JAMES COOK MATHEMATICAL NOTES

Issue Number 4, June 1976: to celebrate the departure of Prof. B.C. Rennie from Australia for overseas study leave.

So that he won't be forgotten, B.C.R. left for J.C.M.N. a new

THEOREM

Almost every convex polyhedron is a tetrahedron

(please submit an elegant proof before his return) and a proof of

GUINAND'S THEOREM (J.C.M.N. No. 3)

If

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix} \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix} \begin{bmatrix} \lambda & c_{12} & c_{13} \\ c_{21} & \mu & c_{23} \\ c_{31} & c_{32} & \nu \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where none of the a's or b's is zero, then if two of λ,μ,ν are zero it follows that the third is zero.

PROOF

Denote the three matrices on the left by A,B and C

$$ABC = I \quad \text{and so} \quad C = (AB)^{-1}$$

$$AB = \begin{bmatrix} a_{12}b_{21} + a_{13}b_{31} & a_{13}\, b_{32} & a_{12}\, b_{23} \\ a_{23}\, b_{31} & a_{21}\, b_{12} + a_{23}\, b_{32} & a_{21}\, b_{13} \\ a_{32}\, b_{21} & a_{31}\, b_{12} & a_{31}\, b_{13} + a_{32} b_{23} \end{bmatrix}$$

The problem is to show that if two of the diagonal elements of the matrix of minors of A B are zero then so is the third. The (1, 1) minor is

$$a_{21}\, b_{12}\, a_{32}\, b_{23} + a_{23}\, b_{32}\, a_{31}\, b_{13} + a_{23}\, b_{32}\, a_{32}\, b_{23}$$

Now put $a_{ij}\, b_{ji} = m_{ij}$. Then the (1, 1) minor is

$$m_{21}\, m_{32} + m_{23}\, m_{31} + m_{23}\, m_{32}$$

The equations are a little simplified by putting

$u_1 = m_{23}$ and $v_1 = m_{32}$ (and so on by cyclic permutation).

The three minors are

$$v_3\, v_1 + u_1\, u_2 + u_1\, v_1$$

$$v_1\, v_2 + u_2\, u_3 + u_2\, v_2$$

$$v_2\, v_3 + u_3\, u_1 + u_3\, v_3$$

If the first two are zero, then multiplying the first by v_2 and the second by u_1 and subtracting gives $u_1 u_2 u_3 = v_1 v_2 v_3$

Then v_1 (third minor) $= u_3(u_1 u_2 + u_1 v_1 + v_1 v_3) = 0$

This proves our results unless $v_1 = 0$.

Now to examine the case $v_1 = 0$.

From the vanishing of the first minor, $u_1 u_2 = 0$; there are two cases, if $u_1 = 0$ then $m_{23} = m_{32} = 0$. That is $a_{23} b_{32} = a_{32} b_{23} = 0$ and so $b_{32} = b_{23} = 0$ and det $B = 0$.
The other case $u_2 = 0$ is disposed of similarly, for it gives $b_{32} = b_{23} = 0$.

This completes the proof and shows that there is a similar theorem where the a's are allowed to be zero but not the b's. The fact that we cannot allow both a's and b's to be zero is shown by

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & -0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Bonus marks were offered for an elegant proof and for any generalisation to higher order matrices. I gave B.C.R. 9 out of 10 (one off for untidiness) on the first score and 0 out of 10 on the second ('putting the theorem into four dimensions baffles me completely').

It is a little hard for Adelaide to compete with the Cook bicentenary but a hundred years ago lectures started at the University of Adelaide, and you might like to try question VI of the first Pure Mathematics I exam set by the first Professor of Mathematics, Horace Lamb, in 1876:

QUESTION VI

A man sets apart £28 a year to be spent in drink, and considers that he requires in the year a quantity of alcohol amounting to 24 (reputed) quarts. He prefers claret to ale, but claret costs 40s. a dozen, ale only 12s. a dozen. The percentage of alcohol in the claret being 10, and in the ale 6, how much does he buy of each. If the price of ale rises, will he drink more ale, or less, than before?

Again, this calls for bonus marks - for proofs with sophistication, abstraction and modern terminology.

Matrices again! Prof. M.N. Brearley (Point Cook) suggests a nostalgic revival of the following THEOREM:

If A,B are symmetric matrices of order n with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0$ and $0, \ldots, 0, \lambda_{r+1}, \ldots, \lambda_n$ respectively, $\lambda_i \neq 0$, and if those of $A + B$ are $\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n$, then $AB = 0$.

No bonus marks (except for Prof. Max Kelly if his filing system is good enough to produce the proof he gave some years ago).

PLEASE send all correspondence to your guest editor:

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who so far hasn't been inundated with mail. A few solutions for other problems in previous issues are being held over - so there's still time to submit yours!