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This is an image of a postage stamp issued by Hungary to commemorate Captain Cook's explorations. The stamp is from Paul Erdős.

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CONTENTS

IRRATIONAL SUMS

Paul Erdös

(a) Let u_n be a sequence of integers satisfying

$2^{-2^n} u_n \to 1$ as $n \to \infty$. Is it then true that

$\sum 1/u_n$ is irrational?

(b) In the Indian Journal in 1948 I proved that

$\sum_{n=1}^{\infty} 1/(2^n-1)$ is irrational. I think that if

$n_1 < n_2 < n_3$ ... is any infinite sequence of positive integers then $\sum_{k=1}^{\infty} 1/(2^{(n_k)} - 1)$ is also irrational. This will perhaps not be easy.

CONGRATULATIONS

Esther Szekeres was given the degree of Doctor of Science (honoris causa) by Macquarie University on the 2nd. May, 1990.

QUOTATION CORNER 30

"You can vote before the election if, on election day, you ... are ill, or ... "

— From a circular distributed by the Australian Election Commission before the Federal Election of March 24th.

LETTERS IN WRONG ENVELOPES (JCMN 51, p.5243)

Mark Kisin

If n letters are put in their envelopes at random, let Y be the random variable representing the number of letters that are in their correct envelopes. Use $E_n(\ )$ to denote the expectation of any random variable, where n is the number of letters. What can be said about the moments of Y?

For $0 \leq k \leq n$ let $p_n(k)$ be the number of ways of putting the letters with exactly k in their right envelopes. Clearly $p_n(k) = \binom{n}{k} p_{n-k}(0)$ .... (i)

Let $q_n(y) = p_n(y)/n!$ be the probability that the random variable Y takes the value y. Using the inclusion-exclusion principle

$p_r(0) = r! - \binom{r}{1}(r-1)! + \binom{r}{2}(r-2)! - \ldots + (-1)^r$

$= r!(1 - 1/1! + 1/2! - \ldots + (-1)^r/r!)$ .... (ii)

Consequently from (i) we have, for $0 \leq k \leq n$:

$p_n(k) = \frac{n!}{k!}(1 - 1/1! + 1/2! - \ldots + (-1)^{n-k}/(n-k)!)$ ... (iii)

$q_n(y) = (1/y!)(1 - 1/1! + 1/2! - \ldots + (-1)^{n-y}/(n-y)!)$ ... (iv)

Note that $q_n(y) = 0$ if either $y < 0$ or $y > n$, and so from (iv) it follows that in all cases

$$y\, q_n(y) = q_{n-1}(y-1) \qquad \ldots \quad (v)$$

At this stage it is possible to calculate a few of the moments of Y, for example $E_n(Y) = \sum y\, q_n(y) = \sum q_{n-1}(y-1) = 1$, and $E_n(Y^2) = \sum y^2 q_n(y) = \sum y\, q_{n-1}(y-1) = \sum (1+y) q_{n-1}(y) = 2$. The summations above are over all integer y. But now we return to more general results.

<u>Theorem</u> If F is a polynomial then $E_n(F(Y))$ is independent of n for $n \geq$ the degree of F.

<u>Proof</u> Use induction on n. Supposing the proposition untrue, take the smallest n for which it fails.

There must be a polynomial f of degree $\leq n$ such that $E_{n+1}(f(Y)) \neq E_n(f(Y))$. Therefore there is $m \leq n$ such that $E_{n+1}(Y^m) \neq E_n(Y^m)$. But

$$\begin{aligned}
E_n(Y^m) &= \sum q_n(y)\, y^m && \text{(now use (v) above)}\\
&= \sum q_{n-1}(y-1)\, y^{m-1} && \text{(now change the dummy variable)}\\
&= \sum q_{n-1}(y)(y+1)^{m-1}\\
&= E_{n-1}((Y+1)^{m-1}) && \text{(now use the induction hypothesis)}\\
&= E_n((Y+1)^{m-1})\\
&= \sum q_n(y)(y+1)^{m-1} && \text{(now change the dummy variable)}\\
&= \sum q_n(y-1)y^{m-1} && \text{(now use (v) above again)}\\
&= \sum q_{n+1}(y)y^m = E_{n+1}(Y^m).
\end{aligned}$$

The proof by induction is now complete.

Corollary If $m \leq n$ then $E_n(Y^m) = E_n((Y+1)^{m-1})$ ... (vi)

A nice way of proceeding from here is to let $n \to \infty$, replacing $q_n(y)$ by $q_\infty(y) = e^{-1}/y!$ In other words we take the distribution of Y to be Poisson with unit mean. The theorem above shows how this gives us the right answers when we calculate expectations of polynomials of degree $\leq n$.

If F is any such polynomial then

$$e\,E(F(Y)) = \sum F(y)/y! \qquad \text{..... (vii)}$$

To any polynomial F corresponds a function $A(x) = \sum_{y=0}^{\infty} F(y)\, x^y/y!$ which is of interest.

Example 1 If $F = 1$ then $A(x) = e^x$, and for any $j \leq n$

$$e^x (d/dx)^j A(x) = \sum_y y(y-1)\ \dots\ (y-j+1)x^{y-j}/y!$$

Putting $x = 1$ gives $E_\infty(Y(Y-1)\dots(Y-j+1)) = 1$ ... (viii)

This is called the factorial moment of order j for the random variable Y. The factorial moments of order $>$ n are all zero. It is now clear that the moments of Y are all positive integers, they can be calculated from the factorial moments.

Example 2 Let $M(t) = \sum_{y=0}^{\infty} y^t/y!$ which, for $t \leq n$ is $1/e$ times the moment of order t for Y. Let $H(x) = \sum_{i=0}^{\infty} M(i)x^i/i!$

$$= \sum\sum (xy)^i/(i!y!) = \sum e^{xy}/y! = e^{(e^x)} \qquad \text{.... (ix)}$$

Hence by Taylor's Theorem M(t) is the value when $x=0$ of $(d/dx)^t e^{(e^x)}$. This gives another way of calculating the moments.

Example 3 Let $L(t) = e^{-1}\sum_{s=0}^{\infty}(s-1)^t/s!$

$$\begin{aligned}
H(x) &= \sum_{t=0}^{\infty} L(t)x^t/t! = e^{-1}\sum\sum \frac{(xs-x)^t}{s!\ t!}\\
&= e^{-1}\sum e^{xs-x}/s! = e^{-1-x}e^{(e^x)} = \exp(e^x-1-x).
\end{aligned}$$

L(t), which is the moment of order t for Y about its mean, is therefore the value at $x=0$ of $(d/dx)^t\exp(e^x-1-x)$.

