Convex Optimization:
from Real-Time Embedded
to Large-Scale Distributed

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Outline

Convex Optimization

Real-Time Embedded Optimization

Large-Scale Distributed Optimization

Summary
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Convex Optimization

Real-Time Embedded Optimization

Large-Scale Distributed Optimization

Summary
Convex optimization — Classical form

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, \quad i = 1, \ldots, m \)
\( Ax = b \)

- variable \( x \in \mathbb{R}^n \)
- \( f_0, \ldots, f_m \) are convex: for \( \theta \in [0, 1] \),
  \[
  f_i(\theta x + (1 - \theta)y) \leq \theta f_i(x) + (1 - \theta)f_i(y)
  \]
i.e., \( f_i \) have nonnegative (upward) curvature
Convex optimization — Cone form

\[
\begin{align*}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & x \in K \\
& Ax = b
\end{align*}
\]

- variable \( x \in \mathbb{R}^n \)
- \( K \subset \mathbb{R}^n \) is a proper cone
  - \( K \) nonnegative orthant \( \rightarrow \) LP
  - \( K \) Lorentz cone \( \rightarrow \) SOCP
  - \( K \) positive semidefinite matrices \( \rightarrow \) SDP
- the ‘modern’ canonical form
Why

- beautiful, nearly complete theory
  - duality, optimality conditions, …
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- effective algorithms, methods (in theory and practice)
  - get **global solution** (and optimality certificate)
  - polynomial complexity
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- conceptual unification of many methods
Why

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- effective algorithms, methods (in theory and practice)
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  - polynomial complexity

- conceptual unification of many methods

- lots of applications (many more than previously thought)
Application areas

- machine learning, statistics
- finance
- supply chain, revenue management, advertising
- control
- signal and image processing, vision
- networking
- circuit design
- combinatorial optimization
- quantum mechanics
Applications — Machine learning

- parameter estimation for regression and classification
  - least squares, lasso regression
  - logistic, SVM classifiers
  - ML and MAP estimation for exponential families

- modern $\ell_1$ and other sparsifying regularizers
  - compressed sensing, total variation reconstruction

- $k$-means, EM, auto-encoders (bi-convex)
Example — Support vector machine

- data \((a_i, b_i), i = 1, \ldots, m\)
  - \(a_i \in \mathbb{R}^n\) feature vectors; \(b_i \in \{-1, 1\}\) Boolean outcomes
- prediction: \(\hat{b} = \text{sign}(w^T a - v)\)
  - \(w \in \mathbb{R}^n\) is weight vector; \(v \in \mathbb{R}\) is offset
Example — Support vector machine

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- SVM: choose \(w, v\) via (convex) optimization problem

  \[
  \text{minimize} \quad L + \frac{\lambda}{2}\|w\|_2^2
  \]

  \[
  L = \frac{1}{m} \sum_{i=1}^{m} (1 - b_i(w^T a_i - v))^+ \quad \text{is avg. loss}
  \]
SVM

\[ w^T z - v = 0 \text{ (solid); } \quad |w^T z - v| = 1 \text{ (dashed)} \]
Sparsity via $\ell_1$ regularization

- adding $\ell_1$-norm regularization

$$\lambda \|x\|_1 = \lambda(|x_1| + |x_2| + \cdots + |x_n|)$$

to objective results in sparse $x$

- $\lambda > 0$ controls trade-off of sparsity versus main objective

- preserves convexity, hence tractability

- used for many years, in many fields
  - sparse design
  - feature selection in machine learning (lasso, SVM, \ldots)
  - total variation reconstruction in signal processing
  - compressed sensing
Example — Lasso

- regression problem with $\ell_1$ regularization:

$$\text{minimize } \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1$$

with $A \in \mathbb{R}^{m \times n}$

- useful even when $n \gg m$ (!!); does feature selection
Example — Lasso

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- useful even when $n \gg m$ (!!); does **feature selection**

- cf. $\ell_2$ regularization (‘ridge regression’):

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\text{minimize} \quad \frac{1}{2} \|Ax - b\|_2^2 + \lambda \|x\|_2^2
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- lasso, ridge regression have same computational cost
Example — Lasso

- \( m = 200 \) examples, \( n = 1000 \) features
- examples are noisy linear measurements of true \( x \)
- true \( x \) is sparse (30 nonzeros)
Example — Lasso

true $x$

$\ell_1$ (lasso) reconstruction
State of the art — Medium scale solvers

- 1000s–10000s variables, constraints
- reliably solved by interior-point methods on single machine
- exploit problem sparsity
- not quite a technology, but getting there
State of the art — Modeling languages

