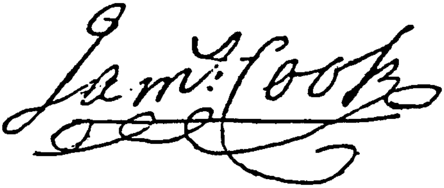
The New

JAMES COOK MATHEMATICAL NOTES

Volume 4, Issue Number 36

February 1985



Editor : B.C. Rennie,
69 Queens Road,
Hermit Park,
Townsville, N.Q. 4812
Australia.

The James Cook Mathematical Notes is published in three issues per year, in February, June and October.

The subscription for 1985 will be the same as for 1984, as follows. The rate for one year (three issues) in Singapore dollars will be

In Singapore (including postage)	\$20
Outside Singapore (including air mail postage)	\$30

Subscribers in countries with no exchange control (such as Australia, the United Kingdom and U.S.A.) may send ordinary cheques in their own currency to the Editor; the amount calculated at the current exchange rate. The rates at the time of writing are approximately

\$1 (Singapore) = 56 cents (Australian)
= 41 pence (U.K.)
= 46 cents (U.S.A.)

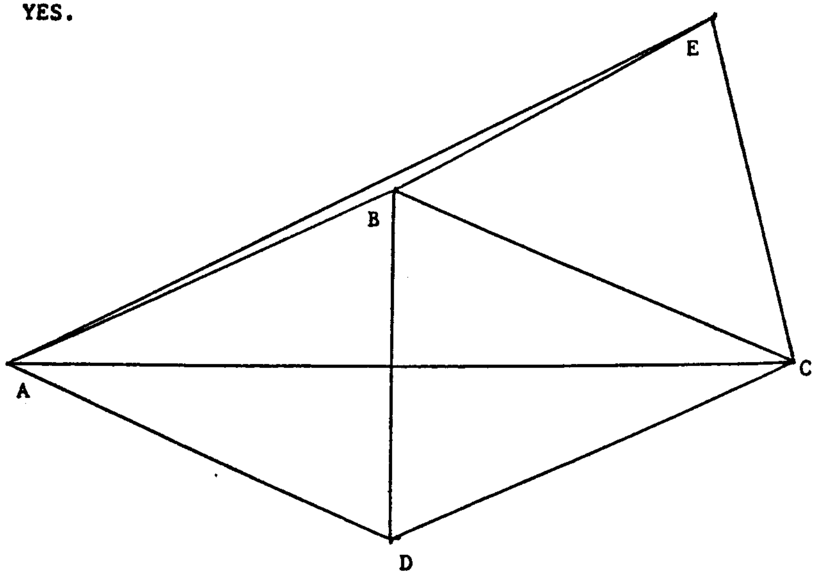
Those in other countries should send whatever currency they can. Please make cheques payable to B.C. Rennie.

Back numbers are \$12 (Singapore) each, sent by sea mail from Australia.

CONTENTS

PROBLEMS (JCMN 38, p.4068)

P. Erdös asked if it is possible to find in the plane five points, no three on a line, no four on a circle, and no one equidistant from three others, such that they determine four distinct distances, one occurring once, one twice, one three times and one four times. The answer is YES.



The points are given in Cartesian coordinates as follows. C and A are $(\pm 1, 0)$, B and D are $(0, \pm t)$ and E is (x, y), where numerically $x = .795775$, $y = .880451$ and $t = .451913$, and exactly $4x^3 + 3x^2 - 20x + 12 = 0$ and $1 - t^2 = x = 2yt$.

$$AB = BC = CD = DA = 1.097372$$
$$DB = BE = EC = 0.903816$$
$$CA = AE = 2$$
$$DE = 1.551918$$

POLYHEDRAL BODIES AND THEIR GRAPHS

J. B. Tabov

In almost all books on the theory of graphs, the puzzle of Hamilton about the dodecahedron is mentioned. This puzzle may be considered for an arbitrary polyhedron, or more abstractly, for any graph; it leads to the concept of Hamiltonian and semi-Hamilitonian graphs. A graph is "Hamiltonian" if there is a circuit going just once through each node, and "semi-Hamiltonian" if there is a path going just once through each node.

On the basis of the above idea, by replacing the vertices of the polyhedron by its faces, one may obtain a new problem, similar to that of Hamilton. More precisely, let P be a polyhedron with faces $F_1, F_2, \ldots, F_n$. By D(P) we denote the graph with nodes $A_1, A_2, \ldots, A_n$, in which two nodes A_i and A_j are connected with an edge exactly when F_i and F_j have a common edge. The question whether D(P) is Hamiltonian or semi-Hamiltonian, or neither, will be considered below.

