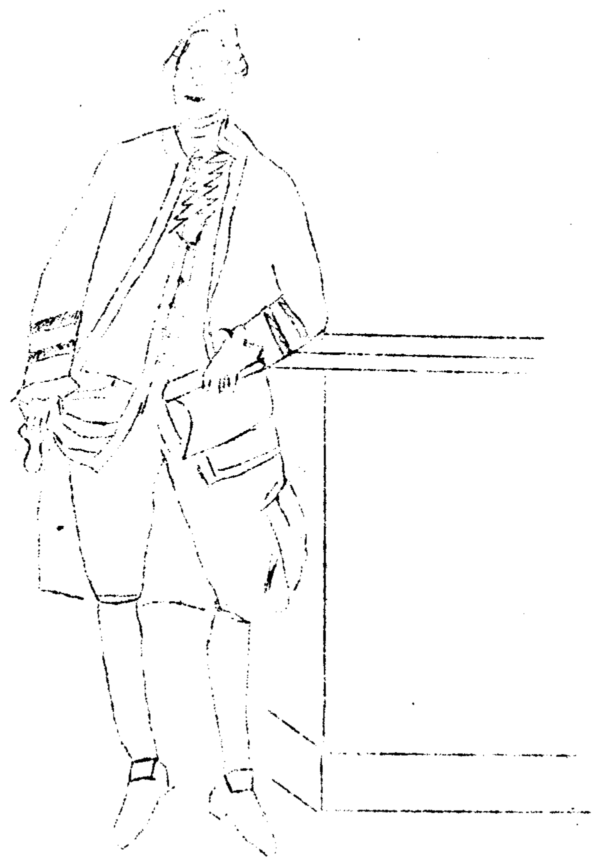
JAMES COOK MATHEMATICAL NOTES

Issue number 7, in January, 1977



This sketch is from Gainsborough's portrait of John Montagu, (1718-1792) fourth Earl of Sandwich. He is best remembered for having beef put between two slices of toast to sustain himself when spending twenty-four hours at gambling in about 1762. The word "sandwich" passed in to the English language soon afterwards. He was First Lord of the Admiralty and James Cook named the Sandwich Islands after him, but afterwards the Polynesian name of Hawaii was adopted for this island group.

From "Eureka" for November 1976 we copy the following clerihew, with thanks to Miss Onymous.

James Cook, Australia-bound
In 1774, on a hunch,
Stopped at the Sandwich Islands
For lunch.

Ann Onymous.

FUNCTIONS OF TWO VARIABLES

From E.R. Love comes this solution to the problem of G.R. Morris in JCMN 6.

<u>To construct $f(x, y)$ differentiable in the whole (x, y)-plane and such that the partial derivative $f_y(x, 0)$ has a jump discontinuity at $x = 0$.</u>

Let $f(x, y) = \dfrac{x^3 y}{x^3 + y^2}$ for $x > 0$, $f(x, y) = 0$ for $x \leqq 0$. (1)

Then $\dfrac{\partial f}{\partial x}(x, y) = \dfrac{3x^2y^3}{(x^3 + y^2)^2}$ for $x > 0$, $\dfrac{\partial f}{\partial x}(x, y) = 0$ for $x < 0$. (2)

Since $\dfrac{f(h, y) - f(0, y)}{h} = \dfrac{h^2 y}{h^3 + y^2} \to 0$ as $h \to 0+$, whether $y = 0$ or not, and since this ratio is 0 for $h < 0$, we have existence of $f_x(0, y)$ and

$$\frac{\partial f}{\partial x}(x, y) = 0 \quad \text{for } x = 0 \text{ and all } y. \tag{3}$$

Clearly $f_x(x, y)$ is continuous in $x > 0$ and in $x < 0$. To prove it continuous on $x = 0$, we have by (2) and (3), for $x > 0$ and fixed y',

$$\left|\frac{\partial f}{\partial x}(x, y) - \frac{\partial f}{\partial x}(0, y')\right| = \frac{3x^2|y|}{x^3 + y^2}\,\frac{y^2}{x^3 + y^2}$$

$$\leqq \frac{3}{2}x^{\frac{1}{2}}\,\frac{2x^{3/2}|y|}{x^3 + y^2} \leqq \frac{3}{2}x^{\frac{1}{2}} \leqq \frac{3}{2}\{x^2 + (y - y')^2\}^{1/4}.$$

The same resultant inequality holds for $x \leqq 0$ because the left side is then 0. The right side tends to 0 as $\{x^2 + (y - y')^2\}^{\frac{1}{2}} \to 0$; and so f_x is continuous at $(0, y')$, for each y'.

