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The James Cook Mathematical Notes has been published in 3 issues per year since 1979, but from this Issue 66 (April 1995) at the start of Volume 7, it will be irregular, appearing when enough contributions are avalable. The history of JCMN is that the first issue (a single foolscap sheet) appeared in September 1975, then others at irregular intervals, to number 17 in November 1978, then JCMN settled into the routine of three issues per year. The issues up to number 31 were produced and sent out free by the Mathematics Department of the James Cook University of North Queensland, of which I was then the Professor. In October 1983 this arrangement was beginning to be unsatisfactory, and since then I have been publishing the JCMN myself.

In October 1992 it had become clear that the paying of subscriptions by readers is an inefficient operation. Bank charges for changing currency and for international transfers, with postage, together absorb most of the initial input of money. Therefore we have abandoned subscriptions as from the beginning of 1993, issue number 60. I ask readers only to tell me every two years if they still want to have JCMN. To those who want to give something in return for the JCMN, I ask them to make a gift to an animal welfare society in their own country. The animals of the world will be grateful and so will I.

Contributors, please tell me if and how you would like your address printed.

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POWER MEAN INEQUALITY (JCNM 42, p.5020 & 65, p.6370)

P. H. Diananda
(Singapore)

The inequality

$$(x_1 + \dots + x_n)^k - (x_1^k + \dots + x_n^k) \geq (n^k - n)(x_1 x_2 \dots x_n)^{k/n}$$

(with n and k positive integers and the x_i all positive) may be proved as follows.

For k = 1 (with any n), for n = 1 (with any k) and for k = n = 2, the inequality clearly holds good, with the two sides equal. For other positive integer values of k and n, the l.h.s. is (using the multinomial theorem)

$$\Sigma \binom{k}{i_1, i_2, \dots, i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n},$$

where the summation is over all $i_1, \dots, i_n$ such that $0 \leq i_j < k$ and $\Sigma\, i_j = k$.
For this weighted sum of the products $x_1^{i_1} \dots x_n^{i_n}$, the sum of the weights is clearly $n^k - n$. Also, by considerations of cyclic symmetry, the weighted geometric mean is clearly $(x_1 \dots x_n)^{k/n}$.

Hence, by the inequality of the weighted arithmetic and geometric means, we have the result above. The two sides are equal if and only if the x_i are all equal (apart from the cases already noted, where k = 1 or n = 1 or n = k = 2).

An essentially similar proof was sent to us by Terry Tao of Princeton University.

Now consider the problem with k not a positive integer.

Suppose that $k < 0$. The original form of the inequality is reversed.

Proof Using the a.m. - g.m. inequality for the variables x_i^k $(i = 1, 2, \dots, n)$, $\Sigma\, x_i^k \geq n\, \Pi\, x_i^{k/n}$.
But also $\Sigma\, x_i \geq n\, \Pi\, x_i^{1/n}$, and therefore

$$(\Sigma\, x_i)^k \leq n^k\, \Pi\, x_i^{k/n} = n\, \Pi\, x_i^{k/n} + (n^k - n)\, \Pi\, x_i^{k/n}$$
$$\leq \Sigma\, x_i^k + (n^k - n)\, \Pi\, x_i^{k/n}.$$

There is equality if and only if the x_i are all equal.

There is equality if $k = 0$. The cases of $0 < k < 1$ and of $1 < k < 2$ and of non-integer $k > 2$ remain to be investigated.

OLD FASHIONED PROBLEM 2

Given three circles in space shew that there is in general one and only one circle that meets each of them twice.

—— Cambridge Mathematical Tripos, Part 1, Problem paper. Saturday, May 31, 1902, 9 - 12. Question 1.

QUOTATION CORNER 49

Society is constructed to defend the second-rater.

—— Nigel Balchin (novelist)
(Contributed by R. A. Lyttleton)

SYMMETRIC SIMULTANEOUS EQUATIONS

(JCMN 59, p.6173, 60, p.6192, 62, p.6276)

Harry Alexiev

(4, Antrim I Street, 4980, Zlatograd, Bulgaria)

The earlier contributions discussed the equations

$$x^2 - yz = a, \qquad y^2 - zx = b, \qquad z^2 - xy = c \qquad \ldots\ldots\ (1)$$

Another approach to (1) is to put $x = uz$ and $y = vz$, then the equations (1) may be regarded as equations for the three unknowns u, v and z. Instead of (1) we have:

$$u^2 - v = a/z^2, \qquad v^2 - u = b/z^2, \qquad 1 - uv = c/z^2 \quad \ldots.\ (2)$$

Hence $\dfrac{u^2 - v}{1 - uv} = \dfrac{a}{c}$ or $v\left(\dfrac{ua}{c} - 1\right) = \dfrac{a}{c} - u^2$.