Example 4 Another way to calculate any moment $E_n(Y^m)$ with $m \leq n$ is as follows. It is $e^{-1}\sum_0^{\infty} y^m/y! = e^{-1}f_m(1)$, where $f_m(x) = \sum_0^{\infty} y^m x^y/y!$ which can be calculated recursively from $f_m(x) = x f'_{m-1}(x)$ and $f_0(x) = e^x$.

Using any of the methods above, we find for $t \leq n$

t	0	1	2	3	4	5	6
$E(Y^t)$	1	1	2	5	15	52	203

These calculations are also related to Terry Tao's Binomial Identity 31.

$$\begin{aligned}
\sum_{m=0}^{n}\sum_{i=0}^{m} \frac{(-1)^i m^2}{i!(n-m)!} &= \sum_{m=0}^{n} m^2 q_n(n-m) = \sum_{m=0}^{n}(n-m)^2 q_n(m)\\
&= n^2\sum_{m=0}^{n} q_n(m) - 2n\sum_{m=0}^{n} m\, q_n(m) + \sum_{m=0}^{n} m^2 q_n(m)\\
&= n^2 - 2n + 2, \text{ as required.}
\end{aligned}$$

Binomial Identity 31 (JCMN 51, p. 5227)

C. C. Rousseau

Note that the identity to be proved can be written

$$\sum_{m=0}^{n} \frac{(n-m)^2}{m!} \sum_{i=0}^{n-m} \frac{(-1)^i}{i!} = n^2 - 2n + 2 \qquad (n \geq 2).$$

We obtain this result as a consequence of a more general identity and show the connection between the latter and well-known formulas for Bell numbers. Following the notation of Graham, Knuth and Patashnik in their recent book **Concrete Mathematics**, we use $x^{\underline{k}}$ to denote the falling factorial function $x(x-1)\cdots(x-k+1)$. On the vector space of all formal series

$$\phi(x) = \sum_{k=0}^{\infty} c_k \, x^{\underline{k}} \tag{1}$$

where

$$\sum_{k=0}^{\infty} |c_k| < +\infty,$$

define the sequence of linear functionals (L_n) by

$$L_n(\phi) = \sum_{m=0}^{n} \frac{\phi(m)}{m!} \sum_{i=0}^{n-m} \frac{(-1)^i}{i!}, \qquad (n = 0, 1, 2 \ldots).$$

We claim that with ϕ given by (1),

$$L_n(\phi) = \sum_{k \leq n} c_k. \tag{2}$$

To see this, first note that

$$\sum_{i=0}^{p} \frac{(-1)^i}{i!} = [t^p] \, \frac{e^{-t}}{1-t},$$

where $[t^p]\, f(t)$ is the coefficient of t^p in the series expansion of f. It is thus apparent that

$$L_n(a^x) = [t^n] \, \frac{e^{(a-1)t}}{1-t}.$$

Differentiating this equation k times with respect to a and setting $a = 1$, we obtain

$$L_n(x^{\underline{k}}) = [t^n] \, \frac{t^k}{1-t} = \begin{cases} 0 & n < k \\ 1 & n \geq k. \end{cases}$$

Thus (2) follows. To evaluate $L_n(\phi)$ for the case in which $\phi(x) = (n-x)^2 = n^2 - 2nx + x^2$, we use

$$1 = x^{\underline{0}}, \qquad x = x^{\underline{1}}, \qquad x^2 = x^{\underline{2}} + x^{\underline{1}},$$

to find

$$L_n(1) = L_n(x) = 1, \qquad L_n(x^2) = 2$$

and thus $L_n(\phi) = n^2 - 2n + 2$ for all $n \geq 2$.

There are other amusing consequences of (2). First, in view of the formula

$$x^r = \sum_{k=1}^{r} S(r,k) \, x^{\underline{k}},$$

$L_n(x^r)$ is a sum of Stirling numbers of the second kind:

$$L_n(x^r) = \sum_{k \leq n} S(r,k).$$

Thus

$$L_n(x^r) = B_r, \qquad (n \geq r),$$

where B_r denotes the rth Bell number. Letting $n \to \infty$, we obtain the limit functional

$$L_\infty(\phi) = e^{-1} \sum_{m=0}^{\infty} \frac{\phi(m)}{m!}.$$

With $\phi(x) = x^{r+1}$, this leads to Dobinski's formula:

$$\begin{aligned} B_{r+1} &= L_\infty(x^{r+1}) \\ &= e^{-1} \sum_{m=1}^{\infty} \frac{m^r}{(m-1)!}. \end{aligned}$$

Also, we can obtain the generating function for Bell numbers using an argument similar to that of Rota [The number of partitions of a set, *Amer. Math. Monthly* **71** (1964), pp. 498-504]:

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{B_r}{r!} t^r &= \sum_{r=0}^{\infty} \frac{L_\infty(x^r)}{r!} t^r \\ &= L_\infty(e^{xt}) \\ &= e^{-1} \sum_{m=0}^{\infty} \frac{(e^t)^m}{m!} \\ &= \exp\left(e^t - 1\right). \end{aligned}$$

THE DERANGED KNIGHTS OF CAMELOT

Marta Sved

It was something of a revolution sparked by Sir Mordred, always the most pugnacious knight. It came at a time after he scored great successes in the spring games. When the knights came in to their round table, he demanded that they form a ranking order, naturally putting himself well in front. He distributed leaflets to the knights, marking the rank of each.

When King Arthur became aware of what was going on he was very angry:

— Here in Camelot we have no ranks or privileges. We serve the common good as equal brethren adhering to our ideals. This is the reason for having a round table. I see that each of you holds a leaflet marking his rank, I want you to keep it so that no one ever claims the place accorded by this rank number. —

— Your majesty, it will be very hard, if not impossible to stick to this rule. — countered Sir Gawain.

Merlin smiled; — It will not be that hard. There are many arrangements that satisfy this new rule, the rule of derangements. By now you should all be familiar with the inclusion-exclusion principle, and able to arrange derangements —

Sir Lancelot volunteered: — We all know that the total number of possible arrangements (when the n seats at the table have been labelled) is $n! = n(n-1)(n-2)\ldots 2.1$. All we have to do is to take away all the arrangements where any one of us is at the place allotted to him by Sir Mordred. There are n such places, and the rest can be arranged in $(n-1)!$ ways, ...

