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JAMES COOK MATHEMATICAL NOTES

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CONTENTS OF JCMN 29

SOME PROBLEMS

P. Erdös (See also p. *3122*)

1. Denote by A_n the least common multiple of all the positive integers not exceeding n. Prove that for every integer k (positive, negative or zero) the sum of the series $\sum_{n=1}^{\infty} n^k/A_n$ is irrational. No doubt if f(x) is a polynomial with integer coefficients then $\sum_{n=1}^{\infty} f(n)/A_n$ is also irrational.

2. Let $a_n \geq 0$ for $n = 1, 2, \ldots$ be a sequence of integers. Let A_n be as in Problem 1 above, the least common multiple of all positive integers not exceeding n. Prove that if a_n tends to infinity but not too fast then $\sum_{n=1}^{\infty} a_n/A_n$ is irrational. Obtain as sharp a result as you can.

3. Take any natural number n. Find the maximum of $\sum_{i=1}^{k} a_i$ for all finite sets of integers $\{a_1, \ldots a_k\}$ such that each $a_i \geq 2$ and $\dfrac{n!}{a_1!\, a_2! \ldots a_k!}$ is an integer.

4. The same problem as 3, but with $a_i \geq t$ instead of $a_i \geq 2$.

5. Prove that only finitely many binomial coefficients $\binom{n}{k}$ $(n \geq 2k > 2)$ are the product of consecutive primes >k, e.g. $\binom{7}{3} = 35 = 5\times7$, and $\binom{14}{4} = 1001 = 7\times11\times13$. Probably there is no solution for n > 14. This should be easy to prove. I think $\binom{n}{k}$ is the product of consecutive primes only finitely often. This I cannot prove, in fact I cannot even prove that $\binom{n}{2} = 2\times3\times5\times\ldots\times p_k$ has only a finite number of solutions.

6. Does the equation

$\binom{n}{k} = p_{i(1)}\, p_{i(2)} \cdots p_{i(r)}$

where $r < k < p_{i(1)} < p_{i(2)} < \cdots p_{i(r)}$ and $n \geq 2k \geq 4$,

have infinitely many solutions? I do not know.

7. Let there be given n points in the plane, not all on a line. Join every two of them. Prove that you get at least n distinct lines.

8. Prove that there is an absolute constant c so that (for any n) if $2n$ points are in the plane, with no $n+1$ of them on a line, then these $2n$ points determine at least cn^2 distinct lines. I could not prove this, and offer a hundred dollars for a proof or disproof.

9. Let $X_1, X_2, \ldots, X_n$ be n points in the plane. Join all possible pairs. This gives the lines $L_1, \ldots L_m$. First observe that if $m > 1$ then $m \geq n$. Denote by y_i the number of the points on the line L_i. Let the lines be labelled so that $y_1 \geq y_2 \geq \cdots \geq y_m$. Estimate as well as you can the number of possible choices of the set $\{y_1, \ldots y_m\}$. Denote this number by $f(n)$. I would like to prove $f(n) < \exp(c\sqrt{n})$ for a certain constant c. I offer a hundred dollars for a proof or disproof.

Determine as accurately as possible the set of possible values of $\sum_{i=1}^{m} y_i$. It is easy to see that $n \leq \sum y_i \leq n(n-1)$ but not all values between n and $n(n-1)$ are possible values of $\sum y_i$.

George Purdy proved that $\sum_{i=1}^{m} y_i < 3m$.

Prove $n^2 \leq \sum y_i^2 \leq 2n(n-1)$.

A DOUBLY-DEFINED INTEGER SEQUENCE

C.J. Smyth

A sequence $S_1 < S_2 < \ldots < S_n < \ldots$ can be defined as follows:

$$S_k = 2\sum_{m=1}^{k-1} (-1)^{k-m-1}\binom{2k}{k-m} S_m + (-1)^{k-1}\binom{2k}{k}/2. \qquad (k \geq 1)$$

So $S_1=1$, $S_2=5$, $S_3=40$, $S_4=437$, $S_5=6046$, $S_6=101192$, $S_7=1986790$, $\ldots$.

From the above definition it is not completely obvious how to prove that the sequence is increasing, and not at all obvious how to obtain a rigorous estimate of the size of S_k. However, S_k satisfies another recurrence, namely

$$S_k = \sum_{m=1}^{k-1} \frac{2k}{k+m}\binom{k+m}{k-m} S_m + 1. \qquad (k \geq 1)$$

Since this recurrence has positive coefficients, the inequality $S_k > 2kS_{k-1}$ is immediate.

