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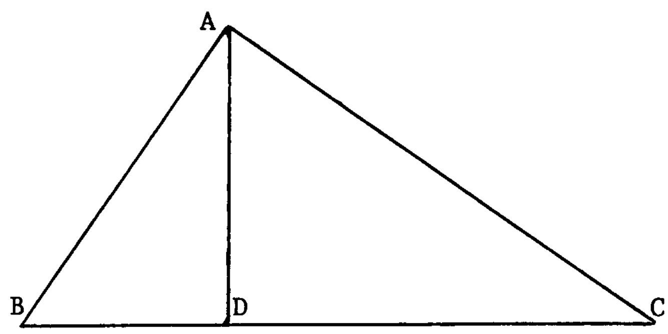
PYTHAGORAS PROVED AGAIN

C.J. Smyth



Let the triangle ABC be right-angled at A. Drop the perpendicular AD from A to the hypotenuse BC. The area of any triangle similar to ABC is proportional to the square of the hypotenuse. In the picture above there are three similar triangles and

$$\text{Area } ABD + \text{Area } ADC = \text{Area } ABC.$$

Therefore $$AB^2 + AC^2 = BC^2 .$$

EXERCISE IN ANALYSIS (JCMN 31, p. 3165)

The same solution (more or less) has come in from A. Brown, J.B. Parker and C.J. Eliezer and D.K. Ross.

If the sequence $\{S_n\}$ is defined by

$$S_{n+1} = \sqrt{S_n + 2} \quad \text{with } S_1 = b \geq -2$$

then $S_n = 2\cos(2^{1-n}\cos^{-1}(b/2)) \qquad \text{if } -2 \leq b \leq 2$

and $\quad = 2\cosh(2^{1-n}\cosh^{-1}(b/2)) \qquad \text{if } 2 \leq b$

From A. Brown is also a comment that an article in the Mathematical Gazette (vol. 66, pp. 296-299) used the same idea.

BINOMIAL IDENTITY NUMBER 16 (JCMN 31, p. 3177)

J.B. Parker

The original identity was the case $p = q$ of the more general

$$\sum_{k=q}^{n} \binom{k}{p}\binom{k}{q} = \sum_{j=0}^{p} \binom{p+q-j}{j}\binom{p+q-2j}{p-j}\binom{n+1}{p+q-j+1}$$

which holds for $0 < p \le q < n$.

Lemma 1 $\quad \binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$

Lemma 2 $\quad \sum_{j=0}^{p} \binom{p}{j}\binom{n-p}{q-j} = \binom{n}{q}$

Note: The product $\binom{p+q-j}{j}\binom{p+q-2j}{p-j}$ on the right hand side of the formula above may, using Lemma 1, be written as $\binom{p}{j}\binom{p+q-j}{p}$.

Main proof. The result is trivial when $n = q$. For $n > q$ denote the right hand side by $F(n)$.

$$F(n) - F(n-1) = \sum_{j=0}^{p} \binom{p}{j}\binom{p+q-j}{p}\left(\binom{n+1}{p+q-j+1} - \binom{n}{p+q-j+1}\right)$$

$$= \sum_{j=0}^{p} \binom{p}{j}\binom{p+q-j}{p}\binom{n}{p+q-j}$$

$$= \sum_{j=0}^{p} \binom{p}{j}\binom{n}{p}\binom{n-p}{q-j} = \binom{n}{p}\binom{n}{q}$$

The result follows by induction on n.

SALES TAX

G.F. Duff

The following problem comes from a true life experience. The price in dollars of a car was a perfect square, and the 7% Ontario sales tax was a perfect cube. What was the total to be paid? Answer: $5243.

CAMELOT AND BINOMIAL IDENTITY 16

Marta Sved

King Arthur returned from a journey with great plans for the merry month of May.

- We have 31 days with no particular obligations, so we could have our fencing and javelin throwing competitions during this month. Each of these competitions will take 2 days, not necessarily consecutive, and since I know that none of you is involved in both pursuits, fencing and javelin throwing could be done on different days or on the same day. -

He turned to Sir Gawain.

- In how many ways could you draw up a timetable for the events? -

- Do not forget that after the competitions we must have a feast before the month ends - said Queen Guinevere.

