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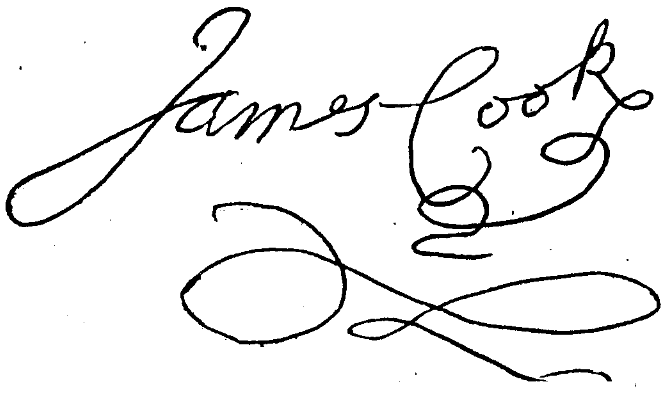

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ORTHOCENTRES (JCMN 50 p.5204)

J. D. E. Konhauser and A. Brown

In the Mathematical Gazette (vol 46, 1962, p.51) John Satterley recounts that the name "orthocentre" "... was invented by two mathematicians, Besant and Ferrers, in 1865, while out for a walk along the Trumpington Road, a road leading out of Cambridge toward London. In those days it was a tree-lined quiet road with a sidewalk, a favourite place for a conversational walk."

The Greek Lexicon of Liddell and Scott tells us that κέντρον comes from a verb meaning to stab or prick, and hence κέντρον meant a sharp point, such as a goad or a spur. Because of this, κέντρον was used for the sharp point of a pair of compasses and, one step further on, as the centre of the circle drawn by using the compasses. So is the orthocentre the sharp point where the altitudes meet?

Perhaps I should add something about the "ortho" part of the word. ἡ ὀρθὴ γωνία was used even in classical times for a right angle, from ὀρθός meaning straight, right, correct ... Sometimes the angle part (γωνία) was left out, or implied, so ἡ ὀρθή became a right angle or a perpendicular line, which would explain how it got into the act in discussing the altitudes of a triangle.

The circumcircle and the 9-point circle of a triangle have the orthocentre as a centre of similitude.

The circle with respect to which the triangle is self--polar is given in trilinear coordinates by $x^2\sin 2A + y^2\sin 2B + z^2\sin 2C = 0$. The orthocentre (sec A, sec B, sec C) is the centre of this circle. This circle is one of the co-axal system including the circumcircle, the 9-point circle and Guinand's critical circle (JCMN 30, p.3128).

CONVERGENCE QUESTION (JCMN 50, p.5215)

C. C. Rousseau

With $g = (\sqrt{5}-1)/2$, the series $\sum n^{-2}/|\sin n\pi g|$ converges. The result is not new, it comes from a 1922 paper of Hardy and Littlewood, using the Thue-Siegel-Roth theorem. However, a simple proof is as follows.

For integer r, let x_r be the non-integer part of rg.

<u>Theorem 1</u> $\sum_{r=1}^{k} 1/x_r$ and $\sum_{r=1}^{k} 1/(1-x_r)$ both $< 4k + 2k\log(2k)$.

Proof Choose n so that $M = F(n-1) \leqslant k < N = F(n)$, where $F(n)$ denotes the n^{th} Fibonacci number. Let x_r^* be the non-integer part of rM/N. The N-1 numbers Nx_r^* $(r = 1, 2, \ldots N-1)$ are unequal integers. (To prove inequality, let $(r-s)M = qN$, then $q < M$, impossible because M and N are coprime) and so the x_r^* are a permutation of $\{1/N, 2/N, \ldots (N-1)/N\}$.

Recall that $N g^n\sqrt{5} = 1 - (-1)^n g^{2n}$, so that

$$\sqrt{5}(N-1)g^n \leqslant 1 - \sqrt{5}\,g^n + g^{2n} < (1-g^n)^2 < 1.$$

Therefore $x_r^* - x_r = (r/N)(M-gN) = r(-g)^n/N = \pm\varepsilon$ where $N\varepsilon = rg^n < r/(\sqrt{5}N - \sqrt{5}) \leqslant 1/\sqrt{5} < 1/2$. For each $r < N$ let s be the integer Nx_r^*, then

$$x_r \geqslant x_r^* - \varepsilon = s/N - \varepsilon > (s-\tfrac{1}{2})/N, \text{ and}$$

$$\sum_{r=1}^{N-1} 1/x_r < N\sum_{s=1}^{N-1} 1/(s-\tfrac{1}{2}) < N(2 + \log N).$$

Also the values of $N(1-x_r^*)$ are a permutation of 1, 2, ... N-1, and a similar argument shows that $\sum_{r=1}^{N-1} 1/(1-x_r) < 2N + N\log N$. Since $\tfrac{1}{2}N < k < N$, the theorem is proved.

<u>Theorem 2</u> The series $\sum n^{-2}/x_n$ and $\sum n^{-2}/(1-x_n)$ both converge.

Proof Recall Abel's formula on partial summation:

$$\sum_1^n a_r b_r = A_n b_n + \sum_{r=1}^{n-1} A_r(b_r - b_{r+1}) \text{ where } A_r = a_1 + a_2 + \ldots + a_r.$$

Put $a_r = 1/x_r$ and $b_r = r^{-2}$. From Theorem 1 we have $A_r < 4r + 2r\log(2r)$, also $b_r - b_{r+1} = (2r+1)/(r^2+r)^2 = O(r^{-3})$. Therefore $\sum n^{-2}/x_n$ converges, the other result is similar.

<u>Theorem 3</u> The series $\sum n^{-2}/|\sin n\pi g|$ converges.

Proof Divide the terms of the series into two classes:

(a) where $x_n < \frac{1}{2}$, and $|\sin n\pi g| = |\sin \pi x_n| > 2x_n$

(b) where $x_n > \frac{1}{2}$, and $|\sin n\pi g| = |\sin \pi(1-x_n)| > 2(1-x_n)$

The result follows from Theorem 2.

