Lecture 7: Holonomy groups

Knowing the importance of groups in mathematics, it is quite natural to try to capture some part of Riemannian geometry in a group.

— Marcel Berger, 2003

In this lecture we will discuss the rudiments of the theory of holonomy groups for principal and vector bundles and in particular the relevant case of the holonomy group of the Levi-Civita connection on a riemannian manifold. As we will see in the next lecture, both the problems of determining the class of manifolds admitting parallel and Killing spinor fields will be solved in terms of riemannian holonomy groups.

7.1 Parallel transport in principal fibre bundles

Let $\pi : P \to M$ be a fixed principal G-bundle with connection $H \subset TP$. Let ω denote the connection 1-form. A smooth curve $\tilde{\gamma} : [0,1] \to P$ is said to be **horizontal** if the velocity vector is everywhere horizontal: $\dot{\tilde{\gamma}}(t) \in H_{\tilde{\gamma}(t)}$ for all t. This is equivalent to $\omega(\dot{\tilde{\gamma}}(t)) = 0$. Let $\gamma(t) = \pi(\tilde{\gamma}(t))$ denote the projection of the curve onto M. Assume that the curve is small enough so that the image of γ lies inside some trivialising neighbourhood U_α. Then $\psi_\alpha(\tilde{\gamma}(t)) = (\gamma(t), g(t))$, where $g(t)$ is a smooth curve on G. The condition $\omega(\dot{\tilde{\gamma}}(t)) = 0$ translates into the following ordinary differential equation for the curve $g(t)$. Indeed, using equation (77) and noticing that $\pi_* \dot{\tilde{\gamma}} = \dot{\gamma}$, we arrive at

$$\mathrm{ad}_{g(t)^{-1}} A_\alpha(\dot{\gamma}(t)) + g(t)^{-1} \dot{g}(t) = 0 \,, \tag{94}$$

where A_α is the gauge field on U_α corresponding to the connection and where again we use notation appropriate to matrix groups. Indeed, for matrix groups we can rewrite this equation further as a matrix differential equation:

$$\dot{g}(t) + A_\alpha(\dot{\gamma}(t)) g(t) = 0 \,. \tag{95}$$

Being a first-order ordinary differential equations with smooth coefficients, equation (94) (equivalently (95)) has a unique solution for specified initial conditions, so that if we specify $g(0)$ then $g(1)$ is determined uniquely. This then defines a map $\Pi_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$ from the fibre over $\gamma(0)$ to the fibre over $\gamma(1)$, associated to the curve $\gamma : [0,1] \to M$. Rephrasing, given the curve γ, there is a unique horizontal lift $\tilde{\gamma}$ once we specify $\tilde{\gamma}(0) \in P_{\gamma(0)}$ and $\Pi_\gamma \tilde{\gamma}(0) = \tilde{\gamma}(1)$ is simply the endpoint of this horizontal curve. The map Π_γ is called **parallel transport along** γ **with respect to the connection** H.

Lemma 7.1. *Parallel transport is* G*-equivariant:* $\Pi_\gamma \circ R_a = R_a \circ \Pi_\gamma$.

Proof. This follows from the observation that if $\tilde{\gamma}(t)$ is a horizontal lift of $\gamma(t)$, then so is $\tilde{\gamma}(t)a$. □

Now let γ be a loop, so that $\gamma(0) = \gamma(1)$. Parallel transport along γ defines a group element $g_\gamma \in G$ defined by $g_\gamma = g(1)g(0)^{-1}$. To show that this element is well-defined, we need to show that it does not depend on the initial point $g(0)$. Indeed, suppose we choose a different starting point $\overline{g}(0)$. Then there is some group element $h \in G$ such that $\overline{g}(0) = g(0)h$. From the lemma $\overline{g}(t) := g(t)h$ is the horizontal lift with initial condition $\overline{g}(0)$. Therefore the final point of the curve is $\overline{g}(1) = g(1)h$, whence $\overline{g}(1)\overline{g}(0)^{-1} = g(1)g(0)^{-1}$ and g_γ is well-defined. This procedure defines a map from piecewise-smooth loops based at $m = \gamma(0)$ to G, whose image is a subgroup of G called the **holonomy group of the connection at** m denoted

$$\mathrm{Hol}(m) = \left\{ g_\gamma \middle| \gamma : [0,1] \to M \,,\ \gamma(1) = \gamma(0) = m \right\} \,. \tag{96}$$