- Let us see - said Sir Gawain. - If we have C-day (celebration) on the 31st, then there are $\binom{30}{2}$ possibilities for the two F-days (fencing) and independently $\binom{30}{2}$ possibilities for the two J-days (javelin), hence $\binom{30}{2}^2$ possible choices. If C-day is on the 30th, there are only $\binom{29}{2}^2$ possibilities, if C-day on the 29th ... -

- You are not going on like this 30 times? - interrupted Sir Mordred impatiently.

- Sir Gawain would not go on 30 times - said Sir Lancelot - C-day cannot be held before the 3rd of the month, the sum would be

$$\sum_{k=2}^{30} \binom{k}{2}^2 \quad -$$

Merlin smiled: - Would it not be simpler to consider three cases:

(a) The events F,J,C take up altogether 5 days

(b) The events take up 4 days

(c) The events take up 3 days.

In case (a) we have $\binom{31}{5}$ choices for the days of the events with $\binom{4}{2} = 6$ choices for the F days in the available 4 days.

In case (b) there are $\binom{31}{4}$ choices for the days, 3 choices for the day in which 2 events F and J are to be held simultaneously, and 2 ways of allocating F and J for the remaining 2 days.

Case (c) is the easiest. We only have to choose 3 days out of the 31 and with that everything is settled, the 3 days being: FJ, FJ and C. Hence, instead of the long sum of Sir Gawain we have

$$6\binom{31}{5} + 6\binom{31}{4} + \binom{31}{3} \quad \text{possibilities.}$$

- Merlin, I am impressed - said King Arthur - I suppose you could solve the problem if we had one event taking p days and the other $q \geq p$, with $n+1$ days for the two events and the celebration.

- Of course - said Merlin - Instead of labouring at the long sum $\sum_{k=q}^{n} \binom{k}{p}\binom{k}{q}$ we could say the number j of days with two events could be anything from 0 to p. For each j we can count the possibilities. There are $\binom{n+1}{p+q-j+1}$ choices for the $p+q-j+1$ days of activity, including the celebration day, and once this choice is made there are $\binom{p+q-j}{j}$ choices of the double-contest days, and then $\binom{p+q-2j}{p-j}$ ways of allocating the $p+q-2j$ single-contest days between the two kinds of sport. -

Thus it came to pass that Merlin's deliberations led to the identity $\sum_{k=q}^{n} \binom{k}{p}\binom{k}{q} = \sum_{j=0}^{p} \binom{p+q-j}{j}\binom{p+q-2j}{p-j}\binom{n+1}{p+q-j+1}$ which may also be written $\sum_{j=0}^{p} \binom{p}{j}\binom{p+q-j}{p}\binom{n+1}{p+q-j+1}$.

POINT IN A TRIANGLE

J.B. Parker

For a triangle in the plane, find a nice geometrical characterization for the point minimizing the sum of squares of the perpendicular distances on to the three sides.

BESSEL FUNCTIONS AND ELLIPTIC INTEGRALS

$$\int_0^\infty J_0(x)\, J_0(Rx)\, dx = \frac{1}{\pi}\int_0^\pi (R^2 - \sin^2\theta)^{-\frac{1}{2}} d\theta \quad \text{for all } R > 1?$$

OLD INDIAN AND ARAB CALCULATIONS

Hayo Ahlburg

It was with surprise that I read the note "Division in Old Germany" in JCMN 29, pages 3102-3103. How can it be that in your hemisphere you have forgotten the original method of division? It has nothing to do with Germany in particular. While the ancient Egyptians and Greeks had very cumbersome methods for multiplication and division, the first convenient and modern method became possible, if you cast your mind back, when the Indians introduced zero and the system where a digit's place in a number is significant; e.g. in the decimal system the position of a digit indicates the power of ten by which it is to be multiplied.

The Indian system of division dominated calculation techniques till the 18th century, down to the details of its arrangement. In fact it was still exclusively taught in J.B. Lechner's "Rechenkunst" (Liegnitz-Leipzig) as recently as 1800.

The Arabs (e.g. Muḥammad ibn Mūsā Alḫwārazmī* at the beginning of the 9th century) had to change the Indian system only insofar as the Indians used a sand-board where they erased digits no longer needed and replaced them by new ones during the calculation, while the Arabs had to cross out used digits and write those produced by each new step above the old ones.

