Lecture 5: Connections on principal and vector bundles

The beauty and profundity of the geometry of fibre bundles were to a large extent brought forth by the (early) work of Chern. I must admit, however, that the appreciation of this beauty came to physicists only in recent years.

— CN Yang, 1979

The aim of this lecture is the construction of a connection on the spin bundle and hence on the associated spinor bundles, but first we will discuss the rudiments of the theory of Ehresmann and Koszul connections on principal and vector bundles, respectively. This is, of course, the language of gauge theory and I will borrow freely from my own preliminary lecture notes on this subject.

5.1 Connections on principal bundles

The push-forward and the pull-back

Let $f : M \to N$ be a smooth map between manifolds. The **push-forward**

$$T f : TM \to TN$$

is the collection of fibre-wise linear maps $f_* : T_m M \to T_{f(m)} N$ defined as follows. Let $v \in T_m M$ be represented as the velocity of a curve $t \mapsto \gamma(t)$ through m; that is, $\gamma(0) = m$ and $\gamma'(0) = v$. Then $f_*(v) \in T_{f(m)}N$ is the velocity at $f(m)$ of the curve $t \mapsto f(\gamma(t))$; that is, $f_*\gamma'(0) = (f\circ\gamma)'(0)$. If $g : N \to Q$ is another smooth map between manifolds, then so is their composition $g \circ f : M \to Q$. The chain rule for derivatives says that $T(g\circ f) = Tg \circ Tf$. Since the push-forward of the identity diffeomorphism 1_M is the identity diffeomorphism 1_{TM}, we see that T is indeed a functor from the category of smooth manifolds and smooth maps to itself.
Dual to the tangent bundle TM is the cotangent bundle T^*M, where $T^*_m M = \mathrm{Hom}(T_m M, \mathbb{R})$. Its sections are called **one-forms** and the space of one-forms on M is denoted $\Omega^1(M)$. The dual of the push-forward is the **pull-back** $f^* : T^*N \to T^*M$, defined for a one-form α by $(f^*\alpha)(v) = \alpha(f_* v)$. Notice that $f^* : T^*_{f(m)}N \to T^*_m M$. It is also functorial, but now reversing the order $(g\circ f)^* = f^* \circ g^*$. (It's a contravariant functor.) Unlike the case of the push-forward, the pull-back defines a map on sections also denoted $f^* : \Omega^1(N) \to \Omega^1(M)$. We also use the notation $\Omega^k(M)$ to denote the sections of the k-th exterior power : $\Lambda^k T^*M$ of the cotangent bundle. If $k = 0$, $\Omega^0(M) = C^\infty(M)$.

Let $\pi : P \to M$ be a principal G-bundle and let $m \in M$ and $p \in \pi^{-1}(m)$. The **vertical** subspace $V_p \subset T_p P$ consists of those vectors tangent to the fibre at p; in other words, $V_p = \ker \pi_* : T_p P \to T_m M$. A vector field $v \in \mathscr{X}(P)$ is **vertical** if $v(p) \in V_p$ for all p. The Lie bracket of two vertical vector fields is again vertical. The vertical subspaces define a G-invariant distribution (in the sense of Frobenius) $V \subset TP$: indeed, since $\pi \circ R_g = \pi$, we have that $\pi_* \circ (R_g)_* \pi_*$, whence $(R_g)_* V_p = V_{pg}$.

We can understand the vertical space also as the image of the Lie algebra $\mathfrak{g}$ of G under the G-action. If we fix $p \in P$, then the action gives a map $G \to P$ defined by $g \mapsto pg$, whose push-forward at the identity defines a map $\sigma_p : \mathfrak{g} \to T_p P$; explicitly,

$$\sigma_p(X) = \frac{d}{dt}\left(p\exp(tX)\right)\Big|_{t=0} .$$

Since $\pi(p\exp(tX)) = \pi(p)$, it follows that $\sigma_p(X) \in V_p$. Since the action of G is free, the map in one-to-one and hence counting dimension we see that $\sigma_p : \mathfrak{g} \to V_p$ is an isomorphism.

