JAMES COOK MATHEMATICAL NOTES

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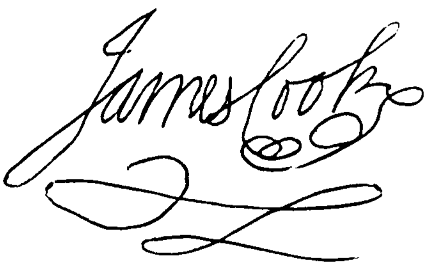

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INTEGRAL INEQUALITY (JCMN 42, p.5023)

The problem was to prove that if $0 \leq \theta \leq \pi/2$ and if $I = I(\theta) =$

$$\int_{\phi=0}^{\pi} \sin^2\phi \, d\phi \int_{\psi=0}^{\pi} \{1 - (\sin\theta \sin\phi \cos\psi + \cos\theta \cos\phi)^2\}^{\frac{1}{2}} \, d\psi$$

then $\qquad I \geq (4\pi/3)\cos\theta.$

In this issue we are glad to be able to print two different solutions, each giving a stronger inequality from which the result above follows.

Further questions might be asked. Clearly $I(\theta)$ is an even positive function with period π. What is $I(\pi/2)$? Is $I(\pi/2)$ a lower bound for the function $I(\theta)$?

FIRST SOLUTION

P. H. Diananda

We shall prove the stronger inequality

$$J = \int_{\phi=0}^{\pi} \sin^2\phi \, d\phi \int_{\psi=0}^{\pi} F^{\frac{1}{2}} \, d\psi \geq \frac{4\pi}{3}\cos\theta \qquad \ldots (*)$$

where $F = 1 - (\sin\theta \sin\phi \cos\psi + \cos\theta \cos\phi)^2 - \sin^2\theta \sin^2\psi$

$$= (\cos\theta \sin\phi - \sin\theta \cos\phi \cos\psi)^2$$

The inner integral in (*) is

$$\int_0^{\pi} |\cos\theta \sin\phi - \sin\theta \cos\phi \cos\psi| \, d\psi$$

(now split the interval into two and use $|a+x| + |a-x| \geq 2|a|$)

$$\geq 2\int_0^{\pi/2} \cos\theta \sin\phi \, d\psi = \pi \cos\theta \sin\phi$$

Let $G = 1 - (\sin\theta \sin\phi \cos\psi + \cos\theta \cos\phi)^2$

Then $0 \leq F \leq G$ and so $F^{\frac{1}{2}} \leq G^{\frac{1}{2}}$ and

$$I \geq J \geq \int_0^{\pi} \pi \cos\theta \sin^3\phi \, d\phi = \frac{4\pi}{3}\cos\theta.$$

SECOND SOLUTION

A. Brown

The form of the integrand suggests that spherical trigonometry is involved. To make this more specific we introduce points (given in Cartesian coordinates) $A = (\sin\theta, 0, \cos\theta)$, $N = (0, 0, 1)$ and $P = (\cos\psi \sin\phi, \sin\psi \sin\phi, \cos\phi)$ on the unit sphere. Then

$$\cos AOP = \cos\theta \cos\phi + \sin\theta \sin\phi \cos\psi$$

and the integral under consideration is

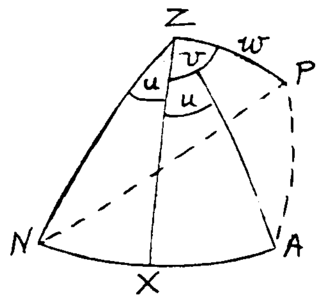
$$I = \int_0^{\pi} \int_0^{\pi} (\sin AOP)(\sin NOP) \sin\phi \, d\phi \, d\psi$$

We can regard ϕ and ψ as spherical polar coordinates with $\phi = NOP$. Thus I is the integral of $(\sin AOP)(\sin NOP)$ over the hemisphere where $0 \leq \phi \leq \pi$ and $0 \leq \psi \leq \pi$. Alternatively, since the integrand is an even function of ψ, we can say that

$2I$ = Integral of $(\sin AOP)(\sin NOP)$ over the whole sphere.

