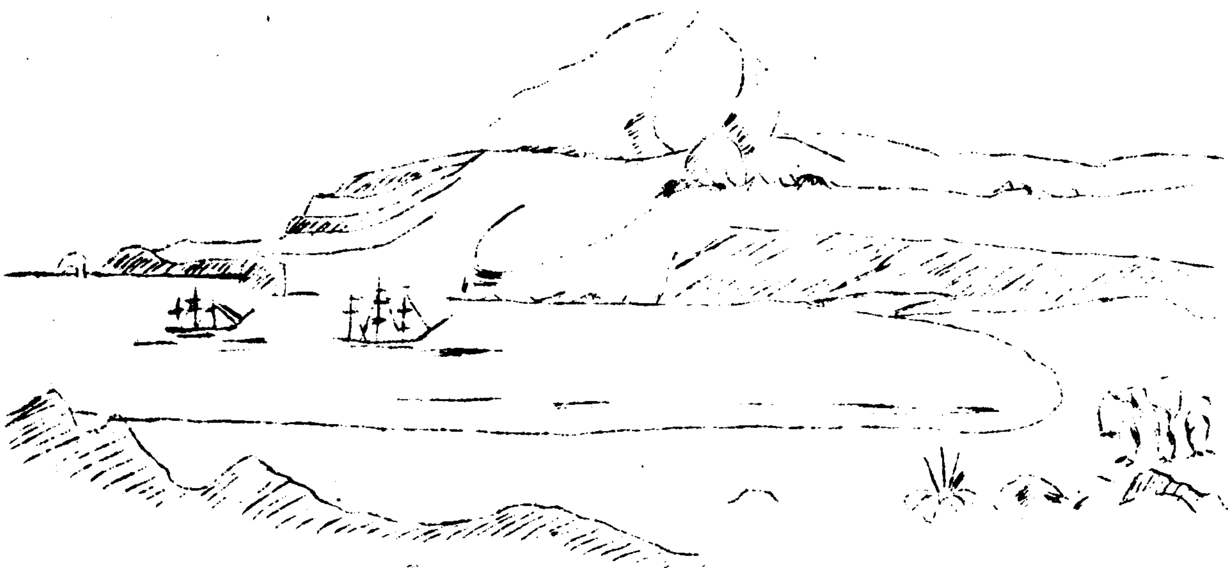
JAMES COOK MATHEMATICAL NOTES

Issue number 6, in November 1976



Captain Cook's two ships, HMS Resolution and HMS Discovery at Kerguelen Land (48° 30'S, 69° 40'E) on Christmas Day 1776. (After a drawing by William Ellis)

ACKNOWLEDGEMENT

On behalf of all readers a big thank you to Ren Potts who took over editing and publishing of the previous two issues.

THE AIRPORT WAITING ROOM (from JCMN 3)

No response has come in to this question. "Can a function in L^2 and its Fourier transform both vanish in an interval?" The substance of what L. Schwartz said to me is as follows.

Let $f(x)$ be an even function whose graph is a smooth hump on a smallish base, say from $-1/4$ to $1/4$, and let $g(x)$ be the function that has similar humps repeated with unit period. Let their Fourier transforms be F and G, the latter is a row of delta functions at integer spacing (we use the now popular convention of defining the transform with 2π in the index, that is $f(x) = \int F(y) \exp 2ixy\, dy$) Then the convolution $f*G$ and its Fourier transform Fg are the required functions, both vanish in the middle half of each unit interval. They are in L^2 because the smoothness of the hump makes F sufficiently small at infinity.

