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The James Cook Mathematical Notes is published in 3 issues per year, dated January, May and September. The history of JCMN is that the first issue appeared in September 1975, and others at irregular intervals, all the issues up to number 31 being produced and sent out by the Mathematics Department of the James Cook University of North Queensland, of which I was then the Professor. In October 1983 this arrangement was beginning to be unsatisfactory, and I changed to publishing the JCMN myself, having three issues per year printed in Singapore and posted from there. I then set a subscription price of 30 Singapore dollars per year. When in 1985 I changed to printing in Australia I kept the same price, for the Singapore dollar is a stable currency.

In October 1992 it had become clear that the paying of subscriptions by readers is an inefficient operation. Bank charges for changing currency and for international transfers, with postage, together absorb most of the initial input of money. Therefore we have abandoned subscriptions as from the beginning of 1993, issue number 60. To those who want to give something in return for the JCMN, I ask them to make a gift to an animal welfare society in their own country. The animals of the world will be grateful and so will I.

Contributors, please tell me if and how you would like your address printed.

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SIMULTANEOUS SYMMETRIC EQUATIONS (JCMN 62, p.6278)

Arthur H Stone and Cedric A B Smith

(North-Eastern University and University College London)

A Cambridge Entrance Scholarship Examination in 1899 proposed the question:

Given $\quad y^3 - z^3 = ayz, \qquad (1)$

$z^3 - x^3 = azx, \qquad (2)$

$x^3 - y^3 = axy, \qquad (3)$

prove that $\quad x^3 + y^3 + z^3 = 3xyz. \qquad (4)$

It was noted that with complex variables and with $a = 0$ the result is not always true. So assume that the examiners intended that $a \neq 0$. Consider two cases:

Case 1: At least one of x, y and z is zero. Then clearly $x = y = z = 0$, satisfying (4).

Case 2: The variables x, y and z are all non-zero. Add the equations (1), (2) and (3) and divide by a, it gives

$$yz + zx + xy = 0 \qquad ..... (5)$$

so that
$$xz + yz = -xy \qquad .... (6)$$

Subtract $y \times (2)$ from $x \times (1)$, giving

$$xy^3 - xz^3 - yz^3 + yx^3 = 0, \qquad \text{or, rearranging,}$$

$$xy(x^2 + y^2) - (xz + yz)z^2 = 0.$$

Using (6), replace $xz + yz$ by $-xy$ and cancel the common factor xy ($\neq 0$), obtaining $\quad x^2 + y^2 + z^2 = 0. \qquad ..... (7)$

But any reasonable scholarship candidate should know that $(x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy) = x^3 + y^3 + z^3 - 3xyz$, so that (5) and (7) together imply (4).

This may be how the examiners thought of the question. But more can be deduced.

Add $2 \times (5)$ to (7) to get $\quad (x + y + z)^2 = 0$, whence $x + y = -z$. Multiply by z and use (6), to get $-xy = -z^2$ which implies $xyz = z^3$. By symmetry also $xyz = y^3$, so $y^3 - z^3 = 0$, whence from (1) $ayz = 0$, contradicting the assumptions of Case 2.

So only Case 1 can hold; giving the trivial solution $x = y = z = 0$. (This conclusion holds in any field)

TRIANGLE PROBLEM

Nigel Tao

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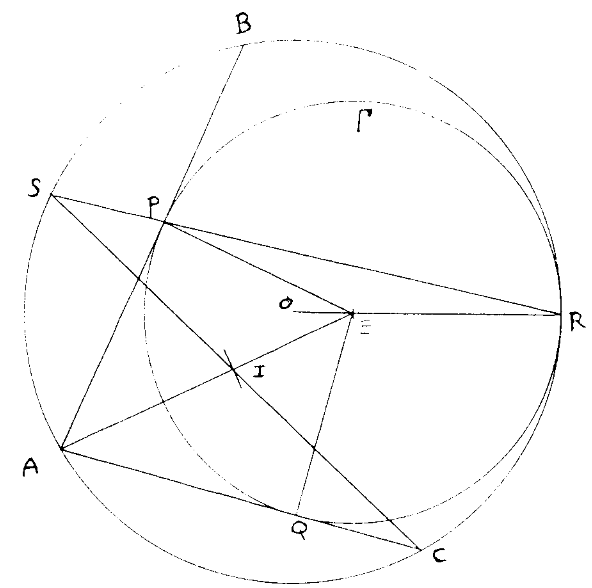
A circle Γ touches the sides AB and AC of a triangle ABC, at points P and Q respectively, and Γ touches the circumcircle internally at R. Denote the incentre by I. Prove (or disprove) that PR must meet CI on the circumcircle.