If $\theta = 0$ then A coincides with N and $2I$ = integral of $\sin^2\phi$ over the whole sphere $= 8\pi/3$. Thus $I = 4\pi/3$

when $\theta=0$, and we can take $0 < \theta \leq \pi/2$ in discussing the general case



Now that we have a geometrical interpretation of the integral it helps to change the coordinate system to take as much advantage as we can of the symmetry properties. Let OX, OY and OZ be rectangular axes with OZ perpendicular to the plane AON, and with OX bisecting the angle AON. If we put $\theta = 2u$ we can take the new coordinates of A as $(\cos u, \sin u, 0)$ and of N as $(\cos u, -\sin u, 0)$. Take P to be $(\cos v \sin w, \sin v \sin w, \cos w)$. Thus v and w are polar coordinates corresponding to the new coordinate axes. With this choice of axes and the integrand (sin AOP)(sin NOP) it is easy to see that XOY and XOZ are planes of symmetry, and (less obviously) YOZ is also a plane of symmetry. For the symmetry about YOZ, let N' be the point diametrically opposite N on the sphere. Then sin N'OP = sin(π − NOP) = sin NOP, and in the same way if A' is diametrically opposite A, sin A'OP = sin AOP. Hence

$$(\sin A'OP)(\sin N'OP) = (\sin AOP)(\sin NOP)$$

and we will get matching contributions from the hemispheres on either side of YOZ.

Because of the symmetry properties we can say that $I = 4I_0$, where I_0 is the integral of (sin AOP)(sin NOP) over the octant where $0 \leq v \leq \pi/2$ and $0 \leq w \leq \pi/2$. In terms of the new coordinates, $\cos AOP = \sin w \cos(v-u)$ and $\cos NOP = \sin w \cos(v+u)$.

$$\sin^2 AOP = 1 - \sin^2 w \cos^2(v-u)$$
$$= \cos^2 w + \sin^2 w \sin^2(v-u) = C - D$$

where $C = \cos^2 w + \sin^2 w\,(\sin^2 v \cos^2 u + \cos^2 v \sin^2 u)$

and $D = 2 \sin^2 w \sin v \cos u \cos v \sin u$.

Similarly, changing the sign of u, $\sin^2 NOP = C + D$.

Also $(\sin^2 AOP)(\sin^2 NOP) = C^2 - D^2 = E^2 + F^2$

where $E = \cos^2 w + \sin^2 w\,(\sin^2 v \cos^2 u - \cos^2 v \sin^2 u)$

and $F = 2 \cos w \sin w \cos v \sin u$

Therefore $E \leq |E| \leq \sqrt{E^2 + F^2} = (\sin AOP)(\sin NOP)$
$$= \sqrt{C^2 - D^2} \leq |C| = C.$$

Any integral over the octant may be calculated as the repeated integral $\int_0^{\pi/2}\int_0^{\pi/2}$ (integrand) $\sin w \, dv \, dw$.

The integral of E is $(\pi/6)(1 + \cos^2 u - \sin^2 u)$
$$= (\pi/6)(1 + \cos\theta)$$

and of C is $(\pi/6)(1 + \cos^2 u + \sin^2 u) = \pi/3$.

Therefore $(\pi/6)(1 + \cos\theta) \leq I_0 \leq \pi/3$.

This gives the required result:

$$(4\pi/3)\cos\theta \leq (2\pi/3)(1 + \cos\theta) \leq I \leq 4\pi/3.$$

SPHERICAL TRIANGLE INEQUALITY

If the sides of a spherical triangle are a, b and c, then is it true that $\sin a \leq \sin b + \sin c$?

QUAINT IDENTITY

(JCMN 42, p. 5020)

Several readers have pointed out that this identity holds only for positive x, and that the transformation $x = \tan\theta$ is useful. The result should be

$$\left\{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-1}\right\}^{\frac{1}{2}} - \left\{\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}-1}\right\}^{\frac{1}{2}} = \begin{cases}\sqrt{2} & \text{for } x > 0\\ -\sqrt{2} & \text{for } x < 0\end{cases}$$

MOUNT WARNING

George Szekeres

In one of the early issues of JCMN, way back in 1977 (JCMN 9), I wrote about Mt. Warning, one of Captain Cook's mountains, and the delights of its top which I called the most beautiful 100 square metres in Australia. This summer, after 10 years of absence, I made another pilgrimage to the top.