Going back to (1),

$\dfrac{\partial f}{\partial y}(x, y) = \dfrac{x^3(x^3 - y^2)}{(x^3 + y^2)^2}$ for $x > 0$, $\dfrac{\partial f}{\partial y}(x, y) = 0$ for $x \leqq 0$. (4)

Thus f has one partial derivative continuous and the other existing, everywhere; so f is differentiable in the whole plane.

Finally (4) shows that

$$f_y(x, 0) = 1 \quad \text{if} \quad x > 0, \qquad f_y(x, 0) = 0 \quad \text{if} \quad x \leqq 0,$$

so that $f_y(x, 0)$ has a jump discontinuity at $x = 0$.

A CAMBRIDGE PROBLEM

If real functions f and g in an interval have derivatives f' and g' such that $f'^2 + g'^2$ is Riemann-integrable, then does it follow that f' is Riemann-integrable?

It was J.C. Burkill who told me about this problem but he did not know the original author.

Perhaps I should mention to readers the general opinion that this problem is difficult.

MORE ANECDOTAGE

Remotely relevant to Dame Mary Cartwright's "Anecdote One" in JCMN 5 is an amusing way to find $\sum 1/n^2$. It was told to me years ago by J.W. Statton of Adelaide University.

A polynomial is determined, apart from a scale factor, by its zeros; consequently the infinite product for sin x is plausible, it gives the right zero and is approximately x for small x. In this way a lecturer should be able to persuade a first year class that

$$x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right) \ldots = \sin x = x - x^3/3! + x^5/5! \ldots$$

Equating coefficients of x^3 on both sides gives:

$$-(1/\pi^2)\sum 1/n^2 = -1/6 .$$

The more rigorous establishment of the infinite product for the sine is one of the unexpectedly difficult parts of elementary analysis. Can any reader give a proof not using complex function theory?

When I. Todhunter wrote his "Plane Trigonometry for the Use of Colleges and Schools" (Third edition 1864) he obtained the infinite product for the sine from the identity (for odd n)

$$\sin(n\theta) = n \sin\theta\,(1 - \sin^2\theta\,\mathrm{cosec}^2\,\pi/n)\,(1 - \sin^2\theta\,\mathrm{cosec}^2\,2\pi/n) \ldots$$

by putting $n\theta = x$ and letting n tend to infinity.

They were giants at manipulation in those days. Here are two "miscellaneous examples" from page 203 of the same book.

4. Find the roots of the equation $x^5 - 10x^3 + 20x - 8 = 0$.

5. A person wishes to ascertain the side BC of a triangular field ABC, but is only able to make measurements of <u>lines</u> within the boundary of a circle which passes through A and touches BC; shew how after measuring four lines he may determine BC.

Solutions are invited. For full marks they should be obtained without modern aids such as electric light, fountain pens or electronic calculators; it is unlikely that the readers for whom Todhunter wrote would have been taught Galois theory, and so perhaps that too should be taboo. Of course the exact answer was expected for the roots of the equation, not a numerical approximation.

Todhunter's book was published by the firm of Macmillan, founded by Daniel and Alexander Macmillan when they bought the bookshop of Bowes and Bowes (then at least 265 years old) in 1846. The shop is still on the corner of Trinity Street and Market Street in Cambridge, it is the oldest bookshop in Great Britain. Daniel Macmillan's grandson Harold as Prime Minister of the U.K. was a predecessor of Edward Heath who retraced parts of Captain Cook's first and third Pacific voyages in reverse to win the Sydney-Hobart yacht race of Christmas 1969.