- (new) high level language support for convex optimization
  - describe problem in high level language
  - description is automatically transformed to cone problem
  - solved by standard solver, transformed back to original form
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- (new) high level language support for convex optimization
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- enables rapid prototyping (for small and medium problems)
- ideal for teaching (can do a lot with short scripts)
CVX

- parser/solver written in Matlab (M. Grant, 2005)
- SVM:
  \[
  \text{minimize} \quad L + \left( \frac{\lambda}{2} \right) \| w \|_2^2
  \]
  \[
  L = \frac{1}{m} \sum_{i=1}^{m} \left( 1 - b_i (w^T a_i - v) \right)_+ \quad \text{is avg. loss}
  \]
- CVX specification:

```matlab
cvx_begin
  variables w(n) v  % weight, offset
  L=(1/m)*sum(pos(1-b.*(A*w-v)));  % avg. loss
  minimize (L+(lambda/2)*sum_square(w))
cvx_end
```
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Motivation

- in many applications, need to solve the same problem repeatedly with different data
  - control: update actions as sensor signals, goals change
  - finance: rebalance portfolio as prices, predictions change
- used now when solve times are measured in minutes, hours
  - supply chain, chemical process control, trading
Motivation

- in many applications, need to solve the same problem repeatedly with different data
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- (using new techniques) can be used for applications with solve times measured in **milliseconds** or **microseconds**
Example — Disk head positioning

- force $F(t)$ moves disk head/arm modeled as 3 masses (2 vibration modes)
- goal: move head to commanded position as quickly as possible, with $|F(t)| \leq 1$
- reduces to a (quasi-) convex problem
Optimal force profile

position

force $F(t)$
Embedded solvers — Requirements

- high speed
  - hard real-time execution limits

- extreme reliability and robustness
  - no floating point exceptions
  - must handle poor quality data

- small footprint
  - no complex libraries
Embedded solvers

- (if a general solver works, use it)
Embedded solvers

- (if a general solver works, use it)
- otherwise, develop custom code
  - by hand
  - automatically via code generation
- can exploit known sparsity pattern, data ranges, required tolerance at solver code development time
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- typical speed-up over general solver: $100-10000 \times$
Parser/solver vs. code generator

![Diagram](Diagram.png)
Parser/solver vs. code generator

Problem instance $\rightarrow$ Parser/solver $\rightarrow$ $x^*$

Problem family description $\rightarrow$ Generator $\rightarrow$ Source code $\rightarrow$ Compiler $\rightarrow$ Custom solver

Problem instance $\rightarrow$ Custom solver $\rightarrow$ $x^*$
CVXGEN code generator

- handles small, medium size problems transformable to QP (J. Mattingley, 2010)
- uses primal-dual interior-point method
- generates flat library-free C source
CVXGEN example specification — SVM

dimensions

\[ m = 50 \] % training examples
\[ n = 10 \] % dimensions

end

parameters

\[ a[i] (n), i = 1..m \] % features
\[ b[i], i = 1..m \] % outcomes
\[ \lambda \) positive

end

variables

\[ w (n) \] % weights
\[ v \] % offset

end

minimize

\[ \frac{1}{m} \sum_{i = 1..m} \text{pos}(1 - b[i](w'*a[i] - v)) + \frac{\lambda}{2} \cdot \text{quad}(w) \]
**CVXGEN sample solve times**

<table>
<thead>
<tr>
<th>problem</th>
<th>SVM</th>
<th>Disk</th>
</tr>
</thead>
<tbody>
<tr>
<td>variables</td>
<td>61</td>
<td>590</td>
</tr>
<tr>
<td>constraints</td>
<td>100</td>
<td>742</td>
</tr>
<tr>
<td>CVX, Intel i3</td>
<td>270 ms</td>
<td>2100 ms</td>
</tr>
<tr>
<td>CVXGEN, Intel i3</td>
<td>230 $\mu$s</td>
<td>4.8 ms</td>
</tr>
</tbody>
</table>
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Summary
Motivation and goal

motivation:

- want to solve arbitrary-scale optimization problems
  - machine learning/statistics with huge datasets
  - dynamic optimization on large-scale networks
Motivation and goal

motivation:

- want to solve **arbitrary-scale** optimization problems
  - machine learning/statistics with huge datasets
  - dynamic optimization on large-scale networks

goal:

