Xpress Non-Linear Solvers

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*Interior Point Solver*

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*Dual Simplex*

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*Mixed Integer*

► Timo Berthold  
*Mixed Integer*
# FICO Snapshot

| **Profile** | The leader in predictive analytics for decision management  
Founded: 1956  
NYSE: FICO  
Annual Global Revenues Approximately: $700 million |
| --- | --- |
| **Products and Services** | Scores and related analytic models  
Analytic Applications for risk management, fraud, marketing, mobility  
Tools for decision management |
| **Clients and Markets** | 5,000+ clients in 80 countries  
Industry focus: Banking, insurance, retail, health care |
| **Recent Rankings** | #1 in services operations analytics (IDC)  
#6 in worldwide analytics software (IDC)  
#7 in Business Intelligence, CPM and Analytic Applications (Gartner)  
#26 in the FinTech 100 (American Banker) |
| **Offices** | 20+ offices worldwide, HQ in San Jose, California, USA  
2,400+ employees  
Regional Hubs: San Rafael (CA); San Diego (CA); New York; Roseville, MN; London; Birmingham (UK); Istanbul; Madrid; Munich; Sao Paulo; Bangalore; Beijing; and Singapore. |
# FICO Product Portfolio

## For Specific Decision Processes

<table>
<thead>
<tr>
<th>Applications</th>
<th>Marketing</th>
<th>Origination</th>
<th>Customer Management</th>
<th>Collections and Recovery</th>
<th>Fraud Management</th>
<th>Mobile</th>
</tr>
</thead>
<tbody>
<tr>
<td>FICO® Customer Dialogue Manager</td>
<td>FICO® Origination Manager</td>
<td>FICO® TRIAD® Customer Manager</td>
<td>FICO® Debt Manager</td>
<td>FICO® Falcon® Fraud Manager</td>
<td>FICO® Adestra® Fraud Resolution</td>
<td></td>
</tr>
<tr>
<td>FICO® Analytic Offer Manager</td>
<td></td>
<td></td>
<td>FICO® Recovery Management System™</td>
<td>FICO® Insurance Fraud Manager</td>
<td></td>
<td>FICO® Adestra® Risk Intervention Manager</td>
</tr>
<tr>
<td>Custom / Embedded Analytics</td>
<td>Targeting Models</td>
<td>Consumer and Small Business Risk Models</td>
<td>Behavior Scorecards</td>
<td>Collections Scores</td>
<td>Consortium Fraud Models</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Time-to-Event Analytics</td>
<td>Economic Impact Models</td>
<td>Transaction Analytics</td>
<td></td>
<td>Custom Fraud Models</td>
<td></td>
</tr>
</tbody>
</table>

## For Any Decision Process

### Scores

- **B2B:**
  - FICO® Score • FICO® Credit Capacity Index™ • FICO® Insurance Risk Scores
- **B2C:**
  - myFICO®

### Tools

- **Business Rules Management:** FICO® Blaze Advisor®
- **Predictive Analytics:** FICO® Model Builder • FICO® Model Central
- **Optimization:** FICO® Optimization Modeler • FICO® Xpress • FICO® Decision Optimizer

### Professional Services

- Custom Analytics
- Operational Best Practices
- Strategy Design and Optimization
A Network of Intelligence
Accelerating the Development of Ideas

CONSUME

Businesses
Researchers

CONTRIBUTE

Entrepreneurs
ISVs

Systems Integrators/
Consultants

COLLABORATE

Governments
Academics
Corporate Developers
FICO Solutions on FICO Solution Stack

- FICO Analytic Marketplace
- FICO Application Studio
  - Analytic Modeler
  - Decision Modeler
  - Optimization Modeler
- FICO Decision Management Platform
- FICO Visual Insights Studio
<table>
<thead>
<tr>
<th>Year</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>1985</td>
<td>MP-Model – modelling language</td>
</tr>
<tr>
<td>1986</td>
<td>Mixed Integer Programming (MIP) solver added</td>
</tr>
<tr>
<td>2001</td>
<td>Mosel modelling language</td>
</tr>
<tr>
<td>2003</td>
<td>XSLP non-linear solver added</td>
</tr>
<tr>
<td>2008</td>
<td>Dash Optimization bought by Fair Isaac (now FICO)</td>
</tr>
<tr>
<td>2012</td>
<td>Xpress Insight added (now Xpress Optimization Modeller)</td>
</tr>
</tbody>
</table>
# Xpress Optimization Suite