The divisor was written under the dividend's leftmost digits, which were divided first. At every following step, the divisor was

*Also known as al-Khwarizmi.

repeated underneath, but moved over one place to the right. The partial products began with the leftmost digits and each was subtracted immediately from the corresponding places in the dividend; only the remainder was then written overhead. Both the digits of the dividend and the digits used of the divisor were crossed out. The first few steps of your first example (dividing 1152450 by 325) would look like this:

(1)

$$\begin{array}{cccccccc}
 & 2 & & & & & & \\
\cancel{1} & \cancel{1} & 5 & 2 & 4 & 5 & 0 & \quad (3 \\
 & \cancel{3} & 2 & 5 & & & &
\end{array}$$

The 3 on the right (first digit of the answer) has been multiplied by the three in the bottom line, which was then crossed out, and the product 9 subtracted from the 11 which was crossed out, the difference 2 being written above.

(2)

$$\begin{array}{cccccccc}
 & 1 & & & & & & \\
 & \cancel{2} & 9 & & & & & \\
\cancel{1} & \cancel{1} & \cancel{5} & 2 & 4 & 5 & 0 & \quad (3 \\
 & \cancel{3} & \cancel{2} & 5 & & & &
\end{array}$$

(3)

$$\begin{array}{cccccccc}
 & 1 & 7 & & & & & \\
 & \cancel{2} & \cancel{9} & 7 & & & & \\
\cancel{1} & \cancel{1} & \cancel{5} & \cancel{2} & 4 & 5 & 0 & \quad (3 \\
 & \cancel{3} & \cancel{2} & \cancel{5} & & & &
\end{array}$$

(4)

$$\begin{array}{cccccccc}
 & & 2 & & & & & \\
 & \cancel{1} & \cancel{7} & & & & & \\
 & \cancel{2} & \cancel{9} & 7 & & & & \\
\cancel{1} & \cancel{1} & \cancel{5} & \cancel{2} & 4 & 5 & 0 & \quad (35 \\
 & \cancel{3} & \cancel{2} & \cancel{5} & 5 & & & \\
 & & \cancel{3} & 2 & & & &
\end{array}$$

Omitting the subsequent stages of the calculation, what appears at the end is

(12)
$$\begin{array}{ccccccc}
 & & \cancel{1} & \cancel{1} & \cancel{1} & & \\
 & & \cancel{2} & \cancel{2} & \cancel{9} & & \\
 & \cancel{1} & \cancel{7} & \cancel{4} & \cancel{1} & & \\
 & \cancel{2} & \cancel{9} & \cancel{7} & \cancel{9} & \cancel{3} & \\
\cancel{1} & \cancel{1} & \cancel{5} & \cancel{2} & \cancel{4} & \cancel{5} & \cancel{0} \\
 & \cancel{3} & \cancel{2} & \cancel{5} & \cancel{5} & \cancel{5} & \cancel{5} \\
 & & \cancel{3} & \cancel{2} & \cancel{2} & \cancel{2} & \\
 & & & \cancel{3} & \cancel{3} & &
\end{array} \qquad (3546$$

A fabulously simple method, at least in India, where after erasing and rewriting in the sand all that remained was the answer 3546. A remainder, if any, would also stay on the board, but nothing else, not even the problem. Bad news for those who are prone to make mistakes and like to check the various steps in the calculation. For them, late in the 15th century, our "modern" system was developed.

We are grieved to note that your example as given is incomplete, the top $\begin{array}{ccc}1&1&1\\&&9\end{array}$ is missing. (Readers will recall Professor Diananda's suggestion in JCMN 30, page 3156, that the calculation should have been finished off with a row of zeros at the top). However, the anonymous author of the manuscript long ago certainly got the right answer.

What kind of division was taught to little James Cook in the schoolroom (still to be seen) in Great Ayton? His navigational calculations preserved in the National Maritime Museum at Greenwich use logarithms.

BINOMIAL IDENTITY NUMBER SEVEN (JCMN 21, p. 78)

C.S. Davis

Under the above heading, A.P. Guinand asks for a simple proof of the identity

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r}^3 = (-1)^n \frac{(3n)!}{(n!)^3}.$$

A combinatorial proof is suggested in Comtet, Advanced Combinatorics (Ex. 42, p. 174). I give here an analytical proof which is straightforward though it can hardly claim to be very simple.

