JAMES COOK MATHEMATICAL NOTES

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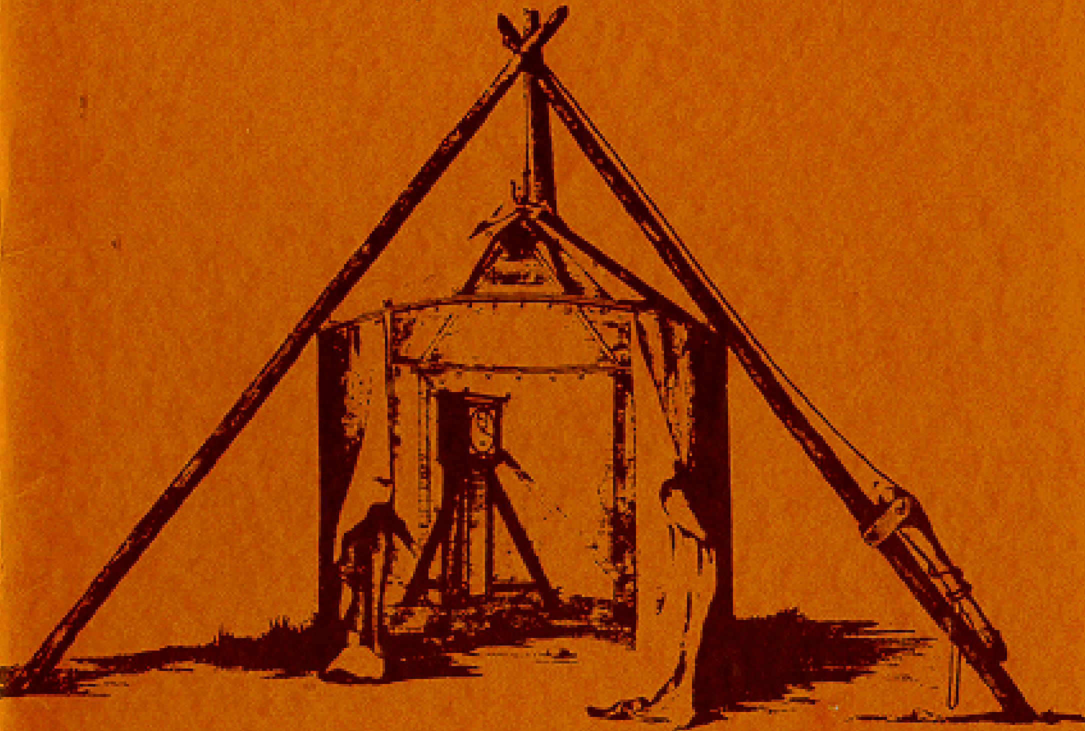

Portable observatory, 1776

A design by William Bayly

Portable observatories were taken on all Cook's voyages for setting up ashore. This is an engraving of the pattern used on the third voyage.

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TRIGONOMETRIC INEQUALITY (JCMN 26, p. 3034)

Jamie Simpson

Find the best possible value of the constant k in the proposition that if y is real and n is a natural number, then at least one of sin y, sin 2y, ..., sin ny has modulus less than k/n.

Use the notation $\|x\|$ (for any real x) to denote the distance from x to the nearest integer. Take any n and any y. By a well-known theorem of Dirichlet there is a natural number $m \leq n$ such that

$$\| my/\pi \| \leq 1/(n+1) \leq \frac{1}{2}$$

and with such an m we have:

$$|\sin m\pi| = |\sin((my/\pi)\pi)| = \sin(\|my/\pi\|\pi)$$

$$\leq \sin(\pi/(n+1)) < \pi/(n+1) < \pi/n.$$

The best possible value for k is therefore π. Note, however, that we have proved the following slightly stronger result. If y is real and n is a natural number then at least one of sin y, sin 2y, ..., sin ny has modulus $\leq \sin \pi/(n+1)$.

BINOMIAL IDENTITY NUMBER TWELVE

$$\sum_{r=1}^{[n/2]} (-1)^{r-1} \binom{n-r-1}{r} 2^{n-2r} = 2^n - 2n.$$

Hall and Knight - Higher Algebra (1889) p. 513.

MATRIX PROBLEMS

H. Kestelman

Q1. Use the notation "semi-positive" for a real matrix or vector with every element ≥ 0, and "positive" for one with every element > 0. Suppose that $A = \begin{pmatrix} B & Q \\ 0 & G \end{pmatrix}$ is semi-positive and non-zero, where B and G are square matrices, and that A has a positive right eigenvector and a positive left eigenvector. Prove (i) that all semi-positive (left or right) eigenvectors of A belong to the same positive eigenvalue λ, and (ii) that $Q = 0$, and (iii) that the nullity of $A - \lambda I$ is at least two.

