Spin Geometry 2010

Tutorial Sheet 6

(Harder problems, if any, are adorned with a ☆.)

Problem 6.1. Let $P \to M$ be a principal G-bundle with connection. Let $\mathrm{Hol}(m)$ denote the holonomy group at $m \in M$ and let $\mathrm{Hol}_0(m)$ denote the restricted holonomy group. Show that $\mathrm{Hol}_0(m)$ is a normal subgroup of $\mathrm{Hol}(m)$. Prove that there is a surjective group homomorphism $\pi_1(M, m) \to \mathrm{Hol}(m)/\mathrm{Hol}_0(m)$. Give an example of a bundle for which this map is not an isomorphism.

Problem 6.2. Let $P \to M$ be a principal G-bundle with connection. Let $p, q \in M$. Show that if p and q can be joined by a smooth curve in M, then the holonomy groups $\mathrm{Hol}(p) \cong \mathrm{Hol}(q)$ are conjugate in G, and hence isomorphic.

Problem 6.3. Let (M, J) be an almost complex manifold; that is, the endomorphism $J : TM \to TM$ satisfies $J^2 = -\mathbf{1}$. Then show that the involutivity of the $+i$-eigenbundle T^+M of $T^{\mathbb{C}}M$ is equivalent to the vanishing of the **Nijenhuis tensor** $N_J : \Lambda^2 TM \to TM$, defined by

$$N_J(X, Y) = J[JX, JY] + [X, JY] + [JX, Y] - J[X, Y] \ .$$

Show that N_J is indeed a tensor; that is, show that N_J is $C^\infty(M)$-bilinear.

Problem 6.4. Let (M, g, J) be a hermitian manifold. This means that J is an orthogonal complex structure. Let ω be the corresponding nondegenerate 2-form. Show that the following conditions on J are equivalent (and equivalent to (M, g, J) being Kähler):

1. $\nabla J = 0$,
2. $\nabla \omega = 0$, and
3. $d\omega = 0$,

where ∇ is the Levi-Civita connection.

Problem 6.5. Let (M, g, I, J) be a hyperkähler manifold. This means that I, J are orthogonal parallel almost complex structures satisfying $IJ = -IJ$. Show that $\alpha I + \beta J + \gamma IJ$ is an integrable complex structure for all $\alpha, \beta, \gamma \in \mathbb{R}$ satisfying $\alpha^2 + \beta^2 + \gamma^2 = 1$.

Problem 6.6.☆ Show that in a quaternionic Kähler manifold, there is a parallel 4-form by investigating the fourth exterior power of the holonomy representation $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4n)$.