Lecture 2: Curvature

In this lecture we will define the curvature of a connection on a principal fibre bundle and interpret it geometrically in several different ways. Along the way we define the covariant derivative of sections of associated vector bundles. Throughout this lecture, $\pi : P \to M$ will denote a principal G-bundle.

2.1 The curvature of a connection

2.1.1 The horizontal projection

Given a connection $H \subset TP$, we define the **horizontal projection** $h : TP \to TP$ to be the projection onto the horizontal distribution along the vertical distribution. It is a collection of linear maps $h_p : T_pP \to T_pP$, for every $p \in P$, defined by

$$h_p(v) = \begin{cases} v & \text{if } v \in H_p, \text{ and} \\ 0 & \text{if } v \in V_p. \end{cases}$$

In other words, $\operatorname{im} h = H$ and $\ker h = V$. Since both H and V are invariant under the the action of G, the horizontal projection is equivariant:

$$h \circ (R_g)_* = (R_g)_* \circ h \,.$$

We will let $h^* : T^*P \to T^*P$ denote the dual map, whence if, say, $\alpha \in \Omega^1(P)$ is a one-form, $h^*\alpha = \alpha \circ h$. More generally if $\beta \in \Omega^k(P)$, then $(h^*\beta)(v_1, \ldots, v_k) = \beta(hv_1, \ldots, hv_k)$. However...

> Despite the notation, h^* is *not* the pull-back by a smooth map! In particular, h^* will *not* commute with the exterior derivative d!

2.1.2 The curvature 2-form

Let $\omega \in \Omega^1(P; \mathfrak{g})$ be the connection one-form for a connection $H \subset TP$. The 2-form $\Omega := h^* d\omega \in \Omega^2(P; \mathfrak{g})$ is called the **curvature (2-form)** of the connection. We will derive more explicit formulae for Ω later on, but first let us interpret the curvature geometrically.

By definition,

$$\begin{aligned} \Omega(u, v) &= d\omega(hu, hv) \\ &= (hu)\omega(hv) - (hv)\omega(hu) - \omega([hu, hv]) \\ \text{(since } h^*\omega = 0) \qquad &= -\omega([hu, hv]) \,; \end{aligned}$$

whence $\Omega(u, v) = 0$ if and only if $[hu, hv]$ is horizontal. In other words, the curvature of the connection measures the failure of integrability of the horizontal distribution $H \subset TP$.

> **Frobenius integrability**
>
> A distribution $D \subset TP$ is said to be integrable if the Lie bracket of any two sections of D lies again in D. The theorem of Frobenius states that a distribution is integrable if every $p \in P$ lies in a unique submanifold of P whose tangent space at p agrees with the subspace $D_p \subset T_pP$. These submanifolds are said to *foliate* P. As we have just seen, a connection $H \subset TP$ is integrable if and only if its curvature 2-form vanishes.
> In contrast, the vertical distribution $V \subset TP$ is always integrable, since the Lie bracket of two vertical vector fields is again vertical, and Frobenius's theorem guarantees that P is foliated by submanifolds whose tangent spaces are the vertical subspaces. These submanifolds are of course the fibres of $\pi : P \to M$.

The integrability of a distribution has a dual formulation in terms of differential forms. A horizontal distribution $H = \ker\omega$ is integrable if and only if (the components of) ω generate a differential ideal, so that $d\omega = \Theta \wedge \omega$, for some $\Theta \in \Omega^1(P; \mathrm{End}(\mathfrak{g}))$. Since Ω measures the failure of integrability of H, the following formula should not come as a surprise.

Proposition 2.1 (Structure equation)**.**

$$\boxed{\Omega = d\omega + \tfrac{1}{2}[\omega, \omega]\ ,}$$

where, as before, $[-,-]$ *is the* symmetric *bilinear product consisting of the Lie bracket on* $\mathfrak{g}$ *and the wedge product of one-forms.*

Proof. We need to show that

$$d\omega(hu, hv) = d\omega(u, v) + [\omega(u), \omega(v)] \tag{9}$$

for all vector fields $u, v \in \mathscr{X}(P)$. We can treat this case by case.