FUNCTIONAL EQUATION (JCMN 44, p.5062)

Terry Tao
(Dept of Math, Princeton University)

In the 1987 note under this title R.L.Agacy asked:-

Let $f(f(x)) = x$, $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$ for all x and y.

(a) If the variables are real, prove that $f(x) = x$.

(b) If the variables are complex, what can you say about f?

In both cases it is easy to see that $f(1) = 1$, $f(2) = 2$, etc. so that $f(r) = r$ for any rational r.

(a) In the real case, $x > 0$ implies $f(x) = f(\surd x.\surd x) = f(\surd x).f(\surd x) > 0$. The mapping preserves order, and it must be the identity.

(b) In the complex case there is little more that can be said about f. The identity function and the complex conjugate both satisfy the given conditions, but they are not the only such functions. This fact may be shown as follows.

<u>Notation</u> We shall use capital letters for fields, they are all subfields of the complex field C. We shall use small letters for the elements (which are complex numbers), and Greek letters for functions (sometimes called mappings, they are either polynomials or homomorphisms, see below).

Some of the jargon of algebra is needed. A *homomorphism* from F to G is a mapping that preserves addition and multiplication, note that the image of F need not be the whole

of G. A *monomorphism* is a homomorphism such that the images of distinct elements are distinct, i.e. it never maps a non-zero element to zero. A monomorphism from F to G is called an *isomorphism* if the image of F is the whole of G. An isomorphism from F to itself is called an *automorphism*.

An element (complex number) b is called *algebraic* over F if b is a zero of a polynomial with coefficients in F; if this is so then of all such polynomials there is one, called the *minimal polynomial* of b, which is monic (leading coefficient = 1) and is a factor of all the others, it is irreducible over F. If an element b is not in F and is not algebraic over F, then it is called *transcendental* over F. If b is not in F (it may be either algebraic or transcendental) there is an *extension field* denoted by F(b) which is the smallest field containing b and F, a typical element of F(b) is $\alpha(b)/\beta(b)$ where α and β are polynomials with coefficients in F, and $\beta(b) \neq 0$.

<u>Lemma 1</u> Suppose that φ is a monomorphism from F to C, and that b is algebraic over F. Then φ can be extended to a monomorphism from F(b) to C.

<u>Proof</u> Let $\alpha(z) = \Sigma_{i=0}^{n} a_i z^i$ be the minimal polynomial for b, so that α is irreducible over F and $\alpha(b) = 0$. Define the polynomial α^* by $\alpha^*(z) = \Sigma_{i=0}^{n} \varphi(a_i)z^i$. Since C is algebraically closed (every equation has a root) there is q in C such that $\alpha^*(q) = 0$. In order to extend φ over F(b), start by putting $\varphi(b) = q$. This fixes the other values. Take any element $\beta(b)/\gamma(b)$ of F(b), the image under φ must be $\varphi(\beta(b))/\varphi(\gamma(b))$, which is known because if $\beta(z) = \Sigma\, b_i z^i$, then

$\varphi(\beta(b))$ must be $\Sigma\,\varphi(b_i)\varphi(b)^i = \Sigma\,\varphi(b_i)q^i$, and similarly for the other polynomial γ.

To be sure that the extended φ is a monomorphism, we must check that the inverse image of zero is zero. Suppose that $\beta(b)/\gamma(b)$ maps to zero, then $\varphi(\beta(b)) = 0$. Therefore $\Sigma\,\varphi(b_i)q^i = 0$, but recall that $\Sigma\,\varphi(a_i)q^i = 0$. We have two equations with coefficients in $\varphi(F)$, both with q as a root. As φ is a monomorphism over F, it is an isomorphism from F to $\varphi(F)$, and it has an inverse. Applying this inverse to the two equations we see that $\Sigma\, b_i z^i$ $(= \beta(z))$ and $\Sigma\, a_i z^i$ $(= \alpha(z))$, both with coefficients in F, have a common zero (in C). But α is irreducible, therefore is a factor of β, and so $\beta(b) = 0$.