Can you prove that the two recurrences do indeed define the same sequence? My proof of this fact is rather roundabout. Also, any closed formula for S_k would be most interesting.

DIVISION IN OLD GERMANY

J. Innes-Reid showed us an old manuscript book which seems to have been used over several generations by a German family for exercises in calligraphy and arithmetic. One feature is that people studying arithmetic were taught to check all their calculations, after adding two numbers to subtract one of them from the sum obtained, etc. It is a pity that this excellent habit has not survived to modern days.

Their methods of setting out addition, subtraction and multiplication are the kind that we still use, but division was performed in a strange way. The following two examples from the book are possibly dated about 1692.

```
      2 2
    1 7 4 1
    2 9 7 9 3        ( 3 5 4 6
  1 1 5 2 4 5 0      \   3 2 5
    3 2 5 5 5 5        1 7 7 3 0
      3 2 2 2          7 0 9 2
        3 3        1 0 6 3 8
                   1 1 5 2 4 5 0
```

The bit on the right is the check, verifying that $3546 \times 325 = 1152450$.


```
          1
          7 2
        2 3 5 5
      1 6 6 3 7
      6 9 1 8 5 3        ( 6 5 8 4
    1 8 1 7 4 3 7        \   2 7 6
      2 7 6 6 6 6          3 9 5 0 4
        2 7 7 7          4 6 0 8 8
          2 2          1 3 1 6 8
                       1 8 1 7 1 8 4
                               2 5 3
                       1 8 1 7 4 3 7
```

This is an example of division with a remainder, and the section on the right verifies that $6584 \times 276 + 253 = 1817437$.

Can anybody explain the working of this method of division?

SPECIAL FUNCTIONS

J.B. Parker

Let $\phi(x) = \int_{-\infty}^{x} \exp(-t^2/2)\,dt$ so that $\phi'(x) = \exp(-x^2/2)$.

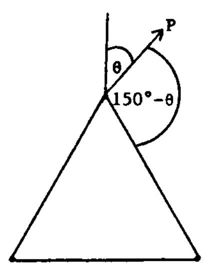
It is easily seen that ϕ'/ϕ is monotonic decreasing, but show that $x + \phi'/\phi$ is positive and increasing for all real x.

TRIANGULAR TEAPOT STAND

Among our household goods there is a teapot stand made of a piece of plywood with three little wooden feet equally spaced from one another.

Apart from its practical uses this teapot stand is a source of problems in mechanics.

Take the feet to be vertices of an equilateral triangle, and suppose that one third of the total weight is carried by each foot. Assume the usual simple laws of friction, that there is a coefficient of friction μ such that if a foot is slipping there is a frictional force on the table in the direction of slip equal to μ times the normal force, and if a foot is not slipping the friction cannot exceed μ times the normal force.

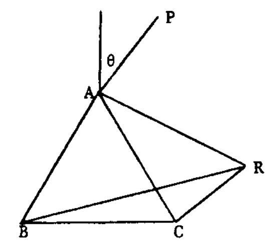


Suppose that in the direction shown a slowly increasing horizontal force P is applied to one of the feet. For what value of P will the stand begin to slip?

For a few values of the angle θ the problem is reasonably simple, suitable for a first year class. Taking the limiting frictional force on each foot as unit of force the answers are:

θ	0°	30°	60°	90°
P	3	$4/\sqrt{3}$	2	2

For other values of θ the question is more difficult. Of course it is sufficient to consider only values between 0° and 90° because considerations of symmetry show that the answer is unchanged by changing the sign of θ or adding 180° to it.



Readers preferring a problem in pure mathematics should find a point R to minimize the ratio $\dfrac{RA + RB + RC}{RA\,|\sin RAP|}$, where ABC is equilateral and θ defined as before (CAP = 150°−θ).

The point R of the geometrical problem is the centre of rotation in the mechanical problem.

GRAM BUT NOT SCHMIDT

H. Kestelman

Let X and Y be m × n matrices, show that $X^*X = Y^*Y$ if and only if $X = \Omega Y$ for some unitary m × m matrix Ω (the asterisk notation denotes the complex conjugate of the transpose).

BINOMIAL IDENTITIES 12 AND 13 (JCMN 28, p. 3071 and p. 3083)

Marta Sved

Both identities can be interpreted by the Principle of Inclusion and Exclusion explained below.