Q2. The off-diagonal elements of the real square matrix B are all ≥ 0 but each row-sum ≤ 0. Prove that every non-zero eigenvalue has negative real part.

Q3. Find the eigenvalues of $C = \begin{pmatrix} 0 & 1 & 0 & 0 & . & . & . \\ 1 & 0 & 1 & 0 & . & . & . \\ 0 & 1 & 0 & 1 & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}$

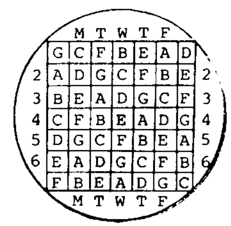
where $c_{rs} = \begin{cases} 1 \text{ if } r - s = \pm 1, \\ 0 \text{ otherwise.} \end{cases}$

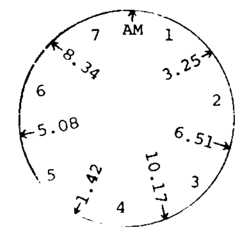
Q4. We call the complex square matrix M "normal" if $M = UDU^*$ for some unitary U and diagonal D. Let $r(M)$ denote the maximum modulus of the eigenvalues of M, and write $\|M\|$ for the maximum (over all $\underline{x}$) of $(\underline{x}^* M^* M\underline{x}/\underline{x}^*\underline{x})^{\frac{1}{2}}$.

Prove that if A and B are both $n \times n$ normal matrices then $|r(A) - r(B)| \leq \|A - B\|$.

MYSTERIOUS MESSAGE

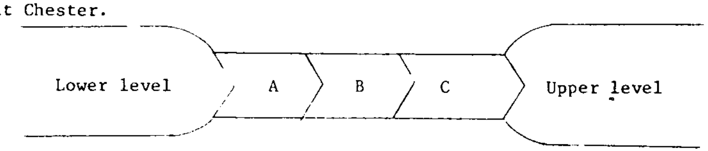
On the first of December 1981 Mr. B. Herd came to the Mathematics Department to show us a brass disc an inch and three quarters in diameter with markings as shown below. Unfortunately nobody was able to suggest what the markings meant or why the disc was made. Does one of our readers have any ideas?





STAIRCASE LOCKS

On a canal, locks are sometimes in "staircase" form. A typical example is the three-chamber staircase on the Shropshire Union Canal at Chester.



The three lock chambers (A, B and C above) are of equal size and are connected by watertight gates which may be opened to allow a boat to pass from one to another. Before opening a pair of gates it is necessary to equalize the water levels, and this is done by opening a valve (known on the canals as a "paddle") to allow water to pass from the upstream side to the downstream side of the pair of gates.

Boats pass through these locks independently at random, each boat going either up or down (only one boat at a time is in the locks).

The changes in water level are illustrated by the following numerical example, where we take the lower level to be constant = 0 and upper level = 24.

	A	B	C	
Let initial levels be	6	16	24	Now a boat wants to go up.
	0	16	24	Boat in to A.
	8	8	24	Boat in to B.
	8	16	16	Boat in to C.
	8	16	24	Boat continues on upper canal.

Next there is another boat going down, it goes in to C.

	A	B	C	
	8	20	20	Boat in to B.
	14	14	20	Boat in to A.
	0	14	20	Boat continues on lower level.

Now another boat begins to go down.

	A	B	C	
	0	14	24	Boat goes in to C.
	0	19	19	Boat goes in to B.
	9½	9½	19	Boat goes in to A.

The lock chambers were cut by hand out of solid rock in the early nineteenth century, and the builders must have asked themselves how deep to make the bottom of chambers B and C. The danger is that, for instance, when a boat locking downwards is in B, and the paddle is opened to equalize the levels in A and B, then the boat might go

aground in B before the levels are equal. The grounding of a heavily loaded narrow boat on a rock bottom could damage the ribs.

Given that it is always unpredictable whether the next boat to appear will be going up or down, can we specify bounds between which the level in chamber B will always be?

ZEROS OF A DETERMINANT (JCMN 22, Vol. 2, p. 89)

If $a_1, a_2, \ldots, a_n$ are all positive and unequal (**we may take them** to be in increasing order), does the determinant

$$D(r) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \cdots & & & \\ a_1^{n-2} & a_2^{n-2} & & \\ a_1^r & a_2^r & \cdots & a_n^r \end{vmatrix}$$

(as a function of the real variable r) have any zeros apart from the obvious ones $r = 0, 1, \ldots, n-2$? The answer is NO.