The holonomy group is indeed a subgroup of G; that is, it is closed under inverses and multiplication. More precisely, if $g_\gamma \in \mathrm{Hol}(m)$, then let $\gamma^{-1}(t) := \gamma(1-t)$ be the curve with the same image as γ but traced backward and let $(\gamma(1-t), g(1-t))$ be its horizontal lift. Then $g_{\gamma^{-1}} = g(0)g(1)^{-1} = (g(1)g(0)^{-1})^{-1} = g_\gamma^{-1}$. Similarly, if g_{γ_1} and g_{γ_2} are elements in $\mathrm{Hol}(m)$, then so is their product. Indeed let (γ_1, g_1) be a horizontal lift of γ_1 (in a trivialisation) and (γ_2, g_2) a horizontal lift of γ_2, so that $g_{\gamma_1} = g_1(1)g_1(0)^{-1}$ and $g_{\gamma_2} = g_2(1)g_2(0)^{-1}$. It then follows that $g_{\gamma_1}g_{\gamma_2} = g_1(1)g_1(0)^{-1}g_2(1)g_2(0)^{-1}$. Let us consider the piecewise smooth curve

$$\gamma(t) = \begin{cases} \gamma_2(2t)\,, & t \in [0, \frac{1}{2}] \\ \gamma_1(2t-1)\,, & t \in [\frac{1}{2}, 1]\,. \end{cases}$$

A continuous horizontal lift of this curve is given (in a trivialisation) by $(\gamma(t), g(t))$ where

$$g(t) = \begin{cases} g_2(2t)\,, & t \in [0, \frac{1}{2}] \\ g_1(2t-1)\,, & t \in [0, \frac{1}{2}] \end{cases}$$

where, for continuity, we choose the horizontal lift of γ_1 in such a way that $g_1(0) = g_2(1)$. Then

$$g_{\gamma_1}g_{\gamma_2} = g_1(1)g_1(0)^{-1}g_2(1)g_2(0)^{-1} = g_1(1)g_2(0)^{-1} = g(1)g(0)^{-1} = g_\gamma\,.$$

Furthermore, if $m, m' \in M$ belong to the same connected component, the holonomy groups $\mathrm{Hol}(m)$ and $\mathrm{Hol}(m')$ are conjugate in G and hence isomorphic. For a manifold the notion of connected component agrees with that of path component, hence there is a curve $\delta : [0,1] \to M$ such that $\delta(0) = m$ and $\delta(1) = m'$. Let $\delta^{-1} : [0,1] \to M$ be the curve $\delta^{-1}(t) = \delta(1-t)$. Then there is a one-to-one correspondence between loops based at m and based at m'. Indeed, if γ' is a loop based at m' then the composition $\gamma = \delta^{-1} \circ \gamma' \circ \delta$ is a loop based at m; and viceversa. Arguments similar to the ones above show that the element g_γ of the holonomy group at m is given by $hg_{\gamma'}h^{-1}$ where h is the group element corresponding to $\delta(0)$ in the trivialisation. This shows that $\mathrm{Hol}(m)$ and $\mathrm{Hol}(m')$ are conjugate subgroups of G, so that if M is connected there is a sense in which we can discuss the holonomy group of the connection, up to isomorphism, without having to specify the base point.

Considering only null-homotopic loops, we arrive at a normal subgroup of the holonomy group called the **restricted holonomy group** and denoted $\mathrm{Hol}_0(m)$. It can be shown that it is the identity component of the holonomy group. We have a surjective homomorphism $\pi_1(M, m) \to \mathrm{Hol}(m)/\mathrm{Hol}_0(m)$, which is not generally an isomorphism: any flat connection on a non-simply connected manifold is a counterexample.