Now two cases arise. If $ua = c$ then $a/c = u^2$, and $u^3 = 1$. If $ua \neq c$ then $v = \dfrac{a - cu^2}{ua - c}$, and from the equation $\dfrac{v^2 - u}{1 - uv} = \dfrac{b}{c}$ we conclude that

$$\left(\frac{a - cu^2}{ua - c}\right)^2 - u = \frac{b}{c} - u\frac{b}{c}\left(\frac{a - cu^2}{ua - c}\right) = \frac{b(u^3 - 1)}{ua - c}$$

This is a fourth degree equation in u,

$$(c^2 - ab)u^4 - (a^2 - bc)u^3 - (c^2 - ab)u + (a^2 - bc) = 0$$

which will factorize as

$$(u^3 - 1)[(c^2 - ab)u - (a^2 - bc)] = 0. \qquad \ldots.\ (3)$$

It is now clear that if a, b and c are such that $c^2 = ab$ and $a^2 = bc$ (and therefore either $a^3 = b^3 = c^3 = abc$ or $a = c = 0$), then equation (3) vanishes, and (1) requires special treatment.

Finally, let me suggest a new problem.

Solve $x^2 + yz = a, \quad y^2 + zx = b, \quad z^2 + xy = c.$

MATRIX INEQUALITY (JCMN 9, p.9-6 & 65 p.6375)

Terry Tao

(Mathematics Dept, Princeton University, NJ 08544, USA)

If a real square matrix M is positive definite (i.e. $x^T Mx > 0$ for all real column vectors $\mathbf{x} \neq 0$), prove that every principal sub-determinant of M is positive. (A principal sub-matrix is one obtained by deleting any subset of the set of rows and deleting the corresponding columns). Is the converse true?

<u>Proof</u> Let N be a principal sub-matrix of M. N is positive definite. Put $N = A + B$, where A is symmetric and B is skew-symmetric (or anti-symmetric, as some people call it). A is positive definite, and by the usual theory all its eigenvalues are > 0, and $\det A > 0$.

If $\det N \le 0$ then $f(\lambda) = \det(A + \lambda B)$, being a polynomial and therefore a continuous function of λ, must have a zero in the closed interval $[0, 1]$. If $\det(A + \lambda B) = 0$ then $A + \lambda B$ must have an eigenvector $\mathbf{e}$ with eigenvalue zero, i.e.

$$(A + \lambda B)\mathbf{e} = 0, \text{ and } \mathbf{e}^T A\mathbf{e} = -\lambda \mathbf{e}^T B\mathbf{e} = 0.$$

This is impossible because A is positive definite; therefore $\det N > 0$. Q.E.D

The converse is untrue.

<u>Disproof of converse</u>

Consider the 3×3 matrix $M = A + B$ where

$$A = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 3 & -2 \\ -2 & -2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The symmetric part A has eigenvectors:

(1, 1, 1) corresponding to eigenvalue -1,

(1, -1, 0) corresponding to eigenvalue 5,

and (1, 1, -2) corresponding to eigenvalue 5.

Thus A is not positive definite, and M is not positive definite. In fact $(1, 1, 1)M(1, 1, 1)^T$ = the sum of the 9 components of the matrix = -3. However,

$$\det M = \begin{vmatrix} 3 & 1 & -2 \\ -5 & 3 & -2 \\ -2 & -2 & 3 \end{vmatrix} = 2,$$

the 2×2 principal sub-determinants are all either

$$\begin{vmatrix} 3 & 1 \\ -5 & 3 \end{vmatrix} = 14 \quad\text{or}\quad \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = 5,$$

and the 1×1 principal sub-determinants (the diagonal elements) are all 3.

QUOTATION CORNER 50

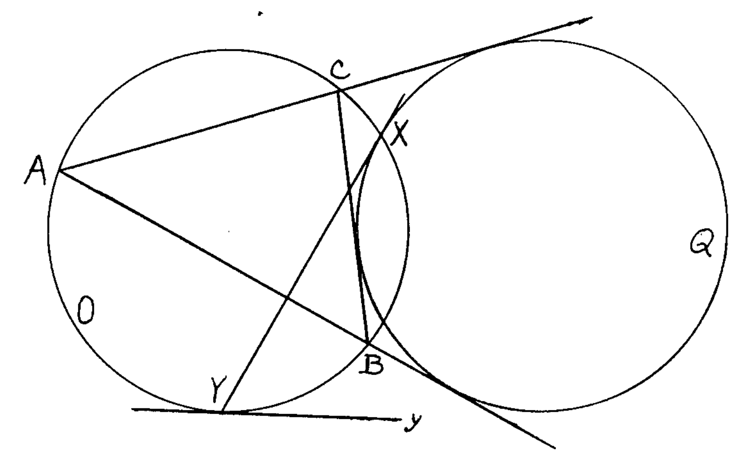
The offender never forgives.