CAPTAIN COOK'S BOOTS BACK

Captain Cook's boots, which went walkabout two years ago, are back in safe-keeping.

They were found recently in a box at Como - the National Trust head-quarters in South Yarra, Melbourne - after being stolen from Captain Cook's cottage in Fitzroy Gardens in June, 1974.

A National Trust spokesman said the boots were in a reasonable condition, but obviously had done "a bit of travelling" since their disappearance.

They were on loan from Commander H.J.P. Adams, whose grandfather brought them to Australia in 1850.

The Trust's administrator, Col. S.R. Birch, said today that in view of the theft, the Trust would have to consult Commander Adams about where the boots would be housed.

Townsville Daily Bulletin,
Sat. Nov. 20th, 1976.

M.N. BREARLEY'S MATRIX PROBLEM

This was in Issue 4. If A and B are real symmetric $n \times n$ matrices with eigenvalues $\lambda_1, \ldots \lambda_r$, 0, 0..0, and 0, 0,..0, $\lambda_{r+1}, \ldots \lambda_n$ respectively, with all $\lambda_i \neq 0$, and if those of A + B are $\lambda_1, \ldots \lambda_n$, then AB = 0. Here is Max Kelly's solution. It depends on the following :-

<u>Theorem</u>. Let $\lambda_1, \ldots \lambda_n$ be real and non-zero. Let $d_1, \ldots d_n$ be real unit column vectors. Suppose that the matrix

$$C = \lambda_1 d_1 d_1^T + \ldots + \lambda_n d_n d_n^T$$

has $\lambda_1, \ldots \lambda_n$ as its eigenvalues, then $d_1, \ldots d_n$ are mutually orthogonal.

Proof: Let D be the matrix $D = (d_1, \ldots d_n)$. Then $C = D \Lambda D^T$ where Λ is the diagonal matrix $\Lambda = \text{diag } (\lambda_1, \ldots \lambda_n)$.

Therefore $\det C = \lambda_1 \lambda_2 \ldots \lambda_n (\det D)^2$ but $\det C = \lambda_1, \ldots \lambda_n$ and so $\det D = \pm 1$. Hadamard's inequality states that $|\det(d_1, \ldots d_n)| \leq ||d_1|| \ldots ||d_n||$, with equality if and only if either some $d_i = 0$ or all d_i are mutually orthogonal. It follows that they are mutually orthogonal. Q.E.D.

Using this theorem the result required comes out fairly simply.

Take orthogonal unit eigenvectors $a_1, \ldots a_r$ for A corresponding to eigenvalues $\lambda_1, \ldots \lambda_r$ respectively; similarly $b_{r+1}, \ldots b_n$ for B. Then $A = \sum \lambda_i a_i a_i^T$ and $B = \sum \lambda_i b_i b_i^T$.

$A + B = \sum_1^r \lambda_i a_i a_i^T + \sum_{r+1}^n \lambda_i b_i b_i^T$ has eigenvalues $\lambda_1, \ldots \lambda_n$ and so by the theorem each $a_i^T b_j = 0$, and therefore $A B = 0$.

GETTING INTO CYCLES

The phenomenon of integer sequences such that a_{n+1} is either $\frac{1}{2}a_n$ or $3a_n + 1$ according to whether a_n is even or odd was mentioned in JCMN 3. It has been extensively discussed in "Eureka" Vol 2 No 7 (Aug-Sept 1976) and is traced back to Lothar O. Collatz in his student days before World War II. H.S.M. Coxeter giving the fourth Felix Behrend Mwmorial Lecture in 1970 offered a prize of \$50 for a proof or \$100 for a counter-example. The fact that the sequence goes into a cycle has been checked by computer for starting values up to 10^9 and for negative starting values as far as -10^8. This information is taken from contributions by Charles W. Trigg and by the Editor (Léo Sauvé) in "Eureka" Vol. 2, pages 144-150. Another reference is "Tomorrow's Math" by C.S. Ogilvy (OUP 1972).