<u>Historical note</u> Given an algebraic irrational x of order n, for which θ is there $c = c(x, \theta) > 0$ such that $|x-p/q| > cq^{-\theta}$? The following answers have been given

$\theta \geq n$	Liouville 1844
$\theta > \frac{1}{2}n+1$	Thue 1909
$\theta > 2\sqrt{n}$	Siegel 1921
$\theta > 2$	Roth 1955

Hardy and Littlewood in 1922 would have known that our Theorem 3 above could be extended to the case of g any quadratic irrational, Liouville's result being sufficient. Now, using Roth's result, we know that Theorem 3 can be extended to the case of g being any algebraic irrational.

Obvious modifications of the calculations above give:

<u>Theorem $1\frac{1}{2}$</u> $\sum_{r=1}^{k} x_r^{-2}$ and $\sum_{r=1}^{k} (1-x_r)^{-2}$ both $< 2\pi^2k^2$.

<u>Theorem $2\frac{1}{2}$</u> The series $\sum n^{-3}x_n^{-2}$ and $\sum n^{-3}(1-x_n)^{-2}$ both converge.

<u>Theorem $3\frac{1}{2}$</u> The series $\sum n^{-3}/\sin^2 n\pi g$ converges.

MONTE CARLO INTEGRATION

(JCMN 46 p.5104, 47 p.5129 and 50 p.5214)

Some of the phenomena indicated by numerical evidence in the previous note can now be proved. Also some more empirical results arise wanting explanation.

As before, let f(x) have its derivative continuous in the closed interval $0 \leq x \leq 1$, and let

$$S^*(N) = \sum_{r=-N}^{N} (f(x_r) - \int_0^1 f(x)dx)$$

where x_r is the non-integer part of $rg = \frac{1}{2}r(\sqrt{5}-1)$, and $f(x_0)$ is to be understood as $\frac{1}{2}f(0) + \frac{1}{2}f(1)$.

We may assume without loss of generality that $f(0) = f(1)$, because if this is not so we may add to f a linear function of x to make it so. This addition will not change $S^*(N)$. On this understanding we express f(x) as the sum of its Fourier series: $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x + b_n \sin 2n\pi x$ in which the coefficients a_n and b_n are $O(n^{-2})$, as may be proved using integration by parts.

<u>Theorem 1</u> $S^*(N) = \sum_{n=1}^{\infty} a_n \sin n(2N+1)\pi g / \sin n\pi g$.

Proof $S^*(N)$ is a sum of 2N+1 terms, and $\int_0^1 f(x)dx = \frac{1}{2}a_0$. The term for $r = 0$ is $\sum_{n=1}^{\infty} a_n$. For any positive or negative r it can be seen that $\cos 2n\pi x_r = \cos 2n\pi(rg+\text{integer}) = \cos 2n\pi rg$ and similarly $\sin 2n\pi x_r = \sin 2n\pi rg$, so that

$$f(x_r) + f(x_{-r}) = a_0 + 2\sum_{n=1}^{\infty} a_n \cos 2n\pi rg$$

$$S^*(N) = \sum_{n=1}^{\infty} a_n(1 + 2\sum_{r=1}^{N} \cos 2n\pi rg)$$

$$= \sum_{n=1}^{\infty} a_n \sin n(2N+1)\pi g / \sin n\pi g.$$

QED

<u>Theorem 2</u> $S^*(N)$ is bounded.

Proof Recall the proof in C.C.Rousseau's note on page 5221 above that $\sum n^{-2}/|\sin n\pi g|$ converges, and the observation above that $a_n = O(n^{-2})$. The result follows from Theorem 1. QED

Now define L^* and U^* as the lower and upper bounds of $S^*(N)$ for $N = 1, 2, \ldots$

<u>Theorem 3</u> $\liminf_{N \to \infty} S^*(N) \leq -U^*$

Proof For simplicity we write c_n for $a_n/\sin n\pi g$, then

$$S^*(N) = \sum_{n=1}^{\infty} c_n \sin n(2N+1)\pi g$$

which is absolutely convergent because $\sum c_n$ is absolutely convergent. Take any $\varepsilon > 0$. Choose m so that $\sum_{m+1}^{\infty} |c_n| < \varepsilon$, and so (for all N) $S^*(N)$ differs by less than ε from $A(N) = \sum_{n=1}^{m} c_n \sin n(2N+1)\pi g$.

Recall that the parity of the Fibonacci numbers follows the sequence odd, odd, even, odd, odd, even, etc. Therefore we may choose arbitrarily large P so that 2P+1 is an odd Fibonacci number preceded by an even one. Recall also that two successive Fibonacci numbers, F(n-1) and F(n) satisfy $F(n-1) = gF(n) + (-g)^n$. Therefore we have arbitrarily large P such that $(2P+1)g$ is arbitrarily near to an even integer, in fact differs from an even integer by less than $\varepsilon / \sum_1^m n|c_n|$.

Take any positive integer $Q < P$.

$$A(P+Q) = \sum_{n=1}^{m} c_n \sin((2P+1)n\pi g + 2Qn\pi g)$$

will differ by less than ε from $\sum_{n=1}^{m} c_n \sin(2Qn\pi g)$.
Similarly $A(P-Q)$ will differ by less than ε from -(the same).
Consequently $A(P+Q) + A(P-Q)$ will be between $\pm 2\varepsilon$, and so $S^*(P+Q) + S^*(P-Q)$ will be between $\pm 4\varepsilon$.
By the definition of U^* as upper bound, there is N_1 such that $S^*(N_1) > U^* - \varepsilon$. For each of our arbitrarily large P we put $Q = P - N_1$. Then

$$S^*(2P-N_1) = S^*(P+Q) < -S^*(P-Q) + 4\varepsilon = -S^*(N_1) + 4\varepsilon < -U^* + 5\varepsilon.$$

This is for arbitrarily large P, and so $\liminf S^*(N) \leq -U^* + 5\varepsilon$.