Let S be a set of N objects, and let S(1), ... S(p) be subsets of S. For any $\{i, j, \dots k\} \subseteq \{1, 2, \dots p\}$ let N(i, j, ... k) be the number of elements in the intersection of the subsets S(i), S(j), ... S(k). Then the number of objects in S that are in none of the subsets is $N_o = N - \sum_{i=1}^{p} N(i) + \sum_{1\le i<j\le p} N(i, j) - \dots + (-1)^p N(1, 2, \dots p)$.

Many combinatorial identities involve terms alternating in sign. The I-E principle provides a way of interpreting alternating sum identities by purely counting methods. On an algebraic level these identities could also be derived by inverting summation formulae. A general treatment is given by G-C Rota: On the Foundations of Combinatorial Theory I. Theory of Möbius Functions, Zeitschrift für Wahrscheinlichkeitstheorie, 2, (1964) 340-368.

Binomial Identity Number Thirteen

$$\sum_{r=0}^{m} (-1)^r \frac{(m+n-r)!}{r!(m-r)!(n-r)!} = 1 \quad \text{for} \quad m \le n.$$

Let S be the set of subsets of {1, 2, ... m + n} that have cardinality m. Let S(i) for i = 1, 2, ... m be the set of all those that do not include the integer i.

Clearly $N = \binom{m+n}{m}$ and each $N(i) = \binom{m+n-1}{m}$, so that $\sum N(i) = m\binom{m+n-1}{m}$. In general there are $\binom{m}{r}$ choices of $\{i_1, i_2, \dots, i_r\}$ and each $N(i_1, \dots, i_r) = \binom{m+n-r}{m}$. There is only one subset, namely {1, 2, ..., m}, not in any S(i), and so the I-E principle gives:

$$1 = \binom{m+n}{m} - m\binom{m+n-1}{m} + \dots + (-1)^r\binom{m}{r}\binom{m+n-r}{m} + \dots (-1)^m\binom{n}{m}.$$

Binomial Identity Number Twelve

It is more appropriate to sum to $r = [(n-1)/2]$ instead of $[n/2]$ because the terms all vanish for $r > n - r - 1$. The equation is then

$$\sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-r-1}{r} 2^{n-2r} = 2n.$$

We need first the following

Lemma: The number of ways in which k objects can be placed in n slots without any two being adjacent is $\binom{n-k+1}{k}$. This can be proved by induction on k or by combinatorial interpretation on a model.

Proof of the identity: Consider the set S of all functions mapping {1, 2, ..., n} into the set {0, 1}, that is all sequences of n numbers each either 0 or 1. First note that the number of monotonic functions is 2n. Now we must count the non-monotonic functions. Let f be one of the functions, and $1 < i < n$. If $f(1) = 0$ we say that "the monotonicity breaks at i" if $f(i) = 1$ and $f(i+1) = 0$, similarly if $f(1) = 1$ we say that the monotonicity breaks at i if $f(i) = 0$ and

$f(i+1) = 1$. For example, in (0, 0, 1, 1, 0, 1) the monotonicity breaks at 4, and nowhere else. In (1, 1, 0, 1, 0, 1) there are two breaks, 3 and 5. A function is monotonic if and only if it has no breaks.

Clearly breaks cannot occur at consecutive points. Let $S(i)$ be the set with a break at i, then $N(i) = 2^{n-2}$. There are $n-2$ possible values of this i and so $\sum N(i) = (n-2)2^{n-2}$. Now consider the general case of r breaks. Let them be at $i_1, i_2, \ldots, i_r$ where $i_{k+1} \geq 2 + i_k$. There are $\binom{n-r-1}{r}$ choices for the set $\{i_1, \ldots, i_r\}$ and for each choice there are 2^{n-2r} possible functions. Therefore $\sum N(i_1, \ldots, i_r) = \binom{n-r-1}{r}2^{n-2r}$. The I-E principle now gives the required formula

$$2n = 2^n - \binom{n-2}{1}2^{n-2} + \ldots + (-1)^r\binom{n-r-1}{r}2^{n-2r} + \ldots$$

BINOMIAL IDENTITY 13 (JCMN 28, p. 3083)

J.B. Parker

This identity $\sum_{r=0}^{m} (-1)^r \frac{(m+n-r)!}{r!(m-r)!(n-r)!} = 1$ for $m \leq n$ arose from substituting $f(x) = x^m$ and $g(x) = x^n$ in the conjectured equation

$$\sum_{r=0}^{\infty} (-1)^r \int_0^{\infty} (x^r/r!)\ f^{(r)}(x)\ g^{(r)}(x)\ e^{-x}dx = \int_0^{\infty} f(x)e^{-x}dx \int_0^{\infty} g(x)e^{-x}dx.$$

See "Inequality for Polynomials", JCMN 27, pp. 3051-3053. Proof of this binomial identity establishes the truth of the conjecture when f and g are polynomials.