QED

<u>Lemma 2</u> There is a field F containing i, such that e is transcendental over F, but there are no complex numbers transcendental over the extension field F(e).

<u>Proof</u> As shown by C. Hermite in 1873, e is transcendental over the field Q of rationals, and therefore over the field Q(i) of complex rationals.

Consider the set of fields that contain i, and over which e is transcendental. By Hermite's result the set is non-empty. By Zorn's Lemma the set has a maximal member, call it F. Is any complex number transcendental over F(e)?

Let y be transcendental over F(e). There exists no two-variable polynomial Ψ over F with $\Psi(y, e) = 0$. Therefore e is transcendental over F(y). This contradicts the maximality of F. Therefore there is no y transcendental over F(e).

QED

<u>Lemma 3</u> Given F as in Lemma 2, there is a monomorphism φ on C such that $\varphi(z) = z$ for all z in F and $\varphi(e) = e + i$.

<u>Proof</u> As e is transcendental over F, every element of F(e) is of the form $\beta(e)/\gamma(e)$ with β and γ coprime polynomials over F. Let φ map this element into $\beta(e+i)/\gamma(e+i)$. Thus we have a monomorphism on F(e), and by Lemma 1 it can be extended to C. Clearly $\varphi(e) = e + i$.

QED

<u>Lemma 4</u> The monomorphism φ of Lemma 3 is an automorphism, and is hence invertible.

<u>Proof</u> Let $G = \varphi(C)$ be the image of C under φ. G is an algebraically closed field because C is. Also F(e) is a subfield of G because $\varphi(e-i) = e$ and $\varphi(F) = F$. Since every complex number is algebraic over F(e) by the definition of F (see Lemma 2), G must equal C, and we are done.

QED

<u>Lemma 5</u> There exists a map θ from C to C that is not the identity or the conjugation function, such that:-

(1) $\theta(x+y) = \theta(x) + \theta(y)$

(2) $\theta(xy) = \theta(x)\theta(y)$

(3) $\theta(\theta(x)) = x$ for all x and y.

<u>Proof</u> Let Γ be the conjugation function, $\Gamma(x+iy) = x-iy$, and take φ as in Lemma 3. Define $\theta(z) = \varphi^{-1}(\Gamma(\varphi(z)))$. The properties (1), (2) and (3) are all immediate. Also $\theta(e) = \varphi^{-1}(\Gamma(e+i)) = \varphi^{-1}(e-i) = e-2i$, so that θ is not either the identity or the conjugation function.

INVERSE PROBABILITY

A. Brown

If you toss four unbiassed coins and count the number of heads, the possible results are 0, 1, 2, 3, 4, with probabilities in the ratio of the binomial coefficients, 1 : 4 : 6 : 4 : 1. You might ask if it is possible to mimic this probability distribution with four biassed coins, and it is not too difficult to show that this cannot be done.

A similar question occurs in *Problems for Mathematicians Young and Old* by Paul R. Halmos; although I should confess that I have not read the book, only a review of it. Two dice are marked 1, 2, ... 6 in the usual way. When they are rolled and the results added, the sum takes the possible values 2, 3, ..., 12 with probabilities in the ratios

1 : 2 : 3 : 4 : 5 : 6 : 5 : 4 : 3 : 2 : 1

which is as it would be if the dice were unbiassed. Does it follow that the two dice are both unbiassed?

The example below illustrates the kind of matching that is possible with two pairs of biassed dice. (In the table P(j) denotes the probability of the number j turning up)

	P(1)	P(2)	P(3)	P(4)	P(5)	P(6)
Die 1A	1/12	1/6	1/4	1/4	1/6	1/12
Die 1B	1/32	5/32	5/16	5/16	5/32	1/32
Die 2A	1/24	1/6	7/24	7/24	1/6	1/24
Die 2B	1/16	3/16	1/4	1/4	3/16	1/16

For pair 1 (die 1A and die 1B), as well as for pair 2, the sum takes the values 2, 3, ... 12 with probabilities in the ratios 1 : 7 : 23 : 48 : 72 : 82 : 72 : 48 : 23 : 7 : 1.