Let $v = (1+t)/(1-t)$ so that $t = (v-1)/(v+1)$ and $(v+1)(1-t) = 2$. Then using the formula of Rodrigues for Legendre polynomials and Leibnitz' theorem on successive differentiation:

$$P_{2n}(v) = \frac{1}{2^{2n}(2n)!} \sum_{r=0}^{2n} \binom{2n}{r} \left(\frac{d}{dv}\right)^r (v+1)^{2n} \left(\frac{d}{dv}\right)^{2n-r} (v-1)^{2n}$$

$$= 2^{-2n} \sum \binom{2n}{r}^2 (v+1)^{2n-r} (v-1)^r \quad ,$$

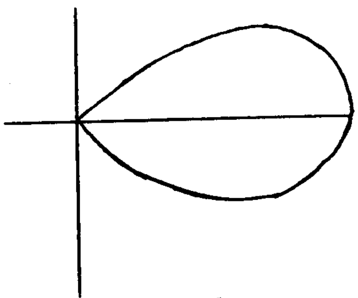
giving

$$(1-t)^{2n} P_{2n}(v) = \sum \binom{2n}{r}^2 t^r .$$

Multiply by $(1-t)^{2n} = \sum_r (-1)^r \binom{2n}{r} t^{2n-r}$ so that the given sum $S = \sum (-1)^r \binom{2n}{r}^3$ is the coefficient of t^{2n} in $(1-t)^{4n} P_{2n}(v)$, and hence

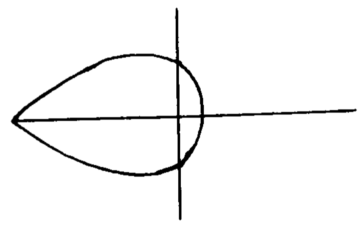
$$S = \frac{1}{2\pi i} \int_C \frac{(1-t)^{4n}}{t^{2n+1}} P_{2n}\left(\frac{1+t}{1-t}\right) dt ,$$

where C is any simple closed contour with the origin in its interior. Now transform this integral by putting $4u = 1 - v^2$ and observe that u, v and t are single-valued functions of one another in the regions shown.



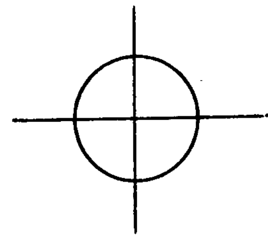
v-plane

$|1 - v^2| < 1$ and real part positive.



t-plane

$4|t| < |1 - t|^2$



u-plane

$|u| < \frac{1}{4}$

We find that $u = -t(1-t)^{-2}$ and $\frac{dt}{t} = \frac{du}{uv}$, and hence

$$S = \frac{1}{2\pi i} \int_{\Gamma} \frac{P_{2n}(v)}{v} \frac{du}{u^{2n+1}} ,$$

where Γ is the circle where $|u| = \rho < \frac{1}{4}$.

Now $P_{2n}(v)/v$ is a regular function of v unless $v = 0$, and so it is a regular function of u for $|u| < \frac{1}{4}$. Thus we may write $g(u) = P_{2n}(v)/v = \sum_0^{\infty} c_k u^k$ for $|u| < \frac{1}{4}$, and then we have $S = c_{2n}$. It remains to determine the coefficients c_k. We have the differential equation for Legendre polynomials:

$$(1 - v^2)\, P''_{2n} - 2vP'_{2n} + 2n(2n+1)P_{2n} = 0.$$

Then writing $\dot{g}$ and $\ddot{g}$ for derivatives with respect to u, the differential equation transforms into

$$u(1-4u)\, \ddot{g} + (1-10u)\, \dot{g} + 2(n+1)(2n-1)g = 0 .$$

Inserting the above power series for g, we find

$$k^2 c_k + 2(n+k)(2n-2k+1)c_{k-1} = 0.$$

Since $c_o = g(0) = P_{2n}(1) = 1$ it follows that

$$c_k = 2^k (n+1)(n+2)\ldots(n+k)(k!)^{-2}(1-2n)(3-2n)\ldots(2k-2n-1).$$

Noting that $(1-2n)(3-2n)\ldots(2n-1) = (-1)^n\, 2^{-2n}\, ((2n)!/(n!))^2$,

$$c_{2n} = 2^{2n} \frac{(3n)!}{n!\ (2n)!^2} (-1)^n\, 2^{-2n} \frac{(2n)!^2}{(n!)^2}$$

$$= (-1)^n (3n)!\ (n!)^{-3}.$$

Since there are simple closed formulae for the sums

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r}^k$$

for $k = -1, 0, 1, 2$ and 3, it is natural to consider the case $k = 4$ (though there may be no such closed formula). The above ideas show that this sum is the coefficient of u^{2n} in the expansion of

$$v^{2n-1}\, P_{2n}(v)\, P_{2n}(1/v),$$

but it is not easy to see how this coefficient may be otherwise determined.