—— Russian proverb.

(Contributed by R. A. Lyttleton)

PROBLEM ON CIRCLES (JCMN 65, p.6366)

Sahib Ram Mandan



In the drawing above, (O) is the circumcircle of ABC, and (Q) is an escribed circle. (Q) meets (O) at X, and the tangent to (Q) at X meets (O) again at Y. Prove that the tangent to (O) at Y is also tangent to (Q).

<u>Proof</u> The tangent y to (O) at Y may be regarded as the line YZ joining two coincident points Y and Z on the circle (O). We may regard XYZ as a degenerate triangle inscribed in the circle (O). The two sides XY and XZ both touch the circle (Q), and hence, by Poncelet's Porism, the third side, YZ, must also touch (Q).

SUMS GIVEN BY ZETA FUNCTIONS (JCMN 65, p.6360)

Harry Braden & Chris Smyth

(University of Edinburgh)

The note entitled THE DREADED ZETA THREE AGAIN, in JCMN 65 asked for a proof of the equation

$$\sum_{n=1}^{\infty} n^{-2} \sum_{m=1}^{n} \frac{1}{m} = 2\sum_{n=1}^{\infty} n^{-3} = 2\zeta(3)$$

because it appeared numerically to be true.

Also it was pointed out that the equation

$$\sum_{n=1}^{\infty} n^{-3} \sum_{m=1}^{n} \frac{1}{m} = \frac{5}{4}\zeta(4) = \pi^4/72$$

could be proved.

Now it appears that these two equations are part of a large family of related results.

SOME NUMERICAL VALUES

$\Sigma n^{-2} = \zeta(2) = 1{\cdot}644934066848 = \pi^2/6$

$\Sigma n^{-3} = \zeta(3) = 1{\cdot}202056903160$

$\Sigma n^{-4} = \zeta(4) = 1{\cdot}082323233711 = \pi^4/90$

$\Sigma n^{-5} = \zeta(5) = 1{\cdot}036927755143$

$\Sigma n^{-6} = \zeta(6) = 1{\cdot}017343061984 = \pi^6/945$

$\Sigma n^{-7} = \zeta(7) = 1{\cdot}008349277382$

$\Sigma n^{-8} = \zeta(8) = 1{\cdot}004077356198 = \pi^8/9450$

$\Sigma n^{-9} = \zeta(9) = 1{\cdot}002008392826$

Write $H(k)$ for $\sum_{n=1}^{\infty} n^{-k}(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n})$.

$H(2) = 2{\cdot}404113806320 = 2\zeta(3)$

$H(3) = 1{\cdot}352904042139 = \pi^4/72$

$H(4) = 1{\cdot}133478915133 = 3\zeta(5) - \zeta(2)\zeta(3)$

$H(5) = 1{\cdot}057879959256 = \pi^6/540 - \zeta(3)^2/2$

$H(6) = 1{\cdot}026705205699 = 4\zeta(7)-\zeta(2)\zeta(5)-\zeta(3)\zeta(4)$

$H(7) = 1{\cdot}012727885298 = \pi^8/4200 - \zeta(3)\zeta(5)$

$H(8) = 1{\cdot}006178634872 = 5\zeta(9) - \zeta(2)\zeta(7) - \zeta(3)\zeta(6) - \zeta(4)\zeta(5)$

These equations can be proved, as follows. If a and b are any two positive integers, put

$$S(a, b) = \sum_{1}^{\infty}\sum_{1}^{\infty} m^{-a}n^{-b}(m + n)^{-1}$$

Lemma 1 $S(1, b) = S(b, 1) = H(b + 1)$

Proof $\Sigma\Sigma\, m^{-1}n^{-b}(m + n)^{-1} = \Sigma\, n^{-b-1} \Sigma \frac{m + n - m}{m(m + n)}$

$$= \sum_{n=1}^{\infty} n^{-b-1} \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m + n}\right) = \sum_{n=1}^{\infty} n^{-b-1} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)$$

Lemma 2 $H(2) = 2\zeta(3)$.