— This is $(n-1)!$ multiplied by n, — exclaimed Sir Mordred triumphantly — if we subtract $n(n-1)! = n!$ from $n!$ we obtain zero. The king demands the impossible if he wants to displace every one of us from his rightful place. —

— It seems, Sir Mordred — laughed Merlin — that you for one forgot the Inclusion-Exclusion principle. —

— Sir Mordred — said Sir Lancelot — you have been too eager to make your point, and did not wait for the end of my story. I was fully aware that I subtracted twice the number of arrangements when both you and I were to occupy the places

allotted by you, with $(n-2)!$ arrangements for the rest of the knights, and this error would occur for each of the possible $\binom{n}{2}$ pairs occupying forbidden places. Hence, to compensate, we add $\binom{n}{2}(n-2)!$ —

— The story does not end here, — added Sir Gareth, — Thinking of 3 knights in illegal positions, we then subtract $\binom{n}{3}(n-3)!$, and going on, using the I-E principle we finally obtain $n! - n(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \ldots$ —

— Ending with +1 or -1, — nodded Merlin. — We could simplify, and write more neatly

$$D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right),$$

using the symbol $D(n)$ for the total number of derangements. —

— Did you say total number, Merlin? — asked Sir Archibald. — Could you have just a partial number of derangements? —

— Of course, — answered Merlin. — It is possible that just 2 or 3 or k people are in unassigned places, and they can be chosen in $\binom{n}{2}$, $\binom{n}{3}$ or $\binom{n}{k}$ ways. —

— What about 1 person? — came a voice from the back of the hall. Queen Guinevere, who had slipped in quietly, listening to the debate, gave the answer:

— Well, I can not see that just one person moves to an unassigned place, and the knight who had occupied it still stays there. —

— Listen to the voice of common sense, — said Merlin, — of course we will have $D(1)=0$, while $D(0)=1$, since there is just one way in which everybody stays in the assigned place, and so none is deranged. Looking then at all these cases, we get all the arrangements:

$$\binom{n}{0}D(0) + \binom{n}{1}D(1) + \binom{n}{2}D(2) + \ldots + \binom{n}{n}D(n) = n!$$

King Arthur produced suddenly a shining helmet and a beautiful sword. — I have intended for some time to give these two presents to one or two deserving knights. Of course, after to-day's happenings, I would not call any knight standing in the place marked by Sir Mordred a deserving knight. —

Sir Mordred was quick to take up the challenge. He did not want to fall out of grace for good.

— Suppose — he said — that there are exactly k deranged knights eligible for the presents, who may be found in $\binom{n}{k}$ different ways, then the presents can be allocated to these in k^2 ways, so this case would give $k^2\binom{n}{k}D(k)$ possibilities. —

— Quite so, — added Merlin — and since k can take any value from 0 to n, the total number of possibilities is

$$\sum_{k=0}^{n} k^2\binom{n}{k}D(k). \text{ —}$$

— You consider $k = 0$ or 1? — asked Sir Gawain. — The contribution of these cases to the sum is zero, — answered Merlin — I see no harm in including them. —

— This gives a very long sum, and you must know $D(k)$ for each k. — objected Sir Lancelot.

— If you are willing to use again the inclusion-exclusion principle (the second time to-day), you could get a simpler answer. — said Merlin.

— You mean — asked Sir Lancelot — that we should begin with counting the possible allocations of the presents by ignoring the restrictions imposed by the King? —

— Correct — said Merlin.

— In that case we would have $n^2 n!$ possibilities — said Sir Lancelot — allowing for a total of $n!$ permutations and n^2 ways of allotting the presents to 1 or 2 of the n knights. —

Merlin continued: — Now, following our usual procedure, we subtract first all the arrangements where the sword goes to the wrong place. — — You mean that the sword goes to a knight taking his assigned place? — asked Sir Archibald.

— Yes,— was the answer — and so we must subtract $n^2(n-1)!$ arrangements, since there are n choices for a possible "wrong" place, $(n-1)!$ arrangements for the remaining knights and n choices for allocating the helmet (without restrictions). —

— But — added Sir Archibald — should you not consider the cases where the helmet goes to a wrong place? —

— Yes, surely,— said Merlin — by the inclusion-exclusion principle we will have to subtract $2n^2(n-1)! = 2n.n!$ arrangements, considering both the sword and the helmet. —

— And this is not the end, of course — said King Arthur.

— No — said Merlin — but the end comes pretty fast, for we must consider for corrections only the case when both sword and helmet go to wrong places. This can occur in two ways. In the first case there are two wrongly placed recipients, chosen in $n(n-1)$ ways, with $(n-2)!$ possible arrangements for the others, hence altogether $n(n-1)(n-2)! = n!$ possibilities. In the second place, when both presents go to one illegally placed knight, we have again $n!$ possibilities, for there are n choices for the placement and $(n-1)!$ arrangements for the other knights. So, adding $n!$ again, we obtain all the arrangements, and get the simple formula

$$(n^2 - 2n + 2)n! \quad \text{instead of} \quad \sum_{k=0}^{n} k^2\binom{n}{k}D(k).$$

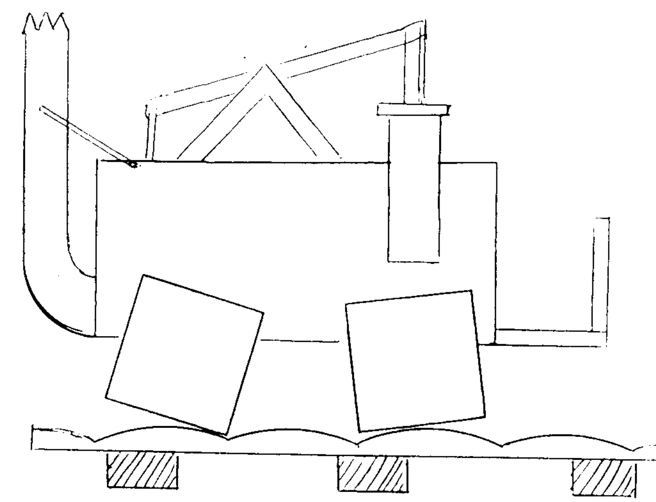
— It is much simpler indeed — said the King. What is more — added Merlin — Binomial Identity 31 on page 327 of JCMN 51 will be $\sum_{m=0}^{n}\sum_{i=0}^{m}\frac{(-1)^i m^2}{i!(n-m)!} = n^2$ [illegible], which comes to this very formula if you multiply both [illegible].