Take any r such that $D(r) = 0$. The function x^r can be fitted at the points $x = a_1, a_2, \ldots, a_n$ by a polynomial $p(x) = c_0 + c_1 x + \ldots + c_{n-1}x^{n-1}$. The determinant D, expanded by its bottom row, may be expressed

$$\begin{aligned} D &= \sum_{s=1}^{n} a_s^r M_s \\ &= \sum_{s=1}^{n} \sum_{t=0}^{n-1} c_t a_s^t M_s \end{aligned}$$

but for $t = 0, 1, 2, \ldots, n-2$ we know that $\sum_{s=1}^{n} a_s^t M_s = 0$ (because it is a determinant with two rows equal).

Therefore $0 = D = c_{n-1}\sum_{s=1}^{n} a_s^{n-1} M_s$, but we know that $\sum a_s^{n-1} M_s$ is the alternant equal to $\prod_{i>j} (a_i - a_j)$ and $\neq 0$, so that $c_{n-1} = 0$, and the polynomial $p(x)$ has degree $\leq n-2$. The difference $x^r - p(x)$ has n distinct zeros and by Rolle's Theorem there is some x between a_1 and a_n such that

$$r(r-1) \ldots (r-n+2)x^{r-n+1} = (d/dx)^{n-1} p(x) = 0 .$$

Therefore r has one of the values $0, 1, \ldots, n-2$. QED

SIMPLE QUESTION FOR UNDERGRADUATES (JCMN 25, p. 3005)

Is the canvas of a fire-hose more likely to split longitudinally or circumferentially?

Sad to say, it has become apparent that this question is not simple enough for undergraduates, for nowadays they are taught nothing about hoses. That is a pity because piped water, oil, gas, petrol and compressed air are inescapable parts of modern living. Could we remedy this bit of undergraduate ignorance? A hose with a constant flow at constant pressure can be studied without going into difficult equations or obscure concepts.

Suppose that in the hose the water speed is v, assumed uniform, density ρ, pressure p (above atmospheric) and area A. If there is tension T in the hose itself then the rate of momentum transfer along the hose

is $A(\rho v^2 + p) - T$, and this has to be zero if the hose lies on the ground in a non-straight line, as usually happens when a fire brigade is putting out a fire. The hose ends in a nozzle from which comes a jet of area = a and speed = $V = Av/a$. Bernoulli's equation tells us that $p = \frac{1}{2}\rho(V^2 - v^2) = \frac{1}{2}\rho v^2 \frac{A^2 - a^2}{a^2}$ and so the tension in the hose is $T = A(p + \rho v^2) = \frac{1}{2}\rho A v^2 \frac{A^2 + a^2}{a^2}$ and the lengthways stress in the canvas of the hose is $T/(2\pi R) = \frac{\rho A v^2}{4\pi R} \frac{A^2 + a^2}{a^2}$. The circumferential stress is $Rp = \frac{\rho R v^2}{2} \frac{A^2 - a^2}{a^2}$.

The hose will split longitudinally if the latter is larger, that is if $2(A^2 - a^2) > A^2 + a^2$ or $A^2 > 3a^2$ or if the hose diameter is more than the fourth root of three (= 1.316...) times the nozzle diameter.

In real life the ratio of diameters is something like 3 or 4 and hoses always split longitudinally.

LITTLE SQUARE MATRIX

H. Kestelman

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ have complex elements. Find what conditions a, b, c and d must satisfy for M to be expressible as $M = S\Omega$, where $S^T = S$ and $\Omega^T\Omega = I$.

PLANE MAPPING PROBLEM

H. Kestelman

Let f map the Euclidean plane into itself in such a way that if A, B and C are points with ABC a right angle, then f(A)f(B)f(C) is also a right angle. Does it follow that f is continuous?

PROBLEM OF IDENTITY

C.J. Smyth

If a function f from the real line to itself is not identically zero, and satisfies $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all x and y, is it necessarily the identity function $f(x) = x$?

QUOTATION CORNER (13)

When Franklin was asked how he could afford the charges of his experiments on electricity, at a time when he was far from being in circumstances of independence, he replied that a man who could not saw with a gimlet and bore with a saw was not fit for an experimental philosopher.

Taken from "A Journal of Natural Philosophy, Chemistry and the Arts" (Edited by William Nicholson), London, 1799, reporting a lecture given by Citizen Guyton at the National Institute of France on the 26th of Brumaire in the year 6.

BEACHCOMBINGS

The following puzzles were found in a bottle on the North Queensland coast by a visiting oceanographer.

(a) $\dfrac{\text{P}}{\text{Venez}}$ à $\dfrac{\text{ci}}{\text{Sans}}$.

was a message sent by Frederick the Great of Prussia to his friend the French author Voltaire, who sent back the answer

J_{a} .

Frederick thought about this and then burst out laughing. Why?

(b) Continue the following sequences:

110, 20, 12, 11, 10, ...