JCMN readers may like to ponder a more general question. Suppose that Y is the sum of M independent random variables, each having the range of values {1, 2, ..., N}, and that the distribution of Y is what it would be if all the M random variables were unbiassed (i.e. with each value having probability 1/N). Does it follow that the random variables are all unbiassed? The two questions mentioned above are the cases (M, N) = (4, 2) and (2, 6).

The possibility of simulating unbiassed random variables by biassed ones is shown by the following example.

The random variable X is described by:

Value	2	3	4
Probability	1/4	1/2	1/4

and Y by:

Value	0	2	4	6	8
Probability	1/9	2/9	1/3	2/9	1/9

It is easily verified that the sum X + Y has the same distribution as the sum of the values given by two unbiassed dice, i.e. 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, with probabilities in the ratios 1 : 2 : 3 : 4 : 5 : 6 : 5 : 4 : 3 : 2 : 1.

NON-DIFFERENTIABLE FUNCTIONS

Terry Tao
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In the work below the variables x and t are always real and integrals are from $-\infty$ to ∞ unless otherwise specified.

Lemma 1 Suppose that a function $f(x)$ is bounded and is differentiable at $x = 0$, and that there is a continuous complex-valued function $u(x)$ satisfying:-

(a) $\int u(t)\,dt = 0$

(b) $\int t\,u(t)\,dt = 0$.

and (c) $|u(t)| < C/(1+|t|)^3$ for some constant $C > 0$.

Then $(1/h)\int f(th)\,u(t)\,dt \to 0$ as $h \to 0$.

Proof Pick any $\delta > 0$.

By the definition of differentiability, for any $\delta > 0$, there exists $H > 0$ such that

$|f(x) - f(0) - xf'(0)| < \delta|x|/(4C)$ if $|x| < H$ .... (1)

From (a), (b) and (c), there is $M > 0$ such that for all $N > M$

$$|f(0)\int_{-N}^{N} u(t)\,dt| < \delta H/(4N)$$

and $|f'(0)\int_{-N}^{N} t\,u(t)\,dt| < \delta H/(4N)$.

Consider any h satisfying $0 < h < H/M$.

$$(1/h)\int_{-H/h}^{H/h} f(th)u(t)dt$$

$$= (1/h)\int_{-H/h}^{H/h}(f(0) + thf'(0) + E)u(t)dt$$

where $|E| < \delta|th|/(4C)$ by (1) above. Noting that $H/h > M$, and that $\int|tu(t)|dt < C$, it follows that

$$(1/h)\left|\int_{-H/h}^{H/h} f(th)u(t)dt\right|$$

$$< \frac{|f(0)|}{h}\left|\int_{-H/h}^{H/h} u(t)dt\right| + |f'(0)|\left|\int_{-H/h}^{H/h} tu(t)dt\right| + \frac{C}{h}\,\frac{\delta h}{4C}$$

$$< \frac{\delta H}{4h(H/h)} + \frac{\delta H}{4h(H/h)} + \frac{\delta h}{4h} = 3\delta/4,$$

also $(1/h)\left|\int_{|t|>H/h} \cdots\right| < \frac{A}{h}\int_{|t|>H/h} C/(1+|t|)^3 dt$

which will become less than $\delta/4$ when h is sufficiently small, in fact when $h < 3\delta H^2/(4AC)$. Therefore (with $h > 0$)

$$\frac{1}{h}\int f(th)\,u(t)\,dt \to 0 \text{ as } h \to 0.$$

the same result follows for $h < 0$, similarly. QED

A note on Fourier transforms

Define the Fourier transform $f^\frown$ of a function $f(x)$ as

$f^\frown(x) = 1/(2\pi)\int \exp(-ixy)\,f(y)\,dy$.

The inverse transform is given by

$f(x) = \int \exp(ixy)\,f^\frown(y)\,dy$.

We describe a function as "smooth" if it is differentiable any number of times, and as "rapidly decreasing" if (for all integer $n > 0$) it is $O(|x|^{-n})$ for large x.