This is for all $\varepsilon > 0$, and so $\liminf S^*(N) \leq -U^*$. QED

<u>Theorem 4</u>

$$\limsup S^*(N) = U^*$$
$$\liminf S^*(N) = L^*$$
and
$$L^* = -U^*.$$

Proof From the previous theorem, and from the same result for the function $-f(x)$,

$$\liminf S^*(N) + U^* \leq 0$$
$$\limsup S^*(N) + L^* \geq 0$$

Subtraction gives $U^* - \limsup S^*(N) \leq L^* - \liminf S^*(N)$, but in this inequality the LHS cannot be negative and the RHS cannot be positive, therefore both sides are zero. This gives the first two results required, the third follows. QED

In consequence of the boundedness of $S^*(N)$ we may observe that $\sum_{r=-N}^{N} f(x_r)/(2N+1)$ will be a numerical estimate for $\int_0^1 f(x)dx$ with error $O(1/N)$. This shows a difference between the pseudo-random numbers x_r and genuine random numbers, for with the latter the error is $O(1/\sqrt{N})$.

As the numbers $S^*(N)$ vary between bounds L^* and U^* equally spaced on the two sides of zero, one might ask if (in any sense) they have mean zero. The answer seems to be yes, in the sense that $S^*(0) + S^*(1) + S^*(2) + \ldots + S^*(N) = O(\log N)$, but this result is only numerical, no proof has yet emerged. If this were true then we would be able to strengthen the comment on numerical integration above to the proposition that

$$\frac{\frac{1}{2}(N+1)(f(0)+f(1)) + \sum_{r=1}^{N}(N+1-r)(f(x_r)+f(x_{-r}))}{(N+1)^2}$$

as an estimate for the integral has error $O(N^{-2}\log N)$. This formula raises another idea about pseudo-random numbers, that a weighted mean over the variables may perform better than the unweighted mean. The idea takes a precise form in the next theorem.

<u>Theorem 5</u> If $f'(0) = f'(1)$ then

$$T(N) = S^*(0) + S^*(1) + \dots + S^*(N)$$

is bounded.

Proof In the notation that we have been using

$$T(N) = \sum_{n=1}^{\infty} a_n (\sin n\pi g + \sin 3n\pi g + \dots + \sin(2N+1)n\pi g)/\sin n\pi g$$
$$= \sum_{n=1}^{\infty} a_n \sin^2(N+1)n\pi g/\sin^2 n\pi g.$$

Therefore $|T(N)| < \sum_{n=1}^{\infty} |a_n|/\sin^2 n\pi g$ for all N.

From the condition $f'(0) = f'(1)$ (remembering that $f(0) = f(1)$) it follows that $a_n = O(n^{-3})$, and by Theorem 3½ of C. C. Rousseau's note (page 5221 above) we see that T(N) is bounded. QED

Theorem 5 shows that in the case where $f'(0) = f'(1)$ the weighted mean:

$$(f(0)+f(1))/(2N+2) + \sum_{r=1}^{N} (f(x_r)+f(x_{-r}))(N+1-r)/(N+1)^2$$

estimates the integral with error $O(N^{-2})$.

<u>More computational results</u>

For the functions $f(x) = (n+1)x^n$ (where n = 1, 2, ..) the upper bound U* of S*(N) seems to be $U^* = \frac{1}{2}(n-1)$.

<u>Thoughts on the topic</u>

There is an analogy between random numbers and rubber. Until the middle of this century random numbers were found by tossing coins, rolling dice or looking at the last ten digits of a value from a book of 20-figure logarithm tables. As computers can not do any of these things we now have to make do with pseudo-random numbers instead of genuine random numbers. Most computers offer a family of pseudo-random numbers, but the book of instructions does not reveal how the numbers are generated.

Likewise, when your Editor was a boy, rubber came from rubber trees, usually in Malaya. During the War, firstly Germany and then Britain were cut off from supplies of natural rubber and had to make and use synthetic rubbers, such as neoprene. Engineers soon discovered that neoprene was better than natural rubber for some purposes, such as making oil seals, though not so good for some other purposes. The latex from a rubber tree is doubtless admirably well suited to playing its proper part in the life of the tree, but it would be rash to jump to the conclusion that it is the best possible material for making tennis shoes or oil seals or garden hoses. The chemists making synthetic rubbers do not try to imitate natural rubber, they try to make a material suitable for the application in which it will be used. Mathematicians, however, are still using pseudo-random numbers that have been designed to be as much like genuine random numbers as possible.

BINOMIAL IDENTITY 31

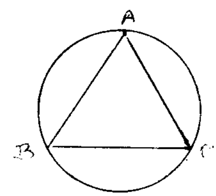
Terry Tao

$$\sum_{m=0}^{n} \sum_{i=0}^{m} \frac{(-1)^i m^2}{i!(n-m)!} = n^2 - 2n + 2$$

ROLLE'S THEOREM IN THE GEOMETRICAL IDENTITY

Jordan Tabov

Recall the geometrical identity mentioned by Mark Kisin in JCMN 48. If A_1, A_2, ... are the vertices of a regular polygon then for certain k the sum $\sum(MA_r)^k$ is constant as M moves round the circumcircle. I mentioned (JCMN 50, p. 5210) that when we have found all the positive integer k for which this is so, it is possible with Rolle's Theorem to deal with the question of non-integer k. The method is best shown by an example. Let ABC be an equilateral triangle in the unit circle.



Suppose that (for some k) the sum $AM^k + BM^k + CM^k$ remains constant as M goes round the circle. The fact that it takes the same value when M is at A as when M is at the mid-point of the arc BC tells us that

$$F(k) = 2(\sqrt{3})^k - 2^k - 2$$

has a zero at these values of k (in fact at k = 2 and k = 4). The derivative $F'(k) = (\sqrt{3})^k \log 3 - 2^k \log 2$ has only one zero for k in $(0, \infty)$. This tells us that the two known integer values, 2 and 4, are the only real positive values for which $F(k) = 0$.