7.2 Parallel transport on vector bundles

Let $E = P \times_G F \to M$ be an associated vector bundle to $P \to M$ and let ∇ be the Koszul connection on sections of E induced from the connection on P. If $\gamma : [0,1] \to M$ is a curve on M, then we define the parallel transport $\Pi_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ as follows. We can use γ to pull the bundle E back to a bundle $\gamma^{-1}E \to [0,1]$, whose fibre at $t \in [0,1]$ is the fibre of E at $\gamma(t)$. Vector bundles over the interval are trivial, so that sections of $\gamma^{-1}E$ are functions $[0,1] \to F$, where F is the typical fibre. Let $f_0 \in E_{\gamma(0)}$ and let $f : [0,1] \to F$ satisfy $\nabla_{\dot\gamma(t)} f = 0$, subject to $f(0) = f_0$. Then $\Pi_\gamma f_0 = f(1) \in E_{\gamma(1)}$. Explicitly, the parallel transport equation $\nabla_{\dot\gamma(t)} f = 0$ becomes the ordinary differential equation

$$\frac{d}{dt} f(t) + \varrho(A(\dot\gamma(t))) f(t) = 0\,, \tag{97}$$

where $\varrho : \mathfrak{g} \to \mathfrak{gl}(F)$ and A is the gauge field, where we have assumed that the image of γ lies inside a trivialising neighbourhood. By considering loops we define the notion of (restricted) holonomy group just as for principal fibre bundles.

We can recover the connection from the parallel transport by the following limiting procedure analogous to the usual definition of the derivative of a real variable:

$$\nabla_{\dot\gamma(t)} f = \lim_{h\to 0} \frac{1}{h} \left(\Pi_{-h} f(\gamma(t+h)) - f(\gamma(t)) \right)\,, \tag{98}$$

where $\Pi_{-h} : E_{\gamma(t+h)} \to E_{\gamma(t)}$ is the parallel transport along γ from $t+h$ to t.

7.3 The holonomy principle

The holonomy principle is arguably the most important conceptual result in the theory of holonomy. Let $E \to M$ be a vector bundle with connection over a connected manifold M. A section σ of $E \to M$ is said to be **invariant under parallel transport** if for every curve $\gamma : [0,1] \to M$ we have that $\Pi_\gamma \sigma(\gamma(0)) = \sigma(\gamma(1))$. Taking γ to be a loop, we see that $\sigma(\gamma(0))$ is invariant under $\mathrm{Hol}(\gamma(0))$. Conversely, given $\sigma(m)$ invariant under $\mathrm{Hol}(m)$, we define $\sigma(m') = \Pi_\gamma \sigma(m)$, where $\gamma : [0,1] \to M$ is a curve with $\gamma(0) = m$ and $\gamma(1) = m'$. This does not depend on the choice of curve γ precisely because $\sigma(m)$ is invariant under the holonomy group. From equation (98), it follows that if σ is invariant under parallel transport, it is covariantly constant: $\nabla\sigma = 0$. If M is simply connected, then the converse also holds. This follows from the following

Theorem 7.2 (Ambrose–Singer)**.** *Let* $E \to M$ *be a vector bundle with connection with* M *connected. Then the Lie algebra* $\mathfrak{hol}(m)$ *of the holonomy group* $\mathrm{Hol}(m)$ *is the Lie subalgebra of* $\mathfrak{gl}(E_m)$ *spanned by the curvature endomorphisms* $R^\nabla(X,Y)$ *and all its covariant derivatives* $\nabla_{Z_1} \cdots \nabla_{Z_k} R^\nabla(X,Y)$ *for* $X, Y, Z_i \in \mathscr{X}(M)$.

Indeed, fix $m \in M$ and suppose that $\nabla\sigma = 0$. Then $R^\nabla(X,Y)\sigma(m) = 0$ for all $X, Y \in T_m M$. Taking a further covariant derivative ∇_Z, say, we see that

$$0 = \nabla_Z (R^\nabla(X,Y)\sigma) = (\nabla_Z R^\nabla(X,Y))\sigma + R^\nabla(X,Y)\nabla_Z \sigma \,,$$

but the last term vanishes because σ is covariantly constant, whence the endomorphism $\nabla_Z R^\nabla(X,Y)$ annihilates $\sigma(m)$. Continuing in this way and using the Theorem we see that $\sigma(m)$ is invariant under the Lie algebra of the holonomy group $\mathrm{Hol}(m)$, whence under the restricted holonomy group $\mathrm{Hol}_0(m)$. If M is simply-connected, then the holonomy group agrees with the restricted holonomy group, and hence $\sigma(m)$ is invariant under $\mathrm{Hol}(m)$.