THE WONDERFUL WHEEL
Mark Kisin

The following is an address given by Rolland Devian, at the celebration of the five billionth anniversary (standard Galactic years) of the discovery of the wheel by the inhabitants of Sol III, the oldest race in the Milky Way:

"All civilizations that are on record at the head office of the Bureau of Galactic Statistics, on Antares VI, have at some stage or other invented the wheel. This has come in various forms, but in all cases the implementation of an object with a round surface, be it sphere or cylinder, was achieved. Research conducted by the now famous archaeologist, Friedrick Kleinbottle (the surname passed down for generations was due to a hereditary deformity of the nose in his family), showed that the stage at which the wheel is discovered is dependent on the shape of the protrusion housing the major cerebral centre (it is of course a well known, if unexplained, fact that such a protrusion is to be found in all the sentient species so far discovered). Thus in the case of the beings of Vega II, who have perfectly spherical cerebral containers, the wheel was discovered before the Vegans had become at all sentient. Indeed the present Vegans do not consider the inventor of the wheel on their planet to have been of their race. Traces have been found, however, of these pre-Vegans, or hemiglobii, as they are locally known, together with small spheres, which seemed to have been grasped by the animals. What is incredible is that no other artifact was being made by the hemiglobii during this period, only the wheels, which would have been quite useless in isolation. This dealt a serious blow to the then popular theory that wheels, like other artifacts, were created when they were needed by the evolving civilization. What remained of the theory was, of course, demolished by the now famous discovery of a planet whose sentient inhabitants were themselves spherical, and on which stone spheres had begun appearing before there was any life at all on the planet.

"It is often assumed (quite wrongly) by species possessed of a reasonably round cranium, that despite Kleinbotttle's theory, a civilization must neverless neeed the wheel in a fairly early stage of development. This mistake is made from the assumption that civilization cannot progress beyond a certain primitive stage without the wheel. The wheel, however, is remarkably deceptive in these matters, and just as there have been many cases of species developing the wheel before there could have been any possible use for it, so civilizations have developed to very advanced stages without a hint of the wheel. We are all aware, gentlebeings, of the case of the inhabitants of Beta Centauri I, affectionately known as 'squareheads'. They went through all the normal phases of civilization, discovering steam, electricity, nuclear energy, and local space travel, but without the need for wheels. Those of us ignorant of the details may well ask how they provided smooth transportation without anything resembling a wheel, which term (let us not forget) covers not only circular cylinders and spheres, but also all shapes of constant width, which can be used as rollers. The remarkable fact, gentlebeings, is that the 'squareheads' used squares to support their vehicles, and the rails on which the vehicles travelled were shaped to give a perfectly smooth ride. All other apparent difficulties were



An early steam locomotive from Beta Centauri I

overcome in equally ingenious and remarkable ways.

"Having developed local space travel, and even seeing their planet from space, the Beta Centaurians failed to get the hint. Finally, when they discovered interstellar space travel, their ships met those of the Galactic Federation. The Beta Centaurian commander, coming on board a Federation ship, was amazed at the 'strange un-square like things', as he put it , that were used throughout the ship. Although he finally acknowledged that the various wheels served their purpose, he did not see any need for them, and maintained that this was indeed a very 'round-about' way of doing things. To this day, gentlebeings, the Beta Centaurians have not adopted the wheel. its use in their society remains extremely limited, and is mainly for the benefit of tourists, who flock to Beta Centauri I to see the 'wavy railways', and other sights

"It is now evident, on this five billionth anniversary of the wheel in our galaxy, that the wheel is no mere common artifact, subject to the whim of a developing civilization. No, gentlebeings, the wheel is something greater than an artifact, it comes and goes as it chooses, blessing some with its presence, abandoning others to their fate. Gentlebeings, I would like to propose a toast: To the wheel, happy birthday and many happy returns!"

K-FOLD FUNCTIONS

(JCMN, 31, p.3180, 39, p.4174 40, p.4188)

H. Burkill

A k-fold function is a real function taking each of its values exactly k times. Define $\lambda(k)$ as the minimum possible number of discontinuities of a k-fold function on an open interval. Similarly $\mu(k)$ for functions on a half-open interval such as (0, 1], and $\nu(k)$ for functions on a compact interval such as [0, 1].

Previous contributions have established the following values and inequalities:

k	1	2	3	4	odd	even	∞
$\lambda(k)$	0	∞	0		0		0
$\mu(k)$	0	1	1	≥ 1	$\leq \frac{1}{2}k-\frac{1}{2}$ and ≥ 1	$\leq \frac{1}{2}k$ and ≥ 1	0
$\nu(k)$	0	∞	1	1	$\leq \frac{1}{2}k-\frac{1}{2}$ and ≥ 1	$\leq \frac{1}{2}k-1$ and ≥ 1	0

A little more can now be proved. We define (for the purposes of this note) a (p, q, r) function as a function on an open interval taking every positive value exactly p times, taking the value zero exactly q times and taking every negative value exactly r times.

It is clear that for any k = 0, 1, 2, ... we can find a (2k+1, 2k+1, 2k+1) function, which is saying that $\lambda(2k+1) = 0$. In figure 1 below is sketched the case k=2. Adapting this idea, we can find a (2k+1, k, 0) function and a (2k+1, 0, 0) function. These are sketched for k=2 in fig. 2 and fig.3, both on the interval $(-\infty, 0)$.

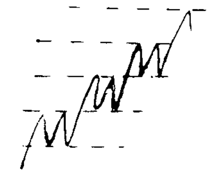


fig. 1

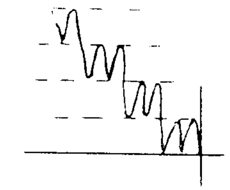


fig. 2

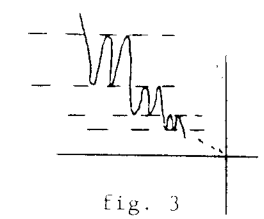


fig. 3

Putting these ideas together, we can construct a (3, 2, 3) function (fig. 4), or more generally a (2k+1, k+1, 2k+1) function, and a (3, 1, 3) function (fig. 5), or more generally a (2k+1, 1, 2k+1) function.