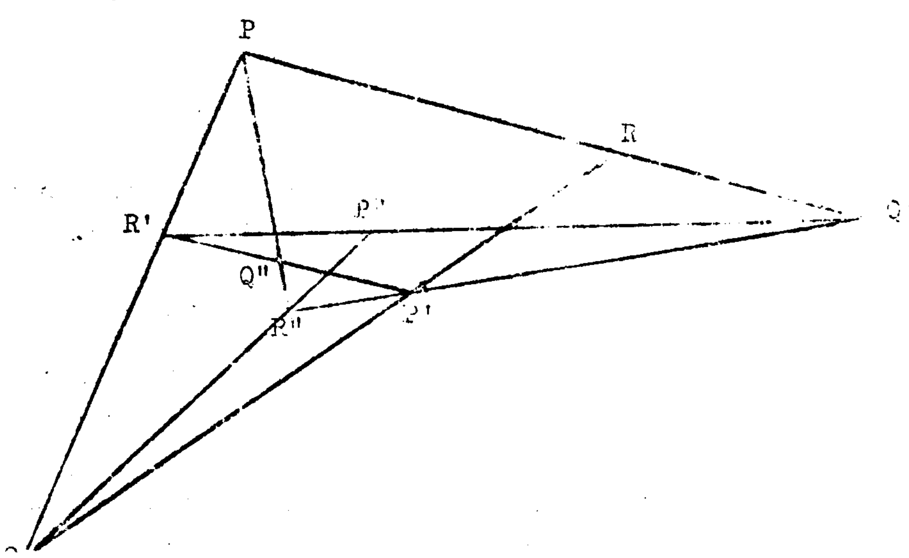
MORE ON GUINAND'S THEOREM.

The question asked in our third issue was as follows. Suppose that A, B and C are 3 x 3 matrices and ABC = I, and both A and B have all diagonal elements zero and all others non-zero. It is required to show that if two of the diagonal elements of C are zero then so is the third.

J.B. Parker's solution invokes the aid of geometry. Represent any triangle XYZ of the projective plane by a matrix M that has the coordinates of each vertex as a column, the first column being the coordinates of X, etc. Then any non-singular matrix S can be regarded as a transformation of triangles, mapping XYZ into X'Y'Z' which is represented by the matrix product MS. With this representation a zero on the diagonal of S has a simple geometrical interpretation. If the top left hand element of S is zero then the first column of the product MS is a linear combination of the second and third columns of M, that is X' is on YZ. The converse is true if M is non-singular (that is if the triangle XYZ is non-degenerate), if X' is on YZ then $s_{11} = 0$.

Having set up our machinery we can tackle the problem. The unit matrix I corresponds to the triangle of reference, denoted by PQR because the traditional letters ABC already have other meanings. Let A represent the triangle P'Q'R', since the diagonal elements are zero, P' is on the side QR (the line $x = 0$) of the triangle of reference, and similarly Q' on PR and R' on PQ. Now the matrix AB represents a triangle P"Q"R" which is the image under B of P'Q'R', and because B has diagonal elements zero it follows that P" is on Q'R', etc.

It is given that ABC = I, that is the image of P"Q"R" under C is PQR. Now suppose that two of the diagonal elements of C are zero, without loss of generality we may take them to be the first and second. Then P is on Q"R" and Q is on P"R". This gives the figure below.



This is the figure for the theorem of Pappus, PQR' and P'Q'R" are straight lines, and the three cross-joins are

QR" meeting Q'R' at P"
PR" meeting P'R' at Q"
and PQ' meeting P'Q at R

The theorem of Pappus tells us that the cross-joins are collinear, that is R is on P"Q" and so the third diagonal element of the matrix C is zero, which proves Guinand's theorem.

DAME MARY CARTWRIGHT'S ANECDOTE (from JCMN 5)

After discussing the matter with K. Stewartson your editor has attempted to reconstruct the examination question as follows.

By considering the integral of $z/\sinh z$ over an indented rectangular contour or otherwise shew that

$$\int_0^\infty x/\sinh x \ dx = \pi^2/4.$$

The examiners' intention was that the candidates should integrate round the rectangle with sides $x = \pm R$, $y = 0$ and $y = \pi$, with a semicircular indentation to exclude the simple pole at $z = i\pi$. As R tends to infinity and the radius of the indentation to zero the contributions from the vertical sides tend to zero and from the horizontal sides to four times the required integral. The small semicircle round the simple pole gives π^2, which leads to the required result.