Lemma 5.1.

$$(R_g)_* \sigma_p(X) = \sigma_{pg}\left(\mathrm{Ad}_{g^{-1}} X\right) .$$

Proof. By definition, at $p \in \mathrm{P}$, we have

$$
\begin{aligned}
(\mathrm{R}_g)_* \sigma_p(\mathrm{X}) &= \frac{d}{dt} \mathrm{R}_g \left(p \exp(t\mathrm{X})\right)\Big|_{t=0} \\
&= \frac{d}{dt} \left(p \exp(t\mathrm{X}) g\right)\Big|_{t=0} \\
&= \frac{d}{dt} \left(p g g^{-1} \exp(t\mathrm{X}) g\right)\Big|_{t=0} \\
&= \frac{d}{dt} \left(p g \exp(t \mathrm{Ad}_{g^{-1}} \mathrm{X})\right)\Big|_{t=0} \\
&= \sigma_{pg}\left(\mathrm{Ad}_{g^{-1}} \mathrm{X}\right) .
\end{aligned}
$$

□

In the absence of any extra structure, there is no natural complement to V_p in $\mathrm{T}_p\mathrm{P}$. This is in a sense what a connection provides.

5.1.1 Connections as horizontal distributions

A **connection** (in the sense of Ehresmann) on P is a smooth choice of **horizontal** subspaces $\mathrm{H}_p \subset \mathrm{T}_p\mathrm{P}$ complementary to V_p:

$$\mathrm{T}_p\mathrm{P} = \mathrm{V}_p \oplus \mathrm{H}_p$$

and such that $(\mathrm{R}_g)_* \mathrm{H}_p = \mathrm{H}_{pg}$. In other words, a connection is a G-invariant distribution $\mathrm{H} \subset \mathrm{TP}$ complementary to V.

Example 5.2. A G-invariant riemannian metric on P gives rise to a connection, simply by defining $\mathrm{H}_p = \mathrm{V}_p^\perp$. This simple observation underlies the Kałuża–Klein programme relating gravity on P to gauge theory on M. It also underlies many geometric constructions, since it is often the case that 'nice' metrics will give rise to 'nice' connections and viceversa.

5.1.2 The connection one-form

The horizontal subspace $\mathrm{H}_p \subset \mathrm{T}_p\mathrm{P}$, being a linear subspace, is cut out by $k = \dim \mathrm{G}$ linear equations $\mathrm{T}_p\mathrm{P} \to \mathbb{R}$. In other words, H_p is the kernel of k one-forms at p, the components of a one-form ω at p with values in a k-dimensional vector space. There is a natural such vector space, namely the Lie algebra $\mathfrak{g}$ of G, and since ω annihilates horizontal vectors it is defined by what it does to the vertical vectors, and we do have a natural map $\mathrm{V}_p \to \mathfrak{g}$ given by the inverse of σ_p. This prompts the following definition.

The **connection one-form** of a connection $\mathrm{H} \subset \mathrm{TP}$ is the $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(\mathrm{P};\mathfrak{g})$ defined by

$$\omega(v) = \begin{cases} \mathrm{X} & \text{if } v = \sigma(\mathrm{X}) \\ 0 & \text{if } v \text{ is horizontal.} \end{cases}$$

Proposition 5.3. *The connection one-form obeys*

$$\mathrm{R}_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega .$$

Proof. Let $v \in \mathrm{H}_p$, so that $\omega(v) = 0$. By the G-invariance of H, $(\mathrm{R}_g)_* v \in \mathrm{H}_{pg}$, whence $\mathrm{R}_g^*\omega$ also annihilates v and the identity is trivially satisfied. Now let $v = \sigma_p(\mathrm{X})$ for some $\mathrm{X} \in \mathfrak{g}$. Then, using Lemma 5.1,

$$\mathrm{R}_g^*\omega(\sigma(\mathrm{X})) = \omega\left((\mathrm{R}_g)_*\sigma(\mathrm{X})\right) = \omega\left(\sigma\left(\mathrm{Ad}_{g^{-1}}\mathrm{X}\right)\right) = \mathrm{Ad}_{g^{-1}}\mathrm{X} .$$

□

Conversely, given a one-form $\omega \in \Omega^1(\mathrm{P};\mathfrak{g})$ satisfying the identity in Proposition 5.3 and such that $\omega(\sigma(\mathrm{X})) = \mathrm{X}$, the distribution $\mathrm{H} = \ker\omega$ defines a connection on P.