BINOMIAL IDENTITIES 12 AND 13 (JCMN 28, p. 3071 and p. 3083)

C.S. Davis

<u>Binomial Identity 12</u>: $\sum_{r=1}^{[n/2]} (-1)^{r-1}\binom{n-r-1}{r} 2^{n-2r} = 2^n - 2n.$

<u>Remark</u> The sum may as well be written $\sum_{r=1}^{\infty}$, and I do this here and elsewhere, if the spirit moves.

Note that $\sum_{s=0}^{\infty} (-1)^s\binom{n-r-1}{s}t^s = (1-t)^{n-r-1}$. Hence

$$\sum_{r=0}^{n-1} \{4t(1-t)\}^{n-r-1} = \sum_{r=0}^{n-1} 2^{2(n-r-1)}t^{n-r-1} \sum_{s=0}^{\infty} (-1)^s\binom{n-r-1}{s}t^s \quad \ldots (1)$$

The coefficient of t^{n-1} in the r.h.s. of (1) is

$$\sum_{r=0}^{n-1} 2^{2(n-r-1)}(-1)^r\binom{n-r-1}{r} = 2^{n-2}(2^n-S),$$

where S is the given sum. Writing $u = 4t(1-t)$ [so $1-u = (1-2t)^2$], the l.h.s. of (1) is

$$\sum_{j=0}^{n-1} u^j = \frac{1-u^n}{1-u} = \frac{1}{(1-2t)^2}(1-u^n) = \sum_{m=1}^{\infty} m(2t)^{m-1}\{1-2^{2n}t^n(1-t)^n\}.$$

Hence $2^{n-2}(2^n-S) = n.2^{n-1}$, so $S = 2^n-2n$.

<u>Further Comment</u> I found a note that the equivalent identity $\sum_{\nu\geq 0}(-1)^{\nu}\binom{n-\nu}{\nu}2^{n-2\nu} = n+1$ appears in Comtet, Advanced Combinatorics, 168. I give the proof, mine (?) or Comtet's (?): from $\frac{1}{1-z} = \sum_0^{\infty} z^n$, we have $\frac{r \sin\theta}{1-2r\cos\theta+r^2} = \sum_1^{\infty} r^n \sin n\theta$ and hence, writing $2\cos\theta = u$,

$$\frac{\sin(n+1)\theta}{\sin\theta} = \sum_{\nu\geq 0} (-1)^{\nu}\binom{n-\nu}{\nu} u^{n-2\nu} \quad \ldots (2)$$

Letting $\theta \to 0$, we have the result. (I would guess that (2), like almost everything else, is to be found in Bromwich, or in Chrystal's Algebra (sic.).)

The result just proved is a special case of

$$\sum_{k\geq 0}(-1)^k\binom{n-k}{k}t^k(1+t)^{n-2k} = \sum_{k=0}^{n} t^k\,, \qquad \ldots (3)$$

which may be established in much the same way, or in a variety of other ways. In passing, I note that this result underlies the 'interesting identities' appearing as a by-product in Carroll and Giola: "On a subgroup of the group of multiplicative arithmetic functions", Jour. Aust. Math. Soc., 20 (1975), 348-358 (355):

$$n+1 = \tau(p^n) = \sum (-1)^{n+i}\binom{n-i}{i}2^{n-2i}\,,$$

$$\sigma(p^n) = \sum (-1)^{n+i}\binom{n-i}{i}(1+p)^{n-2i}p^i\,,$$

where $\tau(k) = \sum_{d|k} 1$, $\sigma(k) = \sum_{d|k} d$ and $\sum = \sum_{0\leq i\leq[\frac{n}{2}]}$.

<u>Binomial Identity 13</u>: $\displaystyle\sum_{r=0}^{m}(-1)^r\frac{(m+n-r)!}{r!(m-r)!(n-r)!} = 1$ for $m \leq n$.