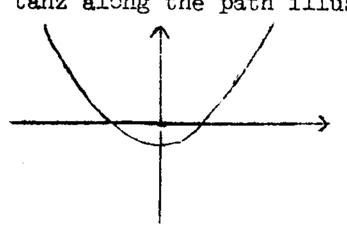
Examiners in those days would put in the phrase "or otherwise" because they thought it more important for the candidate to solve the problem than to follow the method that suggested itself to the examiner. If you are in any danger of having a genius in your class this is a wise precaution.

Another way to evaluate the integral is to put

$$x/\sinh x = 2x\ (\exp{-x} + \exp{-3x} + \ldots)$$

and then integration term by term gives $2(1 + 1/3^2 + 1/5^2 + \ldots)$ but how can this be done with contours and residues?

Back in Townsville I showed a first draft of the note above to B.B. Newman who was sceptical and suggested a different approach. Perhaps the question was:- Evaluate the integral of $z^{-2}\tan z$ along the path illustrated below (cutting the real axis at $z = \pm 1$)



If you make a closed contour by adding a large circular arc at the top then you find inside only one pole which has residue 1, but doing it the harder way with a circular arc coming round the bottom of the picture you find infinitely many poles, with residues $-(4/\pi^2)\ (2n-1)^{-2}$ for all integer n.

FUNCTIONS OF TWO VARIABLES

Construct a function $f(x, y)$ differentiable in the plane and such that $f_y(x, 0)$ has a jump discontinuity at $x = 0$. (Here differentiability means $f(x + h, y + k) - f(x, y) = hf_x(x, y) + kf_y(x, y) + o(\sqrt{h^2+k^2})$ as $h, k \to 0$)

G.R. Morris

100 STATEMENTS (from JCMN 5)

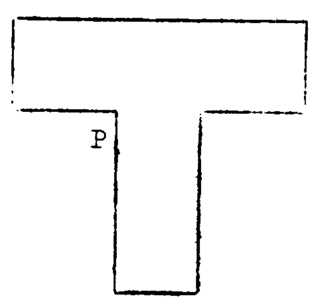
John Hickman from ANU, Canberra writes:
If each statement "k of these statements are incorrect" means "Exactly k of these statements are incorrect" then it is obvious that exactly one of the 100 statements listed is correct, namely the 99th one.

If, however, each is interpreted to mean "At least k of these statements are incorrect" and if furthermore the number 100 is replaced by an arbitrary positive integer n, then the situation becomes more interesting. If such matters as the logical significance of self-referential statements are ignored, it turns out that if n is even then the first $\frac{n}{2}$ statements are correct and the rest incorrect, whilst if n is odd then the n statements form an internally inconsistent system (thus generalizing the classical case of n = 1).

OBJECTIVE TESTING

In our embryonic first issue of late 1975 there was reproduced a question put out by the Australian Council for Educational Research, as follows:

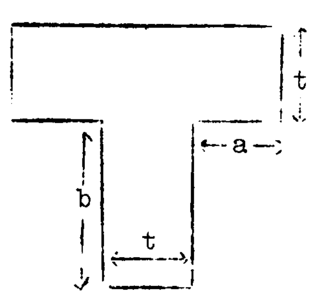
The T-shaped figure has an area of 5 square units. A straight line is drawn through P so as to divide the figure into two parts of equal area. In how many ways can this be done?



(A) 1 (C) 3
(B) 2 (D) It is not possible

Your editor's original comment was that by suitably choosing the proportions of the T the number of ways can be made equal to 1, 2 or 3. More recently it has been pointed out to me by A. Brown that the number can also be made infinite.

If a = b = t, number is 1
If a = 2t and b = 3t number is 2
If a = 2t and b = 4t number is 3
If a = t and b = 3t number is infinite.