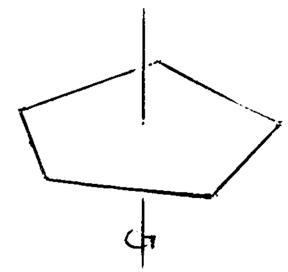
ANALYTIC INEQUALITY

Let f(x) be a positive function of x with positive continuous derivative for $0 \le x \le c$. Show that $\int_0^c x/f(x)\,dx < \int_0^c 4/f'(x)\,dx$. Is the 4 the best possible constant in the inequality?

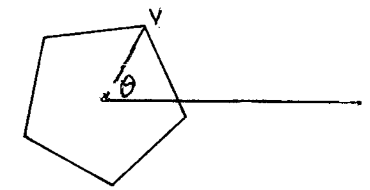
3-DIMENSIONAL GEOMETRICAL IDENTITY

Mark Kisin

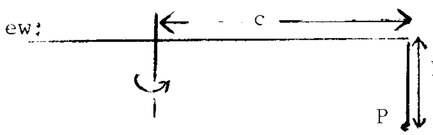
Imagine a regular pentagon (with circumradius = 1) rotating about its axis (the line through the centre perpendicular to its plane). Let V_1, V_2, .. V_5 be the vertices, let V be any one of them and let P be any point in the space.



Plan:

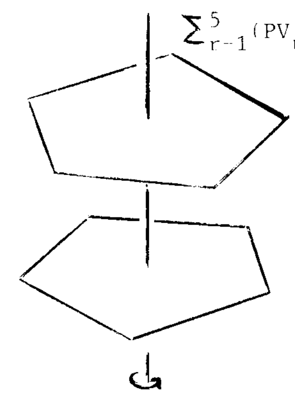


Side view:



Consider $(PV)^{2k} = (b^2 + c^2 + 1 - 2c\cos\theta)^k$ as a function of θ. It is a trigonometric polynomial of degree k. The term of highest degree is $2(-c)^k \cos k\theta$. If $k < 5$ then $\sum_{r=1}^{5}(PV_r)^{2k}$ is a constant, because any term $\cos n\theta$ (n = 1, 2, 3 or 4) when summed over the five vertices gives zero. If k = 5 then the contribution from the fifth degree term depends on θ, in fact $\cos 5\theta$ takes the same value on all the vertices. Therefore

$$\sum_{r=1}^{5}(PV_r)^{10} = \text{Constant} - 10c^5\cos 5\theta.$$



Now imagine another equal pentagon rotating on the same axis, at an angle of $36^\circ = \pi/5$ to the first. We can calculate the sum of powers of distances as before, but with V_r replaced by V_r', b replaced by b', and θ replaced by $\theta' = \theta + 36^\circ$. If $k < 5$ then $\sum (PV_r)^{2k} + (PV_r')^{2k}$ is constant.

If $k = 5$ then the sum $=$ Constant $+ 10c^5(\cos 5\theta + \cos 5\theta')$. This also is independent of θ because $\cos 5\theta' = \cos 5(\theta + 36^\circ) = -\cos 5\theta$. To verify that when $k = 6$ the sum is non-constant, one calculation will suffice. Take $c = 1$ and $b' = 0$.

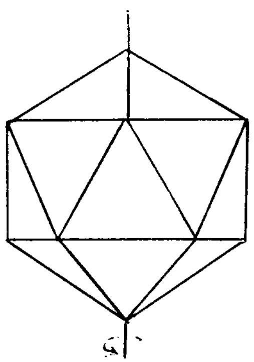
$(PV)^{12} = (b^2 + 2 - 2\cos\theta)^6 = 2\cos 6\theta - 12(b^2+2)\cos 5\theta + \ldots$

Similarly $(PV')^{12} = 2\cos 6\theta' - 24\cos 5\theta' + \ldots$

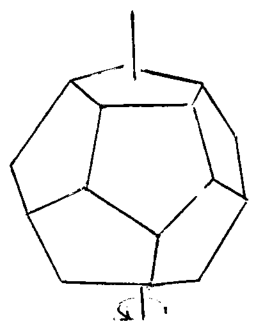
Recall that any term $\cos n\theta$ or $\cos n\theta'$ will vanish on summation unless n is a multiple of 5. Also $\cos 5\theta = -\cos 5\theta'$.

$$(PV_r)^{12} + (PV'_r)^{12} = \text{Constant} - 60\,b^2\cos 5\theta.$$

This has shown that the sum of 12th powers is not constant, except of course when $b = 0$ and the two pentagons are in the same plane.



Icosahedron



Dodecahedron

Now we come to the question raised by Jordan Tabov (in JCMN 50, pp. 5209-5211) about the sums of powers of distances to the vertices of one of the Platonic solids.

An icosahedron has 12 vertices, diametrically opposite to one another in pairs. Each of these 6 pairs of vertices determines an axis about which the icosahedron has rotational symmetry of order 5. The other 10 vertices (those not on the axis) form 2 pentagons, arranged as described above, their planes perpendicular to the axis. Consider the icosahedron

rotating about the axis. If P is any point, the result above shows that $\sum_{r=1}^{12}(PV_r)^{2k}$ is constant if k is 1, 2, 3, 4 or 5, but not if k is 6.

Similar reasoning applies to the dodecahedron. It has 20 vertices and 12 pentagonal faces. It similarly has 6 axes of rotational symmetry, each joining the centres of two parallel faces. The vertices form 4 pentagons, in two pairs, one pair is the two faces whose centres are joined by the axis, this pair has the property considered above, of being displaced from one another by an angle of 36°. The other pair also has this property. The result above shows that $\sum_{r=1}^{20}(PV_r)^{2k}$ is constant if $k = 1, 2, 3, 4$ or 5. To show that it is not constant for $k = 6$, consider the terms in $\cos 5\theta$ that come from the two pairs, a little thought shows that they are of the same sign.