- Let u, v be horizontal. In this case there is nothing to show, since $\omega(u) = \omega(v) = 0$ and $hu = u$ and $hv = v$.
- Let u, v be vertical. Without loss of generality we can take $u = \sigma(X)$ and $v = \sigma(Y)$, for some $X, Y \in \mathfrak{g}$. Then equation (9) becomes

$$\begin{aligned} 0 &\stackrel{?}{=} d\omega(\sigma(X), \sigma(Y)) + [\omega(\sigma(X)), \omega(\sigma(Y))] \\ (\omega(\sigma(X)) = X, \text{ etc}) \qquad &= \sigma(X)Y - \sigma(Y)X - \omega([\sigma(X), \sigma(Y)]) + [X, Y] \\ &= -\omega([\sigma(X), \sigma(Y)]) + [X, Y] \\ ([\sigma(X), \sigma(Y)] = \sigma([X, Y])) \qquad &= -\omega(\sigma([X, Y])) + [X, Y]\ , \end{aligned}$$

  which is clearly true.
- Finally, let u be horizontal and $v = \sigma(X)$ be vertical, whence equation (9) becomes

$$d\omega(hu, \sigma(X)) = 0\ ,$$

  which in turn reduces to

$$\omega([hu, \sigma(X)]) = 0\ .$$

  In other words, we have to show that the Lie bracket of a vertical and a horizontal vector field is again horizontal. But this is simply the infinitesimal version of the G-invariance of H.

□

An immediate consequence of this formula is the

Proposition 2.2 (Bianchi identity)**.**

$$h^* d\Omega = 0\ .$$

Proof. This is simply a calculation using the structure equation:

$$\begin{aligned} h^* d\Omega &= h^* d\left(d\omega + \tfrac{1}{2}[\omega, \omega]\right) \\ &= h^*\left(\tfrac{1}{2}[d\omega, \omega] - \tfrac{1}{2}[\omega, d\omega]\right) \\ &= h^*[d\omega, \omega] \\ &= [h^* d\omega, h^*\omega] \\ &= 0\ . \end{aligned}$$

□

Under a gauge transformation $\Phi : P \to P$, the connection one-form changes by $\omega \mapsto \omega^\Phi = (\Phi^{-1})^*\omega$. The curvature also transforms in this way.

Done? ❑ **Exercise 2.1.** Show that under a gauge transformation $\Phi : P \to P$, the horizontal projections h, h^Φ of H and H^Φ are related by

$$h^\Phi = \Phi_* h \Phi_*^{-1} .$$

Deduce that the curvature 2-form transforms as

$$\Omega \mapsto \Omega^\Phi = (\Phi^{-1})^* \Omega .$$

(This can also be shown directly from the structure equation.)

2.1.3 Gauge field-strengths

Pulling back Ω via the canonical sections $s_\alpha : U_\alpha \to P$ yields the **gauge field-strength** $F_\alpha := s_\alpha^* \Omega \in \Omega^2(U_\alpha; \mathfrak{g})$. It follows from the structure equation that

$$F_\alpha = dA_\alpha + \tfrac{1}{2}[A_\alpha, A_\alpha] . \tag{10}$$

As usual, the natural question to ask is how do F_α and F_β differ on $U_{\alpha\beta}$. From equation (4), using the Maurer–Cartan structure equation $d\theta = -\frac{1}{2}[\theta, \theta]$ and simplifying, we find

$$F_\alpha = \mathrm{ad}_{g_{\alpha\beta}} \circ F_\beta \tag{11}$$

or, for matrix groups,

$$F_\alpha = g_{\alpha\beta} F_\beta g_{\alpha\beta}^{-1} .$$

In other words, the $\{F_\alpha\}$ define a global 2-form $F \in \Omega^2(M; \mathrm{ad}\, P)$ with values in $\mathrm{ad}\, P$. We may sometimes write F_A if we want to make the dependence on the gauge fields manifest.

Done? ❑ **Exercise 2.2.** Show that the gauge-transformed field-strength is given by

$$F_\alpha^\Phi = \mathrm{ad}_{\phi_\alpha} \circ F_\alpha .$$

2.2 The covariant derivative

A connection allows us to define a "covariant" derivative on sections of associated vector bundles to $P \to M$, but first we need to understand better the relation between forms on P and forms on M.

2.2.1 Basic forms

A k-form $\alpha \in \Omega^k(P)$ is **horizontal** if $h^*\alpha = \alpha$. A horizontal form which in addition is G-invariant is called **basic**. It is a basic fact (no pun intended) that α is basic if and only if $\alpha = \pi^*\bar{\alpha}$ for some k-form $\bar{\alpha}$ on M (hence the name). This story extends to forms on P taking values in a vector space V admitting a representation $\varrho : G \to GL(V)$ of G. Let α be such a form. Then α is **horizontal** if $h^*\alpha = \alpha$ and it is **invariant** if for all $g \in G$,

$$R_g^* \alpha = \varrho(g^{-1}) \circ \alpha .$$

If α is both horizontal and invariant, it is said to be **basic**. Basic forms are in one-to-one correspondence with forms on M with values in the associated bundle $P \times_G V$. Indeed, let

$$\Omega_G^k(P; V) = \left\{ \bar{\zeta} \in \Omega^k(P; V) \,\middle|\, h^*\bar{\zeta} = \bar{\zeta} \quad \text{and} \quad R_g^*\bar{\zeta} = \varrho(g^{-1}) \circ \bar{\zeta} \right\} \tag{12}$$