Proof Taking the sum over the reverse diagonals, where $m + n$ is constant, $S(1, 1) = \Sigma\, (m + n)^{-2}\, \Sigma\, (1/m + 1/n)$

(now put t for m + n) $= \sum_{t=2}^{\infty} t^{-2} \sum_{m=1}^{t-1} 2/m = 2(H(2) - \zeta(3))$

Thus $H(2) = 2H(2) - 2\zeta(3)$. QED

Lemma 3 $H(3) = \pi^4/72$.

Proof $\zeta(2)^2 = \Sigma\Sigma\, m^{-2}n^{-2} = \Sigma\Sigma\, (m + n)^{-1}(m^{-1}n^{-2} + m^{-2}n^{-1})$,

i.e. $(\pi^2/6)^2 = S(1, 2) + S(2, 1) = 2H(3)$ QED

Lemma 4 $$H(k) = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^{k-2} + n^{k-2}}{m^{k-1}n^{k-1}(m+n)}$$

Proof This follows at once from Lemma 1.

Corollary For odd k, observing that m+n divides $m^{k-2}+n^{k-2}$, we can easily express H(k) as a sum of products of zeta functions.

<u>Lemma 5</u> $$\sum_{\substack{m=1\\ m\neq n}}^{\infty} \frac{1}{m^2-n^2} = \frac{3}{4n^2}$$

<u>Proof</u> $$\text{L.H.S.} = \sum_{k=2}^{\infty} \frac{1}{(kn)^2-n^2} + \sum_{j=1}^{n-1}\sum_{k=0}^{\infty} \frac{1}{(kn+j)^2-n^2}$$

$$= \frac{1}{2n^2}\sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) + \sum_{j=1}^{n-1}\frac{1}{2n}\sum_{k=0}^{\infty}\left(\frac{1}{kn-n+j} - \frac{1}{kn+n+j}\right)$$

$$= \frac{1}{2n^2}\left\{\left(\frac{1}{1}-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \dots\right\} \quad \text{(which telescopes)}$$

$$+ \frac{1}{2n}\sum_{j=1}^{n-1}\left\{\left(\frac{1}{-n+j} - \frac{1}{n+j}\right) + \left(\frac{1}{j} - \frac{1}{2n+j}\right) + \left(\frac{1}{n+j} - \frac{1}{3n+j}\right) + \dots\right\}$$

(which also telescopes)

$$= \frac{1}{2n^2}\left\{\frac{1}{1}+\frac{1}{2}\right\} + \frac{1}{2n}\sum_{j=1}^{n-1}\left\{\frac{1}{-n+j}+\frac{1}{j}\right\}$$

$$= \frac{3}{4n^2} + \frac{1}{4n}\sum_{j=1}^{n-1}\left\{\left(\frac{1}{-n+j}+\frac{1}{j}\right) + \left(\frac{1}{-j}+\frac{1}{n-j}\right)\right\} \quad \text{(using } j \mapsto n-j \text{ symmetry)}$$

$$= 3/(4n^2).$$

<u>Lemma 6</u> $$H(k) = \frac{k+2}{2}\zeta(k+1) - \frac{1}{2}\sum_{r=2}^{k-1}\zeta(r)\zeta(k+1-r)$$

<u>Proof</u> $$H(k) = \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{k-2}+n^{k-2}}{m^{k-1}n^{k-1}(m+n)} \quad \text{(Lemma 4)}$$

$$H(k) + \frac{1}{2}\sum_{r=2}^{k-1}\zeta(r)\zeta(k+1-r)$$

$$= \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{m^{k-2}+n^{k-2}}{m^{k-1}n^{k-1}(m+n)} + \sum_{r=2}^{k-1}\frac{m^{k-1-r}n^{r-2}(m+n)}{m^{k-1}n^{k-1}(m+n)}\right\}$$

$$= \sum\sum\frac{\left\{m^{k-2}+n^{k-2} + (m^{k-2}+m^{k-3}n) + (m^{k-3}n+m^{k-4}n^2) + \dots + (mn^{k-3}+n^{k-2})\right\}}{2\,m^{k-1}\,n^{k-1}\,(m+n)}$$

$$= \sum\sum\frac{m^{k-2}+m^{k-3}n+\dots+mn^{k-3}+n^{k-2}}{m^{k-1}\,n^{k-1}\,(m+n)}$$