We say that a form on P is **horizontal** if it annihilates the vertical vectors. Notice that if ω and ω' are connection one-forms for two connections H and H′ on P, their difference $\omega - \omega' \in \Omega^1(\mathrm{P};\mathfrak{g})$ is horizontal. We will see later that this means that it defines a section through a bundle on M associated to P.

5.1.3 The horizontal projection

Given a connection $H \subset TP$, we define the **horizontal projection** $h : TP \to TP$ to be the projection onto the horizontal distribution along the vertical distribution. It is a collection of linear maps $h_p : T_pP \to T_pP$, for every $p \in P$, defined by

$$h_p(v) = \begin{cases} v & \text{if } v \in H_p, \text{ and} \\ 0 & \text{if } v \in V_p. \end{cases}$$

In other words, $\operatorname{im} h = H$ and $\ker h = V$. Since both H and V are invariant under the the action of G, the horizontal projection is equivariant:

$$h \circ (R_g)_* = (R_g)_* \circ h \, .$$

We will let $h^* : T^*P \to T^*P$ denote the dual map, whence if, say, $\alpha \in \Omega^1(P)$ is a one-form, $h^*\alpha = \alpha \circ h$. More generally if $\beta \in \Omega^k(P)$, then $(h^*\beta)(v_1, \ldots, v_k) = \beta(hv_1, \ldots, hv_k)$. However...

Despite the notation, h^* is *not* the pull-back by a smooth map! In particular, h^* will *not* commute with the exterior derivative d!

5.1.4 The curvature 2-form

Let $\omega \in \Omega^1(P;\mathfrak{g})$ be the connection one-form for a connection $H \subset TP$. The 2-form $\Omega := h^* d\omega \in \Omega^2(P;\mathfrak{g})$ is called the **curvature (2-form)** of the connection. We will derive more explicit formulae for Ω later on, but first let us interpret the curvature geometrically.

By definition,

$$\begin{aligned} \Omega(u,v) &= d\omega(hu,hv) \\ &= (hu)\omega(hv) - (hv)\omega(hu) - \omega([hu,hv]) \\ \text{(since } h^*\omega = 0) \quad &= -\omega([hu,hv]) \, ; \end{aligned}$$

whence $\Omega(u,v) = 0$ if and only if $[hu,hv]$ is horizontal. In other words, the curvature of the connection measures the failure of integrability of the horizontal distribution $H \subset TP$.

Frobenius integrability

A distribution $D \subset TP$ is said to be integrable if the Lie bracket of any two sections of D lies again in D. The theorem of Frobenius states that a distribution is integrable if every $p \in P$ lies in a unique submanifold of P whose tangent space at p agrees with the subspace $D_p \subset T_pP$. These submanifolds are said to *foliate* P. As we have just seen, a connection $H \subset TP$ is integrable if and only if its curvature 2-form vanishes.
In contrast, the vertical distribution $V \subset TP$ is always integrable, since the Lie bracket of two vertical vector fields is again vertical, and Frobenius's theorem guarantees that P is floated by submanifolds whose tangent spaces are the vertical subspaces. These submanifolds are of course the fibres of $\pi : P \to M$.

The integrability of a distribution has a dual formulation in terms of differential forms. A horizontal distribution $H = \ker\omega$ is integrable if and only if (the components of) ω generate a differential ideal, so that $d\omega = \Theta \wedge \omega$, for some $\Theta \in \Omega^1(P;\operatorname{End}(\mathfrak{g}))$. Since Ω measures the failure of integrability of H, the following formula should not come as a surprise.

Proposition 5.4 (Structure equation)**.**

$$\Omega = d\omega + \tfrac{1}{2}[\omega,\omega] \, ,$$

where $[-,-]$ *is the* symmetric *bilinear product consisting of the Lie bracket on* $\mathfrak{g}$ *and the wedge product of one-forms.*

Proof. We need to show that

$$d\omega(hu, hv) = d\omega(u, v) + [\omega(u), \omega(v)] \tag{72}$$

for all vector fields $u, v \in \mathscr{X}(P)$. We can treat this case by case.