To keep to the geometrical aspect of the problem, let us limit our considerations to graphs D(P) generated by some polyhedron P, or by some <u>polyhedral body</u> P. Here, by "polyhedral body" (or p-body), we mean any finite part of the space bounded by parts of planes. About 200 years ago such objects were called polyhedra, but now the usual definition of "polyhedron" is rather narrower. In this connection the famous book "Proofs and Refutations" by I. Lakatos (Cambridge University Press, 1976) contains interesting discussions.

Our purpose is to mention and to discuss some extremal questions about Hamiltonian (denoted by H) and semi-Hamiltonian (denoted by semi-H) graphs D(P).

<u>Example 1</u>. Let $P_1, P_2, \ldots, P_9$ be the mid-points of the edges of a triangular prism, and let P be

the convex hull of the set $\{P_1, P_2, \ldots, P_9\}$. P is a convex polyhedron with 9 vertices and 11 faces. The graph D(P) is shown in Fig.1. The nodes A and B correspond to (the parts of) the bases of the triangular prism. D(P) is semi-H, but it is not H.

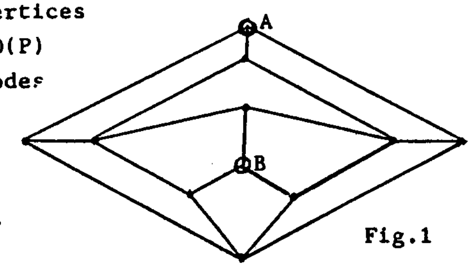


Fig.1

Example 2. Consider the polyhedral body P, made by cutting two notches in one of the edges of a tetrahedron - see Fig.2. P has 12 vertices and 8 faces. D(P) is shown in Fig.3. It is semi-H, but not H.

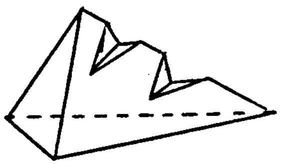


Fig.2

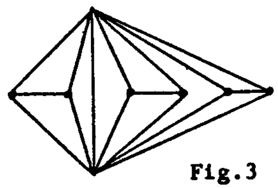


Fig.3

Example 3. Consider the polyhedral body P, made by pasting a small tetrahedron on a face of a big tetrahedron (Fig.4). P has 8 vertices and 7 faces. D(P) is shown in Fig.5. It is semi-H, but not H.

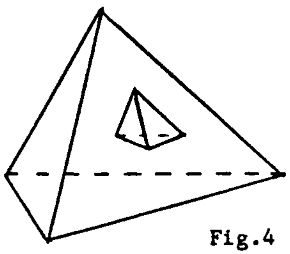


Fig.4

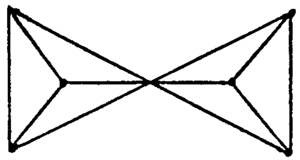


Fig.5

Example 4. Consider the polyhedral body P, made by cutting three notches through one of the edges of a tetrahedron (Fig.6). P has 16 vertices and 10 faces. D(P) is shown in Fig.7. It is not semi-H. (This example is by my colleague V. Popov.)

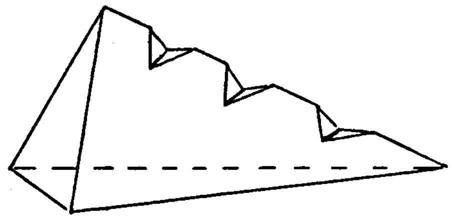


Fig.6

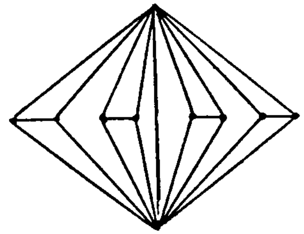


Fig.7

Example 5. Consider the p-body P, made by pasting two small tetrahedra on a face of a big tetrahedron - see Fig.8. P has 12 vertices and 10 faces. D(P) is shown in Fig.9. It is not semi-H.

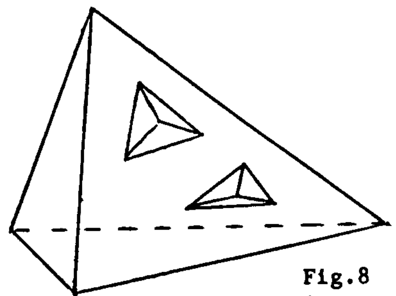


Fig.8

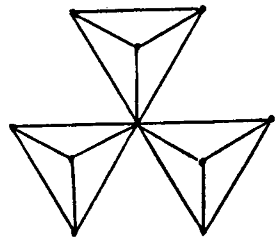


Fig.9

And now let us discuss the above examples and some related questions.

1. Example 1 shows that:

1(a) There exists a convex polyhedron P with 9 vertices and 11 faces, such that D(P) is not H.

I think that the following conjectures are true:

1(b) If P is a polyhedron with less than 9 vertices, then D(P) is H.