<u>Remark</u> The condition $m \leq n$ is superfluous. The sum may be written as $\sum_{r=0}^{m}$, $\sum_{r=0}^{n}$ or $\sum_{r=0}^{\infty}$, since the terms vanish for $r > m$ or $r > n$.

Note that $\displaystyle\frac{(m+n-r)!}{r!(m-r)!(n-r)!} = \frac{m!}{r!(m-r)!}\,\frac{(m+n-r)!}{m!(n-r)!} = \binom{m}{r}\binom{m+n-r}{n-r}$.

Since $\displaystyle\sum_{r=0}^{\infty}(-1)^r\binom{m}{r}t^r = (1-t)^m$ and $\displaystyle\sum_{s=0}^{\infty}\binom{m+s}{s}t^s = (1-t)^{-m-1}$,

$\displaystyle\sum_{r=0}^{\infty}(-1)^r\binom{m}{r}\binom{m+n-r}{n-r}$ is the coefficient of t^n in the expansion of $\frac{1}{1-t}$, i.e. 1.

BINOMIAL IDENTITY 12 (JCMN 28, p. 3071)

J.B. Parker

Consider the two families of propositions (for N = 1, 2, ...)

$$\text{(aN)} \quad \sum_{r=1}^{N}(-1)^{r-1}\binom{2N-r+1}{r}2^{2N-2r} = 2^{2N} - N - 1$$

$$\text{(bN)} \quad \sum_{r=1}^{N+1}(-1)^{r-1}\binom{2N-r+2}{r}2^{2N-2r+2} = 2^{N+2} - 2N - 3$$

They can be proved inductively.

From (aN) and (bN) we can deduce (a(N+1)) and (b(N+1)), using the identity $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$.

BINOMIAL IDENTITY NUMBER 14

C.J. Smyth

Show that for $k > 0$ and $0 \leq m \leq k$

$$\sum_{j=m}^{k}\binom{j}{m}\binom{2k}{2j} = 4^{k-m}\frac{k}{2k-m}\binom{2k-m}{m}.$$

TRANSPOSES SIMILAR

H. Kestelman

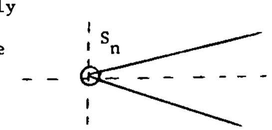
Show that any square matrix M is similar to its transpose M^T, in the sense that $M = X^{-1}M^TX$ for some X.

COMPLEX FUNCTION THEORY (JCMN 27, p. 3059, 28, p. 3080)

I.N. Baker

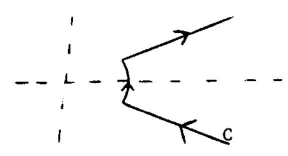
What can you say about a sequence of entire functions converging pointwise? In Walter Rudin's *"Real and Complex Analysis"* Chapter 13, Exercise 3 asks - 'Is there a sequence of polynomials P_n such that $P_n(0) = 1$ for $n = 1, 2, \ldots$ but $P_n(z) \to 0$ for every $z \neq 0$?' The answer is YES. This result can be derived from Runge's approximation theorem, but you might like to have a solution depending upon only the most elementary parts of complex function theory.

<u>Lemma</u> For each $n = 1, 2, \ldots$ there is an entire function F_n such that $F_n(0) = 1$ and $F_n(z) \to 0$ uniformly as $z \to \infty$ in the sector S_n where the phase (alias "amplitude" or "argument") of z is between $1/n$ and $2\pi - 1/n$, i.e.



$S_n = \{r\, e^{i\theta};\ 1/n \leq \theta \leq 2\pi - 1/n,\ r > 0\}$.

<u>Proof of lemma</u> For any n and any radius ρ, consider the function $F_n(z) = \frac{n}{2\pi i} \int_C \frac{\exp(t^n)}{t - z}\, dt$ defined on the left of the infinite path C shown, consisting of a circular arc $\{\rho \exp i\theta;\ -\beta \leq \theta \leq \beta\}$ and two half-infinite lines $\{r \exp \pm i\beta;\ r \geq \rho\}$, with β such that $\pi/(2n) < \beta \leq \pi/n$. This choice of β ensures that the (improper) integral exists. The convergence of the integral is uniform with respect to z in any compact set not meeting the path C. The function



F_n is therefore analytic to the left of C. By Cauchy's theorem the radius ρ may be increased without changing the value of the function at the points where it is already defined. Therefore the continuation of the function F_n above is an entire function.