- Let u, v be horizontal. In this case there is nothing to show, since $\omega(u) = \omega(v) = 0$ and $hu = u$ and $hv = v$.
- Let u, v be vertical. Without loss of generality we can take $u = \sigma(X)$ and $v = \sigma(Y)$, for some $X, Y \in \mathfrak{g}$. Then equation (72) becomes

$$\begin{aligned} 0 &\overset{?}{=} d\omega(\sigma(X), \sigma(Y)) + [\omega(\sigma(X)), \omega(\sigma(Y))] \\ (\omega(\sigma(X)) = X, \text{etc}) \quad &= \sigma(X)Y - \sigma(Y)X - \omega([\sigma(X), \sigma(Y)]) + [X, Y] \\ &= -\omega([\sigma(X), \sigma(Y)]) + [X, Y] \\ ([\sigma(X), \sigma(Y)] = \sigma([X, Y])) \quad &= -\omega(\sigma([X, Y])) + [X, Y] , \end{aligned}$$

  which is clearly true.
- Finally, let u be horizontal and $v = \sigma(X)$ be vertical, whence equation (72) becomes

$$d\omega(u, \sigma(X)) = 0 ,$$

  which in turn reduces to

$$\omega([u, \sigma(X)]) = 0 .$$

  In other words, we have to show that the Lie bracket of a vertical and a horizontal vector field is again horizontal. But this is simply the infinitesimal version of the G-invariance of H.

□

An immediate consequence of this formula is the

Proposition 5.5 (Bianchi identity)**.**

$$h^* d\Omega = 0 .$$

Proof. This is simply a calculation using the structure equation:

$$\begin{aligned} h^* d\Omega &= h^* d\left(d\omega + \tfrac{1}{2}[\omega, \omega]\right) \\ &= h^* \left(\tfrac{1}{2}[d\omega, \omega] - \tfrac{1}{2}[\omega, d\omega]\right) \\ &= h^* [d\omega, \omega] \\ &= [h^* d\omega, h^* \omega] \\ &= 0 . \end{aligned}$$

□

5.2 Connections on vector bundles

A connection on a principal bundle allows us to define a covariant derivative (a.k.a. a Koszul connection) on sections of any associated vector bundle. If $E \to M$ is a vector bundle, we let $C^\infty(M, E)$ denote the space of smooth sections. If $s \in C^\infty(M, E)$ and $f \in C^\infty(M)$, then $fs \in C^\infty(M, E)$, where $(fs)(m) = f(m)s(m)$. This makes $C^\infty(M, E)$ into a $C^\infty(M)$-module. In fact, a celebrated theorem of Swann's (based on a theorem of Serre's in algebraic geometry) says that the category of smooth vector bundles on a (compact) manifold M is equivalent to the category of finitely-generated projective $C^\infty(M)$-modules.

5.2.1 Koszul connections

Notation

If $E \to M$ is a vector bundle, we let $\Omega^k(M, E)$ denote the space of sections of the vector bundle $\Lambda^k T^* M \otimes E$. If F is a vector space then $\Omega^k(M, F)$ denotes the F-valued k-forms on M, but they can also be interpreted as an example of the previous notation, where $E = M \times F$ is a trivial bundle.

Definition 5.6. A **Koszul connection** on a vector bundle $\pi : E \to M$ is a map $\nabla : C^\infty(M, E) \to \Omega^1(M, E)$ satisfying the following property:

$$\nabla(fs) = df \otimes s + f\nabla s \qquad \text{for all } f \in C^\infty(M) \text{ and } s \in C^\infty(M, E).$$

In other words, if $\xi \in \mathscr{X}(M)$ is a vector field then $\nabla_\xi : C^\infty(M, E) \to C^\infty(M, E)$ satisfies the following properties:

$$\nabla_{f\xi} s = f \nabla_\xi s \qquad \nabla_{\xi+\chi} s = \nabla_\xi s + \nabla_\chi s \qquad \text{and} \qquad \nabla_\xi(fs) = \xi(f)s + f\nabla_\xi s\,,$$

for all $\xi, \chi \in \mathscr{X}(M)$, $f \in C^\infty(M)$ and $s \in C^\infty(M, E)$.

We will now show how a connection on a principal bundle $P \to M$ defines a Koszul connection on any associated vector bundle $P \times_G F \to M$, but first we need to understand better the relation between forms on P and forms on M.