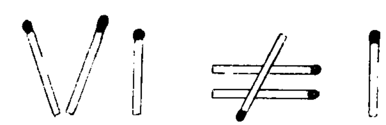
63, 94, 46, ...

(c) A function f(x) is defined for all positive integers x, a few of the values are given below, what is the function?

x	1	3	4	8	10	11	20
f(x)	3	5	4	5	3	6	6

(d) What constant is obviously not named after Captain Cook?

(e) Seven matchsticks are arranged like this:-



Move one matchstick to make a good equation.

COMPLEX FUNCTION THEORY (JCMN 27, p. 3059)

If a sequence of entire complex functions $f_n(z)$ converges for all z to some finite $\phi(z)$, what can you say about the function ϕ? The function ϕ is analytic on an open set everywhere dense.

Lemma 1 In any closed disc there is a closed disc in which the functions f_n are uniformly bounded.

Proof Start with a closed disc D_0. If the functions are uniformly bounded in D_0 there is no more to prove. If not, then there is $n(1)$ such that $|f_{n(1)}(z)| > 1$ at some point in D_0, and therefore (because $f_{n(1)}$ is continuous) $|f_{n(1)}(z)| > 1$ in some closed disc $D_1 \subseteq D_0$. If the functions are uniformly bounded in D_1 there is no more to prove, and if not there exists $n(2) > n(1)$ and $D_2 \subseteq D_1$ such that $|f_{n(2)}(z)| > 2$ in D_2. We continue in the same way. Either we find at some stage a disc in which the functions are uniformly bounded or we find an infinite sequence $n(1) < n(2) < \ldots$ and an infinite nest of discs $D_1 \supseteq D_2 \supseteq \ldots$ such that $|f_{n(j)}(z)| > j$ in D_j. In the latter case the Heine-Borel theorem gives a point w in all D_j, and there is a contradiction in the facts that $|f_n(w)| \to |\phi(w)|$ and $|f_{n(j)}(w)| > j$ for all $j = 1, 2, \ldots$. The lemma is therefore proved.

Lemma 2 If the functions f_n are uniformly bounded in a closed disc then the limit ϕ is analytic in the interior.

Proof Noting that ϕ is L-measurable and using Cauchy's integral and Lebesgue's theorem on dominated convergence, for any z in the interior of the disc we have:-

$$\phi(z) = \lim f_n(z) = \oint \frac{\lim f_n(t)}{2\pi i(t - z)}dt = \oint \frac{\phi(t)}{2\pi i(t - z)}dt$$

where the contour integrals are round the boundary. Also

$$\lim_{h\to 0} \frac{\phi(z+h) - \phi(z)}{h} = \lim_{h\to 0} \oint \frac{\phi(t)}{2\pi i(t-z)(t-z-h)}dt = \oint \frac{\phi(t)}{2\pi i(t-z)^2}dt$$

so that ϕ is differentiable.

Our two lemmata establish that any disc contains an open disc in which the limit function ϕ is analytic.

CYCLIC HEXAGON (JCMN 25, p. 3006)

E. Szekeres

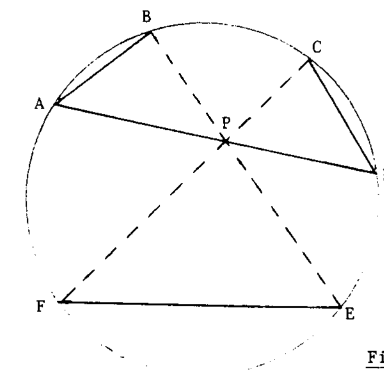


Figure 1

The hexagon ABCDEF is on a circle of unit radius and the diagonals AD, BE and CF meet at P, show that AB + CD + EF $\leq$ 4. In proving this result we may re-name the points so that AB and CD both $\leq$ EF.

Lemma 1 If EF is parallel to AD then AB + CD $\leq$ 2.

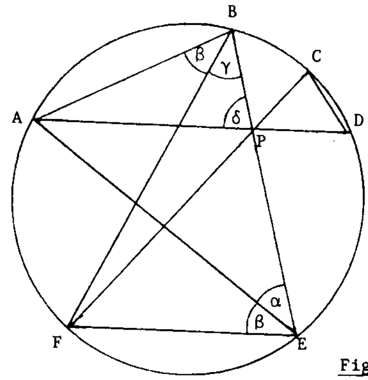


Figure 2

Since $AB \leq EF$ it follows that $\alpha \leq \gamma$ and so $\delta = \alpha + \beta \leq \beta + \gamma$ and therefore $AB \leq AP$. Similarly $CD \leq PD$ and so

$$AB + CD \leq AD \leq 2 \quad .$$

Lemma 2 (due to G.M. Kelly) In figure 1 if the points A, B, C and D are fixed and if P, F and E are variable, then EF has its maximum value when EF is parallel to AD.