- ideally, a system that
  - has CVX-like interface
  - targets modern large-scale computing platforms
  - scales arbitrarily

…not there yet, but there’s promising progress
Distributed optimization

- devices/processors/agents coordinate to solve large problem, by passing relatively small messages
- can split variables, constraints, objective terms among processors
- variables that appear in more than one processor called ‘complicating variables’ (same for constraints, objective terms)
Example — Distributed optimization

\[
\text{minimize } f_1(x_1, x_2) + f_2(x_2, x_3) + f_3(x_1, x_3)
\]
Distributed optimization methods

- dual decomposition (Dantzig-Wolfe, 1950s–)
- subgradient consensus
  (Tsitsiklis, Bertsekas, Nedić, Ozdaglar, Jadbabaie, 1980s–)
Distributed optimization methods

- dual decomposition (Dantzig-Wolfe, 1950s–)
- subgradient consensus
  (Tsitsiklis, Bertsekas, Nedić, Ozdaglar, Jadbabaie, 1980s–)

- alternating direction method of multipliers (1980s–)
  - equivalent to many other methods
    (e.g., Douglas-Rachford splitting)
  - well suited to modern systems and problems
Consensus optimization

- want to solve problem with $N$ objective terms
  \[
  \text{minimize } \sum_{i=1}^{N} f_i(x)
  \]
e.g., $f_i$ is the loss function for $i$th block of training data

- consensus form:
  \[
  \text{minimize } \sum_{i=1}^{N} f_i(x_i)
  \]
  subject to $x_i - z = 0$

  - $x_i$ are **local variables**
  - $z$ is the **global variable**
  - $x_i - z = 0$ are **consistency** or **consensus** constraints
Consensus optimization via ADMM

with $\bar{x}^k = (1/N) \sum_{i=1}^{N} x_i^k$ (average over local variables)

$x_i^{k+1} := \arg\min_{x_i} \left( f_i(x_i) + (\rho/2)\|x_i - \bar{x}^k + u_i^k\|_2^2 \right)$

$u_i^{k+1} := u_i^k + (x_i^{k+1} - \bar{x}^{k+1})$

- get **global** minimum, under very general conditions
- $u^k$ is running sum of inconsistencies (PI control)
- minimizations carried out independently and in parallel
- coordination is via averaging of local variables $x_i$
Statistical interpretation

- $f_i$ is negative log-likelihood (loss) for parameter $x$ given $i$th data block
- $x_i^{k+1}$ is MAP estimate under prior $\mathcal{N}(\bar{x}^k - u_i^k, \rho I)$
- Processors only need to support a Gaussian MAP method
  - Type or number of data in each block not relevant
  - Consensus protocol yields global ML estimate
- **Privacy preserving**: agents never reveal data to each other
Example — Consensus SVM

- baby problem with $n = 2$, $m = 400$ to illustrate
- examples split into $N = 20$ groups, in worst possible way: each group contains only positive or negative examples
Iteration 1

Large-Scale Distributed Optimization
Iteration 5
Iteration 40
Example — Distributed lasso

- example with dense $A \in \mathbb{R}^{400000 \times 8000}$ ($\sim$30 GB of data)
  - distributed solver written in C using MPI and GSL
  - no optimization or tuned libraries (like ATLAS, MKL)
  - split into 80 subsystems across 10 (8-core) machines on Amazon EC2

- computation times
  
<table>
<thead>
<tr>
<th>Task</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>loading data</td>
<td>30s</td>
</tr>
<tr>
<td>factorization (5000 × 8000 matrices)</td>
<td>5m</td>
</tr>
<tr>
<td>subsequent ADMM iterations</td>
<td>0.5–2s</td>
</tr>
<tr>
<td>total time (about 15 ADMM iterations)</td>
<td>5–6m</td>
</tr>
</tbody>
</table>
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convex optimization problems

- arise in many applications

- can be solved effectively
  - small problems at microsecond/millisecond time scales
  - medium-scale problems using general purpose methods
  - arbitrary-scale problems using distributed optimization
References

- *Convex Optimization* (Boyd & Vandenberghe)

- *CVX: Matlab software for disciplined convex programming* (Grant & Boyd)

- *CVXGEN: A code generator for embedded convex optimization* (Mattingley & Boyd)

- *Distributed optimization and statistical learning via the alternating direction method of multipliers* (Boyd, Parikh, Chu, Peleato, & Eckstein)

  all available (with code) from stanford.edu/~boyd