Now to consider the implications of these results for the two solids. From now on we use "solid" to mean either the icosahedron or the dodecahedron. If $k = 1, 2, 3, 4$ or 5, and P is any fixed point, then the mean (over the vertices V of the solid) of $(PV)^{2k}$ is unchanged by any rotation of the solid about its axis of symmetry. Also it is unchanged by rotation about any of the other 5 axes of symmetry. Therefore it is unchanged by any (3-dimensional) rotation of the solid about its centre. Averaging this result over the 3-dimensional orthogonal group, it follows that the mean (over the vertices) of $(PV)^{2k}$ is equal to the integral mean of $(PX)^{2k}$ over points X of the circumsphere of the solid. This is easily calculated. It is $((R+r)^{2k+2} - (R-r)^{2k+2})/(4Rr(k+1))$ where R is the distance of P from the centre and r is the circumradius.

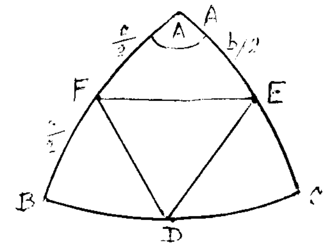
SPHERICAL TRIANGLE PROBLEM

(JCMN 50, p.5204)

A. Brown

If ABC is a spherical triangle and the mid-points D, E and F of the sides form an equilateral triangle, then must ABC be equilateral? NO.

Use the standard notation, with the sides a, b, c, and the angles A, B, C all between 0 and π. We shall find the answer that DEF is equilateral when either $A = B = C$ or $A + B + C = 2\pi$. If ABC is any spherical triangle then by the cosine rule



$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos EF = \cos\tfrac{1}{2}b \cos\tfrac{1}{2}c + \sin\tfrac{1}{2}b \sin\tfrac{1}{2}c \cos A$$

From the second of these equations

$$4 \cos EF \cos\frac{b}{2} \cos\frac{c}{2} = (1 + \cos b)(1 + \cos c) + \sin b \sin c \cos A$$

$$= 1 + \cos a + \cos b + \cos c = K$$

Similarly $\quad 4 \cos DF \cos\frac{a}{2} \cos\frac{c}{2} = 4 \cos DE \cos\frac{a}{2} \cos\frac{b}{2} = K.$

Given that EF = FD = DE, there are two possibilities:-

Case 1. $\quad K \neq 0. \quad \cos\frac{a}{2} = \cos\frac{b}{2} = \cos\frac{c}{2}$ and ABC is equilateral.

Case 2. $\quad K = 0. \quad EF = FD = DE = \pi/2$ and the angles A, B and C are all obtuse, because $\cos A = -\cot\frac{b}{2} \cot\frac{c}{2} < 0$, etc.

For any triangle it can be shown that $K = 0$ if and only if $A + B + C = 2\pi$, i.e. the triangle ABC covers 1/4 of the area of the sphere. To prove this, add the cosine rule equations for the sides b and c. This gives

$$(\cos b + \cos c)(1 - \cos a) = \sin a (\sin c \cos B + \sin b \cos C)$$

Multiply by $(1 + \cos a)/\sin a$, and then use the sine rule.

$$(\cos b + \cos c) \sin A = (1 + \cos a) \sin(B + C)$$

Now the "if" and the "only if" are both clear.

NON-BINOMIAL IDENTITY (JCMN 50, p.5200) (1)

Mark Kisin

For real positive x, $\quad \sum_{n=1}^{\infty} [2^{-n}x + \tfrac{1}{2}] = [x].$

Proof: First note that $[y] + [y + \tfrac{1}{2}] = [2y]$, which is clear by considering the two cases $y - [y] > \tfrac{1}{2}$ and $\leq \tfrac{1}{2}$.

$$\sum_1^{\infty} [2^{-n}x + \tfrac{1}{2}] = \sum_0^{\infty} [2^{-n}x] - \sum_1^{\infty} [2^{-n}x] = [x].$$

NON-BINOMIAL IDENTITY (2)

P. H. Diananda

Let x be a real number, there is no need for it to be positive as it was in the original problem. Then

$$[x] = [\tfrac{1}{2}x] + [\tfrac{1}{2}x + \tfrac{1}{2}]$$

Hence $\quad \sum_1^N [2^{1-n}x] = \sum_1^N [2^{-n}x] + \sum_1^N [2^{-n}x + \tfrac{1}{2}]$

Thus $\quad [x] - [2^{-N}x] = \sum_{n=1}^N [2^{-n}x + \tfrac{1}{2}]$

For large n, $\quad [2^{-n}x + \tfrac{1}{2}] = 0$

and $\quad [2^{-n}x] = \begin{cases} 0 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$

Hence $\quad \sum_{n=1}^{\infty} [2^{-n}x + \tfrac{1}{2}] = \begin{cases} [x] & \text{if } x \geq 0 \\ [x] + 1 & \text{if } x < 0 \end{cases}$

Similarly, since $[x] = [x/s] + \sum_{t=1}^{s-1} [x/s + t/s]$ for integral $s \geq 2$, it follows that

$$\sum_{n=1}^{\infty} \sum_{t=1}^{s-1} [s^{-n}x + t/s] = \begin{cases} [x] & \text{if } x \geq 0 \\ [x] + 1 & \text{if } x < 0. \end{cases}$$

More generally, with integers $s_1, s_2, \ldots \geq 2$,

$$\sum_{n=1}^{\infty} \sum_{t=1}^{s_n - 1} \left[\frac{x}{s_1 s_2 \ldots s_n} + \frac{t}{s_n}\right] = \begin{cases} [x] & \text{if } x \geq 0 \\ [x] + 1 & \text{if } x < 0. \end{cases}$$

PROBABILITY PROBLEM (JCMN 50, p.5208)

Mark Kisin

I have a desk with drawers D_1, D_2, ... and I have left my car keys in one, but cannot remember which. The probability of the keys being in D_i is 2^{-i}. Each time I look in a drawer there is probability ½ of finding the keys if they are there. What is the best search strategy? and the expected number of looks in drawers? Do the answers change if I must start with the first drawer and if after looking in D_i I must go to one of D_{i-1}, D_i and D_{i+1}?