Proof

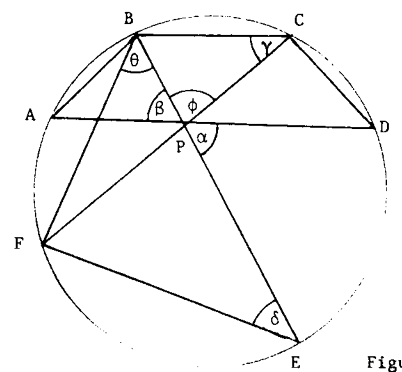


Figure 3

The length EF is maximum when θ is maximum, that is when ϕ (= θ + BFC) is maximum, that is when AD is tangent at P to the circle BPC. When this happens $\beta = \gamma$, and therefore $\alpha = \beta = \gamma = \delta$.

Lemma 3 The result of Lemma 1 holds without the condition that EF and AD must be parallel.

Proof We may choose E' and F' on the circle so that E'F' is parallel to AD and so that E'B meets F'C at P' on AD. Then by Lemma 2 if follows that $E'F' \geq EF$. By Lemma 1, $AB + CD \leq 2$. Because $EF \leq$ diameter, we have the required result

$$AB + CD + EF \leq 4 \quad .$$

BINOMIAL IDENTITY NUMBER THIRTEEN

J.B. Parker

$$\sum_{r=0}^{m} (-1)^r \frac{(m + n - r)!}{r!(m - r)!(n - r)!} = 1 \qquad \text{for } m \leq n.$$

ROLL A BALL (JCMN 27, p. 3047)

J.M. Hammersley

A ball rolls on a plane, the instantaneous axis of rotation being always in the plane. How should one choose the path on the table of minimum length to give a prescribed re-orientation of the ball?

Let B be a smooth convex body that rolls geodesically on a flat horizontal table T. Suppose that, at every point of the surface of B, the Gaussian curvature K is finite and strictly positive. Then, at any instant, there will be a unique point P_T on the table and a unique point P_B on the surface of B such that P_T and P_B are in contact. As the body rolls, P_T will describe a curve C_T on T, and P_B will describe a curve C_B on the surface of B. Then

(i) C_B and C_T have equal lengths, and

(ii) the geodesic curvature of C_B at any point is equal to the ordinary curvature of C_T at the corresponding point.

Let the initial and final orientations of B be prescribed. Then the initial and final points of C_B are fixed: call them P_B^0 and P_B^1 respectively. Let C_B^0 be the geodesic on B from P_B^0 to P_B^1, let C_B be any other curve on B from P_B^0 to P_B^1, and let Δ be the region on the surface of B between C_B and C_B^0. If we roll B so that P_B describes <u>either</u> C_B <u>or</u> C_B^0, starting with the given initial orientation, the final orientations will differ only by a rotation about the vertical and this difference will be $\iint_\Delta K\,dS$, by virtue of the Gauss-Bonnet theorem. Hence, to get B into its prescribed final orientation, we wish to choose a path C_B of shortest length subject to a prescribed value for $\iint_\Delta K\,dS$. Call this problem 1.

Now consider problem 2, defined as choosing a path C_B of shortest length subject to a prescribed value for $\iint_\Delta dS$. This is a known problem [Weatherburn, <u>Differential Geometry</u> (1927) Vol. 1, p. 122]; and its solution is that C_B shall have constant geodesic curvature. If B is a unit sphere, K = 1; and problems 1 and 2 coincide. Thus, by (i) and (ii) above, the shortest path C_T on the table that will deliver

a unit sphere to a prescribed final orientation is an arc of a circle or a straight line.

When B is a unit sphere, C_B^0 is an arc of a great circle, and C_B is an arc of a small circle. To show that the length of the minimal C_B never exceeds $\pi\sqrt{3}$, we have to calculate the area of Δ. This is a bit messy, but not too difficult.

OF THE EARTH MURPHY (JCMN 27, p. 3055)

J.B. Parker

Recall the theory of Thue sequences on two symbols (JCMN 20, p. 51). Take the symbols as +1 and -1, or as + and - for brevity. There are three ways of constructing the Thue sequence.

(a) The finite sequences

+
+ -
+ - - +
+ - - + - + + - etc.

may be constructed by writing below each row a copy followed by the result of changing all the signs. There is an infinite Thue sequence such that each of the finite sequences is an initial segment.

(b) The same finite sequences may be constructed by putting + - beneath each + and - + beneath each -.

(c) The infinite sequence s(1), s(2), ... may be defined by the conditions that s(1) = 1 and s(n) = s(2n - 1) = -s(2n).

The equivalence of the three methods is easily shown. The moment of order m for the sequence of length N is $M(m) = \sum_{r=1}^{N} r^m\, s(r)$.