The transform of a rapidly decreasing function is smooth, and of a smooth function is rapidly decreasing.

By equating the two repeated integrals of

$$f^\frown(s)g^\frown(t)\exp(ihst)$$

we find $\int f(th)g^\frown(t)dt = \int f^\frown(t)g(th)dt$ for any h, which is a form of Parseval's identity.

Lemma 2 Suppose that $0 < \delta < 1$. There exists a complex function u that satisfies (a), (b) and (c) of Lemma 1, and is the Fourier transform $g^\frown$ of a smooth function g zero outside the interval $(1-\delta, 1+\delta)$, and such that $g(1) = 1$.

Proof Choose a smooth function g, zero outside $(1-\delta, 1+\delta)$ and with $g(1) = 1$. Define u as the transform of g, so that $g^\frown = u$. Requirements (a) and (b) follow from the observation that $g(0) = g'(0) = 0$. Requirement (c) follows because the transform of a smooth function with compact support is bounded and rapidly decreasing. QED

Lemma 3 Suppose that:-

$$0 < Cc_n < c_{n+1} \text{ for some } C > 1 \text{ and for all } n = 1, 2, \ldots$$

and $\Sigma_{n=1}^{\infty} (|a_n|+|b_n|)/c_n = A < \infty$,

and $f(x) = \Sigma_{n=1}^{\infty}(a_n/c_n)\exp(ic_nx) + (b_n/c_n)\exp(-ic_nx)$,

and $f(x)$ is differentiable at $x = 0$.

Then (a_n) and (b_n) both $\to 0$ as $n \to \infty$.

Proof Choose δ so that $0 < \delta < 1-1/C$, then C and $1/C$ are both outside the interval $(1-\delta, 1+\delta)$. Take the functions u and g as in lemma 2.

$$f^\frown(x) = \Sigma(a_n/c_n)\delta(x-c_n) + (b_n/c_n)\delta(x+c_n)$$

If m is any positive integer, consider $h = 1/c_m$.

$$\text{Then } \int f^\frown(x)g(hx)dx = \Sigma(a_n/c_n)g(c_n/c_m) + (b_n/c_n)g(-c_n/c_m)$$
$$= a_m/c_m \quad \text{(all other terms vanishing)}$$

Now use the form of Parseval identity mentioned above

$$c_m \int f(t/c_m)u(t)dt = a_m.$$

Since $1/c_m \to 0$, Lemma 1 shows that $a_m \to 0$. Similarly, by taking $h = -1/c_m$, it follows that $b_m \to 0$. QED

Theorem Suppose that:-

a_n and b_n are complex, and c_n is real for $n = 1, 2, 3, \ldots,$

and $C > 1$ and $0 < Cc_n < c_{n+1}$ for all $n = 1, 2, \ldots,$

and $\Sigma\,(|a_n|+|b_n|)/c_n$ converges, and

the sequences (a_n) and (b_n) do not both converge to zero.

Then the functions $\Sigma_{n=1}^{\infty}(a_n/c_n)\exp(ic_nx)+(b_n/c_n)\exp(-ic_nx)$

and $\Sigma_{n=1}^{\infty}(a_n/c_n)\cos c_nx + (b_n/c_n)\sin c_nx$

(though both clearly continuous) are both non-differentiable for all x.

Proof Lemma 3 tells us that the first function above is non-differentiable at $x = 0$. It can then be shown to be non-differentiable everywhere, using multiplication of the coefficients by suitable factors with unit modulus. The result for the second function then follows easily.

The theorem above reminds us of a result of Hadamard:

Theorem Suppose that a_n are complex numbers, and c_n are positive integers for $n = 1, 2, 3, \ldots\ldots,$ and that $c_{n+1}/c_n > C > 1$, and that the power series $\Sigma\, a_n z^{c_n} = f(z)$ has unit radius of convergence. Then the unit circle is a natural boundary (i.e. there is no point on the circle from which analytic continuation of the function $f(z)$ is possible).