Again by Cauchy's theorem we may choose any β in the given range, say $\beta = (3/4)\pi/n$. For z in S_n the distance from z to any point of C is at least $k|z|$ and so

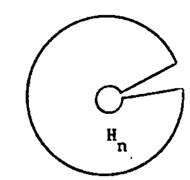
$$|F_n(z)| < \frac{n}{2\pi} \int_C \frac{|\exp(t^n)\,dt|}{k|z|} < K_n/|z| \text{ for some constant } K_n.$$

This shows that as $z \to \infty$ in S_n, $|F_n(z)| \to 0$ uniformly. Finally, to evaluate $F_n(0)$ take $\beta = \pi/n$ and change the variable of integration to $u = t^n$.

$$F_n(0) = \frac{n}{2\pi i} \int_C (1/t)\exp(t^n)\,dt = \frac{1}{2\pi i} \int u^{-1} \exp u\, du = 1$$

(the last integral being round the unit circle, $|u| = 1$)

<u>Main proof</u> Define A_n (for each positive integer n) so that $|F_n(z)| < 1/n$ for all z in S_n with $|z| \geq A_n$. Then $|F_n(n A_n z)| < 1/n$ for all z in S_n with $|z| \geq 1/n$. Put $G_n(z) = F_n(n\, A_n\, z \exp(-2i/n))$, it is entire and takes the value 1 at the origin. Also $|G_n(z)| < 1/n$ in the region H_n shown here.



$H_n = \{r \exp i\theta;\ 1/n \leq r \leq n,\ 3/n \leq \theta \leq 2\pi + 1/n\}$

Because G_n is entire we may put $G_n(z) = 1 + \sum_{\nu=1}^{\infty} g_\nu z^\nu$ and choose $N(n)$ so that the polynomial $P_n(z) = 1 + \sum_{\nu=1}^{N(n)} g_\nu z^\nu$ satisfies $|P_n(z)| < 2/n$ in H_n. Every point $z \neq 0$ is in H_n for all sufficiently large n and so we have a sequence of polynomials, each = 1 at the origin but converging to zero at every other point. This gives the answer to Rudin's exercise and the question in JCMN, that the limit function need not be entire.

One form of Runge's approximation theorem is as follows. Suppose that K is a compact set in the plane and that in the sphere (the extended complex plane) the complement of K is connected, and that f is analytic on some open set containing K. Then f can be approximated arbitrarily closely on K by a polynomial.

This theorem gives a quick proof of our result. For any n let $K_n = \{0\} \cup H_n$ and put $f(z) = 1$ for $|z| < 1/(3n)$ and $f(z) = 0$ for $|z| > 2/(3n)$. The conditions for Runge's theorem are satisfied and there is a polynomial Q_n such that

$$|f(z) - Q_n(z)| < 1/n \text{ for all } z \text{ in } K_n.$$

Put $P_n(z) = Q_n(z)/Q_n(0)$, then $P_n(0) = 1$ and for all z in H_n, $|P_n(z)| \leq \frac{n}{n-1} |Q_n(z)| < 1/(n-1)$, giving the result as before.

DISTINGUISHED INVERSE (JCMN 26, p. 3040)

H. Kestelman

Suppose that $m < n$ and that the $m \times n$ matrix A has linearly independent rows, then AA^* is invertible and $X = A^*(AA^*)^{-1}$ is a right inverse of A, in the sense that $AX = I$. What distinguishes X from all the other right inverses of A?

The answer is that XA is Hermitean. To show that X is the only right inverse with this property, take any $n \times m$ matrix Y such that $AY = I$ and YA is Hermitean,

$YAA^* = (YA)^*A^* = A^*Y^*A^* = A^*(AY)^* = A^*$, and so

$Y = A^*(AA^*)^{-1} = X$.

SOLVING ALGEBRAIC EQUATIONS

With a simple programmable calculator it is fairly easy to find the real zeros of any real polynomial. You just set the machine to calculate the polynomial p(x) for any x, and then keep trying different values for x. But is there a way to find complex zeros? Often it is only the real parts of the complex roots that matter; is there a way of finding them?

BEACHCOMBINGS (JCMN 28, p. 3079)

(a) $\frac{\text{P}}{\text{Venez}}$ à $\frac{\text{ci}}{\text{Sans}}$ One reader suggested that by cross-multiplying this it could be regarded as a French pun - "paysans venez ici", or "peasants come here". Voltaire's answer Ja could then be interpreted as meaning that Germans were peasants but French were not. Another theory also interprets the first message as a French pun - "Venez souper à Sans Souci" and Voltaire's answer not as a German word but as two letters, one large and one small, making another French pun, "J'ai grand appetit."