1(c) If P is a polyhedron with less than 11 faces then D(P) is H.

The following questions may be of interest:

1(d) What is the minimal n for which some polyhedron P has n faces and D(P) is not semi-H?

1(e) What is the minimal n for which some polyhedron P has n vertices and D(P) is not semi-H?

2. Example 2 shows that:

2(a) There exists a p-body P with 8 faces, whose faces are polygons, and D(P) is not H.

I think that the following conjecture is true:

2(b) If P is a p-body whose faces are polygons and the number of its faces is less than 8, then D(P) is H.

3. Example 3 shows that:

3(a) There exists a p-body P with 7 faces, whose D(P) is not H.

Since the degree of each node of D(P) is not less than 3, then:

3(b) For every p-body P with less than 7 faces, D(P) is H.

4. Example 4 shows that there exists a p-body P with 10 faces, whose faces are polygons and whose D(P) is not semi-H.

5. Example 5 shows that there exists a p-body P with 12 vertices, whose graph D(P) is not semi-H.

6. Since the degree of each node of D(P) is not less than 3, then for every p-body with less than 9 faces, D(P) is semi-H.

One of my students told me that he has proved (partially by computer) that for every p-body P with 9 faces, D(P) is semi-H. If we suppose that this is true, then, having in mind 4 and 6, we can conclude that the minimal n for which there exists a p-body P with n faces, whose D(P) is not semi-H, is n = 10.

7. Numerous problems about the minimal number of vertices, faces and edges of a p-body P with non-semi-H and non-H D(P) are related with the above examples and comments. We may restrict P to some family of p-bodies (polyhedra, convex polyhedra, etc.)

8. It is interesting to note that according to the definition of a polyhedron in Hadamard's "Geometry", the p-bodies in examples 2 and 4 are polyhedra. But the more usual definitions require the faces to be convex, so that (in the notation of this article) they are not polyhedra.

QUOTATION CORNER 17

1. The question "What is Mathematics?" is as unavoidable and as unanswerable as "What is Life?". In actual fact I think it's almost the same question. - E.W. Dijkstra in "The Correctness Problem in Computer Science" (R. Boyer and J. Moore, editors), Academic Press, 1981.

2. Mathematics (and Statistics, a subset of mathematics) is always true because it is completely irrelevant. Editorial, IEEE Transactions on Reliability, R - 32 (1983) p.337.

(The quotations above both sent in by C.J. Smyth)

A DIFFERENCE EQUATION EXAMPLE

A. Brown

I did some work last year on the difference equation

$$x_{n+1} = F(x_n) = ax_n^3 + (1-a)x_n, \quad (0 < a \leqslant 4)$$

which had been put forward as a simple model for a population dynamics problem. The limitations on the parameter a ensure that if $-1 \leqslant x_n \leqslant 1$ then $-1 \leqslant x_{n+1} \leqslant 1$, so F gives a mapping of $[-1, 1]$ into itself.

I obtained some results for periodic solutions of this difference equation and when I sent a copy to a friend he raised the question:

Given that f(x) is continuous, with $-1 \leqslant f(x) \leqslant 1$ for $-1 \leqslant x \leqslant 1$, must the equation $x_{n+1} = f(x_n)$ always have a stable periodic solution?

The answer is that you can have cases where all the periodic solutions are unstable.

If you take the equation above and put a = 4, then

$$x_{n+1} = 4x_n^3 - 3x_n$$

and if $-1 \leqslant x_0 \leqslant 1$ you can write $x_0 = \cos\phi$, with $0 \leqslant \phi \leqslant \pi$. It follows that

$$x_1 = 4\cos^3\phi - 3\cos\phi = \cos 3\phi$$

and, in the same way, $x_2 = \cos 9\phi, \ldots, x_n = \cos 3^n\phi$. Thus the solution is periodic, with period N, if $\cos(3^N\phi) = \cos\phi$. This gives

$$\phi = 2M\pi/(3^N \pm 1),$$

for any integer M. In practice you can restrict the choice of M and pick out the solutions with minimum period N. (It is possible to have a solution which settles into periodic behaviour after a finite number of terms but ultimately all the periodic solutions are given by the

argument above.) So with this example there are periodic solutions of any given length, N, and these periodic solutions can be identified.

For a solution of period N, the stability criterion is

$$S_N = \frac{dx_{m+N}}{dx_m} = \frac{dx_N}{dx_0} = \frac{(dx_N/d\phi)}{(dx_0/d\phi)}$$

$$= 3^N(\sin 3^N\phi)/(\sin\phi) = \pm 3^N,$$

since $\sin 3^N\phi = \pm\sin\phi$ when $\cos 3^N\phi = \cos\phi$. Thus $|S_N| \geqslant 3$ and all periodic solutions are unstable.