ORTHIC LIMITS OF TRIANGLES

(JCMN 58, p.6138; 60, p.6209; 61, p.6250 and 62, p.6296)

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Let g(z) be the function

$$\Sigma_{n=1}^{\infty} z^{(-2)^n}/(-2)^n \quad = - z^{-2}/2 + z^4/4 - z^{-8}/8 + \ldots$$

By the article in the previous issue (JCMN 62), if A, B and C are thought of either as complex numbers or as points, and if O = O(A, B, C) is the circumcenter of the triangle ABC, then the orthic limit of the triangle ABC is

$$L(A, B, C) = O + k[g((A-O)/k) + g((B-O)/k) + g((C-O)/k)]$$

where k = k(A, B, C) is a scaling factor such that

$$\frac{A-O}{k}\,\frac{B-O}{k}\,\frac{C-O}{k} = 1.$$

Unless A = B = C, a special case which will be discussed later, the functions O and k, thought of as functions of 3 complex variables, can be chosen to be smooth, and even holomorphic (i.e. differentiable) in each variable in a neighbourhood of (A, B, C). Questions about the differentiability of the orthic limit function L(A, B, C) are thus reduced to questions about the differentiability of the function g(a) + g(b) + g(c), where a = (A-O)/k, etc. Note that a, b and c are restricted to be on the unit circle and that abc is constrained to be 1. (Amusingly, the locus of such points is a torus). Note that a, b and c are kept fixed if the triangle ABC is dilated, rotated or translated (four of the six degrees of freedom), and thus L has directional derivatives in these directions.

Replacing a by exp(ix) and b by exp(iy), we see that differentiability questions reduce to consideration of the function $H(x, y) = h(x) + h(y) + h(-x-y)$ where x and y are real, and $h(x) = \Sigma_{n=1}^{\infty}(-2)^{-n} \exp((-2)^n ix)$

The theorem in NON-DIFFERENTIABLE FUNCTIONS (pages 6312 - 6315 above in this issue) shows that h is nowhere differentiable. But what about the function H? Does it have a directional derivative at any point? More specifically, do there exist x, y, u and v such that the single-variable function H(x+ut, y+vt) as a function of t, is differentiable at t = 0?

Returning to the case A=B=C, one can see that L does in fact have a directional derivative in every direction, arising somewhat trivially from the homogeneity of L. But L is not differentiable at this point, for (as can easily be proved), if it were the homogeneity of L would then imply that L was a linear function of A, B and C, which is not true (as can be shown from the non-differentiability in the case of isosceles triangles).

SEQUENCES WITHOUT ARITHMETIC PROGRESSIONS

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Whenever a letter such as A, B, A′ or B_n, etc. is used it denotes a set of positive integers such that no three elements of the set are in arithmetic progression.

A special case of a conjecture of Erdös is:

Conjecture 1 $\quad \Sigma_{n\epsilon A} 1/n < \infty$ for all A.

In JCMN 46, p.5107 this conjecture was proven to be equivalent to

Conjecture 2 There is a constant $C < \infty$ such that $\Sigma_{n\epsilon A} 1/n < C$ for all A.

For all natural numbers N define c(N) to be the largest possible cardinality of a set $A \subseteq \{1, 2, \ldots, N\}$. Alternatively c(N) is the supremum (over all A) of $\#\{n \epsilon A: \ n \le N\}$.

One easily sees that $c(N) \le c(N+1) \le c(N) + 1$, $c(N + M) \le c(N) + c(M)$ and $c(NM) \le Nc(M)$. This shows for example that if $N = O(M)$ then $c(N) = O(c(M))$

I first show that Conjecture 2 is equivalent to the following:

Conjecture 3 $\quad \Sigma_{n=1}^{\infty} c(3^n)/3^n < \infty$.

Proof that Conjecture 3 implies Conjecture 2.

Let A be a set as above, $A = A_0 \cup A_1 \cup \ldots\ldots$, where $A_n = \{m \epsilon A: \ 3^n \le m < 3^{n+1}\}$. The cardinality of each A_n is at most $c(3^{n+1} - 3^n) \le 2\,c(3^n)$, and for any m in A_n $1/m \le 3^{-n}$. Thus $\Sigma_{m\epsilon A} 1/m \le 2\Sigma c(3^n)/3^n$, giving Conjecture 2.