(Now separating the terms with $m = n$ from those with $m \neq n$)

$$= \sum_{m=1}^{\infty}\frac{(k-1)m^{k-2}}{m^{2k-2}(m+m)} + \sum_{m\neq n}\sum\frac{m^{k-2}+m^{k-3}n+\dots+n^{k-2}}{m^{k-1}\,n^{k-1}\,(m+n)}$$

$$= \frac{k-1}{2}\zeta(k+1) + \sum_{m\neq n}\sum\frac{m^{k-1}-n^{k-1}}{m^{k-1}n^{k-1}(m^2-n^2)}$$

$$= \frac{k-1}{2}\zeta(k+1) + \sum_{m\neq n}\sum\left(n^{1-k}-m^{1-k}\right)\frac{1}{m^2-n^2}$$

$$= \frac{k-1}{2}\zeta(k+1) + 2\sum_{m\neq n}\sum n^{1-k}\frac{1}{m^2-n^2} \quad \text{(now use lemma 5)}$$

$$= \frac{k-1}{2}\zeta(k+1) + \frac{3}{2}\sum n^{-1-k} = \frac{k+2}{2}\zeta(k+1).$$

Thus, in terms of zeta functions, we know H(k) for all k, so that we know S(a, b) for all a and b, because

$$S(a, b-1) + S(a-1, b) = \zeta(a)\zeta(b).$$

For example:

$S(2, 2) = -3\zeta(5) + 2\zeta(2)\zeta(3)$

$S(3, 3) = 4\zeta(7) - 2\zeta(2)\zeta(5)$

$S(4, 4) = -5\zeta(9) + 2\zeta(2)\zeta(7) + 2\zeta(4)\zeta(5)$

We leave as an exercise the investigation of how when k is odd the result of Lemma 6 relates to that mentioned as a corollary to Lemma 4, and of how the result for even k relates to the formula:-

$$H(2k) = (k+1)\zeta(2k+1) - \sum_{r=1}^{k-1}\zeta(2r)\zeta(2k+1-2r)$$

OTHER FORMULAE

$$\sum_1^\infty n^{-3}(1 + 2^{-2} + 3^{-2} + \ldots + n^{-2}) = 3\zeta(2)\zeta(3) - 9\zeta(5)/2.$$

$$\sum_1^\infty n^{-4}(1 + 2^{-2} + 3^{-2} + \ldots + n^{-2}) = \zeta(3)^2 - \zeta(6)/3.$$

$$\sum_1^\infty n^{-5}(1+2^{-2}+3^{-2}+ \ldots +n^{-2}) = 5\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4) - 10\zeta(7)$$

$$\sum_1^\infty n^{-2}(1 + 2^{-3} + \ldots + n^{-3}) = 11\zeta(5)/2 - 2\zeta(2)\zeta(3).$$

Write $\mathrm{HD}(k) = \sum_1^\infty n^{-k}(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n})$

$\mathrm{HD}(2) = 3\cdot305656483692 = 11\zeta(3)/4$

$\mathrm{HD}(3) = 1\cdot974627536842$ (One might perhaps hope for this to be equal to a simple rational multiple of π^4, but it does not seem to be).

$\mathrm{HD}(4) = 1\cdot682364333887 = 37\zeta(5)/4 - 4\zeta(2)\zeta(3)$

$$\sum_1^\infty (-1)^{n+1}\, n^{-2}(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}) = (5/8)\zeta(3).$$

This last equation can be deduced from those for HD(2) and H(2).

$$\sum_1^\infty \frac{1}{n(n+1)}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) = \zeta(2).$$

$$\sum_1^\infty \frac{1}{n(n+1)(n+2)}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) = \zeta(2)/2 - 1/2.$$

$$\sum_1^\infty \frac{1}{n(n+1)(n+2)(n+3)}\left(1 + \frac{1}{2} + \ldots \frac{1}{n}\right) = \zeta(2)/6 - 5/24.$$

$$\sum_1^\infty \frac{1}{n(n+1)}(1 + 2^{-2} + 3^{-2} + \ldots + n^{-2}) = \zeta(3).$$

The four equations above barely qualify for mention in this note, they may be regarded as exercises in the formula

$$\Sigma\, a_n(b_1+b_2+ \ldots +b_n) = \Sigma(a_n+a_{n+1}+a_{n+2}+ \ldots)b_n$$

for partial summation of a series of products.

OLD FASHIONED PROBLEM (JCMN 64 p.6342)

The problem (obtained from Hall & Knight's *Higher Algebra*) was to evaluate the infinite continued fraction

$$\frac{1}{2+}\;\frac{2}{3+}\;\frac{3}{4+}\;\frac{4}{5+}\;\cdots\quad \left(= \frac{3-e}{e-2}\right)$$

In fact it is Question 21 of Examples XXXI.a. on page 369 of my copy (4th edition, 1891. reprinted 1932).

Firstly, recall the following classical result:

<u>Theorem</u> Take any $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, they may in fact be elements of any field, but think of them as real numbers. Define p_n, q_n, and c_n for $n = 1, 2, \ldots$ as follows.