(b) (Continuing the given sequences). The rule of formation of the sequences will be apparent when each is continued with two more members as follows:

110, 20, 12, 11, 10, 6, 6, ...

63, 94, 46, 18, 001, ...

(c) The function f(x) is the number of letters in the English name for the number x.

(d) What constant is not named after Captain Cook? This is a difficult question and two theories have been put forward, (i) A pie is defined as a quantity of fruit or meat, already cooked, which is to be placed under a crust of suitably prepared flour. As you cannot cook what is already cooked, π is not named after Captain Cook. (ii) In a part of the

world that shall be nameless, a scientist who obtains a result not acceptable may modify it either by adding a number or multiplying by a number, after making a suitable choice for the number and calling it "Cook's constant". Captain Cook never did this (see JCMN 10, page 8) and so Cook's constant must have been named after somebody else.

(e) Rearranging the matchsticks. The least implausible suggestion is to make $\sqrt{I} = 1$. Another candidate is VI = II, for an equation does not have to be true to be good (see the quotation from Vilfredo Pareto in JCMN 10, page 4).

PROBLEM OF IDENTITY (JCMN 28, p. 3078)

We are given that f is a real function of the real variable, not identically zero, and that $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$. From A. van der Poorten and H. Kestelman we have essentially the same solution. Firstly $f(1) = 1$ because f is not identically zero. Secondly $f(rx) = r\,f(x)$ for any integer r. Thirdly the function must map a rational number on to itself. Now the tricky part; the image of a positive number is positive because if $x \geq 0$ then $x = y^2$ and $f(x) = (f(y))^2 \geq 0$. It follows that the function is monotonic increasing, and therefore continuous, and so it is the identity function.

None of our readers has commented yet upon the relation between this problem and the one (PLANE MAPPING PROBLEM) that was printed just above it on page 3078.

LITTLE SQUARE MATRIX (JCMN 28, p. 3077)

H. Kestelman

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ have complex elements. Under what conditions is M expressible as $M = S\Omega$ with $S^T = S$ and $\Omega^T\Omega = I$? The answer is "unless $ad - bc = a^2 + b^2 + c^2 + d^2 = 0$". To prove this we first have to dispose of four simple special cases, where $a = \pm d$ and where $b = \pm c$. For each of these it is clear that either $\Omega = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ or $\Omega = \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}$ gives the required result. Now suppose $b^2 \neq c^2$ and $a^2 \neq d^2$.

Every Ω with $\Omega\Omega^T = I$ is either

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

for some complex θ and the factorization of M is possible if and only if either

$$(a+d)\sin\theta = (b-c)\cos\theta \quad \text{or} \quad (a-d)\sin\theta = (b+c)\cos\theta.$$

This means that $\tan\theta$ = either $(b-c)/(a+d)$ or $(b+c)/(a-d)$. The only values not taken by the complex function $\tan\theta$ are $\pm i$ and so the factorization is impossible when (and only when)

$$(a+d)^2 = -(b-c)^2 \quad \text{and} \quad (a-d)^2 = -(b+c)^2.$$

THE FALKLAND ISLANDS

After James Cook's first great Pacific exploration in H.M.S. Endeavour, he reached England in July 1771. The ship was refitted and used for four voyages carrying stores to the Falkland Islands between October 1771 and September 1774. She was sold out of the Royal Navy in March 1775.

The naval base at Port Stanley in the Falkland Islands emerged from obscurity again in December 1914. Admiral Sturdee's squadron was refuelling there when Admiral Graf Spee's squadron appeared. In the battle that followed all but one of the German ships were sunk.

To anybody lecturing on mechanics it is of interest to recall that the first salvos from the twelve-inch guns of the battle-cruisers HMS Invincible and HMS Inflexible seemed badly aimed; in the rushed journey from the North Sea to the South Atlantic, the gunners had forgotten to adjust the fire control system for the change in Coriolis force.

MATHEMATICS IN LONDON

B.B. Newman

(Extract from a letter sent back home to Townsville)

I was consulted on a project with a program not working very well. There was one error in the program that I am particularly proud of finding. They defined PI = 3.14... and I spotted an error in the eighth decimal place. You see, as a joke, I once learnt π to 40 decimal places, and I have remembered this absolutely useless piece of information for many years (perhaps not completely useless, as I always manage to get a laugh from students when I quote it and add that 22/7 is good enough for our purposes).