FORWARD DIFFERENCES

Dieter K. Ross

Let $(\alpha)_\beta = \Gamma(\alpha+1)/\Gamma(\alpha-\beta+1)$ where $\Gamma(x)$ is Euler's gamma function. If $\Delta^n u_\beta$ is the usual forward difference operator defined iteratively by $\Delta u_\beta = u_{\beta+1} - u_\beta$ and $\Delta^n u_\beta = \Delta(\Delta^{n-1} u_\beta)$ find $\Delta^n(\alpha)_\beta$. This problem arose in the study of consecutive moments of k-times completely monotone functions.

TRANSPOSES SIMILAR (JCMN 29, p. 3111)

This problem, of showing every square matrix to be similar to its transpose, was sent by the late H. Kestelman without a solution, but later he sent the comment that it depended ultimately on proving the result for the super-diagonal $n \times n$ matrix

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & & \\ 0 & 0 & 1 & \cdots & & \\ \cdots & & & & & \\ 0 & \cdots & & & 0 & 1 \\ 0 & \cdots & & & 0 & 0 \end{pmatrix}$$

In fact for the problem of finding X so that $J^T = XJX^{-1}$ we can without much difficulty find the general solution, which is

$$X = \begin{pmatrix} 0 & 0 & \cdots & & a \\ 0 & 0 & \cdots & a & b \\ 0 & 0 & \cdots & b & c \\ & & \cdots & & \\ a & b & c & \cdots & \end{pmatrix}$$

where $a \neq 0$. More concisely, the i, j component of X is a function of $i+j$ only, and is zero for $i+j \leq n$ and non-zero for $i+j = n+1$.

EIGENVALUE CALCULATION (JCMN 31, p. 3177)

A. Brown

If a matrix has elements all $= a$ on the diagonal and all $= b$ elsewhere then what are the eigenvalues? There is a simple eigenvalue $= a-b+nb$ and a value $a-b$ with multiplicity $n-1$. To prove this it is sufficient to write the matrix as $(a-b)1+bK$ where K has every element $= 1$. Since K has rank $n-1$ it has eigenvalue zero with multiplicity $n-1$, and its other eigenvalue is therefore the sum of the diagonal elements.

GENERALIZED HADAMARD INEQUALITY (JCMN 31, p. 3165)

This was Kestelman's last problem for the JCMN, asking for a proof of Theorem G below.

Theorem G. Let A be the $n \times n$ positive definite Hermitean matrix

$A = \begin{pmatrix} P & N \\ N^* & Q \end{pmatrix}$ where P and Q are square and N^* is the Hermitean conjugate of N. Then $\det A \le \det P \det Q$ with equality if and only if $N = 0$.

Proof. The sub-matrices P and Q are both positive definite Hermitean, and so we may set $P = S^*S$ and $Q = T^*T$ where S and T are invertible.

$$\text{Put } M = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} I & Z \\ Z^* & I \end{pmatrix}$$

where $Z = (S^*)^{-1} N T^{-1}$. Then $A = M^*WM$ and $\det (M^*M) = \det P \det Q$ so that we have to prove $\det W \le 1$ with equality if and only if $Z = 0$.

Note that W, like A, is positive definite with all eigenvalues real and positive. The sum of the eigenvalues of W is the sum of the diagonal elements, which is n. Therefore the arithmetic mean is 1 and the geometric mean ≤ 1, with equality if and only if all eigenvalues are 1. Therefore $\det W$ = product of eigenvalues ≤ 1 with equality if and only if all eigenvalues are 1, in which case W is the unit matrix and $Z = 0$ and therefore $N = S^*ZT = 0$.

This result leads to a proof for the simple version of Hadamard's inequality, as follows:

Theorem H. If B is any complex $n \times n$ matrix and k_r (for $r = 1, 2, \ldots n$) is the length of the vector represented by column r, then $|\det B| \le \Pi k_r$.

Proof. Let $A = B^*B$, then a typical diagonal element of A is $a_{rr} = \sum |b_{sr}|^2 = k_r^{\,2}$. If B is singular then the result is trivial, therefore suppose B to be non-singular, then A is strictly positive definite and Theorem G applies, with P of size $n-1 \times n-1$ and Q the 1×1 matrix $= a_{nn} = k_n^{\,2}$. By induction it follows that

$$|\det B|^2 = \det A \le k_n^{\,2} \det P \le \ldots \le \Pi k_r^{\,2}.$$

Theorem G also shows which are the cases of equality in H, namely when one column is zero and when all columns are orthogonal.