Alas, in the intervening years much of the beauty of the place has gone. Mind you, the track is still in excellent shape and the view from the top is still there, but what used to be a lonely meadow of indescribable loveliness and charm, with strange butterflies and beetles flitting in and out from the surrounding bushland, is now a dusty wasteland with not a blade of grass in sight and not a clean square metre to sit down and dream back to the time in May 1770 when Captain Cook's "Endeavour" appeared on the distant horizon.

Was it the New South Wales Premier's much publicised champagne party a few weeks before (which could explain some of the garbage lying about) or the trampling of tens of thousands of feet(nay, pairs of feet) that make it to the top every year? Whatever the cause, I would certainly not call it the most beautiful 100 square metres in Australia any more. The ground is perfectly well-prepared now to have the ubiquitous lollipop T.V. tower or some such objet d'art to be erected on the spot, which apparently gets the first rays of sunlight on mainland Australia. This desecration happened in numerous instances in the past; most painfully (for me) on top of Mt. William, the highest peak of the Grampians, named after Major William Mitchell who first set foot on that very beautiful spot way back in the thirties of the last century. No more beauty there. Sic transit gloria mundi.

FOUR SQUARES

It is easy to find three squares in arithmetic progression, such as 1, 25, 49, or 49, 169, 289, but is it possible to find four?

HOLIDAY PROBLEM

Seven people each made three visits to a holiday resort. During these visits each met each of the others. Prove that at one time there were three of them at the resort.

H. M. BARK ENDEAVOUR

From Jim Douglas has come a newspaper cutting about Captain Cook's ship "Endeavour".

Mr. Laurence Gruzman, a Sydney barrister, has been investigating the fate of the ship. His research enables us to add a little to the story told in these pages (JCMN, 30 p. 3147 and JCMN31, p.3176). After the hull was grounded in the harbour of Newport, Rhode Island in May 1794 it was bought by Captain John Cahoone of the U.S. Revenue cutter Vigilant. Some of the materials were sold to Mr. Seth Russel and used for the construction of a new ship called the "Wareham", and other parts went into the "Concord" being built by Captain Cahoone and his brother Stephen. The latter ship was broken up in 1796 and her remaining timbers were left in the mud where they were visible for many years at low tide. In 1815 a gale broke up the old hull and a piece of the timber was presented to James Fenimore Cooper, author of "The Last of the Mohicans". It is this piece of which Mr. Gruzman is now the proud owner, and he also has a letter written by Mr. George Howard dated September 1828 about the piece of timber.

Other bits of the old ship are in many museums all over the world, and a small piece is now on the Moon, where it was put by the Apollo astronauts.

BINOMIAL IDENTITY 18
(JCMN 40, p.4190)

Vichian Laohakosol

$$\sum_{j=1}^{n} \binom{n}{j}(-1)^j j^n = (-1)^n n!$$

In order to prove this, consider the polynomial

$$G(x) = (1-x)^n = \sum_{s=0}^{n} (-1)^s \binom{n}{s} x^s \qquad (n \geqslant 1)$$

and the sum $F(j, n) = \sum_{s=0}^{n} (-1)^s \binom{n}{s} s^j \qquad (j=1, 2, \dots)$

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} s\, x^s = (x\, d/dx)\, G(x) = -nx(1-x)^{n-1}$$

Putting x=1 tells us that $F(1, n) = 0$ (if $n > 1$)

because $F(j, n)$ is the value of $(x\, d/dx)^j G(x)$ when $x = 1$.

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} s^2 x^s = (x\, d/dx)^2 G(x) =$$

$$n(n-1)\, x^2 (1-x)^{n-2} - nx(1-x)^{n-1}$$

Putting $x = 1$ tells us that $F(2, n) = 0$ if $n > 2$.

Generally $(x\, d/dx)^j G(x)$ for any $j < n$ is

$(-x)^j n(n-1)\dots(n-j+1)(1-x)^{n-j}$ + terms with higher powers of $(1-x)$.