Some people may be tempted to think that a computer check up to 10^9 is almost conclusive. They could take warning from an example given by Golovina and Yaglom (Induction in Geometry, English translation published by D.C. Heath, Boston, Mass., page 2). Consider the hypothesis that $991x^2 + 1$ is never a perfect square. It would probably pass any computer verification because the smallest x for which the hypothesis fails is $x = 12,005,735,790,331,359,447,442,538,767$.

HAVE A GUESS

This question from Issue 3 was to find in what circumstances the equations $A(x) = \inf(B(y) + xy)$ and $B(y) = \sup(A(x) - xy)$ give a reciprocal pair of transforms. The answer is as follows:

If (i) $B(y)$ is finite for all y,
and (ii) $B(y)$ is convex, that is if $x < y < z$ then
$(z-x)\,B(y) \leq (z-y)\,B(x) + (y-x)\,B(z)$,
and (iii) $B(y)/(1 + |y|)$ tends to infinity with $|y|$,
then (i) $A(x)$ given by the first equation above is finite for all x,
and (ii) $A(x)$ is concave,
and (iii) $B(y)$ is given by the second equation above.

As to the history of these equations J.M. Hammersley writes "G-C. Rota once told me that the pair of reciprocal transforms are apparently 'well-known' and go under the name of 'maximum transform' but I cannot cite a reference."

SPANNING BY COLUMNS

H. Kestelman of University College London sends the following problem.

Let $p_1(x), \dots p_q(x)$ be q real polynomials in one variable, with no common divisor. If A is any real $n \times n$ matrix show that the columns of $p_1(A), \dots p_q(A)$ together span R^n.

NEW THEOREM (from JCMN 4).

"Almost every convex polyhedron is a tetrahedron."

The first reaction of some people to a proposition like this is to point out that as no prior probability distribution is specified the "almost every" is meaningless and the problem is ill posed. One of the faults of modern mathematical education is that we turn out graduates unable to cope with ill-posed problems, while most problems in real life are ill-posed.

Take a simple example. Can we give a meaning to the assertion that a triangle is almost certainly not isoceles? Any triangle XYZ in the plane can be represented in the obvious way as the point $(x_1, y_1, x_2, y_2, x_3, y_3)$ in six-dimensional space R^6. Any prior probability is a probability distribution on the space R^6, it could of course be concentrated at a single point, but when we think of taking a triangle at random that is not the sort of distribution that we are thinking of sampling. The sort of probability that we envisage in the context of taking a triangle at random must satisfy two axioms; it must be absolutely continuous and non-zero, so that the probability of choosing a point in a measurable subset E of R^6 is zero if and only if the Lebesgue measure of E is zero. On this understanding we may assert that for instance a triangle taken at random is almost certainly not right-angled, and that a random triangle has non-zero probability of being obtuse.

Now to get down to the problem. We may represent an n-hedron as a point in 3n-dimensional space, and the convex n-hedrons correspond to a subset (of positive L-measure) in R^{3n}. Given any set of four of the n vertices, the set of representative points for which these four vertices are coplanar is an algebraic variety of dimension 3n-1, and so has L-measure zero. Because the probability distribution is absolutely continuous it follows that the four vertices are almost certainly not coplanar. A similar argument may be applied to each face of the tetrahedron and so we may say that almost certainly every face is a triangle.

Let the number of faces, vertices and edges be F, V and E respectively. If every face is a triangle then $3F = 2E$.

Euler's formula is $F + V = E + 2$

Putting together these two results we have almost certainly

$$2V = F + 4$$

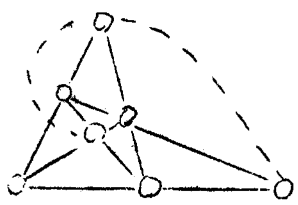
The theory of duality gives an automorphism of period two in the space of all convex polyhedra (actually there are several theories of duality but it is easy enough to pick one and make the assertion above precise) and duality interchanges faces with vertices, that is F with V, and so almost certainly $2F = V + 4$

Solving the pair of equations, $F = V = 4$, so that almost every convex polyhedron is a tetrahedron, Q.E.D.