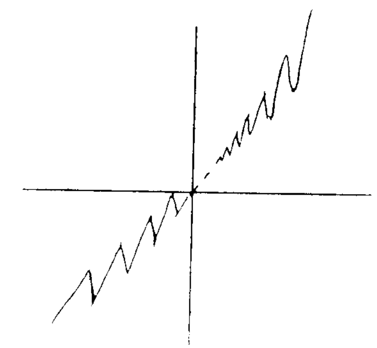


fig. 4

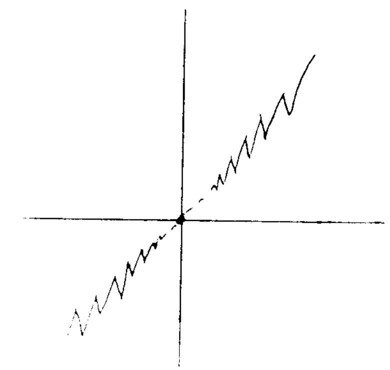


fig. 5

<u>Theorem 1</u> For any k = 0, 1, 2, ... $\lambda(2k+4) \leq 1$.

Proof Take a (2k+1, 2k+1, 2k+1) function on (0, 1), and a (3, 2, 3) function on (1, 2), putting f(1) = 0. This gives a 2k+4-fold function on the open interval (0, 2) with a discontinuity only at 1.

<u>Theorem 2</u> For k = 0, 1, 2, ... , $\mu(2k+4) \leq 2$.

Proof Take a (2k+1, 2k+1, 2k+1) function on (0, 1), and a (3, 1, 3) function on (1, 2), putting f(1) = f(2) = 0. This gives a 2k+4-fold function on the half-open interval (0, 2], with discontinuities only at 1 and 2.

<u>Theorem 3</u> For k = 0, 1, 2, ... , $\mu(2k+9) \leq 3$.

Proof Take a (0, 3, 7) function on (0, 1)
and a (1, 1, 1) - - - - (1, 2)
and a (2k+1, 2k+1, 2k+1) - - - - (2, 3)
and a (7, 0, 0) - - - - (3, 4)
with f(1) = f(2) = f(3) = f(4) = 0. See fig. 6 below.
This is a 2k+9-fold function on the half-open interval (0, 4] with discontinuities only at 1, 2 and 3.

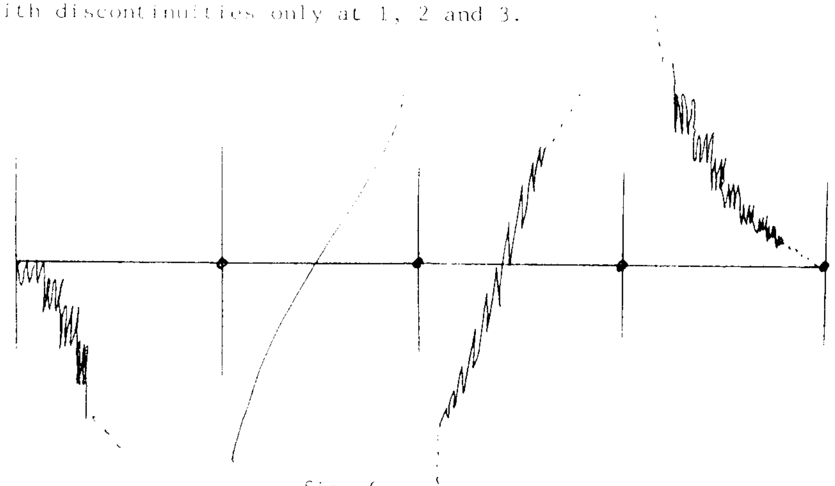


fig. 6

Note that this theorem tells us nothing about $\mu(7)$, but previous results give $\mu(7) \leq 3$.

<u>Theorem 4</u> $\nu(5) = 1$.

Proof Take a (5, 2, 0) function on (0, 1) and a (0, 0, 5) function on (1, 2), putting f(0) = f(1) = f(2) = 0, see fig. 7. This gives a 5-fold function on the compact interval [0, 2], with only one discontinuity, proving $\nu(5) \leq 1$. For the opposite inequality see p.4176 in JCMN 39.

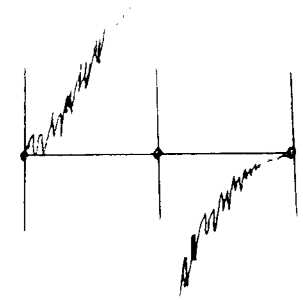


fig. 7

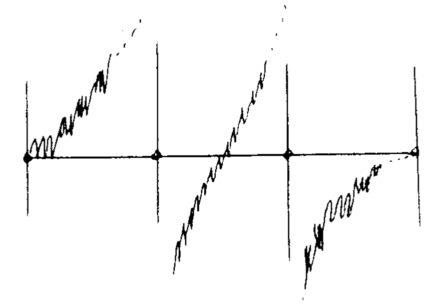


fig. 8

<u>Theorem 5</u> For k = 0, 1, 2, .. , $\nu(2k+8) \leq 2$

Proof Take a (7, 3, 0) function on (0, 1)
and a (2k+1, 2k+1, 2k+1) - - - - - - (1, 2)
and a (0, 0, 7) - - - - - - (2, 3)
putting f(0) = f(1) = f(2) = f(3) = 0. This gives a 2k+8-fold function on the compact interval [0, 3], with discontinuities only at 1 and 2, see fig. 8.

<u>Theorem 6</u> For k = 0, 1, 2, ... , $\nu(2k+7) \leq 3$

Proof Take a (5, 0, 0) function on (0, 1)
and a (1, 1, 1) - - - - - (1, 2)
and a (2k+1, 2k+1, 2k+1) - - - - - (2, 3)
and a (0, 0, 5) - - - - - (3, 4)
with f(0) = f(1) = f(2) = f(3) = f(4) = 0. See fig. 9.

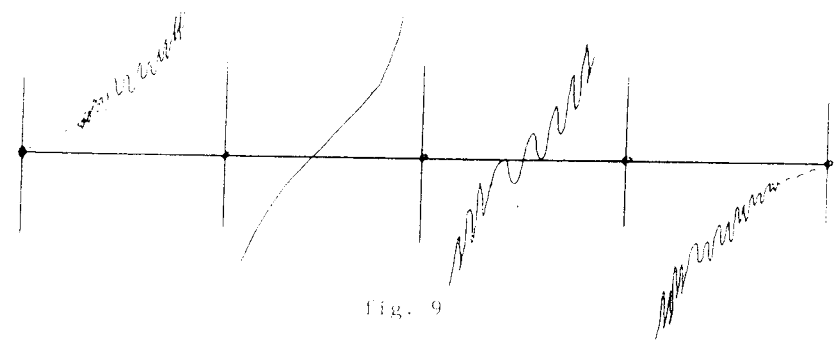


fig. 9

<u>Theorem 7</u> For $k = 0, 1, 2, \ldots$, $\lambda(2k+4) = 1$.