5.2.2 Basic forms

A k-form $\alpha \in \Omega^k(P)$ is **horizontal** if $h^*\alpha = \alpha$. A horizontal form which in addition is G-invariant is called **basic**. It is a basic fact (no pun intended) that α is basic if and only if $\alpha = \pi^*\bar{\alpha}$ for some k-form $\bar{\alpha}$ on M (hence the name). This story extends to forms on P taking values in a vector space F admitting a representation $\varrho : G \to GL(F)$ of G. Let α be such a form. Then α is **horizontal** if $h^*\alpha = \alpha$ and it is **invariant** if for all $g \in G$,

$$R_g^*\alpha = \varrho(g^{-1}) \circ \alpha\,.$$

If α is both horizontal and invariant, it is said to be **basic**. Basic forms are in one-to-one correspondence with forms on M with values in the associated bundle $P \times_G F$. Indeed, let

$$\Omega^k_G(P, F) = \left\{ \bar{\zeta} \in \Omega^k(P, F) \middle| h^*\bar{\zeta} = \bar{\zeta} \quad \text{and} \quad R_g^*\bar{\zeta} = \varrho(g^{-1}) \circ \bar{\zeta} \right\} \tag{73}$$

denote the basic forms on P with values in F. Then we have an isomorphism $\Omega^k_G(P, F) \cong \Omega^k(M, P \times_G F)$. The case $k = 0$ is particularly important. This is the an isomorphism between G-equivariant functions $P \to F$ (which are vacuously horizontal) and sections of $P \times_G F$.

It is instructive to prove the general result, though. To this end we need to introduce one more object.

Every principal fibre bundle admits local sections. In fact, a local trivialisation $\mathfrak{U} = \{U_\alpha\}$,

$$\begin{array}{ccc} \pi^{-1}U_\alpha & \xrightarrow{\ \psi_\alpha\ } & U_\alpha \times G \\ & \searrow^{\pi} \quad {}^{pr_1}\swarrow & \\ & U_\alpha & \end{array}$$

defines local sections $s_\alpha : U_\alpha \to \pi^{-1}U_\alpha$ by $\psi_\alpha(s_\alpha(m)) = (m, e)$, where e is the identity in G. Conversely, local sections $s_\alpha : U_\alpha \to \pi^{-1}U_\alpha$ define a trivialisation by $\psi_\alpha(s_\alpha(m)g) = (m, g)$. On overlaps, these sections are related by the transition functions of the bundle. Indeed, if $m \in U_\alpha \cap U_\beta$, then

$$\psi_\alpha(s_\beta(m)) = (\psi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta)(s_\beta(m)) = (\psi_\alpha \circ \psi_\beta^{-1})(m, e) = (m, g_{\alpha\beta}(m))\,, \tag{74}$$

whence $s_\beta(m) = s_\alpha(m) g_{\alpha\beta}(m)$. Using the local sections we can now prove the following:

Proposition 5.7. *We have an isomorphism of* $C^\infty(M)$*-modules*

$$\Omega^k_G(P,F) \cong \Omega^k(M, P\times_G F)\ .$$

Proof. We will only give the construction and let the verification to the reader. If $\bar\zeta \in \Omega^k_G(P,F)$, let $\zeta_\alpha = s_\alpha^*\bar\zeta \in \Omega^k(U_\alpha, F)$. Then one shows that on $U_{\alpha\beta}$, $\zeta_\beta(m) = \varrho(g_{\alpha\beta}(m)^{-1})\zeta_\alpha(m)$, whence the $\{\zeta_\alpha\}$ define a section of $\Omega^k(M, P\times_G F)$. Conversely, if $\{\zeta_\alpha \in \Omega^k(U_\alpha, F)\}$ satisfy $\zeta_\beta(m) = \varrho(g_{\alpha\beta}(m)^{-1})\zeta_\alpha(m)$ for $m\in U_{\alpha\beta}$, we define $\bar\zeta_\alpha(p) = \varrho(g_\alpha^{-1})\circ\pi^*\zeta_\alpha$, where $g_\alpha : \pi^{-1}(m) \to G$ is defined by $\psi_\alpha(p) = (\pi(p), g_\alpha(p))$. The $\bar\zeta_\alpha$ are basic by construction and one simply checks that on $\pi^{-1}U_{\alpha\beta}$, $\bar\zeta_\alpha = \bar\zeta_\beta$. □

5.2.3 The covariant derivative

The exterior derivative $d : \Omega^k(P,F) \to \Omega^{k+1}(P,F)$ obeys $d^2 = 0$ and defines a complex: the F**-valued de Rham complex**. The invariant forms do form a subcomplex, but the basic forms do not, since $d\alpha$ need not be horizontal even if α is. Projecting onto the horizontal forms defines the **exterior covariant derivative**

$$d^\nabla : \Omega^k_G(P,F) \to \Omega^{k+1}_G(P,F) \qquad \text{by} \qquad d^\nabla\alpha = h^* d\alpha\ .$$

The price we pay is that $(d^\nabla)^2 \neq 0$ in general, so we no longer have a complex. Indeed, the failure of d^∇ defining a complex is again measured by the curvature of the connection.