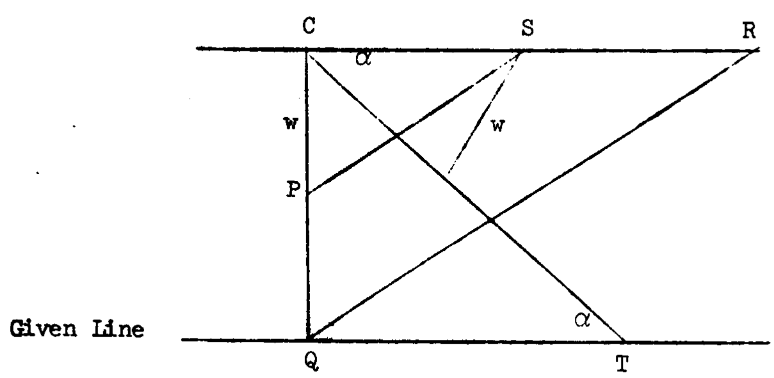
NEW THEOREM (from JCMN 4)

"Almost every convex polyhedron is a tetrahedron". No proof or disproof has yet been sent in; please do not send the editor boxes of assorted wooden blocks, for we do not find that sort of evidence convincing. Answer promised for JCMN 7.

GEOMETRY WITHOUT COMPASSES (from JCMN 3)

The surprising result is that a ruler with parallel sides is a very good substitute for the compasses of the traditional geometer, it enables you to make any Euclidean construction except actually drawing circles. This can be shown in six steps. Each except the last is almost obvious using the ones before.

1. You can erect a perpendicular from a given point on a given line.
2. You can double a given length, that is on PQ find R so that RP = PQ.
3. You can draw a parallel from a given point P to a given line.
4. You can bisect an angle.
5. You can cut off a given length from a given point along a given line.
6. Finding the intersection of a given line and the circle with given centre C and radius r.



Draw CQ perpendicular and CR parallel to the given line, with CR = r. Make CP equal to the ruler width w and PS parallel to QR. Put your ruler with one edge on S and the other on C (possible in two ways, one shown above) and draw CT meeting the given line in T, the point required.

Proof $\sin\alpha = w/CS = CQ/r$

THE CENTRE OF AUSTRALIA

In "Wildlife" for September there is a letter to the editor on this subject saying:-

"I venture to state that the geographic centre is really the centroid of the area. This is the point on the area where a line in any direction through it, but on the same plane as the area, will bisect this area... However by finding the centre of gravity of a thin card, of uniform thickness, in the shape of a given area we can find its centroid" On a picture is shown how the writer finds the centroid to be at Lilla Creek just north of the S.A. - N.T. border.

This method seems to ignore Magnetic Island (charted by James Cook in June 1770) and other offshore islands, but let that pass. The writer's ideas about centroids give us a problem - How many ways are there of bisecting Australia by a line through the centroid? And as so often happens the applied mathematical problem suggests one in pure - for what numbers n can you find a plane set bisected in just n ways by lines through the centroid?

NEW QUESTION (from JCMN 5)

If $f'(x)$ is bounded then is $(f'(x))^2$ a derivative? G.R. Morris answers NO. His proof depends on the fact that any derivative has the intermediate value property - in an interval it takes all values between those at the end points. (This is clear if you sketch a graph of a function with positive derivative at one point and negative at another.)

$$\text{Put } f(x) = x^2 \sin 1/x \quad (x \neq 0) \text{ and } = 0 (x = 0)$$
$$g(x) = x^2 \sin 2/x \quad (x \neq 0) \text{ and } = 0 (x = 0)$$

These functions are both differentiable and the function

$$g'(x)/4 + (f'(x))^2 = \tfrac{1}{2} - (3/2)x \sin 2/x + 4x^2 \sin^2 1/x \quad (x \neq 0)$$
$$= 0 \quad \text{if} \quad x = 0$$

does not have the intermediate value property and so is not a derivative. Therefore $(f')^2$ is not a derivative.

A merry Christmas to everybody. Your editor would like to hear from you anything connected with mathematics or with James Cook.

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