Proof By Theorem 1 we need to prove only that $\lambda(2k+4) \neq 0$.

Use reductio ad absurdum. Suppose that f is a continuous 2k+4-fold function on $(-1, 1)$. Without loss of generality we may assume f bounded (because if f is continuous then $f/(1+|f|)$ is continuous and bounded). Also we may assume that l.u.b. $= 1$ and g.l.b. $= -1$. The function cannot attain these bounds, because if it attained its l.u.b. at 2k+4 distinct points then each would have to be a local maximum, and consideration of the 2k+3 intervals between these points shows that the function would have to take some value at least 4k+6 times, impossible.

Note that f must tend to some limit as $x \uparrow 1$, because if lim sup and lim inf were unequal then every value between them would have to be taken by $f(x)$ infinitely many times. Similar comments apply at $x = -1$. Therefore we extend the function f to be continuous on the closed interval $[-1, 1]$. The function must attain its bounds, and cannot attain them in the open interval, and so the bounds must be $f(1)$ and $f(-1)$. Without loss of generality we may suppose that

$$-1 = f(-1) < f(x) < f(1) = 1 \quad \text{for all } x \text{ in } (-1, 1).$$

Let $x_1 < x_2 < \ldots < x_{2k+4}$ be the 2k+4 points where f takes the value zero. In each of the 2k+3 open intervals (x_r, x_{r+1}) $f(x)$ cannot change sign. Either there are k+2 (or more) of these intervals in which $f(x) > 0$, or there are k+2 in which $f(x) < 0$. Take the first case, the other may be dealt with similarly. Take any positive y less than all the upper bounds of $f(x)$ in these intervals. The function f must take the value y at least twice in each interval (making 2k+4 times) and must also take the value y in the interval

$(x_{2k+4}, 1)$. This contradicts the fact that f is 2k+4-fold, and so the theorem is proved.

<u>Theorem 8</u> A real function on an interval has no more than countably many local maxima and minima.

Proof A local maximum of a function $f(x)$ is a number c such that, for some $b > 0$, $f(x) < f(c)$ in the two open intervals $(c-b, c)$ and $(c, c+b)$.

If c_1 and c_2 are two local maxima, and if b_1 and b_2 are the corresponding interval-lengths, then b_1 and b_2 are not both $> |c_1-c_2|$. For each local maximum c let us choose the corresponding b as the largest suitable member of the set $\{1, \frac{1}{2}, \frac{1}{4}, \ldots\}$. This choice is always possible, and in fact unique. Now consider all the maxima of a function in a finite interval. There are only finitely many for which the chosen value of b is 1, because they must be at distance 1 or more apart. Similarly there are only finitely many for which $b = \frac{1}{2}$, etc. Therefore the total number is finite or countable. The result extends to a function on an infinite interval, and similar reasoning applies to the minima.

<u>Corollary</u> Suppose that a continuous function $f(x)$ on an interval takes values $f(a) \neq f(b)$, and has the property of not taking any value more than k times (for some finite k). Then there is y between $f(a)$ and $f(b)$ such that $f(x) - y$ changes sign at each point where $f(x) = y$.

<u>Theorem 9</u> It is impossible to have a 4-fold function $f(x)$ on the half-open interval $(-1, 1]$ with only one discontinuity.

Proof We may take f to be bounded, with g.l.b. $= -M$ and l.u.b $= M$. Note that f cannot attain either of its bounds.

For example, if f took the value M at one point, it would have to do so at 4 points, at least 3 would be points of continuity, with 2-sided continuity at 2 or more of the 3; this would lead to f taking some smaller value at 5 or more points.

Case 1 The discontinuity is at $x = 1$. We can define a continuous function $F(x)$ on the closed interval $[-1, 1]$, with $F(x) = f(x)$ at all the interior points and $F(-1) = f(-1+)$ and $F(1) = f(1-)$. $F(x)$ must have the same bounds $\pm M$ as $f(x)$, and must attain them, therefore $F(1)$ and $F(-1)$ must between them have the two values $\pm M$. By Theorem 8 we may choose $y \neq f(1)$ so that the 4 points x where $f(x) = y$ are not maxima or minima. Then (as in the corollary) $F(x) - y$ changes sign 4 times between -1 and 1. This is the required contradiction.

Case 2 The discontinuity is at $x = d$, with $-1 < d < 1$. Define continuous functions $F(x)$ on $[-1, d]$ and $G(x)$ on $[d, 1]$, both $= f(x)$ in the interior of their intervals, with $G(1) = f(1)$. The 3 values $F(-1)$, $F(d)$ and $G(d)$ must include both $-M$ and M. If $F(d)$ is not one of the two bounds we may reverse the function on $(-1, d)$, putting $f(d-1-x)$ instead of $f(x)$. Then $F(d)$ and $G(d)$ are the two bounds. By Theorem 8 we may choose y so that firstly $y \neq f(d)$, secondly $y - f(1)$ and $y - F(-1)$ have the same sign, and thirdly the 4 points where $f(x) = y$ are not maxima or minima. Then the difference $f(x) - y$ changes sign 5 times between $-1+$ and 1, 4 times at its zeros and once at d. This is the required contradiction.

Finally, a summary of the results.

k	1	2	3	4	5	odd	even	∞
$\lambda(k)$	0	∞	0	1	0	0	1	0
$\mu(k)$	0	1	1	2	1 or 2	1, 2 or 3	1 or 2	0
$\nu(k)$	0	∞	1	1	1	1, 2 or 3	1 or 2	0

OVERARM AND UNDERARM THROWING (JCMN49, p.5182)

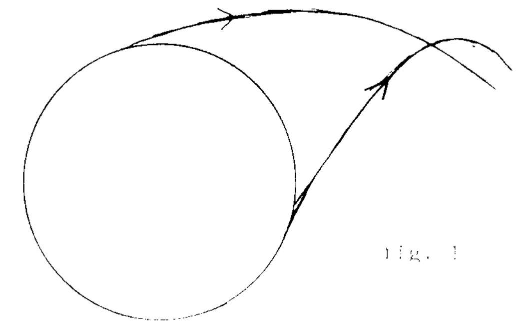


fig. 1

Can you throw further overarm or underarm? The answer (in our simple model) is that the two ways are equally good. To investigate the question, write the equation of the trajectory in vector form.