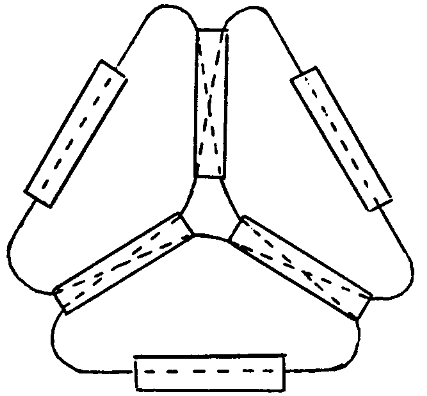
Putting x=1 shows that $F(j, n) = 0$.

Finally $(x\, d/dx)^n G(x) = (-x)^n n! +$ powers of $(1-x)$

and putting x=1 shows that $F(n, n) = (-1)^n n!$

COLLAPSING POLYHEDRA

For my grandson Daniel's sixth birthday I decided to send him a regular tetrahedron, so that he could ponder on some arithmetical properties of the number 6, illustrated by the tetrahedron. As he lives on the opposite side of the world, it seemed desirable to make a small collapsible tetrahedron which could be posted inside a birthday card, and which could be erected by pulling on a string. I did this by cutting a plastic drinking straw into six equal lengths, threading string through them as shown below and tying the ends of the string.



Since then I have wondered about making an octohedron for his eighth birthday. How could it best be done?

QUOTATION CORNER 23

When Captain Cook and Sir Joseph Banks sailed up the east coast of Australia back in 1770, they collected a wide variety of plants from the mainland which they planned to examine once they arrived back in England. On their return voyage, however, they ran out of tea. The true Englishman that he was, Captain Cook couldn't do without his daily pot of tea, so he ordered one of his sailors to go down below and from the plants he and Banks had collected to find one with a pleasant aroma. The sailor picked out one he fancied and turned it into a brew which Captain Cook noted in his journals made a refreshing drink. Since then the plant has been known as coming from the "tea-tree".

The quotation above is from a book "Practicalities" published by the Australian Broadcasting Corporation, consisting of edited versions of radio talks.

Orthodox opinion is that the name tea-tree or ti-tree is derived from the Polynesian name ti for a certain kind of small tree.

QUESTION ABOUT A NUMBER

V. Laohakosol

Is $\sum_{0}^{\infty} 2^{-(n^2)}$ algebraic or transcendental?

NON-LINEAR SUMMATION OF SERIES

Consider an infinite series $t_1 + t_2 + t_3 + \dots$ Can we estimate the sum S by calculating the first thousand (or million) terms? Pure mathematicians will insist that it is quite impossible to do so, but many scientists would regard it as a routine sort of calculation. Elementary theory tells us that the sum S is the limit of the partial sum $S_n = t_1 + t_2 + \dots + t_n$

and consequently we are tempted to think that if we calculate the partial sum S_n for some very large n we might reasonably hope to have a good approximation for S. Is there a better way? Should we consider some other function S_n^* instead of S_n?

For the geometric series $a + ar + ar^2 + \dots$ the sum is $a/(1-r) = t_1^{\,2}/(t_1-t_2)$. This fact guides us towards trying

$$S_n^* = S_n + t_{n+1}^2 \,/(t_{n+1} - t_{n+2}) \qquad \dots\dots (1)$$

This formula (1) must be exact (that is $S_n^* = S$) for any geometric series, and so we hope that S_n^* may be a good approximation for S in the case of other series, specially if the series tends to become nearly geometric, (i.e. if the ratio of successive terms tends to a constant) For any series let $R_n = S - S_n$ and let $E_n = S - S_n^*$.

Example 1 The series $1 + 2r + 3r^2 + \dots$ has sum $S = (1-r)^{-2}$ We may calculate that $R_n = r^n(1 + n - nr)/(1 - r)^2$ and $E_n = -r^{n+2}(1 - r)^{-2}(1 + n - nr - 2r)^{-1}$, so that S_n^* is an appreciably better estimate than S_n.

Example 2 The series $x + x^2/2 + x^3/3 + \dots$ which for $|x| < 1$ converges to the sum $S = -\log(1 - x)$.

$$R_n \sim \frac{x^{n+1}}{n(1-x)} - \frac{x^{n+1}}{n^2(1-x)^2} + \frac{x^{n+1}(1+x)}{n^3(1-x)^3} - \dots$$

$$S_n^* - S_n \sim \frac{x^{n+1}}{n(1-x)} - \frac{x^{n+1}}{n^2(1-x)^2} + \frac{x^{n+1}(1+x-x^2)}{n^3(1-x)^3} - \dots$$

so that $E_n = R_n - S_n^* + S_n \sim \frac{x^{n+3}}{n^3(1-x)^3}$, smaller than R_n by a factor $O(1/n^2)$.