A simpler example where you can use the same type of argument is

$$W_{n+1} = -1 + 2W_n^2.$$

If $|W_0| \leqslant 1$, put $W_0 = \cos\phi$. Then $W_1 = \cos 2\phi$, $W_2 = \cos 4\phi, \ldots$ and $W_n = \cos 2^n\phi$. Here again there are periodic solutions of period N, for any positive integer N, and they can all be identified. In this case, $S_N = \pm 2^N$ and all the periodic solutions are unstable.

If you go back to the first example and write $x_0 = \cos\phi_0$, then for each N there are only a finite number of values of ϕ_0 in $[0, \pi]$ which make x_0 an element of a solution with period N. If ϕ_0^* is one of these values, then $\phi_0 = \phi_0^*/3^m$ will lead to a periodic solution eventually, for any positive integer m. This means that there is an enumerable infinity of values of ϕ_0 which lead to a periodic solution with period N. I should expect this to be a set of measure zero, like the rational numbers, so despite all the exceptions almost all initial values x_0 should lead to aperiodic sequences.

The same argument should apply for W_0 in the second example.

BERNOULLI STILL RIGHT

For years I used to tell my class, whenever I lectured on gas flow, that Bernoulli's equation (asserting that in steady flow $\frac{1}{2}q^2 + \int dp/\rho$ is constant on each streamline) failed when a streamline passed through a shock wave, because then the functional relation between pressure and density (which is necessary for the existence of the integral) failed. For a perfect gas p/ρ^{γ} is constant except for a discontinuous increase on passing through a shock wave.

However, Bernoulli's equation for a perfect gas can be rewritten so that it holds even with shock waves, as follows.

Let p = pressure, ρ = density, T = absolute temperature, q = gas speed and $c = (\gamma p/\rho)^{\frac{1}{2}}$ = the speed of sound. The gas laws tell us that $p = R\rho T$ (for some constant R) and that p/ρ^{γ} is constant for adiabatic (i.e. isentropic) changes, where γ (= 1.4 for air) is the ratio of specific heats. The indefinite integral $\int dp/\rho$ may be written as $\frac{\gamma}{\gamma-1}\frac{p}{\rho}$ or as $\frac{\gamma}{\gamma-1}RT$. Using suffix zero for some arbitrary state of the gas, and writing $M = q/c_0$ (a kind of Mach number) we may rewrite Bernoulli's equation as the assertion that $\frac{1}{2}q^2 + \frac{\gamma}{\gamma-1}P/\rho$ or (equivalently) $\frac{\gamma-1}{2}M^2 + T/T_0$ is constant on each streamline. The fact that the constant does not change when the streamline passes through a shock wave may be verified as follows.

It is sufficient to consider one-dimensional flow.

	Speed	shock	Speed	
Pressure $= p$	$\leftarrow$		$\leftarrow$	Pressure $= p'$
Density $= \rho$	v		u	Density $= \rho'$
Temperature $= T$				Temp. $= T'$

The equation for conservation of energy is

$$\frac{u^2 - v^2}{2} + \frac{p'}{\rho'} - \frac{p}{\rho} = \frac{R}{\gamma-1}(T - T') = \frac{1}{\gamma - 1}\left(\frac{p}{\rho} - \frac{p'}{\rho'}\right)$$

because $R/(\gamma - 1)$ is the specific heat at constant volume. This shows that $\frac{q^2}{2} + \frac{\gamma}{\gamma-1}\frac{p}{\rho}$ is the same on both sides of the shock wave.

TIMES AND DATES

Since Christmas your Editor has had a copy of Captain Cook's Journal of his first Pacific voyage of 1768-71, and in consequence can now clarify his footnote on p.4080 of Carl Moppert's "Captain Cook and the Moon". In Cook's time (as now) the Civil day went from midnight to midnight, but in the Navy they used "Ship's Time" in which the day went from noon to noon, and the "Ship's Day" started twelve hours earlier than the Civil day. Astonomers also reckoned the day to be from noon to noon, but they counted the day as starting twelve hours after the Civil day. The difference between Ship's Time and Astonomical Time must be remembered when comparing the journals of Captain Cook and of Charles Green the astronomer who went on the voyage: for instance, according to Cook's journal, the ship left Plymouth at 2 p.m. on Friday, 26th August, 1768, and at 6 a.m. on the same day was 4 or 5 leagues from the Lizard. In Charles Green's journal, kept in Astonomical Time, all this was regarded as happening on Thursday, 25th. According to Civil Time (now used by astronomers as well as seamen) the ship left Plymouth in the afternoon of Thursday, 25th, and passed the Lizard early in the morning of Friday, 26th.

Unfortunately I have not been able to find whether the Nautical Almanac used Astronomical Time or Ship's Time.