Solution Consider one drawer certainly containing the keys. The situation can be represented by a new model, a sequence of boxes B_1, B_2, ... with probability 2^{-j} that the keys are in B_j, such that one look in the box will certainly show if the keys are in it, and under the constraint that I must look in the boxes in the right order, i.e. B_j before B_{j+1}.

Transforming the problem in this way, we have a doubly infinite set of boxes B_{ij} (i, j = 1, 2, ...) as shown below.

1/16	1/32	1/64
1/8	1/16	1/32
1/4	1/8	1/16

(In each box is shown the probability that the keys are in it)

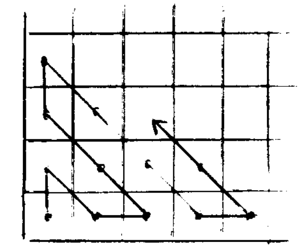
The optimal strategies are now clear — first look in B_{11}, then in B_{12} and B_{21} (in either order), then in B_{13}, B_{22} and B_{31} (in any order), etc. The expected number of looks needed is

$$\frac{1}{4} + \frac{2+3}{8} + \frac{4+5+6}{16} + \frac{7+8+9+10}{32} + \dots = \sum_1^{\infty} n(n^2+1)/2^{n+2}$$

To find the sum (= 7) of this series an easy way is to use four times the lemma that

$$\sum_{n=1}^{\infty} f(n)/2^n = f(1) + \sum_{n=1}^{\infty}(f(n+1) - f(n))/2^n$$

The answer to the last question is no, for one possible optimal search strategy is that indicated below.



EDGE-COLOURED GRAPHS (JCMN 42, p.5015 and 50, p.5216)

Mark Kisin

Let f(n) be the largest m such that the complete m-graph can have an edge-colouring of n colours with no monochromatic circuit. We shall see that f(n) = 2n.

Theorem 1 Any graph with the number of edges $\geq$ the number of nodes contains a circuit.

Proof Use induction. If all nodes are of degree 2 or more then there is a circuit; if not then remove a node of lower degree and the edge (if any) to it.

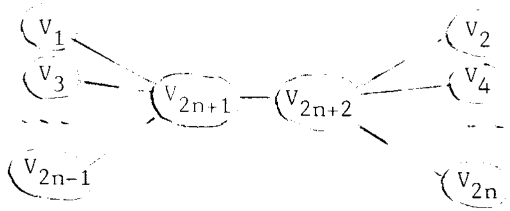
Theorem 2 Any complete 2n+1-graph with n edge-colours must contain a monochromatic circuit, i.e. $f(n) \leq 2n$.

Proof There are n(2n+1) edges, therefore in the subgraph made up of the edges of one colour there are $\geq$ 2n+1 edges and $\leq$ 2n+1 nodes, so that by Theorem 1 there is a circuit in this colour.

Theorem 3 For any n there is an edge-colouring of the complete 2n-graph with n colours giving no monochromatic circuit, i.e. $f(n) \geqslant 2n$.

Proof We use induction on n. The case $n = 1$ is trivial, and cases $n = 2$ and 3 were given in the previous contributions. Take a complete graph with 2n vertices V_1, V_2, ... V_{2n}, the edges coloured in n colours, c_1, c_2, ... c_n, with no monochromatic circuit. Add two more vertices, V_{2n+1} and V_{2n+2} joined by an edge in the new colour c_{n+1}. Join V_{2n+1} to the original vertices as follows; with edges of the new colour c_{n+1} to V_1, V_3, ... V_{2n-1}, and (for each $i = 1, 2, ..n$) with an edge of colour c_i to V_{2i}. Now join V_{2n+2} with edges of colour c_{n+1} to V_2, V_4, ... V_{2n}, and (for each $i \leqslant n$) with an edge of colour c_i to V_{2i-1}.

Suppose that the enlarged graph were to contain a monochromatic circuit. This circuit would have to contain one of the two new nodes, V_{2n+1} and V_{2n+2} (by the induction hypothesis). The circuit could not be in one of the original colours, because each of the two new nodes is on only one edge in each of these colours c_1, c_2, ... c_n. The circuit could not be in the new colour c_{n+1} because the subgraph made up of the edges in this colour is a tree as sketched below.



The enlarged graph is a complete 2n+2-graph with n+1 edge colours and no monochromatic circuit, and so the theorem is proved by induction.

Theorems 2 and 3 show that $f(n) = 2n$.

BINOMIAL IDENTITY 25 (JCMN 47 p.5123)

A. Brown (see also p.5243 below)

$$\sum_{r=0}^{2m+j}(-1)^r\binom{4m+p}{2r}/(2r+1) = (-4)^m k/(4m+p+1)$$

where for p = 0, 1, 2 or 3, j and k are given by

p	0	1	2	3
j	0	0	1	1
k	1	2	2	0

The origins of this problem are in an article "What they don't teach you about integration at school" by A. D. Fitt in the Mathematical Gazette, 72 (1988) pp.11-15.

Let $J_p = \int_0^{\pi/4} \cos p\theta/(\cos\theta)^{p+2}\,d\theta$.

Integration by parts shows that

$$(p+2)J_{p+1} = 2(p+1)J_p - 2^{1+\frac{1}{2}p}\sin p\pi/4.$$

This recursion gives the following few values:

p	0	1	2	3	4	5	6	7	8	9	10
J_p	1	1	2/3	0	-4/5	-4/3	-8/7	0	16/9	16/5	32/11

From these data we may guess $J_{4m-1} = 0$, and deduce that $J_{4m} = (-4)^m/(4m+1)$, $J_{4m+1} = (-4)^m/(2m+1)$ and $J_{4m+2} = 2(-4)^m/(4n+3)$.
The guess is then proved by induction.