<u>Theorem</u> For the Thue sequence defined above of length N the moment of order m is zero for $N = 2^{m+1}$ or 2^{m+2} or ... .

<u>Proof</u> Use induction on m. The result is clear for m = 0 or 1. Now suppose that the moments of order 0, 1, ..., m - 1 are all zero for $N = 2^{m+1}$ or 2^{m+2} or ... . Investigate the moment M(m) of order m. Note that $(2r - 1)^m = (2r)^m + f(r)$ where f is a polynomial of degree m - 1.

$$M(m) = \sum_{r=1}^{N} r^m s(r) = \text{sum over odd } r + \text{sum over even } r.$$

$$= \sum_{r=1}^{N/2} s(2r - 1)(2r - 1)^m + s(2r)(2r)^m$$

$$= \sum_{r=1}^{N/2} (s(2r - 1) + s(2r))(2r)^m + s(2r - 1)f(r).$$

Since s(2r - 1) + s(2r) = 0 (see (c) above) and

$$\sum s(2r - 1)f(r) = \sum s(r)f'(r) = 0$$

by the induction hypothesis, it follows that M(m) = 0 and the theorem is proved.

The problem of page 3055 above was for any m to find the length n(m) of the shortest possible sequence $s(r) = \pm 1$, r = 1, 2,...,n(m) such that the moments of order 0, 1, 2, ..., m are all zero. What we have shown is that $n(m) \leq 2^{m+1}$. Can this be improved to equality? It can in the cases m = 0, 1, 2, see "The Rig of a Rowing Boat" (JCMN 26, pp. 3037-3040). To have M(0) = M(1) = 0 it is necessary that N be a multiple of 4, (for $\sum 1$ and $\sum r$ must be even), but from M(0) = M(1) = M(2) = 0 does it follow that N must be a multiple of 8?

COMMUTING MATRICES (JCMN 24, p. 143)

H. Kestelman

(a) Commuting matrices have a common eigenvector. This result may be proved as follows.

We say that a linear subspace L is "stable for" a matrix M if $M\underline{x}$ is in L for all $\underline{x}$ in L.

<u>Lemma 1</u> If A and B commute then every eigenspace V of A is stable for B.

<u>Proof</u> Take any $\underline{x}$ in V. Let $A\underline{x} = \lambda\underline{x}$. Then $A(B\underline{x}) = B(A\underline{x}) = B(\lambda\underline{x}) = \lambda(B\underline{x})$ and $B\underline{x}$ is also in V.

<u>Lemma 2</u> If a linear subspace W is stable for A, then W contains an eigenvector of A.

<u>Proof</u> Let M be a rectangular $(n \times k)$ matrix whose columns form a basis for W. Then AM is $n \times k$ and has columns all in W, and so there is a $k \times k$ matrix B such that AM = MB. Take any eigenvector $\underline{v}$ of B, with $B\underline{v} = \lambda\underline{v}$, then $M\underline{v} \neq 0$ and $AM\underline{v} = MB\underline{v} = M\lambda\underline{v}$. Therefore $M\underline{v}$ is an eigenvector of A and is in W.

<u>Main Proof</u> Let E be a set of commuting n x n matrices. Some linear subspaces of C^n are stable for all X in E (for example the whole space). Let W be one of these subspaces of minimal (non-zero) dimension. Take any A in E. By Lemma 2, W contains an eigenvector of A, let V be the eigenspace of A for the corresponding eigenvalue. Let U be the intersection of W with V, it has dimension at least one. By Lemma 1, V is stable for every X in E, so is W, therefore

so is U. But W is minimal with this property, and so U = W. Therefore W is a subspace of V, and every non-zero $\underline{x}$ in W is in V and is an eigenvector of A. Since A was arbitrary we have shown that every non-zero element of W is a common eigenvector for all the matrices of E.

(b) Commuting $n \times n$ matrices, if each is diagonalizable, are simultaneously diagonalizable.

Proof Trivial if $n = 1$. Suppose the proposition proved for all $n < k$. Take a set E of commuting $k \times k$ matrices. If all are diagonal there is nothing to prove, therefore suppose that one of them, A, is not diagonal. Let $\lambda_1, \lambda_2, \ldots \lambda_q$ be the distinct eigenvalues of A, note that q is at least 2 because A is not diagonal. For each $r = 1, \ldots q$ there is a rectangular matrix M_r whose columns span the eigenspace corresponding to λ_r. Put $M = (M_1, \ldots M_q)$, it is $k \times k$ and non-singular because its columns span C^k. Also $AM_r = \lambda_r M_r$ for all r. Take any $r = 1, 2, \ldots q$ and any X in E.