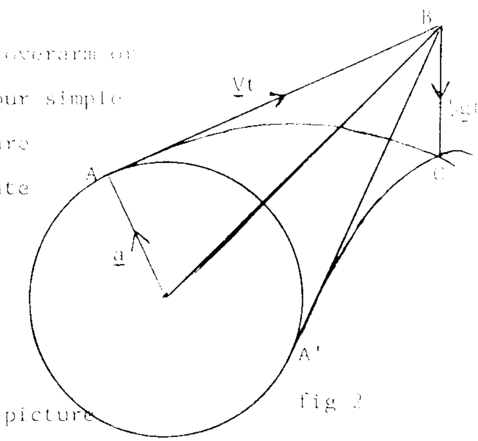


fig 2

$$\underline{x} = \underline{a} + \underline{V}t + \tfrac{1}{2}\underline{g}t^2$$

Then naturally one draws the picture as a vector diagram (fig. 2), showing the point C reached by an overarm throw from A. It is clear that C can also be reached by an underarm throw from A', and that the two throws have the same time of flight, t, because the two tangents from B to the circle are of equal length. Conversely, the same diagram shows how any point that can be reached by an underarm throw can also be reached by an overarm throw.

Now, what are the accessible points? Roughly speaking, the answer is the same as in the simple case of projection from a point, when the accessible points are those under the "enveloping parabola". More precisely, there is such an enveloping parabola in this case also, and all the accessible points are in it; if $V^2 \geq ag$ then every point under the parabola is accessible. To obtain these results see fig. 3.

Use Cartesian coordinates with origin at the centre of the circle.

Let C be (x, y). Using the theorem of Pythagoras in two different ways to find the square of OB gives the relation

$$a^2+V^2t^2 = x^2+(y+\tfrac{1}{2}gt^2)^2$$

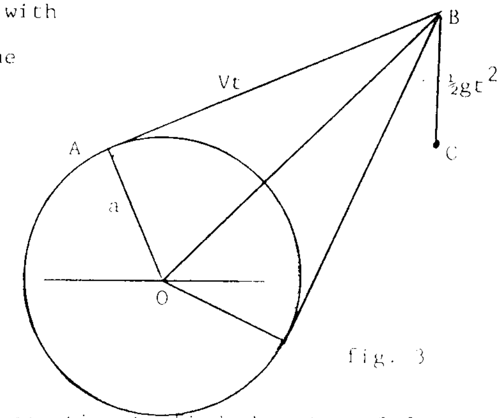


fig. 3

This can be regarded as an equation to find the time of flight of a ball thrown to (x, y). Write it as a quadratic in t^2.

$$\tfrac{1}{4}g^2t^4+(yg-V^2)t^2+(x^2+y^2-a^2) = 0 \qquad (1)$$

The roots (in t^2) must be real,

$$(yg-V^2)^2 \geq g^2(x^2+y^2-a^2)$$

or

$$2yV^2/g \leq V^4/g^2+a^2-x^2 \qquad (2)$$

This formula (2) describes the enveloping parabola, we have shown that it contains all the accessible points. The circle is in the parabola, because any (x, y) in the circle satisfies $a^2-x^2 \geq y^2 \geq y^2-(y-V^2/g)^2 = 2yV^2/g-V^4/g^2$. But note that if $V^2 \leq ag$ then the circle and parabola touch at the two points where $y = V^2/g$ (see fig. 4)

in this case not all the points in the parabola are accessible, only those (shown shaded) where either $y \leq V^2/g$ or $x^2+y^2 \leq a^2$. This follows from the requirement that (1) must have a root $t^2 \geq 0$.

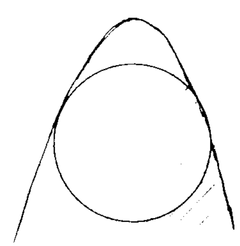


fig. 4

In the case where $V^2 > ag$ (the case of practical interest) the points in the parabola (2) but outside the circle are all accessible in four ways, two overarm and two underarm, because equation (1) for t^2 has two positive roots.

It is no accident that (1) is of degree 4 in t. Because our equations of motion are invariant under reversal of time, negative values of t indicate reversal of rotation of the throwing arm. The structure of the equations does not distinguish between overarm and underarm, but rather between clockwise and anticlockwise rotations, according to the sign of the root t of equation (1). If we want to throw the ball to a point inside the circle, we find that (1) gives one positive and one negative value for t^2, because the last term is negative. There are two real roots for t, of opposite signs; they are both overarm in the sense that the ball is released when the arm is above the horizontal, but one is with clockwise rotation, and one with anticlockwise.

QUOTATION CORNER 31

Grandmother's warning to young mothers not to give milk to children when they have a cold because it chokes them up with mucus is to be put to the test by an Adelaide medical research team. "It is a popular belief without any basis at all, and up to 30 per cent of the community hold the view, saying they have personal experience." Dr Carole Pinnock said yesterday. Dr Pinnock, clinical lecturer in medicine at Flinders Medical Centre, wants to contact 300 people aged between 16 and 90 ...

— Adelaide Advertiser (newspaper) April 1990.

(Anyone lecturing on Bayesian statistics might find this example useful)

MORE NUMBERS

Readers will recall the set of numbers H_n mentioned by Gerry Myerson (see JCMN 47, p.5129). These numbers are related to another family, K_n, and to the Fibonacci numbers F_n and the Lucas numbers G_n. The first few values are as follows

n	0	1	2	3	4	5	6	7	8	9	10	11
F_n	0	1	1	2	3	5	8	13	21	34	55	89
G_n	2	1	3	4	7	11	18	29	47	76	123	199
H_n	0	0	1	2	3	5	9	15	24	39	64	104
K_n	1	1	3	5	8	13	22	36	58	94	153	248

They may be expressed in terms of $g = \frac{1}{2}\sqrt{5}-\frac{1}{2}$ or in terms of one another as follows

$$5F_n = (1+2g)(g^{-n} - (-g)^n) = 2G_{n-1} + G_n$$
$$G_n = g^{-n} + (-g)^n = 2F_{n-1} + F_n$$