denote the basic forms on P with values in V. The k-forms on M with values in the associated bundle $P \times_G V$ are best described relative to a trivialisation of P as a family $\zeta_\alpha \in \Omega^k(U_\alpha; V)$ subject to the gluing condition

$$\zeta_\alpha = \varrho(g_{\alpha\beta}) \circ \zeta_\beta \tag{13}$$

on nonempty overlaps $U_{\alpha\beta}$. Let $\Omega^k(M; P \times_G V)$ denote the space of such bundle-valued forms. We will now construct isomorphisms

$$\Omega^k_G(P;V) \rightleftarrows \Omega^k(M; P \times_G V)$$

as follows in terms of local data.

Let $\bar{\zeta} \in \Omega^k_G(P;V)$ and define $\zeta_\alpha = s_\alpha^* \bar{\zeta} \in \Omega^k(U_\alpha; V)$.

Done? ❑ **Exercise 2.3.** Show that the $\{\zeta_\alpha\}$ define a form in $\Omega^k(M; P \times_G V)$, by showing that equation (13) is satisfied on nonempty overlaps.

Conversely, if $\zeta_\alpha \in \Omega^k(U_\alpha; V)$ define a form in $\Omega^k(M; P \times_G V)$, then define

$$\bar{\zeta}_\alpha := \varrho(g_\alpha^{-1}) \circ \pi^* \zeta_\alpha \in \Omega^k(\pi^{-1} U_\alpha; V) .$$

Done? ❑ **Exercise 2.4.** Show that $\bar{\zeta}_\alpha$ is the restriction to $\pi^{-1}U_\alpha$ of a basic form $\bar{\zeta} \in \Omega^k_G(P;V)$.

Finally we observe that these two constructions are mutual inverses, hence they define the desired isomorphism. This isomorphism is very useful: it allows us to work with bundle-valued forms on M either locally in terms of a trivilisation or globally on P subject to an equivariance condition.

2.2.2 The covariant derivative

The exterior derivative $d : \Omega^k(P;V) \to \Omega^{k+1}(P;V)$ obeys $d^2 = 0$ and defines a complex: the V-**valued de Rham complex**. The invariant forms do form a subcomplex, but the basic forms do not, since $d\alpha$ need not be horizontal even if α is. Projecting onto the horizontal forms defines the **exterior covariant derivative**

$$d^H : \Omega^k_G(P;V) \to \Omega^{k+1}_G(P;V) \qquad \text{by} \qquad d^H\alpha = h^* d\alpha .$$

The price we pay is that $(d^H)^2 \neq 0$ in general, so we no longer have a complex. Indeed, the failure of d^H defining a complex is again measured by the curvature of the connection.

Let us start by deriving a more explicit formula for the exterior covariant derivative on sections of $P \times_G V$. Every section $\zeta \in \Omega^0(M; P \times_G V)$ defines an equivariant function $\bar{\zeta} \in \Omega^0_G(P;V)$ obeying $R_g^* \bar{\zeta} = \varrho(g^{-1}) \circ \bar{\zeta}$ and whose exterior covariant derivative is given by $d^H\bar{\zeta} = h^* d\bar{\zeta}$. Applying this to a vector field $u = u_V + hu \in \mathscr{X}(P)$,

$$(d^H\bar{\zeta})(u) = d\bar{\zeta}(hu) = d\bar{\zeta}(u - u_V) = d\bar{\zeta}(u) - u_V(\bar{\zeta}) .$$

The derivative $u_V\bar{\zeta}$ at a point p only depends on the value of u_V at that point, whence we can take $u_V = \sigma(\omega(u))$, so that

$$u_V\bar{\zeta} = \sigma(\omega(u))\bar{\zeta} = \frac{d}{dt}\Big|_{t=0} R^*_{g(t)}\bar{\zeta} \qquad \text{for } g(t) = e^{t\omega(u)}.$$

By equivariance,

$$u_V\bar{\zeta} = \frac{d}{dt}\Big|_{t=0} \varrho(g(t)^{-1}) \circ \bar{\zeta} = -\varrho(\omega(u)) \circ \bar{\zeta} ,$$

where we also denote by $\varrho : \mathfrak{g} \to \mathrm{End}(V)$ the representation of the Lie algebra. In summary,

$$(d^H\bar{\zeta})(u) = d\bar{\zeta}(u) + \varrho(\omega)(u) \circ \bar{\zeta}$$

or, abstracting u,

$$d^H\bar{\zeta} = d\bar{\zeta} + \varrho(\omega) \circ \bar{\zeta} . \tag{14}$$

This form is clearly horizontal by construction, and it is also invariant:

$$\begin{aligned} R_g^* d^H\bar{\zeta} &= R_g^* h^* d\bar{\zeta} \\ &= h^* R_g^* d\bar{\zeta} && \text{(since H is invariant)} \\ &= h^* d R_g^* \bar{\zeta} && \text{(since } d \text{ commutes with pull-backs)} \\ &= h^* d\left(\varrho(g^{-1}) \circ \bar{\zeta}\right) && \text{(equivariance of } \bar{\zeta}) \\ &= \varrho(g^{-1}) \circ h^* d\bar{\zeta} \\ &= \varrho(g^{-1}) \circ d^H\bar{\zeta} \,. \end{aligned}$$