BOUNDING TAILS OF PROBABILITY DISTRIBUTION

C. J. Smyth

General Bounds.

Let X be a real-valued random variable with distribution function $F(t)$. We seek upper and lower bounds for the tail $\int_B^\infty dF(t)$. Here B is assumed to be greater than the mean of X. We assume that $F(t)$ is such that $M(s) = \int_{-\infty}^\infty e^{st}dF(t)$ is finite for some $s > 0$.
Then

$$\int_B^\infty dF \leqslant \min_{s:\ M'(s)/M(s)<B} e^{-sB}\left(\frac{M''M - M^2}{M'' - 2BM' + B^2M}\right) \qquad \text{(U)}$$

$$\int_B^\infty dF \geqslant \max_{s:\ M'(s)/M(s)>B} \exp\left(-s\left(\frac{M'' - BM'}{M' - BM}\right)\right)\frac{(M' - BM)^2}{M'' - 2BM' + B^2M} \qquad \text{(L)}$$

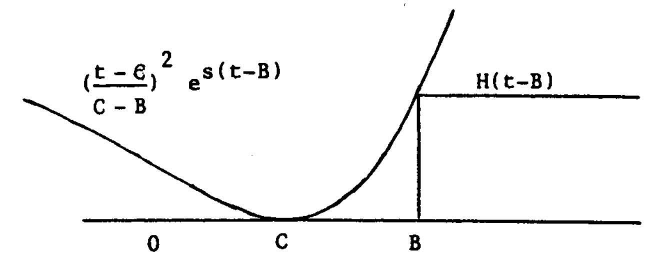
Remark: Since $M'(0)/M(0)$ = mean of $X < B$, and $M'(s)/M(s)$ is an increasing function of s, there will always be some s such that $M'(s)/M(s) \leqslant B$. If the tail $\int_B^\infty dF$ is zero then $M'(s)/M(s) \leqslant B$ for all s. If the tail is positive, however, then for s sufficiently large, $M'(s)$ will be $> B$ (provided $M(s)$ and $M'(s)$ are finite), so that (L) will give a positive lower bound for the tail.

Let $H(t)$ be the Heaviside function (1 for positive t and 0 otherwise), then we bound $H(t - B)$ above and below, and use the fact that

$$\int_B^\infty dF(t) = \int_{-\infty}^\infty H(t - B)\ dF(t).$$

1. The Upper Bound.

For this case we note that



$$H(t - B) \leqslant \left(\frac{t - C}{C - B}\right)^2 e^{s(t-B)} \quad \text{for any } C < B$$

so that

$$\int_B^\infty dF \leqslant \int_{-\infty}^\infty \left(\frac{t - C}{B - C}\right)^2 e^{s(t-B)}dt = e^{-sB}\,\frac{M'' - 2CM' + C^2M}{(C - B)^2}.$$

If $M'/M \geqslant B$, this function has a minimum of M at $C = -\infty$, i.e., we get Chernoff's upper bound $\int_B^\infty dF \leqslant e^{-sB}M$.

If $M'/M < B$, however, the function has a minimum of:

$$e^{-sB}\,\frac{MM'' - M^2}{M'' - 2BM' + B^2M} \quad \text{at } C = \frac{BM' - M''}{BM - M'}.$$

(Note that $C - B = -\dfrac{M'' - 2BM' + B^2M}{BM - M'} < 0$, and

$$\frac{MM'' - M'^2}{M'' - 2BM' + B^2M} = M - \frac{(BM - M')^2}{M'' - 2BM' + B^2M} < M,$$

so that the upper bound is an improvement upon Chernoff's.)

2. The Lower Bound.

Here we bound $H(t-B)$ below by

$$H(t - B) \geqslant \frac{(t - B)(t - C)}{(u - B)(u - C)}\,e^{s(t-u)} = f(t) \text{ say,}$$

where $C > B$ and $u = u(C)$ is the root between B and C of

$$s(u - B)(u - C) + 2u - (B + C) = 0$$

(u is where $f(t)$ has its maximum of 1). Then

$$\int_B^{\infty} dF \geqslant \int_{-\infty}^{\infty} f(t)dF = e^{-su}\,\frac{(M''-(B+C)M'+BCM)}{(u-B)(u-C)} = h(C) \text{ say,}$$