COMPLEX INEQUALITY

Suppose that we are given three complex numbers, all of unit modulus, denote them by u^2, v^2, and w^2. Then each of u, v and w is defined only up to an ambiguity of sign, there are eight possible choices for (u, v, w). Show that there are two of these choices such that:-

$$|2u^2 + 2v^2 + 2w^2 + 3uv + 3vw + 3wu| \le |u^2 + v^2 + w^2|.$$

Which are the cases of equality?

VARIATION ON EULER-MACLAURIN

The Euler-Maclaurin formula can be written

$$\int_{-h}^{h} f(x)\,dx = h(f(-h)+f(h)) + \sum_{m=1}^{\infty} \frac{(-4h^2)^m}{(2m)!} B_m (f^{(2m-1)}(h) - f^{(2m-1)}(-h))$$

$$= h(f(-h)+f(h)) - (h^2/3)(f'(h) - f'(-h)) + (h^4/45)(f'''(h) - f'''(-h)) + \ldots$$

where the B_m are the Bernoulli numbers defined by

$$1 - (z/2)\cot(z/2) = \sum_{1}^{\infty} B_m z^{2m}/(2m)!$$

For polynomial f the series is finite and the formula is exact, otherwise it may be modified to a finite series with a remainder term (see Whittaker and Watson, page 128) or it may be regarded as an asymptotic expansion for small h. From this last point of view we can regard the Euler-Maclaurin formula as an asymptotic error estimate for the trapezoidal formula of numerical integration. As the mid-point formula is both more accurate and more convenient than the trapezoidal formula, it is interesting to look at the corresponding asymptotic error estimate. It is easy to find the first few terms:

$$\int_{-h}^{h} f(x)\,dx = 2h\, f(0) + (h^2/6)(f'(h) - f'(-h)) - (7h^4/360)(f'''(h) - f'''(-h)) + (31h^6/15120)(f^{(5)}(h) - f^{(5)}(-h)) - \ldots$$

What is the general term?

DIVERGENT MACLAURIN SERIES

Can a function f(x) of the real variable on a non-empty open interval $(-\varepsilon, \varepsilon)$ have a formal expansion $\sum x^n f^{(n)}(0)/n!$ with zero radius of convergence? In other words can f have derivatives of all orders but $|f^{(n)}(0)/n!|^{1/n}$ be unbounded as n tends to infinity?

EVENSONG

Marta Sved

He is a student, Steven Todd
Whose work is even, brains are odd,
Thoughts from them are far remote,
Learning everything by rote.

Even times odd - quotes Steven Todd
Is even, while their sum is odd -
Next he's telling (and believin')
"The product of two odds is even".

$(-)^n$ by Steven Todd
Is + (even if n is odd)
Yet he's learned without compunction
What's an odd or even function.

It is a mystery to Steven
Why sine is odd and cos is even
And it has not made him wiser
That e^x is not either.

It was explained to Steven Todd
When's a permutation odd
But it's bewildering for Steven
That an odd cycle is even.

With all (?) possible combinations
Crammed in for examinations
I am anxious for our Steven,
Are his chances odd or even?

BINOMIAL IDENTITY 15 (JCMN 30, p. 3141)

Marta Sved

After King Arthur was carried away to Avalon the k knights remaining at Camelot dined daily at the Round Table, awaiting his return. They decided to use each day a different set of k places round the table (which would seat n). It would take $\binom{n}{k}$ days to try out all the possibilities and each day there would be n-k vacant seats. Some of these empty seats, they decided, could be used for guests. Any knight wanting to invite a guest would arrange to sit where he had a vacant place on his right.

For each $i = 1, 2, \dots n$ let $S(i)$ be the set of days in the cycle (of $\binom{n}{k}$ days) when there was a knight in seat i with a vacant place (or a guest) on his right, and in general let $S(i_1, i_2, \dots i_r)$ be the intersection of the sets $S(i_1), \dots S(i_r)$. The number of days when seat number $i+1$ (that is the seat to the right of seat i, numbering anticlockwise) was available for a guest, was $|S(i)|$, the number of elements in the set $S(i)$. In general $|S(i_1, \dots i_r)|$ is the number of days when seats $i_1+1, i_2+1, \dots i_r+1$ are all available for guests. The inclusion-exclusion principle tells us that the number of days when no guest could be invited, which is the number of days not in any of the sets $S(i)$ is

$$\binom{n}{k} - \sum_i |S(i)| + \sum_{i<j} |S(i, j)| - \sum_{i<j<k} |S(i, j, k)| + \dots$$

This total is of course zero.