Let us start by deriving a more explicit formula for the exterior covariant derivative on sections of $P\times_G F$. Every section $\zeta\in\Omega^0(M, P\times_G F)$ defines an equivariant function $\bar\zeta\in\Omega^0_G(P,F)$ obeying $R_g^*\bar\zeta = \varrho(g^{-1})\circ\bar\zeta$ and whose covariant derivative is given by $d^\nabla\bar\zeta = h^*d\bar\zeta$. Applying this to a vector field $u = u_V + hu \in \mathscr{X}(P)$,

$$(d^\nabla\bar\zeta)(u) = d\bar\zeta(hu) = d\bar\zeta(u - u_V) = d\bar\zeta(u) - u_V(\bar\zeta)\ .$$

The derivative $u_V\bar\zeta$ at a point p only depends on the value of u_V at that point, whence we can take $u_V = \sigma(\omega(u))$, so that

$$u_V\bar\zeta = \sigma(\omega(u))\bar\zeta = \frac{d}{dt}\Big|_{t=0} R^*_{g(t)}\bar\zeta \qquad \text{for } g(t) = \exp(t\omega(u)).$$

By equivariance,

$$u_V\bar\zeta = \frac{d}{dt}\Big|_{t=0}\varrho(g(t)^{-1})\circ\bar\zeta = -\varrho(\omega(u))\circ\bar\zeta\ ,$$

where we also denote by $\varrho : \mathfrak{g}\to\mathfrak{gl}(F)$ the representation of the Lie algebra. In summary,

$$(d^\nabla\bar\zeta)(u) = d\bar\zeta(u) + \varrho(\omega)(u)\circ\bar\zeta$$

or, abstracting u,

$$d^\nabla\bar\zeta = d\bar\zeta + \varrho(\omega)\circ\bar\zeta\ . \tag{75}$$

This form is clearly horizontal by construction, and it is also invariant:

$$\begin{aligned}
R_g^* d^\nabla\bar\zeta &= R_g^* h^* d\bar\zeta \\
\text{(since H is invariant)} \quad &= h^* R_g^* d\bar\zeta \\
\text{(since } d \text{ commutes with pull-backs)} \quad &= h^* d R_g^*\bar\zeta \\
\text{(equivariance of } \bar\zeta) \quad &= h^* d\left(\varrho(g^{-1})\circ\bar\zeta\right) \\
&= \varrho(g^{-1})\circ h^* d\bar\zeta \\
&= \varrho(g^{-1})\circ d^\nabla\bar\zeta\ .
\end{aligned}$$

As a result, it is a basic form and hence comes from a 1-form $\nabla\zeta\in\Omega^1(M, P\times_G F)$. In this way, we have defined a Koszul connection

$$\nabla : C^\infty(M, P\times_G F) \to \Omega^1(M, P\times_G F)\ .$$

This story extends to k-forms in the obvious way. Let $\alpha \in \Omega^k(M, P \times_G F)$ and represent it by a basic form $\bar\alpha \in \Omega^k_G(P,F)$. Define $d^\nabla \bar\alpha = h^* d\bar\alpha$. Then one can show that

$$d^\nabla \bar\alpha = d\bar\alpha + \varrho(\omega) \wedge \bar\alpha \in \Omega^{k+1}_G(P,F) ,$$

where $\wedge$ denotes both the wedge product of forms and the composition of the components of $\varrho(\omega)$ with $\bar\alpha$, whence it defines an element $d^\nabla \alpha \in \Omega^{k+1}(M, P \times_G F)$. Contrary to the exterior derivative, $(d^\nabla)^2 \bar\zeta \neq 0$ in general. Instead, for $\bar\zeta \in \Omega^0_G(P,F)$, we have

$$\begin{aligned}
(d^\nabla)^2 \bar\zeta &= h^* d h^* d\bar\zeta \\
&= h^* d\left(d\bar\zeta + \varrho(\omega) \circ \bar\zeta\right) \\
&= h^* \left(\varrho(d\omega) \circ \bar\zeta - \varrho(\omega) \wedge d\bar\zeta\right) \\
\text{(since } h^*\omega = 0\text{)} \qquad &= \varrho(h^* d\omega) \circ \bar\zeta \\
&= \varrho(\Omega) \circ \bar\zeta .
\end{aligned}$$