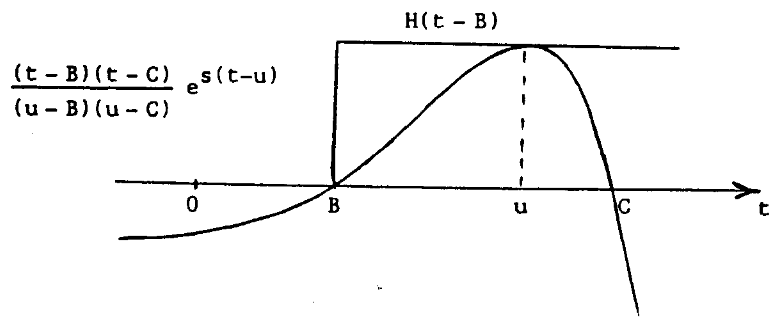
Now $(u-B)(u-C) < 0$, so we need

$$M'' - (B+C)M' + BCM = M'' - BM' - C(M'-BM)$$

to be < 0 for a non-trivial bound. This function is positive for C just greater than B, so in order for it to be negative for some $C > B$ we need $M' - BM > 0$, i.e., $M'/M > B$. Assuming this, it turns out that h(C) has its minimum value of

$$\frac{M'-BM}{M''-2BM''+B^2M}\exp\left(-s\,\frac{M''-BM'}{M'-BM}\right) \quad \text{when } u = \frac{M''-BM'}{M'-BM}\,.$$

The corresponding value of C is $C = u + \dfrac{u-B}{1+s(u-B)}$



3. <u>Results for X = Y + Z.</u>

We now restrict our attention to $X = Y + Z$, where Y and Z are independent, and Z is gaussian (mean 0, variance σ^2). We give these upper bounds, in increasing order of strength (and of complication).

$$\int_B^{\infty} dF \leqslant \min\left(1, \frac{1}{s\sigma\sqrt{2\pi}}\right) e^{-sB}\, M_Y(s)\, M_Z(s). \tag{U1}$$

$$\int_B^{\infty} dF \leqslant \frac{1}{\exp(-0.38s\sigma) + s\sigma\sqrt{2\pi}}\, e^{-sB} M_Y(s)\, M_Z(s) \tag{U2}$$

$$\int_B^{\infty} dF \leqslant k(s\sigma)\; e^{-sB} M_Y(s)\, M_Z(s) \tag{U3}$$

where $k(x) = \dfrac{1}{\sqrt{2\pi}} \exp(-x^2/2) \max\limits_{\alpha} e^{\alpha x} \displaystyle\int_{\alpha}^{\infty} \exp(-t^2/2)dt$ for $x > 0$.

The "Modified Chernoff Bound" (U1) is a consequence of (U2) but we give a straightforward independent derivation. Let Y have the distribution function F_Y. Then

$$\int_B^{\infty} dF = \int_{-\infty}^{\infty} dF_Y(u)\,\frac{1}{\sigma\sqrt{2\pi}} \int_{B-u}^{\infty} \exp-(t/\sigma)^2/2\, dt. \tag{3.1}$$

Now $(t - s\sigma^2)^2 > 0$ and so it follows that

$$\exp-(t/\sigma)^2/2 < \exp(-st + \sigma^2 s^2/2).$$

$$\int_{B-u}^{\infty} \exp-(t/\sigma)^2/2\, dt < \exp\frac{s^2\sigma^2}{2} \int_{B-u}^{\infty} e^{-st}dt$$

$$= \frac{1}{s}\exp(s^2\sigma^2/2 - sB + su).$$

On substitution in (3.1) we get

$$s\sigma\sqrt{2\pi}\int_B^{\infty} dF < \exp(s^2\sigma^2/2 - sB)\int_{-\infty}^{\infty} e^{su}\, dF_Y(u) = e^{-sB} M_Y(s)\, M_Z(s).$$

Combining this with Chernoff's bound gives (U1). Inequality (U2) is derived from (U3) using the inequality

$$k(x) \leqslant 1/(\exp(-0.38x) + x\sqrt{2\pi}) \tag{3.2}$$

which is proved by asymptotic analysis of k(x) as x tends to zero and to infinity, combined with numerical verification.

Equation (3.2) is not valid if 0.38 is replaced by 0.375.

The derivation of (U3) is similar to that of (U1). From (3.1) we have

$$\sigma\sqrt{2\pi}\int_B^\infty dF = e^{-sB}\int_{-\infty}^{\infty} e^{su}\, e^{s(B-u)}\int_{B-u}^{\infty}\exp-(t/\sigma)^2/2\,dt\,dF_Y(u)$$

$$\leqslant e^{-sB}\int_{-\infty}^{\infty} e^{su}\,dF_Y(u)\,\max_{\alpha} e^{s\alpha\sigma}\int_{\alpha\sigma}^{\infty}\exp-(t/\sigma)^2/2\,dt$$

$$= e^{-sB}M_Y(s)\,\max_{\alpha} e^{s\alpha\sigma}\sigma\int_{\alpha}^{\infty}\exp-t^2/2\,dt$$