PAPPUS, DESARGUES, ETC.

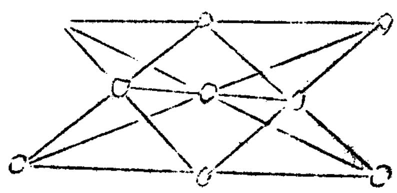
Start with a purely combinatorial concept. If p elements are in p sets so that each element is in precisely 3 sets and each set contains precisely 3 elements and the intersection of two sets contains at most one element, then let us call this a p - configuration. For what $p \geqq 4$ does there exist a p - configuration? and is there any p for which two distinct p - configurations exist?

Some geometrical figures exhibit p - configurations when the elements are interpreted as points and the sets as lines or curves.



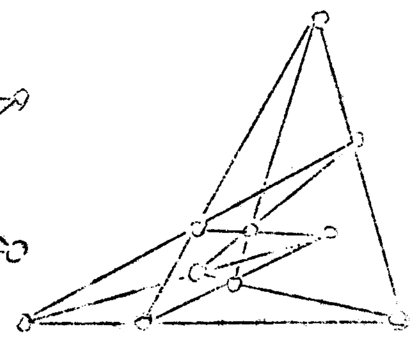
Complete quadrilateral

$p = 7$



Pappus

$p = 9$



Desargues

$p = 10$

Is your geometry rusty? Pappus says that if two lines each contain three points then the three cross-joins are collinear. Desargues says that if two triangles are in perspective then corresponding sides meet in collinear points. Note the rich symmetry in the three figures drawn above, for instance any one of the ten points in the Desargues figure may be seen as the centre of perspective of two triangles.

Now suppose that for some p we have a p - configuration, then we may formulate a proposition in plane geometry as follows. If p points are elements of the p - configuration and if p - 1 of the sets of the configuration consist of three collinear points, then the points of the other set are collinear. The figures above show that the proposition is false for $p = 7$ but true for $p = 9$ or 10, provided that we use the appropriate configuration (but are the configurations shown the only possible ones?)

Are there p - configurations for any other p that make the proposition true?

AN EASY PROBLEM

If $f(x)$ is positive, increasing and concave on $(0, \infty)$, is $g(x) = f(1/x)$ necessarily convex on $(0, \infty)$?

E.R. Love

RATIONAL ROOTS

From "Eureka" by kind permission of the editor Léo Sauvé we copy the following problem by M.G. Dworschak of Algonquin College, Ottawa.

Prove that for any positive integer $n > 1$ the equation

$$1 + 2x + 3x^2 + \dots + nx^{n-1} = n^2$$

has a rational root between 1 and 2.

C.F. MOPPERT'S ANECDOTE (from JCMN 5)

This was about the problem of finding a quadratic function

$$z = ax^2 + bxy + cy^2 + dx + ey + f$$

to take given values at six given (distinct) points of the plane.

One answer is as follows. If the points do not lie on any conic then given any values at the points it is possible to find a quadratic to fit these values. If the points are on some conic then in general it is not possible to fit a quadratic to six given values.

This suggests another problem. What conditions on five points are necessary or sufficient to ensure that a quadratic can be fitted to any set of values at the points?

SOCIAL SCIENCE

The Sydney Bulletin recently published the following quotation from "Triviata - A Compendium of Useless Information" by Timothy T. Fullerton.

A French magazine conducted a survey to investigate sexual behaviour in France. If the responses of those interviewed are to be believed, the average Frenchman sleeps with 11.8 women in his life, while the average Frenchwoman shares her bed with only 1.8 men.

PENNY GEOMETRY

Given a straight edge and a penny can you draw a perpendicular from a given point to a given straight line? A penny is a bronze disc, once a valuable bit of money, for our purposes the significance is that given two points not too far apart you can put the edge of the penny to the two points and draw a circle through them, and in fact there are in general two such circles through the given points.

Your editor would like to hear from you anything connected with mathematics or with James Cook.

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