We can summarise the above in the following

Theorem 7.3 (Holonomy principle)**.** *Let* M *be a 1-connected manifold and* $E \to M$ *be a vector bundle with connection. Then there is a one-to-one correspondence between*

1. *sections of* E *which are invariant under parallel transport,*
2. $\mathrm{Hol}(m)$*-invariant vectors in* E_m*, for some* $m \in M$*, and*
3. *covariantly constant sections of* E.

If M *is connected but not simply-connected, then there may be covariantly constant sections which are only* $\mathrm{Hol}_0(m)$*-invariant, but not* $\mathrm{Hol}(m)$*-invariant.*

The holonomy principle allows us to turn questions concerning covariantly constant objects into algebraic questions about the holonomy representation.

7.4 Riemannian holonomy groups

Let (M, g) be a connected riemannian manifold of signature (s,t). Let ∇ denote the Levi-Civita connection on the tangent bundle TM. Since g is covariantly constant, it follows from the holonomy principle that the holonomy group is contained inside the orthogonal group, or more precisely, $\mathrm{Hol}(m) \subset O(T_m M) \cong O(s,t)$ and in particular at the level of the Lie algebras, $\mathfrak{hol}(m) \cong \mathfrak{so}(T_m M) \cong \mathfrak{so}(s,t)$. A natural question is whether any Lie subalgebra of $\mathfrak{so}(s,t)$ can appear as the holonomy Lie algebra of a riemannian manifold. To this day this problem has only been solved in the positive-definite and lorentzian signatures. In this section we will recall the positive-definite case and explain why the indefinite case is so much harder.

The vector space $T_m M$ is naturally a representation of $\mathrm{Hol}(m)$, called the **holonomy representation**. It is clear that for a riemannian product $(M, g) = (M_1, g_1) \times (M_2, g_2)$ the holonomy representation is reducible. (Since M is assumed connected, the (ir)reducibility of the holonomy representation does not depend on the point m.) The de Rham decomposition theorem below provides a partial converse to this *in positive-definite signature.*

Theorem 7.4 (De Rham)**.** *Let* (M, g) *be a 1-connected, complete, positive-definite riemannian manifold. If its holonomy representation is reducible, then* (M, g) *is a riemannian product.*

A sketch of a proof can be found in [Bes87, §10.44]. This result essentially reduces the classification of positive-definite riemannian holonomy groups to representation theory. The classification problem was was eventually solved by Berger, although later refined by a number of people including Simons, Alekseevsky and Bryant. A recent survey of this story can be found in [Bry00], which also describes the more general problem for torsion-free affine connection (not necessarily metric), recently solved by Merkulov and Schwachhöfer. The torsion-free condition is what makes this problem nontrivial, since a classical theorem of Nomizu's states that any group can appear if we drop the torsion-free condition.

Back to the riemannian holonomy problem, the difference in indefinite signature is that reducibility is not enough to decompose the space. We say that a subspace $W \subset T_m M$ is **nondegenerate** if the restriction of the metric to W is non-degenerate, and **degenerate** otherwise. Clearly in positive-definite signature all subspaces are nondegenerate, but this is not the case in indefinite signature: a null line, for instance, provides an example of a degenerate subspace. In a riemannian product $(M, g) = (M_1, g_1) \times (M_2, g_2)$, the embedding at $m_2 \in M_2$ of the tangent space $T_{m_1} M_1$ into $T_{(m_1,m_2)} M$ is a nondegenerate subspace, and similarly for the embedding $T_{m_2} M_2 \subset T_{(m_1,m_2)} M$ at $m_1 \in M_1$. Hence it may happen that the holonomy representation is reducible, yet the manifold is not a riemannian product. Let us say that the holonomy representation is **decomposable** if it is reducible and if each invariant subspace is nondegenerate. We can then state the following extension of the de Rham decomposition theorem due to Wu [Wu64].