As a result, it is a basic form and hence comes from a 1-form $d^H\zeta \in \Omega^1(M; P \times_G V)$. In this way, we have defined a covariant exterior derivative

$$d^H : \Omega^0(M; P \times_G V) \to \Omega^1(M; P \times_G V) \,.$$

Contrary to the exterior derivative, $(d^H)^2\bar{\zeta} \neq 0$ in general. Instead,

$$\begin{aligned} (d^H)^2\bar{\zeta} &= h^* d h^* d\bar{\zeta} \\ &= h^* d\left(d\bar{\zeta} + \varrho(\omega) \circ \bar{\zeta}\right) \\ &= h^*\left(\varrho(d\omega) \circ \bar{\zeta} - \varrho(\omega) \wedge d\bar{\zeta}\right) \\ &= \varrho(h^* d\omega) \circ \bar{\zeta} && \text{(since } h^*\omega = 0) \\ &= \varrho(\Omega) \circ \bar{\zeta} \,. \end{aligned}$$

In other words, the curvature measures the obstruction of the exterior covariant derivative to define a de-Rham-type complex.

This story extends to k-forms in the obvious way. Let $\alpha \in \Omega^k(M; P \times_G V)$ and represent it by a basic form $\bar{\alpha} \in \Omega^k_G(P; V)$. Define $d^H\bar{\alpha} = h^* d\bar{\alpha}$.

Done? ❑ **Exercise 2.5.** Show that

$$d^H\bar{\alpha} = d\bar{\alpha} + \varrho(\omega) \wedge \bar{\alpha} \in \Omega^{k+1}_G(P; V) \,,$$

where $\wedge$ denotes both the wedge product of forms and the composition of the components of $\varrho(\omega)$ with $\bar{\alpha}$, whence it defines an element $d^H\alpha \in \Omega^{k+1}(M; P \times_G V)$. Furthermore, show that

$$(d^H)^2\bar{\alpha} = \varrho(\Omega) \wedge \bar{\alpha} \,.$$

Let us derive a formula for the covariant derivative of a section $\zeta \in \Omega^k(M; P \times_G V)$ defined locally by a family of forms $\zeta_\alpha \in \Omega^k(U_\alpha; V)$, such that on every nonempty overlap $U_{\alpha\beta}$,

$$\zeta_\alpha = \varrho(g_{\alpha\beta}) \circ \zeta_\beta \,.$$

As seen before, $\zeta_\alpha = s_\alpha^* \bar{\zeta}$ for $\bar{\zeta} \in \Omega^k(P; V)$. We define the covariant derivative $d^H\zeta_\alpha$ by pulling back $d^H\bar{\zeta}$ via the canonical section s_α:

$$\begin{aligned} d^H\zeta_\alpha := s_\alpha^* d^H\bar{\zeta} &= s_\alpha^*\left(d\bar{\zeta} + \varrho(\omega) \wedge \bar{\zeta}\right) \\ &= d s_\alpha^* \bar{\zeta} + \varrho(s_\alpha^*\omega) \wedge s_\alpha^* \bar{\zeta} \\ &= d\zeta_\alpha + \varrho(A_\alpha) \wedge \zeta_\alpha \,. \end{aligned}$$

It is not hard to see, using the transformation properties of A_α and ζ_α on overlaps that on $U_{\alpha\beta}$,

$$d^H\zeta_\alpha = \varrho(g_{\alpha\beta}) \circ d^H\zeta_\beta \,.$$

This result justifies the name "covariant derivative" as used in the Physics literature.

Notation

We will change notation and write the exterior covariant derivative on basic forms as

$$d^{\omega} : \Omega^k_G(P; V) \to \Omega^{k+1}_G(P; V) ,$$

to make manifest the dependence on the connection one-form, and the one on bundle-valued forms on M by

$$d_A : \Omega^k(M; P \times_G V) \to \Omega^{k+1}(M; P \times_G V) ,$$

to make manifest the dependence on the gauge field. For example, in this notation, the Bianchi identity for the curvature can be rewritten as $d_A F_A = 0$.