## Applications Services Optimization Modeler

<table>
<thead>
<tr>
<th>FEATURES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adapt data and parameters to create and compare scenarios</td>
</tr>
<tr>
<td>Understand trade-offs and sensitivities</td>
</tr>
<tr>
<td>Visualize data and results for analysis</td>
</tr>
<tr>
<td>Collaborate in a multi-user environment</td>
</tr>
<tr>
<td>Works in a rich client and a web browser — on premise and in the cloud</td>
</tr>
<tr>
<td>Fully featured APIs including web</td>
</tr>
</tbody>
</table>

## Modeling Mosel

<table>
<thead>
<tr>
<th>FEATURES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flexible, modular, easy-to-learn and use</td>
</tr>
<tr>
<td>Development IDE</td>
</tr>
<tr>
<td>Distributed modeling and cloud enablement</td>
</tr>
<tr>
<td>Data connections (file, excel, databases, web services)</td>
</tr>
<tr>
<td>Precompiled for efficiency and IP protection</td>
</tr>
<tr>
<td>Fully featured APIs</td>
</tr>
</tbody>
</table>

## Optimization NonLinear Kalis

<table>
<thead>
<tr>
<th>FEATURES</th>
</tr>
</thead>
<tbody>
<tr>
<td>High-performance, scalable and robust LP (Simplex</td>
</tr>
<tr>
<td>Great out-of-the-box performance — advanced users have full control over solution process</td>
</tr>
<tr>
<td>Utilizes multi-core/CPU machines, automatic tuning</td>
</tr>
<tr>
<td>N-best solutions capabilities and advanced infeasibility handling</td>
</tr>
<tr>
<td>Fully featured APIs</td>
</tr>
</tbody>
</table>

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FICO Xpress Solver Performance

- FICO has the most complete optimization offering and all solvers are very competitive
- Robust / (MI)SOCP solver dominates the competition
- FICO has the leading nonlinear offering with applications in particular in finance, insurance, and power/gas/oil industries

Results as of March 30, geometric means of time to optimality, LP/QP geometric mean computed by FICO, MIP 12 threads, nonlinear numbers directly computed from the logs and computed by FICO
## Basic Problem Types

<table>
<thead>
<tr>
<th>Type</th>
<th>Objective Function</th>
<th>Constraints</th>
<th>Methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>$\min cx$</td>
<td>$a^i x \geq b_i, i \in M$</td>
<td>Primal simplex, Dual simplex, Interior point</td>
</tr>
<tr>
<td>QP</td>
<td>$\min cx + xQx$</td>
<td>$a^i x \geq b_i, i \in M$</td>
<td>Quadratic primal simplex, Quadratic dual simplex, Interior point</td>
</tr>
<tr>
<td>QCQP</td>
<td>$\min cx + xQx$</td>
<td>$a^i x + xQ^i x \geq b_i, i \in M$</td>
<td>Interior point</td>
</tr>
<tr>
<td>SOCP</td>
<td>$\min cx + xQx$</td>
<td>$a^i x + xQ^i x \geq b_i, i \in M$</td>
<td>Interior point</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$</td>
<td>D^k x + f_k</td>
</tr>
</tbody>
</table>
## Basic Problem Types