An alternative calculation for J_p is obtained by putting $C = \cos\theta$ and $S = \sin\theta$, so that

$\cos p\theta$ = Real part of $(C+iS)^p = C^p - \binom{p}{2}S^2C^{p-2} + \binom{p}{4}S^4C^{p-4} - ..$

$\cos p\theta \sec^{p+2}\theta = \sec^2\theta(1 - \binom{p}{2}\tan^2\theta + \binom{p}{4}\tan^4\theta - \ldots)$

Integrating from 0 to $\pi/4$ gives

$$(p+1)J_p = \binom{p+1}{1} - \binom{p+1}{3} + \binom{p+1}{5} - \ldots$$

Comparison of the two calculated values for J_p leads to the binomial identity.

BINOMIAL IDENTITY 30 (JCMN 50 p.5208)

C. C. Rousseau

$$\sum_{k=0}^{n} (-4)^k \binom{n+k}{2k} = (-1)^n(2n+1)$$

To prove this, let C be the circle $z = Re^{it}$ where $R > 2$. Firstly $\binom{n+k}{2k} = \frac{1}{2\pi i}\int_C (z+1)^{n+k}/z^{2k+1}\,dz$, and the required sum is therefore $\frac{1}{2\pi i}\int_C \frac{(z+1)^n}{z} \sum_{k=0}^{n} (-4(z+1)/z^2)^k\,dz$.

Summing the geometric series gives

$$\frac{1}{2\pi i}\int_C z(z+1)^n(z+2)^{-2}\left\{1-(-4(z+1)/z^2)^{n+1}\right\}dz$$

$$= \frac{1}{2\pi i}\int_C z(z+1)^n(z+2)^{-2}dz - \frac{(-4)^n}{2\pi i}\int_C \frac{(z+1)^{2n+1}}{z^{2n+1}(z+2)^2}\,dz.$$

The second integral is zero, as can be seen by letting $R \to \infty$. The first term can be evaluated from the residue of the function at the double pole where $z = -2$.

Sum $= (-1)^n(2n+1)$.

BINOMIAL IDENTITY 32

$$\sum_{k=0}^{[n/2]} (-4)^{-k}\binom{n-k}{k} = 2^{-n}(n+1)$$

CONGRATULATIONS

On 29th June 1989 Professor Paul Erdös was elected a Foreign Member of the Royal Society of London.

At the 1989 International Mathematical Olympiad in Braunschweig, Mark Kisin won a silver medal.

POLYNOMIAL PROBLEM

(JCMN 46, p.5103)

In the polynomial $Q_n(x)$ the term of highest degree is $(-1)^n x^{2n}$. All the other terms are of odd degree. Also $Q_n(x) = Q_n(1-x)$. Show that $Q_n(x) = P_n(x-x^2)$ where P_n is a polynomial of degree n with all the coefficients positive integers except the constant term, which is zero.

Put $x = \frac{1}{2}+t$, then $Q_n(x)$ is a polynomial in t, say R(t), with the property that $R(t) = R(-t)$, so that R is a polynomial in $t^2 = x^2 - x + \frac{1}{4}$. If we put $y = x - x^2$ then $Q_n(x) = R(t)$ is a polynomial in y, say $P_n(y)$, in which the term of highest degree is y^n. Clearly $P(0) = Q(0) = 0$. Therefore put $P_n(y) = \sum_{r=1}^{n} a_{rn} y^r$, where $a_{nn} = 1$. The first few cases are:

$P_1(y) = y = x-x^2 = Q_1(x)$

$P_2(y) = y+y^2 = x-2x^3+x^4 = Q_2(x)$

$P_3(y) = 3y+3y^2+y^3 = 3x-5x^3+3x^5-x^6 = Q_3(x)$

$P_4(y) = 17y+17y^2+6y^3+y^4 = 17x-28x^3+14x^5-4x^7+x^8 = Q_4(x)$

$P_5(y) = 155y+155y^2+55y^3+10y^4+y^5$

In $P_n(x-x^2) = \sum a_{rn} x^r(1-x)^r$ the coefficient of x^{2m} (if $m < n$) is zero. Therefore

$$\sum (-1)^r \binom{r}{2m-r} a_{rn} = 0 \quad \text{(for } m < n) \qquad ..(1)$$

This implies the matrix equation $BA = 1$, where

$$B = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & .. \\ 0 & 1 & -3 & 1 & 0 & .. \\ 0 & 0 & 1 & -6 & 5 & .. \\ 0 & 0 & 0 & 1 & -10 & .. \\ 0 & 0 & 0 & 0 & 1 & .. \\ & & \cdots & & & \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 & 3 & 17 & 155 & .. \\ 0 & 1 & 3 & 17 & 155 & .. \\ 0 & 0 & 1 & 6 & 55 & .. \\ 0 & 0 & 0 & 1 & 10 & .. \\ 0 & 0 & 0 & 0 & 1 & .. \\ & & \cdots & & & \end{pmatrix}$$

The i, j component of B is $(-1)^{i+j}\binom{j}{2i-j}$, and of A is a_{ij}.