Let $p_1 = a_1$, $p_2 = a_1b_2$, $p_n = b_np_{n-1} + a_np_{n-2}$, $q_1 = b_1$, $q_2 = b_1b_2 + a_2$, $q_n = b_nq_{n-1} + a_nq_{n-2}$ (for $n > 2$) and

$$c_n = \frac{a_1}{b_1+}\;\frac{a_2}{b_2+}\;\cdots\;\frac{a_n}{b_n}. \quad \text{Then } c_n = \frac{p_n}{q_n}.$$

<u>Proof</u> Use induction on n. The result is clear when $n = 1$ and when $n = 2$. Suppose that $c_1 = p_1/q_1$, $c_2 = p_2/q_2$, ... and $c_n = p_n/q_n$. We must try to prove that $c_{n+1} = p_{n+1}/q_{n+1}$.

The equation $$\frac{a_1}{b_1+}\;\frac{a_2}{b_2+}\;\cdots\;\frac{a_n}{b_n} = \frac{b_np_{n-1} + a_np_{n-2}}{b_nq_{n-1} + a_nq_{n-2}}$$

is an identity in the $2n$ variables $a_1, \ldots a_n, b_1, \ldots b_n$, and so it remains true if b_n is changed to $b_n + a_{n+1}/b_{n+1}$. Note that p_{n-1}, p_{n-2}, q_{n-1} and q_{n-2} do not depend on b_n.

$$c_{n+1} = \frac{(b_n + a_{n+1}/b_{n+1})p_{n-1} + a_n\, p_{n-2}}{(b_n + a_{n+1}/b_{n+1})q_{n-1} + a_n\, q_{n-2}}$$
$$= \frac{b_{n+1}(b_n\, p_{n-1} + a_n\, p_{n-2}) + a_{n+1}\, p_{n-1}}{b_{n+1}(b_n\, q_{n-1} + a_n\, q_{n-2}) + a_{n+1}\, q_{n-1}}$$

$$= \frac{b_{n+1}\ p_n + a_{n+1}\ p_{n-1}}{b_{n+1}\ q_n + a_{n+1}\ q_{n-1}} = \frac{p_{n+1}}{q_{n+1}}$$

which completes the proof by induction.

Apply the theorem to our problem, with $a_n = n$ and $b_n = n+1$.

n	1	2	3	4	5
p_n	1	3	15	87	597
q_n	2	8	38	222	1522

We see that $p_n = (n+1)p_{n-1} + np_{n-2}$, while the q_n satisfy a similar recursion with different initial values.

To solve the recursion, put $p_n = (n+1)!\ r_n$. Then $r_n = r_{n-1} + r_{n-2}/(n+1)$; we expect two linearly independent solutions. One obvious solution is $r_n = n+3$, and so we reduce the problem from second order to first order by putting $r_n = (n+3)s_n$; this gives $(n+3)s_n = (n+2)s_{n-1} + s_{n-2}$.

$$(n+3)(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) = 0, \text{ so that}$$

$$s_n - s_{n-1} = (-1)^n \text{ Constant } / (n+3)!$$

Now look at the initial conditions for the sequence.

n	1	2	3	4	5
s_n	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{5}{48}$	$\frac{29}{280}$	$\frac{199}{1920}$
$s_n - s_{n-1}$		$-\frac{1}{40}$	$\frac{1}{240}$	$-\frac{1}{1680}$	$\frac{1}{13440}$

From the values above, $s_n - s_{n-1} = 3(-1)^{n+1}/(n+3)!$

and $s_n = 1/8 - 3/5! + 3/6! - 3/7! + \ldots + 3(-1)^{n+1}/(n+3)!$

so that $p_n = (n+1)!r_n = (n+3)(n+1)!(1/8 - 3/5! + 3/6! - \ldots)$

and similarly $q_n = (n+3)(n+1)!(1/4 + 2/5! - 2/6! + \ldots)$.

Since $1/8 - 3/5! + 3/6! - 3/7! + \ldots$ (to ∞) $= 3/e - 1$

and $1/4 + 2/5! - 2/6! + 2/7! - \ldots$ (to ∞) $= 1 - 2/e$,

we find that $p_n/q_n \to \frac{3 - e}{e - 2}$.

PARADOX IN PROBABILITY (1)

Two gamblers, A and B, sit at a table in a Mississippi stern-wheel paddle steamer. A tosses two silver dollars, they lie on the table hidden from B. Two other passengers, C and D, pass by. C says to D "When I see the head on a silver dollar it reminds me of what a wonderfully good likeness of George Washington the Mint has got for their coins." A looks up and says "Excuse me, sir, but you have rather upset our game; my friend B here was going to bet on whether the two coins were both showing the same face or showing one head and one tail. We have always thought that the probability of the coins showing the same was exactly a half; but now you have given him the extra information that there is at least one head showing. And that must surely change the odds."