<table>
<thead>
<tr>
<th>Type</th>
<th>Formulation</th>
<th>Branching Method</th>
</tr>
</thead>
</table>
| **MILP**   | $\min \ cx$  
$s.t. \ a^i x \geq b_i, \ i \in M$  
$x_j \in \mathbb{Z}, \ j \in I$ | Branch and bound                        |
| **MIQP**   | $\min \ cx + xQx$  
$s.t. \ a^i x \geq b_i, \ i \in M$  
$x_j \in \mathbb{Z}, \ j \in I$ | Branch and bound + quadratic dual simplex |
| **MIQCQP** | $\min \ cx + xQx$  
$s.t. \ a^i x + xQ^i x \geq b_i, \ i \in M$  
$x_j \in \mathbb{Z}, \ j \in I$ | Branch and bound + outer approximation  |
| **MISOCQ** | $\min \ cx + xQx$  
$s.t. \ a^i x + xQ^i x \geq b_i, \ i \in M$  
$|D^k x + f_k| \leq g^k x + h_k, \ k \in C$  
$x_j \in \mathbb{Z}, \ j \in I$ | Branch and bound + outer approximation  |
Non-linear Problems

Standard non-linear formulation:

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \geq 0, \ i \in M
\end{align*}
\]

Xpress non-linear formulation:

\[
\begin{align*}
\min & \quad cx + xQx + f(x) \\
\text{s.t.} & \quad a^i x + xQ^i x + g_i(x) \geq b_i, \ i \in M \\
& \quad |D^k x + f_k| \leq g^k x + h_k, \ k \in C
\end{align*}
\]

- Solvers: SLP or Knitro
SLP – Sequential Linear Programming

Create linear approximation

Might cut off feasible regions

Solve LP

Repeat
Sequential Linear Programming

Non-linear problem:
(assume non-linear objective moved into constraints)

\[
\begin{align*}
\min & \quad cx \\
\text{s.t.} & \quad a^i x + g_i(x) \geq b_i, \ i \in M
\end{align*}
\]

Linearize non-linear functions around a solution \(x^k\):

\[
g_i(x) \approx g_i(x^k) + \nabla g_i(x - x^k)
\]

to create LP

\[
\begin{align*}
\min & \quad cx \\
\text{s.t.} & \quad a^i x + g_i(x^k) + \nabla g_i(x - x^k) \geq b_i, \ i \in M
\end{align*}
\]

Solve LP to get next iterate solution: \(x^{k+1}\)
Sequential Linear Programming

Trust regions

Solution will bounce between $x^1$ and $x^2$. 
Sequential Linear Programming
Trust regions

Solution will bounce between $x^1$ and $x^2$

Use trust region

Smaller movement
Sequential Linear Programming
Infeasibility

Non-convex non-linear with feasible region.

Iterate solution $x^k$ in infeasible region.

Can result in infeasible LP

Use penalty variables to make LP feasible.
Sequential Linear Programming
Extended LP

\[ \begin{align*}
\min & \quad cx \\
\text{s.t.} & \quad a^i x + g_i(x) \geq b_i, i \in M
\end{align*} \]

Linearize:

\[ \begin{align*}
\min & \quad cx \\
\text{s.t.} & \quad a^i x + g_i(x^k) + \nabla g_i(x - x^k) \geq b_i, i \in M
\end{align*} \]

Add trust region:

\[ \begin{align*}
\min & \quad cx \\
\text{s.t.} & \quad a^i x + g_i(x^k) + \nabla g_i \Delta x \geq b_i, i \in M \\
& \quad x = x^k + \Delta x \\
& \quad L \leq \Delta x \leq U
\end{align*} \]

Add penalty variables:

\[ \begin{align*}
\min & \quad cx + Py \\
\text{s.t.} & \quad a^i x + g_i(x^k) + \nabla g_i \Delta x + y \geq b_i, i \in M \\
& \quad x = x^k + \Delta x \\
& \quad L \leq \Delta x \leq U, y \geq 0
\end{align*} \]
Sequential Linear Programming
Optimality

Non-linear problem:

\[ \begin{align*}
\text{min} & \quad cx \\
\text{s.t.} & \quad a^i x + g_i(x) \geq b_i, \; i \in M
\end{align*} \]

