T_n(z)$, $U_n(z)$ are the Chebyshev polynomials of the first and second kinds respectively show that $T_n(z) = \sqrt{z^2-1}\,U_{n-1}(z) + 0(z^{-n})$ for large $|z|$, and find an explicit form for the remainder. Hence find an explicit form for the Pade approximation for $\sqrt{1-x^2}$

$$\sqrt{1-x^2} = \frac{P_n(x)}{Q_n(x)} + R(x)$$

where $P_n(x)$, $Q_n(x)$ are polynomials of degree n and $R(x) = 0(x^{2n})$ for small x.. How can these results be generalised?

INEQUALITY WANTED (JCMN 10)

There has been considerable interest in this problem by Kenneth S. Williams which we took from "Eureka". The first solution to come in was from G. Szekeres and the second from D.I. Cartwright and M.J. Field of the University of Sydney.

If $0 < a_1 \leq \ldots \leq a_n$ then is there an inequality

$$(k/a_n) \sum_{1\leq i<j\leq n} (a_i-a_j)^2 \leq \frac{a_1+\ldots+a_n}{n} - (a_1\ldots a_n)^{1/n} \leq (k/a_1) \sum_{1\leq i<j\leq n} (a-a_j)^2$$

where k is a function of n?

As the editor of Eureka gave his readers until the first of October to work on this problem we shall keep our contributions unpublished until after then.

PROBLEM UNSUITABLE FOR UNDERGRADUATES

J.M. Hammersley contributes the following, which arose in the Oxford Finals examination for 1976.

The real function f(x) is defined for all real x and has finite derivative at every point of the set A which has Lebesgue measure zero. Does the image set f(A) also have measure zero?

PROBLEM FOR SECOND YEAR STUDENTS

If θ is an irrational number and I is any real interval then

$\dfrac{\cos(m+\frac{1}{2})\pi\theta}{\cos(m\theta\pi)}$ belongs to I for infinitely many integers m.

H. Kestelman.

BYZANTINE MATHEMATICS

The description of the church of the Holy Apostles at Constantinople by Nikolaos Mesarites written between 1198 and 1203 throws some light on teaching and attitudes towards mathematics in those days. In the courts on one side of the church there were groups being taught grammar, reckoning and singing; beyond the church the higher arts including the relations of numbers (odd and even, male and female) and geometry. It is to the last two that Mesarites refers in saying that 'Mathematics is the highest of the sciences', while as regards those who teach reckoning he thinks that 'their natures are not disciplined and because their calling by which they have lived and in which they have grown old is commonplace and rude wherefore they always look upon their pupils with a wild, angry and bitter eye; and all those who are under them are dejected and tremble at them.' Mesarites seems somewhat contradictory about reckoning; at first he sees it through rose-coloured spectacles; 'you will see those who are busy with the art of reckoning. How do they close their fingers so continuously and as constantly open them, quickly curling them next to each other and even more quickly sending them off again, learning so to speak the art of dancing with their hands.' However he goes on <u>immediately</u> 'and fearing the rod, lest, if the hand makes a mistake along with the memory, it comes to linger fondly on their palms, which spread open unwillingly and secretly hollow themselves, when the rod like a bird of prey comes down on them with a great whistle and bends them back as it strikes them, and takes off the skin and flesh and does not leave without tasting the bones. For these men who teach them with their hands are a violent race, brutal and ungovernable. Indeed you may see most of them cutting into the children's shoulders with a whip of bull's sinews. This happens for no other reason, I think, than that', as I quoted above, 'their natures are not disciplined, etc.,'

It seems as if one must make considerable allowances for the way Mesarites is carried away by his own language. For in the section <u>immediately</u> following the remark that 'all those who are under them are dejected and tremble at them', he writes 'However every passer-by seeing the school life of the student as described here wishes to become a student and to be child for his whole life and a learner. And all fathers who have tender affection for their children send their children to these studies.'

I may say that the grammarians also get beaten, but the hymn singers <u>seem</u> to come off better.

Nikolaos Mesarites, Description of the Church of the Holy Apostles, ed. Glanville Downey, Trans. Amer. Phil. Soc. (1957). Some bits are quoted by Gervase Matthews in Byzantine Aesthetics, John Murray, London (1963), but Matthews does not seem to realise that the number theory, number lore aspect, including the male and female ideas, is considered on the same high level as geometry.

M.L. Cartwright.

(JCMN 11)

COMBINATORIAL GEOMETRY PROBLEM (JCMN 10)

The question from A.J. Dobson was as follows, given $\frac{1}{2}n(n-1)$ positive real numbers $r(i, j)$ for $1 \leq i < j \leq n$, can you find n points $P_1, \ldots P_n$ in the plane such that the $\frac{1}{2}n(n-1)$ distances $d(i, j) = P_iP_j$ are ordered in the same way as $r(i, j)$?