SURFACE AREA OF AN ELLIPSOID (JCMN, pp. 3031 and 3056)

Murray Klamkin writes that there has been a considerable amount published on inequalities for the surface of an ellipsoid.

G. Polya, Approximations to the area of the ellipsoid. Publ. Inst. Mat. Rosario, 5 (1943) 1-13.

D.H. Lehmer, Approximations to the area of an n-dimensional ellipsoid, Canad. J. Math. 2, (1950) 267-282.

B.C. Carlson, Some inequalities for hypergeometric functions. Proc. Amer. Math. Soc., 17 (1966) 32-39.

M.S. Klamkin, Elementary approximations to the area of N-dimensional ellipsoids. Amer. Math. Monthly 78 (1971) 280-283.

MATRIX PROBLEMS (JCMN 28, p. 3072)

A. Brown

In Question 3 H. Kestelman asks for the eigenvalues of the matrix with components $c_{rs} = 1$ if $r = s \pm 1$, and $= 0$ otherwise. If we use C_n for the $n \times n$ case of this matrix and I_n for the unit $n \times n$ matrix then $P_n(\lambda) = \det (C_n - \lambda I_n)$ satisfies the recurrence relation $P_{n+1} + \lambda P_n + P_{n-1} = 0$ $(n = 2, 3, \ldots)$ with $P_1 = -\lambda$ and $P_2 = \lambda^2 -1$. The n roots of $P_n(\lambda) = 0$ are the eigenvalues of C_n, and each satisfies $-2 < \lambda < 2$ because the two matrices $2 I_n \pm C_n$ are positive definite, and so we put $\lambda = 2 \cos \phi$ and note that $\sin \phi \neq 0$.

It is easy to verify that

$$P_n = (-1)^n (\sin(n+1)\phi)\operatorname{cosec} \phi$$

satisfies the recurrence relationship. The zeros are $\lambda = 2 \cos(N\pi/(n+1))$ with $N = 1, 2, \ldots n$.

This problem is essentially the same as that of finding the frequencies of vibration for small transverse vibrations of n equal particles which are attached at equal distances, a , along a light elastic string of length $(n+1)a$, with the end points of the string fixed. As such it is discussed in detail in D.E. Rutherford's "Classical Mechanics", pp 183-186 (Oliver and Boyd, 1951) and the solution to the eigenvalue problem above simply omits the mechanics. A large number of books on mechanics discuss particular cases of this problem (n = 1, 2 or 3) and there is a tradition that Euler obtained the frequencies of vibration of a uniform string by letting n tend to infinity. So it is a problem with a respectable history in applied mathematics.

LATEST NEWS FROM HUNGARY

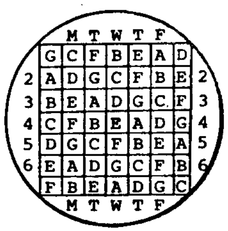
P. Erdős

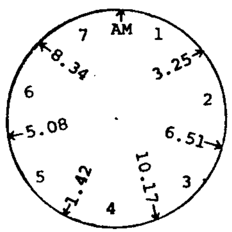
My problem number 9 (on page *3100* above) has been proved by Szemerédi and Trotter, and I think that a young mathematician here has proved 8.

Here is another problem. Denote by $A(n; k)$ the least common multiple of $n+1, n+2, \ldots, n+k$. Probably the equation $A(n; k) = A(m; k)$ with $m \geq n+k$ and $k \geq 3$ has only a finite number of solutions in n and k. If true - this will not be easy. There are surprisingly many relatively small solutions. It is easy to see that $A(n; k) = A(n+1; k)$ has only a finite number of solutions in n for fixed k. Prove this (not hard) and estimate the largest solution n_k as well as you can.

MYSTERIOUS MESSAGE (JCMN 28, p. 3073)

Comments from A. Brown and J.B. Parker shed a little light.





There must have been either shift-works or watchkeeping duties, by seven squads or sentries, labelled A, B, C, D, E, F and G. Each one was allocated seven periods, totalling 24 hours, in the week. G, for example, was on duty in early Sunday morning for period number one from midnight to 3.25 a.m., then A took over for the second period from 3.25 to 6.51, while G was off until Monday afternoon, period 5, from 1.42 p.m. to 5.08 p.m.

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EDITORIAL

We would like to hear from you anything connected with mathematics or with Capt. James Cook.

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