$$= \sigma e^{-sB}M_Y(s)\,\sqrt{2\pi}\,\exp(s^2\sigma^2/2)\,k(s\sigma)$$

$$\int_B^\infty dF \leqslant k(s\sigma)\,e^{-sB}M_Y(s)\,M_Z(s)\quad .$$

We now show that $k(x) < \min(1, 1/(x\sqrt{2\pi}))$ so that (U3) is stronger than (U1). By calculus the α for which $e^{\alpha x}\int_{\alpha}^{\infty}\exp(-t^2/2)dt$ is minimal satisfies

$$x\int_{\alpha}^{\infty}\exp(-t^2/2)dt = \exp(-\alpha^2/2)$$

and so for this α it follows that

$$\sqrt{2\pi}\,k(x) = \exp(-x^2/2+\alpha x-\alpha^2/2)/x \leqslant 1/x.$$

But also $k(x) = \dfrac{\exp(x\alpha - x^2/2)}{\sqrt{2\pi}}\displaystyle\int_{\alpha}^{\infty}\exp(-t^2/2)dt.$

By inserting the factor $\exp x(t-\alpha)$, clearly $\geqslant 1$ in the integrand we obtain the inequality

$$\sqrt{2\pi}\,k(x) \leqslant \int_{\alpha}^{\infty}\exp(-t^2/2+xt-x^2/2)dt < \int_{-\infty}^{\infty} = \sqrt{2\pi}$$

so that $k(x) < 1$.

GOAT AND COMPASSES

(JCMN 33, p.4036 and 34, p.4060)

A. P. Guinand

The problem may be changed from two dimensions to three.

A space-goat tethered to a point on the surface of a spherical space-ship of unit radius can browse over a volume equal to half the volume of the ship. Is the length of the tether rational, algebraic or transcendental?

DIVERGENT MACLAURIN SERIES

(JCMN 32, p.4018 and 33, p.4040)

R. Vyborny

There is a theorem due to E. Borel:

Let $\{a_n ;\ n=0, 1, ..\}$ be an arbitrary sequence of real numbers.

There exists a function f of the real variable, differentiable any number of times, with n th derivative at the origin equal to a_n.

A proof may be found in "An Introduction to Classical Real Analysis" by Karl A. Stromberg.

APOLOGY

To George Berzsenyi whose name was spelt wongly in the previous issue (pages 4067 and 4084).

EXPANSION FROM RAMANUJAN (JCMN 35, p.4083)

Let y be defined as a function of the positive integer n by

$$e^n/2 = 1 + n + n^2/2 + \ldots + n^{n-1}/(n-1)! + yn^n/n!$$

How can y (for large n) be expanded in negative powers of n? Why was it clear to Ramanujan that no half-integer powers were needed?

First note three simple results:-

Lemma 1 $\int_0^\infty e^{-x} x^n \, dx = n!$

Lemma 2 $\int_n^\infty e^{-t}(1+t/n)^n \, dt < 2^{n+1} e^{-n}$

Lemma 3 $\int_0^\infty s^{2k+1} \exp(-ns^2/2) ds = k! \, 2^k n^{-k-1}$

Now to begin the calculation, express y in terms of an integral.

$$\int_0^\infty e^{-t}(1+t/n)^n \, dt = \sum_{r=0}^n \binom{n}{r} r! \, n^{-r}$$

$$= n^{-n} n! (1 + n + n^2/2 + \ldots + n^n/n!)$$

$$= n^{-n} n! (\tfrac{1}{2}e^n + (1-y)\, n^n/n!).$$ This has established

(1) $$2 - 2y = 2\int_0^\infty e^{-t}(1+t/n)^n \, dt - n!(e/n)^n$$

Secondly we may observe

(2) $$\int_{-n}^\infty e^{-t}(1+t/n)^n \, dt = \int_0^\infty e^{n-x}(x/n)^n \, dx = n!(e/n)^n$$

Adding (1) and (2) gives

$$2 - 2y = \left(\int_0^\infty - \int_{-n}^0\right) e^{-t}(1+t/n)^n \, dt.$$

We may discard the contribution from the interval (n,∞) because by Lemma 2 it is exponentially small for large n.

From here onwards we must therefore replace the equality sign by $\sim$ which denotes that the difference of the two sides when multiplied by any power of n will tend to zero as n tends to infinity.

$$2 - 2y \sim \left(\int_0^n - \int_{-n}^0\right) e^{-t}(1+t/n)^n \, dt$$

$$(2-2y)/n \sim \left(\int_0^1 - \int_{-1}^0\right)(g(s))^n \, ds$$

where $g(s) = e^{-s}(1+s) = 1 - s^2/2 + s^3/3 - s^4/8 + \ldots$

This function g(s) has a maximum at s = 0 and since $\exp(-s^2/2)$ looks like a good approximation to g(s) in the interval (-1, 1) we write

$$f(s) = g(s)\exp(s^2/2) = (1+s)\exp(-s+s^2/2)$$

$$= 1 + \frac{s^3}{3} - \frac{s^4}{4} + \frac{s^5}{5} - \frac{s^6}{9} + \frac{5s^7}{84} - \frac{13s^8}{480} + \ldots$$