C answers "My dear chap, you have no need to worry at all. B knows only that one of the coins shows George Washington's head, and that the other one is equally likely to show head or tail. And so you can safely just carry on as usual." But D broke in "When the coins were tossed, there were four possible outcomes, HH, HT, TH and TT, each with probability one quarter. The extra information that there was at least one head simply eliminates the possibility TT, leaving the other three outcomes all having probability one third. With the extra information, B's estimate for the probability of the two coins showing the same is therefore reduced from a half to one third."

From the next table comes another voice "I am sorry to see you youngsters disagreeing over such a simple question. I never set eyes on George Washington myself and I cannot say if the silver dollar gives a good likeness; but my old schoolmaster used to tell us how when he was at Yale in 1781 there was a great occasion when they gave honorary degrees to George Washington and to an English Presbyterian minister called Richard Price, who was asked to give a lecture to the mathematics department. He talked about his old friend Thomas Bayes who had died in 1761, leaving some unpublished mathematical work, in which he (Price) had found a theorem which would help to solve the difficulties about how information changes probabilities. My old schoolmaster would have been able to tell you the answer to your question, he was clever. I am sorry I never paid proper attention in his classes."

We now know that Price was right about the importance of that theorem of Bayes. Price had arranged for it to be printed in the *Philosophical Transactions of the Royal Society*. You find it in the probability text-books these days, usually in something like this form:

Write $P(X|Y)$ for the probability of X when we know Y. Then $P(X|Y) = P(X\&Y|)/P(Y|) = P(X|) \times P(Y|X)/P(Y|)$, or posterior probability = prior probability × likelihood. The usefulness of the theorem is where X is the event of a parameter in a mathematical model having a certain value, and Y is an experimental observation, so that $P(Y|X)$ is usually easy to find, and $P(X|Y)$ is what the statistician wants to know.

What about A and B? Looking back from the twentieth

century, could we give advice on what they should do?

(a) Agree that the probability of the two coins showing the same was 1/2?

or (b) Agree that it was 1/3?

or (c) Learn mathematics at Yale?

or (d) Resolve to give up gambling?

To those concerned about historical accuracy, some comments may be appropriate. George Washington and Robert Price did indeed both receive honorary degrees of LL.D. at Yale in 1781, Price had crossed the Atlantic after being asked by the U.S. Congress to give advice on financial matters. That Robert Price gave a lecture for the Mathematics Department of Yale on the occasion of the graduation ceremony is only supposition, but if he had given a lecture it might well have been on the work of his old friend Thomas Bayes. The big historical inaccuracy in the story is that George Washington's head was never on a silver dollar; in fact nobody else's head (apart from a bald eagle and the anonymous young lady representing Liberty) was ever on a silver dollar.

There is a long tradition of minting coins with an image of a head on one side. Recall the passage in St Matthew's Gospel, chapter 22, verses 19 - 22: —— Shew me the tribute money. And they brought unto him a penny. And he saith unto them, Whose is this image and superscription? They say unto him, Caesar's. Then saith he unto them, Render therefore unto Caesar the things which are Caesar's and unto God the things that are God's.

PARADOX IN PROBABILITY (2)

At the *Goldfinger* casino there is a new game. An assistant, A, on the stage shows the patrons three cards, a King, a Queen and a Jack, he shuffles them and then puts one in a box and the other two face down on a table. Another assistant, B, looks at these two cards and then holds one up for all the patrons to see, it is (in a typical game) the Queen. The patrons are then invited to place bets on whether the card in the box is the King.

The method of betting is that the house offers, at $8 each, "King" tickets, which will pay $18 if the card in the box is the King. Also they offer "non-King" tickets at $11 each, these will pay $18 if the card turns out not to be the King.

Some of the patrons think that the probability of the King is 1/3, and so they cheerfully buy "non-King" tickets, thinking that they have a 2/3 probability of a return of $18 for the $11 bet, an expectation of $1 gain. Others think that King and Jack are equally likely, and they cheerfully buy "King" tickets, thinking that they have a 1/2 probability of a return of $18 for the $8 bet, an expectation of $1 gain. The proprietor knows that by selling one ticket of each kind he is certain of a gain of $1.

Another patron, Z. is more cautious. He approaches B and bribes him for the full story of what goes on. B tells him that A's shuffling is perfectly genuine, and that his (B's) instructions are to look at both cards on the table and to show one of them, but if one is the King he must show the other, because to show that the King was not the card in the box would make the betting pointless. How does Z bet?