Proof that Conjecture 2 implies Conjecture 3.

For each natural number n choose $A_n \subseteq \{1, 2, \ldots 3^n\}$ with cardinality $c(3^n)$. Now consider the union of all the sets $A_n + 3^{n+1}$. It is a routine matter to check that the union contains no arithmetic progression of length 3, and hence by Conjecture 2 the sum of the reciprocals is at most C. However, since the sum of reciprocals of elements in $A_n + 3^{n+1}$ is at least $c(3^n)/(3^{n+1}+3^n)$, we thus have that $\frac{1}{4}\Sigma_{n=1}^{\infty} c(3^n)/3^n \le C$, thus proving Conjecture 3. QED

From the elementary properties of c, it can be seen that $c(3^n)/3^n$ is a decreasing sequence. However it is certainly not apparent that $c(3^n)/3^n$ tends to 0, let alone that Conjecture 3 holds. On the other hand, the obvious lower bounds for $c(3^n)$ are not much more than 2^n. This leads us to conjecture:

Conjecture 4 There is $\epsilon > 0$ such that $c(n) = O(n^{1-\epsilon})$.

This would easily imply Conjecture 3, though Conjecture 3 may hold without it. I will show in a moment that Conjecture 4 follows from an even stronger conjecture. But first we need a lemma.

Lemma Suppose that a_{ij} and b_{ij} are real numbers in [0, 1) for i = 1, 2, ..., r and j = 1, 2, ... s. Suppose also that $\Sigma_{i=1}^{r} a_{ij}b_{ij} < 1$ for all j, and that

$$c(n) < \max_j(\Sigma_{i=1}^{r} a_{ij}c(b_{ij}n)) + O(1),$$

where if x is not an integer c(x) is understood as c([x]). Then c satisfies Conjecture 4.

Proof Let b_0 be the smallest non-zero b_{ij}, and let b′ be the largest. Write $\max_j \Sigma_{i=1}^{r} a_{ij}b_{ij} = 1 - \delta$, and choose $\epsilon > 0$ such that $1 - \delta < (b_0)^{\epsilon}$. Then one can see readily that for a sufficiently large C and n, if $c(m) \le Cm^{1-\epsilon}$ for all $m \le b'n$, then $c(m) \le Cm^{1-\epsilon}$ for all $m \le n$. Thus we can prove Conjecture 4 by induction.

For the statement of the next conjecture we need another function. We define d(n) to be the largest cardinality of a union of two sets A and B that are both subsets of the set {1, 2, ..., n}. Alternatively, d(n) is the maximum of $|A| + |B|$, where A and B are disjoint subsets of {1, ..., n}. Clearly $d(n) \le 2c(n)$, with equality or near-equality holding only if one can find two disjoint, nearly-maximal subsets A and B of {1, 2, ..., n}. Because of the nature of these sets (those given by the greedy algorithm tend to look like Cantor sets), it seems reasonable to conjecture:

<u>Conjecture 5</u> There is $\epsilon > 0$ such that $d(n) \le (2-\epsilon)c(n)+O(1)$.

<u>Theorem</u> If $0 < \delta < 1$, $c(2n) \le \max(2c(n-n\delta), d(n+n\delta)) + O(1)$.

<u>Proof</u> Let $A \subseteq \{1, 2, ..., 2n\}$. We divide into two cases.
Case 1. There is no element of A in $[n-n\delta, n+n\delta]$. Then, since A is contained inside two intervals each of length $n - n\delta + O(1)$, we have $|A| \le 2c(n-n\delta) + O(1)$, as desired.
Case 2. There is an element m of A inside $[n-n\delta, n+n\delta]$. The remaining elements of A can be divided into the set A_1 of elements $< m$, and the set A_2 of elements $> m$. Since A contains no arithmetic progression of length 3, the sets A_1 and $m - A_2$ are disjoint and are contained in an interval of

length at most $n + n\delta + O(1)$, as desired.

<u>Corollary</u> Conjecture 5 implies Conjectures 1, 2, 3 and 4.
<u>Proof</u> With ϵ as in Conjecture 5, choose δ such that $(1 + \delta)(2 - \epsilon) < 2$, and apply the Theorem to the Lemma.