$$A(XM_r) = X(AM_r) = X(\lambda_r M_r) = \lambda_r (XM_r)$$

and the columns of XM_r are in the eigenspace of A corresponding to λ_r and so are spanned by the columns of M_r. Therefore $XM_r = M_r X_r^*$ for some square matrix X_r^*, and

$$XM = M \operatorname{diag}(X_1^*, \ldots X_q^*)$$

and $$M^{-1}XM = \operatorname{diag}(X_1^*, \ldots X_q^*).$$

Now take any other member Y of E, and define the Y_r^* similarly. Since X and Y commute, so do X_r^* and Y_r^*. Since $q \geq 2$, each X_r^* is smaller than $k \times k$ and so the induction hypothesis can be applied to it. Because X is diagonalizable there is a polynomial without multiple roots that annihilates X, the same polynomial will annihilate X_r^*, and so X_r^* is diagonalizable. Apply the induction hypothesis to each set $E_r^* = \{X_r^*;\ X \text{ in } E\}$. There is N_r such that, for all X in E, $X_r^* = N_r^{-1} X_r^{**} N_r$, with X_r^{**} diagonal. Finally put $N = \operatorname{diag}(N_1, \ldots N_q)$, then for all X in E we have:

$$\begin{aligned} N^{-1}M^{-1}XMN &= N^{-1}\operatorname{diag}(X_1^*, \ldots X_q^*)N \\ &= \operatorname{diag}(N_1^{-1}X_1^*N_1, \ldots N_q^{-1}X_q^*N_q) = \operatorname{diag}(X_1^{**}, \ldots X_q^{**}) \end{aligned}$$

(c) New Problem Given a set E of commuting matrices, is there T such that, for all X in E, $T^{-1} X T$ is upper triangular?

QUOTATION CORNER (14)

APOLOGY. The Australian National Rail advertisement which appeared in Saturday's Advertiser headed: "Fly to Perth for $80 without leaving the ground" was incorrect. It should have read "Fly to Perth for $90 without leaving the ground". Departures are FROM Perth every Monday and Wednesday, not Tuesday and Wednesday as reported. Pensioners, students and children one-way fare is $64, not $56. We sincerely apologise for any inconvenience caused.

– From the Adelaide Advertiser newspaper.

HIGHER DIMENSIONAL ROTATIONS (JCMN 27, p. 3047)

Terry Gagen

This problem of Kestelman was to prove that every real orthogonal matrix is either involutary (equal to its own inverse) or the product of two involutary matrices.

Every real orthogonal matrix is orthogonally equivalent (in the sense $B = P\,A\,P^{-1}$ with P orthogonal) to a matrix of the form

$$\begin{pmatrix} \pm 1 & & & & & \\ & \pm 1 & & & & \\ & & \ddots & & & \bigcirc \\ & & & \boxed{} & & \\ \bigcirc & & & & \boxed{} & \\ & & & & & \ddots \end{pmatrix}$$

where the 2×2 blocks are of the form $u = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Consequently it suffices to prove that u can be written as a product of two involutions. In fact

$$u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

Alternatively $$u = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ -\sin\theta/2 & -\cos\theta/2 \end{pmatrix} \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ \sin\theta/2 & -\cos\theta/2 \end{pmatrix}$$

CAPTAIN COOK'S MOUNTAINS

G. Szekeres

Some years ago I told you about our visit to two of Captain Cook's mountains, Mt. Warning and Pigeon House. We have now added a third one to our collection, Mt. Dromedary on the South Coast of N.S.W. According to a pamphlet provided by the Forestry Department, Captain Cook made the following entry in his journal on the 21st April, 1770:

> "We were abreast of a pretty high mountain laying near the shore which on account of its figure I named Mt. Dromedary."

Actually the mountain is not all that high, barely 800 metres, but it dominates the landscape for quite a stretch as you drive South towards Bega. The last kilometre of the track to the top is particularly enjoyable as it is mercifully free from motorcyclists who cannot (or are not allowed to) negotiate this last bit. Fortunately it was also spared by the early gold-miners who during a brief bonanza cleared most of the rainforests covering the slopes. The top is a snake-lover's delight, particularly if you are fond of the red-bellied black variety.

We were wondering how many more of the coastal mountains were named by Captain Cook in such a charming fashion - we stumbled on this one by accident.

SOCIAL MECHANICS

Theoretical science makes progress by setting up models and comparing them with observations of the real world. The "Worked Examples in Social Mechanics" (JCMN 27, p. 3057), dealing with how industry meets its need for capital, might be supplemented by the following story recently printed in Australian newspapers. A business in Western Australia wanted to develop and build a prototype for a new variety of bicycle to be built in China. An engineering firm would do the job for $40,000 but the metalwork department of a State High School took it on for the cost ($8,000) of materials plus royalties on future sales and are expecting to get a million dollars. A spokesman for the W.A. Department of Education has said that all the money would go to the school, not to the State Government. The principle is not new, Engineering departments in universities have been doing this sort of thing for years, but what will be the effect if such activity becomes widespread?