For our expansion of y we have

(3) $$(2-2y)/n \sim \left(\int_0^1 - \int_{-1}^0\right) f(s)^n \exp(-ns^2/2) \, ds$$

Any term in s^r in the expansion of $f(s)^n$ will give a term in $\int_0^1 s^r \exp(-ns^2/2) \, ds$ which is proportional to $n^{-(r+1)/2}$. However, if r is even the contributions from $\int_0^1$ and from $\int_{-1}^0$ will cancel. This shows that half-integer powers of n will not occur in our expansion of y. To find the first few terms we need the first few odd powers of s in the expansion of

$$f(s)^n = 1 + ns^3\left(\frac{1}{3} - \frac{s}{4} + \frac{s^2}{5} - \ldots\right) + \frac{n(n-1)}{2} s^6\left(\frac{1}{3} - \frac{s}{4} + \ldots\right)^2 + \ldots$$

Noting the result:

$$\text{Lemma 4} \quad \int_1^{\infty} s^r \exp(-ns^2/2)ds < \exp(-n/2)/(n-r),$$

we are able to replace formula (3) for the expansion of y by

$$(4) \qquad (1-y)/n \sim$$

$$\int_0^{\infty} (\text{Sum of terms of odd degree in } f(s)^n) \exp(-ns^2/2)ds.$$

We can pick out the coefficients of powers of s in $f(s)^n$ as follows: (The multiplier for formula (4) is given by Lemma 3)

(of s^3) $\quad n/3 \quad$ (Multiply by $2n^{-2}$)

(of s^5) $\quad n/5 \quad$ (Multiply by $8n^{-3}$)

(of s^7) $\quad \dfrac{5n}{84} - \dfrac{n(n-1)}{2}\dfrac{1}{6} \quad$ (Multiply by $48n^{-4}$)

(of s^9) (lower powers of n) +

$$\frac{n(n-1)(n-2)}{6}\,\frac{1}{27} \quad (\text{Multiply by } 384n^{-5})$$

These terms lead to

$$\frac{1-y}{n} \sim \frac{2}{3n} + \frac{8}{5n^2} + \left(\frac{5n}{84} - \frac{n(n-1)}{12}\right)\frac{48}{n^4} + \frac{n(n-1)(n-2)}{162}\,\frac{384}{n^5} + \cdots$$

$$\sim \frac{2}{3n} + \frac{1}{n^2}\left(\frac{8}{5} - 4 + \frac{64}{27}\right) + \text{higher powers of } 1/n.$$

$$1-y \sim \frac{2}{3} - \frac{4}{135n} + \text{higher powers of } 1/n.$$

This gives the first two terms of Ramanujan's result

$$y \sim \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{16}{8505n^3} + \cdots.$$

"CURIOUSER AND CURIOUSER!"

C. J. Smyth

Have you ever seen this curiosity before? I was shown it by someone in the Computing Department.

Start with	1	2	3	4	5	6	7	8	9	
Cross out every 2nd number	1		3		5		7		9	
Form partial sums	1		4		9		16		25	squares (well-known)

Start with	1	2	3	4	5	6	7	8	9	10	11	
Cross out every 3rd number	1	2		4	5		7	8		10	11	
Form partial sums	1	3		7	12		19	27		37	48	
Cross out every 2nd number	1			7			19			37		
Form partial sums	1			8			27			64		cubes!

Start with	1	2	3	4	5	6	7	8	9	10	11	12	13
Cross out every 4th number	1	2	3		5	6	7		9	10	11		13
Form partial sums	1	3	6		11	17	24		33	43	54		67
Cross out every 3rd number	1	3			11	17			33	43			67
Form partial sums	1	4			15	32			65	108			175
Cross out every 2nd number	1				15				65				175
Form partial sums	1				16				81				256

fourth powers!

EDITORIAL

Contributions will be welcomed. They should be written so as to be clear to all mathematicians.

Since Issue 32 (October 1983) the JCMN has been published by me (the Editor). Issues 1 to 31 were published by

Mathematics Department,
James Cook University of North Queensland,
Post Office James Cook,
Townsville, N.Q., 4811,
Australia.

These issues have been reprinted as paperback volumes

Volume 1 (Issues 1 - 17)
Volume 2 (Issues 18 - 24)
Volume 3 (Issues 25 - 31)

and they are on sale for $10 (Australian) per volume (including postage by surface mail); cheques for these volumes should be made payable to the James Cook University.

My address is either at the University (address above) or at home (see page 4094).

Basil Rennie