The number $|S(i_1, \ldots i_r)|$ is the number of ways of choosing k seats round the table so that $i_1, i_2, \ldots i_r$ are all chosen but $i_1+1, i_2+1, \ldots i_r+1$ are not chosen. In the episode of the fat knights (JCMN 31, page 3170) Merlin explained that there are $\frac{n}{n-r}\binom{n-r}{r}$ ways of choosing r seats with no two adjacent. These are the seats $i_1, \ldots i_r$ for knights who can each invite a guest; the other k–r knights can sit in the remaining $n-2r$ seats in $\binom{n-2r}{k-r}$ ways. It follows that (for each r)

$$\sum |S(i_1, \ldots i_r)| = \frac{n}{n-r}\binom{n-r}{r}\binom{n-2r}{k-r}$$

and application of the inclusion-exclusion principle above leads to

$$\sum (-1)^r \frac{n}{n-r}\binom{n-r}{r}\binom{n-2r}{k-r} = 0$$

where the sum may be over all integer r for which $0 \le 2r \le n$, or perhaps more simply over all integer r. This is Binomial Identity 15 (JCMN 30, p. 3141 and 31, p. 3175) rearranged and generalized. In fact the original identity was the case of even n. To see this, put 2n for n in the equation above, then put $n-k$ for k.

$$\sum_r (-1)^r \frac{2n}{2n-r}\binom{2n-r}{r}\binom{2n-2r}{n-k-r} = 0$$

Now change the variable of summation from r to $j = n-r$, and omit the factor $(-1)^{n+k}$ to arrive at the original identity:

$$\sum_j \frac{2n}{n+j}\binom{n+j}{n-j}\binom{2j}{j-k}(-1)^{j-k} = 0\ .$$

CONTINUED FRACTIONS AND RANDOM VARIABLES

Find a random variable on $(1, \infty)$ such that the reciprocal of the non-integer part has the same distribution.

The motivation for this question is that if you take a random irrational number greater than 1 and express it as a continued fraction $a_1 + \frac{1}{a_2 +}\ \frac{1}{a_3 +}\ \cdots$ then the operations performed on the number are alternately taking the non-integer part and taking the reciprocal. It might be supposed that if the initial random variable had any reasonably smooth probability distribution then, after many iterations of the two operations, the distribution would converge to some stable limit.

The problem may be put as a functional equation in classical analysis, find $f(x)$ positive on the interval $(1, \infty)$ such that $x^{-2} f(1/x) = \sum_1^\infty f(x+n)$ for $0 < x < 1$. This f suitably normalized would be the probability density.

When you have solved the functional equation, try to find a result about the distribution of the values of quotients $a_1, a_2, \ldots$ in the continued fractions for "most" irrational numbers. Do the quotients 1 and 2 occur with mean densities $41\frac{1}{2}\%$ and 17% respectively?

SOLAR HOT WATER

A story is told about a Government organization wanting hot water for a large building; on investigating prices they decided that it would be cheaper not to have a single large solar water heater on the roof, but to have several of the easily available, mass-produced, domestic sized units. When all the plumbing was installed and being used it appeared that the supply of hot water was not as good as it should be. Experts were called in and they pronounced that some of the solar heating units seemed to be "lazy" and that further investigations were needed. A good exercise for an armchair detective is to say why the system did not work well.

EDITORIAL

Reprints of earlier issues as paperback volumes are now available from the Mathematics Department, James Cook University of North Queensland, Post Office James Cook, Townsville, N.Q. 4812, Australia.

Volume 1	(Issues 1-17)
Volume 2	(Issues 18-24)
Volume 3	(Issues 25-31)

Price is $10 (Australian) for each volume (including sea mail postage). Cheques should be payable to James Cook University.

From this issue onwards the journal will be published by me (the editor). Of course we still want contributions, but, now that publication is no longer paid for out of University funds, we also want as many subscribers as possible. My address is either at the University (see above) or at home (see page 4002).

Basil Rennie.