Each a_{ij} can be expressed as a determinant in elements of B, and so it is an integer. But is it positive? The matrix equation BA = 1 may be written AB = 1, giving another recursion $\sum a_{mk} (-1)^{k+n} \binom{n}{2k-n} = 0$ .. (2)
where summation is over the k for which the binomial coefficient is non-zero. Now take m as fixed, and simplify the notation by putting $a_{mr} = c_r$. We know that $c_m = 1$, and $c_r = 0$ for all $r < m$. The other values are determined by the recursion

$$c_n = \binom{n}{2} c_{n-1} - \binom{n}{4} c_{n-2} + \cdots \qquad .. (3)$$

Now use induction, assume that for $m+1 \leq r < n$

$$0 \leq 2c_r \leq r(r-1)c_{r-1} \leq 3c_r \qquad ... (H)$$

This (H) can be seen to be true for $n < 5$. Assuming (H), the ratio of magnitudes of two successive non-trivial terms of the series in (3) is

$$\binom{n}{2r} c_{n-r} \Big/ \left\{ \binom{n}{2r+2} c_{n-r-1} \right\} \geq \frac{(2r+1)(2r+2)}{(n-2r-1)(n-2r)} \; \frac{(n-r)(n-r-1)}{3} > 4$$

Therefore the series in (3) is of the familiar kind with terms alternating in sign and decreasing in magnitude. The sum is between the first and second partial sums, therefore less than the first term but greater than three quarters of it.

$$c_n \leq \tfrac{1}{2} n(n-1)c_{n-1}, \qquad c_n \geq (3/8)n(n-1)c_{n-1} \geq n(n-1)c_{n-1}/3$$

This has established (H) also for $r = n$, completing the proof by induction. Now that we have shown all the $b_{mn} = c_n$ to be non-negative, it follows without difficulty from the inequalities above that they are strictly positive.

BOOK REVIEW (1)

"A brief history of time" by Stephen Hawking in a bookshop was a temptation not to be resisted. I do not regret buying it, but in some ways the book disappointed me. The author explains how the publisher had urged him not to have equations in the book, on the grounds that the public does not understand equations. I think that trying to write a book on theoretical physics without equations is like trying to play football without a ball, but the book was not written for JCMN readers, it is a brave and probably successful attempt to communicate difficult ideas to the general public. The author makes an exception to the rule against equations by quoting $E = mc^2$. Every journalist knows this equation and thinks it is something to do with the atomic bomb, but of the many people who can quote the equation, few understand what E means, what m means, or what c means. The 2 up above the line is the easiest symbol to understand, which brings your reviewer to another comment — the author wants to give readers some idea of the size of the universe, and to that end writes numbers like 10^{19}, and each time adds the information that 10^{19} means 1 with 19 noughts after it, perhaps this was on the advice of the publisher who aims at a readership of people who know what 10000000000000000000 means but not what 10^{19} means. Perhaps the trouble is that we of the late 20th century have lost the art of using appropriate units for measuring things. For describing the universe, the parsec or the light-year used to be used, they avoid these difficulties of very large numbers.

Professor Hawking is an academic, and we should not hold it against him if he shows symptoms of the occupational disease of that profession — jealousy. But a good publisher would have cut out from the book the few embittered remarks about two of the author's great predecessors in Cambridge, Newton and Eddington. Mathematicians often find it necessary to point out that the work of others is inaccurate or obscure or misguided, but we should try to do it politely, avoiding personal defamation. To sum up, this is an important book, you should read it if only to cope with questions about it from your non-mathematical friends.

B.C.R.

BOOK REVIEW (2)

"Fourier analysis" by T.W.Körner (Cambridge University Press, 1988, 591 pages, paperback £20 in U.K.) Do not be misled by the title, this is not a rehash of Zygmund or of Carslaw. As the author writes in his preface, it "is meant neither as a drill book for the successful nor as a lifebelt for the unsuccessful student." About half the book is on Fourier analysis, the rest is on many other topics, often including diversions into history, and quotations from the writings and sayings of great mathematicians. There are delightful little bits about Monte Carlo integration, stability, statistics, prime numbers, Brownian motion, ... Indeed the author describes the book as a series of interlinked essays. The book is divided into 110 chapters, each more or less self-contained, so that it is easy to open the book at random and become fascinated. Highly recommended.

B.C.R.

LETTERS IN WRONG ENVELOPES

Someone wrote n letters and addressed the corresponding n envelopes, but put the letters in the envelopes(one in each) at random. The number of letters in their correct envelopes is the random variable Y. It can be seen that $E(Y)$ = the expectation of $Y = 1$. What can you say about the distribution of Y? If $Z = Y - 1$ then

$E(Z^k)$ = Moment of order k of Y about its mean

has the following values for the first few n and k

	k = 1	2	3	4	5	6	7	8
n = 1	0	0	0	0	0	0	0	0
n = 2	0	1	0	1	0	1	0	1
n = 3	0	1	1	3	5	11	21	43
n = 4	0	1	1	4	10	31	91	274
n = 5	0	1	1	4	11	40	147	568

Are these moments all integers? Do they converge as n tends to infinity?

BINOMIAL IDENTITY 25

(JCMN 47, p.5123, 49 p.5183 and p.5237 above)

Observe that $\int_0^1 (1+ix)^{n-1} + (1-ix)^{n-1} dx$

$= -(i/n)((1+i)^n - (1-i)^n) = n^{-1} 2^{\frac{1}{2}n+1} \sin n\pi/4.$

Also by expanding the integrand in powers of x we find the value $\sum 2(-1)^r \binom{n-1}{2r}/(2r+1)$ where summation is over all r for which the binomial coefficient is non-zero. The required results may be obtained by equating these values, treating separately the four possible remainders of n mod 4.

FACTORIZING COMPLETE GRAPHS

Terry Tao

Consider the conjecture C(n) that the complete graph on 2n nodes is a union (or "product") of n simple paths. If this is so then each path is of length 2n-1 and is Hamiltonian (contains all the nodes). Also every node is an end of just one path, and the paths are disjoint, no two have an edge in common.

Theorem 1 C(n) is true if and only if the complete graph on 2n+1 nodes is a union of n simple circuits.
Proof is trivial.

Theorem 2 C(n) is true if $p = 2n + 1$ is prime.
Proof Consider the complete p-graph, with the nodes labelled by the residue classes mod p. For each r = 1, 2, ... n, the edges xy for which $x - y \equiv \pm r \mod p$ form a circuit, with nodes 0, r, 2r, ... 2nr, (mod p). These n circuits are disjoint, and their union is the complete graph.

Is C(n) true for all n?

The question may be put in terms of the story of King Arthur and his knights of the Round Table — if there were an odd number, 2n+1, could they sit at the Round Table on n different days so that each pair sat together just once?