Theorem 7.5 (Wu)**.** *Let* (M, g) *be a 1-connected, complete, riemannian manifold. If its holonomy representation is decomposable, then* (M, g) *is a riemannian product.*

This means that it is not enough to restrict to irreducible holonomy representations in order to classify indefinite riemannian holonomy groups. Indeed, a result of Bérard-Bergery and Ikemakhen [BI93] says that the only irreducible lorentzian holonomy group is the Lorentz group itself, yet there exist indecomposable lorentzian manifolds with reduced holonomy. It is this which makes the indefinite case much harder. To date only the lorentzian problem has been solved completely. It is described in a recent survey by Leistner and Galaev [GL08].

Let us review the positive-definite classification, since this will play an important rôle in the rest of the lectures. The classification breaks up naturally into two classes of irreducible manifolds. The first class consists of those for which the curvature is parallel with respect to the Levi-Civita connection: $\nabla R = 0$. Such manifolds are said to be **locally symmetric** and if simply connected, they are (riemannian) **symmetric spaces**. Symmetric spaces were classified by Élie Cartan. He found two types, each type being a pair consisting of a compact and a noncompact space and all indexed by simple Lie algebras. Pairs of the first type are $(G, G^{\mathbb{C}})$, where G is a 1-connected simple Lie group and $G^{\mathbb{C}}$ is the corresponding complex Lie group. Typical examples are (the simply-connected version of) $(SO(n), SO(n, \mathbb{C}))$ and $(SU(n), SL(n, \mathbb{C}))$. The second type consists of pairs $(G/H, G^*/H)$ where G is a 1-connected noncompact simple Lie group, H the connected maximal compact subgroup and G^* the compact form. A typical example is (the simply-connected version of) $(SL(n, \mathbb{R})/SO(n), SU(n)/SO(n))$. The holonomy group is G in the first type and H in the second type.

The second class, which in some sense is the most interesting for our purposes, consists of a finite list known as **Berger's table**; although the original list contained one more case which Alekseevsky showed was necessarily symmetric.

Theorem 7.6. *Let* (M, g) *be a complete, 1-connected, non-symmetric positive-definite riemannian manifold. Then its holonomy representation is one of the following:*

$n = \dim M$	$H \subset SO(n)$	*Geometry*
n	$SO(n)$	*generic*
$2m$	$U(m)$	*Kähler*
$2m$	$SU(m)$	*Calabi–Yau*
$4m$	$Sp(m)$	*hyperkähler*
$4m$	$Sp(m)\cdot Sp(1)$	*quaternionic Kähler*
7	G_2	*exceptional*
8	$Spin(7)$	*exceptional*

where $Sp(m)\cdot Sp(1)$ *is the image of* $Sp(m)\times Sp(1) \subset Spin(4m)$ *under* $Spin(4m) \to SO(4m)$.

One can understand these subgroups better in terms of the objects that they leave invariant which, by the holonomy principle, translates into the existence of covariantly constant fields on a riemannian manifold with that holonomy. We will look in detail at the Kähler, Calabi–Yau and G_2-holonomy cases.

7.4.1 Kähler manifolds

Every $U(m)$ subgroup of $SO(2m)$ arises as the subgroup of automorphisms of $\mathbb{R}^{2m}$ which commute with an orthogonal complex structure. Recall that a complex structure on $\mathbb{R}^{2m}$ is any endomorphism $J : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ obeying $J^2 = -\mathbf{1}$. The reason for the name is that J allows us to multiply by complex numbers: $a + ib \in \mathbb{C}$ acts like $a\mathbf{1} + bJ$. This makes $\mathbb{R}^{2m}$ into a complex vector space. A linear transformation which commutes with J commutes with complex multiplication, whence it is complex linear. The complex structure determines an embedding $GL(m,\mathbb{C})$ in $GL(2m,\mathbb{R})$: namely, those invertible linear transformations commuting with J.