AN EXAMPLE

There is one well-structured example of a sequence of positive integers with this property of not containing any three in arithmetic progression. It is as follows.

The sequence $A = \{a(n)\} = \{1, 3, 4, ...\}$ is constructed as shown. The binary representation of n is the ternary representation of a(n).

n		a(n)
1	1	1
2	10	3
3	11	4
4	100	9
5	101	10
6	110	12
7	111	13
8	1000	27
9	1001	28
10	1010	30
11	1011	31
12	1100	36
13	1101	37
14	1110	39
15	1111	40
16	10000	81
17	10001	82
18	10010	84
19	10011	85
20	10100	90

There are three remarks to be made about the sequence:
(a) It contains no three numbers in arithmetic progression (as noted in JCMN 47, p.5125)
(b) The sum of reciprocals converges.
(c) The power series $f(z) = \Sigma z^{a(n)}$ has the unit circle as a natural boundary, i.e. no analytic extension across the unit circle is possible.

To prove (c), take any positive integer k and integer p, and consider the radial line $z = r \exp(2\pi ip/3^k)$ for $0 < r < 1$. $z^{(3^k)} = r^{(3^k)}$ and $a(2^k + n) = a(n) + 3^k$, and therefore

$\exp(2\pi ia(2^k+n)p/3^k) = (z/r)^{a(n)}$ for $n = 1, 2, \ldots, 2^k-1$.

$$f(z) = (z^{a(0)}+ z^{a(1)}+ \ldots + z^{a(2^k-1)})(1 + r^{(3^k)}+ r^{(2.3^k)}+ \ldots)$$
$$= (z^{a(0)}+ z^{a(1)}+ \ldots + z^{a(2^k-1)})/(1 - r^{(3^k)})$$

which $\to \infty$ as $r \to 1$. This means that the point $z = \exp(2\pi ip/3^k)$ is a singularity of the function $f(z)$. But such points are dense on the unit circle, which is therefore a natural boundary of the function. QED

Problem —— I wonder if we can add (d):

(d) $\Sigma_{n=1}^{\infty} \frac{1}{a(n)} \exp(ia(n)\theta)$ is non-differentiable everywhere.

DEMOCRACY

In setting up a mathematical model for parliamentary government, one question arising is how to model the voting policies of members of Parliament. The question may be expressed —— what objective function is the member optimizing when deciding how to vote? It is tempting to assume that members vote according to what they regard as the best interests of the country, but this supposition is not in accord with the available evidence.

In July this year the ruling party of the United Kingdom parliament was split on an important question; 24 members disagreed with the rest of the party and announced their intention of voting against the Government. The Prime Minister responded by threatening that those who voted against the Government would not be sponsored by the Party at the next General Election, (and so probably would lose their seats). Then 23 of the rebels changed their minds and voted with the Party, the other abstained.

In the absence of any other good evidence, it seems established that most members vote purely to promote their own political careers.

SUM OF A SERIES (JCMN 57, p.6098)

The problem was to find $\Sigma_{n=0}^{\infty} P_n(x)\, y^n/n!$ and the answer is $J_0(y\sqrt{1-x^2})\, e^{xy}$.

One possible proof is as follows.

In three dimensions we have Cartesian coordinates (x, y, z), cylindrical polars (ρ, φ, z), and spherical polars (r, θ, φ), connected by:-

$x = \rho \cos\varphi$, $y = \rho \sin\varphi$, $\rho = r \sin\theta$, $z = r\cos\theta$.

The function $F = J_0(\rho)\, e^z$ satisfies Laplace's equation $\partial^2F/\partial x^2 + \partial^2F/\partial y^2 + \partial^2F/\partial z^2 = 0$, and has axial symmetry about the line $x = y = 0$, and takes the value e^z on the axis. But $\Sigma_{n=0}^{\infty} r^n P_n(\cos\theta)/n!$ also satisfies Laplace's equation, has the same axial symmetry, and takes the same values on the axis where $\theta = 0$ (and so $P_n(\cos\theta) = 1$).

Therefore $J_0(\rho) \exp z = \Sigma_{n=0}^{\infty} r^n P_n(\cos\theta)/n!$

and a change of variables gives the result required.