From Esther Szekeres comes the answer NO if n is 6 (or more).

Take $r(i, j)$ such that $r(1, 2)$ is the largest and $r(i, k) > r(i, k+1)$ if $1 \leq i < k \leq 5$. Suppose if possible that points $P_1, \ldots P_6$ have the required properties. Firstly none of P_3, P_4, P_5 can be on the line P_1P_2 because a typical one of them, say P_m, if on the line must be between P_1 and P_2, and

$$d(1, 2) \leq d(1, 6) + d(2, 6) < d(1, m) + d(2, m),$$

Take a pair of integers (m, n) which may be (3, 4) or (3, 5) or (4, 5), that is $3 \leq m < n \leq 5$. Draw the circles through P_m with centres at P_1 and P_2 (fig. 1). The points P_r for $r > m$ must all be in the shaded area.

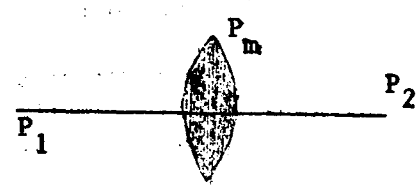


fig. 1.

Now two cases must be considered, the point P_n may be on the same side of $P_1 P_2$ as P_m or it may be on the opposite side. Take the first case.

The point P_6 must be in the shaded area of fig. 2. Since $d(m, n) > d(m, 6)$ this gives a contradiction. Therefore the case of fig. 2 is impossible and the points P_m and P_n must be on opposite sides of the line $P_1 P_2$. Since the pair of numbers (m, n) may be (3, 4) or (3, 5) or (4, 5) this leads to a contradiction.

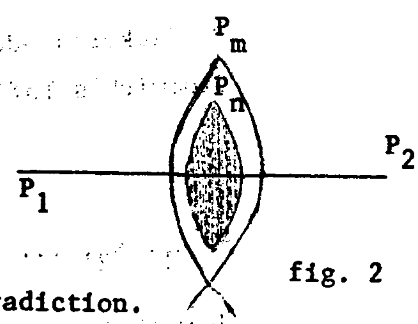


fig. 2

NORMS

The norm of a real n × n matrix is defined to be max $\|Ax\|$ for all vectors x with $\|x\| = 1$; it depends on the definition of $\|x\|$ in R^n, and this is restricted by the axioms: $\|x\| \geq 0$ for all x, $\|x\| > 0$ whenever $x \neq 0$, $\|tx\| = |t| \, \|x\|$ for all real t, and the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$. Prove that the norm of

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

exceeds 1 however $\|x\|$ is defined.

H. Kestelman

BICENTENNIAL PROBLEM

To celebrate (though it is a few months late) the birthday of C.F. Gauss, prove or disprove that every complex function of the form (polynomial in z) + exp z has a zero. *unless the polynomial is identically zero.*

MORE TRIGONOMETRY (JCMN 10)

Several solutions have been sent in. This one is from G.M.L. Gladwell.

From de Moivre's theorem -

$$(\cos\theta + i\sin\theta)^{2n+1} = \cos(2n+1)\theta + i\sin(2n+1)\theta$$

so that equating imaginary parts we find

$$\sin(2n+1)\theta = (-)^n \sin\theta\cos^{2n}\theta\{\tan^{2n}\theta - n(2n+1)\tan^{2n-2}\theta \ldots + (-)^n(2n+1)\}.$$

The roots of $\sin(2n+1)\theta = 0$ are $\theta = 0,\ r\pi/(2n+1)$, $r = 1, 2, \ldots$ so that the n non-zero values of $\tan^2\theta$ for which $\sin(2n+1)\theta = 0$ are the roots of the factpr in brackets. The sum of the values of $\tan^2\theta$ is $n(2n+1)$. Thus

$$\sum_{r=1}^{n} \tan^2|r\pi/(2n+1)| = n(2n+1) \qquad n = 1,2,\ldots \quad .$$

In particular n = 4 gives

$$\tan^2 20 + \tan^2 40 + \tan^2 60 + \tan^2 80 = 36 \;.$$

G. Szekeres adds the comment that an equivalent result is to be found in Bromwich's Infinite Series, page 211.

EQUATIONS

If $a_1, a_2, \ldots a_n$ are the complex zeros of a polynomial $f(z)$, and are distinct,

show that $\quad \sum_1^n a_r^{\,k}/f'(a_r)$

is zero if $k = 0, 1, \ldots n-2$ and 1 if $k = n-1$.

H. Kestelman

Your editor would like to hear from you anything connected with mathematics or with James Cook.

Prof. B.C. Rennie, Mathematics Department,
James Cook University of North Queensland,
Post Office James Cook University, Q.4811,
Townsville, Australia.