For the other two families we need the values r_n defined in terms of the remainder class of n modulo 4, as follows

$n \equiv$	[illegible]	[illegible]	[illegible]	[illegible]
$r_n =$	[illegible]	[illegible]	[illegible]	1

Define H_n and K_n by:

$$5H_n = g^{-n-2} + (-g)^{n+2} - r_n = 2F_{n+1} + F_{n+2} - r_n = G_{n+2} - r_n$$
$$5K_n = 2g^{-n-1} + 3g^{-n} + 3(-g)^n + 2(-g)^{n+1} - r_n$$
$$= F_n + 8F_{n+1} - r_n = 3G_n + 2G_{n+1} - r_n$$

The following relations are clear

$$H_{n+4} = H_{n+3} + H_{n+1} + H_n + 1 \quad \text{and} \quad K_{n+4} = K_{n+3} + K_{n+1} + K_n + 1$$
$$H_n + H_{n+2} + 1 = F_{n+3} \quad \text{and} \quad K_n + K_{n+2} + 1 = 3F_{n+1} + 2F_{n+2}$$
$$K_n = H_n + F_{n+1}$$

Recalling the duplication formulae $F_{2n} = F_nG_n$, $F_{4n+1} = F_{2n+1}G_{2n} - 1$ and $F_{4n+3} = F_{2n+1}G_{2n+2} - 1$, we look for analogies.

$$H_{4n-2} = F_{2n}^2 \qquad H_{4n-1} = F_{2n}F_{2n+1}$$
$$H_{4n} = F_{2n}F_{2n+2} \qquad H_{4n+1} = F_{2n}F_{2n+3}$$

$$K_{4n} = F_{2n+1}(F_{2n+1} + G_{2n}) - 2 = F_{2n+1}F_{2n+2} + 2F_{2n}^2$$
$$K_{4n+1} = F_{2n+2}(F_{2n+1} + G_{2n}) - 2 = F_{2n+2}^2 + 2F_{2n}F_{2n+1}$$
$$K_{4n+2} = F_{2n+1}(F_{2n+3} + G_{2n+2}) - 2 = F_{2n+1}^2 + 2F_{2n+2}^2$$
$$K_{4n+3} = F_{2n+1}(F_{2n+4} + G_{2n+3}) - 2 = F_{2n+2}(F_{2n+1} + 2F_{2n+3})$$

All the relations above apply equally when $n < 0$, but if we ignore the case of negative n we may describe the numbers by their generating functions, $f(t) = \sum_0^\infty F_n t^n$, etc.

$$f(t) = t/(1 - t - t^2) \qquad g(t) = (2 - t)/(1 - t - t^2)$$
$$h(t) = \frac{t^2}{(1-t-t^2)(1-t)(1+t^2)} \qquad k(t) = \frac{1-t+2t^2-t^3}{(1-t-t^2)(1-t)(1+t^2)}$$

Our interest in the numbers H_n and K_n is concerned with the sum $S(N) = \sum_{n=1}^{N}(x_n - \frac{1}{2})$ where x_n is the non-integer part of ng. It seems that $S(H_n)$ is a bound of the set $\{S(N): 1 \leq N \leq H_n\}$, and similarly for K_n, they are upper bounds if n is even and lower bounds if n is odd. Numerical evidence indicates that

$$S(H_{2n}) = n/10 + (3-4g)/50 + G_{n+1}(-g)^{3n+1}(3g-1)/50$$
$$S(H_{2n+1}) = (2n+1)/20 + 3(3-4g)/100 + o(1)$$
$$S(K_n) = (-1)^n n/20 + \frac{3-4g}{40}(1+(-1)^n(4+5g)(3-g)/5) + o(1)$$
$$S(K_n) - S(H_n) = (-1)^n g^3/5 + o(1).$$

TWO-DIMENSIONAL FOURIER TRANSFORMS IN POLAR COORDINATES

Function	Transform
$f(r,\ \theta)$ $\iint e^{2\pi i rs\cos(\theta-\phi)} F(s,\ \phi)\, s\,ds\,d\phi$	$\iint e^{-2\pi i rs\cos(\theta-\phi)} f(s,\ \phi)\, s\,ds\,d\phi$ $F(r,\ \theta)$
$f(r/k,\ \theta)$	$k^2 F(kr,\ \theta)$
$f(r,\ \theta+\alpha)$	$F(r,\ \theta+\alpha)$
$\partial f/\partial\theta$	$\partial F/\partial\theta$
$\nabla^2 f \quad (=f_{rr}+f_r/r+f_{\theta\theta}/r^2)$	$-4\pi^2 r^2 F$
$f + r\partial f/\partial r$	$-F - r\partial F/\partial r$
$g(r)e^{in\theta}$	$2\pi(-ie^{i\theta})^n \int_0^\infty g(s)J_n(2\pi rs)\, s\,ds$
$1/r$	$1/r$
$\exp(-\pi r^2/a^2)$	$a^2\exp(-\pi a^2 r^2)$
$\delta(r-c)$	$2\pi c J_0(2\pi cr)$
$\delta(r-c)e^{i\theta}$	$-2\pi i c e^{i\theta} J_1(2\pi cr)$
$\left.\begin{matrix} 1 & \text{if } r < c \\ 0 & \text{if } r > c \end{matrix}\right\}$	$(c/r)J_1(2\pi cr)$
$\left.\begin{matrix} r^2 & \text{if } r < c \\ 0 & \text{if } r > c \end{matrix}\right\}$	$(c^3/r)J_2(2\pi cr)$
$\left.\begin{matrix} r^n e^{in\theta} & \text{if } r < c \\ 0 & \text{if } r > c \end{matrix}\right\}$	$(-ie^{i\theta})^n(c^{n+1}/r)J_{n+1}(2\pi cr)$
$\left.\begin{matrix} J_n(ar)e^{in\theta} & \text{if } r < c \\ 0 & \text{if } r > c \end{matrix}\right\}$	$\dfrac{2\pi c(-ie^{i\theta})^n}{a^2-4\pi^2r^2}\left\{aJ_{n+1}(ac)J_n(2\pi cr) - 2\pi r J_n(ac)J_{n+1}(2\pi cr)\right\}$
$(1/r)\log r$	$-(1/r)\log(2\pi r)$

Corrections and additions would be welcome.

QUOTATION CORNER #2

" ... as good as the bread you had as a child."

- Printed on the wrapper of a loaf of bread from a shop.