What is the moral of this story?

YEAST MIXING (JCMN 65 p.6369)

The following question was in the previous issue.

This is a model for what might happen in making bread. The yeast is a fine powder suspended in a fluid which may be thought of as water, though in fact it is a mixture containing milk, sugar and other additives. This yeast mixture is initially at a temperature of 0°; the scale of the temperature is not relevant though you may like to think of it as on the Réaumur scale. Yeast is inactive when cold, and for it to work in the bread dough we want it to be at 20°. To accomplish this we mix three parts of the cold yeast mixture with one part of boiling water at 80°, stirring them together. Assume that there is no conduction of heat, the hot and cold fluids mix by diffusion. It is an unfortunate fact that yeast is killed by temperatures over 40°. What proportion of our original yeast will be killed in the mixing process?

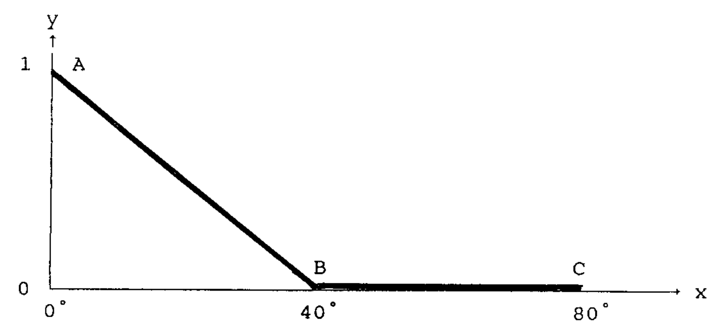
The answer is 1/3.

It looks a tricky problem at first, the difficulty is in seeing that it is not biology, not fluid mechanics and not thermodynamics, but just a combinatorial question.

The state of the system at any time may be represented as a distribution in an affine plane. To be more precise, let x be temperature and let y be (in some units) the proportion of (live) yeast in the mixture. Then let $f(x, y)$ be a "density", such that $f(x, y)\, dx\, dy$ is the amount of the yeast mixture that has temperature between x and $x+dx$ and has its proportion of yeast between y and $y+dy$. Those interested in the finer points of integration theory may prefer to think of an additive interval function f(I) defined on intervals I, where I might be where a < x < b and c < y < d, instead of using the (possibly improper) point function f(x, y) defined at points (x, y).

Now consider the process of diffusion by which f changes with time, ignoring for the moment the possible killing of yeast by heat. If amounts m at the point (x, y) and m′ at (x′, y′) mix together, they result in an amount m+m′ at the point ((mx+m′x′)/(m+m′), (my+m′y′)/(m+m′)), so that the centroid of the distribution is unchanged by diffusion, and the convex hull of the set carrying the distribution can change only by getting smaller in the diffusion process.

Take units for y such that y = 1 for the cold yeast mixture, the diagram is as shown below.



The situation is complicated by the fact of yeast being killed by temperatures over 40°. Initially the distribution consists of one point mass (or delta function) of magnitude 3/4 at (0, 1), representing the cold yeast mixture, and one mass of magnitude 1/4 at (80, 0), representing the hot water. As mixing starts the hot part becomes diluted and greater in mass

and its distribution begins to spread over the line BC. The part of the mixture in which the temperature is between 0° and 40° follows the rules of the paragraph above, and is represented by a distribution on the line AB. The part above 40° always has its centroid moving to the left, and is always on the line y = 0, i.e. is on the line segment BC.

The distribution can never extend outside the set consisting of the two line segments AB and BC. Its centroid is initially at the point (20, 3/4) and can move (as yeast is killed) only vertically down, so that when mixing is complete the system is represented by a single unit mass at the point (20, 1/2). The total amount of live yeast changes from 3/4 to 1/2 of our units, so that 1/3 if it must have been killed by the heat.

NON-SYMMETRIC SIMULTANEOUS EQUATIONS

If $$\frac{a}{w^3} = \frac{2b}{x(zw-xy)} = \frac{2c}{w(zw+3xy)} = \frac{d}{xw^2}$$

prove $$\frac{x}{d^3} = \frac{2y}{a(cd-ab)} = \frac{2z}{d(cd+3ab)} = \frac{w}{ad^2}.$$

(From the Cambridge Mathematical Tripos, Part 1, 9 - 12, Friday, May 16, 1902. Question vii)

By Tripos standards it is an easy question, substitution gives the answer at once; the interest for us is in how the question was created, and in whether the equations have a geometrical significance when regarded as a self-inverse mapping of projective 3-dimensional space into itself.

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