Now take an example where we cannot expect (1) to work well, because the series converges quite differently from a geometric series.

Example 3 $1/2 + 1/6 + 1/12 + 1/20 + \dots$ where $t_n = 1/(n(n+1))$. It is easy to find that

$S_n = n/(n+1)$ $\quad S = 1$ $\quad S_n^* = (2n+3)/(2n+4)$

$R_n = 1/(n+1)$ $\quad E_n = 1/(2n+4)$

The use of S_n^* instead of S_n in this case is only moderately successful, roughly speaking it halves the error in estimating S, or halves the number of terms that need to be calculated for a given accuracy.

We may refine the formula (1) by introducing a "fudge factor" F, that is, put $S_n^* = S_n + F\,t_{n+1}^2/(t_{n+1} - t_{n+2})$ where F is determined by putting $S_{n+1}^* = S_n^*$, and therefore

$$F\left\{t_{n+1}^2/(t_{n+1} - t_{n+2}) - t_{n+2}^2/(t_{n+2} - t_{n+3})\right\} = S_{n+1} - S_n = t_{n+1}$$

The new formula is

$$S_n^* = S_n + \frac{t_{n+1}^3(t_{n+2} - t_{n+3})}{t_{n+1}^2(t_{n+2} - t_{n+3}) - t_{n+2}^2(t_{n+1} - t_{n+2})} \quad \dots. (2)$$

This formula (2) must of course be exact for a geometric series, and so we can expect it, like formula (1), to give a good approximation in the case of a series that is nearly geometric, such as examples 1 and 2.

Applying formula (2) to our example 3, we find

$$S_n^* = S_n + \frac{(n+3)^2}{(n+1)(n+2)(n+5)} = \frac{n^2+7n+9}{(n+2)(n+5)} \quad \text{and } E_n = \frac{1}{(n+2)(n+5)}$$

so that E_n is smaller than R_n by a factor $O(1/n)$.

Experiment suggests that formula (2) gives similarly good results whenever t_n is a rational function of n.

Formulae (1) and (2) have their limitations. Just as (1) was ineffective when $t_n \sim n^{-2}$, so (2) will be found ineffective when $t_n \sim n^{-1}(\log n)^{-2}$.

For a trigonometric series with slowly converging coefficients we must try to find another formula, for neither (1) nor (2) is appropriate. For guidance we start by looking at a series with constant coefficients. The series $\sum_1^{\infty} a \sin(nx+b)$ does not converge in the ordinary sense, but has Abel or Cesaro sum $\frac{a\cos(\frac{1}{2}x+b)}{2\sin(\frac{1}{2}x)}$

$$\sum_{r=n+1}^{\infty} a\sin(rx+b) = \frac{a\cos((n+\frac{1}{2})x+b)}{2\sin(\frac{1}{2}x)} = \frac{t_{n+1}-t_n}{4\sin^2(\frac{1}{2}x)}$$

We may verify that $t_{n+1} - 2t_{n+2} + t_{n+3} = -4\,t_{n+2}\sin^2(\frac{1}{2}x)$, and so we put:

$$S_n^* = S_n + \frac{t_{n+2}(t_n - t_{n+1})}{t_{n+1} - 2t_{n+2} + t_{n+3}} \qquad \ldots (3)$$

and we have a formula which we know to be exact for a trigonometric series with constant coefficients, and we therefore hope it to be good in the case of slowly varying coefficients. Rather unexpectedly (3) turns out to be exact for any geometric series.

MODERN COOKING

A microwave oven takes 30 seconds to heat one cup of coffee, 45 seconds to heat two cups, and 70 seconds to heat three cups. How long does it take to heat five cups?

BACK NUMBERS

The JCMN for its first eight years (1975-1983) was published by the Mathematics Department of the James Cook University. Issues 1-31 of this period have been reprinted in three paperback volumes, on sale for 10 Australian dollars each, postage included, from

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Since 1983 the JCMN has had no connection with the James Cook University. For issues from 32 onwards, back numbers are available from me at

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