SLP problem:

\[ \begin{align*}
\text{min} & \quad cx + Py \\
\text{s.t.} & \quad a^i x + g_i(x^k) + \nabla g_i \Delta x + y \geq b_i, \; i \in M \\
& \quad x = x^k + \Delta x \\
& \quad L \leq \Delta x \leq U, \; y \geq 0
\end{align*} \]

- Trust bounds and penalty variables bound primal and dual variables.
- SLP solution feasible if penalties are zero.
- SLP solution optimal if dual multipliers for trust bounds are zero.
- Optimal, feasible LP solution satisfies KKT complementarity.

KKT optimality conditions:
(under regularity conditions)

\[ \begin{align*}
a^i x^* + g_i(x^*) & \geq b_i, \; i \in M \\
c & = \sum_{i \in M} \mu_i (a^i + \nabla g_i(x^*)) \\
\mu_i (a^i x^* + g_i(x^*)) & = 0, \; i \in M \\
\mu_i & \geq 0, \; i \in M
\end{align*} \]
Sequential Linear Programming

Convergence

Strong convergence

Extended convergence
Sequential Linear Programming

Local Optima
Liquids $A$ and $B$ with densities $\delta(A)$ and $\delta(B)$.  
Create blend $C$ with density $L_C \leq \delta(C) \leq U_C$ in amount $V(C)$.  
Select amounts $V(A)$ and $V(B)$ to blend.

\[
V(C) = V(A) + V(B)
\]
\[
\delta(C)V(C) = \delta(A)V(A) + \delta(B)V(B)
\]

Linear problem if $A$ and $B$ given.  
Bi-linear if $A$ and $B$ are also blends.

Refinery problems can contain 1000s of blends.  
Blends can be returned to earlier stages!
\( \delta(C)V(C) = \delta(A)V(A) + \delta(B)V(B) \)

- \( V(C) = 0 \) results in \( \delta(C) \) undefined.
  - Causes "infinite" LP coefficients
  - XSLP detects such edge cases.
- Non-convex problem.
  - Use restarts to find better solutions.
  - XSLP provides parallel multi-start feature with different starting points.
- Other solutions iterate between fixing \( \delta \) or \( V \).
  - XSLP approximates both at the same time.
Sequential Linear Programming
Overview

► Solves 1st order approximations.
► Builds on top of a strong LP solver.
► Highly efficient for bi-linear or problems with a large amount of linear constraints.
► Local solver: Global optimality guaranteed only for convex problems.
► Applications:
  ► Petro-chemical industry.
  ► Finance.
  ► Price optimization.
  ► …
► Comes with a Mixed Integer Programming solver.
Knitro Non-linear Solver

- Knitro licensed from ziena
- 2\textsuperscript{nd} order interior point solver.
- Strong for highly non-linear, medium sized problems.
- Integrated with Xpress non-linear solver
  - Usable with any non-linear model.
  - Xpress non-linear solver decides whether to use Knitro or XSLP.
- Xpress calculates 1\textsuperscript{st} and 2\textsuperscript{nd} order derivatives.
  - Numerical derivatives
  - Symbolic differentiation
  - Automatic differentiation.
## XSLP vs. Knitro

<table>
<thead>
<tr>
<th>XSLP</th>
<th>Knitro</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\textsuperscript{st} order</td>
<td>2\textsuperscript{nd} order</td>
</tr>
<tr>
<td>Bilinear or highly linear problems</td>
<td>Highly non-linear problems</td>
</tr>
<tr>
<td>Large sized problems</td>
<td>Medium sized problems</td>
</tr>
<tr>
<td>Local solver</td>
<td>Local solver</td>
</tr>
</tbody>
</table>
XNLP Automatic Solver Selection

\[ \min f(x) \]
\[ \text{s.t. } g(x) \geq 0 \]

Problem recognition

Non-linear

XSLP
Knitro

Optimizer

LP
QP
QCQP
SOCP
Example!
Thank You

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