A complex structure J is said to be orthogonal if it preserves the inner product $\langle -,-\rangle$ defining the $SO(2m)$ subgroup; that is, $\langle Jx, Jy\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^{2m}$. In particular, we have that $\omega(x, y) := \langle Jx, y\rangle$ is a symplectic structure. Any two of $\langle -,-\rangle$, J and ω determines the third. It also defines a positive-definite hermitian structure on $\mathbb{R}^{2m}$ by

$$H(x, y) = \langle x, y\rangle + i\omega(x, y) . \tag{99}$$

It follows that $H(Jx, y) = -iH(x, y)$ and $H(x,Jy) = iH(x, y)$ and $\overline{H(x, y)} = H(y, x)$, whence H is indeed a hermitian structure. It is positive definite since so is $\langle -,-\rangle$. Any orthogonal linear transformation commuting with J preserves H and, conversely, if it preserves H, then it preserves its real and imaginary parts separately, whence it is orthogonal and commutes with J. Hence the $U(m)$ subgroup defined above is precisely the subgroup leaving H invariant. It can be thought of as the intersection with $SO(2m)$ of the $GL(m,\mathbb{C})$ subgroup defined by J.

On a $2m$-dimensional riemannian manifold with $U(m)$ holonomy, the holonomy principle guarantees the existence of a parallel complex structure J and a parallel symplectic form ω, called the **Kähler form**. Since $J^2 = -\mathbf{1}$, its eigenvalues are $\pm i$. The complexified tangent bundle $T^{\mathbb{C}}M$ decomposes into a direct sum of eigenbundles of J:

$$T^{\mathbb{C}}M = T^+M \oplus T^-M , \tag{100}$$

where $T^{\pm}M$ is the J-eigenbundle with eigenvalue $\pm i$. It follows from the fact that J is parallel, that $T^{\pm}M$ are integrable distributions in the sense of Frobenius. In other words, if X, Y are smooth sections of T^+M, then so is their Lie bracket [X, Y], and similarly for T^-M. A hard theorem of Newlander and Nirenberg [NN57] then shows that M is a complex manifold; that is, there are coordinate charts homeomorphic to open subsets of $\mathbb{C}^m$ such that the transition functions on nonempty overlaps are biholomorphic. That means that we have local complex coordinates z^α and that there is a well-defined notion of holomorphicity. In turn this refines the de Rham complex, allowing us to define a notion of (p, q)**-form**, as a differential form of the form (summation convention in force!)

$$f_{\alpha_1\dots\alpha_p\beta_1\dots\beta_q} dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q} , \tag{101}$$

where $f_{\alpha_1\dots\alpha_p\beta_1\dots\beta_q}$ are *smooth* functions on the domain of definition of the coordinates z^α. The notion of being a (p, q)-form is preserved on overlaps due to the holomorphicity of the transition functions. For example, the Kähler form $\omega(X,Y) = g(JX,Y)$ is a $(1,1)$-form. Its normalised powers $\frac{1}{k!}\omega^k$ are (k,k)-forms and, in particular, $\frac{1}{m!}\omega^m$ is the volume form corresponding to the metric g.

Let $\Omega^{(p,q)}(M)$ denote the $C^\infty(M)$-module of (p,q)-forms. This is a bigraded refinement of the de Rham complex in that

$$\Omega^r(M) = \bigoplus_{p+q=r} \Omega^{(p,q)}(M) . \tag{102}$$

If $f \in C^\infty(M)$, then $df \in \Omega^1(M) = \Omega^{(1,0)}(M) \oplus \Omega^{(0,1)}(M)$. Let us denote the component in $\Omega^{(1,0)}(M)$ by ∂f and the component in $\Omega^{(0,1)}(M)$ by $\bar\partial f$. More generally one has that $d = \partial + \bar\partial$, where $\partial : \Omega^{(p,q)}(M) \to \Omega^{(p+1,q)}(M)$ and $\bar\partial : \Omega^{(p,q)}(M) \to \Omega^{(p,q+1)}(M)$. Since $d^2 = 0$, it follows by looking at degrees that $\partial^2 = 0$, $\bar\partial^2 = 0$ and $\partial\bar\partial + \bar\partial\partial = 0$. A great deal more could be said about Kähler manifolds, but this is not the place.