RULER AND COMPASSES

C.F. Moppert

Two intersecting lines, a point P on one of them, and a length c are given. Is there a Euclidean construction for points A on one line and B on the other so that AP = BP and AB = c?

REAL FUNCTION THEORY (JCMN 26, p. 3035)

H. Kestelman

If the continuous real function $f(x)$ has derivative zero for all irrational x, does it follow that f is a constant? YES.

<u>Lemma</u> Suppose that on the closed interval $[a, b]$ there is a continuous real function $\phi(x)$, and that E is a subset with the two properties that $\phi(E)$ has no interior points and that to any x not in E corresponds $y > x$ with $\phi(y) \geq \phi(x)$. Then $\phi(b)$ is the maximum of ϕ in the interval.

<u>Proof</u> Suppose the result false. There is c in the interval such that $\phi(c) > \phi(b)$. Because $\phi(E)$ does not contain the interval $(\phi(b), \phi(c))$ there is g not in $\phi(E)$ such that $\phi(b) < g < \phi(c)$.

Take w to be the largest x for which $\phi(x) \geq g$ (because ϕ is continuous this w exists and $\phi(w) = g$). Since $\phi(w) = g$ is not in $\phi(E)$ it follows that w is not in E. By hypothesis there is y such that $y > w$ and $\phi(y) \geq \phi(w) = g$. This contradicts the definition of w and so the lemma is proved.

<u>Theorem</u> If f is continuous on an interval and $f'(x) = 0$ except on a countable set E, then f is constant.

<u>Proof</u> Take any $a < b$ in the interval and take any $\varepsilon > 0$. Consider $\phi(x) = f(x) + \varepsilon x$ in the interval $[a, b]$. Let E* consist of the part of E in $[a, b]$ together with b. For any x not in E*, since $\phi'(x) = \varepsilon > 0$, there is $y > x$ with $y < b$ and $\phi(y) \geq \phi(x)$. Since ϕ is

continuous and $\phi(E^*)$, being countable, has no interior, the lemma applies, and $\phi(b) \geq \phi(a)$. Since $f(b) - f(a) \geq \varepsilon(b-a)$ for all $\varepsilon > 0$, it follows that $f(b) \geq f(a)$. Similarly (considering $-f(x)$ instead of $f(x)$) $f(b) \leq f(a)$, and we conclude that $f(b) = f(a)$.

This applies for any $a < b$ in the original interval, and so $f(x)$ is constant.

PHILATELIC NUMBER THEORY

C. Silverbach

A little while ago I had a letter from London with a 29 pence stamp, and wondered if this were the largest prime number to occur as the face value of a stamp. Then today an air letter came bearing stamps of denomination 4, 4½ and 15½ pence. If the British Government printed only stamps of value $p/2$ pence where p is prime, would that make it possible to send any letter with no more than two stamps?

CASE FOR CONTROL THEORY

We are indebted to P.F. Brownell for the following observation.

A pamphlet handed out by the Australia and New Zealand Bank urges members of the public to look in to any bank window as they walk along the street, so that if a robbery is taking place they can inform the police. The pamphlet goes on to add that it would also be desirable to report anyone looking suspiciously through a bank window.

ADDITION PROBLEM

J.B. Parker

C.J. Smyth's "Easy Algebra" in JCMN 16 suggests the following:

Find the smallest set of numbers, non-zero and of unequal modulus, such that each is the sum of two (distinct) other members of the set.

FAREWELL TO SECRETARY

Best wishes from everybody to Michele Askin who has left the position of Secretary.

BOUND VOLUMES

The reprint of Volume 1 has just now (7th April) been finished, and proved unexpectedly expensive, so that now we must charge \$10 (Australian) for it, but Volume 2 (Issues 18-24) still sells for \$5. For customers in Australia the preferred method of payment is a cheque payable to James Cook University.

MAILING LIST

One of the difficulties in our present system of free distribution is that those who do not want JCMN seldom write and tell the editor. Likewise those who have died, and those who have changed their addresses. Consequently in order to keep down our postage expenses we have to try to enumerate the complementary set, those who are interested. Our thanks go to the readers who have so far responded to the appeal about this in JCMN 27; as some of you get your copies by sea mail, there may well be more answers to come. We can assume that all our contributors are interested, but if you have not communicated with the editor in the last two years, and you want to stay on the mailing list, do write.

Your editor would like to hear from you anything connected with mathematics or with Capt. James Cook. We hope for contributions to be either significant or entertaining, preferably both.

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