More generally, if $\bar\alpha \in \Omega^k_G(P,F)$, we have

$$(d^\nabla)^2 \bar\alpha = \varrho(\Omega) \wedge \bar\alpha ,$$

whence the curvature measures the obstruction of the exterior covariant derivative to define a complex.

5.2.4 Gauge fields

We often need to do explicit calculations with objects in the base manifold of a fibre bundle and we need to have an expression for the covariant derivative of, say, a section of $P \times_G F$ explicitly and not just in terms of the G-equivariant functions $P \to F$. This requires the introduction of locally defined 1-forms which go by the name of gauge fields. More precisely, the connection 1-form ω on a principal fibre bundle pulls back to the base via any local section. In particular we can use the local sections s_α associated to a trivialisation to define $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ by $A_\alpha = s_\alpha^* \omega$. One can show that on overlaps $U_{\alpha\beta}$,

$$A_\alpha(m) = g_{\alpha\beta}(m) A_\beta(m) g_{\alpha\beta}(m)^{-1} - d g_{\alpha\beta} g_{\alpha\beta}^{-1} , \tag{76}$$

in a notation appropriate to matrix groups. Conversely given $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ subject to equation (76) on overlaps, we can define $\omega_\alpha \in \Omega^1(\pi^{-1} U_\alpha, \mathfrak{g})$ by

$$\omega_\alpha = \mathrm{Ad}_{g_\alpha^{-1}} \circ \pi^* A_\alpha + g_\alpha^{-1} d g_\alpha , \tag{77}$$

where the second term on the right-hand side is the pullback by g_α of the left-invariant Maurer–Cartan 1-form on G, again in a notation appropriate to matrix groups. One checks that on $\pi^{-1} U_{\alpha\beta}$, $\omega_\alpha = \omega_\beta$, whence it does define a global one-form $\omega \in \Omega^1(P, \mathfrak{g})$. One finally verifies that it is a connection one-form.

This means that we have now three ways to think of connections on a principal fibre bundle: as invariant horizontal distributions, as connection one-forms or as gauge fields. Each way has its virtue and it's convenient to understand all three and how they are related.

Back to the covariant derivative, letting $E = P \times_G F$, we define $d^\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E)$ by the commutativity of the following diagram:

$$\begin{array}{ccc}
\Omega^k_G(P,F) & \xrightarrow{h^* d} & \Omega^{k+1}_G(P,F) \\
\updownarrow & & \updownarrow \\
\Omega^k(M,E) & \xrightarrow{d^\nabla} & \Omega^{k+1}(M,E)
\end{array}$$

For example, if $\{\sigma_\alpha : U_\alpha \to F\}$ defines a section $\sigma \in C^\infty(M,E)$, then on U_α, $d^\nabla \sigma$ is represented by

$$d^\nabla \sigma_\alpha = d\sigma_\alpha + \varrho(A_\alpha) \sigma_\alpha .$$

Then on overlaps, we have

(78) $$d^{\nabla}\sigma_{\alpha} = \varrho(g_{\alpha\beta})d^{\nabla}\sigma_{\beta} \ ,$$

which earns the derivative d^{∇} the adjective 'covariant'.

Often we write simply $\nabla\sigma$ for $d^{\nabla}\sigma$ when σ is a section. The curvature 2-form R^{∇} associated to ∇ is the section of $\Omega^2(M, \operatorname{End} E)$ given by $R^{\nabla} = d^{\nabla} \circ \nabla$, or explicitly,

(79) $$R^{\nabla}(X,Y)\sigma = \nabla_{[X,Y]}\sigma - \nabla_X\nabla_Y\sigma + \nabla_Y\nabla_X\sigma \ ,$$

for all $X, Y \in \mathscr{X}(M)$.