7.4.2 Calabi–Yau manifolds

Since $SU(m) \subset U(m)$, manifolds with $SU(m)$ holonomy are Kähler. They are Ricci-flat and, in fact, they are characterised also in this way. Calabi–Yau manifolds have a parallel complex structure J and Kähler form ω, but in addition also a parallel **complex volume form** $\Theta \in \Omega^{(m,0)}(M)$. This follows from the fact that $SU(m)$ is the intersection of $U(m)$ with $SL(m,\mathbb{C})$ and $SL(m,\mathbb{C})$ is the subgroup of $GL(m,\mathbb{C})$ which acts trivially on the top exterior power of $\mathbb{C}^m$. The complex volume form Θ is in particular holomorphic: $\bar\partial\Theta = 0$. Let $\overline{\Theta} \in \Omega^{(0,m)}$ be its complex conjugate. Then $\Theta \wedge \overline{\Theta}$ is the volume form corresponding to the metric g.

7.4.3 Manifolds of G_2 holonomy

The group G_2 can be defined in several ways. It is the subgroup of Spin(8) fixed under the triality automorphism. Therefore any two representations of Spin(8) related by triality are equivalent when restricted to G_2. In particular, consider the three 8-dimensional representations: the vector V and the two half-spinor representations $S_\pm$. They are equivalent as representations of G_2. With hindsight, let us denote this representation (which is not irreducible) by $\mathbb{O}$. The Clifford action $V \otimes S_+ \to S_-$ becomes, under G_2, a *non-associative* multiplication $\mathbb{O} \otimes \mathbb{O} \to \mathbb{O}$, turning $\mathbb{O}$ into an $\mathbb{R}$-algebra. This is nothing but the algebra of **octonions** and G_2 is the group of automorphisms of the octonions: the map $V \otimes S_+ \to S_-$ is Spin(8) equivariant, hence $\mathbb{O} \times \mathbb{O} \to \mathbb{O}$ is G_2-equivariant. Since G_2 acts under automorphisms, it preserves the identity $1 \in \mathbb{O}$. Thus we see that $\mathbb{O}$ breaks up under G_2 into a direct sum of irreducible representations $\mathbb{R}1 \oplus \operatorname{Im}\mathbb{O}$. The holonomy representation $G_2 \subset SO(7)$ is precisely the action of G_2 on $\operatorname{Im}\mathbb{O}$. Octonion multiplication restricts to a bilinear map $\operatorname{Im}\mathbb{O} \otimes \operatorname{Im}\mathbb{O} \to \mathbb{O}$ and this, in turn, defines a G_2-invariant tensor $\varphi : \operatorname{Im}\mathbb{O} \otimes \operatorname{Im}\mathbb{O} \otimes \operatorname{Im}\mathbb{O} \to \mathbb{R}$ by $\varphi(x,y,z) = \langle xy, z\rangle$ for all $x,y,z \in \operatorname{Im}\mathbb{O}$. Here $\langle -,-\rangle$ is the norm on $\mathbb{O}$, defined by $\langle x,y\rangle = \operatorname{Re}(\bar{x}y)$, where $\bar{x}$ is the octonionic conjugate of x. When $x,y \in \operatorname{Im}\mathbb{O}$, then $\langle x,y\rangle = -\operatorname{Re} xy$. The octonion algebra is not associative, but it is alternating and this means that φ is totally skewsymmetric. The holonomy principle guarantees that on a manifold of G_2 holonomy, there is a parallel 3-form, also denoted φ. Its Hodge dual $\star\varphi$ is a parallel 4-form. These are the only parallel forms on a generic manifold of G_2 holonomy. Manifolds of G_2 holonomy are also Ricci-flat.

7.4.4 Ricci-flatness

We saw in the previous lecture that Ricci-flatness was an integrability condition for the existence of parallel spinors in a positive-definite riemannian manifold. It is thus natural to ask whether a reduction of the holonomy group implies Ricci-flatness. We saw above that Calabi–Yau and G_2-holonomy manifolds are Ricci flat. It turns out that hyperkähler and Spin(7)-holonomy manifolds are also Ricci-flat. In contrast, quaternionic Kähler manifolds are Einstein but never Ricci-flat – indeed, a Ricci-flat quaternionic Kähler manifold is hyperkähler. Similarly, the Calabi conjecture (proved by Yau) says that a Ricci-flat Kähler manifold is Calabi–Yau. There are examples of noncompact positive-definite riemannian manifolds with $SO(n)$ holonomy, but to this day there is no known Ricci-flat compact manifold which has $SO(n)$-holonomy. This is perhaps the last remaining mystery in the holonomy of positive-definite riemannian manifolds.